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Two-phase Flows Involving Discontinuities on the Capillary Pressure

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ABSTRACT. *We consider a immiscible two-phase flow through an one-dimensional heterogeneous porous medium made of an apposition of several homogeneous porous media. This leads to a nonlinear degenerate parabolic problem, with monotonous transmission conditions between the different homogeneous subdomains. We give an implicit finite volume scheme for such a two-phase flow, and we prove the convergence of the inducted discrete solutions to a weak solution. Under assumption on the initial condition, i.e. if it yields bounded flux, and if the total flow-rate belongs to $BV(0,T)$, then the discrete solution obtained via the scheme converges toward a solution with bounded flux. We prove a L^1-contraction principle for such a bounded flux solution. This uniqueness result is extended using a SOLA approach in the case where there are no particular regularity assumptions on the initial data, and we check that the discrete solution obtained via the implicit finite volume scheme converges to this SOLA.*

KEYWORDS: *flows in porous media, finite volume method, discontinuous capillarity pressures, nonlinear PDE of parabolic type.*

1. Introduction: the model

We consider a heterogeneous porous medium Ω, which is an apposition of homogeneous porous media Ω_i, representing the different geological layers, so that the physical properties of the medium only depend of the rock type and are piece-wise constant. We restrict our study to the one-dimensional case, even if all the results stated in section 2 can be quite easily adapted to larger dimensions (see [ENC 06]). For the sake of simplicity, we only deal with two geological layers with same size, because a generalisation to an arbitrary finite number of geological layers would only lead to notation difficulties. In the sequel, we denote by $\Omega =]-1,1[$ the heterogeneous

porous medium, and by $\Omega_1 =]-1,0[$, $\Omega_2 =]0,1[$ the two homogeneous layers. The interface between the layers is thus $\{x=0\}$. T is a positive integer.

The study of the flow in each Ω_i, $i = 1,2$ leads to the following degenerated parabolic equation:

$$\phi_i \partial_t u + \partial_x \left(q f_i(u) - \lambda_i(u) \partial_x \pi_i(u) \right) = 0, \qquad [1]$$

where, for $i = 1,2$:

– ϕ_i is the porosity of Ω_i, $0 < \phi_i < 1$,
– $u(x,t)$ is the water-saturation, $0 \le u \le 1$,
– $q(t)$ is the total flow rate, $q \in L^1(0,T)$, $q \ge 0$,
– f_i is a continuous increasing function, $f_i(0) = 0$, $f_i(1) = 1$,
– λ_i is a continuous function, with $\lambda_i(0) = \lambda_i(1) = 0$, $\lambda_i(s) > 0$ if $0 < s < 1$,
– the capillary pressure $u \mapsto \pi_i(u) \in C^1([0,1], \mathbf{R})$ is increasing.

Focusing on the interface $\{x = 0\}$, we get the following transmission conditions: denoting u_i the restriction of u to Ω_i and $\varphi_i(s) = \int_0^s \lambda_i(a)\pi_i'(a)da$,

$$q(t) f_1(u_1)(0,t) - \partial_x \varphi_1(u_1)(0,t) = q(t) f_2(u_2)(0,t) - \partial_x \varphi_2(u_2)(0,t), \qquad [2]$$

$$\tilde{\pi}_1(u_1(0,t)) \cap \tilde{\pi}_2(u_2(0,t)) \neq \emptyset, \qquad [3]$$

where $\tilde{\pi}_i(s) = \begin{cases} \pi_i(s) & \text{if } s \in]0,1[, \\ ]-\infty, \pi_i(0)] & \text{if } s = 0, \\ [\pi_i(1), +\infty[& \text{if } s = 1. \end{cases}$

We choose the boundary conditions:

$$q(t) f_1(u_1)(-1,t) - \partial_x \varphi_1(u_1)(-1,t) = g(t) \in [0, q(t)], \qquad [4]$$

$$\partial_x \varphi_2(u_2)(1,t) = 0. \qquad [5]$$

and an initial data $u_0 \in L^\infty(\Omega)$.

Definition 1.1 (weak solution) *A function u is said to be a weak solution if it fulfills:*

1) $u \in L^\infty(\Omega \times (0,T))$, with $0 \le u \le 1$,
2) for $i = 1,2$, $\varphi_i(u) \in L^2((0,T); H^1(\Omega_i))$,
3) for a.e. $t \in (0,T)$, $\tilde{\pi}_1(u_1(0,t)) \cap \tilde{\pi}_2(u_2(0,t)) \neq \emptyset$,
4) for all $\psi \in \mathcal{D}(\overline{\Omega} \times [0,T[)$,

$$\begin{aligned} &\int_0^T \sum_{i=1,2} \int_{\Omega_i} \phi_i u(x,t) \partial_t \psi(x,t) dx dt + \sum_{i=1,2} \int_{\Omega_i} \phi_i u_0(x) \psi(x,0) dx \\ &\quad + \int_0^T \sum_{i=1,2} \int_{\Omega_i} \left[q(t) f_i(u)(x,t) - \partial_x \varphi_i(u)(x,t) \right] \partial_x \psi(x,t) dx dt \\ &\qquad + \int_0^T g(t) \psi(-1,t) dt - \int_0^T q(t) f_2(u_2)(1,t) \psi(1,t) dt = 0. \end{aligned} \qquad [6]$$

2. Convergence of an implicit finite volume scheme

For the sake of simplicity, we choose to deal only with uniform discretization of $\Omega \times (0,T)$, which are defined by two positive integers M, N.

Discretization of Ω*:* we denote by $\delta x = 1/N$ and

$$\begin{cases} x_j = j/N, & \forall j \in [\![-\mathrm{N}, \mathrm{N}]\!], \\ x_{j+1/2} = \dfrac{j+1/2}{N}, & \forall j \in [\![-\mathrm{N}, \mathrm{N}-1]\!]. \end{cases}$$

Discretization of $[0,T]$*:* we denote by $\delta t = T/M$, and $t^n = n\delta t$, for all $n \in [\![0, \mathrm{M}]\!]$.

Discretization of the data:

– $\forall j \in [\![-\mathrm{N}, \mathrm{N}-1]\!]$, $u_{0,\mathcal{D}}(x_{j+1/2}) = u^0_{j+1/2} = \frac{1}{\delta x}\int_{x_j}^{x_{j+1}} u_0(x)dx$,

– $q^0 = q^1 = \frac{1}{\delta t}\int_0^{\delta t} q(t)dt, \quad q^{n+1} = \frac{1}{\delta t}\int_{t^n}^{t^{n+1}} q(t)dt$,

– $g^0 = g^1 = \frac{1}{\delta t}\int_0^{\delta t} g(t)dt, \quad g^{n+1} = \frac{1}{\delta t}\int_{t^n}^{t^{n+1}} g(t)dt$,

Equation [1] can be rewritten $\phi_i \partial_t u + \partial_x F(x,t) = 0$, with $F(x,t) = q(t) f_i(u) - \partial_x \varphi_i(u)$. We consider the following implicit scheme: $\forall j \in [\![-\mathrm{N}, \mathrm{N}-1]\!]$, $\forall n \in [\![0, \mathrm{M}-1]\!]$,

$$\phi_i \frac{u^{n+1}_{j+1/2} - u^n_{j+1/2}}{\delta t}\delta x + F^{n+1}_{j+1} - F^{n+1}_j = 0, \qquad [7]$$

where the discrete fluxes are given by the following two-points scheme:

– $\forall j \in J_{int} = [\![-\mathrm{N}+1, -2]\!] \cup [\![1, \mathrm{N}-2]\!]$, $\forall n \in [\![0, \mathrm{M}-1]\!]$,

$$F^{n+1}_j = q^{n+1} f_i(u^{n+1}_{j-1/2}) - \frac{\varphi_i(u^{n+1}_{j+1/2}) - \varphi_i(u^{n+1}_{j-1/2})}{\delta x}, \qquad [8]$$

– $\forall n \in [\![0, \mathrm{M}-1]\!]$,

$$F^{n+1}_{-N} = g^{n+1}, \qquad [9]$$

$$F^{n+1}_N = q^{n+1} f_2(u^{n+1}_{N-1/2}). \qquad [10]$$

– $\forall n \in [\![0, \mathrm{M}-1]\!]$,

$$\tilde{\pi}_1(u^{n+1}_{0,1}) \cap \tilde{\pi}_2(u^{n+1}_{0,2}) \neq \emptyset \qquad [11]$$

$$F^{n+1}_0 = q^{n+1} f_1(u^{n+1}_{-1/2}) - \frac{2(\varphi_1(u^{n+1}_{0,1}) - \varphi_1(u^{n+1}_{-1/2}))}{\delta x} \qquad [12]$$

$$= q^{n+1} f_2(u^{n+1}_{0,2}) - \frac{2(\varphi_2(u^{n+1}_{1/2}) - \varphi_2(u^{n+1}_{0,2}))}{\delta x}. \qquad [13]$$

The definition of F^{n+1}_0 is justified by the following lemma, which is a consequence of the monotony of the transmission conditions.

Lemma 2.1 (discrete transmission conditions) *Let $(a,b) \in \mathbf{R}^2$, let $n \in [\![0, \mathrm{M}-1]\!]$, then there exists a unique couple $(c,d) \in [0,1]^2$ such that:*

$$\begin{cases} q^{n+1} f_1(a) - \dfrac{2(\varphi_1(c) - \varphi_1(a))}{\delta x} = q^{n+1} f_2(d) - \dfrac{2(\varphi_2(b) - \varphi_2(d))}{\delta x}, \\ \tilde{\pi}_1(c) \cap \tilde{\pi}_2(d) \neq \emptyset, \end{cases}$$

where for all $s \leq 0,\ f_i(s) = 0,\ \varphi_i(s) = 0$, and for all $s \geq 1,\ f_i(s) = 1,\ \varphi_i(s) = \varphi_i(1)$. Furthermore, $(a,b) \mapsto c$ and $(a,b) \mapsto d$ are continuous and non-decreasing w.r.t. each one of their arguments.

We deduce from lemma 2.1 that [7] is a monotonous scheme, and thus we can apply a classical study (see e.g. [EYM 00, ENC 06]) to claim the following.

Proposition 2.2 (discrete solution) *There exists a unique solution $(u^{n+1}_{j+1/2})_{j,n}$ to the scheme, fulfilling furthermore: $\forall j \in [\![-\mathrm{N}, \mathrm{N}-1]\!]$, $\forall n \in [\![0, \mathrm{M}]\!]$, $0 \leq u^n_{j+1/2} \leq 1$.*
We define the unique discrete solution $u_{\mathcal{D}}$ almost everywhere in $\Omega \times (0,T)$ by:

$$u_{\mathcal{D}}(x,t) = u^{n+1}_{j+1/2} \quad \textit{if}\ (x,t) \in]x_j, x_{j+1}[\times]t^n, t^{n+1}].$$

If $u_{0,\mathcal{D}}, v_{0,\mathcal{D}}$ are two discrete initial data, and if $u_{\mathcal{D}}, v_{\mathcal{D}}$ are the inducted discrete solution, we obtain: $\forall t \in]0,T]$,

$$\sum_{i=1,2} \int_{\Omega_i} \phi_i |u_{\mathcal{D}}(x,t) - v_{\mathcal{D}}(x,t)| dx \leq \sum_{i=1,2} \int_{\Omega_i} \phi_i |u_{0,\mathcal{D}}(x) - v_{0,\mathcal{D}}(x)| dx. \qquad [14]$$

The discrete solution fulfils a discrete $L^2(0,T;H^1(\Omega_i))$ estimate, i.e. there exists $C(q,T,\phi_i,\pi_i)$ depending on the data, but not on the discretization of $\Omega \times (0,T)$ such that: $\forall i = 1,2$, $\forall (N,M) \in (\mathbf{N}^\star)^2$,

$$\int_0^T \left[\sum_{j \in J_{int}} \delta x \left(\frac{\varphi_i(u_{\mathcal{D}})(x_{j+1/2},t) - \varphi_i(u_{\mathcal{D}})(x_{j-1/2},t)}{x_{j+1/2} - x_{j-1/2}} \right)^2 \right] dt \leq C. \qquad [15]$$

We deduce from [15] some estimates on the space and time translates (see [EYM 00]), and so we obtain a compactness result on the set of the discrete solutions. Letting the discretization step tend to 0, one gets the following convergence result.

Proposition 2.3 (convergence to a weak solution) *Let $(M_p)_{p \in \mathbf{N}}, (N_p)_{p \in \mathbf{N}}$ be two sequences of positive integers tending to $+\infty$, and let $(u_{\mathcal{D}_p})_p$ be the sequence of the inducted discrete solutions. Then there exists a weak solution u in the sense of definition 1.1 such that, up to a subsequence: $\forall r \in [1, +\infty[$,*

$$u_{\mathcal{D}_p} \to u \textit{ in } L^r(\Omega \times (0,T)).$$

All the results of this section still hold in larger dimensions. This can be proven by adapting the proof of [ENC 06] to transmission conditions [2],[3].

3. Uniform bound on the flux

As will be stated in section 4, we are not able to prove the uniqueness of the weak solution. Some additional regularity on the weak solution is necessary, so we define the notion of the bounded-flux solution.

Definition 3.1 (bounded-flux solution) *A function u is said to be a bounded-flux solution if:*

1) u is a weak solution in the sense of definition 1.1,
2) $\partial_x \varphi_i(u)$ belongs to $L^\infty(\Omega_i \times (0,T))$.

In order to get an existence result for bounded-flux solutions, we need some more regularity on the data, as stated below.

Assumption 1 (additional regularity on the data) *We assume that:*

1) $\partial_x \varphi_i(u_0) \in L^\infty(\Omega_i)$, $0 \leq u_0 \leq 1$,
2) $\tilde{\pi}_1(u_{0,1}) \cap \tilde{\pi}_2(u_{0,2}) \neq \emptyset$, where $u_{0,i}$ is the trace of $u_{0|\Omega_i}$ on $\{x = 0\}$,
3) $q \in BV(0,T)$, $q \geq 0$,
4) $g \in L^\infty(0,T)$, $0 \leq g \leq q$.

Let us define $(F_j^0)_{|j| \leq N}$ as in [8], [9], [10], [11], [12] and [13] where $(n+1)$ must be replaced by 0. Assumption 1 allows us to claim that:

$$\max_{j=-N,\dots,N} |F_j^0| \leq q^0 + 2 \max_{i=1,2} \left(\|\partial_x \varphi_i(u_0)\|_{L^\infty(\Omega_i)} \right) \leq C. \qquad [16]$$

Lemma 3.1 (uniform bound on the discrete fluxes) *Under assumption 1, there exists $C > 0$ only depending on u_0, φ_i, q, g and T such that:*

$$\max_{j \in [\![-\mathrm{N},\mathrm{N}]\!]} \left(\max_{n \in [\![0,\mathrm{M}]\!]} |F_j^n| \right) \leq C.$$

Sketch of the proof of lemma 3.1

We here perform an adaptation to the discrete case of the proof stated in [CAN], which uses a maximum principle on the fluxes. For $i = 1, 2$, equation [1] can be rewritten

$$\phi_i \partial_t u + \partial_x F(x,t) = 0, \qquad [17]$$

with

$$F(x,t) = q(t) f_i(u)(x,t) - \partial_x \varphi_i(u)(x,t). \qquad [18]$$

Formally, deriving [18] w.r.t. t and using [17] yields:

$$\partial_t F + \frac{q f_i'(u)}{\phi_i} \partial_x F - \partial_x \left(\frac{\varphi_i'(u)}{\phi_i} \partial_x F \right) = q' f_i(u). \qquad [19]$$

This computations on the discrete case gives a monotonous scheme with $(F_j^{n+1})_{j,n}$ as unknowns: we deduce from [7] and [8] that for all $j \in J_{int}$, for all $n \in [\![0, \mathrm{M}-1]\!]$, there exists $(a_{j,j-1}^{n+1}, a_{j,j+1}^{n+1}) \in (\mathbf{R}_+)^2$ such that:

$$\left[F_{j-1}^{n+1}, F_j^{n+1}, F_{j+1}^{n+1}\right] \cdot \begin{bmatrix} -a_{j,j-1}^{n+1} \\ 1 + a_{j,j-1}^{n+1} + a_{j,j+1}^{n+1} \\ -a_{j,j+1}^{n+1} \end{bmatrix} \leq F_j^n + \left(q^{n+1} - q^n\right)^+. \quad [20]$$

Using strongly the monotony of the discrete transmission conditions [11], [12] and [13], we can also show that there exist $(a_{0,-1}^{n+1}, a_{0,1}^{n+1}) \in (\mathbf{R}_+)^2$ such that [20] also holds for $j = 0$.
Since the discrete boundary conditions [9] and [10] insure that F_N^{n+1} and F_{-N}^{n+1} stay uniformly bounded by $\|q\|_\infty$, we can claim that

$$\max_j F_j^{n+1} \leq \max\left(\|q\|_\infty, \max_j F_j^n + \left(q^{n+1} - q^n\right)^+\right),$$

and thus

$$\max_n \max_j F_j^{n+1} \leq \max\left(\|q\|_\infty, \max_j F_j^0\right) + 2TV(q).$$

Thanks to assumptions 1 and estimate [16], the right-hand-side member is uniformly bounded. We conclude the proof of lemma 3.1 by showing the corresponding under-estimate:

$$\min_n \min_j F_j^{n+1} \geq \min\left(0, \min_j F_j^0\right) - 2TV(q).$$

Proposition 3.2 (convergence to a bounded-flux solution) *Let $(M_p)_p, (N_p)_p$ be two sequences of positive integers tending to $+\infty$, and let $(u_{\mathcal{D}_p})_p$ be the inducted sequence of discrete solutions. Under assumption 1 there exists a bounded-flux solution u in the sense of defintion 1.1 such that, up to a subsequence: $\forall r \in [1, +\infty[$,*

$$u_{\mathcal{D}_p} \to u \text{ in } L^r(\Omega \times (0,T)).$$

4. Uniqueness of the bounded-flux solution

Theorem 4.1 (L^1-contraction for bounded-flux solution) *Suppose that for $i = 1, 2$, $f_i \circ \varphi_i^{-1} \in C^{0,\frac{1}{2}}([0, \varphi_i(1)], \mathbf{R})$. Let u,v be two bounded-flux solutions in the sense of definition 3.1 associated to initial data u_0,v_0. Then u and v belong to $C([0,T]; L^1(\Omega))$, and for all $t \geq 0$,*

$$\sum_{i=1,2} \int_{\Omega_i} \phi_i \left(u(x,t) - v(x,t)\right)^\pm dx \leq \sum_{i=1,2} \int_{\Omega_i} \phi_i \left(u_0(x) - v_0(x)\right)^\pm dx.$$

Corollary 4.2 *Keeping the notations of proposition 3.2, and under assumptions 1, the whole sequence $(u_{\mathcal{D}_p})_p$ converges in $L^r(\Omega \times (0,T))$ towards the unique bounded flux solution.*

Sketch of the proof of theorem 4.1

The proof is based on a doubling variable method w.r.t. the time variable, first in each Ω_i, and on a transmission lemma 4.3 to deal with the interface $\{x = 0\}$.

Since u, v are weak solutions, they belong to $C([0,T]; L^1(\Omega))$ (see lemma 7.41 in [M` 96]). Some classical computations (see e.g. [OTT 96]) give: $\forall \theta \in BV(0,T)$, $\theta \geq 0$, $\theta(T) = 0$, $\forall \psi \in W^{1,1}(\Omega)$, $\psi \geq 0$, $\psi(0) = 0$,

$$\begin{aligned} &\textstyle\int_0^T \sum_i \int_{\Omega_i} \phi_i (u(x,t) - v(x,t))^+ \partial_t \theta(t) \psi(x) dx dt \\ &\quad + \textstyle\sum_i \int_{\Omega_i} \phi_i (u_0(x) - v_0(x))^+ \psi(x) \theta(0) dx \\ &+ \textstyle\int_0^T \theta(t) \sum_i \int_{\Omega_i} q(t)(f_i(u)(x,t) - f_i(v)(x,t))^+ \partial_x \psi(x) dx dt \\ &- \textstyle\int_0^T \theta(t) \sum_i \int_{\Omega_i} \partial_x(\varphi_i(u)(x,t) - \varphi_i(v)(x,t))^+ \partial_x \psi(x) dx dt \geq 0. \end{aligned} \qquad [21]$$

We need the following key lemma, which is a consequence of the uniform bound on the flux and of the monotony of the transmission conditions [2] and [3].

Lemma 4.3 *If $\varepsilon > 0$, we denote $\xi_\varepsilon(x) = \max\left(0, 1 - \frac{|x|}{\varepsilon}\right)$. $\forall \theta \in BV(0,T)$, $\theta \geq 0$, $\theta(T) = 0$,*

$$\liminf_{\varepsilon \to 0} \int_0^T \theta(t) \sum_{i=1,2} \int_{\Omega_i} \begin{bmatrix} q(t)(f_i(u)(x,t) - f_i(v)(x,t))^+ \\ -\partial_x(\varphi_i(u)(x,t) - \varphi_i(v)(x,t))^+ \end{bmatrix} \partial_x \xi_\varepsilon(x) dx dt \geq 0.$$

Let $t \in]0,T[$. Taking $\psi(x) = (1 - \xi_\varepsilon(x))$, $\theta_t(s) = \chi_{[0,t]}(s)$ in [21], and letting ε tend to 0 gives:

$$\sum_{i=1,2} \int_{\Omega_i} \phi_i \left(u(x,t) - v(x,t)\right)^+ dx \leq \sum_{i=1,2} \int_{\Omega_i} \phi_i \left(u_0(x) - v_0(x)\right)^+ dx.$$

Similar calculations leads to the same inequality with $(\cdot)^-$ instead of $(\cdot)^+$.

5. Convergence to the unique SOLA

The goal of this part is to get the convergence of the whole sequence of discrete solutions despite u_0 "only" belongs to $L^\infty(\Omega)$.

Definition 5.1 (SOLA to the problem) *A function u is said to be a SOLA if it fulfils:*

– *u is a weak solution in the sense of definition 1.1,*

– *there exists a sequence $(u_n)_{n \in \mathbf{N}}$ of bounded flux solutions such that*

$$u_n \to u \text{ in } C([0,T]; L^1(\Omega)), \text{ as } n \to +\infty.$$

Theorem 5.1 (Convergence to the unique SOLA) *Let $u_0 \in L^\infty(\Omega)$, $0 \leq u_0 \leq 1$, and $q \in BV(0,T)$, $g \in L^\infty(0,T)$, $0 \leq g \leq q$. There exists a unique SOLA u in the sense of definition 5.1.*
Let $(M_p)_p, (N_p)_p$ be two sequences of positive integers tending to $+\infty$, and let $(u_{\mathcal{D}_p})_p$ be the sequence of the inducted discrete solutions. Then the whole sequence $(u_{\mathcal{D}_p})_p$ converges to the unique SOLA u in $L^r(\Omega \times (0,T))$, $r \in [1, +\infty[$ as p tends to $+\infty$.

Sketch of the proof of theorem 5.1

Let $(u_{0,\nu})_{\nu \in \mathbf{N}}$ be a sequence of approximate initial data such that: for all $\nu \in \mathbf{N}$, $(u_{0,\nu}, q, g)$ fulfils assumptions 1, and $\lim_{\nu \to +\infty} \|u_{0,\nu} - u_0\|_{L^1(\Omega)} = 0$. Let $(u_\nu)_\nu$ be the inducted sequence of bounded flux solutions. The L^1-contraction principle stated in theorem 4.1 allows us to claim that $(u_\nu)_\nu$ is a Cauchy sequence in $C([0,T]; L^1(\Omega))$. It is then easy to check that u is a weak solution, so it is a SOLA. The L^1-contraction principle still holds for the SOLAs, which is thus unique.

Let $(M_p)_p, (N_p)_p$ be two sequences of positive integers tending to $+\infty$, $(u_{\mathcal{D}_p})_p$ (resp.$(u_{\nu,\mathcal{D}_p})_p$) be the sequence of the inducted discrete solutions for u_0 (resp. $u_{0,\nu}$) as initial data. From [14], we obtain:

$$\sum_{i=1,2} \int_{\Omega_i} \phi_i |u_{\mathcal{D}_p}(x,t) - u_{\nu,\mathcal{D}_p}(x,t)| dx \leq \sum_{i=1,2} \int_{\Omega_i} \phi_i |u_{0,\mathcal{D}_p}(x) - u_{0,\nu,\mathcal{D}_p}(x)| dx.$$

Letting p tend to $+\infty$, $u_{\mathcal{D}_p}(x,t)$ tends to a weak solution u thanks to proposition 2.3. $u_{\nu,\mathcal{D}_p}$ tends to a bounded flux solution u_ν thanks to proposition 3.2, and so:

$$\sum_{i=1,2} \int_{\Omega_i} \phi_i |u(x,t) - u_\nu(x,t)| dx \leq \sum_{i=1,2} \int_{\Omega_i} \phi_i |u_0(x) - u_{0,\nu}(x)| dx.$$

Letting ν tend to $+\infty$ shows that u is a SOLA in the sense of definition 5.1.

6. References

[CAN] CANCÈS C., GALLOUËT T., PORRETTA A., "Two-phase flows involving capillary barriers in heterogeneous porous media", submitted.

[ENC 06] ENCHÉRY G., EYMARD R., MICHEL A., "Numerical approximation of a two-phase flow in a porous medium with discontinuous capillary forces", *SIAM J. Numer. Anal.*, vol. 43, num. 6, 2006, p. 2402-2422.

[EYM 00] EYMARD R., GALLOUËT T., HERBIN R., *Finite Volume Methods*, North-Holland, 2000.

[M` 96] MÀLEK J., NEČAS J., ROKYTA M., RUŽIČKA M., *Weak and measure valued solutions to evolutionary PDEs*, Chapman & Hall, London, 1996.

[OTT 96] OTTO F., "L^1-contraction and uniqueness for quasilinear elliptic-parabolic equations", *J. Differential Equations*, vol. 131, 1996, p. 20-38.