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THE BOUNDARY OF RANDOM PLANAR MAPS VIA LOOPTREES

by

Igor Kortchemski & Loïc Richier

Abstract. — We study the scaling limits of the boundary of Boltzmann planar maps conditioned on having a large perimeter. We first deal with the non-generic critical regime, where the degree of a typical face falls within the domain of attraction of a stable law with parameter $\alpha \in (1, 2)$. In the so-called dense phase $\alpha \in (1, 3/2)$, it was established in [Ric18] that the scaling limit of the boundary is a stable looptree. In this work, we complete the picture by proving that in the dilute phase $\alpha \in (3/2, 2)$ (as well as in the generic critical regime), the scaling limit is a multiple of the unit circle. This establishes the first evidence of a phase transition for the topology of the boundary: in the dense phase, large faces are self-intersecting while in the dilute phase, they are self-avoiding. The subcritical regime is also investigated. In this case, we show that the scaling limit of the boundary is a multiple of the Brownian CRT instead.

The strategy consists in studying scaling limits of looptrees associated with specific Bienaymé–Galton–Watson trees. In the first case, it relies on an invariance principle for random walks with negative drift, which is of independent interest. In the second case, we obtain the more general result that the Brownian CRT is the scaling limit of looptrees associated with BGW trees whose offspring distribution is critical and in the domain of attraction of a Gaussian distribution, confirming thereby a prediction of [CK14].

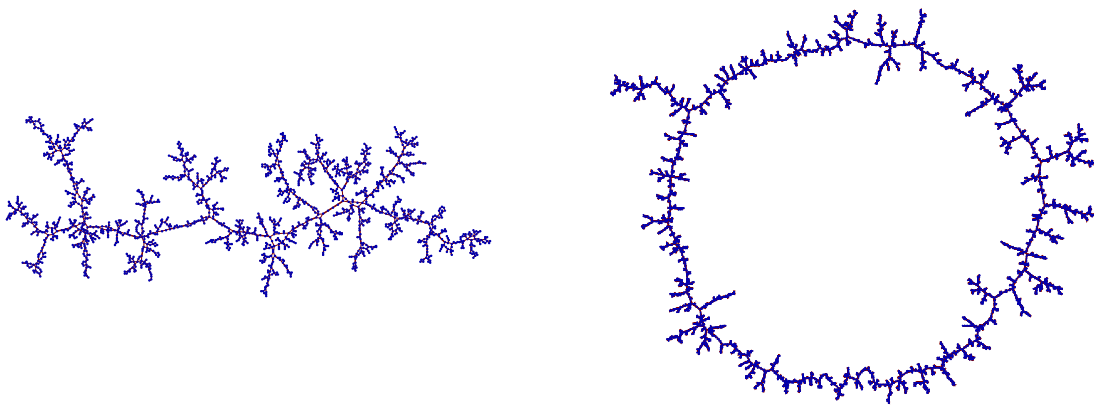


Figure 1. The boundary of a Boltzmann planar map conditioned on having a large perimeter in the subcritical regime (left) and the dilute regime (right).

1. Introduction

Motivation. — This work deals with scaling limits (in the Gromov–Hausdorff sense) of the boundary of bipartite Boltzmann planar maps whose perimeter goes to infinity.

The Boltzmann measures are defined out of a weight sequence $\mathbf{q} = (q_1, q_2, \dots)$ of nonnegative real numbers assigned to the faces of the maps. Precisely, the Boltzmann weight of a bipartite planar map \mathbf{m} (that is, with faces of even degree) is given by

$$w_{\mathbf{q}}(\mathbf{m}) := \prod_{f \in \text{Faces}(\mathbf{m})} q_{\deg(f)/2}.$$

The sequence \mathbf{q} is called *admissible* when these weights form a finite measure on the set of rooted bipartite maps (i.e. with a distinguished oriented edge called the *root edge*). The resulting probability measure $\mathbb{P}_{\mathbf{q}}$ is the Boltzmann measure with weight sequence \mathbf{q} . We say that \mathbf{q} is *critical* if the expected squared number of vertices of a map is infinite under $\mathbb{P}_{\mathbf{q}}$, and *subcritical* otherwise (see Section 5.2.1 for precise definitions).

The scaling limits of Boltzmann bipartite maps conditioned to have a large number of faces have been actively studied. The first milestone was the proof by Le Gall [LG07] of the subsequential convergence of uniform quadrangulations. Later on, Le Gall [LG13] and Miermont [Mie13] identified the limit as the *Brownian map*, a random compact metric space previously introduced by Marckert & Mokkadem [MM06]. his result has been extended to critical sequences \mathbf{q} such that the degree of a typical face has small exponential moments [MM07, LG13], and even finite variance [Mar18] (we say that \mathbf{q} is *generic critical*). A different scaling limit appears when we assume that the critical sequence \mathbf{q} is such that the degree of a typical face is in the domain of attraction of a stable law with parameter $\alpha \in (1, 2)$ (we say that \mathbf{q} is *non-generic critical* with parameter α). Under slightly stronger assumptions, Le Gall and Miermont [LGM11] proved the subsequential convergence of such Boltzmann maps. There is a natural candidate for the limiting compact metric space, called the *stable map* with parameter α . The geometry of the stable maps is dictated by large faces that remain present in the scaling limit. The behaviour of these faces is believed to differ in the dense phase $\alpha \in (1, 3/2)$, where they are supposed to be self-intersecting, and in the dilute phase $\alpha \in (3/2, 2)$, where it is conjectured that they are self-avoiding. Our work is a first step towards this dichotomy.

The strategy initiated in [Ric18] consists in studying Boltzmann maps *with a boundary*. This means that we view the face on the right of the root edge as the boundary $\partial\mathbf{m}$ of the map \mathbf{m} . Consequently, this face receives no weight, and its degree is called the *perimeter* of the map. Then, for every $k \geq 0$, we let \mathcal{M}_k be a Boltzmann map with weight \mathbf{q} conditioned to have perimeter $2k$. The boundary $\partial\mathcal{M}_k$ of this map can be thought of as a typical face of degree $2k$ of a Boltzmann map with weight \mathbf{q} . The main result of [Ric18] deals with the dense regime. It shows that if \mathbf{q} is a non-generic critical weight sequence with parameter $\alpha \in (1, 3/2)$, there exists a slowly varying function L such that in distribution for the Gromov–Hausdorff topology,

$$\frac{L(k)}{(2k)^{\alpha-1/2}} \cdot \partial\mathcal{M}_k \xrightarrow[k \rightarrow \infty]{} \mathcal{L}_{\beta},$$

where \mathcal{L}_β is the *random stable looptree* with parameter $\beta := (\alpha - 1/2)^{-1} \in (1, 2)$, introduced in [CK14]. Recall that a function $L : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is slowly varying (at infinity) if for every $\lambda > 0$ we have $L(\lambda x)/L(x) \rightarrow 1$ as $x \rightarrow \infty$. In this statement, $\partial\mathcal{M}_k$ is viewed as a metric space, and equipped with its graph distance. Moreover, for every $\lambda > 0$ and every metric space (E, d) , $\lambda \cdot E$ stands for $(E, \lambda \cdot d)$. We refer to [BBI01, Chapter 7.3] for details on the Gromov–Hausdorff topology.

The purpose of this work is to investigate the subcritical, dilute and generic critical regimes that were left untouched in [Ric18]. Thanks to the results of [Ric18], this problem boils down to the study of scaling limits of *discrete looptrees*, in two specific regimes that we now discuss. Afterwards, we will present the applications concerning random planar maps.

Random discrete looptrees: scaling limits. — We establish scaling limits of discrete looptrees associated with large conditioned Bienaymé–Galton–Watson trees. By tree, we always mean *plane tree*, that is a rooted ordered tree (with a distinguished corner and an ordering on vertices incident to each vertex). Given a probability measure μ on $\mathbb{Z}_{\geq 0}$, a Bienaymé–Galton–Watson tree with offspring distribution μ (in short, BGW_μ) is a random plane tree in which vertices have a number of offspring distributed according to μ all independently of each other. Details are postponed to Section 2.1.

Following [CK14], with every plane tree τ we associate a planar map denoted by $\text{Loop}(\tau)$ and called looptree. This map has the same set of vertices as τ , and for every vertices $u, v \in \tau$, there is an edge between u and v in $\text{Loop}(\tau)$ if and only if u and v are consecutive children of the same parent in τ , or if v is the first or the last child of u in τ (see Figure 2 for an example). Again, we view $\text{Loop}(\tau)$ as a compact metric space by endowing its vertices with the graph distance.

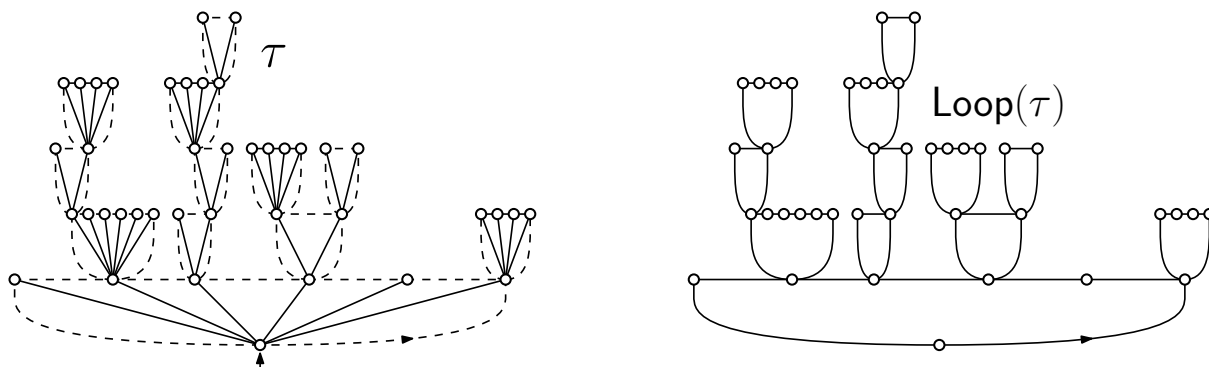


Figure 2. A plane tree τ and its associated looptree $\text{Loop}(\tau)$.

Scaling limits of looptrees (circle regime). — Our first main result deals with looptrees associated to *non-generic subcritical* BGW trees, meaning that the offspring distribution μ is subcritical (i.e., has mean less than 1) and is heavy-tailed.

THEOREM 1. — Let μ be a offspring distribution with mean $m_\mu < 1$. We assume that there exists $\beta > 1$ and a slowly varying function L such that, for every $i \geq 1$,

$$\mu([i, \infty)) = \frac{L(i)}{i^\beta}.$$

Let also J be the real-valued random variable such that $\mathbb{P}(J \geq x) = (\frac{x}{1-m_\mu})^\beta$ for $x \geq 1 - m_\mu$. Finally, for every $n \geq 1$, let $\mathcal{T}_{\geq n}$ be a BGW_μ tree conditioned on having at least n vertices. Then the convergence

$$\frac{1}{n} \cdot \text{Loop}(\mathcal{T}_{\geq n}) \xrightarrow[n \rightarrow \infty]{} J \cdot \mathbb{S}_1$$

holds in distribution, where \mathbb{S}_1 is the circle $[0, 1]_{/0 \sim 1}$ of unit length.

This theorem roughly says that in the tree $\mathcal{T}_{\geq n}$, for large n , there is a unique vertex with degree proportional to the total number of vertices. This phenomenon, known as *condensation*, has already been observed under various forms in [JS10, Jan12, Kor15]. One may hope to obtain a stronger result by considering BGW trees conditioned on having a fixed size. The rub is that without additional regularity assumptions on the offspring distribution μ , it is not clear whether the result of Theorem 1 remains true or not. More precisely, if we assume that there exists a slowly varying function L such that for every $i \geq 1$,

$$\mu(i) = \frac{L(i)}{i^\beta},$$

then the result indeed holds for BGW trees conditioned on having a fixed size, as proved in [Kor15]. The strategy of [Kor15] is based on the so-called “one big jump principle” of [AL11], that holds only if μ satisfies a *sub-exponentiality* assumption. However, we do not know if the probability measure involved in our application to random planar maps satisfies this assumption (see Remark 2 in Section 5 for more on this). While our result is weaker than that of [Kor15], our assumptions are more general: roughly speaking, Theorem 1 trades the weaker conditioning for the stronger regularity assumption on μ .

The proof of Theorem 1 is based on an invariance principle for random walks with negative drift (Theorem 7), which extends a result of Durrett [Dur80] and is of independent interest.

Scaling limits of looptrees (CRT regime). — The next result involves Aldous’ *Brownian Continuum Random Tree* (CRT) [Ald93], whose construction we now recall. Let \mathfrak{e} be a normalized Brownian excursion (which can be seen as Brownian motion conditioned to return to 0 at time 1 and to stay positive on $(0, 1)$, see [LG05, Section 2]). Introduce a pseudo-distance on $[0, 1]$ by setting

$$d_{\mathfrak{e}}(s, t) = \mathfrak{e}_s + \mathfrak{e}_t - 2 \min_{s \wedge t \leq u \leq s \vee t} \mathfrak{e}_u, \quad s, t \in [0, 1].$$

We also let

$$s \approx t \quad \text{if and only if} \quad d_{\mathfrak{e}}(s, t) = 0, \quad s, t \in [0, 1].$$

Then, the CRT is the quotient space $\mathcal{T}_{\mathfrak{e}} := [0, 1]_{/\approx}$, equipped with the distance $d_{\mathfrak{e}}$.

We now present our second main result, that deals with looptrees associated to *generic critical* BGW trees, meaning that the offspring distribution μ is critical (i.e., has mean 1) and finite positive variance σ_μ^2 . We also set $\mu(2\mathbb{Z}_+) = \mu(0) + \mu(2) + \mu(4) + \dots$.

THEOREM 2. — *Let μ be an offspring distribution with mean $m_\mu = 1$ and finite positive variance σ_μ^2 . For every $n \geq 1$, let \mathcal{T}_n be a BGW_μ tree conditioned on having n vertices. Then the convergence*

$$\frac{1}{\sqrt{n}} \cdot \text{Loop}(\mathcal{T}_n) \xrightarrow{n \rightarrow \infty} \frac{2}{\sigma_\mu} \cdot \frac{1}{4} (\sigma_\mu^2 + 4 - \mu(2\mathbb{Z}^+)) \cdot \mathcal{T}_e$$

holds in distribution for the Gromov–Hausdorff topology.

We refer to [AM08, Bet15, CHK15, JS15, PSW16, Car16, Stu16, Stu17] for a zoology of random discrete structures which are not trees, but whose scaling limits are the Brownian CRT.

The result of Theorem 2 was already established in [CHK15, Theorem 13] under the existence of $\lambda > 0$ such that $\sum_{k \geq 0} \mu(k)e^{\lambda k} < \infty$. The improvement in Theorem 2 is important in two directions. First, the existence of small exponential moments does not hold *a priori* in our application to random planar maps. Second, it is often challenging to relax an assumption involving a finite exponential moment condition to a finite variance condition: in particular, the proof of Theorem 2 uses different techniques than in [CHK15, Theorem 13], and new ideas. We emphasize that until now, convergence towards the Brownian CRT of similar rescaled discrete weighted tree-like structures has only been obtained under finite exponential moment conditions (see [CHK15, Theorems 1, 13 and 14], [PSW16, Theorem 5.1], [Stu16, Theorem 6.60], and in particular the discussion in [SS, Section 3.3]).

Furthermore, the method is robust. As a prime example, it allows to establish the following result left as an open question in [CK14], saying that the Brownian CRT is the scaling limit of looptrees associated with critical BGW trees whose offspring distribution is in the domain of attraction of a Gaussian distribution, but has *infinite* variance.

THEOREM 3. — *Let μ be an offspring distribution with mean $m_\mu = 1$ and infinite variance. We assume that there exists a slowly varying function L such that*

$$\mu([i, \infty)) = \frac{L(i)}{i^2}, \quad i \geq 1.$$

For every $n \geq 1$, let \mathcal{T}_n be a BGW_μ tree conditioned to have n vertices. Then, there exists an increasing sequence $(B_n : n \geq 1)$ tending to infinity such that the convergence

$$\frac{1}{B_n} \cdot \text{Loop}(\mathcal{T}_n) \xrightarrow{n \rightarrow \infty} \frac{1}{\sqrt{2}} \cdot \mathcal{T}_e$$

holds in distribution for the Gromov–Hausdorff topology.

The scaling sequence $(B_n : n \geq 1)$ is characterized by the following fact: if X_1, X_2, \dots are i.i.d. random variables with distribution μ , $(X_1 + \dots + X_n - n)/B_n$ converges in distribution to $\sqrt{2}$ times a standard Gaussian random variable (we use this normalization to keep the same definition of B_n as in [CK14]; see [BGT89] for background concerning slowly varying functions and [IL71] for background on domains of attraction of stable variables). Actually, there exists a slowly varying function ℓ such that $\ell(n) \rightarrow \infty$ and $B_n = \ell(n)\sqrt{n}$. As we will see in the proof of Theorem 3, the height of \mathcal{T}_n (of order $\frac{n}{B_n}$) is negligible compared to the typical size of loops

in $\text{Loop}(\mathcal{T}_n)$ (of order B_n), so that asymptotically distances in \mathcal{T}_n do not contribute to distances in $\text{Loop}(\mathcal{T}_n)$, in stark contrast with the finite variance case of Theorem 2.

Note that the classification of the scaling limit of looptrees associated with conditioned critical BGW trees whose offspring distribution is in the domain of attraction of a stable distribution of index $\alpha \in (1, 2]$ is now complete: [CK14, Theorem 4.1.] covers the case $\alpha \in (1, 2)$, Theorem 2 covers the case $\alpha = 2$ with finite variance and Theorem 3 covers the case $\alpha = 2$ with infinite variance.

Applications to random planar maps. — Let us now state the applications of Theorems 1 and 2 to the scaling limits of the boundary of Boltzmann planar maps. We start with the dilute and generic critical regimes.

COROLLARY 4. — *Let \mathbf{q} be a critical weight sequence which is either generic, or non-generic with parameter $\alpha \in (3/2, 2)$ (dilute phase). For every $k \geq 0$, let $\mathcal{M}_{\geq k}$ be a Boltzmann map with weight sequence \mathbf{q} conditioned to have perimeter at least $2k$. Then, there exists a non-degenerate random variable $J_{\mathbf{q}}$ such that the convergence*

$$\frac{1}{2k} \cdot \partial \mathcal{M}_{\geq k} \xrightarrow[k \rightarrow \infty]{} J_{\mathbf{q}} \cdot \mathbb{S}_1$$

holds in distribution for the Gromov-Hausdorff topology.

This result is consistent with the conjecture that large faces are self-avoiding in the dilute phase. It is also related to [BM17, Theorem 8], which states that some generic critical Boltzmann maps (called *regular*) conditioned to have large perimeter converge towards the so-called *free Brownian disk*, that has the topology of the unit disk. Of course, the reason why we need to consider maps having perimeter *at least* $2k$ is the same as in Theorem 1 (see Remark 2 in Section 5 for details).

We now deal with the subcritical regime. Intuitively, when conditioning a subcritical Boltzmann map to have a large face, this face "folds" onto itself, forcing it to become tree-like.

COROLLARY 5. — *Let \mathbf{q} be a subcritical weight sequence. For every $k \geq 0$, let \mathcal{M}_k be a Boltzmann map with weight sequence \mathbf{q} conditioned to have perimeter $2k$. Then, there exists a constant $K_{\mathbf{q}} > 0$ such that the convergence*

$$\frac{1}{\sqrt{2k}} \cdot \partial \mathcal{M}_k \xrightarrow[k \rightarrow \infty]{} K_{\mathbf{q}} \cdot \mathcal{T}_{\mathbf{e}}$$

holds in distribution for the Gromov-Hausdorff topology.

This result should be compared with [JS15, Theorem 1.1] (see also [Bet15, Theorem 5]), where the convergence of subcritical Boltzmann maps conditioned to have large volume towards the Brownian CRT is proved. On the one hand, [JS15] deals with the whole map (not only its boundary), but on the other hand, our assumptions are more general (in [JS15], it is assumed beyond subcriticality that typical faces have a heavy-tailed distribution in a quite strong sense).

Together with the results of [Ric18], Corollaries 4 and 5 complete the study of scaling limits of the boundary of Boltzmann maps (see Figure 3 for an illustration). In particular, Corollary 4 together with [Ric18, Theorem 1.1] establish the phase transition in the parameter α for

the topology of large faces in Boltzmann maps, that was only overviewed through local limits in [Ric18, Theorem 1.2], and *via* volume growth exponents in [BC17]. Our results also have an interpretation in terms of large loops in the rigid $\mathcal{O}(n)$ loop model on quadrangulations, by applying the results of [BBG12] (see [Ric18] for more on this).

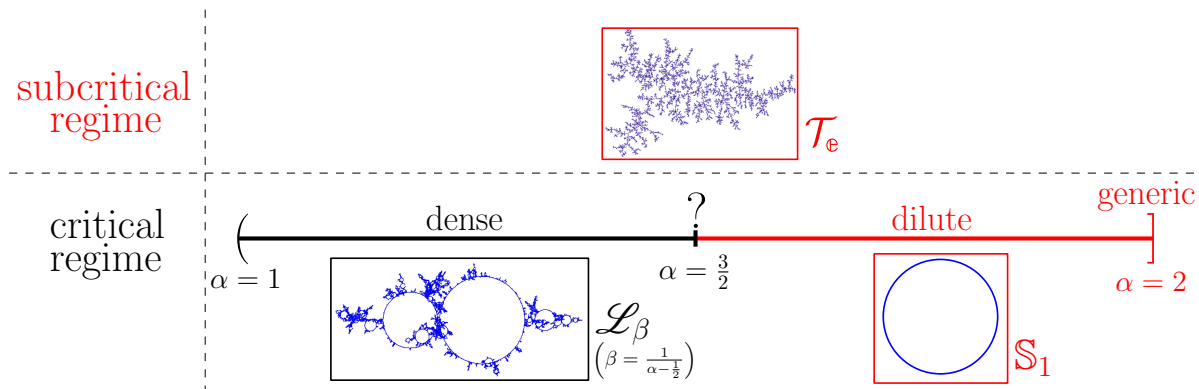


Figure 3. A summary of the scaling limits of the boundary of Boltzmann planar maps. The contributions of this paper are indicated in red. The generic critical weight sequences are usually identified with the parameter $\alpha = 2$, because the distribution of a typical face has finite variance and thus belongs to the domain of attraction of a Gaussian distribution.

REMARK 1 (SPECIAL CASE $\alpha = 3/2$). — This case is still open. As proved in [Ric18], the BGW tree structure describing the boundary of non-generic critical Boltzmann maps with parameter $3/2$ can be either subcritical, or critical. As suggested in [Ric18, Remark 6.3], we expect the scaling limit to be a multiple of a loop in both cases, although the condensation phenomenon could occur at a scale smaller than the total number of vertices in the critical tree setting. This will be investigated in future work.

The paper is organized as follows. In Section 2, we recall definitions and fundamental results about (random) plane trees. Then, Sections 3 and 4 are devoted to the proofs of Theorem 1 and Theorems 2 and 3, respectively. The first is based on a limit theorem for random walks with negative drift, while the others use a spinal decomposition and a tightness argument. Finally, we discuss the applications to random planar maps in Section 5.

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2. Trees

2.1. Plane trees. — We use Neveu’s formalism [Nev86] to define plane trees: let $\mathbb{N} = \{1, 2, \dots\}$ be the set of all positive integers, set $\mathbb{N}^0 = \{\emptyset\}$ and consider the set of labels $\mathbb{U} = \bigcup_{n \geq 0} \mathbb{N}^n$. For $v = (v_1, \dots, v_n) \in \mathbb{U}$, we denote by $|v| = n$ the length of v . If $n \geq 1$, we define $\text{pr}(v) = (v_1, \dots, v_{n-1})$ and for $i \geq 1$, we let $vi = (v_1, \dots, v_n, i)$. More generally, for $w = (w_1, \dots, w_m) \in \mathbb{U}$, we let $vw = (v_1, \dots, v_n, w_1, \dots, w_m) \in \mathbb{U}$ be the concatenation of v and w . We endow \mathbb{U} with the lexicographical order (denoted by \prec): given $v, w \in \mathbb{U}$, if $z \in \mathbb{U}$ is their longest common prefix (so that $v = z(v_1, \dots, v_n)$ and $w = z(w_1, \dots, w_m)$ with $v_1 \neq w_1$), then we have $v \prec w$ if $v_1 < w_1$.

A (locally finite) *plane tree* is a nonempty subset $\tau \subset \mathbb{U}$ such that (i) $\emptyset \in \tau$; (ii) if $v \in \tau$ with $|v| \geq 1$, then $\text{pr}(v) \in \tau$; (iii) if $v \in \tau$, then there exists an integer $k_v(\tau) \geq 0$ such that $vi \in \tau$ if and only if $1 \leq i \leq k_v(\tau)$.

We may view each vertex v of a tree τ as an individual of a population for which τ is the genealogical tree. For $v, w \in \tau$, we let $[[v, w]]$ be the vertices belonging to the shortest path from v to w in τ . Accordingly, we use $[[v, w[$ for the same set, excluding w . The vertex \emptyset is called the *root* of the tree and for every $v \in \tau$, $k_v(\tau)$ is the number of children of v (if $k_v(\tau) = 0$, then v is called a *leaf*), $|v|$ is its *generation*, $\text{pr}(v)$ is its *parent* and more generally, the vertices $v, \text{pr}(v), \text{pr} \circ \text{pr}(v), \dots, \text{pr}^{|v|}(v) = \emptyset$ belonging to $[[\emptyset, v]]$ are its *ancestors*. If τ is a tree and v a vertex of τ , $\theta_v(\tau) = \{vw \in \tau : w \in \mathbb{U}\}$ denotes the tree made of v together with its descendants in τ . We also let $\text{Cut}_v(\tau) = \{v\} \cup \tau \setminus \theta_v(\tau)$ be tree obtained from τ by removing all the (strict) descendants of v in τ . Finally, we let $|\tau|$ be the total number of vertices (or size) of the plane tree τ . For every $n \geq 1$, we let \mathbb{A}_n be the set of plane trees with n vertices.

2.2. Bienaymé–Galton–Watson trees and their codings. — Let μ be a probability measure on $\mathbb{Z}_{\geq 0}$ (called the *offspring distribution*) such that $\mu(0) > 0$ and $\mu(0) + \mu(1) < 1$ (to avoid trivial cases). We also assume that μ has mean $m_\mu := \sum_{i \geq 0} i\mu(i) \leq 1$. The Bienaymé–Galton–Watson (BGW) measure with offspring distribution μ is the probability measure BGW_μ characterized by

$$(1) \quad \text{BGW}_\mu(\tau) = \prod_{u \in \tau} \mu(k_u(\tau)),$$

for every plane tree τ , see e.g. [LG05, Prop. 1.4]. We say that the offspring distribution μ (or a tree with law BGW_μ) is *critical* (resp. *subcritical*) if $m_\mu = 1$ (resp. $m_\mu < 1$). For the sake of simplicity, we always assume that μ is *aperiodic*, meaning that $\text{gcd}(\{k \geq 1 : \mu(k) > 0\}) = 1$. However, all the results can be adapted to the periodic setting without effort.

Consider a tree τ with its vertices listed in lexicographical order: $\emptyset = u_0 \prec u_1 \prec \dots \prec u_{|\tau|-1}$. The *height function* $\mathbf{H}(\tau) = (\mathbf{H}_n(\tau) : 0 \leq n < |\tau|)$ is defined, for $0 \leq n < |\tau|$, by $\mathbf{H}_n(\tau) = |u_n|$. The *Lukasiewicz path* $\mathbf{W}(\tau) = (\mathbf{W}_n(\tau) : 0 \leq n < |\tau|)$ of a tree τ is defined by $\mathbf{W}_0(\tau) = 0$, and $\mathbf{W}_{n+1}(\tau) = \mathbf{W}_n(\tau) + k_{u_n}(\tau) - 1$ for $0 \leq n < |\tau|$.

Finally, the *contour function* $\mathbf{C}(\tau) = (\mathbf{C}_n(\tau) : 0 \leq n \leq 2(|\tau| - 1))$ of a tree τ is defined by considering a particle that starts from the root and visits continuously all edges at unit speed (assuming that every edge has unit length), going backwards as little as possible and respecting

the lexicographical order of vertices. If we let $\varnothing = x_0, \dots, x_{2(|\tau|-1)}$ be the ordered list of vertices of τ visited by the particle (with repetition), then we have $C_n(\tau) = |x_n|$ for $0 \leq n \leq 2(|\tau| - 1)$ (so that $C_t(\tau)$ is the distance to the root of the position of the particle at time t , see [Duq03, Section 2] for more on this).

For technical reasons, we let $H_n(\tau) = W_n(\tau) = 0$ for $n > |\tau|$ and $C_n(\tau) = 0$ for $n > 2(|\tau| - 1)$. We also extend $H(\tau)$, $W(\tau)$ and $C(\tau)$ to \mathbb{R}_+ by linear interpolation.

We finally recall a functional invariance principle which will be of main importance. We denote by $\mathcal{C}([0, 1])$ the space of continuous real-valued functions on $[0, 1]$ equipped with the uniform norm. In the remaining part of this paper, when dealing with BGW_μ tree conditioned to have n vertices, we always implicitly assume that we work along a subsequence on which $BGW_\mu(\mathbb{A}_n) > 0$.

PROPOSITION 6. — *Let μ be a critical offspring distribution with finite positive variance. For every $n \geq 1$, let \mathcal{T}_n be a BGW_μ tree conditioned on having n vertices. Then the convergence*

$$\left(\frac{1}{\sqrt{n}} W_{nt}(\mathcal{T}_n), \frac{1}{\sqrt{n}} H_{nt}(\mathcal{T}_n), \frac{1}{\sqrt{n}} C_{2nt}(\mathcal{T}_n) \right)_{0 \leq t \leq 1} \xrightarrow{n \rightarrow \infty} \left(\sigma_\mu \cdot \mathfrak{e}_t, \frac{2}{\sigma_\mu} \cdot \mathfrak{e}_t, \frac{2}{\sigma_\mu} \cdot \mathfrak{e}_t \right)_{0 \leq t \leq 1}$$

holds in distribution in the space $\mathcal{C}([0, 1])^3$, where $(\mathfrak{e}_t : 0 \leq t \leq 1)$ is the Brownian excursion.

This result was established in [MM03] when μ has small exponential moments, and in [Duq03] in the general case (see also [Kor13]). In view of future use, we record the following simple consequence of Proposition 6:

$$(2) \quad \frac{1}{\sqrt{n}} \sup_{v \in \mathcal{T}_n} k_v(\mathcal{T}_n) \xrightarrow{n \rightarrow \infty} 0, \quad \text{in probability.}$$

Indeed, the maximum out-degree of \mathcal{T}_n is the largest jump of $W(\mathcal{T}_n)$ (plus one). In addition, Proposition 6 entails that the convergence $(\frac{1}{\sqrt{n}} W_{[nt]}(\mathcal{T}_n) : 0 \leq t \leq 1) \rightarrow \sigma_\mu \cdot (\mathfrak{e}_t : 0 \leq t \leq 1)$ holds in distribution in the space $\mathcal{D}([0, 1])$ of real-valued càdlàg functions on $[0, 1]$ equipped with the Skorokhod J_1 topology. The claim follows from the continuity of the functional “largest jump” for the Skorokhod J_1 topology (see e.g. [JS03, Proposition 2.4 in Chapter VI]).

3. Looptrees: the non-generic subcritical case

3.1. Invariance principle for random walks with negative drift. — The roadmap to Theorem 1 is based on a limit theorem for random walks with negative drift that we now state. Let $(X_i : i \geq 1)$ be an i.i.d. sequence of real-valued random variables such that:

- $\mathbb{E}[X_1] = -\gamma < 0$
- $\mathbb{P}(X_1 \geq x) = L(x)x^{-\beta}$ with $\beta > 1$ and L a slowly varying function at infinity.

We set $W_0 = 0$, $W_n = X_1 + \dots + X_n$ for every $n \geq 1$ and let

$$\zeta = \inf\{i \geq 1 : W_i < 0\}.$$

We also set $W_i = 0$ for $i < 0$ by convention. In this section, our goal is to study the behaviour of the random walk $(W_i^{(n)} : i \geq 0)$ under the conditional probability $\mathbb{P}(\cdot | \zeta \geq n)$, as $n \rightarrow \infty$. More precisely, we shall couple with high probability the trajectory $(W_i^{(n)} : i \geq 0)$ with that

of a random walk conditioned to be nonnegative for a random number of steps, followed by an independent “big jump”, and then followed by an independent unconditioned random walk.

Statement of the main result. — For $n \geq 1$, we consider the process $(Z_i^{(n)} : i \geq 0)$ whose distribution is specified as follows. First, let I be a random variable with law given by

$$\mathbb{P}(I = j) = \frac{\mathbb{P}(\zeta \geq j)}{\mathbb{E}[\zeta]}, \quad j \geq 1.$$

Note that $\mathbb{E}[\zeta] < \infty$ since W_n has negative drift (see e.g. [Fel71, Theorem 1, Sec. XII.2]). Then, for every $j \geq 1$, conditionally given $\{I = j\}$, the three random variables $(Z_i^{(n)} : 0 \leq i < j)$, $Y_j^{(n)} := Z_j^{(n)} - Z_{j-1}^{(n)}$ and $(Z_{i+j}^{(n)} - Z_j^{(n)} : i \geq 0)$ are independent and distributed as follows:

- $(Z_i^{(n)} : 0 \leq i < j) \stackrel{(d)}{=} (W_i : 0 \leq i < j)$ under $\mathbb{P}(\cdot | \zeta \geq j)$
- $Y_j^{(n)} \stackrel{(d)}{=} X_1$ under $\mathbb{P}(\cdot | X_1 \geq \gamma n)$
- $(Z_{i+j}^{(n)} - Z_j^{(n)} : i \geq 0) \stackrel{(d)}{=} (W_i : i \geq 0)$.

We refer to [Lin92] or [dH, Section 2] for background concerning the total variation distance.

THEOREM 7. — *Let $(W_i^{(n)} : i \geq 0)$ be distributed as the random walk $(W_i : i \geq 0)$ under the conditional probability $\mathbb{P}(\cdot | \zeta \geq n)$. Then, we have*

$$d_{\text{TV}} \left(\left(W_i^{(n)} : i \geq 0 \right), \left(Z_i^{(n)} : i \geq 0 \right) \right) \xrightarrow[n \rightarrow \infty]{} 0,$$

where d_{TV} denotes the total variation distance on $\mathbb{R}^{\mathbb{Z}_+}$ equipped with the product topology.

Intuitively speaking, this means that under the conditional probability $\mathbb{P}(\cdot | \zeta \geq n)$, as $n \rightarrow \infty$, the random walk $(W_i : i \geq 0)$ first behaves as conditioned to stay nonnegative for a random number I of steps, then makes a jump distributed as $\mathbb{P}(\cdot | X_1 \geq \gamma n)$, and finally evolves as a non-conditioned walk.

3.2. Proof of Theorem 7. — In order to establish Theorem 7, our main input is a result describing the asymptotic behavior of the tail distribution $\mathbb{P}(\zeta \geq n)$ as $n \rightarrow \infty$:

$$(3) \quad \mathbb{P}(\zeta \geq n) \underset{n \rightarrow \infty}{\sim} \mathbb{E}[\zeta] \mathbb{P}(X_1 \geq \gamma n).$$

This follows from [BB08, Theorem 8.2.4]. Indeed, in the notation of [BB08], X_1 belongs to the class \mathcal{R} of distributions with regularly varying right tails, see [BB08, Equation 8.2.3]. See also [DS13, Theorem 3.2, Remark 3.3].

We now turn to the proof of Theorem 7.

Proof of Theorem 7. — The proof starts from the following simple observation. If $(\mu_n : n \geq 1)$ and $(\nu_n : n \geq 1)$ are sequences of probability measures on a measurable space (Ω, \mathcal{A}) and if $(\varepsilon_n : n \geq 1)$ is a sequence of real numbers tending to zero as $n \rightarrow \infty$, then

$$(4) \quad \left(\forall A \in \mathcal{A}, \forall n \geq 1, \mu_n(A) - \nu_n(A) \geq \varepsilon_n \right) \implies d_{\text{TV}}(\mu_n, \nu_n) \xrightarrow[n \rightarrow \infty]{} 0.$$

To see this, let $n \geq 1$ and $A \in \mathcal{A}$, and note that $\nu_n(A) - \mu_n(A) = \mu_n(A^c) - \nu_n(A^c) \geq \varepsilon_n$. This entails $\varepsilon_n \leq \mu_n(A) - \nu_n(A) \leq -\varepsilon_n$, hence the result.

Now, denote by \mathcal{A} be the Borel σ -algebra on $\mathbb{R}^{\mathbb{N}}$ associated with the product topology. If $\mathbf{x} = (x_1, \dots, x_i) \in \mathbb{R}^i$ and $A \in \mathcal{A}$, we write $\mathbf{x} \in A$ if $(x_1, \dots, x_i, 0, \dots) \in A$. For every $x \in \mathbb{R}$, let us introduce the stopping time

$$U_x := \inf\{i \geq 1 : X_i > x\},$$

and let μ_n be the probability measure on $\mathbb{Z} \times (\mathbb{R}^{\mathbb{N}})^3$ defined by

$$\mu_n(j, A, B, C) := \mathbb{P}\left(U_{n\gamma} = j, (W_i : 0 \leq i < j) \in A, X_j \in B, (W_{i+j} - W_j : i \geq 0) \in C \mid \zeta \geq n\right),$$

where $j \in \mathbb{Z}$ and $A, B, C \in \mathcal{A}$. Then, let $(\eta_n : n \geq 1)$ be a sequence of positive real numbers (which will be specified below), and fix $j \in \mathbb{Z}$ and $A, B, C \in \mathcal{A}$. We have

$$\begin{aligned} \mu_n(j, A, B, C) &\geq \mathbb{P}\left(\max_{1 \leq i < j} X_i < \gamma n, (W_i : 0 \leq i < j) \in A, \zeta \geq j\right) \mathbb{P}(X_j \in B, X_j \geq \gamma n + \eta_n) \\ &\quad \cdot \mathbb{P}\left((W_{i+j} - W_j : i \geq 0) \in C, \min_{1 \leq i \leq n} W_{i+j} - W_j > -\gamma n - \eta_n\right) \frac{1}{\mathbb{P}(\zeta \geq n)} \\ &= \mathbb{P}\left(\max_{1 \leq i < j} X_i < \gamma n, (W_i : 0 \leq i < j) \in A \mid \zeta \geq j\right) \mathbb{P}(\zeta \geq j) \\ &\quad \cdot \mathbb{P}(X_1 \in B, X_1 \geq \gamma n + \eta_n \mid X_1 \geq \gamma n) \frac{\mathbb{P}(X_1 \geq \gamma n)}{\mathbb{P}(\zeta \geq n)} \\ &\quad \cdot \mathbb{P}\left((W_i : i \geq 0) \in C, \min_{1 \leq i \leq n} W_i > -\gamma n - \eta_n\right). \end{aligned}$$

We now define a probability measure on $\mathbb{Z} \times (\mathbb{R}^{\mathbb{N}})^3$ by

$$\begin{aligned} \nu_n(j, A, B, C) &:= \mathbb{P}\left(I = j, (Z_i^{(n)} : 0 \leq i < j) \in A, Y_j^{(n)} \in B, (Z_{i+j}^{(n)} - Z_j^{(n)} : i \geq 0) \in C\right) \\ &= \mathbb{P}((W_i : 0 \leq i < j) \in A \mid \zeta \geq j) \mathbb{P}(X_1 \in B \mid X_1 \geq \gamma n) \mathbb{P}((W_i : i \geq 0) \in C) \\ &\quad \cdot \frac{\mathbb{P}(\zeta \geq j)}{\mathbb{E}[\zeta]}. \end{aligned}$$

By the previous lower bound on μ_n , we get, for every $j \in \mathbb{Z}$ and $A, B, C \in \mathcal{A}$:

$$\mu_n(j, A, B, C) - \nu_n(j, A, B, C) \geq \varepsilon_n(j, A, B, C)$$

where

$$\begin{aligned} &|\varepsilon_n(j, A, B, C)| \\ &\leq \left| \mathbb{P}\left(\max_{1 \leq i < j} X_i < \gamma n, (W_i : 0 \leq i < j) \in A \mid \zeta \geq j\right) - \mathbb{P}((W_i : 0 \leq i < j) \in A \mid \zeta \geq j) \right| \\ &\quad + \left| \mathbb{P}(X_1 \in B, X_1 \geq \gamma n + \eta_n \mid X_1 \geq \gamma n) - \mathbb{P}(X_1 \in B \mid X_1 \geq \gamma n) \right| + \left| \frac{\mathbb{P}(X_1 \geq \gamma n)}{\mathbb{P}(\zeta \geq n)} - \frac{1}{\mathbb{E}[\zeta]} \right| \\ &\quad + \left| \mathbb{P}\left((W_i : i \geq 0) \in C, \min_{1 \leq i \leq n} W_i > -\gamma n - \eta_n\right) - \mathbb{P}((W_i : i \geq 0) \in C) \right| \\ &\leq \mathbb{P}\left(\max_{1 \leq i < j} X_i \geq \gamma n, \mid \zeta \geq j\right) + \mathbb{P}(X_1 < \gamma n + \eta_n \mid X_1 \geq \gamma n) + \left| \frac{\mathbb{P}(X_1 \geq \gamma n)}{\mathbb{P}(\zeta \geq n)} - \frac{1}{\mathbb{E}[\zeta]} \right| \\ &\quad + \mathbb{P}\left(\min_{1 \leq i \leq n} W_i \leq -\gamma n - \eta_n\right). \end{aligned}$$

We now show that for every fixed $j \geq 1$, each one of the four terms of the last sum tends to 0 as $n \rightarrow \infty$.

Last term. — Using Doob's inequality (see e.g. [Dur10, Theorem 5.4.2]), write

$$\mathbb{P}\left(\min_{1 \leq i \leq n} W_i \leq -\gamma n - \eta_n\right) \leq \frac{\mathbb{E}[|W_n + \gamma n|]}{\eta_n}.$$

Since $\theta_n := \mathbb{E}[|W_n + \gamma n|]/n \rightarrow 0$ as $n \rightarrow \infty$ by the law of large numbers, this leads us to set $\eta_n = n\sqrt{\theta_n}$, so that $\mathbb{P}(\min_{1 \leq i \leq n} W_i \leq -\gamma n - \eta_n) \rightarrow 0$ as $n \rightarrow \infty$.

Third term. — This follows from (3).

Second term. — Write

$$\mathbb{P}(X_1 < \gamma n + \eta_n \mid X_1 \geq \gamma n) = 1 - \frac{\mathbb{P}(X_1 \geq \gamma n + \eta_n)}{\mathbb{P}(X_1 \geq \gamma n)} = 1 - \frac{L(\gamma n + \eta_n)}{L(\gamma n)} \frac{1}{(1 + \eta_n/(\gamma n))^\beta}$$

which tends to 0 as $n \rightarrow \infty$ since $\frac{\eta_n}{n} = \sqrt{\theta_n} \rightarrow 0$.

First term. — This follows from the fact that $\max_{1 \leq i < j} X_i$ is almost surely finite.

We finally get that for every fixed $J \geq 1$,

$$\sup_{\substack{A, B, C \in \mathcal{A} \\ 1 \leq j \leq J}} |\varepsilon_n(j, A, B, C)| \xrightarrow{n \rightarrow \infty} 0.$$

Now, observe that for every $n \geq 1$, $\nu_n(j, A, B, C) \leq \mathbb{P}(\zeta \geq j)/\mathbb{E}[\zeta] \rightarrow 0$ as $j \rightarrow \infty$ since ζ is almost surely finite. Finally, for every $j \in \mathbb{Z}$ and $A, B, C \in \mathcal{A}$, $\mu_n(j, A, B, C) - \nu_n(j, A, B, C) \geq \varepsilon_n$, with

$$|\varepsilon_n| \leq \sup_{\substack{A', B', C' \in \mathcal{A} \\ j' \geq 1}} (\varepsilon_n(j', A', B', C') \wedge \nu_n(j', A', B', C')) \xrightarrow{n \rightarrow \infty} 0,$$

and we complete the proof by applying (4). \square

3.3. Application to looptrees. — We first state a straightforward consequence of Theorem 7 (details are left to the reader). If I is an interval, we denote by $\mathbb{D}(I, \mathbb{R})$ the set of real-valued càdlàg functions on I equipped with the Skorokhod J_1 topology (see [JS03, Chapter VI] for background).

PROPOSITION 8. — *Let $(W_i^{(n)} : i \geq 0)$ be distributed as the random walk $(W_i : i \geq 0)$ under the conditional probability $\mathbb{P}(\cdot \mid \zeta \geq n)$. Let also J be the real-valued random variable such that $\mathbb{P}(J \geq x) = (\gamma/x)^\beta$ for $x \geq \gamma$. Then, the convergence*

$$\left(\frac{W_{\lfloor nt \rfloor}^{(n)}}{n} : t \geq -1\right) \xrightarrow{n \rightarrow \infty} ((J - \gamma t) \mathbb{1}_{t \geq 0} : t \geq -1)$$

holds in distribution in $\mathbb{D}([-1, \infty), \mathbb{R})$. In addition, the convergence

$$\frac{1}{n} \inf \{i \geq 1 : W_i^{(n)} < 0\} \xrightarrow{n \rightarrow \infty} \frac{J}{\gamma}$$

holds jointly in distribution.

Observe that instead of working as usual with $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$, we work with $\mathbb{D}([-1, \infty), \mathbb{R})$ by extending our function with value 0 on $[-1, 0)$. The reason is that our limiting process almost surely takes a positive value in 0 (it “starts with a jump”), while $(W_i^{(n)} : i \geq 0)$ stays small for a positive time.

Let us mention that this result extends [Dur80, Theorem 3.2] in two directions: it allows infinite variance for the step distribution of the random walk, and includes the convergence of the absorption time (this is not a mere consequence of the scaling limit result, since absorption times - if they exist - are in general not continuous functionals for Skorokhod’s topology). The possibility of the first extension is already mentioned in [Dur80], but the proof of Theorem 3.2 in [Dur80] uses a finite variance condition (see in particular the estimates at the top of page 285 in [Dur80]).

We finally prove Theorem 1 using Proposition 8.

Proof of Theorem 1. — The proof is similar to that of [CK15, Theorem 1.2]. For every $n \geq 1$, let $\mathcal{T}_{\geq n}$ be a BGW $_{\mu}$ tree conditioned to have at least n vertices, such that the offspring distribution μ is subcritical and satisfies $\mu([i, \infty)) = L(i)i^{-\beta}$.

Let us consider the Łukasiewicz path $(W_i(\mathcal{T}_{\geq n}) : i \geq 0)$ of the random tree $\mathcal{T}_{\geq n}$. By construction (see for instance [Duq03, Section 2]), $(W_i(\mathcal{T}_{\geq n}) : i \geq 0)$ has the same distribution as a random walk $(W_i : i \geq 0)$ with step distribution ξ defined by

$$\xi(i) = \mu(i + 1), \quad i \geq -1,$$

conditionally on the event $\{\zeta \geq n\}$. Thus, the requirements of Proposition 8 are met by the process $(W_i(\mathcal{T}_{\geq n}) : i \geq 0)$. First, by the correspondence between jumps of the Łukasiewicz path and the degrees of the vertices in $\mathcal{T}_{\geq n}$, there exists an asymptotically unique vertex $v_n^* \in \mathcal{T}_{\geq n}$ with maximal degree, such that

$$\frac{k_{v_n^*}(\mathcal{T}_{\geq n})}{n} \xrightarrow[n \rightarrow \infty]{} J \quad \text{in distribution,}$$

where J is the random variable defined in Proposition 8. Moreover, if $\{T_j^* : 0 \leq j \leq k_{v_n^*}(\mathcal{T}_{\geq n})\}$ are the connected components of $\mathcal{T}_{\geq n} \setminus \{v_n^*\}$, we have

$$\frac{1}{n} \sup_{0 \leq j \leq k_{v_n^*}(\mathcal{T}_{\geq n})} |T_j^*| \xrightarrow[n \rightarrow \infty]{} 0, \quad \text{in probability,}$$

see also [?]ollary 1]Kor15 for a similar statement. Now, recall the construction of the looptree $\text{Loop}(\tau)$ from a plane tree τ , detailed in Section 1. By construction, $\text{Loop}(\mathcal{T}_{\geq n})$ has a unique face of degree $k_{v_n^*}(\mathcal{T}_{\geq n}) + 1$, and the largest connected component of $\text{Loop}(\mathcal{T}_{\geq n})$ deprived of this face has size $o(n)$. This completes the proof. \square

4. Looptrees: the generic critical case

We now study the asymptotic behavior of looptrees associated with large generic critical BGW trees (that is, whose offspring distribution has mean one and finite positive variance). Specifically, if τ is a plane tree, recall from Section 1 the definition of its associated looptree $\text{Loop}(\tau)$ (and see Figure 2 for an example). Recall also from Section 2.2 that $(C_t(\tau) : t \geq 0)$ and

$(H_t(\tau) : t \geq 0)$ denote respectively the contour and height function of τ . Starting from now, we denote by d_τ° the graph metric on $\text{Loop}(\tau)$. If $\varnothing = u_0, u_1, \dots, u_{|\tau|-1}$ denote the vertices of τ listed in lexicographical order, we set

$$(5) \quad H_i^\circ(\tau) = d_\tau^\circ(\varnothing, u_i), \quad 0 \leq i < |\tau|,$$

and $H_i^\circ(\tau) = 0$ for $i \geq |\tau|$. We again extend $H^\circ(\tau)$ to \mathbb{R}_+ by linear interpolation.

Finally, for every offspring distribution μ with variance σ_μ^2 , we set

$$(6) \quad c_\mu := \frac{1}{4} (\sigma_\mu^2 + 4 - \mu(2\mathbb{Z}+)).$$

The goal of this section is to establish Theorem 2, which will follow from the next result (see Section 4.3 below for the proof of Theorem 2 using Proposition 9).

PROPOSITION 9. — *Let μ be a critical offspring distribution with finite positive variance. For every $n \geq 1$, let \mathcal{T}_n be a BGW_μ tree conditioned on having n vertices. Then, the convergence*

$$(7) \quad \left(\frac{1}{\sqrt{n}} C_{2nt}(\mathcal{T}_n), \frac{1}{\sqrt{n}} H_{nt}(\mathcal{T}_n), \frac{1}{\sqrt{n}} H_{nt}^\circ(\mathcal{T}_n) \right)_{0 \leq t \leq 1} \xrightarrow{n \rightarrow \infty} \left(\frac{2}{\sigma_\mu} \cdot \mathfrak{e}_t, \frac{2}{\sigma_\mu} \cdot \mathfrak{e}_t, \frac{2c_\mu}{\sigma_\mu} \cdot \mathfrak{e}_t \right)_{0 \leq t \leq 1}$$

holds jointly in distribution in the space $\mathcal{C}([0, 1])^3$.

To prove this convergence, we establish the following result which, roughly speaking, shows tightness and identifies the finite dimensional marginals.

LEMMA 10. — *Let μ be a critical offspring distribution with finite positive variance. For every $n \geq 1$, let \mathcal{T}_n be a BGW_μ tree conditioned on having n vertices. Then, the following assertions hold.*

- (i) *The sequence $(\frac{1}{\sqrt{n}} H_{nt}^\circ(\mathcal{T}_n) : 0 \leq t \leq 1)$ is tight in $\mathcal{C}([0, 1])$.*
- (ii) *For every $n \geq 1$, let U^n be a random variable uniformly distributed on $\{0, 1, \dots, n-1\}$, independent of \mathcal{T}_n . Then,*

$$\frac{1}{\sqrt{n}} |H_{U^n}^\circ(\mathcal{T}_n) - c_\mu H_{U^n}(\mathcal{T}_n)| \xrightarrow{n \rightarrow \infty} 0 \quad \text{in probability.}$$

Before proving this, let us explain why it implies Proposition 9.

Proof of Proposition 9 using Lemma 10. — By Proposition 6 and Lemma 10, up to extraction and using Skorokhod's representation theorem (see e.g. [Bil99, Theorem 6.7]), we may assume that the convergence

$$(8) \quad \left(\frac{1}{\sqrt{n}} C_{2nt}(\mathcal{T}_n), \frac{1}{\sqrt{n}} H_{nt}(\mathcal{T}_n), \frac{1}{\sqrt{n}} H_{nt}^\circ(\mathcal{T}_n) \right)_{0 \leq t \leq 1} \xrightarrow{n \rightarrow \infty} \left(\frac{2}{\sigma_\mu} \cdot \mathfrak{e}_t, \frac{2}{\sigma_\mu} \cdot \mathfrak{e}_t, Z_t \right)_{0 \leq t \leq 1}$$

holds almost surely in the space $\mathcal{C}([0, 1])^3$ for a certain continuous random function Z , and we aim at showing that

$$(Z_t : 0 \leq t \leq 1) = \left(\frac{2c_\mu}{\sigma_\mu} \cdot \mathfrak{e}_t : 0 \leq t \leq 1 \right) \quad \text{almost surely.}$$

For every $n \geq 1$, we let $(U_i^n : i \geq 1)$ be an i.i.d. sequence of uniform random variables on $\{0, 1, \dots, n-1\}$, $(U_i : i \geq 1)$ an i.i.d. sequence of uniform random variables on $[0, 1]$, all independent of $(\mathcal{T}_n : n \geq 1)$. We may also assume that for every $i \geq 1$, the convergence $\frac{1}{n}U_i^n \rightarrow U_i$ holds almost surely as $n \rightarrow \infty$.

Now, let us fix $k \geq 1$. We claim that

$$Z_{U_k} = \frac{2c_\mu}{\sigma_\mu} \cdot \mathbb{e}_{U_k} \quad \text{almost surely.}$$

Indeed, by Lemma 10 (ii), we may find an extraction ϕ such that

$$\frac{1}{\sqrt{\phi(n)}} \left| \mathbf{H}_{U_k^{\phi(n)}}^\circ(\mathcal{T}_{\phi(n)}) - c_\mu \mathbf{H}_{U_k^{\phi(n)}}(\mathcal{T}_{\phi(n)}) \right| \xrightarrow{n \rightarrow \infty} 0 \quad \text{almost surely.}$$

But we also have the almost sure convergences

$$\frac{1}{\sqrt{\phi(n)}} \mathbf{H}_{U_k^{\phi(n)}}^\circ(\mathcal{T}_{\phi(n)}) \xrightarrow{n \rightarrow \infty} Z_{U_k} \quad \text{and} \quad \frac{c_\mu}{\sqrt{\phi(n)}} \mathbf{H}_{U_k^{\phi(n)}}(\mathcal{T}_{\phi(n)}) \xrightarrow{n \rightarrow \infty} \frac{2c_\mu}{\sigma_\mu} \cdot \mathbb{e}_{U_k},$$

which entails our claim.

It follows that almost surely, the two continuous functions $(Z_t : 0 \leq t \leq 1)$ and $\left(\frac{2c_\mu}{\sigma_\mu} \cdot \mathbb{e}_t : 0 \leq t \leq 1\right)$ coincide on the set $\{U_i : i \geq 1\}$ which is dense in $[0, 1]$. This completes the proof. \square

4.1. Tightness. — The goal of this section is to establish the tightness statement (i) of Lemma 10. We start with a (deterministic) upper bound for the distance d_τ° .

LEMMA 11. — *Let τ be a plane tree and denote by $u_0, u_1, \dots, u_{|\tau|-1}$ its vertices listed in lexicographical order. Then, for every $0 \leq i < j < |\tau|$, if u_i is an ancestor of u_j in τ we have*

$$|\mathbf{H}_i^\circ(\tau) - \mathbf{H}_j^\circ(\tau)| \leq \mathbf{W}_j(\tau) - \mathbf{W}_i(\tau) + \mathbf{H}_j(\tau) - \mathbf{H}_i(\tau).$$

Proof. — To simplify, assume that $i \neq 0$ (the case $i = 0$ is treated in the same way). Observe that u_i disconnects $\text{Loop}(\tau)$ into two connected components, one containing u_j and the other containing u_0 . We consider the first of these two components, which is actually $\text{Loop}(\theta_{u_i}(\tau))$ (where we recall that $\theta_{u_i}(\tau)$ is the tree made of the descendants of u_i in τ). Since the shortest path from u_j to u_i in $\text{Loop}(\tau)$ stays in $\text{Loop}(\theta_{u_i}(\tau))$, the distance between u_i and u_j in $\text{Loop}(\theta_{u_i}(\tau))$ is $\mathbf{H}_j^\circ(\tau) - \mathbf{H}_i^\circ(\tau)$. On the other hand, the number of vertices branching (weakly) to the right of the ancestral line $\llbracket u_i, u_j \rrbracket$ in $\theta_{u_i}(\tau)$ is $\mathbf{W}_j(\tau) - \mathbf{W}_i(\tau) + \mathbf{H}_j(\tau) - \mathbf{H}_i(\tau)$. Therefore the path in $\text{Loop}(\tau)$ which goes from u_j to u_i by only using the vertices of τ belonging to $\llbracket u_i, u_j \rrbracket$, and their children grafted on the right of $\llbracket u_i, u_j \rrbracket$ has length $\mathbf{W}_j(\tau) - \mathbf{W}_i(\tau) + \mathbf{H}_j(\tau) - \mathbf{H}_i(\tau)$ (see Figure 4 for an illustration). This entails the desired result. \square

We can now prove the tightness statement (i) of Lemma 10.

Proof of Lemma 10 (i). — By a standard tightness criterion (see e.g. [Dur10, Theorem 8.10.5]) it suffices to check that for every $\varepsilon > 0$,

$$\limsup_{n \rightarrow \infty} \mathbb{P} \left(\sup_{|i-j| \leq \delta n} |\mathbf{H}_i^\circ(\mathcal{T}_n) - \mathbf{H}_j^\circ(\mathcal{T}_n)| > \varepsilon \sqrt{n} \right) \xrightarrow{\delta \rightarrow 0} 0.$$

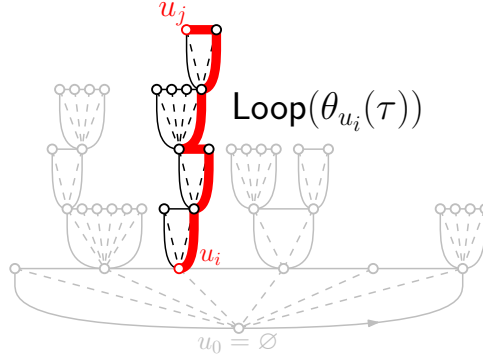


Figure 4. A plane tree τ , the associated looptree $\text{Loop}(\theta_{u_i}(\tau))$ (in black) and the path of length $W_j(\tau) - W_i(\tau) + H_j(\tau) - H_i(\tau)$ between u_j and u_i (in bold).

Since $\frac{1}{\sqrt{n}} \sup H(\mathcal{T}_n)$ converges in distribution as $n \rightarrow \infty$ by Proposition 6, it is enough to check that

$$\limsup_{n \rightarrow \infty} \mathbb{P} \left(\sup_{|i-j| \leq \delta n} |H_i^\circ(\mathcal{T}_n) - H_j^\circ(\mathcal{T}_n)| > \varepsilon \sqrt{n}, \quad \sup H(\mathcal{T}_n) \leq n^{3/4} \right) \xrightarrow{\delta \rightarrow 0} 0.$$

To this end, we start with an identity inspired from [BM14, Equation (11)], see the proof of Proposition 7 in [Mar18] for a similar argument in a different context. Let us introduce some notation. We fix $n \geq 1$, and let $\tilde{\mathcal{T}}_n$ be the mirror image of the tree \mathcal{T}_n (see Figure 5). We claim that on the event $\{\sup H(\mathcal{T}_n) \leq n^{3/4}\}$, for every $\delta > 0$ and every n sufficiently large,

$$(9) \quad \sup_{|p-q| \leq \delta n} |H_p^\circ(\mathcal{T}_n) - H_q^\circ(\mathcal{T}_n)| \leq \sup_{\substack{|i-j| \leq \delta n \\ u_i \prec u_j}} |H_i^\circ(\mathcal{T}_n) - H_j^\circ(\mathcal{T}_n)| + \sup_{\substack{|i-j| \leq 2\delta n \\ \tilde{u}_i \prec \tilde{u}_j}} |H_i^\circ(\tilde{\mathcal{T}}_n) - H_j^\circ(\tilde{\mathcal{T}}_n)| + \sup_{v \in \mathcal{T}_n} k_v(\mathcal{T}_n),$$

where $\tilde{u}_0 = \emptyset, \dots, \tilde{u}_{n-1}$ denote the vertices of $\tilde{\mathcal{T}}_n$ listed in the lexicographical order.

To establish (9), we fix $p, q \in \{0, \dots, n-1\}$ and assume, without loss of generality, that $p < q$. We denote by $m(p, q)$ the index of the most recent common ancestor between u_p and u_q (in the lexicographical order of \mathcal{T}_n). We also let p' and q' be the indices of the children of $u_{m(p, q)}$ that are ancestors of respectively u_p and u_q . By the triangular inequality,

$$|H_p^\circ(\mathcal{T}_n) - H_q^\circ(\mathcal{T}_n)| \leq |H_p^\circ(\mathcal{T}_n) - H_{p'}^\circ(\mathcal{T}_n)| + |H_q^\circ(\mathcal{T}_n) - H_{q'}^\circ(\mathcal{T}_n)| + \sup_{v \in \mathcal{T}_n} k_v(\mathcal{T}_n).$$

One has now to be careful because u_q and $u_{q'}$ are close in the lexicographical order of \mathcal{T}_n (since $|q - q'| \leq |p - q|$, see Figure 5), but u_p and $u_{p'}$ may not be. However, the indices of their mirror images $l(u_p)$ and $l(u_{p'})$ are at distance at most $|p - q| + |H_p(\mathcal{T}_n) - H_{p'}(\mathcal{T}_n)|$ in the lexicographical order of $\tilde{\mathcal{T}}_n$ by construction (see Figure 5 for an illustration). Hence, on the event $\{\sup H(\mathcal{T}_n) \leq n^{3/4}\}$, for every $\delta > 0$ and every n sufficiently large, if $|p - q| \leq \delta n$, then the mirror images $l(u_p)$ and $l(u_{p'})$ are at distance less than $2\delta n$ in the lexicographical order of $\tilde{\mathcal{T}}_n$. This entails (9).

Since $\tilde{\mathcal{T}}_n$ and \mathcal{T}_n have the same distribution, by (2), it suffices to check that

$$(10) \quad \limsup_{n \rightarrow \infty} \mathbb{P} \left(\sup_{\substack{|i-j| \leq \delta n \\ u_i \prec u_j}} |H_i^\circ(\mathcal{T}_n) - H_j^\circ(\mathcal{T}_n)| > \varepsilon \sqrt{n} \right) \xrightarrow{\delta \rightarrow 0} 0.$$

But by Lemma 11, if u_i is an ancestor of u_j , we have

$$|H_i^\circ(\mathcal{T}_n) - H_j^\circ(\mathcal{T}_n)| = H_j^\circ(\mathcal{T}_n) - H_i^\circ(\mathcal{T}_n) \leq W_j(\mathcal{T}_n) - W_i(\mathcal{T}_n) + H_j(\mathcal{T}_n) - H_i(\mathcal{T}_n),$$

so that the probability in (10) is bounded from above by

$$\mathbb{P} \left(\sup_{|i-j| \leq \delta n} |W_i(\mathcal{T}_n) - W_j(\mathcal{T}_n)| > \frac{\varepsilon}{2} \sqrt{n} \right) + \mathbb{P} \left(\sup_{|i-j| \leq \delta n} |H_i(\mathcal{T}_n) - H_j(\mathcal{T}_n)| > \frac{\varepsilon}{2} \sqrt{n} \right).$$

Recall that by Proposition 6, $(\frac{1}{\sqrt{n}}W_{nt}(\mathcal{T}_n) : 0 \leq t \leq 1)$ and $(\frac{1}{\sqrt{n}}H_{nt}(\mathcal{T}_n) : 0 \leq t \leq 1)$ are both tight in $\mathcal{C}([0, 1])$. The desired result then follows. \square

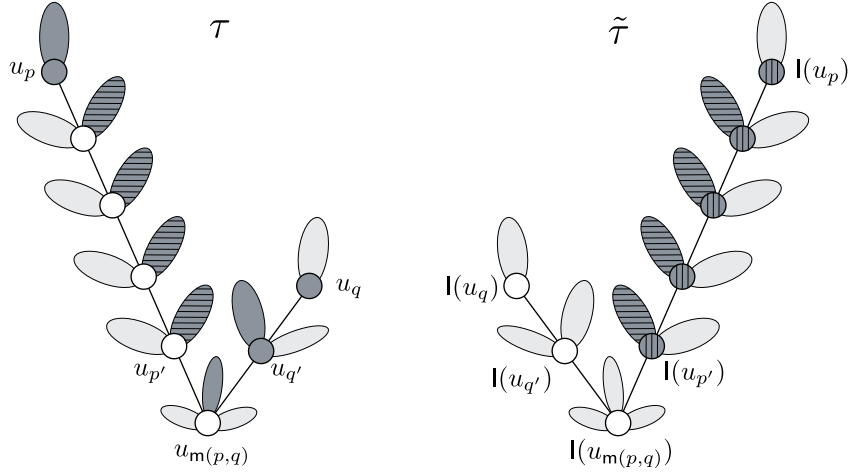


Figure 5. Left: a plane tree τ , where the dark gray region encompasses the vertices that contribute to the lexicographical distance between u_p and u_q in τ . Right: the mirror $\tilde{\tau}$ of τ , where the dark gray region encompasses this time the vertices that contribute to the lexicographical distance between $l(u_{p'})$ and $l(u_p)$ in $\tilde{\tau}$. The indices of $l(u_{p'})$ and $l(u_p)$ in the lexicographical order of $\tilde{\tau}$ are at distance at most $|p - q| + |H_p(\mathcal{T}_n) - H_{p'}(\mathcal{T}_n)|$. Indeed, the dark gray region on the right can be decomposed into two parts: the first one (horizontal hatches), which is a subset of the dark gray region on the left, yields to the contribution $|p - q|$, and the second one (vertical hatches) gives the contribution $|H_p(\mathcal{T}_n) - H_{p'}(\mathcal{T}_n)|$.

4.2. Convergence of finite-dimensional distributions. — The goal of this section is to establish the convergence of finite dimensional marginals statement (ii) of Lemma 10. In what follows, given a tree τ and a random variable U uniformly distributed on $\{0, 1, \dots, |\tau| - 1\}$, we may interpret the U -th vertex of τ in lexicographical order as a vertex V chosen uniformly at random in τ . Recalling the definition (5), we then have

$$H_U^\circ(\tau) = d_\tau^\circ(\emptyset, V).$$

We shall use the following result.

LEMMA 12. — *Let μ be a critical offspring distribution with finite positive variance. For every $n \geq 1$, let \mathcal{T}_n be a BGW $_\mu$ tree conditioned on having n vertices. Conditionally given \mathcal{T}_n , let V_n be a vertex of \mathcal{T}_n chosen uniformly at random. Then for every $\varepsilon > 0$ and every $\delta \in (0, 1)$, we have*

$$\mathbb{P} \left(\left| \frac{d_{\mathcal{T}_n}^{\circ}(\emptyset, V_n)}{|V_n|} - c_\mu \right| > \varepsilon, |V_n| \in [\delta\sqrt{n}, \frac{1}{\delta}\sqrt{n}], R(V_n) \geq \delta\sqrt{n} \right) \xrightarrow{n \rightarrow \infty} 0,$$

where $R(V_n)$ is the number of children of vertices of $[[\emptyset, V_n[[$ branching (strictly) on the right of $[[\emptyset, V_n[[$ in \mathcal{T}_n .

Before proving this result, let us explain why it implies Lemma 10 (ii).

Proof of Lemma 10 (ii) using Lemma 12. — We interpret the U^n -th vertex of \mathcal{T}_n in lexicographical order as a vertex V_n chosen uniformly at random in \mathcal{T}_n . Then, if u_0, \dots, u_{n-1} are the vertices of \mathcal{T}_n listed in lexicographical order, it is well known that for every $0 \leq i < n$, $W_i(\mathcal{T}_n)$ is equal to the number of number of children of vertices of $[[\emptyset, u_i[[$ branching on the right of $[[\emptyset, u_i[[$, so that $|V_n| = H_{U^n}(\mathcal{T}_n)$ and $R(V_n) = W_{U^n}(\mathcal{T}_n)$. Therefore, by Proposition 6, the convergence

$$\left(\frac{\sigma_\mu |V_n|}{2\sqrt{n}}, \frac{R(V_n)}{\sigma_\mu \sqrt{n}} \right) \xrightarrow{n \rightarrow \infty} (\mathbb{e}_U, \mathbb{e}_U)$$

holds in distribution, where U is uniform on $[0, 1]$ and independent of $(\mathbb{e}_t : 0 \leq t \leq 1)$. Since $\mathbb{e}_U > 0$ almost surely, we may find $\delta > 0$ such that if $E_n(\delta)$ stands for the event $E_n(\delta) = \{|V_n| \in [\delta\sqrt{n}, \frac{1}{\delta}\sqrt{n}], R(V_n) \geq \delta\sqrt{n}\}$, then $\mathbb{P}(E_n(\delta)) \geq 1 - \varepsilon$ for n sufficiently large.

Then write, for every $n \geq 1$,

$$\begin{aligned} \mathbb{P}(|H_{U^n}^{\circ}(\mathcal{T}_n) - c_\mu H_{U^n}(\mathcal{T}_n)| > \varepsilon\sqrt{n}) &= \mathbb{P}(|d_{\mathcal{T}_n}^{\circ}(\emptyset, V_n) - c_\mu |V_n|| > \varepsilon\sqrt{n}) \\ &\leq \mathbb{P} \left(\left| \frac{d_{\mathcal{T}_n}^{\circ}(\emptyset, V_n)}{|V_n|} - c_\mu \right| > \frac{\varepsilon\sqrt{n}}{|V_n|}, E_n(\delta) \right) + 1 - \mathbb{P}(E_n(\delta)) \end{aligned}$$

which is less than 2ε for n large enough by Lemma 12. This completes the proof. \square

The key ingredient to prove Lemma 12 is a spinal decomposition of the underlying BGW tree. It implicitly appears in [LPP95], see also [BM14, Mar18] for similar statements in a different context. First, let us introduce some notation. If μ is a critical offspring distribution, denote by $\bar{\mu}$ the size-biased version of μ defined by

$$\bar{\mu}(j) = j\mu(j), \quad j \geq 0.$$

We will use in the next part the *local limit* \mathcal{T}_∞ of BGW $_\mu$ trees conditioned to survive, that was first defined in [Kes86] (see also [Jan12, AD14, Ste18]). The random locally finite tree \mathcal{T}_∞ can be constructed as follows. It has almost surely a unique infinite *spine* $(v_0^* = \emptyset, v_1^* \dots)$, whose vertices have offspring distribution $\bar{\mu}$. Among all children of a vertex of the spine, the child belonging to the spine is uniform. Finally, outside of the spine, vertices have offspring distribution μ , and all the numbers of offspring are independent. See Figure 6 for an illustration and [Duq09, Definition 3.1] for a more formal definition. The construction of looptrees can be adapted verbatim to \mathcal{T}_∞ , and we denote by $d_{\mathcal{T}_\infty}^{\circ}$ the graph distance in $\text{Loop}(\mathcal{T}_\infty)$ (see Figure 6,

and [Ric18, Section 5.1] for details). Recall that for every $j \geq 0$, $\text{Cut}_{v_j^*}(\mathcal{T}_\infty) = \{v_j^*\} \cup \mathcal{T}_\infty \setminus \theta_{v_j^*}(\mathcal{T}_\infty)$ is the tree obtained from \mathcal{T}_∞ by removing all the (strict) descendants of the j -th vertex of the spine in \mathcal{T}_∞ . We will also need the tree $\mathcal{T}_\infty^{(j)}$, that is defined out of $\text{Cut}_{v_j^*}(\mathcal{T}_\infty)$ by keeping only the vertices of the spine and their children (see Figure 6 for an example).

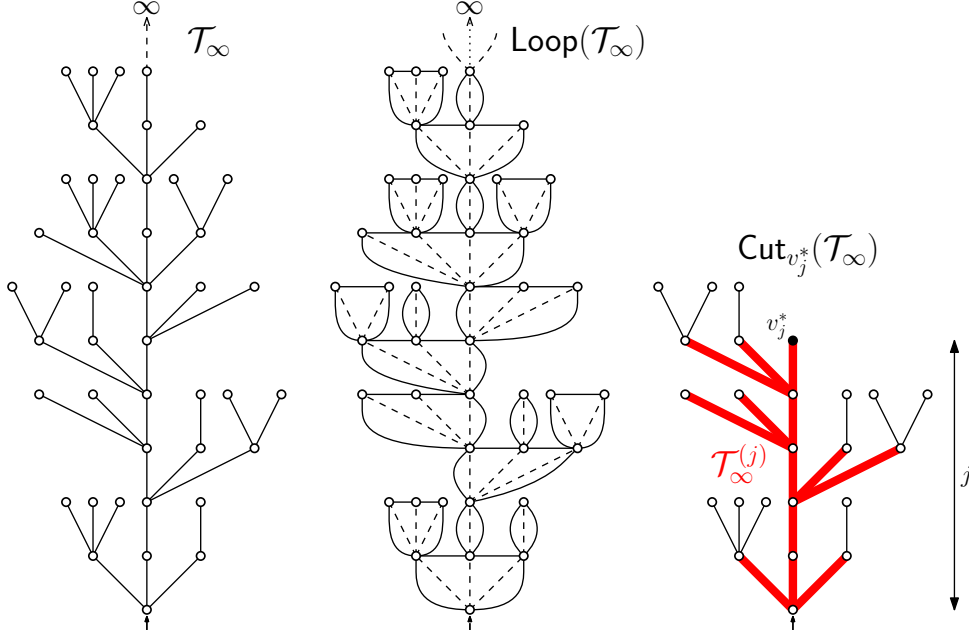


Figure 6. A schematic representation of \mathcal{T}_∞ , and of the associated looptree $\text{Loop}(\mathcal{T}_\infty)$. On the right, the tree $\text{Cut}_{v_j^*}(\mathcal{T}_\infty)$, and its subtree $\mathcal{T}_\infty^{(j)}$ (in bold red).

Proof of Lemma 12. — Let \mathcal{T} be a BGW_μ tree, and conditionally given \mathcal{T} , let V be a uniformly chosen vertex in \mathcal{T} . The proof is based on a classical absolute continuity relation between \mathcal{T} and \mathcal{T}_∞ (see e.g. [Duq09, Equation (24)]) that we now recall. For every $h \geq 0$ and every nonnegative functions G_1 and G_2 , we have

$$(11) \quad \mathbb{E} \left[\sum_{\substack{v \in \mathcal{T} \\ |v|=h}} G_1(\text{Cut}_v(\mathcal{T}), v) G_2(\theta_v(\mathcal{T})) \right] = \mathbb{E} \left[G_1(\text{Cut}_{v_h^*}(\mathcal{T}_\infty), v_h^*) G_2(\mathcal{T}') \right],$$

where \mathcal{T}' is an independent BGW_μ tree. Let us fix $\varepsilon > 0$ and $\delta \in (0, 1)$. In order to simplify the notation, we set

$$P(n, h) := \mathbb{P} \left(|V| = h, \left| \frac{d_{\mathcal{T}}^\circ(\emptyset, V)}{|V|} - c_\mu \right| > \varepsilon, R(V) \geq \delta \sqrt{n}, |\mathcal{T}| = n \right),$$

where $R(V)$ is the number of children of vertices of $[[\emptyset, V[[$ branching on the right of $[[\emptyset, V[[$. Thus, we have that

$$(12) \quad \mathbb{P} \left(\left| \frac{d_{\mathcal{T}_n}^\circ(\emptyset, V_n)}{|V_n|} - c_\mu \right| > \varepsilon, E_n(\delta) \right) = \frac{1}{\mathbb{P}(|\mathcal{T}| = n)} \sum_{h=\lceil \delta \sqrt{n} \rceil}^{\lfloor \sqrt{n}/\delta \rfloor} P(n, h),$$

where we recall that $E_n(\delta) = \{|V_n| \in [\delta\sqrt{n}, \frac{1}{\delta}\sqrt{n}], R(V_n) \geq \delta\sqrt{n}\}$. Then, we have

$$P(n, h) = \frac{1}{n} \mathbb{E} \left[\sum_{\substack{v \in \mathcal{T} \\ |v|=h}} \mathbb{1}_{\{|d_{\mathcal{T}}^{\circ}(\emptyset, v)/|v| - c_{\mu}| > \varepsilon, R(v) \geq \delta\sqrt{n}, |\mathcal{T}|=n\}} \right].$$

But the event $\{|\mathcal{T}| = n\}$ can be rewritten as $\{|\text{Cut}_v(\mathcal{T})| + |\theta_v(\mathcal{T})| = n + 1\}$, so that (11) gives

$$P(n, h) = \frac{1}{n} \mathbb{P} \left(\left| \frac{d_{\text{Cut}_{v_h^*}(\mathcal{T}_{\infty})}^{\circ}(\emptyset, v_h^*)}{|v_h^*|} - c_{\mu} \right| > \varepsilon, R(v_h^*) \geq \delta\sqrt{n}, |\text{Cut}_{v_h^*}(\mathcal{T}_{\infty})| + |\mathcal{T}'| = n + 1 \right),$$

where \mathcal{T}' is an independent BGW $_{\mu}$ tree. Note that by construction,

$$d_{\text{Cut}_{v_h^*}(\mathcal{T}_{\infty})}^{\circ}(\emptyset, v_h^*) = d_{\mathcal{T}_{\infty}^{(h)}}^{\circ}(\emptyset, v_h^*), \quad h \geq 0,$$

so that

$$(13) \quad P(n, h) = \frac{1}{n} \mathbb{P} \left(\left| \frac{d_{\mathcal{T}_{\infty}^{(h)}}^{\circ}(\emptyset, v_h^*)}{|v_h^*|} - c_{\mu} \right| > \varepsilon, R(v_h^*) \geq \delta\sqrt{n}, |\text{Cut}_{v_h^*}(\mathcal{T}_{\infty})| + |\mathcal{T}'| = n + 1 \right).$$

Now, set $K_{\infty}(0) = 0$ and, for $j \geq 1$, let $K_{\infty}(j) := |\mathcal{T}_{\infty}^{(j)}| - (j + 1) = \sum_{i=0}^{j-1} (k_{v_i^*}(\mathcal{T}_{\infty}) - 1)$ be number of children of vertices of the spine with height smaller than j , and that do not belong to the spine. Since $R(v_j^*) \leq K_{\infty}(j)$, it follows that $P(n, h) \leq \frac{1}{n} B(n, h)$, where

$$B(n, h) := \mathbb{P} \left(\left| \frac{d_{\mathcal{T}_{\infty}^{(h)}}^{\circ}(\emptyset, v_h^*)}{|v_h^*|} - c_{\mu} \right| > \varepsilon, K_{\infty}(h) \geq \delta\sqrt{n}, |\text{Cut}_{v_h^*}(\mathcal{T}_{\infty})| + |\mathcal{T}'| = n + 1 \right).$$

Observe that conditionally given $\mathcal{T}_{\infty}^{(h)}$, we have the identity in distribution

$$|\text{Cut}_{v_h^*}(\mathcal{T}_{\infty})| \stackrel{(d)}{=} h + 1 + \sum_{i=1}^{K_{\infty}(h)} |\mathcal{T}(i)|,$$

where $(\mathcal{T}(i) : i \geq 1)$ is a sequence of i.i.d. BGW $_{\mu}$ trees. (In this identity, we think of $(\mathcal{T}(i) : 1 \leq i \leq K_{\infty}(h))$ as the subtrees grafted on all the leaves of $\mathcal{T}_{\infty}^{(h)}$ except v_h^*). Therefore, if we set $\Phi_k(m) = \mathbb{P}(\sum_{i=1}^k |\mathcal{T}(i)| = m)$, by conditioning with respect to $\mathcal{T}_{\infty}^{(h)}$, we get

$$B(n, h) = \mathbb{E} \left[\mathbb{1}_{\left\{ \left| \frac{d_{\mathcal{T}_{\infty}^{(h)}}^{\circ}(\emptyset, v_h^*)}{|v_h^*|} - c_{\mu} \right| > \varepsilon, K_{\infty}(h) \geq \delta\sqrt{n} \right\}} \Phi_{K_{\infty}(h)+1}(n - h) \right].$$

Now, observe that there exists a constant $C > 0$ such that for every $k, m \geq 1$,

$$(14) \quad \Phi_k(m) = \mathbb{P} \left(\sum_{i=1}^k |\mathcal{T}(i)| = m \right) \leq \frac{C}{k^2}.$$

Indeed, it is well known that $\mathbb{P}(|\mathcal{T}| = n) \sim cn^{-3/2}$ as $n \rightarrow \infty$ for a certain constant $c > 0$ (this follows for instance from the cyclic lemma [Pit06, Lemma 6.1] combined with the local limit theorem [IL71, Theorem 4.2.1], see e.g. [MM03, Section 3.2]). Therefore $|\mathcal{T}|$ falls in the domain of attraction of a spectrally positive stable law of index $1/2$, and (14) follows also from the local limit theorem [IL71, Theorem 4.2.1].

Back to $B(n, h)$, the estimate (14) guarantees the existence of a constant $C > 0$ such that on the event $\{\delta\sqrt{n} \leq K_\infty(h)\}$, $\Phi_{K_\infty(h)+1}(n-h) \leq \frac{C}{(K_\infty(h)+1)^2} \leq \frac{C}{\delta^2 n}$. Consequently, for every $n, h \geq 1$,

$$(15) \quad B(n, h) \leq \frac{C}{\delta^2 n} \mathbb{P} \left(\left| \frac{d_{\mathcal{T}_\infty}^{\circ}(\emptyset, v_h^*)}{|v_h^*|} - c_\mu \right| > \varepsilon \right) = \frac{C}{\delta^2 n} \mathbb{P} \left(\left| \frac{d_{\mathcal{T}_\infty}^{\circ}(\emptyset, v_h^*)}{|v_h^*|} - c_\mu \right| > \varepsilon \right).$$

Back to (12), recalling that $\mathbb{P}(|\mathcal{T}| = n) \sim cn^{-3/2}$ as $n \rightarrow \infty$ and using (15) we find that

$$\limsup_{n \rightarrow \infty} \frac{1}{\mathbb{P}(|\mathcal{T}| = n)} \sum_{h=\lceil \delta\sqrt{n} \rceil}^{\lfloor \sqrt{n}/\delta \rfloor} P(n, h) \leq C_\delta \limsup_{n \rightarrow \infty} \mathbb{P} \left(\left| \frac{d_{\mathcal{T}_\infty}^{\circ}(\emptyset, v_n^*)}{|v_n^*|} - c_\mu \right| > \varepsilon \right),$$

for a certain constant $C_\delta > 0$. We then observe that by construction, $d_{\mathcal{T}_\infty}^{\circ}(\emptyset, v_n^*)$ is a sum of n i.i.d. random variables whose distribution is that of $\min(U_{\bar{X}}, \bar{X} - U_{\bar{X}} + 1)$, where \bar{X} has the size-biased law $\bar{\mu}$ and conditionally on \bar{X} , $U_{\bar{X}}$ is uniform on $\{1, \dots, \bar{X}\}$. One sees that such a variable has mean c_μ , so that by the law of large numbers

$$(16) \quad \frac{d_{\mathcal{T}_\infty}^{\circ}(\emptyset, v_n^*)}{n} \xrightarrow[n \rightarrow \infty]{} c_\mu \text{ almost surely.}$$

We conclude that

$$\mathbb{P} \left(\left| \frac{d_{\mathcal{T}_n}^{\circ}(\emptyset, V_n)}{|V_n|} - c_\mu \right| > \varepsilon, |V_n| \in [\delta\sqrt{n}, \frac{1}{\delta}\sqrt{n}], R(V_n) \geq \delta\sqrt{n} \right) \xrightarrow[n \rightarrow \infty]{} 0,$$

as desired. \square

4.3. Convergence to a multiple of the CRT. — We are finally in position to establish Theorem 2. Before that, let us recall a basic fact about the Gromov–Hausdorff topology (see [BBI01, Theorem 7.3.25]). If (E_1, d_1) and (E_2, d_2) are metric spaces, a *correspondence* between E_1 and E_2 is a subset \mathcal{R} of $E_1 \times E_2$ such that for every $x_1 \in E_1$, there exists $x_2 \in E_2$ such that $(x_1, x_2) \in \mathcal{R}$, and conversely. Now, the *distorsion* $\text{dis}(\mathcal{R})$ of the correspondence \mathcal{R} is defined by

$$\text{dis}(\mathcal{R}) := \sup\{|d_1(x_1, y_1) - d_2(x_2, y_2)| : (x_1, x_2), (y_1, y_2) \in \mathcal{R}\}.$$

Then, the Gromov–Hausdorff distance between (E_1, d_1) and (E_2, d_2) is given by

$$d_{\text{GH}}((E_1, d_1), (E_2, d_2)) = \frac{1}{2} \inf_{\mathcal{R}} \text{dis}(\mathcal{R}),$$

where the supremum is over all correspondences between (E_1, d_1) and (E_2, d_2) .

Proof of Theorem 2. — First of all, by Skorokhod’s representation theorem, we may assume that the convergence of Proposition 9 holds almost surely, and we aim at proving that the convergence of Theorem 2 also holds in this sense. Recall that u_0, \dots, u_{n-1} are the vertices of \mathcal{T}_n listed in lexicographical order. We let \mathbf{p}_e be the canonical projection from $[0, 1]$ onto \mathcal{T}_e , and build a correspondence \mathcal{R}_n between $\frac{1}{\sqrt{n}} \cdot \text{Loop}(\mathcal{T}_n)$ and $\frac{2c_\mu}{\sigma_\mu} \cdot \mathcal{T}_e$ as follows:

$$\mathcal{R}_n := \{(u_i, \mathbf{p}_e(s)) \in \text{Loop}(\mathcal{T}_n) \times \mathcal{T}_e : i = \lfloor (n-1)s \rfloor, s \in [0, 1], 0 \leq i < n\}.$$

Let us show that the distorsion of \mathcal{R}_n vanishes as $n \rightarrow \infty$. We argue by contradiction and assume that there exists $\varepsilon > 0$, sequences $(i_n : n \geq 1)$ and $(j_n : n \geq 1)$ with $i_n, j_n \in \{0, \dots, n-1\}$ and

$(s_n : n \geq 1)$ and $(t_n : n \geq 1)$ with $s_n, t_n \in [0, 1]$ such that for every $n \geq 1$, $(u_{i_n}, \mathbf{p}_e(s_n)) \in \mathcal{R}_n$, $(u_{j_n}, \mathbf{p}_e(t_n)) \in \mathcal{R}_n$ and

$$\left| \frac{1}{\sqrt{n}} d_{\mathcal{T}_n}^\circ(u_{i_n}, u_{j_n}) - \frac{2c_\mu}{\sigma_\mu} d_e(s_n, t_n) \right| > \varepsilon.$$

By compactness, up to extraction, we may assume that $\frac{i_n}{n} \rightarrow s \in [0, 1]$ and $\frac{j_n}{n} \rightarrow t \in [0, 1]$ as $n \rightarrow \infty$, so that $s_n \rightarrow s$ and $t_n \rightarrow t$ as well. Without loss of generality, we can also assume that $s \leq t$. By the construction of $\text{Loop}(\mathcal{T}_n)$, following [CK14, Equation (4.3)] we have for every $n \geq 1$

$$(17) \quad \left| d_{\mathcal{T}_n}^\circ(u_{i_n}, u_{j_n}) - \left(H_{i_n}^\circ(\mathcal{T}_n) + H_{j_n}^\circ(\mathcal{T}_n) - 2H_{\mathfrak{m}(i_n, j_n)}^\circ(\mathcal{T}_n) \right) \right| \leq k_{u_{\mathfrak{m}(i_n, j_n)}}(\mathcal{T}_n),$$

where $\mathfrak{m}(i_n, j_n)$ is the index of the most recent common ancestor of u_{i_n} and u_{j_n} in the lexicographical order of \mathcal{T}_n .

Now, recall that $x_0, \dots, x_{2(n-1)}$ are the vertices of \mathcal{T}_n listed in contour order. We follow the guidelines of [Duq03, Section 2] to compare the lexicographical and contour orders of vertices in \mathcal{T}_n . We set

$$b_n(i) := 2i - H_i(\mathcal{T}_n), \quad 0 \leq i < n,$$

so that $b_n(i)$ is the index of the first visit of the vertex u_i in the contour order of \mathcal{T}_n . As a consequence, we have

$$(18) \quad u_i = x_{b_n(i)}, \quad 0 \leq i < n.$$

Moreover, since $\frac{1}{\sqrt{n}} H(\mathcal{T}_n)$ converges in virtue of Proposition 9, the convergence

$$(19) \quad \sup_{0 \leq t \leq 1} \left| \frac{1}{2n} b_n(\lfloor t(n-1) \rfloor) - t \right| = \sup_{0 \leq t \leq 1} \left| \frac{1}{2n} (2\lfloor t(n-1) \rfloor - H_{2\lfloor t(n-1) \rfloor}(\mathcal{T}_n)) - t \right| \xrightarrow[n \rightarrow \infty]{} 0,$$

holds almost surely. But the quantity $\left| \frac{1}{\sqrt{n}} H_{\mathfrak{m}(i_n, j_n)}^\circ(\mathcal{T}_n) - \frac{2c_\mu}{\sigma_\mu} \inf_{s \leq u \leq t} e_u \right|$ is bounded from above by

$$\frac{1}{\sqrt{n}} \sup_{0 \leq i < n} |H_i^\circ(\mathcal{T}_n) - c_\mu H_i(\mathcal{T}_n)| + c_\mu \left| \frac{1}{\sqrt{n}} H_{\mathfrak{m}(i_n, j_n)}(\mathcal{T}_n) - \frac{2}{\sigma_\mu} \inf_{s \leq u \leq t} e_u \right|.$$

Since $u_{\mathfrak{m}(i_n, j_n)}$ is the most recent common ancestor between u_{i_n} and u_{j_n} in \mathcal{T}_n , using (18), we get

$$H_{\mathfrak{m}(i_n, j_n)}(\mathcal{T}_n) = C_{b_n(\mathfrak{m}(i_n, j_n))}(\mathcal{T}_n) = \inf_{b_n(i_n) \leq k \leq b_n(j_n)} C_k(\mathcal{T}_n) = \inf_{\frac{b_n(i_n)}{2n} \leq t \leq \frac{b_n(j_n)}{2n}} C_{2nt}(\mathcal{T}_n).$$

From Proposition 9 and (19) we deduce that

$$\frac{1}{\sqrt{n}} H_{\mathfrak{m}(i_n, j_n)}^\circ(\mathcal{T}_n) \xrightarrow[n \rightarrow \infty]{} \frac{2c_\mu}{\sigma_\mu} \inf_{s \leq u \leq t} e_u \quad \text{almost surely.}$$

By Equation (2), we get by passing to the limit into (17) that

$$\frac{1}{\sqrt{n}} d_{\mathcal{T}_n}^\circ(u_{i_n}, u_{j_n}) \xrightarrow[n \rightarrow \infty]{} \frac{2c_\mu}{\sigma_\mu} d_e(s, t) \quad \text{almost surely,}$$

thus a contradiction, and the proof is complete. \square

4.4. Robustness of the approach. — Let us discuss the robustness of our approach, based on a tightness argument and on the identification of the limit by using a spinal decomposition.

First, it is possible to similarly extend the main result of [CHK15] concerning dissections in a straightforward way, and more precisely to show that Theorem 1 in [CHK15] is more generally true when the finite exponential moment is relaxed to a finite variance condition (we leave details to the reader). As we already mentioned in the Introduction, convergence to the Brownian CRT of discrete weighted tree-like structures is usually obtained under finite exponential moment conditions.

Second, this approach allows to establish Theorem 3, which was left open in [CK14] (in the notation of the latter reference, $\mathcal{T}_2 = \sqrt{2} \cdot \mathcal{T}_e$). The rest of this section is devoted to the proof of this result. In this direction, we fix a critical offspring distribution μ with *infinite* variance such that

$$\mu([i, \infty)) = \frac{L(i)}{i^2}, \quad i \geq 1.$$

If X_1, X_2, \dots are i.i.d. random variables with distribution μ , let B_n be an increasing sequence such that $(X_1 + \dots + X_n - n)/B_n$ converges in distribution to a random variable with Laplace exponent $\lambda \mapsto e^{\lambda^2}$, that is $\sqrt{2}$ times a standard Gaussian random variable (we use the same convention as in [CK14]). We refer to [BGT89, IL71] for a proof of the existence of such a function, that additionally satisfies $B_n = \ell(n)\sqrt{n}$ for a slowly varying function ℓ tending to infinity.

As we will see, the proof of Theorem 3 is similar to that of Theorem 2, except at one crucial point: the analog of (16) in our infinite variance setting, which is the following Lemma.

LEMMA 13. — *Let $((L_i, R_i) : i \geq 1)$ be a sequence of independent random variables with distribution given by $\mathbb{P}((L_1, R_1) = (i, j)) = \mu(i + j + 1)$ for $i, j \geq 0$. Then the convergence*

$$\frac{\sum_{i=1}^n \min(L_i + 1, R_i + 1)}{\sum_{i=1}^n R_i} \xrightarrow[n \rightarrow \infty]{} \frac{1}{2}$$

holds in probability.

In the previous statement, we think of L_i (resp. R_i) as the number of children of v_i^* branching on the left (resp. right) of $[[\emptyset, v_i^*[[$ (so that $\min(L_i + 1, R_i + 1)$ is $d_{\mathcal{T}_\infty}^c(v_{i-1}^*, v_i^*)$).

The remaining of this section is organized as follows. We explain how Theorem 3 follows from Lemma 13 in Section 4.4.1, and then establish Lemma 13 in Section 4.4.2.

4.4.1. Towards the proof of Theorem 3. — The starting point is the analog of Proposition 6 (see [Duq03, Kor13] for a proof). For every $n \geq 1$, let \mathcal{T}_n be a BGW_μ tree conditioned to have n vertices. Then, we have

$$(20) \quad \left(\frac{1}{B_n} W_{nt}(\mathcal{T}_n), \frac{B_n}{n} H_{nt}(\mathcal{T}_n), \frac{B_n}{n} C_{2nt}(\mathcal{T}_n) \right)_{0 \leq t \leq 1} \xrightarrow[n \rightarrow \infty]{} \sqrt{2} \cdot (\mathfrak{e}_t, \mathfrak{e}_t, \mathfrak{e}_t)_{0 \leq t \leq 1}.$$

where the convergence holds in distribution in the space $\mathcal{C}([0, 1])^3$, and $(\mathfrak{e}_t : 0 \leq t \leq 1)$ is the Brownian excursion.

First step. — One starts by establishing the analog of Lemma 10:

- (i) The sequence $(\frac{1}{B_n} \mathbf{H}_{nt}^\circ(\mathcal{T}_n) : 0 \leq t \leq 1)$ is tight in $\mathcal{C}([0, 1])$.
- (ii) For every $n \geq 1$, if U^n be a random variable uniformly distributed on $\{0, 1, \dots, n-1\}$ independent of \mathcal{T}_n , then

$$\frac{1}{B_n} \left| \mathbf{H}_{U^n}^\circ(\mathcal{T}_n) - \frac{1}{2} \mathbf{W}_{U^n}(\mathcal{T}_n) \right| \xrightarrow[n \rightarrow \infty]{} 0 \quad \text{in probability.}$$

The first assertion is proved exactly as Lemma 10 (i) (the main difference with the finite variance case, where $\mathbf{W}(\mathcal{T}_n)$ and $\mathbf{H}(\mathcal{T}_n)$ were of the same order \sqrt{n} , is that here $\mathbf{W}(\mathcal{T}_n)$ is of order B_n and $\mathbf{H}(\mathcal{T}_n)$ is of order $\frac{n}{B_n} = o(B_n)$).

The second assertion is proved by used the size-biasing argument of Section 4.2. The main difference is that the analog of Lemma 12 is the following result, whose proof is similar.

LEMMA 14. — *Let V_n be the U^n -th vertex of \mathcal{T}_n in lexicographical order. Then, for every $\varepsilon > 0$ and every $\delta \in (0, 1)$, we have*

$$(21) \quad \mathbb{P} \left(\left| \frac{d_{\mathcal{T}_n}^\circ(\emptyset, V_n)}{R(V_n)} - \frac{1}{2} \right| > \varepsilon, |V_n| \in \left[\delta \frac{n}{B_n}, \frac{1}{\delta} \frac{n}{B_n} \right], R(V_n) \geq \delta B_n \right) \xrightarrow[n \rightarrow \infty]{} 0,$$

where $R(V_n)$ is the number of children of vertices of $[\![\emptyset, V_n]\!]$ branching on the right of $[\![\emptyset, V_n]\!]$.

This result then implies (ii), since by definition $(|V_n|, R(V_n)) = (H_{U^n}(\mathcal{T}_n), W_{U^n}(\mathcal{T}_n))$, so that (20) entails

$$\limsup_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P} \left(|V_n| \notin \left[\delta \frac{n}{B_n}, \frac{1}{\delta} \frac{n}{B_n} \right] \text{ or } R(V_n) < \delta B_n \right) = 0.$$

Observe the presence of the quantity $d_{\mathcal{T}_n}^\circ(\emptyset, V_n)/R(V_n)$ in Lemma 14, in contrast with the finite variance case (where $|V_n|$ appears instead of $R(V_n)$). Indeed, in this new context, $d_{\mathcal{T}_n}^\circ(\emptyset, V_n)$ is of order $R(V_n)$ and not of order $|V_n|$ anymore. This change comes from the fact that distances in $\text{Loop}(\mathcal{T}_n)$ asymptotically come only from the number of children grafted on branches of \mathcal{T}_n (of order B_n) and not from the height of \mathcal{T}_n (of order $\frac{n}{B_n} = o(B_n)$).

Proof of Lemma 14. — One starts exactly as in the proof of Lemma 12 to obtain

$$(22) \quad \mathbb{P} \left(\left| \frac{d_{\mathcal{T}_n}^\circ(\emptyset, V_n)}{R(V_n)} - \frac{1}{2} \right| > \varepsilon, |V_n| \in \left[\delta \frac{n}{B_n}, \frac{1}{\delta} \frac{n}{B_n} \right], R(V_n) \geq \delta B_n \right) = \frac{1}{\mathbb{P}(|\mathcal{T}| = n)} \sum_{h=\lceil \delta n/B_n \rceil}^{\lfloor n/(\delta B_n) \rfloor} P(n, h),$$

with

$$P(n, h) = \frac{1}{n} \mathbb{P} \left(\left| \frac{d_{\mathcal{T}_\infty}^\circ(\emptyset, v_h^*)}{R(v_h^*)} - \frac{1}{2} \right| > \varepsilon, R(v_h^*) \geq \delta B_n, |\text{Cut}_{v_h^*}(\mathcal{T}_\infty)| + |\mathcal{T}'| = n + 1 \right),$$

where $R(v_h^*)$ is the number of children of vertices of $[\![\emptyset, v_h^*]\!]$ branching on the right of $[\![\emptyset, v_h^*]\!]$.

Recall that $K_\infty(j)$ is the number of children of vertices of the spine with height smaller than j , and that do not belong to the spine. In particular, $K_\infty(j) \geq R(v_j^*)$. As a consequence, keeping the notation $\Phi_k(m) = \mathbb{P}(\sum_{i=1}^k |\mathcal{T}(i)| = m)$, we get

$$P(n, h) \leq \frac{1}{n} \mathbb{E} \left[\mathbb{1} \left\{ \left| \frac{d_{\mathcal{T}_\infty}^\circ(\emptyset, v_h^*)}{R(v_h^*)} - c_\mu \right| > \varepsilon, K_\infty(h) \geq \delta B_n \right\} \Phi_{K_\infty(h)+1}(n-h) \right].$$

We now use the following estimates (see [Kor17, Sec 4.3]): we have $\mathbb{P}(|\mathcal{T}| = n) \sim C(nB_n)^{-1}$ as $n \rightarrow \infty$ for a certain constant $C > 0$, and there exists a constant $C' > 0$ and an increasing regularly varying sequence $(B'_n : n \geq 1)$ tending to infinity such that for every $k, m \geq 1$,

$$(23) \quad \Phi_k(m) \leq \frac{C'}{B'_k},$$

with $B'_{B_n} \sim n$ as $n \rightarrow \infty$. Thus

$$P(n, h) \leq \frac{C'}{nB'_{\delta B_n}} \mathbb{P} \left(\left| \frac{d_{\mathcal{T}_\infty}^\circ(\emptyset, v_h^*)}{R(v_h^*)} - \frac{1}{2} \right| > \varepsilon \right) \leq \frac{C''}{n^2} \mathbb{P} \left(\left| \frac{d_{\mathcal{T}_\infty}^\circ(\emptyset, v_h^*)}{R(v_h^*)} - \frac{1}{2} \right| > \varepsilon \right)$$

for a certain constant $C''_\delta > 0$. Therefore,

$$\begin{aligned} \limsup_{n \rightarrow \infty} \mathbb{P} \left(\left| \frac{d_{\mathcal{T}_n}^\circ(\emptyset, V_n)}{R(V_n)} - \frac{1}{2} \right| > \varepsilon, |V_n| \in \left[\delta \frac{n}{B_n}, \frac{1}{\delta} \frac{B_n}{n} \right], R(V_n) \geq \delta B_n \right) \\ \leq \limsup_{n \rightarrow \infty} nB_n \cdot \frac{n}{\delta B_n} \cdot \frac{C''_\delta}{n^2} \sup_{h \geq \delta n/B_n} \mathbb{P} \left(\left| \frac{d_{\mathcal{T}_\infty}^\circ(\emptyset, v_h^*)}{R(v_h^*)} - \frac{1}{2} \right| > \varepsilon \right) \end{aligned}$$

The desired result then follows from Lemma 13, since $d_{\mathcal{T}_\infty}^\circ(\emptyset, v_h^*)/R(v_h^*)$ has the same distribution as $\sum_{i=1}^h \min(L_i + 1, R_i + 1) / (\sum_{i=1}^h R_i)$. \square

Second step. — Using (i) and (ii), one then shows a similar result to Proposition 9 with the same proof (only by replacing the occurrences of \mathbf{H} with $\frac{\mathbf{W}}{2}$ and of \sqrt{n} with B_n): the convergence

$$(24) \quad \left(\frac{B_n}{n} \mathbf{C}_{2nt}(\mathcal{T}_n), \frac{B_n}{n} \mathbf{H}_{nt}(\mathcal{T}_n), \frac{1}{B_n} \mathbf{H}_{nt}^\circ(\mathcal{T}_n) \right)_{0 \leq t \leq 1} \xrightarrow[n \rightarrow \infty]{} \sqrt{2} \cdot \left(\mathbf{e}_t, \mathbf{e}_t, \frac{1}{2} \mathbf{e}_t \right)_{0 \leq t \leq 1}$$

holds jointly in distribution in the space $\mathcal{C}([0, 1])^3$.

Third step. — Finally, Theorem 3 is obtained from (24) in the same way Theorem 2 was obtained from Proposition 9 in Section 4.3. The only difference is that one bounds the quantity $\left| \frac{1}{B_n} \mathbf{H}_{m(i_n, j_n)}^\circ(\mathcal{T}_n) - \frac{\sqrt{2}}{2} \inf_{s \leq u \leq t} \mathbf{e}_u \right|$ from above by

$$\sup_{0 \leq i < n} \left| \frac{1}{B_n} \mathbf{H}_i^\circ(\mathcal{T}_n) - \frac{B_n}{2n} \mathbf{H}_i(\mathcal{T}_n) \right| + \frac{1}{2} \left| \frac{B_n}{n} \mathbf{H}_{m(i_n, j_n)}(\mathcal{T}_n) - \sqrt{2} \inf_{s \leq u \leq t} \mathbf{e}_u \right|.$$

This completes the sketch of the proof of Theorem 3.

4.4.2. Proof of Lemma 13. — We first recall some results concerning random variables falling within the domain of attraction of a stable law of index $\alpha = 1$ (see e.g. [Ber]). Assume that $(Z_i : i \geq 1)$ are i.i.d. integer valued random variables such that

$$\mathbb{P}(Z_1 \geq k) = \frac{\ell(k)}{k}, \quad k \geq 1,$$

where ℓ is a slowly varying function such that $\sum_{k \geq 1} \frac{\ell(k)}{k} = \infty$ (so that $\mathbb{E}(Z_1) = \infty$). Then the convergence

$$\frac{Z_1 + \cdots + Z_n}{n \cdot \sum_{k=1}^{a_n} \frac{\ell(k)}{k}} \xrightarrow[n \rightarrow \infty]{} 1$$

holds in probability, where a_n satisfies $\frac{\ell(a_n)}{a_n} \sim \frac{1}{n}$ as $n \rightarrow \infty$.

Back to the proof of Lemma 13, observe that for every $k \geq 0$, $\mathbb{P}(R_1 = k) = \mu([k+1])$ and $\mathbb{P}(\min(L_1, R_1) = k) = 2\mu([2k+1]) - \mu(2k+1)$. Therefore, by standard integration properties of slowly varying functions (see e.g. [BGT89, Proposition 1.5.10])

$$\mathbb{P}(R_1 \geq k) \underset{k \rightarrow \infty}{\sim} \frac{L(k)}{k}, \quad \mathbb{P}(\min(L_1, R_1) \geq k) \underset{k \rightarrow \infty}{\sim} \frac{L(k)}{2k}.$$

As a consequence if we choose a_n so that $\frac{L(a_n)}{a_n} \sim \frac{1}{n}$, by the previous paragraph the convergences

$$\frac{\sum_{i=1}^n R_i}{n \cdot \sum_{k=1}^{a_n} \frac{L(k)}{k}} \xrightarrow[n \rightarrow \infty]{} 1 \quad \text{and} \quad \frac{\sum_{i=1}^n \min(L_i, R_i)}{n \cdot \sum_{k=1}^{a_n/2} \frac{L(k)}{2k}} \xrightarrow[n \rightarrow \infty]{} 1$$

hold in probability. Since $\sum_{k=1}^{a_n} \frac{L(k)}{k} \rightarrow \infty$ as $n \rightarrow \infty$, it follows that

$$\frac{\sum_{i=1}^n \min(L_i, R_i)}{\sum_{i=1}^n \min(L_i + 1, R_i + 1)} \xrightarrow[n \rightarrow \infty]{} 1 \quad \text{in probability.}$$

It therefore remains to check that

$$\sum_{k=1}^{a_n} \frac{L(k)}{k} \underset{n \rightarrow \infty}{\sim} 2 \sum_{k=1}^{a_n/2} \frac{L(k)}{2k}.$$

But this simply follows from [BGT89, Proposition 1.5.9 a], which ensures that the function $n \mapsto \sum_{k=1}^n \frac{L(k)}{k}$ is slowly varying at infinity. The proof is now complete. \square

5. Applications to random planar maps

5.1. A modified looptree. — In view of our applications to random planar maps, we need to extend Theorems 1 and 2 to a modified version of the looptree $\text{Loop}(\tau)$, that was first introduced in [CK15] and whose definition we now recall.

With every plane tree τ , we associate a planar map $\overline{\text{Loop}}(\tau)$, that is obtained from $\text{Loop}(\tau)$ by contracting the edges (u, v) such that v is the last child of u in lexicographical order in τ (see Figure 7 for an example). We still view $\overline{\text{Loop}}(\tau)$ as a compact metric space by endowing its vertices with the graph distance.

It is a simple matter to check that a result similar to Theorem 1 holds with Loop replaced by $\overline{\text{Loop}}$, with almost the same proof. However, for Theorem 2, distances are changed by a constant

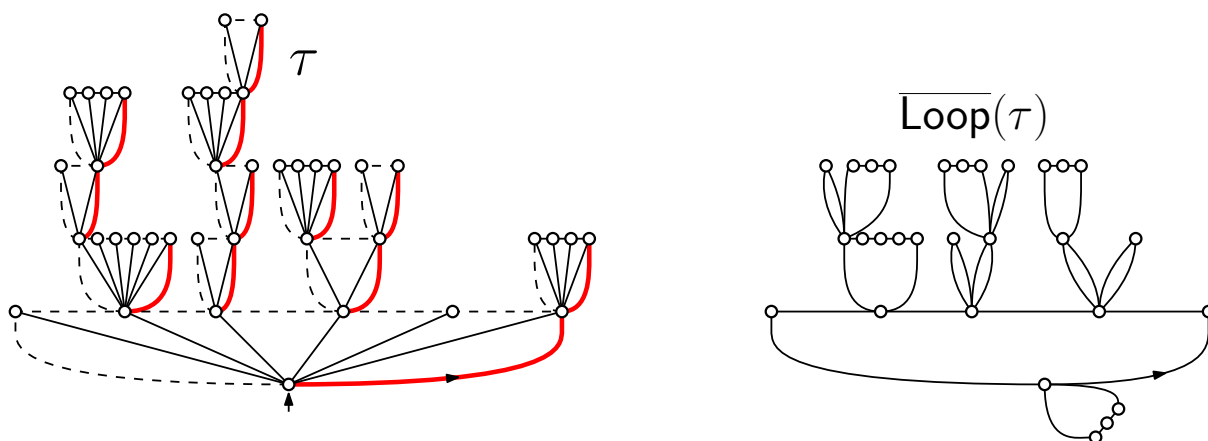


Figure 7. A discrete tree τ (with $\text{Loop}(\tau)$ in dashed edges) and its associated planar map $\overline{\text{Loop}}(\tau)$. The contracted edges are in bold red.

factor when replacing Loop by $\overline{\text{Loop}}$. The proof of the following theorem goes along the same lines as that of Theorem 2, and we leave details to the reader.

THEOREM 15. — *Let μ be a critical offspring distribution with finite positive variance σ_μ^2 . For every $n \geq 1$, let \mathcal{T}_n be a BGW_μ tree conditioned on having n vertices. Then the convergence*

$$\frac{1}{\sqrt{n}} \cdot \overline{\text{Loop}}(\mathcal{T}_n) \xrightarrow{n \rightarrow \infty} \frac{2}{\sigma_\mu} \cdot \frac{1}{4} (\sigma_\mu^2 + \mu(2\mathbb{Z}^+)) \cdot \mathcal{T}_e$$

holds in distribution for the Gromov–Hausdorff topology.

5.2. Applications to random planar maps. —

5.2.1. Maps. — Recall that a *planar map* is a proper embedding of a finite connected graph in the sphere \mathbb{S}^2 , viewed up to orientation-preserving homeomorphisms. The faces are the connected components of the complement of the embedding, and the degree $\deg(f)$ of the face f is the number of oriented edges incident to this face. We systematically consider *rooted* maps, i.e., with a distinguished oriented edge called the *root edge*. The face f_* on the right of the root edge is the *root face*. We focus on planar maps *with a boundary*, meaning that the root face is an *external face* whose incident edges and vertices form the *boundary* of the map. The boundary of a map \mathbf{m} is denoted by $\partial\mathbf{m}$ and the degree $\#\partial\mathbf{m}$ of the external face is called the *perimeter* of \mathbf{m} . For technical reasons, it is sometimes simpler to deal with the *scooped-out* map $\text{Scoop}(\mathbf{m})$, which is obtained from $\partial\mathbf{m}$ by duplicating the edges whose both sides belong to the root face (see Figure 8). Note that $\partial\mathbf{m}$ and $\text{Scoop}(\mathbf{m})$ define the same metric space.

We also restrict ourselves to *bipartite* maps, in which all the faces have even degree. The set of bipartite maps is denoted by \mathcal{M} , and the set of bipartite maps with perimeter $2k$ by \mathcal{M}_k . By convention, the map \dagger made of a single vertex is the only element of \mathcal{M}_0 . We may also consider *rooted pointed* maps, that have a distinguished vertex additionally to the root edge and whose set is denoted by \mathcal{M}_\bullet .

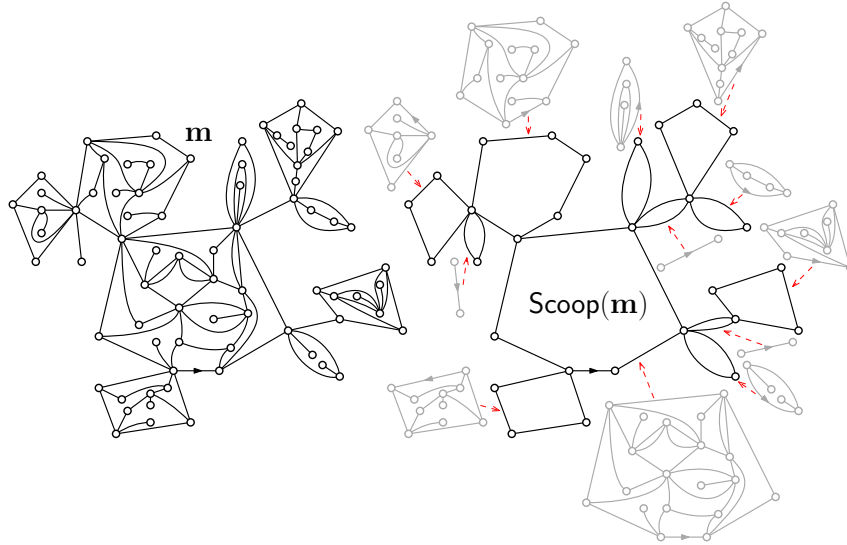


Figure 8. A rooted planar map \mathbf{m} and its scooped-out map $\text{Scoop}(\mathbf{m})$.

5.2.2. Boltzmann distributions. — Let us recall the construction of the Boltzmann distributions on bipartite maps and their main properties. We first fix a *weight sequence* $\mathbf{q} = (q_k : k \geq 1)$ of nonnegative real numbers, and define the *Boltzmann weight* of a bipartite map \mathbf{m} by

$$w_{\mathbf{q}}(\mathbf{m}) := \prod_{f \in \text{Faces}(\mathbf{m})} q_{\deg(f)/2},$$

with the convention $w_{\mathbf{q}}(\dagger) = 1$. We say that \mathbf{q} is admissible if the *partition function*

$$Z_{\mathbf{q}} := 1 + \frac{w_{\mathbf{q}}(\mathcal{M}_{\bullet}) - 1}{2} \in [1, \infty]$$

is finite. Then, the *Boltzmann distribution with weight sequence* \mathbf{q} is the probability measure defined by

$$\mathbb{P}_{\mathbf{q}}(\mathbf{m}) = \frac{w_{\mathbf{q}}(\mathbf{m})}{w_{\mathbf{q}}(\mathcal{M})}, \quad \mathbf{m} \in \mathcal{M}.$$

We will use the following function introduced in [MM07]:

$$f_{\mathbf{q}}(x) := \sum_{k=1}^{\infty} \binom{2k-1}{k-1} q_k x^{k-1}, \quad x \geq 0.$$

The partition function for bipartite maps with a fixed perimeter and the associated generating function

$$F_k := \frac{1}{q_k} \sum_{\mathbf{m} \in \mathcal{M}_k} w_{\mathbf{q}}(\mathbf{m}), \quad k \geq 0, \quad \text{and} \quad F(x) := \sum_{k=0}^{\infty} F_k x^k, \quad x \geq 0,$$

will also play a role (here, the factor $1/q_k$ stands for the fact that the root face receives no weight). The radius of convergence of the latter power series is $r_{\mathbf{q}} := (4Z_{\mathbf{q}})^{-1}$.

A powerful tool to study Boltzmann distributions is the Bouttier–Di Francesco–Guitter bijection [BDFG04] that associates to every (pointed) map a tree with labels associated to vertices

at even height. The tree associated to a (pointed) Boltzmann map by this bijection is then a so-called *two-type* BGW tree, whose distribution is given in [MM07, Proposition 7].

The weight sequences \mathbf{q} can then be classified throughout the distribution of this tree, following [MM07, LGM11, BBG12]. This classification can be rephrased as follows: we say that \mathbf{q} is critical if the expected number of vertices of the tree (or, equivalently, of the associated Boltzmann map) is infinite, and subcritical otherwise. Moreover, we say that \mathbf{q} is generic if the offspring distribution of vertices at odd height in the tree (which one can think of as the law of the degrees of the faces in the map) has finite variance, and \mathbf{q} is non-generic with parameter $\alpha \in (1, 2)$ if this offspring distribution falls in the domain of attraction of a stable law with parameter α . As we mentioned in the Introduction, non-generic critical sequences with parameter $\alpha \in (1, 3/2)$ are often called *dense*, while for $\alpha \in (3/2, 2)$ they are called *dilute*.

5.2.3. Proof of Corollaries 4 and 5. — The following result is a direct consequence of [Ric18, Corollaries 3.4 and 3.7], combined with [CK15, Lemma 4.3].

LEMMA 16. — *Let \mathbf{q} be an admissible weight sequence. For every $k \geq 0$, let \mathcal{M}_k (resp. $\mathcal{M}_{\geq k}$) be a Boltzmann map with weight sequence \mathbf{q} conditioned to have perimeter $2k$ (resp. at least $2k$). Then, there exists an offspring distribution ν such that the following identities hold in distribution*

$$\text{Scoop}(\mathcal{M}_k) = \overline{\text{Loop}}(\mathcal{T}_{2k+1}) \quad \text{and} \quad \text{Scoop}(\mathcal{M}_{\geq k}) = \overline{\text{Loop}}(\mathcal{T}_{\geq 2k+1}),$$

where \mathcal{T}_{2k+1} (resp. $\mathcal{T}_{\geq 2k+1}$) is a BGW $_{\nu}$ tree conditioned to have $2k + 1$ vertices (resp. at least $2k + 1$ vertices).

The offspring distribution ν is given explicitly in [Ric18, Corollary 3.4], and we also have the following information.

- If \mathbf{q} is subcritical (case $a = 3/2$ in [Ric18]), then ν has mean $m_{\nu} = 1$ and finite variance

$$\sigma_{\nu}^2 = \left(\frac{F(r_{\mathbf{q}})}{1 - Z_{\mathbf{q}}^2 f'_{\mathbf{q}}(Z_{\mathbf{q}})} \right)^2,$$

see [Ric18, Lemma 3.5 and (41)].

- If \mathbf{q} is generic critical (case $a = 5/2$ in [Ric18]) or dilute (case $a \in (3/2, 2)$ in [Ric18]), then ν has mean

$$m_{\nu} = \frac{1}{1 + \frac{F(r_{\mathbf{q}})}{2r_{\mathbf{q}}F'(r_{\mathbf{q}})}} < 1,$$

and falls in the domain of attraction of a stable distribution with parameter $3/2$ (in the generic critical regime) or $\alpha - 1/2 \in (1, 3/2)$ (in the dilute regime with parameter $\alpha \in (3/2, 2)$).

Corollaries 4 and 5 then immediately follow, with

$$K_{\mathbf{q}} = \frac{2}{\sigma_{\nu}} \cdot \frac{1}{4} (\sigma_{\nu}^2 + 1),$$

and $J_{\mathbf{q}}$ a random variable defined by $\mathbb{P}(J_{\mathbf{q}} \geq x) = \left(\frac{1-m_{\nu}}{x}\right)^{\alpha-1/2}$ for $x \geq 1 - m_{\nu}$ (in the dilute regime with parameter $\alpha \in (3/2, 2)$) or $\mathbb{P}(J_{\mathbf{q}} \geq x) = \left(\frac{1-m_{\nu}}{x}\right)^{3/2}$ (in the generic critical regime).

REMARK 2. — We can now discuss more precisely the assumption that the BGW tree has *at least* n vertices in Theorem 1, rather than exactly n vertices. In the above application, the offspring distribution ν is given in terms of the partition function for maps with a *simple* boundary (that is, with no self-intersections) of fixed perimeter (see [Ric18, Corollary 3.4]). However, we are not able to obtain an asymptotic behaviour for these quantities (only for the remainder of their sum) as explained in [Ric18, Remark 2.8]. For this reason, the assumptions of [Kor15] are a priori not satisfied by the offspring distribution ν , which forces us to use a weaker conditioning (so that the weaker regularity assumption is fulfilled by ν).

Finally, in the non-generic critical regime with parameter $\alpha = 3/2$, the probability measure ν can be either subcritical or critical, and is expected to be in the domain of attraction of a Cauchy distribution. However, the last assertion is only established in [Ric18, Section 6] for a particular weight sequence \mathfrak{q} (and then, ν is critical, so that Theorem 1 does not apply).

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