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# Radial extensions in fractional Sobolev spaces

Haim Brezis<sup>(1),(2)</sup>, Petru Mironescu<sup>(3)</sup><sub>fi</sub>, Itai Shafrir<sup>(4)</sup>

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## Abstract

Given  $f : \partial(-1, 1)^n \rightarrow \mathbb{R}$ , consider its radial extension  $Tf(X) := f(X/\|X\|_\infty)$ ,  $\forall X \in [-1, 1]^n \setminus \{0\}$ . In “On some questions of topology for  $S^1$ -valued fractional Sobolev spaces” (RACSAM 2001), the first two authors (HB and PM) stated the following auxiliary result (Lemma D.1). If  $0 < s < 1$ ,  $1 < p < \infty$  and  $n \geq 2$  are such that  $1 < sp < n$ , then  $f \mapsto Tf$  is a bounded linear operator from  $W^{s,p}(\partial(-1, 1)^n)$  into  $W^{s,p}((-1, 1)^n)$ . The proof of this result contained a flaw detected by the third author (IS). We present a correct proof. We also establish a variant of this result involving higher order derivatives and more general radial extension operators. More specifically, let  $B$  be the unit ball for the standard Euclidean norm  $|\cdot|$  in  $\mathbb{R}^n$ , and set  $U_a f(X) := |X|^a f(X/|X|)$ ,  $\forall X \in \overline{B} \setminus \{0\}$ ,  $\forall f : \partial B \rightarrow \mathbb{R}$ . Let  $a \in \mathbb{R}$ ,  $s > 0$ ,  $1 \leq p < \infty$  and  $n \geq 2$  be such that  $(s-a)p < n$ . Then  $f \mapsto U_a f$  is a bounded linear operator from  $W^{s,p}(\partial B)$  into  $W^{s,p}(B)$ .

In [1], the first two authors stated the following

**Lemma 1.** ([1, Lemma D.1]) Let  $0 < s < 1$ ,  $1 < p < \infty$  and  $n \geq 2$  be such that  $1 < sp < n$ . Let

$$Q := (-1, 1)^n. \tag{1}$$

Set

$$Tf(X) := f(X/\|X\|_\infty), \forall X \in \overline{Q} \setminus \{0\}, \forall f : \partial Q \rightarrow \mathbb{R}; \tag{2}$$

here,  $\|\cdot\|_\infty$  is the sup norm in  $\mathbb{R}^n$ . Then  $f \mapsto Tf$  is a bounded linear operator from  $W^{s,p}(\partial Q)$  into  $W^{s,p}(Q)$ .

The argument presented in [1] does not imply the conclusion of Lemma 1. Indeed, it is established in [1] (see estimate (D.3) there) that

$$|Tf|_{W^{s,p}(Q)}^p \leq C \int_{\partial Q} \int_{\partial Q} \frac{|f(x) - f(y)|^p}{\|x - y\|_\infty^{n+sp}} d\sigma(x) d\sigma(y).$$

However, this does not imply the desired conclusion in Lemma 1, for which we need the stronger estimate

$$|Tf|_{W^{s,p}(Q)}^p \leq C \int_{\partial Q} \int_{\partial Q} \frac{|f(x) - f(y)|^p}{\|x - y\|_\infty^{n-1+sp}} d\sigma(x) d\sigma(y).$$

In what follows, we establish the following slight generalization of Lemma 1.

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**Lemma 2.** Let  $0 < s \leq 1$ ,  $1 \leq p < \infty$  and  $n \geq 2$  be such that  $sp < n$ . Let  $Q, T$  be as in (1)–(2). Then  $f \mapsto Tf$  is a bounded linear operator from  $W^{s,p}(\partial Q)$  into  $W^{s,p}(Q)$ .

Lemma 2 can be generalized beyond one derivative, but for this purpose it is necessary to work on unit spheres arising from norms smoother than  $\|\cdot\|_\infty$ . We consider for example maps  $f : \partial B \rightarrow \mathbb{R}$ , with

$$B := \text{the Euclidean unit ball in } \mathbb{R}^n. \quad (3)$$

For  $a \in \mathbb{R}$ , set

$$U_a f(X) := |X|^a f(X/|X|), \quad \forall X \in \overline{B} \setminus \{0\}, \quad \forall f : \partial B \rightarrow \mathbb{R}; \quad (4)$$

here,  $\|\cdot\|$  is the standard Euclidean norm in  $\mathbb{R}^n$ .

We will prove the following

**Lemma 3.** Let  $a \in \mathbb{R}$ ,  $s > 0$ ,  $1 \leq p < \infty$  and  $n \geq 2$  be such that  $(s-a)p < n$ . Then  $f \mapsto U_a f$  is a bounded linear operator from  $W^{s,p}(\partial B)$  into  $W^{s,p}(B)$ .

It is possible to establish directly Lemma 2 by adapting some arguments presented in Step 3 in the proof of Lemma 4.1 in [2]. However, we will derive it from Lemma 3.

*Proof of Lemma 2 using Lemma 3* Let

$$\Phi : \mathbb{R}^n \rightarrow \mathbb{R}^n, \quad \Phi(X) := \begin{cases} \frac{|X|}{\|X\|_\infty} X, & \text{if } X \neq 0 \\ 0, & \text{if } X = 0 \end{cases}, \quad \Lambda := \Phi|_{\overline{B}} \text{ and } \Psi := \Phi|_{\partial B}.$$

Clearly,

$$\Lambda : \overline{B} \rightarrow \overline{Q}, \quad \Psi : \partial B \rightarrow \partial Q \text{ are bi-Lipschitz homeomorphisms} \quad (5)$$

and

$$Tf = [U_0(f \circ \Psi)] \circ \Lambda^{-1}. \quad (6)$$

Using (5) and the fact that  $0 < s \leq 1$ , we find that

$$f \mapsto f \circ \Psi \text{ is a bounded linear operator from } W^{s,p}(\partial Q) \text{ into } W^{s,p}(\partial B) \quad (7)$$

and

$$g \mapsto g \circ \Lambda^{-1} \text{ is a bounded linear operator from } W^{s,p}(B) \text{ into } W^{s,p}(Q). \quad (8)$$

We obtain Lemma 2 from (6)–(8) and Lemma 3 (with  $a = 0$ ). The same argument shows that the conclusion of Lemma 2 holds for the unit sphere and ball of any norm in  $\mathbb{R}^n$ .  $\square$

*Proof of Lemma 3* Consider  $a, s, p$  and  $n$  such that

$$a \in \mathbb{R}, \quad s > 0, \quad 1 \leq p < \infty, \quad n \geq 2 \text{ and } (s-a)p < n. \quad (9)$$

Considering spherical coordinates on  $B$ , we obtain that

$$\begin{aligned} \|U_a f\|_{L^p(B)}^p &= \int_0^1 \int_{\partial B} r^{n-1} |U_a f(rx)|^p d\sigma(x) dr \\ &= \int_0^1 \int_{\partial B} r^{n-1+a p} |f(x)|^p d\sigma(x) dr = \frac{1}{n+ap} \|f\|_{L^p(\partial B)}^p. \end{aligned} \quad (10)$$

Here, we have used the fact that, by (9), we have  $n + ap > n - (s - a)p > 0$ .

In view of (10), it suffices to establish the estimate

$$|U_a f|_{W^{s,p}(B)}^p \leq C \|f\|_{W^{s,p}(\partial B)}^p, \quad \forall f \in W^{s,p}(\partial B), \quad (11)$$

for some appropriate  $C = C_{a,s,p,n}$  and semi-norm  $|\cdot|_{W^{s,p}}$  on  $W^{s,p}(B)$ .

*Step 1. Proof of (11) when  $0 < s < 1$ .* We consider the standard Gagliardo semi-norm on  $W^{s,p}(B)$ . We have

$$\begin{aligned} |U_a f|_{W^{s,p}(B)}^p &= \int_B \int_B \frac{|U_a f(X) - U_a f(Y)|^p}{|X - Y|^{n+sp}} dX dY \\ &= \int_0^1 \int_0^1 \int_{\partial B} \int_{\partial B} r^{n-1} \rho^{n-1} \frac{|U_a f(rx) - U_a f(\rho y)|^p}{|rx - \rho y|^{n+sp}} d\sigma(x) d\sigma(y) dr d\rho \\ &= \int_0^1 \int_0^1 \int_{\partial B} \int_{\partial B} r^{n-1} \rho^{n-1} \frac{|r^\alpha f(x) - \rho^\alpha f(y)|^p}{|rx - \rho y|^{n+sp}} d\sigma(x) d\sigma(y) dr d\rho \\ &= 2 \int_{\partial B} \int_{\partial B} \int_0^1 \int_0^r r^{n-1} \rho^{n-1} \frac{|r^\alpha f(x) - \rho^\alpha f(y)|^p}{|rx - \rho y|^{n+sp}} d\rho dr d\sigma(x) d\sigma(y). \end{aligned}$$

With the change of variable  $\rho = tr$ ,  $t \in [0, 1]$ , we find that

$$\begin{aligned} |U_a f|_{W^{s,p}(B)}^p &= 2 \int_0^1 r^{n-(s-a)p-1} dr \int_{\partial B} \int_{\partial B} \int_0^1 t^{n-1} \frac{|f(x) - t^\alpha f(y)|^p}{|x - ty|^{n+sp}} dt d\sigma(x) d\sigma(y) \\ &= \frac{2}{n - (s-a)p} \int_{\partial B} \int_{\partial B} \int_0^1 k(x, y, t) dt d\sigma(x) d\sigma(y), \end{aligned}$$

with

$$k(x, y, t) := t^{n-1} \frac{|f(x) - t^\alpha f(y)|^p}{|x - ty|^{n+sp}}, \quad \forall x, y \in \partial B, \quad \forall t \in [0, 1].$$

In order to complete this step, it thus suffices to establish the estimates

$$I_1 := \int_{\partial B} \int_{\partial B} \int_0^{1/2} k(x, y, t) dt d\sigma(x) d\sigma(y) \leq C \|f\|_{L^p(\partial B)}^p, \quad (12)$$

$$I_2 := \int_{\partial B} \int_{\partial B} \int_{1/2}^1 \frac{|f(x) - f(y)|^p}{|x - ty|^{n+sp}} dt d\sigma(x) d\sigma(y) \leq C \|f\|_{W^{s,p}(\partial B)}^p, \quad (13)$$

$$I_3 := \int_{\partial B} \int_{\partial B} \int_{1/2}^1 \frac{|(1 - t^\alpha) f(y)|^p}{|x - ty|^{n+sp}} dt d\sigma(x) d\sigma(y) \leq C \|f\|_{L^p(\partial B)}^p; \quad (14)$$

here,  $|\cdot|_{W^{s,p}(\partial B)}$  is the standard Gagliardo semi-norm on  $\partial B$ .

In the above and in what follows,  $C$  denotes a generic finite positive constant independent of  $f$ , whose value may change with different occurrences.

Using the obvious inequalities

$$\begin{aligned} |x - ty| &\geq 1 - t \geq 1/2, \quad \forall x, y \in \partial B, \quad \forall t \in [0, 1/2], \\ |f(x) - t^\alpha f(y)| &\leq (1 + t^\alpha)(|f(x)| + |f(y)|), \end{aligned}$$

and the fact that, by (9), we have  $n + ap > 0$ , we find that

$$I_1 \leq C \int_0^{1/2} (t^{n-1} + t^{n-1+ap}) dt \|f\|_{L^p(\partial B)}^p \leq C \|f\|_{L^p(\partial B)}^p,$$

so that (12) holds.

In order to obtain (13), it suffices to establish the estimate

$$\int_{1/2}^1 \frac{1}{|x-ty|^{n+sp}} dt \leq \frac{C}{|x-y|^{n-1+sp}}, \quad \forall x, y \in \partial B. \quad (15)$$

Set  $A := \langle x, y \rangle \in [-1, 1]$ . If  $A \leq 0$ , then  $|x-ty| \geq 1$ ,  $\forall t \in [1/2, 1]$ , and then (15) is clear. Assuming  $A \geq 0$  we find, using the change of variable  $t = A + (1-A^2)^{1/2} \tau$ ,

$$\begin{aligned} \int_{1/2}^1 \frac{1}{|x-ty|^{n+sp}} dt &\leq \int_{\mathbb{R}} \frac{1}{|x-ty|^{n+sp}} dt = \int_{\mathbb{R}} \frac{1}{(t^2+1-2At)^{(n+sp)/2}} dt \\ &= \frac{1}{(1-A^2)^{(n-1+sp)/2}} \int_{\mathbb{R}} \frac{1}{(\tau^2+1)^{(n+sp)/2}} d\tau \\ &= \frac{C}{(1-A^2)^{(n-1+sp)/2}} \leq \frac{C}{(2-2A)^{(n-1+sp)/2}} = \frac{C}{|x-y|^{n-1+sp}}, \end{aligned}$$

and thus (15) holds again. This completes the proof of (13).

In order to prove (14), we note that

$$|1-t^a|^p \leq C(1-t)^p, \quad \forall t \in [1/2, 1],$$

and that the integral

$$J := \int_{1/2}^1 \int_{\partial B} \frac{(1-t)^p}{|x-ty|^{n+sp}} d\sigma(x) dt$$

does not depend on  $y \in \partial B$ .

By the above, we have

$$I_3 \leq C \int_{1/2}^1 \int_{\partial B} \int_{\partial B} \frac{(1-t)^p |f(y)|^p}{|x-ty|^{n+sp}} d\sigma(x) d\sigma(y) dt = C J \|f\|_{L^p(\partial B)}^p,$$

and thus (14) amounts to proving that  $J < \infty$ . Since  $J$  does not depend on  $y$ , we may assume that  $y = (0, \dots, 0, 1)$ . Expressing  $J$  in spherical coordinates and using the change of variable  $t = 1 - \tau$ ,  $\tau \in [0, 1/2]$ , we find that

$$J = C \int_{1/2}^1 \int_0^\pi \frac{\tau^p \sin^{n-1} \theta}{(\tau^2 + 4(1-\tau) \sin^2 \theta/2)^{(n+sp)/2}} d\theta d\tau.$$

When  $\tau \in [0, 1/2]$  and  $\theta \in [0, \pi]$ , we have

$$\begin{aligned} \frac{\tau^p \sin^{n-1} \theta}{(\tau^2 + 4(1-\tau) \sin^2 \theta/2)^{(n+sp)/2}} &\leq C \frac{\tau^p \sin^{n-1} \theta}{(\tau + \sin \theta/2)^{n+sp}} \leq C \frac{\tau^p \sin^{n-1} \theta/2 \cos \theta/2}{(\tau + \sin \theta/2)^{n+sp}} \\ &\leq C (\tau + \sin \theta/2)^{p-sp-1} \cos \theta/2. \end{aligned}$$

Inserting the last inequality into the formula of  $J$ , we find that

$$\begin{aligned} J &\leq C \int_0^{1/2} \int_0^\pi (\tau + \sin \theta/2)^{p-sp-1} \cos \theta/2 d\theta d\tau \\ &= C \int_0^{1/2} \int_0^1 (\tau + \xi)^{p-sp-1} d\xi d\tau < \infty, \end{aligned}$$

the latter inequality following from  $p - sp > 0$ . This completes the proof of (14) and Step 1.

*Step 2. Proof of (11) when  $s \geq 1$ .* We will reduce the case  $s \geq 1$  to the case  $0 \leq s < 1$ . Using the linearity of  $f \mapsto U_a f$  and a partition of unity, we may assume with no loss of generality that  $\text{supp} f$  is contained in a spherical cap of the form  $\{x \in \partial B; |x - \mathbf{e}| < \varepsilon\}$  for some  $\mathbf{e} \in \partial B$  and sufficiently small  $\varepsilon$ . We may further assume that  $\mathbf{e} = (0, 0, \dots, 0, 1)$ , and thus

$$f \in W^{s,p}(\partial B; \mathbb{R}), \text{supp} f \subset \mathcal{E} := \{x \in \partial B; |x - (0, 0, \dots, 0, 1)| < \varepsilon\}. \quad (16)$$

Let

$$\mathcal{S} := \{x \in \partial B; |x - (0, 0, \dots, 0, 1)| \leq 2\varepsilon\} \text{ and } \mathcal{H} := \mathbb{R}^{n-1} \times \{1\}.$$

Consider the projection  $\Theta$  with vertex 0 of

$$\mathbb{R}_+^n := \{X = (X', X_n) \in \mathbb{R}^{n-1} \times \mathbb{R}; X_n > 0\}$$

onto  $\mathcal{H}$ , given by the formula  $\Theta(X', X_n) = (X'/X_n, 1)$ . The restriction  $\Pi$  of  $\Theta$  to  $\mathcal{S}$  maps  $\mathcal{S}$  onto  $\mathcal{N} := \mathcal{B} \times \{1\}$ , with

$$\mathcal{B} := \{X' \in \mathbb{R}^{n-1}; |X'| \leq r := 2\varepsilon \sqrt{1 - \varepsilon^2}/(1 - 2\varepsilon^2)\},$$

and is a smooth diffeomorphism between these two sets. We choose  $\varepsilon$  such that  $r = 1/2$ , and thus  $\mathcal{B} \subset \{X' \in \mathbb{R}^{n-1}; \|X'\|_\infty \leq 1/2\}$ .

Set

$$g(X') := \begin{cases} |X'|^\alpha f(\Pi^{-1}(X', 1)), & \text{if } X' \in \mathcal{B} \\ 0, & \text{otherwise} \end{cases}. \quad (17)$$

By the above, there exist  $C, C' > 0$  such that for every  $f \in W^{s,p}(\partial B)$  satisfying (16), the function  $g$  defined in (17) satisfies

$$C \|g\|_{W^{s,p}(\mathbb{R}^{n-1})} \leq \|f\|_{W^{s,p}(\partial B)} \leq C' \|g\|_{W^{s,p}(\mathbb{R}^{n-1})}. \quad (18)$$

On the other hand, set  $\mathcal{C} := \{(tY', t); Y' \in \mathcal{B}, t > 0\}$  and

$$V_a g(X', X_n) := \begin{cases} (X_n)^\alpha g(X'/X_n), & \text{if } (X', X_n) \in \mathcal{C} \\ 0, & \text{otherwise} \end{cases}.$$

Then we have  $U_a f(X', X_n) = V_a g(X', X_n)$ ,  $\forall (X', X_n) \in \overline{\mathcal{B}} \setminus \{0\}$ .

Write now  $s = m + \sigma$ , with  $m \in \mathbb{N}$  and  $0 \leq \sigma < 1$ . When  $s = m$ , we consider, on  $W^{s,p}(B)$ , the semi-norm

$$|F|_{W^{s,p}(B)}^p = \sum_{\substack{\alpha \in \mathbb{N}^n \setminus \{0\} \\ |\alpha| \leq m}} \|\partial^\alpha F\|_{L^p(B)}^p. \quad (19)$$

When  $s$  is not an integer, we consider the semi-norm

$$|F|_{W^{s,p}(B)}^p = \sum_{\substack{\alpha \in \mathbb{N}^n \setminus \{0\} \\ |\alpha| \leq m}} \|\partial^\alpha F\|_{L^p(B)}^p + \sum_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha| = m}} |\partial^\alpha F|_{W^{\sigma,p}(B)}^p \quad (20)$$

(the semi-norm on  $W^{\sigma,p}(B)$  is the standard Gagliardo one.)

By the above discussion, in order to obtain (11) it suffices to establish the estimate

$$\|V_a g\|_{W^{s,p}(B)}^p \leq C \|g\|_{W^{s,p}(\mathbb{R}^{n-1})}^p, \quad \forall g \in W^{s,p}(\mathbb{R}^{n-1}) \text{ with } \text{supp } g \subset \mathcal{B}. \quad (21)$$

Let  $\alpha \in \mathbb{N}^n \setminus \{0\}$  be such that  $|\alpha| \leq m$ . By a straightforward induction on  $|\alpha|$ , the distributional derivative  $\partial^\alpha[V_a g]$  satisfies

$$\partial^\alpha[V_a g](X', X_n) = \sum_{|\beta'| \leq |\alpha|} V_{a-|\alpha|} [P_{\alpha,\beta'} \partial^{\beta'} g](X', X_n) \text{ in } \mathcal{D}'(B \setminus \{0\}), \quad (22)$$

for some appropriate polynomials  $P_{\alpha,\beta'}(Y')$ ,  $Y' \in \mathbb{R}^{n-1}$ , depending only on  $a \in \mathbb{R}$ ,  $\alpha \in \mathbb{N}^n$  and  $\beta' \in \mathbb{N}^{n-1}$ .

Thanks to the fact that  $g(X'/X_n) = 0$  when  $(X', X_n) \notin \mathcal{C}$ , we find that for any such  $\alpha$  we have

$$\begin{aligned} \int_B |\partial^\alpha[V_a g]|^p dx &\leq C \sum_{|\beta'| \leq |\alpha|} \int_{\mathcal{C} \cap \mathcal{Q}} (X_n)^{(a-|\alpha|)p} |\partial^{\beta'} g(X'/X_n)|^p dX' dX_n \\ &= \frac{C}{n + (a - |\alpha|)p} \sum_{|\beta'| \leq |\alpha|} \int_{\mathcal{B}} |\partial^{\beta'} g(Y')|^p dY'. \end{aligned} \quad (23)$$

Here, we rely on

$$\int_0^1 (X_n)^{n-1+(a-|\alpha|)p} dX_n = \frac{1}{n + (a - |\alpha|)p} < \infty,$$

thanks to the assumption (9), which implies that  $(|\alpha| - a)p < n$ .

Using (23), the fact that  $V_a g \in W_{loc}^{m,p}(B \setminus \{0\})$  and the assumption that  $n \geq 2$ , we find that the equality (22) holds also in  $\mathcal{D}'(B)$ , that  $V_a g \in W^{m,p}(B)$  and that

$$\|V_a g\|_{W^{m,p}(B)}^p \leq C \|g\|_{W^{m,p}(\mathbb{R}^{n-1})}^p, \quad \forall g \in W^{m,p}(\mathbb{R}^{n-1}) \text{ with } \text{supp } g \subset \mathcal{B}. \quad (24)$$

In particular, (21) holds when  $s$  is an integer.

Assume next that  $s$  is not an integer. In view of (18), (22) and (24), estimate (21) will be a consequence of

$$\begin{aligned} \|V_b[Ph]\|_{W^{\sigma,p}(B)}^p &\leq C \|h\|_{W^{\sigma,p}(\mathbb{R}^{n-1})}^p, \quad \forall h \in W^{\sigma,p}(\mathbb{R}^{n-1}) \\ &\text{with } \text{supp } h \subset \mathcal{B}, \end{aligned} \quad (25)$$

under the assumptions

$$0 < \sigma < 1, \quad 1 \leq p < \infty, \quad n \geq 2, \quad (\sigma - b)p < n \quad (26)$$

and

$$P \in C^\infty(\mathbb{R}^{n-1}). \quad (27)$$

(Estimate (25) is applied with  $b := a - m$ ,  $P := P_{\alpha,\beta'}$  and  $h := \partial^{\beta'} g$ .)

In turn, estimate (25) follows from Step 1. Indeed, consider  $k : \partial B \rightarrow \mathbb{R}$  such that  $\text{supp} k \subset \mathcal{B}$  and  $U_b k = V_b[Ph]$ . (The explicit formula of  $k$  can be obtained by “inverting” the formula (17).) By Step 1 and (18), we have

$$\begin{aligned} |V_b[Ph]|_{W^{s,p}(B)}^p &= |U_b k|_{W^{s,p}(B)}^p \leq C \|k\|_{W^{s,p}(\partial B)}^p \leq C \|Ph\|_{W^{s,p}(\mathbb{R}^{n-1})}^p \\ &\leq C \|h\|_{W^{s,p}(\mathbb{R}^{n-1})}^p. \end{aligned}$$

This completes Step 2 and the proof of Lemma 3.  $\square$

Finally, we note that the assumptions of Lemma 3 are optimal in order to obtain that  $U_a f \in W^{s,p}(B)$ .

**Lemma 4.** Let  $a \in \mathbb{R}$ ,  $s > 0$ ,  $1 \leq p < \infty$  and  $n \geq 2$ . Assume that for some measurable function  $f : \partial B \rightarrow \mathbb{R}$  we have  $U_a f \in W^{s,p}(B)$ . Then:

1.  $f \in W^{s,p}(\partial B)$ .
2. If, in addition,  $U_a f$  is not a polynomial, we deduce that  $(s-a)p < n$ .

*Proof.* 1. Let  $G : (1/2, 1) \times \partial B \rightarrow \mathbb{R}$ ,  $G(r, x) := r^{-a} U_a f(rx)$ . If  $U_a f \in W^{s,p}(B)$ , then  $G \in W^{s,p}((1/2, 1) \times \partial B)$ . In particular, we have  $G(r, \cdot) \in W^{s,p}(\partial B)$  for a.e.  $r$ . Noting that  $G(r, x) = f(x)$ , we find that  $f \in W^{s,p}(\partial B)$ .

2. Let

$$\Omega_j := \{X \in \mathbb{R}^n; 2^{-j-1} < |X| < 2^{-j}\}, j \in \mathbb{N}.$$

We consider on each  $\Omega_j$  a semi-norm as in (19)–(20). Assuming that  $U_a f$  is not a polynomial, we have  $|U_a f|_{W^{s,p}(\Omega_0)} > 0$ . By scaling and the homogeneity of  $U_a f$ , we have

$$|U_a f|_{W^{s,p}(\Omega_j)}^p = 2^{j[(s-a)p-n]} |U_a f|_{W^{s,p}(\Omega_0)}^p.$$

Assuming that  $U_a f \in W^{s,p}(B)$ , we find that

$$\infty > |U_a f|_{W^{s,p}(B)}^p \geq \sum_{j \geq 0} |U_a f|_{W^{s,p}(\Omega_j)}^p = \sum_{j \geq 0} 2^{j[(s-a)p-n]} |U_a f|_{W^{s,p}(\Omega_0)}^p > 0,$$

so that  $(s-a)p < n$ .  $\square$   $\square$

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