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A Convergent Finite Volume Scheme for Two-Phase Flows in Porous Media with Discontinuous Capillary Pressure Field *

K. Brenner, C. Cancès and D. Hilhorst

Abstract We consider an immiscible incompressible two-phase flow in a porous medium composed of two different rocks. The flows of oil and water are governed by the Darcy–Muskat law and a capillary pressure law, where the capillary pressure field may be discontinuous at the interface between the rocks. Using the concept of multi-valued phase pressures, we introduce a notion of weak solution for the flow, and prove the convergence of a finite volume approximation towards a weak solution.

Key words: Finite volume, two phase flow, discontinuous capillary pressures
MSC2010: 65M112, 76M12, 35K65, 76S05

1 The Continuous Problem

1.1 Multivalued Phase Pressures

Consider a heterogeneous porous medium, represented by a polygonal domain $\Omega \subset \mathbb{R}^d$, built of two homogeneous and isotropic subdomains, represented by polygonal domains $\Omega_1, \Omega_2 \subset \mathbb{R}^d$. We assume that $\overline{\Omega_1 \cup \Omega_2} = \overline{\Omega}$ and $\Omega_1 \cap \Omega_2 = \emptyset$, and we denote by Γ the interface between the two rocks, i.e. $\overline{\Gamma} = \partial\Omega_1 \cap \partial\Omega_2$. We consider two immiscible incompressible phases (e.g. water and oil), whose flows

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within Ω_i are described by the conservation of mass equations together with the Darcy–Muskat law:

$$\phi_i \partial_t s - \nabla \cdot (\eta_{o,i}(s)(\nabla p_o - \rho_o \mathbf{g})) = 0, \tag{1}$$
$$-\phi_i \partial_t s - \nabla \cdot (\eta_{w,i}(s)(\nabla p_w - \rho_w \mathbf{g})) = 0, \tag{2}$$

where s denotes the oil saturation of the fluid, $\phi_i > 0$ the porosity of Ω_i, the oil mobility $\eta_{o,i}$ is a Lipschitz continuous increasing function on $[0,1]$ satisfying $\eta_{o,i}(0) = 0$, while the water mobility $\eta_{w,i}$ is Lipschitz continuous, decreasing on $[0,1]$ and such that $\eta_{w,i}(1) = 0$. The density of the phase α ($\alpha \in \{o,w\}$) is denoted by ρ_α, and $\mathbf{g}$ is the gravity vector. Assume first that both phases coexist, i.e. $s \in (0,1)$, then each phase has its own pressure denoted by p_α. Classically, they are supposed to be linked by the capillary pressure relation

$$p_o - p_w = \pi_i(s), \tag{3}$$

where the capillary pressure function π_i is supposed to be increasing and to belong to $\mathscr{C}^1((0,1);\mathbb{R}) \cap L^1(0,1)$. Since the equation (1) degenerates, there is no control on the oil pressure p_o on $\{s=0\} \cap \Omega_i$, excepted that, because of (3), one has $p_o \leq p_w + \pi_i(0)$. Similarly, on $\{s=1\} \cap \Omega_i$, $p_w \leq p_o - \pi_i(1)$. In these cases, the pressure has to be considered as multivalued, i.e.

$$s = 0 \Leftrightarrow p_o = [-\infty, p_w + \pi_i(0)], \qquad s = 1 \Leftrightarrow p_w = [-\infty, p_o - \pi_i(1)]. \tag{4}$$

We deduce from (4) that the capillary pressure function π_i has to be extended into the monotone graph $\tilde{\pi}_i$, already introduced in [3, 5], defined by

$$\tilde{\pi}_i(s) = \begin{cases} [-\infty, \pi_i(0)] & \text{if } s = 0, \\ \pi_i(s) & \text{if } s \in (0,1), \\ [\pi_i(1), +\infty] & \text{if } s = 1. \end{cases} \tag{5}$$

Note that there exists a continuous non-decreasing reciprocal function on $\mathbb{R}$, which we denote by $\tilde{\pi}_i^{-1}$.

At the interface Γ, we prescribe the continuity of the multivalued phase pressures

$$p_{\alpha,1} \cap p_{\alpha,2} \neq \emptyset, \qquad (\alpha \in \{o,w\}) \tag{6}$$

where $p_{\alpha,i}$ denote the trace of the pressure of the phase α on Γ from Ω_i. It is worth noticing that the condition (6) is equivalent to the continuity of the mobile phases prescribed in [8]. The volume conservation of each phase yields

$$\sum_{i=1,2} \eta_{\alpha,i}(s_i)(\nabla p_{\alpha,i} - \rho_\alpha \mathbf{g}) \cdot \mathbf{n}_i = 0, \tag{7}$$

where $\mathbf{n}_i$ denote the outward normal to $\partial\Omega_i$ w.r.t. Ω_i. In order to close the problem, we prescribe the initial condition

$$s_0 \in L^\infty(\Omega), \quad 0 \le s_0 \le 1 \text{ a.e. in } \Omega, \tag{8}$$

and the null-flux boundary condition on $\partial\Omega_i \cap \partial\Omega$:

$$\eta_{\alpha,i}(s_i)(\nabla p_{\alpha,i} - \rho_\alpha \mathbf{g}) \cdot \mathbf{n}_i = 0. \tag{9}$$

1.2 Reformulation of the Problem

We define the fractional flow function $f_i(s) = \frac{\eta_{o,i}(s)}{\eta_{o,i}(s)+\eta_{w,i}(s)}$. We introduce the Kirchhoff transform $\varphi_i(s)$ and the global pressure P defined by

$$\varphi_i(s) = \int_0^s f_i(a)\eta_{w,i}(a)\pi_i'(a)da, \tag{10}$$

$$P = p_w + \lambda_{w,i}(\pi) = p_o + \lambda_{o,i}(\pi) \quad \text{for some } \pi \in \tilde{\pi}_i(s), \tag{11}$$

where $\lambda_{w,i}(\pi) = \int_0^\pi f_i \circ \tilde{\pi}_i^{-1}(p)dp$ and $\lambda_{o,i}(\pi) = \lambda_{w,i}(\pi) - \pi$. Classical computations (see e.g. [7]) allow the rewrite the equations (1) as

$$\phi_i \partial_t s - \nabla \cdot (\eta_{o,i}(s)(\nabla P - \rho_o \mathbf{g}) + \nabla \varphi_i(s)) = 0, \tag{12}$$

while the sum of the equations (1) and (2) yields

$$-\nabla \cdot (M_i(s)\nabla P - \zeta_i(s)\mathbf{g}) = 0, \tag{13}$$

where $M_i(s) = \eta_{o,i}(s) + \eta_{w,i}(s) \ge \alpha_M > 0$ and $\zeta_i(s) = \eta_{o,i}(s)\rho_o + \eta_{w,i}(s)\rho_w$. At the interface, the relations (6) have to be replaced (see [6]) by

$$\exists \pi \in \tilde{\pi}_1(s_1) \cap \tilde{\pi}_2(s_2) \text{ s.t. } P_1 - \lambda_{w,1}(\pi) = P_2 - \lambda_{w,2}(\pi). \tag{14}$$

We solve the problem on the domain $Q = \Omega \times (0,T)$ for some $T > 0$, and we define $Q_i = \Omega_i \times (0,T)$.

Definition 1 (weak solution). A pair (s,P) is said to be a weak solution of the problem if

1. $s \in L^\infty(Q)$ with $0 \le s \le 1$ a.e. in Q, $\varphi_i(s)$ and P belong to $L^2((0,T);H^1(\Omega_i))$;
2. there exists a measurable function π mapping $\Gamma \times (0,T)$ to $\overline{\mathbb{R}}$ such that

$$\pi \in \tilde{\pi}_1(s_1) \cap \tilde{\pi}_2(s_2) \text{ and } P_1 - \lambda_{w,1}(\pi) = P_2 - \lambda_{w,2}(\pi);$$

3. for all $\psi \in C_c^\infty(\overline{\Omega} \times [0,T))$,

$$\iint_Q \phi s \partial_t \psi dxdt + \int_\Omega \phi s_0 \psi(\cdot,0) dxdt$$
$$- \sum_{i=1,2} \iint_{Q_i} (\eta_{o,i}(s)(\nabla P - \rho_o \mathbf{g}) + \nabla \varphi_i(s)) \cdot \nabla \psi dxdt = 0, \tag{15}$$

$$\iint_Q (M_i(s)\nabla P - \zeta_i(s)\mathbf{g}) \cdot \nabla \psi dxdt = 0. \tag{16}$$

Because of the choice of the boundary conditions, the global pressure P is only defined up to a constant. In order to eliminate this degree of freedom, we prescribe that

$$\int_\Omega P(x,t)dx = 0, \quad \forall t > 0. \tag{17}$$

The equation (13) can be reformulated as

$$\nabla \cdot \mathbf{q} = 0, \text{ with } \mathbf{q} = -M_i(s)\nabla P + \zeta_i(s)\mathbf{g}, \tag{18}$$

while (12) can be rewritten under the form

$$\phi_i \partial_t s + \nabla \cdot (\mathbf{q} f_i(s) + \gamma_i(s)\mathbf{g} - \nabla \varphi_i(s)) = 0, \tag{19}$$

with $\gamma_i(s) = (\rho_o - \rho_w)\eta_{w,i}(s) f_i(s)$.

2 The Finite Volume Scheme

Since nonlinear test functions are necessary for proving the convergence of the scheme, we must restrict our study to spatial discretizations satisfying an orthogonality condition, as developed in [9].

Definition 2 (admissible discretization of Q).

1. An admissible discretization of Ω is given by $(\mathscr{T}, \mathscr{E}, (x_K)_{K\in\mathscr{T}})$ where for all $K \in \mathscr{T}$, K is an open polygonal subset of Ω such that $K \subset \Omega_i$ for some i. We define $\mathscr{T}_i = \{K \subset \Omega_i\}$, and we assume that $\overline{\Omega}_i = \overline{\bigcup_{K\in\mathscr{T}_i} K}$. For $K, L \in \mathscr{T}$ with $K \neq L$, then either the $(d-1)$-Lebesgue measure of $\overline{K} \cap \overline{L}$ is 0, or there exists $\sigma \in \mathscr{E}_K \cap \mathscr{E}_L$ (denoted by $\sigma = K|L$) such that $\overline{\sigma} = \overline{K} \cap \overline{L}$. For all $K \in \mathscr{T}$, there exists $\mathscr{E}_K \subset \mathscr{E}$ such that $\partial K = \bigcup_{\sigma\in\mathscr{E}_K} \overline{\sigma}$. Moreover, $\mathscr{E} = \bigcup_{K\in\mathscr{T}} \mathscr{E}_K$. We define $\mathscr{E}_\Gamma = \{\sigma \in \mathscr{E} : \sigma \subset \Gamma\}$, $\mathscr{E}_i = \{\sigma \in \mathscr{E} : \sigma \subset \Omega_i\}$ and $\mathscr{E}_{\text{ext}} = \{\sigma \in \mathscr{E} : \sigma \subset \partial\Omega\}$, and set $\mathscr{E}_{K,\Gamma} = \mathscr{E}_K \cap \mathscr{E}_\Gamma$, $\mathscr{E}_{K,i} = \mathscr{E}_K \cap \mathscr{E}_i$. The family of points $(x_K)_{K\in\mathscr{T}}$ is such that $x_K \in K$ and if $\sigma = K|L$, the straight line $(x_K x_L)$ is orthogonal to σ. We denote by $d_{K,L}$ the distance between x_K and x_L, and by $d_{K,\sigma}$ the distance between x_K and $\sigma \in \mathscr{E}_K$. For all $K \in \mathscr{T}$ and $\sigma \in \mathscr{E}$ we denote by $m(K)$ and $m(\sigma)$ the corresponding Lebesgue measures.
2. Let N be a positive integer, and $\delta t = T/N$; then a uniform discretization of $(0,T)$ is given by the family $(t^n)_{n\in\{0,\dots,N\}}$, where $t^n = n\delta t$.

3. A discretization $\mathscr{D} = \left(\mathscr{T}, \mathscr{E}, (x_K)_{K\in\mathscr{T}}, (t^n)_{n\in\{0,\dots,N\}}\right)$ of Q is said to be admissible if $(\mathscr{T}, \mathscr{E}, (x_K)_{K\in\mathscr{T}})$ is an admissible discretization of Ω and $(t^n)_n$ is a uniform discretization of $(0,T)$.

For a given admissible discretization $\mathscr{D} = (\mathscr{T}, \mathscr{E}, (x_K)_{K\in\mathscr{T}}, (t^n)_{n\in\{0,\dots,N\}})$ of Q, we define the quantities

$$\mathrm{size}(\mathscr{T}) = \max_{K\in\mathscr{T}} \mathrm{diam}(K), \qquad \mathrm{reg}(\mathscr{T}) = \max_{i=1,2} \max_{K\in\mathscr{T}} \left(\sum_{\sigma=K|L\in\mathscr{E}_{K,\mathrm{i}}} \frac{m(\sigma)d_{K,L}}{m(K)} \right),$$

and

$$\mathrm{size}(\mathscr{D}) = \max\left(\mathrm{size}(\mathscr{T}), \delta t\right), \qquad \mathrm{reg}(\mathscr{D}) = \mathrm{reg}(\mathscr{T}).$$

Remark 1. The choice of uniform time steps is not necessary, and all the results presented here can be adapted to the case of nonuniform time steps.

For $K \in \mathscr{T}_i$, we define $g_K(s) = g_i(s)$ for all functions g whose definition depends on the subdomain Ω_i, as for example $\phi_i, \varphi_i, M_i, f_i, \dots$.

We propose a fully implicit cell-centered finite volume scheme for the problem, whose unknowns at each time step are $(s_K^n, P_K^n)_{K\in\mathscr{T}}$ and an interface unknown $(\pi_\sigma^n)_{\sigma\in\mathscr{E}_\Gamma}$. For all $\sigma \in \mathscr{E}_{K,\Gamma}$, we define $s_{K,\sigma}^n = \tilde{\pi}_K^{-1}(\pi_\sigma^n)$, so that, if $\sigma = K|L$, one directly has that

$$\pi_\sigma^n \in \tilde{\pi}_K(s_{K,\sigma}^n) \cap \tilde{\pi}_L(s_{L,\sigma}^n).$$

The total flux balance equation (18) is discretized by

$$\sum_{\sigma\in\mathscr{E}_K} m(\sigma) Q_{K,\sigma}^n = 0, \qquad \forall n \in \{1,\dots,N\}, \forall K \in \mathscr{T}, \tag{20}$$

with

$$Q_{K,\sigma}^n = \begin{cases} \frac{M_{K,L}(s_K^n, s_L^n)}{d_{K,L}} \left(P_K^n - P_L^n\right) + \mathscr{R}\left(Z_{K,\sigma}; s_K^n, s_L^n\right) & \text{if } \sigma = K|L \in \mathscr{E}_{K,i}, \\ \frac{M_K(s_K^n)}{d_{K,\sigma}} \left(P_K^n - P_{K,\sigma}^n\right) + \mathscr{R}\left(Z_{K,\sigma}; s_K^n, s_{K,\sigma}^n\right) & \text{if } \sigma \in \mathscr{E}_{K,\Gamma}, \\ 0 & \text{if } \sigma \in \mathscr{E}_{K,\mathrm{ext}}, \end{cases} \tag{21}$$

where $M_{K,L}(s_K^n, s_L^n) = M_{L,K}(s_L^n, s_K^n)$ is an average of $M_K(s_K^n)$ and $M_L(s_L^n)$. For example, we can suppose, as in [10] that it is given by the harmonic mean

$$M_{K,L}(s_K^n, s_L^n) = \frac{M_K(s_K^n) M_K(s_L^n) d_{K,L}}{d_{L,\sigma} M_K(s_K^n) + d_{K,\sigma} M_K(s_L^n)}.$$

The function $Z_{K,\sigma}$ is defined by $Z_{K,\sigma}(s) = \zeta_K(s)\mathbf{g}\cdot\mathbf{n}_{K,\sigma}$, where $\mathbf{n}_{K,\sigma}$ denotes the outward normal to σ with respect to K. For a function f, we denote by $\mathscr{R}(f; a, b)$ the Riemann solver

$$\mathscr{R}(f; a, b) = \begin{cases} \min_{c\in[a,b]} f(c) & \text{if } a \le b, \\ \max_{c\in[b,a]} f(c) & \text{if } b \le a. \end{cases}$$

The oil-flux balance equation (19) is discretized in the form

$$\phi_K \frac{s_K^n - s_K^{n-1}}{\delta t} m(K) + \sum_{\sigma \in \mathscr{E}_K} m(\sigma) F_{K,\sigma}^n = 0, \qquad \forall n \in \{1,\dots,N\}, \forall K \in \mathscr{T}, \tag{22}$$

with

$$F_{K,\sigma}^n = \begin{cases} Q_{K,\sigma}^n \; f_K(\overline{s}_{K,\sigma}^n) + \mathscr{R}(G_{K,\sigma}; s_K^n, s_L^n) + \frac{\varphi_K(s_K^n) - \varphi_K(s_L^n)}{d_{K,L}} \text{ if } \sigma = K|L \in \mathscr{E}_{K,i}, \\ Q_{K,\sigma}^n \; f_K(\overline{s}_{K,\sigma}^n) + \mathscr{R}(G_{K,\sigma}; s_K^n, s_{K,\sigma}^n) + \frac{\varphi_K(s_K^n) - \varphi_K(s_{K,\sigma}^n)}{d_{K,\sigma}} \text{ if } \sigma \in \mathscr{E}_{K,\Gamma}, \\ 0 \text{ if } \sigma \in \mathscr{E}_{K,\text{ext}}, \end{cases} \tag{23}$$

where $G_{K,\sigma}(s) = \gamma_K(s)\mathbf{g} \cdot \mathbf{n}_{K,\sigma}$ and $\overline{s}_{K,\sigma}^n$ is the upstream value defined by

$$\overline{s}_{K,\sigma}^n = \begin{cases} s_K^n & \text{if } Q_{K,\sigma}^n \geq 0, \\ s_L^n & \text{if } Q_{K,\sigma}^n < 0 \text{ and } \sigma = K|L \in \mathscr{E}_{K,i}, \\ s_{K,\sigma}^n & \text{if } Q_{K,\sigma}^n < 0 \text{ and } \sigma \in \mathscr{E}_{K,\Gamma}. \end{cases} \tag{24}$$

The interface values $\left(\pi_\sigma^n, P_{K,\sigma}^n, P_{L,\sigma}^n\right)$ for $\sigma = K|L \in \mathscr{E}_\Gamma$ are defined by the following nonlinear system:

$$P_{K,\sigma}^n - \lambda_{w,K}(\pi_\sigma^n) = P_{L,\sigma}^n - \lambda_{w,L}(\pi_\sigma^n). \tag{25}$$

$$Q_{K,\sigma}^n + Q_{L,\sigma}^n = 0, \tag{26}$$

$$F_{K,\sigma}^n + F_{L,\sigma}^n = 0. \tag{27}$$

Note that since the equations (25) and (26) are linear with respect to $P_{K,\sigma}^n$ and $P_{L,\sigma}^n$, one can eliminate these interface values, only keeping π_σ^n. We impose the discrete counterpart of (17), that is

$$\sum_{K \in \mathscr{T}} m(K) P_K^n = 0, \qquad \forall n \in \{1,\dots,N\}. \tag{28}$$

The discrete initial data is given by:

$$s_K^0 = \frac{1}{m(K)} \int_K s_0(x)dx, \quad \forall K \in \mathscr{T},$$

so that $0 \leq s_K^0 \leq 1$.

Proposition 1 (existence of a discrete solution). *For all $n \in \{1,\dots,N\}$, there exists $\left((s_K^n)_{K\in\mathscr{T}}, (P_K^n)_{K\in\mathscr{T}}, (\pi_\sigma^n)_{\sigma\in\mathscr{E}_\Gamma}\right)$ satisfying the relations* (20)–(28)*. Moreover,*

$$0 \leq s_K^n \leq 1, \quad \forall K \in \mathscr{T}. \tag{29}$$

The proof of Proposition 1 will be given in the forthcoming paper [2].

For an admissible discretization $\mathscr{D}$ of Q, we denote by $s_{\mathscr{D}}$ and $P_{\mathscr{D}}$ the piecewise constant functions defined almost everywhere by

$$s_{\mathscr{D}}(x,t) = s_K^n, \quad P_{\mathscr{D}}(x,t) = P_K^n \quad \text{if } (x,t) \in K \times (t^{n-1}, t^n].$$

We consider now a sequence $(\mathscr{D}_m)_{m\geq 0}$ of admissible discretizations of Q in the sense of Definition 2 such that size$(\mathscr{D}_m)$ tends to 0 and reg$(\mathscr{D}_m)$ remains uniformly bounded as m tends to ∞. We denote by $(s_{\mathscr{D}_m}, P_{\mathscr{D}_m})_m$ a corresponding sequence of discrete solutions, whose existence has been stated in Proposition 1.

Theorem 1 (main result). *There exists a weak solution (s,P) in the sense of Definition 1 such that, up to a subsequence,*

$$s_{\mathscr{D}_m} \to s \text{ and a.e. in } Q \text{ as } m \to \infty,$$

$$P_{\mathscr{D}_m} \to P \text{ weakly in } L^2(Q) \text{ as } m \to \infty.$$

The proof of Theorem 1 that we will present in the forthcoming paper [2] is based on compactness arguments, using the material developed in [9, 10]. The proof adapts the steps that are given in [6] for the continuous frame.

3 Numerical Results

We consider a model porous medium $\Omega = (0,1)^2$ composed of two layers $\Omega_1 = \{(x,y) \in \Omega \mid y < \Gamma(x)\}$ and $\Omega_2 = \{(x,y) \in \Omega \mid y > \Gamma(x)\}$, which have different capillary pressure laws. The fluid densities are given by $\rho_o = 0.81$, $\rho_w = 1$, and $\mathbf{g} = -9.81\mathbf{e}_y$. We suppose that the porosity is such that $\phi_i = 1, i \in \{1,2\}$, and we define the oil and water mobilities by

$$\eta_{o,i}(s) = 0.5s^2 \text{ and } \eta_{w,i} = (1-s)^2,\ i \in \{1,2\}.$$

Moreover we suppose that the capillary pressure curves have the form

$$\pi_1(s) = s \text{ and } \pi_2(s) = 0.5 + s.$$

and that the initial saturation is given by

$$s_0(x) = \begin{cases} 0.3 & \text{if } x \in \Omega_1, \\ 0 & \text{otherwise.} \end{cases}$$

The flow is driven by buoyancy, making the oil move along $\mathbf{e}_y$ until it reaches the interface Γ. For $t \leq 0.11$, oil can not access the domain Ω_2, since the capillary pressure $\pi_1(s_1)$ is lower than the threshold value $\pi_2(0) = 0.5$, which is called *the entry pressure*. Hence the saturation below the interface s_1 increases, as well as the capillary pressure $\pi_1(s_1)$. As soon as the capillary pressure $\pi_1(s_1)$ reaches the entry pressure $\pi_2(0)$, oil starts to penetrate in the domain Ω_2. Nevertheless, as pointed out in [1, 4], a finite quantity of oil remains trapped under the rock discontinuity. This phenomenon is called *oil trapping*.

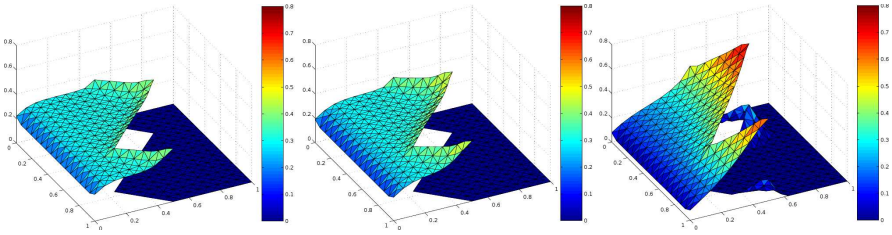

Fig. 1 Saturation for $t = 0.06$, $t = 0.11$ and $t = 0.6$

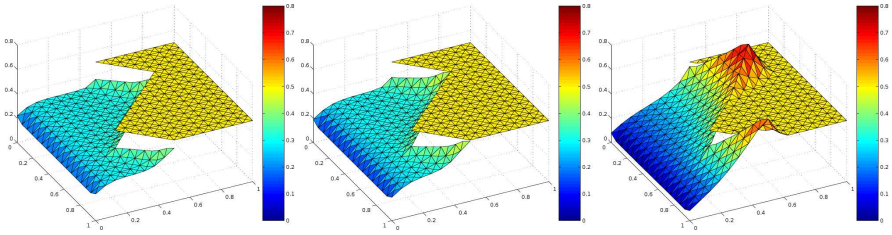

Fig. 2 Capillary pressure for $t = 0.06$, $t = 0.11$ and $t = 0.6$

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