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Nonlinear Parabolic Equations with Spatial Discontinuities

Clément Cancès *

Abstract

We consider a two phase flow involving no capillary barriers in a heterogeneous porous media, composed by an apposition of several homogeneous porous media. We prove the existence of a weak solution for such a flow using the convergence of a finite volume approximation. Then under the assumption that the equations governing the flows in each homogeneous porous media degenerate in not too different ways, we prove the uniqueness of the weak solution, using a doubling variable method. We also prove that such a solution belongs to $C([0, T], L^p(\Omega))$ for any $p \in [1, +\infty)$.

Keywords. finite volume methods, porous media flow, spatial discontinuity.

AMS subject classification. 35B05, 35K65, 35R05, 65M12

1 Presentation of the problem and main results

1.1 Presentation of the problem

In this paper, we are interested by the parabolic equation obtained by modeling a two phase flow in a heterogeneous porous media. Let Ω be an open polygonal subset of \mathbb{R}^d ($d \leq 3$), let $T > 0$. One assumes that there exists a finite number of polygonal open subsets $\Omega_i \subset \Omega$, $i \in \{1, \dots, N\}$ such that:

$$\bigcup_i \overline{\Omega_i} = \overline{\Omega} \quad \text{and} \quad \Omega_i \cap \Omega_j = \emptyset \quad \text{if } i \neq j. \quad (1)$$

For all $(i, j) \in \{1, \dots, N\}^2$, $i \neq j$, one defines $\Gamma_{i,j}$ the subset of Ω given by $\overline{\Gamma_{i,j}} = \overline{\Omega_i} \cap \overline{\Omega_j}$.

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Each Ω_i will represent a porous media with its own physical characteristics, u will represent the saturation of the oily phase. We aim to solve the problem:

$$\left\{ \begin{array}{ll} \partial_t u - \operatorname{div}(\lambda_i(u)\nabla\bar{\pi}_i(u)) = 0 & \text{in } \Omega_i \times (0, T), \\ \bar{\pi}_i(u) = \bar{\pi}_j(u) & \text{on } \Gamma_{i,j} \times (0, T), \\ \lambda_i(u)\nabla\bar{\pi}_i(u) \cdot \mathbf{n}_i + \lambda_j(u)\nabla\bar{\pi}_j(u) \cdot \mathbf{n}_j = 0 & \text{on } \Gamma_{i,j} \times (0, T), \\ \lambda_i(u)\nabla\bar{\pi}_i(u) \cdot \mathbf{n}_i = 0 & \text{on } \partial\Omega \times (0, T), \\ u(x, 0) = u_0(x) & \text{in } \Omega, \end{array} \right. \quad (2)$$

where $\bar{\pi}_i$ is an increasing Lipschitz continuous function associated to Ω_i , λ_i is a non negative continuous function with $\lambda_i|_{]0,1[} > 0$, the initial data $u_0 \in L^\infty(\Omega)$ with $0 \leq u_0 \leq 1$ a.e. in Ω .

Remark 1.1 *We choose to consider only homogeneous Neumann boundary conditions on $\partial\Omega$, but this work can be easily generalized for non-homogeneous Dirichlet condition, the same way as in [5, 7].*

1.2 Mathematical definition of the problem

In this part, one defines all the functions necessary to explicit the problem (2) and the notion of weak solution.

Particularly, one can associate to each Ω_i two functions, the capillary pressure $\bar{\pi}_i$ and the global mobility λ_i , on which we do the following assumptions:

Assumption 1

1. For all $i \in \{1, \dots, N\}$, the function $\bar{\pi}_i : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and satisfies:

- $\forall s \leq 0, \quad \bar{\pi}_i(s) = \bar{\pi}_i(0)$
- $\forall s \geq 1, \quad \bar{\pi}_i(s) = \bar{\pi}_i(1)$
- $\bar{\pi}_i|_{]0,1[} \in C^1([0, 1], \mathbb{R})$ is an increasing function
- $\forall (i, j) \in \{1, \dots, N\}^2, \quad \bar{\pi}_i(0) = \bar{\pi}_j(0)$
- $\forall (i, j) \in \{1, \dots, N\}^2, \quad \bar{\pi}_i(1) = \bar{\pi}_j(1)$.

Let $m_0 \geq 1$. For all $i \in \{1, \dots, N\}$, for all $s \in [0, 1]$, we might choose $\bar{\pi}_i(s) = \beta_i s^{m_0} + (1 - \beta_i) s^{m_i}$ for any $\beta_i \in]0, 1]$ and any $m_i > m_0$.

2. For all $i \in \{1, \dots, N\}$, the function $\lambda_i : \mathbb{R} \rightarrow \mathbb{R}^+$ is continuous and satisfies:

- $\forall s \leq 0, \quad \lambda_i(s) = \lambda_i(0)$
- $\forall s \geq 1, \quad \lambda_i(s) = \lambda_i(1)$
- $\forall s \in]0, 1[, \quad \lambda_i(s) > 0$.

One can now define a function $\lambda : \bigcup_i \Omega_i \times \mathbb{R} \rightarrow \mathbb{R}^+$ by $(x, s) \mapsto \lambda_i(s)$, for all $x \in \Omega_i$, for all $s \in \mathbb{R}$. One denotes $C_\lambda = \max_i(\sup_{s \in \mathbb{R}}(|\lambda_i(s)|))$. A classical choice for λ_i is: $\forall i \in \{1, \dots, N\}, \forall s \in [0, 1], \lambda_i(s) = \alpha_i s(1 - s)$ with $\alpha_i > 0$.

We can now define the functions φ_i and Π_i , $i \in \{1, \dots, N\}$.

Definition 1.1 Under Assumptions 1, for all $i \in \{1, \dots, N\}$, we define:

$$\varphi_i : \begin{cases} \mathbb{R} & \rightarrow \mathbb{R}^+ \\ s & \mapsto \int_0^s \lambda_i(a) \bar{\pi}'_i(a) da \end{cases} \quad (3)$$

- $\forall i \in \{1, \dots, N\}$, $\varphi_i|_{[0,1]}$ is a derivable increasing function
- $\forall s \leq 0$, $\varphi_i(s) = \varphi_i(0) = 0$
- $\forall s \geq 1$, $\varphi_i(s) = \varphi_i(1)$.

We denote by L_φ a Lipschitz constant for all φ_i , $i \in \{1, \dots, N\}$.

We also define the function Π_i :

$$\Pi_i : \begin{cases} \mathbb{R} & \rightarrow \mathbb{R}^+ \\ s & \mapsto \int_{\bar{\pi}_i(0)}^{\bar{\pi}_i(s)} \sqrt{\min_{j \in \{1, \dots, N\}} (\lambda_j \circ \bar{\pi}_j^{(-1)}(a))} da \end{cases} \quad (4)$$

- $\forall i \in \{1, \dots, N\}$, $\Pi_i|_{[0,1]}$ is a derivable increasing function
- $\forall s \leq 0$, $\Pi_i(s) = \Pi_i(0) = 0$
- $\forall s \geq 1$, $\Pi_i(s) = \Pi_i(1)$
- $\forall (i, j) \in \{1, \dots, N\}, \forall (s_i, s_j) \in \mathbb{R}, \bar{\pi}_i(s_i) = \bar{\pi}_j(s_j) \Leftrightarrow \Pi_i(s_i) = \Pi_j(s_j)$.

We denote by $\Pi(s, x) = \Pi_i(s)$ and $\varphi(s, x) = \varphi_i(s)$ if $x \in \Omega_i$.

Remark 1.2 The last point seen in the previous definition allows us to connect Π_i and Π_j instead of $\bar{\pi}_i$ and $\bar{\pi}_j$ on $\Gamma_{i,j}$.

The definition of φ_i implies that $\partial_t u - \Delta \varphi_i(u) = 0$ in Ω_i , and we can rewrite the transmission conditions on $\Gamma_{i,j}$: $\bar{\pi}_i(u) = \bar{\pi}_j(u)$ and $\nabla \varphi_i(u) \cdot \mathbf{n}_i + \nabla \varphi_j(u) \cdot \mathbf{n}_j = 0$, where \mathbf{n}_i represents the outward normal to Ω_i . We can now define the notion of weak solution for the problem (2):

Definition 1.2 (weak solution) Under assumptions 1, a function u is said to be a weak solution to problem (2) if it satisfies:

1. $u \in L^\infty(\Omega \times (0, T))$, $0 \leq u \leq 1$ a.e. in $\Omega \times (0, T)$,

2. $\forall i \in \{1, \dots, N\}, \varphi_i(u) \in L^2(0, T; H^1(\Omega_i)),$
3. $\Pi(u, \cdot) \in L^2(0, T; H^1(\Omega)),$
4. for all $\psi \in \mathcal{D}(\Omega \times [0, T]),$

$$\begin{aligned} & \int_{\Omega} \int_0^T u(x, t) \partial_t \psi(x, t) dx dt + \int_{\Omega} u_0(x) \psi(x, 0) dx \\ & - \sum_{i=1}^N \int_{\Omega_i} \int_0^T \nabla \varphi_i(u(x, t)) \cdot \nabla \psi(x, t) dx dt = 0. \end{aligned} \quad (5)$$

Remark 1.3 *The transmission condition $\bar{\pi}_i(u) = \bar{\pi}_j(u)$ on the interface $\Gamma_{i,j}$ is now replaced by the point 3 in the previous definition. Because of the lack of regularity on the solution, we cannot write $\nabla \varphi_i(u) \cdot \mathbf{n}_i + \nabla \varphi_j(u) \cdot \mathbf{n}_j = 0$ on $\Gamma_{i,j}$ in a strong sense. But it is easy to check that, if u is a regular enough weak solution of (2), this condition is imposed by point 4 of the previous definition.*

1.3 Finite volume approximation and main convergence result

Let us first define space and time discretization of $\Omega \times (0, T)$.

Definition 1.3 (Admissible mesh of Ω) *An admissible mesh of Ω is given by a set \mathcal{T} of open bounded convex subsets of Ω called control volumes, a family \mathcal{E} of subsets of $\bar{\Omega}$ contained in hyperplanes of \mathbb{R}^d with strictly positive measure, and a family of points $(x_K)_{K \in \mathcal{T}}$ (the “centers” of control volumes) satisfying the following properties:*

1. $\exists i \in \{1, \dots, N\}, K \subset \Omega_i.$ We denote by $\mathcal{T}_i = \{K \in \mathcal{T} / K \subset \Omega_i\}.$
2. $\overline{\bigcup_{K \in \mathcal{T}_i} K} = \bar{\Omega}_i.$ Thus, $\overline{\bigcup_{K \in \mathcal{T}} K} = \bar{\Omega}.$
3. For any $K \in \mathcal{T},$ there exists a subset \mathcal{E}_K of \mathcal{E} such that $\partial K = \bar{K} \setminus \bigcup_{\sigma \in \mathcal{E}_K} \bar{\sigma}.$ Furthermore, $\mathcal{E} = \bigcup_{K \in \mathcal{T}} \mathcal{E}_K.$
4. For any $(K, L) \in \mathcal{T}^2$ with $K \neq L,$ either the “length” (i.e. the $(d-1)$ Lebesgue measure) of $\bar{K} \cap \bar{L}$ is 0 or $\bar{K} \cap \bar{L} = \bar{\sigma}$ for some $\sigma \in \mathcal{E}.$ In the latter case, we shall write $\sigma = K|L,$ and
 - $\mathcal{E}_i = \{\sigma \in \mathcal{E}, \exists (K, L) \in \mathcal{T}_i^2, \sigma = K|L\}$
 - $\mathcal{E}_{ext} = \{\sigma \in \mathcal{E}, \sigma \subset \partial \Omega\}, \mathcal{E}_{ext,i} = \{\sigma \in \mathcal{E}, \sigma \subset \partial \Omega_i \cap \partial \Omega\}$
 - $\mathcal{E}_{\Gamma_{i,j}} = \{\sigma \in \mathcal{E}, \exists (K, L) \in \mathcal{T}_i \times \mathcal{T}_j, \sigma = K|L\}$
 - $\mathcal{F} = \bigcup_{i,j} \mathcal{E}_{\Gamma_{i,j}}.$

For any $i \in \{1, \dots, N\},$ for any $K \in \mathcal{T}_i,$ we shall denote by:

- $\mathcal{N}_K = \{L \in \mathcal{T}, \exists \sigma \in \mathcal{E}, \sigma = K|L\}$
- $\mathcal{N}_{K,i} = \{L \in \mathcal{T}_i, \exists \sigma \in \mathcal{E}_i, \sigma = K|L\}$
- $\mathcal{F}_K = \{L \in \mathcal{T}, \exists j \neq i, \exists \sigma \in \mathcal{E}_{\Gamma_{i,j}}, \sigma = K|L\}$
- $\mathcal{E}_{K,i} = \mathcal{E}_K \cap \mathcal{E}_i$
- $\mathcal{E}_{K,ext} = \mathcal{E}_K \cap \mathcal{E}_{ext}$.

5. The family of points $(x_K)_{K \in \mathcal{T}}$ is such that $x_K \in K$ (for all $K \in \mathcal{T}$) and, if $\sigma = K|L$, it is assumed that the straight line (x_K, x_L) is orthogonal to σ .

For a control volume $K \in \mathcal{T}$, we will denote by $m(K)$ its measure and $\mathcal{E}_{ext,K} = \mathcal{E}_K \cap \partial\Omega$. For all $\sigma \in \mathcal{E}$, we denote by $m(\sigma)$ the $(d-1)$ -Lebesgue measure of σ . If $\sigma \in \mathcal{E}_K$, we denote by $\tau_{K,\sigma}$ the transmissibility of K through σ , defined by $\tau_{K,\sigma} = \frac{m(\sigma)}{d(x_K,\sigma)}$. We also define $\tau_{K|L} = \frac{m(\sigma)}{d(x_K,x_L)}$. The size of the mesh is defined by:

$$size(\mathcal{T}) = \max_{K \in \mathcal{T}} diam(K),$$

and a geometrical factor, linked with the regularity of the mesh, is defined by

$$reg(\mathcal{T}) = \max_{K \in \mathcal{T}} (card(\mathcal{E}_K), \max_{\sigma \in \mathcal{E}_K} \frac{diam(K)}{d(x_K,\sigma)}).$$

Remark 1.4 For all $\sigma \in \mathcal{E}$, $\frac{1}{\tau_{K|L}} = \frac{1}{\tau_{K,\sigma}} + \frac{1}{\tau_{L,\sigma}}$.

Definition 1.4 (Uniform time discretization of $(0, T)$) A uniform time discretization of $(0, T)$ is given by an integer value M and a sequence of real values $(t^n)_{n=0, \dots, M+1}$. We define $\delta t = \frac{T}{M+1}$ and, $\forall n \in \{0, \dots, M\}$, $t^n = n\delta t$. Thus we have $t^0 = 0$ and $t^{M+1} = T$.

Remark 1.5 We can easily prove all the results of this paper for a general time discretization, but for the sake of simplicity, we choose to consider only uniform time discretization.

Definition 1.5 (Space-time discretization of $\Omega \times (0, T)$) A finite volume discretization \mathcal{D} of $\Omega \times (0, T)$ is the family

$$\mathcal{D} = (\mathcal{T}, \mathcal{E}, (x_K)_{K \in \mathcal{T}}, N, (t^n)_{n \in \{0, \dots, M\}}),$$

where $(\mathcal{T}, \mathcal{E}, (x_K)_{K \in \mathcal{T}})$ is an admissible mesh of Ω in the sense of Definition 1.3 and $(N, (t^n)_{n \in \{0, \dots, M\}})$ is a discretization of $(0, T)$ in the sense of Definition 1.4. For a given mesh \mathcal{D} , one defines:

$$size(\mathcal{D}) = \max(size(\mathcal{T}), \delta t), \quad \text{and } reg(\mathcal{D}) = reg(\mathcal{T}).$$

We may now define the finite volume discretization of problem (2). Let \mathcal{D} be a finite volume discretization of $\Omega \times (0, T)$ in the sense of Definition 1.5. The initial condition is discretized by:

$$U_K^0 = \frac{1}{m(K)} \int_K u_0(x) dx, \quad \forall K \in \mathcal{T}. \quad (6)$$

An **implicit finite volume scheme** for the discretization of problem (2) is given by the following set of nonlinear equations, whose discrete unknowns are $U = (U_K^{n+1})_{K \in \mathcal{T}, n \in \{0, \dots, M\}}: \forall K \in \mathcal{T}, \forall n \in \{0, \dots, M\}$

$$\begin{aligned} m(K) \frac{U_K^{n+1} - U_K^n}{\delta t} + \sum_{\sigma \in \mathcal{E}_{K,i}} \tau_{K,\sigma} (\varphi(U_K^{n+1}, x_K) - \varphi(U_{K,\sigma}^{n+1}, x_K)) \\ + \sum_{\sigma \in \mathcal{F}_K} \tau_{K,\sigma} (\varphi(U_K^{n+1}, x_K) - \varphi(U_{K,\sigma}^{n+1}, x_K)) = 0. \end{aligned} \quad (7)$$

where $\forall L \in \mathcal{N}_K, U_{K,K|L}^{n+1}, U_{L,K|L}^{n+1}$ are the only values in $[0, 1]$ that satisfy the transmission conditions:

$$\begin{cases} \tau_{K,\sigma} (\varphi(U_K^{n+1}, x_K) - \varphi(U_{K,\sigma}^{n+1}, x_K)) \\ \quad + \tau_{L,\sigma} (\varphi(U_L^{n+1}, x_L) - \varphi(U_{L,\sigma}^{n+1}, x_L)) = 0 \\ \Pi(U_{K,\sigma}^{n+1}, x_K) = \Pi(U_{L,\sigma}^{n+1}, x_L). \end{cases} \quad (8)$$

Definition 1.6 Let \mathcal{D} be an admissible discretization of $\Omega \times (0, T)$ in the sense of Definition 1.5. The approximate solution of problem (2) associated to the discretization \mathcal{D} is defined almost everywhere in $\Omega \times (0, T)$ by:

$$\begin{aligned} \forall x \in K, \forall t \in (t^n, t^{n+1}), \forall K \in \mathcal{T}, \forall n \in \{0, \dots, M\}, \\ u_{\mathcal{D}}(x, t) = U_K^{n+1}, \end{aligned} \quad (9)$$

where $(U_K^{n+1})_{K \in \mathcal{T}, n \in \{0, \dots, M\}}$ is the unique solution to (7).

We will now state an assumption which will be useful to prove the uniqueness of the weak-solution in section 3.

Assumption 2 For all $i \in \{1, \dots, N\}$, $(\varphi_i \circ \Pi_i^{(-1)})'$ is a Lipschitz continuous function on $[0, 1]$.

We will now state our main result:

Theorem 1.1 (Convergence to the weak solution) Let $\xi \in \mathbb{R}$, consider a family of admissible discretizations of $\Omega \times (0, T)$ in the sense of Definition 1.5 such that, for all \mathcal{D} in the family, one has $\xi \geq \text{reg}(\mathcal{D})$. For a given admissible

discretization \mathcal{D} of this family, let $u_{\mathcal{D}}$ denote the associated approximate solution as defined in Definition 1.6. Then, under assumptions 1-2:

$$u_{\mathcal{D}} \rightarrow u \in L^p(\Omega \times (0, T)) \text{ as } \text{size}(\mathcal{D}) \rightarrow 0, \forall p \in [1, +\infty)$$

where u is the unique weak solution to problem (2) in the sense of Definition 1.2. Furthermore, we have the following regularity result:

$$u \in C^0([0, T], L^p(\Omega)), \quad \forall p \in [1, +\infty)$$

All this paper will be devoted to the proof of Theorem 1.1. First, we will use the work of [3] to prove the existence of a weak-solution. Then, in subsection 2.5, we will prove the existence of a time continuous solution, applying Ascoli theorem to a family of approximate solutions for a regular enough initial data. The uniqueness of the weak solution will be proven in the section 3 by using a doubling variable method inspired from [2, 6, 8].

2 Existence of a weak solution

The main work of this section has already been done in [3]. We only need to get enough results to prove the time-continuity in section 2.5. This proof will need estimates obtained by working on the scheme, so we prefer to give the whole proof of convergence. In this whole part, any sequence $(\mathcal{D}_m)_{m \in \mathbb{N}}$ of admissible discretizations of $\Omega \times (0, T)$ in the sense of Definition 1.5 will be supposed to have a bounded regularity.

$$\exists \zeta \in \mathbb{R}, \forall m \in \mathbb{N}, \quad \zeta \geq \text{reg}(\mathcal{D}_m). \quad (10)$$

2.1 Existence, uniqueness of the approximate solution

We state here the properties and estimates which are satisfied by the scheme (7) which we introduced in the previous section and prove existence and uniqueness of the solution to this scheme. First, we will take some notations for the convenience of the reader:

Notations 1 for all $K \in \mathcal{T}$, for all $\sigma \in \mathcal{E}_K \cap \mathcal{F}_K$, for all $n \in \{0, \dots, M+1\}$,

$$\left\{ \begin{array}{l} \varphi_K^n = \varphi(U_K^n, x_K), \\ \varphi_{K,\sigma}^n = \varphi(U_{K,\sigma}^n, x_K), \\ \Pi_K^n = \Pi(U_K^n, x_K), \\ \Pi_{K,\sigma}^n = \Pi(U_{K,\sigma}^n, x_K), \\ \bar{\pi}_K^n = \bar{\pi}(U_K^n, x_K), \\ \bar{\pi}_{K,\sigma}^n = \bar{\pi}(U_{K,\sigma}^n, x_K). \end{array} \right.$$

We will need the following lemma:

Lemma 2.1 For all $K \in \mathcal{T}$, for all $L \in \mathcal{N}_K$, for all $n \in \{0, \dots, M\}$, for all $(U_K^{n+1}, U_L^{n+1}) \in \mathbb{R}^2$, there exists a unique $(U_{K,\sigma}^{n+1}, U_{L,\sigma}^{n+1}) \in [0, 1]^2$ solution of (8).

Proof

Suppose that there exists $i \in \{1, \dots, N\}$ such that $(x_K, x_L) \in \Omega_i^2$. Then (8) can be written:

$$\begin{cases} \tau_{K,\sigma}(\varphi_K^{n+1} - \varphi_{K,\sigma}^{n+1}) + \tau_{L,\sigma}(\varphi_L^{n+1} - \varphi_{L,\sigma}^{n+1}) = 0 \\ U_{K,\sigma}^{n+1} = U_{L,\sigma}^{n+1} \end{cases}$$

which leads to

$$\begin{cases} U_{K,\sigma}^{n+1} = U_{L,\sigma}^{n+1} \\ \varphi_{K,\sigma}^{n+1} = \frac{1}{\tau_{K,\sigma} + \tau_{L,\sigma}} (\tau_{K,\sigma} \varphi_K^{n+1} + \tau_{L,\sigma} \varphi_L^{n+1}) \end{cases}$$

and $\frac{1}{\tau_{K,\sigma} + \tau_{L,\sigma}} (\tau_{K,\sigma} \varphi_K^{n+1} + \tau_{L,\sigma} \varphi_L^{n+1})$ admits an unique antecedent through φ_i in $[0, 1]$.

Let us now suppose that $(x_K, x_L) \in \Omega_i \times \Omega_j$ with $j \neq i$

$$(8) \Leftrightarrow \begin{cases} U_{K,\sigma}^{n+1} = \bar{\pi}_i^{-1}(\bar{\pi}_{L,\sigma}^{n+1}) \\ \tau_{K,\sigma}(\varphi_K^{n+1} - \varphi_i(\bar{\pi}_i^{-1}(\bar{\pi}_{L,\sigma}^{n+1}))) + \tau_{L,\sigma}(\varphi_L^{n+1} - \varphi_{L,\sigma}^{n+1}) = 0 \end{cases}$$

then

$$\begin{cases} U_{K,\sigma}^{n+1} = \bar{\pi}_i^{-1}(\bar{\pi}_{L,\sigma}^{n+1}) \\ \tau_{K,\sigma} \varphi_i(\bar{\pi}_i^{-1}(\bar{\pi}_{L,\sigma}^{n+1})) + \tau_{L,\sigma} \varphi_{L,\sigma}^{n+1} = \tau_{K,\sigma} \varphi_K^{n+1} + \tau_{L,\sigma} \varphi_L^{n+1}. \end{cases}$$

Since $\bar{\pi}_i(0) = \bar{\pi}_j(0)$ and $\bar{\pi}_i(1) = \bar{\pi}_j(1)$ (Assumption 1), then:

$$\forall U_{L,\sigma}^{n+1} \in [0, 1], U_{K,\sigma}^{n+1} \in [0, 1].$$

The application $\theta : z \mapsto \tau_{K,\sigma} \varphi_i(\bar{\pi}_i^{-1}(\bar{\pi}_j(z))) + \tau_{L,\sigma} \varphi_j(z)$ is increasing on $[0, 1]$, ensuring this way the uniqueness of the solution of (8). Furthermore, it satisfies:

$$\begin{cases} \theta(0) = 0 \\ \theta(1) = \tau_{K,\sigma} \varphi_i(1) + \tau_{L,\sigma} \varphi_j(1). \end{cases}$$

We conclude the proof by obtaining the existence of the solution by using the intermediate value theorem. \blacksquare

L^∞ -stability of the scheme

One assumes $u_0 \in L^\infty(\Omega)$, $0 \leq u_0 \leq 1$ a.e in Ω , then for all $K \in \mathcal{T}$, $U_K^0 \in [0, 1]$ where U_K^0 is given by (6).

Proposition 2.2 Let \mathcal{D} an admissible discretization of $\Omega \times (0, T)$ in the sense of Definition 1.5, let $(U_K^{n+1})_{K \in \mathcal{T}, n \in \{0, \dots, M\}}$ be a solution of the scheme (7), then for all $K \in \mathcal{T}$, for all $n \in \{0, \dots, M\}$:

$$0 \leq U_K^{n+1} \leq 1. \quad (11)$$

Proof

Let us first remark that the scheme can be written:

$$\begin{aligned} \frac{U_K^{n+1} - U_K^n}{\delta t} m(K) &+ \sum_{L \in \mathcal{N}_{K,i}} \tau_{K|L} (\varphi_K^{n+1} - \varphi_L^{n+1}) \\ &+ \sum_{\sigma \in \mathcal{F}_K} \tau_{K,\sigma} (\varphi_K^{n+1} - \varphi_{K,\sigma}^{n+1}) = 0, \end{aligned}$$

with $\tau_{K|L} = \frac{m(K|L)}{d(x_K, x_L)}$. Let us now rewrite it once again. $\forall K \in \mathcal{T}_i, \forall n \in \{0, \dots, M\}$:

$$U_K^{n+1} = H_K \left(U_K^n, (U_L^{n+1})_{L \in \mathcal{T}} \right)$$

with:

$$H_K \left(a, (a_L)_{L \in \mathcal{T}} \right) = \frac{a + \lambda_K a_K + \frac{\delta t}{m(K)} \left[\sum_{L \in \mathcal{N}_{K,i}} \tau_{K|L} (\varphi_i(a_L) - \varphi_i(a_K)) + \sum_{\sigma \in \mathcal{F}_K} \tau_{K,\sigma} (\varphi_i(a_{K,\sigma}) - \varphi_i(a_K)) \right]}{1 + \lambda_K}$$

where λ_K is given by:

$$\lambda_K = \frac{\delta t L_\varphi}{m(K)} \left(\sum_{L \in \mathcal{N}_{K,i}} \tau_{K|L} + \sum_{\sigma \in \mathcal{F}_K} \tau_{K,\sigma} \right)$$

and L_φ is a Lipschitz constant for all φ_i . However, $a_{K,\sigma} = g(a_K, a_L)$ where g is a non-decreasing function. We deduce from it that H_K is a nondecreasing function of all its arguments.

Let $n \in \{0, \dots, M\}$, let us assume $0 \leq U_K^n \leq 1, \forall K \in \mathcal{T}$. Let us assume that there exists $K_{max} \in \mathcal{T}$ such that:

$$U_{K_{max}}^{n+1} = \max_{K \in \mathcal{T}} (U_K^{n+1}) > 1,$$

then:

$$1 < U_{K_{max}}^{n+1} \leq H_K \left(1, (U_{K_{max}}^{n+1})_{L \in \mathcal{T}} \right) = \frac{1 + \lambda_K U_{K_{max}}^{n+1}}{1 + \lambda_K} < U_{K_{max}}^{n+1},$$

a contradiction. Therefore:

$$U_{K_{max}}^{n+1} \leq 1.$$

We can prove exactly in the same way that

$$U_{K_{min}}^{n+1} = \min_{K \in \mathcal{T}} (U_K^{n+1}) \geq 0.$$

■

Proposition 2.3 *Let \mathcal{D} be an admissible discretization of $\Omega \times (0, T)$ in the sense of Definition 1.5. There exists a unique solution to the scheme (7).*

Proof

Existence of the discrete solution

Let $n \in \{0, \dots, M\}$. Since $(U_{K,\sigma}^{n+1})_{\sigma \in \mathcal{E}}$ is a function of $(U_L^{n+1})_{L \in \mathcal{T}}$, we shall see the scheme as a non linear system of equations only depending on $(U_L^{n+1})_{L \in \mathcal{T}}$.

Let us consider the application $\Psi : \begin{cases} \mathbb{R}^{\#\mathcal{T}} \times [0, 1] & \rightarrow \mathbb{R}^{\#\mathcal{T}} \\ ((U_K^{n+1})_{K \in \mathcal{T}}, \lambda) & \mapsto (V_K)_{K \in \mathcal{T}} \end{cases}$

where:

$$V_K = \frac{U_K^{n+1} - \lambda U_K^n}{\delta t} m(K) + \lambda \sum_{L \in \mathcal{N}_{K,i}} \tau_{K|L} (\varphi_K^{n+1} - \varphi_L^{n+1}) + \lambda \sum_{\sigma \in \mathcal{F}_K} \tau_{K,\sigma} (\varphi_K^{n+1} - \varphi_{K,\sigma}^{n+1})$$

The linear system of equation $\Psi((U_K^{n+1})_{K \in \mathcal{T}}, 0) = (0)_{K \in \mathcal{T}}$ admits a unique trivial solution. The continuity of $\lambda \mapsto \Psi(\cdot, \lambda)$ and the L^∞ -estimate (11) allows us to use a topological degree argument, insuring the existence of a solution for $\lambda = 1$.

Uniqueness of the discrete solution

Assume that, for a given value of n , there exist two solutions to the scheme (7), $(U_K^{n+1})_{K \in \mathcal{T}}$ and $(V_K^{n+1})_{K \in \mathcal{T}}$. Then, for all $K \in \mathcal{T}$, using the monotony of H_K :

$$\max(U_K^{n+1}, V_K^{n+1}) \leq H_K(U_K^n, (\max(U_L^{n+1}, V_L^{n+1}))_{L \in \mathcal{T}}), \quad (12)$$

$$\min(U_K^{n+1}, V_K^{n+1}) \leq H_K(U_K^n, (\min(U_L^{n+1}, V_L^{n+1}))_{L \in \mathcal{T}}). \quad (13)$$

One multiplies (12) and (13) by $(1 + \lambda_K)m(K)$, subtracts (13) to (12) and sum on $K \in \mathcal{T}$. Remarking that all the exchange terms between neighboring control volume disappear, we get:

$$\sum_{K \in \mathcal{T}} |U_K^{n+1} - V_K^{n+1}| m(K) \leq 0.$$

■

2.2 Discrete $L^2(0, T; H^1(\Omega))$ estimates

In this section, we will prove some estimates on the approximate solution. We first have to define the space $\mathcal{X}(\mathcal{D})$ the solution belongs to.

Definition 2.1 *Let \mathcal{D} be an admissible discretization of $\Omega \times (0, T)$ in the sense of the Definition 1.5. We denote by $\mathcal{X}(\mathcal{D})$ the functional space:*

$$\mathcal{X}(\mathcal{D}) = \left\{ \begin{array}{l} v \in L^\infty(\Omega \times (0, T)), \forall K \in \mathcal{T}, \forall n \in \{0, \dots, M\}, \\ \exists V_K^{n+1}, v(x, t) = V_K^{n+1} \text{ a.e. in } K \times (t^n, t^{n+1}) \end{array} \right\}.$$

Definition 2.2 (Discrete $L^2(0, T; H^1(\Omega_i))$ semi-norm) We define the discrete $L^2(0, T; H^1(\Omega_i))$ semi norm on $\mathcal{X}(\mathcal{D})$ by:

$$|v|_{1, \mathcal{D}, i}^2 = \sum_{n=0}^M \delta t \sum_{K|L \in \mathcal{E}_i} \tau_{K, \sigma} (V_K^{n+1} - V_L^{n+1})^2.$$

We will need the following lemma:

Lemma 2.4 Let \mathcal{D} be an admissible discretization of $\Omega \times (0, T)$ in the sense of the Definition 1.5. Let $u_{\mathcal{D}}$ be the discrete solution to (7). Let $\sigma \in \Gamma_{i, j}$ for some $i, j \in \{1, \dots, N\}^2$, $\sigma = K|L, K \in \mathcal{T}_i, L \in \mathcal{T}_j$. Then:

$$0 \leq (\varphi_K^{n+1} - \varphi_{K, \sigma}^{n+1})(\bar{\pi}_K^{n+1} - \bar{\pi}_{K, \sigma}^{n+1}) \leq (\varphi_K^{n+1} - \varphi_{K, \sigma}^{n+1})(\bar{\pi}_K^{n+1} - \bar{\pi}_L^{n+1}), \quad (14)$$

$$0 \leq (\varphi_K^{n+1} - \varphi_{K, \sigma}^{n+1})(\Pi_K^{n+1} - \Pi_{K, \sigma}^{n+1}) \leq (\varphi_K^{n+1} - \varphi_{K, \sigma}^{n+1})(\Pi_K^{n+1} - \Pi_L^{n+1}). \quad (15)$$

Proof

$\forall K \in \mathcal{T}$, $\varphi(\cdot, x_K)$, and $\bar{\pi}(\cdot, x_K)$ are increasing functions on $[0, 1]$, thus

$$(\varphi_K^{n+1} - \varphi_{K, \sigma}^{n+1})(\bar{\pi}_K^{n+1} - \bar{\pi}_{K, \sigma}^{n+1}) \geq 0.$$

Furthermore, $(\bar{\pi}_K^{n+1} - \bar{\pi}_L^{n+1}) = (\bar{\pi}_K^{n+1} - \bar{\pi}_{K, \sigma}^{n+1} + \bar{\pi}_{L, \sigma}^{n+1} - \bar{\pi}_L^{n+1})$ then:

$$\begin{aligned} (\varphi_K^{n+1} - \varphi_{K, \sigma}^{n+1})(\bar{\pi}_K^{n+1} - \bar{\pi}_L^{n+1}) &= (\varphi_K^{n+1} - \varphi_{K, \sigma}^{n+1})(\bar{\pi}_K^{n+1} - \bar{\pi}_{K, \sigma}^{n+1}) \\ &\quad + (\varphi_K^{n+1} - \varphi_{K, \sigma}^{n+1})(\bar{\pi}_L^{n+1} - \bar{\pi}_{L, \sigma}^{n+1}) \\ &= (\varphi_K^{n+1} - \varphi_{K, \sigma}^{n+1})(\bar{\pi}_K^{n+1} - \bar{\pi}_{K, \sigma}^{n+1}) \\ &\quad + \frac{\tau_{L, \sigma}}{\tau_{K, \sigma}} (\varphi_L^{n+1} - \varphi_{L, \sigma}^{n+1})(\bar{\pi}_L^{n+1} - \bar{\pi}_{L, \sigma}^{n+1}) \\ &\geq (\varphi_K^{n+1} - \varphi_{K, \sigma}^{n+1})(\bar{\pi}_K^{n+1} - \bar{\pi}_{K, \sigma}^{n+1}). \end{aligned}$$

This proof is also true with Π instead of $\bar{\pi}$. ■

One introduces the function

$$\eta_i : s \mapsto \int_0^s \sqrt{\lambda_i(a)} \bar{\pi}'_i(a) da,$$

which fulfills thanks to Cauchy-Schwarz inequality, for all $(a, b) \in [0, 1]^2$, for all $i \in \{1, \dots, N\}$,

$$(\eta_i(a) - \eta_i(b))^2 \leq (\varphi_i(a) - \varphi_i(b))(\bar{\pi}_i(a) - \bar{\pi}_i(b)). \quad (16)$$

Proposition 2.5 (Discrete $L^2(0, T; H^1(\Omega_i))$ estimate) Under Assumption 1, let \mathcal{D} be an admissible discretization of $\Omega \times (0, T)$ in the sense of Definition 1.5, let $u_{\mathcal{D}}$ be the solution of the scheme (7). Then there exists C only depending on $\bar{\pi}_i, \Omega_i, i \in \{1, \dots, N\}$ such that:

$$\sum_{i=1}^N |\eta_i(u_{\mathcal{D}})|_{1,\mathcal{D},i}^2 \leq C. \quad (17)$$

$$0 \leq \sum_{n=0}^M \delta t \sum_{\substack{\sigma \in \mathcal{F} \\ \sigma = K|L}} \tau_{K,\sigma} (\varphi_K^{n+1} - \varphi_{K,\sigma}^{n+1}) (\bar{\pi}_K^{n+1} - \bar{\pi}_L^{n+1}) \leq C \quad (18)$$

Proof

Accumulation term:

Let us multiply the equations (7) by $\delta t \bar{\pi}_K^{n+1}$ and sum on $K \in \mathcal{T}, n \in \{0, \dots, M\}$. We get:

$$\sum_{n=0}^M \sum_{K \in \mathcal{T}} \left(\begin{array}{l} (m(K)(U_K^{n+1} - U_K^n) + \delta t \sum_{L \in \mathcal{N}_{K,i}} \tau_{K|L} (\varphi_K^{n+1} - \varphi_L^{n+1}) + \\ \sum_{s \in \mathcal{F}_K} \tau_{K,\sigma} (\varphi_K^{n+1} - \varphi_{K,\sigma}^{n+1})) \bar{\pi}_K^{n+1} \end{array} \right) = 0.$$

Let i belong to $\{1, \dots, N\}$, let $K \in \mathcal{T}_i$. Since $\bar{\pi}_i$ is an increasing function, $g_i : s \mapsto \int_0^s \bar{\pi}_i(a) da$ is a convex function. Then:

$$(U_K^{n+1} - U_K^n) \bar{\pi}_K^{n+1} \geq g_i(U_K^{n+1}) - g_i(U_K^n).$$

Thus:

$$\begin{aligned} \sum_{n=0}^M \sum_{K \in \mathcal{T}} m(K) (U_K^{n+1} - U_K^n) \bar{\pi}_K^{n+1} &\geq \sum_{K \in \mathcal{T}} m(K) (g_i(U_K^{M+1}) - g_i(U_K^0)) \\ &\geq -m(\Omega) \int_0^1 \max_{i \in \{1, \dots, N\}} |\bar{\pi}_i(a)| da. \end{aligned}$$

Diffusion term:

One gets, thanks to (16)

$$\begin{aligned} &\sum_{n=0}^M \delta t \sum_{i=1}^N \sum_{K|L \in \mathcal{E}_i} [\tau_{K|L} (\varphi_K^{n+1} - \varphi_L^{n+1}) (\bar{\pi}_K^{n+1} - \bar{\pi}_L^{n+1})] \\ &\geq \sum_{n=0}^M \delta t \sum_{i=1}^N \sum_{K|L \in \mathcal{E}_i} \tau_{K|L} (\eta_K^{n+1} - \eta_L^{n+1})^2, \end{aligned}$$

where η_K^{n+1} denotes $\eta_i(U_K^{n+1})$ if $K \subset \Omega_i$. Furthermore, for all $\sigma = K|L \in \mathcal{F}$, Lemma 2.4 implies:

$$(\varphi_K^{n+1} - \varphi_{K,\sigma}^{n+1}) (\bar{\pi}_K^{n+1} - \bar{\pi}_L^{n+1}) \geq 0,$$

then, we have the following estimates:

$$\sum_{i=1}^N |\eta_i(u_{\mathcal{D}})|_{1,\mathcal{D},i}^2 \leq m(\Omega) \int_0^1 \max_{i \in \{1, \dots, N\}} |\bar{\pi}_i(a)| da,$$

$$0 \leq \sum_{n=0}^M \delta t \sum_{\sigma=K|L \in \mathcal{F}} (\varphi_K^{n+1} - \varphi_{K,\sigma}^{n+1})(\bar{\pi}_K^{n+1} - \bar{\pi}_L^{n+1}) \leq m(\Omega) \int_0^1 \max_{i \in \{1, \dots, N\}} |\bar{\pi}_i(a)| da.$$

■

Definition 2.3 (Discrete $L^2(0, T; H^1(\Omega))$ semi-norm) Let \mathcal{D} be an admissible discretization of $\Omega \times (0, T)$ in the sense of the Definition 1.5. One defines the discrete $L^2(0, T; H^1(\Omega))$ semi norm of $v \in \mathcal{X}(\mathcal{D})$ by:

$$|v|_{1,\mathcal{D}}^2 = \sum_{i=1}^N |v|_{1,\mathcal{D},i}^2 + \sum_{\sigma=K|L \in \mathcal{F}} \left[\tau_{K|L} (v(x_K, t^{n+1}) - v(x_L, t^{n+1}))^2 \right].$$

We will need the following lemma:

Lemma 2.6 Under assumptions 1, for all i in $\{1, \dots, N\}$, the function $\Pi_i \circ \eta_i^{(-1)}$ admits 1 as Lipschitz constant.

Proof

Let $i \in \{1, \dots, N\}$, let $a \in]0, \varphi_i(1)[$, let $b \in]0, \eta_i(1)[$ with $b \neq a$.

We set $A = \eta_i^{(-1)}(a)$ and $B = \eta_i^{(-1)}(b)$. One has:

$$\frac{\bar{\pi}_i \circ \eta_i^{(-1)}(b) - \bar{\pi}_i \circ \eta_i^{(-1)}(a)}{b - a} = \frac{\bar{\pi}_i(B) - \bar{\pi}_i(A)}{\eta_i(B) - \eta_i(A)}.$$

One denote by $I(A, B)$ the interval $[A, B]$ if $B \geq A$, and $[B, A]$ if $A \geq B$. The definition of the function η_i implies:

$$\min_{C \in I(A, B)} \sqrt{\lambda_i(C)} (\bar{\pi}_i(B) - \bar{\pi}_i(A)) \leq \eta_i(B) - \eta_i(A) \leq \max_{C \in I(A, B)} \sqrt{\lambda_i(C)} (\bar{\pi}_i(B) - \bar{\pi}_i(A)).$$

Then, there exists $C \in I(A, B)$ such that:

$$\eta_i(B) - \eta_i(A) = \sqrt{\lambda_i(C)} (\bar{\pi}_i(B) - \bar{\pi}_i(A)).$$

So one gets:

$$\frac{\bar{\pi}_i \circ \eta_i^{(-1)}(b) - \bar{\pi}_i \circ \eta_i^{(-1)}(a)}{b - a} = \frac{1}{\sqrt{\lambda_i(C)}}.$$

Letting b tend to a , we get, using the continuity of $\eta_i^{(-1)}$:

$$(\bar{\pi}_i \circ \eta_i^{(-1)})'(a) = \frac{1}{\sqrt{\lambda_i \circ \eta_i^{(-1)}(a)}}.$$

Remarking that $\Pi_i = \Psi \circ \bar{\pi}_i$ with

$$\Psi : \begin{cases} [\bar{\pi}_i(0), \bar{\pi}_i(1)] & \rightarrow \mathbb{R}^+ \\ p & \mapsto \int_{\bar{\pi}_i(0)}^p \min_{j \in \{1, \dots, N\}} \left(\sqrt{\lambda_j \circ \bar{\pi}_j^{(-1)}(a)} \right) da \end{cases}$$

we may obtain:

$$(\Pi_i \circ \eta_i^{(-1)})'(a) = \Psi'(\bar{\pi}_i \circ \eta_i^{(-1)}(a)) (\bar{\pi}_i \circ \eta_i^{(-1)})'(a) = \frac{\Psi'(\bar{\pi}_i \circ \eta_i^{(-1)}(a))}{\sqrt{\lambda_i \circ \eta_i^{(-1)}(a)}}.$$

Remarking that the definition of Ψ implies $\Psi'(\bar{\pi}_i(y)) \leq \sqrt{\lambda_i(y)}$, we get that

$$(\Pi_i \circ \eta_i^{(-1)})'(a) \leq 1.$$

■

Proposition 2.7 (Discrete $L^2(0, T; H^1(\Omega))$ estimate) *Let \mathcal{D} be an admissible discretization of $\Omega \times (0, T)$ in the sense of Definition 1.5, let $u_{\mathcal{D}} \in \mathcal{X}(\mathcal{D})$ be the approximate solution given by the scheme (7). There exists a constant C only depending on $\Omega, \bar{\pi}_i, \forall i \in \{1, \dots, N\}$ such that:*

$$|\Pi(u_{\mathcal{D}}, \cdot)|_{1, \mathcal{D}}^2 \leq C.$$

Proof

Using inequality (17) proven in Proposition 2.5 and Lemma 2.6, we immediately get that:

$$\sum_{i=1}^N |\Pi_i(u_{\mathcal{D}})|_{1, \mathcal{D}, i}^2 \leq C.$$

Let us now consider the case $\sigma = K|L \in \mathcal{F}$. Using $\bar{\pi}_{K, \sigma}^{n+1} = \bar{\pi}_{L, \sigma}^{n+1}$, inequality (18) together with (8) leads to:

$$\sum_{n=0}^M \delta t \sum_{\sigma=K|L \in \mathcal{F}} \left[\begin{array}{c} \tau_{K, \sigma}(\varphi_K^{n+1} - \varphi_{K, \sigma}^{n+1})(\bar{\pi}_K^{n+1} - \bar{\pi}_{K, \sigma}^{n+1}) + \\ \tau_{L, \sigma}(\varphi_L^{n+1} - \varphi_{L, \sigma}^{n+1})(\bar{\pi}_L^{n+1} - \bar{\pi}_{L, \sigma}^{n+1}) \end{array} \right] \leq C. \quad (19)$$

For all $\sigma \in \mathcal{F}$, for all K such that $\sigma \in \mathcal{E}_K$, we have thanks to (16):

$$(\eta_K^{n+1} - \eta_{K, \sigma}^{n+1})^2 \leq (\varphi_K^{n+1} - \varphi_{K, \sigma}^{n+1})(\bar{\pi}_K^{n+1} - \bar{\pi}_{K, \sigma}^{n+1}).$$

Lemma 2.6 implies:

$$(\Pi_K^{n+1} - \Pi_{K,\sigma}^{n+1})^2 \leq (\eta_K^{n+1} - \eta_{K,\sigma}^{n+1})^2$$

and

$$(\Pi_L^{n+1} - \Pi_{L,\sigma}^{n+1})^2 \leq (\eta_L^{n+1} - \eta_{L,\sigma}^{n+1})^2,$$

thus (19) leads to

$$\sum_{n=0}^M \delta t \sum_{\sigma=K|L \in \mathcal{F}} \left[\tau_{K,\sigma} (\Pi_K^{n+1} - \Pi_{K,\sigma}^{n+1})^2 + \tau_{L,\sigma} (\Pi_L^{n+1} - \Pi_{L,\sigma}^{n+1})^2 \right] \leq C. \quad (20)$$

The convexity of the function $x \mapsto x^2$ together with the relation $\frac{1}{\tau_{K|L}} = \frac{1}{\tau_{K,\sigma}} + \frac{1}{\tau_{L,\sigma}}$ leads to:

$$\sum_{n=0}^M \delta t \sum_{\sigma=K|L \in \mathcal{F}} \tau_{K|L} (\Pi_K^{n+1} - \Pi_L^{n+1})^2 \leq C.$$

■

2.3 Some compactness results

We aim in this section to get enough compactness results to be able to let $size(\mathcal{D})$ tend to 0. Proposition 2.2 insures that, up to a subsequence, $\exists u \in L^\infty(\Omega \times (0, T))$ such that $u_{\mathcal{D}} \rightharpoonup u$ in the $L^\infty(\Omega \times (0, T))$ -weak \star topology.

Let us now turn to the Kolmogorov compactness criterion (see e.g. [1]) which will allow us to pass to the limit in the nonlinear second order terms.

Theorem 2.8 (Kolmogorov) *Let \mathcal{Q} be an open bounded subset of \mathbb{R}^k , and $(v_n)_n$ be a bounded sequence in $L^2(\mathbb{R}^k)$ such that:*

$$\lim_{\delta \rightarrow 0} [\sup_{n \in \mathbb{N}} \|v_n(\cdot + \delta) - v_n(\cdot)\|] = 0,$$

then there exists $v \in L^2(\mathcal{Q})$ such that, up to a subsequence,

$$v_n \rightarrow v \text{ in } L^2(\mathcal{Q}) \text{ as } n \rightarrow \infty.$$

Let us now show that we are in position to apply the Kolmogorov compactness criterion to $(\eta_i(u_{\mathcal{D}_m}))_{m \in \mathbb{N}}$ where $(\mathcal{D}_m)_{m \in \mathbb{N}}$ is a sequence of admissible discretization of $\Omega \times (0, T)$, with $\lim_{m \rightarrow \infty} size(\mathcal{D}_m) = 0$.

Space translates estimates

We will now state a proposition proven in [3], which is a consequence of Proposition 2.5.

Proposition 2.9 *Let \mathcal{D} be an admissible discretization of $\Omega \times (0, T)$ in the sense of Definition 1.5, let $u_{\mathcal{D}} \in \mathcal{X}(\mathcal{D})$ be the approximate solution given by the scheme (7), let $i \in \{1, \dots, N\}$, let $\xi \in \mathbb{R}^d$, and $\Omega_{i,\xi}$ the open subset of Ω_i defined by:*

$$\Omega_{i,\xi} = \{x \in \Omega_i / [x, x + \xi] \subset \Omega_i\}.$$

Then there exists C only depending on Ω_i such that:

$$\int_0^T \int_{\Omega_{i,\xi}} |\eta_i(u_{\mathcal{D}}(x + \xi, t) - \eta_i(u_{\mathcal{D}}(x, t)))|^2 dx dt \leq |\xi|(|\xi| + C \text{size}(\mathcal{D})) |\eta_i(u_{\mathcal{D}})|_{1,\mathcal{D},i}. \quad (21)$$

Time translates estimates

We state here a first result on the time translates of $\varphi_i(u)$, already proven in [3].

Proposition 2.10 *Let \mathcal{D} be an admissible discretization of $\Omega \times (0, T)$. Let $u_{\mathcal{D}} \in \mathcal{X}(\mathcal{D})$ be the approximate solution obtained with the scheme (7). Let $\omega_i \subset \bar{\omega}_i \subset \Omega_i$ be an open subset of Ω_i . We assume that $\text{size}(\mathcal{T})$ is small enough to ensure that:*

$$\omega_i \subset \Omega_{i,\text{size}(\mathcal{T})} = \{x \in \Omega_i / \overline{B(x, \text{size}(\mathcal{T}))} \subset \Omega_i\}.$$

We set:

$$\eta_{\mathcal{D},\omega_i} = \begin{cases} \eta_i(u_{\mathcal{D}}) & \text{on } \omega_i \times (0, T) \\ 0 & \text{on } \mathbb{R}^{d+1} \setminus (\omega_i \times (0, T)) \end{cases}$$

then, for all $\tau \in \mathbb{R}$, we get the following inequality:

$$\|\eta_{\mathcal{D},\omega_i}(\cdot, \cdot + \tau) - \eta_{\mathcal{D},\omega_i}\|^2 \leq C\tau, \quad (22)$$

where C is a constant which only depends on $T, \Omega, d, \varphi_i, \bar{\pi}_i, \Theta_i$.

Thus we can apply the Kolmogorov compactness criterion, and claim that there exists a function f such that, up to a subsequence, $\eta(u_{\mathcal{D}}, \cdot)$ converges to f in $L^2(\Omega \times (0, T))$ as $\text{size}(\mathcal{D})$ tends to 0. Furthermore, letting $\text{size}(\mathcal{D})$ tend to 0 in estimates (21) ensures that $f \in L^2(0, T; H^1(\Omega_i))$ for all $i \in \{1, \dots, N\}$. The same way, we can prove that $\Pi(u_{\mathcal{D}}, \cdot)$ converges to some $g \in L^2(0, T; H^1(\Omega))$ in the $L^2(\Omega \times (0, T))$ -topology.

Remark 2.1 *Since the functions $\eta_i \circ \varphi_i^{-1}$ are Lipschitz continuous, estimates (21) and (22) still hold with φ_i instead of η_i . Then $\varphi(u_{\mathcal{D}}, \cdot)$ also converges to a function h in $L^2(\Omega \times (0, T))$.*

2.4 Convergence to a weak solution

In this section, we aim to prove that, for an admissible sequence (\mathcal{D}_m) of discretization of $\Omega \times (0, T)$ in the sense of the Definition 1.5, with $\lim_{m \rightarrow +\infty} \text{size}(\mathcal{D}_m) = 0$ the solution of the scheme $u_{\mathcal{D}}$ tends to a weak solution of the problem (2).

We have proven in the previous section that $\varphi(u_{\mathcal{D}_m}, \cdot)$ (resp. $\Pi(u_{\mathcal{D}_m}, \cdot)$) converges in $L^2(\Omega \times (0, T))$ to a function f (resp. g). Proposition 2.2 allows us to assume that $u_{\mathcal{D}_m}$ converges to u in $L^\infty(\Omega \times (0, T))$ for the weak \star topology. Using Minty Lemma, stated below, let us show that $f = \eta(u, \cdot)$, $g = \Pi(u, \cdot)$ and $h = \varphi(u, \cdot)$.

Lemma 2.11 (Minty lemma) *Let Ω be an open subset of \mathbb{R}^k , let $\Psi : \mathbb{R} \rightarrow \mathbb{R}$ be a monotonous continuous function. Let $(u_n)_{n \in \mathbb{N}}$ such that:*

$$\begin{cases} u_n \rightharpoonup u & \text{in } L^\infty(\Omega) \text{ weak-}\star \\ \Psi(u_n) \rightarrow f & \text{in } L^1(\Omega) \end{cases}$$

then

$$\Psi(u) = f.$$

From the Lemma 2.11, we can deduce that, for all $i \in \{1, \dots, N\}$:

$$\eta_i(u_{\mathcal{D}}) \rightarrow \eta_i(u) \quad \text{in } L^1(\Omega_i \times (0, T)),$$

thus:

$$\eta(u_{\mathcal{D}}, \cdot) \rightarrow \eta(u, \cdot) \quad \text{in } L^1(\Omega \times (0, T)).$$

The same way, $\varphi(u_{\mathcal{D}}, \cdot)$ and $\Pi(u_{\mathcal{D}}, \cdot)$ converge in $L^2(\Omega \times (0, T))$ to $\varphi(u, \cdot)$ and $\Pi(u, \cdot)$, respectively. Estimate (21) insures that, for all $i \in \{1, \dots, N\}$, $\eta_i(u)$ belongs to $L^2(0, T; H^1(\Omega_i))$, and thus $\varphi_i(u)$ too. A straightforward adaptation of Lemma 2.9 with $\Pi(u_{\mathcal{D}}, \cdot)$ instead of $\eta_i(u_{\mathcal{D}})$ allows us to claim that $\Pi(u, \cdot)$ belongs to $L^2(0, T; H^1(\Omega))$.

Proposition 2.12 (Convergence to a weak solution) *Under assumption 1, let (\mathcal{D}_m) be a sequence of admissible discretizations in the sense of Definition 1.5 which fulfill the assumption (10) and such that $\text{size}(\mathcal{D}_m) \rightarrow 0$. Let $(u_{\mathcal{D}_m})$ be the sequence of approximate solutions given by the scheme (7). Then there exists a subsequence of approximate solutions still denoted $(u_{\mathcal{D}_m})$ which converges to u in $L^q(\Omega \times (0, T))$ for all $q \in [1, +\infty)$. Furthermore, u is a weak solution to the problem (2) in the sense of Definition 1.2.*

The proof of this proposition is a straightforward adaptation of the proof stated in [3].

2.5 Time continuity of the approximation limit

The aim of this section is to prove the existence of a time continuous solution. This result will be fundamental to prove the uniqueness of the weak solution to the problem (2).

In order to prove the continuity of a time continuous solution to the problem (2), we will apply the Ascoli theorem on a family of approximate solutions obtained through the scheme (7). We need a classical CFL assumption on the family of space-time discretizations.

Assumption 3 *Let $(\mathcal{D}_m)_{m \in \mathbb{N}}$ be an admissible space-time discretization of $\Omega \times (0, T)$ in the sense of Definition 1.5. In all this subsection, we furthermore assume that there exists $S_1 > 0$ which does not depend on m such that:*

$$\max_m \frac{\delta t_m}{(\text{size}(\mathcal{T}_m))^2} \leq S_1.$$

Remark 2.2 *Assumption 3 and (10) ensure us that the quantity*

$\max_{K \in \mathcal{T}} (\max_{L \in \mathcal{N}_{K,i}} (\frac{\delta t \tau_{K|L}}{m(K)}))$ *stays bounded as m tends to $+\infty$. The assumption 3 is not hard to fulfill in the theoretical framework. One just has to choose a convenient time step. Nevertheless, this assumption is very demanding in the numerical framework, but we will be able to relax it in the sequel of this work.*

We also need to make an assumption on the regularity of the initial data, but once again this assumption will be relaxed in the sequel, thanks to a density argument.

Assumption 4 *The initial data u_0 belongs to $L^\infty(\Omega)$, $0 \leq u_0 \leq 1$, and furthermore fulfills: $\varphi(u_0)$ is a piecewise Lipschitz function.*

We will first need the following lemmas:

Lemma 2.13 (Discrete $H^1(0, T; L^2(\omega_i))$ estimate) *Let $(\mathcal{D}_m)_{m \in \mathbb{N}}$ be a sequence of admissible discretizations of $\Omega \times (0, T)$ fulfilling assumption 3. Let $(u_{\mathcal{D}_m})$ the sequence of approximate solutions given by the scheme (7). Let O_i be an open subset of Ω_i such that $\varphi_i(u_0)|_{O_i}$ is a Lipschitz continuous function. Let ω_i be an open subset of O_i , with $\bar{\omega}_i \subset O_i$. Then there exists C only depending on $O_i, \omega_i, \lambda_i, \bar{\pi}_i, u_0, S_1, \zeta$ such that:*

$$\sum_{n=0}^M \sum_{\substack{K \in \mathcal{T} \\ K \subset \omega_i}} m(K) (\varphi_i(u_{\mathcal{D}_m}(x_K, t^{n+1})) - \varphi_i(u_{\mathcal{D}_m}(x_K, t^n)))^2 \leq C \delta t.$$

Proof

We use the following notations:

$$\mathcal{E}_{O_i} = \{\sigma \in \mathcal{E}_i, \sigma = K|L/ K \subset O_i, L \subset O_i\},$$

$$\mathcal{E}_{\omega_i} = \{\sigma \in \mathcal{E}_i, \sigma = K|L/ K \subset \omega_i, L \subset \omega_i\}.$$

Let $\Theta_i \in C_c^\infty(\Omega)$ such that $\text{supp}(\Theta_i) \subset O_i$, $\Theta_i|_{\omega_i} = 1$, $1 \geq \Theta_i \geq 0$. For all $K \in \mathcal{T}$, we denote $\Theta_{i,K} = \Theta_i(x_K)$. We multiply the scheme (7) by $(\varphi_K^{n+1} - \varphi_K^n)\Theta_{i,K}^2 \delta t$:

$$\frac{U_K^{n+1} - U_K^n}{\delta t} m(K) (\varphi_K^{n+1} - \varphi_K^n) \Theta_{i,K}^2 + \sum_{L \in \mathcal{N}_{K,i}} \left[\frac{\tau_{K|L} (\varphi_K^{n+1} - \varphi_L^{n+1})}{(\varphi_K^{n+1} - \varphi_K^n) \Theta_{i,K}^2} \right] = 0,$$

thus:

$$m(K) (\varphi_K^{n+1} - \varphi_K^n)^2 \Theta_{i,K}^2 \leq L_{\varphi_i} \delta t \sum_{L \in \mathcal{N}_{K,i}} \tau_{K|L} (\varphi_L^{n+1} - \varphi_K^{n+1}) (\varphi_K^{n+1} - \varphi_K^n) \Theta_{i,K}^2.$$

Let $M_1 \in \{0, \dots, M\}$. We sum on $K \in \mathcal{T}$ and on $n \in \{0, M_1\}$:

$$\begin{aligned} & \sum_{n=0}^{M_1} \sum_{K \in \mathcal{T}} m(K) (\varphi_K^{n+1} - \varphi_K^n)^2 \Theta_{i,K}^2 \\ & \leq L_{\varphi_i} \sum_{n=0}^{M_1} \delta t \sum_{K|L \in \mathcal{E}_i} \left[((\varphi_K^{n+1} - \varphi_K^n) \Theta_{i,K}^2 - (\varphi_L^{n+1} - \varphi_L^n) \Theta_{i,L}^2) \right] \\ & \leq \left(\begin{aligned} & L_{\varphi_i} \sum_{n=0}^{M_1} \delta t \sum_{K|L \in \mathcal{E}_i} \left[\frac{\tau_{K|L} (\varphi_L^{n+1} - \varphi_K^{n+1}) \left(\frac{\Theta_{i,K}^2 + \Theta_{i,L}^2}{2} \right)}{((\varphi_L^n - \varphi_K^n) - (\varphi_L^{n+1} - \varphi_K^{n+1}))} \right] \\ & + \frac{L_{\varphi_i}}{2} \sum_{n=0}^{M_1} \delta t \sum_{K|L \in \mathcal{E}_i} \left[\frac{\tau_{K|L} (\varphi_L^{n+1} - \varphi_K^{n+1}) (\Theta_{i,L}^2 - \Theta_{i,K}^2)}{((\varphi_K^{n+1} - \varphi_K^n) + (\varphi_L^{n+1} - \varphi_L^n))} \right] \end{aligned} \right) \\ & \sum_{n=0}^{M_1} \sum_{K \in \mathcal{T}} m(K) (\varphi_K^{n+1} - \varphi_K^n)^2 \Theta_{i,K}^2 \leq A_1 + A_2, \end{aligned} \quad (23)$$

with

$$\begin{cases} A_1 = L_{\varphi_i} \sum_{n=0}^{M_1} \delta t \sum_{K|L \in \mathcal{E}_i} \left[\frac{\tau_{K|L} (\varphi_L^{n+1} - \varphi_K^{n+1}) \left(\frac{\Theta_{i,K}^2 + \Theta_{i,L}^2}{2} \right)}{((\varphi_L^n - \varphi_K^n) - (\varphi_L^{n+1} - \varphi_K^{n+1}))} \right] \\ A_2 = \frac{L_{\varphi_i}}{2} \sum_{n=0}^{M_1} \delta t \sum_{K|L \in \mathcal{E}_i} \left[\frac{\tau_{K|L} (\varphi_L^{n+1} - \varphi_K^{n+1}) (\Theta_{i,L}^2 - \Theta_{i,K}^2)}{((\varphi_K^{n+1} - \varphi_K^n) + (\varphi_L^{n+1} - \varphi_L^n))} \right]. \end{cases}$$

Using $xy \leq \frac{1}{2}(x^2 + y^2)$ in A_1 leads to:

$$A_1 \leq \frac{L_{\varphi_i} \delta t}{2} \sum_{n=0}^{M_1} \sum_{K|L \in \mathcal{E}_i} \left[((\varphi_K^n - \varphi_L^n)^2 - (\varphi_K^{n+1} - \varphi_L^{n+1})^2) \right] \quad (24)$$

$$\leq \frac{L_{\varphi_i} \delta t}{2} \sum_{K|L \in \mathcal{E}_{O_i}} \tau_{K|L} (\varphi_K^0 - \varphi_L^0)^2. \quad (25)$$

Since $\varphi_i(u_0)$ is a continuous Lipschitz function on O_i , there exists C_{O_i, u_0} , which does not depend on the mesh such that:

$$|\varphi_K^0 - \varphi_L^0| \leq C_{O_i, u_0} d(x_K, x_L). \quad (26)$$

Thus, using (26) in inequality (25) leads to:

$$A_1 \leq \frac{C_{O_i, u_0} dm(O_i) L_{\varphi_i} \delta t}{2}. \quad (27)$$

We will now prove a similar estimate for A_2 :

$$A_2 = \frac{\delta t L_{\varphi_i}}{2} \sum_{n=0}^{M_1} \sum_{K|L \in \mathcal{E}_i} \left[\tau_{K|L} (\varphi_L^{n+1} - \varphi_K^{n+1}) (\Theta_{i,K} + \Theta_{i,L}) (\Theta_{i,L} - \Theta_{i,K}) \right].$$

For all $\alpha > 0$, we deduce from Young inequality that:

$$\begin{aligned} A_2 &\leq \alpha \delta t L_{\varphi_i} \sum_{n=0}^{M_1} \sum_{K|L \in \mathcal{E}_i} \left[\frac{\tau_{K|L} (\Theta_{i,K} + \Theta_{i,L})^2}{((\varphi_K^{n+1} + \varphi_L^{n+1}) - (\varphi_K^n + \varphi_L^n))^2} \right] \\ &\quad + \frac{L_{\varphi_i} \delta t}{4\alpha} \sum_{n=0}^{M_1} \sum_{K|L \in \mathcal{E}_i} \tau_{K|L} (\varphi_L^{n+1} - \varphi_K^{n+1})^2 (\Theta_{i,L} - \Theta_{i,K})^2 = A_{21} + A_{22}. \end{aligned}$$

Since for all $(x, y) \in \mathbb{R}^2$, $(x + y)^2 \leq 2x^2 + 2y^2$, we can write:

$$\begin{aligned} A_{21} &\leq 2\alpha \delta t L_{\varphi_i} \sum_{n=0}^{M_1} \sum_{K|L \in \mathcal{E}_i} \left[\frac{\tau_{K|L} (\Theta_{i,K} + \Theta_{i,L})^2}{((\varphi_K^{n+1} - \varphi_K^n)^2 + (\varphi_L^{n+1} - \varphi_L^n)^2)} \right] \\ &\leq 2\alpha \delta t L_{\varphi_i} \sum_{n=0}^{M_1} \sum_{K \in \mathcal{T}} \sum_{L \in \mathcal{N}_{K,i}} \tau_{K|L} (\Theta_{i,K} + \Theta_{i,L})^2 (\varphi_K^{n+1} - \varphi_K^n)^2 \\ &\leq 2\alpha L_{\varphi_i} \sum_{n=0}^{M_1} \sum_{K \in \mathcal{T}} \sum_{L \in \mathcal{N}_{K,i}} \left[\frac{\tau_{K|L} \delta t}{m(K)} m(K) (\varphi_K^{n+1} - \varphi_K^n)^2 \right]. \end{aligned}$$

The Remark 2.2 allows us to take a constant S_2 not depending on m such that, for all $K \in \mathcal{T}$, for all $L \in \mathcal{N}_{K,i}$:

$$\frac{\tau_{K|L} \delta t}{m(K)} \leq S_2,$$

thus, using once again $(x + y)^2 \leq 2x^2 + 2y^2$, and that the number of edges of K is not bigger than $reg(\mathcal{T})$:

$$\begin{aligned} A_{21} &\leq 16\alpha L_{\varphi_i} reg(\mathcal{T}) S_2 \sum_{n=0}^{M_1} \sum_{K \in \mathcal{T}} m(K) (\varphi_K^{n+1} - \varphi_K^n)^2 \Theta_{i,K}^2 \\ &\quad + 4\alpha L_{\varphi_i} \sum_{n=0}^{M_1} \sum_{K|L \in \mathcal{E}_i} \tau_{K|L} (\Theta_{i,L} - \Theta_{i,K})^2 (\varphi_K^{n+1} - \varphi_K^n)^2. \end{aligned} \quad (28)$$

One applies exactly the same method with a regular function Ψ_i instead of Θ_i , with $supp(\Psi_i) \subset O_i$, $\exists \varepsilon > 0$, $\Psi_i|_{supp(\Theta_i) + \varepsilon} = 1$, $\Psi_i \geq 0$, where:

$$supp(\Theta_i) + \varepsilon = \{x \in \Omega_i / d(x, supp(\Theta_i)) < \varepsilon\}$$

Then, for $size(\mathcal{T})$ small enough, we obtain that there exists $H > 0$ such that :

$$\sum_{n=0}^{M_1} \sum_{K \subset (supp(\Theta_i) + \varepsilon)} m(K) (\varphi_K^{n+1} - \varphi_K^n)^2 \leq \sum_{n=0}^{M_1} \sum_{K \in \mathcal{T}} m(K) (\varphi_K^{n+1} - \varphi_K^n)^2 \Psi_K^2 \leq H. \quad (29)$$

Denoting by C_{Θ_i} the Lipschitz constant of the function Θ_i , using (29) in (28) and Remark 2.2, one gets:

$$\begin{aligned} A_{21} &\leq 16\alpha L_{\varphi_i} reg(\mathcal{T}) S_2 \sum_{n=0}^{M_1} \sum_{K \in \mathcal{T}} m(K) (\varphi_K^{n+1} - \varphi_K^n)^2 \Theta_{i,K}^2 \\ &\quad + 4\alpha L_{\varphi_i} (reg(\mathcal{D}))^2 C_{\Theta_i}^2 H S_2 (size(\mathcal{T}))^2. \end{aligned} \quad (30)$$

A similar estimate on A_{22} is obvious.

$$A_{22} \leq \left(\frac{L_{\varphi_i} C_{\Theta_i}^2}{4\alpha} |\varphi_i(u_{\mathcal{D}})|_{1, \mathcal{D}, i} \right) (reg(\mathcal{D}))^2 (size(\mathcal{T}))^2.$$

Assumption 3 ensures that there exist constants C_1, C', C_{21}, C_{22} such that:

$$\begin{cases} A_1 \leq C_1 \delta t, \\ A_{21} \leq \alpha C' \sum_{n=0}^{M_1} \sum_{K \in \mathcal{T}} m(K) (\varphi_K^{n+1} - \varphi_K^n)^2 \Theta_{i,K}^2 + \alpha C_{21} \delta t, \\ A_{22} \leq \frac{C_{22}}{\alpha} \delta t. \end{cases} \quad (31)$$

We can now choose $\alpha = \frac{1}{2C'}$ and claim that inequalities (31) together with (23) lead to the existence of a constant C such that

$$\sum_{n=0}^{M_1} \sum_{\substack{K \in \mathcal{T} \\ K \subset \omega_i}} m(K) (\varphi_K^{n+1} - \varphi_K^n)^2 \leq \sum_{n=0}^{M_1} \sum_{K \in \mathcal{T}} m(K) (\varphi_K^{n+1} - \varphi_K^n)^2 \Theta_{i,K}^2 \leq C \delta t. \quad (32)$$

■

Lemma 2.14 (Discrete $L^\infty(0, T, H^1(\omega_i))$ estimate) *With the same assumptions and notations as in Lemma 2.13, there exists C such that:*

$$\sup_{m \in \mathbb{N}} \left(\sup_{t \in [0, T]} (|\varphi_i(u_{\mathcal{D}_m})(\cdot, t)|_{1, \mathcal{T}_m, \omega_i}) \right) \leq C,$$

where:

$$|\varphi_i(u_{\mathcal{D}_m})(\cdot, t)|_{1, \mathcal{T}_m, \omega_i}^2 = \sum_{K|L \subset \omega_i} \tau_{K|L} (\varphi_i(u_{\mathcal{D}_m})(x_K, t) - \varphi_i(u_{\mathcal{D}_m})(x_L, t))^2.$$

Proof

Keeping the notations of the previous proof, inequality (24) leads to:

$$A_1 \leq \frac{L_{\varphi_i} \delta t}{2} \left[\sum_{K|L \in \mathcal{E}_{\omega_i}} \tau_{K|L} (\varphi_K^0 - \varphi_L^0)^2 - \sum_{K|L \in \mathcal{E}_{\omega_i}} \tau_{K|L} (\varphi_K^{M_1+1} - \varphi_L^{M_1+1})^2 \right].$$

So we can deduce from (32) the following estimate:

$$\sum_{n=0}^{M_1} \sum_{\substack{K \in \mathcal{T} \\ K \subset \omega_i}} m(K) (\varphi_K^{n+1} - \varphi_K^n)^2 + \frac{L_{\varphi_i} \delta t}{2} \sum_{K|L \in \mathcal{E}_{\omega_i}} \tau_{K|L} (\varphi_K^{M_1+1} - \varphi_L^{M_1+1})^2 \leq C \delta t. \quad (33)$$

Dividing by δt leads to:

$$\frac{L_{\varphi_i}}{2} \sum_{K|L \in \mathcal{E}_{\omega_i}} \tau_{K|L} (\varphi_K^{M_1+1} - \varphi_L^{M_1+1})^2 \leq C.$$

This estimates holds for any $M_1 \in \{0, \dots, M\}$ and also for $M_1 = -1$ because of assumption 4. \blacksquare

Proposition 2.15 (time continuity of a weak solution) *One supposes that assumptions 1 and 4 is fulfilled. Then there exists a weak solution to the problem (2) in the sense of Definition 5 satisfying:*

$$\forall p \in [1, +\infty), \quad u \in C([0, T], L^p(\Omega)).$$

Proof

Let (\mathcal{D}_m) be a sequence of admissible discretizations of $\Omega \times (0, T)$ in the sense of Definition 1.5 fulfilling assumption 3. We will apply Ascoli theorem to the family of approximate solutions obtained through the scheme (7). We will first build another sequence of approximate solutions $(v_m)_m$, whose terms will be continuous

in respect to the time variable. We denote by v_m the function defined almost everywhere in Ω for all $t \in [0, T]$ by:

$$v_m(x, t) = \frac{t^{n+1} - t}{\delta t} (\varphi_K^n - \varphi_K^{n+1}) + \varphi_K^{n+1} \quad \text{if } (x, t) \in K \times [t^n, t^{n+1}].$$

Let $i \in \{1, \dots, N\}$. Let O_i be an open subset of Ω_i such that $\varphi_i(u_0)$ is a Lipschitz continuous function on O_i . Let U_i be an open subset of Ω_i such that $\overline{U_i} \subset O_i$. Let ω_i be an open subset of Ω_i such that $\overline{\omega_i} \subset U_i$. Let $\Theta_i \in \mathcal{D}(U_i)$ such that $\Theta_i|_{\omega_i} = 1$ and $0 \leq \Theta_i \leq 1$. We suppose that m is large enough to ensure that:

- $size(\mathcal{T}_m) < d(U_i, \partial O_i)$,
- $size(\mathcal{T}_m) < d(supp(\Theta_i), \partial U_i)$.

We denote by $\mathcal{E}_{U_i} = \{\sigma \in \mathcal{E}, \sigma = K|L / K \subset U_i, L \subset U_i\}$. Then, for all $t \in [0, T]$:

$$\begin{aligned} & \|v_m(\cdot, t) - \varphi_i(u_{\mathcal{D}_m}(\cdot, t))\|_{L^2(\omega_i)}^2 \\ & \leq \int_{O_i} \Theta_i(x) (v_m(x, t) - \varphi_i(u_{\mathcal{D}_m}(x, t)))^2 dx \\ & \leq \frac{(t^{n+1} - t)^2}{\delta t^2} \sum_{K \in \mathcal{T}} \Theta_{i,K} m(K) (\varphi_K^{n+1} - \varphi_K^n)^2 \\ & \leq \frac{(t^{n+1} - t)^2}{\delta t^2} L_{\varphi_i} \sum_{K \in \mathcal{T}} \Theta_{i,K} m(K) (\varphi_K^{n+1} - \varphi_K^n) (U_K^{n+1} - U_K^n) \\ & \leq \frac{(t^{n+1} - t)^2}{\delta t} L_{\varphi_i} \sum_{K \in \mathcal{T}} \sum_{L \in \mathcal{N}_{K,i}} \Theta_{i,K} (\varphi_K^{n+1} - \varphi_K^n) \tau_{K|L} (\varphi_K^{n+1} - \varphi_L^{n+1}) \\ & \leq \frac{(t^{n+1} - t)^2}{\delta t} L_{\varphi_i} \sum_{K|L \in \mathcal{E}_{U_i}} \left[\begin{array}{c} (\Theta_{i,K} (\varphi_K^{n+1} - \varphi_K^n) - \Theta_{i,L} (\varphi_L^{n+1} - \varphi_L^n)) \\ \tau_{K|L} (\varphi_K^{n+1} - \varphi_L^{n+1}) \end{array} \right], \end{aligned}$$

then we have:

$$\|v_m(\cdot, t) - \varphi_i(u_{\mathcal{D}_m}(\cdot, t))\|_{L^2(\omega_i)}^2 \leq A_1(t) + A_0(t),$$

with

$$\left\{ \begin{array}{l} A_0(t) = \frac{(t^{n+1} - t)^2}{2\delta t} L_{\varphi_i} \sum_{K|L \in \mathcal{E}_{U_i}} \left[\begin{array}{c} \tau_{K|L} (\varphi_K^{n+1} - \varphi_L^{n+1}) (\Theta_{i,K} + \Theta_{i,L}) \\ (\varphi_K^{n+1} - \varphi_K^n - \varphi_L^{n+1} + \varphi_L^n) \end{array} \right], \\ A_1(t) = \frac{(t^{n+1} - t)^2}{2\delta t} L_{\varphi_i} \sum_{K|L \in \mathcal{E}_{U_i}} \left[\begin{array}{c} \tau_{K|L} (\varphi_K^{n+1} - \varphi_L^{n+1}) (\Theta_{i,K} - \Theta_{i,L}) \\ (\varphi_K^{n+1} - \varphi_K^n + \varphi_L^{n+1} - \varphi_L^n) \end{array} \right]. \end{array} \right.$$

We apply Cauchy-Schwarz on A_1 , so we get:

$$|A_1(t)| \leq \delta t \varphi_i(1) L_{\varphi_i} \left[\begin{array}{c} \left(\sum_{K|L \in \mathcal{E}_{U_i}} \tau_{K|L} (\Theta_{i,K} - \Theta_{i,L})^2 \right)^{1/2} \\ \left(\sum_{K|L \in \mathcal{E}_{U_i}} \tau_{K|L} (\varphi_K^{n+1} - \varphi_L^{n+1})^2 \right)^{1/2} \end{array} \right].$$

Using Lemma 2.14, we can claim that there exists C_1 only depending on the data and on the regularity of the mesh such that, for all $t \in [0, T]$,

$$|A_1(t)| \leq C_1 \delta t.$$

Let us now have a look on A_0 .

$$\begin{aligned} |A_0(t)| &\leq L_{\varphi_i} \delta t \sum_{K|L \in \mathcal{E}_{U_i}} \tau_{K|L} |\varphi_K^{n+1} - \varphi_L^{n+1}| (|\varphi_K^{n+1} - \varphi_K^n| + |\varphi_L^{n+1} - \varphi_L^n|) \\ &\leq \delta t \sqrt{\sum_{K|L \in \mathcal{E}_{U_i}} \tau_{K|L} (\varphi_K^{n+1} - \varphi_L^{n+1})^2} \left[\begin{array}{c} \sqrt{\sum_{K|L \in \mathcal{E}_{U_i}} \tau_{K|L} (\varphi_K^{n+1} - \varphi_K^n)^2} \\ + \sqrt{\sum_{K|L \in \mathcal{E}_{U_i}} \tau_{K|L} (\varphi_L^{n+1} - \varphi_L^n)^2} \end{array} \right]. \end{aligned}$$

Using Remark 2.2 and lemmas 2.13 and 2.14, we can find C_0 , not depending on m such that:

$$|A_0(t)| \leq C_0 \delta t.$$

So, we have shown that there exists C , only depending on the data and the regularity of the mesh, such that:

$$\forall m \in \mathbb{N}, \forall t \in [0, T], \quad \|v_m(\cdot, t) - \varphi_i(u_{\mathcal{D}_m}(\cdot, t))\|_{L^2(\omega_i)}^2 \leq C \delta t. \quad (34)$$

We are now able to prove the relative compactness of the family $(v_m)_{m \in \mathbb{N}}$ in $C([0, T]; L^2(\omega_i))$.

Uniform equicontinuity: Let $\varepsilon > 0$, let $m \in \mathbb{N}$, let $t \in [0, T]$, let $\tau \in (0, T - t)$. We denote $N_1 = \lceil \frac{t}{\delta t} \rceil$, $N_2 = \lceil \frac{t+\tau}{\delta t} \rceil$.

$$\begin{aligned} \|v_m(\cdot, t + \tau) - v_m(\cdot, t)\|_{L^2(\omega_i)} &\leq \|v_m(\cdot, t + \tau) - \varphi_i(u_{\mathcal{D}_m}(\cdot, t + \tau))\|_{L^2(\omega_i)} \\ &\quad + \|v_m(\cdot, t) - \varphi_i(u_{\mathcal{D}_m}(\cdot, t))\|_{L^2(\omega_i)} \\ &\quad + \|\varphi_i(u_{\mathcal{D}_m}(\cdot, t + \tau)) - \varphi_i(u_{\mathcal{D}_m}(\cdot, t))\|_{L^2(\omega_i)}. \end{aligned}$$

Using estimate (34), we can choose $m_1 \in \mathbb{N}$ large enough so that:

$$\forall m \geq m_1, \forall t \in [0, T], \quad \|v_m(\cdot, t) - \varphi_i(u_{\mathcal{D}_m}(\cdot, t))\|_{L^2(\omega_i)} \leq \varepsilon/3. \quad (35)$$

There exists $m_2 \in \mathbb{N}$ such that, for all $m \geq m_2$, $size(\mathcal{D}_m) \leq d(\omega_i, \partial U_i)$. Then, for all $m \geq m_2$, one has:

$$\begin{aligned} \|\varphi_i(u_{\mathcal{D}_m}(\cdot, t + \tau)) - \varphi_i(u_{\mathcal{D}_m}(\cdot, t))\|_{L^2(\omega_i)}^2 &\leq \sum_{K \subset U_i} m(K) (\varphi_K^{N_2} - \varphi_K^{N_1})^2 \\ &\leq \sum_{K \subset U_i} m(K) \left(\sum_{n=N_1}^{N_2-1} (\varphi_K^{n+1} - \varphi_K^n) \right)^2 \end{aligned}$$

$$\leq (N_2 - N_1) \sum_{K \subset U_i} m(K) \sum_{n=N_1}^{N_2-1} (\varphi_K^{n+1} - \varphi_K^n)^2.$$

Lemma 2.13 ensures that there exists C , not depending on m , such that:

$$\sum_{K \subset U_i} m(K) \sum_{n=N_1}^{N_2-1} (\varphi_K^{n+1} - \varphi_K^n)^2 \leq C\delta t,$$

thus

$$\|\varphi_i(u_{\mathcal{D}_m}(\cdot, t + \tau)) - \varphi_i(u_{\mathcal{D}_m}(\cdot, t))\|_{L^2(\omega_i)}^2 \leq C(N_2 - N_1)\delta t.$$

The definition of N_1 and N_2 implies that $(N_2 - N_1)\delta t \leq \tau + dt$. So we can claim that: $\forall \varepsilon > 0, \exists m_3 \in \mathbb{N}, \exists \alpha > 0, \forall t \in [0, T - \alpha], \forall \tau \in (0, \alpha), \forall m \geq m_3$,

$$\|v_m(\cdot, t + \tau) - v_m(\cdot, t)\|_{L^2(\omega_i)} \leq \varepsilon.$$

Local relative compactness: We state the following lemma which is a straightforward generalization of Lemma 3.3 of [4] together with Lemma 2.14.

Lemma 2.16 *Let $(\mathcal{D}_m)_{m \in \mathbb{N}}$ be a sequence of admissible discretizations of $\Omega \times (0, T)$ fulfilling assumption 3. Let $(u_{\mathcal{D}_m})$ the sequence of approximate solutions given by the scheme (7). Let O_i be an open subset of Ω_i such that $\varphi_i(u_0)|_{O_i}$ is a Lipschitz continuous function. Let ω_i be an open subset of O_i , with $\bar{\omega}_i \subset O_i$. Then there exist C_1, C_2 and an integer m_0 such that, for all $m \geq m_0$, for all $t \in [0, T[$, for all $\eta \in \mathbb{R}^d$ such that $|\eta| \leq \frac{1}{2}d(\omega_i, \partial O_i)$:*

$$\|\varphi_i(u_{\mathcal{D}_m}(x + \eta, t)) - \varphi_i(u_{\mathcal{D}_m}(x, t))\|_{L^2(\omega_i)} \leq C_1|\eta|(|\eta| + C_2 \text{size}(\mathcal{T}))$$

where C_1 only depends on $O_i, \omega_i, u_0, \zeta, S_1, \lambda_i, \bar{\pi}_i$ for all $i \in \{1, \dots, N\}$, and C_2 only depends on Ω .

It is easy to check that, for all $t > 0$:

$$\begin{aligned} \|v_m(x + \eta, t) - v_m(x, t)\|_{L^2(\omega_i)} &\leq \|\varphi_i(u_{\mathcal{D}_m}(x + \eta, t)) - \varphi_i(u_{\mathcal{D}_m}(x, t))\|_{L^2(\omega_i)} \\ &\quad + \|\varphi_i(u_{\mathcal{D}_m}(x + \eta, t - \delta t)) - \varphi_i(u_{\mathcal{D}_m}(x, t - \delta t))\|_{L^2(\omega_i)} \end{aligned}$$

with the convention $\varphi_i(u_{\mathcal{D}_m}(x, t)) = \varphi_i(u_{\mathcal{D}_m}(x, 0))$ if $t < 0$. Then using lemma 2.16

$$\|v_m(x + \eta, t) - v_m(x, t)\|_{L^2(\omega_i)} \leq 2C_1|\eta|(|\eta| + C_2 \text{size}(\mathcal{T})).$$

Then we can apply Theorem 2.8 to state that, for all $t \geq 0$, $(v_m(\cdot, t))_m$ is relatively compact in $L^2(\omega_i)$.

Ascoli theorem implies that the sequence $(v_m)_m$ is relatively compact in $C([0, T], L^2(\omega_i))$, so, up to a subsequence, it converges to $v \in C([0, T], L^2(\omega_i))$. It is now obvious that $\varphi_i(u)|_{\omega_i \times [0, T]} = v$. Thus $\varphi_i(u) \in C([0, T]; L^1(\omega_i))$ for all $\omega_i \subset \bar{\omega}_i \subset O_i$, then $\varphi_i(u)$ belongs to $C([0, T]; L^1(O_i))$.

Assumption 4 implies that, for all $i \in \{1, \dots, N\}$, there exists a family $(O_{i,j})_j$ of open subsets of Ω_i such that $\bar{\Omega}_i = \bigcup_j \bar{O}_{i,j}$. So $\varphi_i(u) \in C([0, T]; L^1(\Omega_i))$, and:

$$\varphi_i(u) \in C([0, T]; L^1(\Omega)). \quad (36)$$

We deduce from (36) that, for almost every $x \in \Omega$, for all $t \in [0, T]$:

$$\lim_{\tau \rightarrow 0} \varphi(u(x, t + \tau), x) = \varphi(u(x, t), x).$$

The continuity of φ_i^{-1} for all $i \in \{1, \dots, N\}$ leads to: for all $t \in [0, T]$, for almost every $x \in \Omega$,

$$\lim_{\tau \rightarrow 0} |u(x, t + \tau) - u(x, t)| = 0.$$

Let $p \in [1, +\infty)$. The continuity of $s \mapsto s^p$ leads to: for all $t \in [0, T]$, for almost every $x \in \Omega$,

$$\lim_{\tau \rightarrow 0} |u(x, t + \tau) - u(x, t)|^p = 0.$$

Since $|u(x, t + \tau) - u(x, t)|^p \leq 1$ a.e. $x \in \Omega$, $\forall t \in [0, T]$, the dominated convergence theorem leads to:

$$u \in C([0, T]; L^p(\Omega)).$$

■

3 Uniqueness of the weak solution

In this section, we aim to prove the following L^1 -contraction principle, which directly implies the uniqueness of the weak solution to the problem (2) under Assumption 2. The method is inspired from [2, 6, 8].

Theorem 3.1 *Let u_0, v_0 belong to $L^\infty(\Omega)$, $0 \leq u_0, v_0 \leq 1$, and let u, v be weak solutions associated to the initial data u_0, v_0 . Then under assumptions 1 and 2, u and v belong to $C([0, T], L^p(\Omega))$ for all $p \in [1, +\infty[$. Furthermore, for all $t \in [0, T]$,*

$$\int_{\Omega} |u(x, t) - v(x, t)|^{\pm} dx dt \leq \int_{\Omega} |u_0(x) - v_0(x)|^{\pm} dx$$

where $|\cdot|^+$ (resp. $|\cdot|^-$) denotes the positive (resp. negative) part.

Proof

Let u be a weak solution to the problem (2) in the sense of Definition 1.2. It is easy to check that $\partial_t u \in L^2(0, T; (H^1(\Omega))')$, and that for any $\theta \in L^2(0, T; H^1(\Omega))$,

$$\int_0^T \left\langle \partial_t u(\cdot, t), \theta(\cdot, t) \right\rangle dt = - \sum_{i=1}^N \int_0^T \int_{\Omega_i} \nabla \varphi_i(u(x, t)) \cdot \nabla \theta(x, t) dx dt. \quad (37)$$

Let S_n^\pm be a Lipschitz continuous non-decreasing functions fulfilling:

$$S_n^+(a) = \begin{cases} 0 & \text{if } a < 0, \\ 1 & \text{if } a > \frac{1}{n}, \end{cases} \quad S_n^-(a) = -S_n^+(-a) \quad (38)$$

Let $\kappa(x)$ such that $\Pi(\kappa(x), x) \in H^1(\Omega)$, then for almost every $x \in \Omega$, the functions $a \mapsto S_n^\pm(\Pi(a, x) - \Pi(\kappa(x), x))$ is non-decreasing, and so $\mu_{n,x}^\pm : a \mapsto \int_0^a S_n^\pm(\Pi(s, x) - \Pi(\kappa(x), x)) ds$ is convex. One defines for $u(x, t) = u_0(x)$ if $t < 0$, then for almost every $(x, t) \in \Omega \times (0, T)$, for almost every $\tau > 0$,

$$\mu_{n,x}^\pm(u(x, t)) - \mu_{n,x}^\pm(u(x, t - \tau)) \leq S_n^\pm(\Pi(u(x, t), x) - \Pi(\kappa(x), x))(u(x, t) - u(x, t - \tau))$$

One multiplies the previous inequality by $\psi(x, t) \geq 0$, with $\psi \in \mathcal{D}^+(\bar{\Omega} \times [0, T])$, one divides by τ , on integrates on $\Omega_i \times (0, T)$ and sums for $i \in \llbracket 1, N \rrbracket$, so one gets:

$$\begin{aligned} & \frac{1}{\tau} \sum_{i=1}^N \int_{T-\tau}^T \int_{\Omega_i} \mu_{n,x}^\pm(u(x, t)) \psi(x, t) dx dt \\ & - \frac{1}{\tau} \sum_{i=1}^N \int_0^\tau \int_{\Omega_i} \mu_{n,x}^\pm(u_0(x)) \psi(x, t) dx dt \\ & + \frac{1}{\tau} \sum_{i=1}^N \int_0^T \int_{\Omega_i} \mu_{n,x}^\pm(u(x, t - \tau)) (\psi(x, t - \tau) - \psi(x, t)) dx dt \\ & \leq \frac{1}{\tau} \sum_{i=1}^N \int_0^T \int_{\Omega_i} \left[S_n^\pm(\Pi_i(u(x, t)) - \Pi_i(\kappa(x))) \right. \\ & \quad \left. (u(x, t) - u(x, t - \tau)) \right] \psi(x, t) dx dt. \end{aligned} \quad (39)$$

One can let τ tend to 0. Since $(x, t) \mapsto S_n^\pm(\Pi(u(x, t), x) - \Pi(\kappa(x), x))$ belongs to $L^2(0, T; H^1(\Omega))$, one gets:

$$\begin{aligned} & \sum_{i=1}^N \int_{\Omega_i} \mu_{n,x}^\pm(u_0(x)) \psi(x, 0) dx + \sum_{i=1}^N \int_0^T \int_{\Omega_i} \mu_{n,x}^\pm(u(x, t)) \partial_t \psi(x, t) dx dt \\ & \geq - \int_0^T \left\langle \partial_t u(\cdot, t), S_n^\pm(\Pi(u(x, t), x) - \Pi(\kappa(x), x)) \psi(x, t) \right\rangle dt. \end{aligned} \quad (40)$$

Thus using (37) in (40) leads to:

$$\begin{aligned} & \sum_{i=1}^N \int_{\Omega_i} \mu_{n,x}^\pm(u_0(x)) \psi(x, 0) dx + \sum_{i=1}^N \int_0^T \int_{\Omega_i} \mu_{n,x}^\pm(u(x, t)) \partial_t \psi(x, t) dx dt \\ & - \sum_{i=1}^N \int_0^T \int_{\Omega_i} S_n^\pm(\Pi_i(u(x, t)) - \Pi_i(\kappa(x))) \nabla \varphi_i(u(x, t)) \cdot \nabla \psi(x, t) dx dt \\ & - \sum_{i=1}^N \int_0^T \int_{\Omega_i} \left[\begin{array}{c} S_n^{\pm'}(\Pi_i(u(x, t)) - \Pi_i(\kappa(x))) \\ \nabla \varphi_i(u(x, t)) \cdot \nabla (\Pi_i(u(x, t)) - \Pi_i(\kappa(x))) \end{array} \right] \psi(x, t) dx dt \geq 0. \end{aligned} \quad (41)$$

Let ξ belong to $\mathcal{D}^+(\overline{\Omega} \times [0, T] \times (0, T))$. Let v be a weak solution for the problem (2) for an initial data v_0 regular enough to insure $v \in C([0, T]; L^1(\Omega))$. For almost every $s \in (0, T)$, one has $\Pi(v(x, s), x) \in H^1(\Omega)$, and so we can substitute $v(x, s)$ to $\kappa(x)$ in (41), and integrate for $s \in (0, T)$.

$$\begin{aligned}
& \int_0^T \sum_{i=1}^N \int_{\Omega_i} \left(\int_0^{u_0(x)} S_n^\pm(\Pi_i(a) - \Pi_i(v(x, s))) da \right) \xi(x, 0, s) dx ds \\
& + \sum_{i=1}^N \int_0^T \int_0^T \int_{\Omega_i} \left(\int_0^{u(x, t)} S_n^\pm(\Pi_i(a) - \Pi_i(v(x, s))) da \right) \partial_t \xi(x, t, s) dx dt ds \\
& \quad - \sum_{i=1}^N \int_0^T \int_0^T \int_{\Omega_i} \left[\begin{array}{c} S_n^\pm(\Pi_i(u(x, t)) - \Pi_i(v(x, s))) \\ \nabla \varphi_i(u(x, t)) \cdot \nabla \xi(x, t, s) \end{array} \right] dx dt ds \\
& \quad - \sum_{i=1}^N \int_0^T \int_0^T \int_{\Omega_i} \left[\begin{array}{c} S_n^{\pm'}(\Pi_i(u(x, t)) - \Pi_i(v(x, s))) \\ \nabla(\Pi_i(u(x, t)) - \Pi_i(v(x, s))) \cdot \\ \nabla \varphi_i(u(x, t)) \xi(x, t, s) \end{array} \right] dx dt ds \geq 0.
\end{aligned} \tag{42}$$

Inverting the roles of $u(x, t)$ and $v(x, s)$, using $\xi(\cdot, \cdot, 0) = 0$, one gets:

$$\begin{aligned}
& \sum_{i=1}^N \int_0^T \int_0^T \int_{\Omega_i} \left(\int_0^{v(x, s)} S_n^\mp(\Pi_i(a) - \Pi_i(u(x, t))) da \right) \partial_s \xi(x, t, s) dx dt ds \\
& \quad - \sum_{i=1}^N \int_0^T \int_0^T \int_{\Omega_i} \left[\begin{array}{c} S_n^\mp(\Pi_i(v(x, s)) - \Pi_i(u(x, t))) \\ \nabla \varphi_i(v(x, s)) \cdot \nabla \xi(x, t, s) \end{array} \right] dx dt ds \\
& \quad - \sum_{i=1}^N \int_0^T \int_0^T \int_{\Omega_i} \left[\begin{array}{c} S_n^{\mp'}(\Pi_i(v(x, s)) - \Pi_i(u(x, t))) \\ \nabla(\Pi_i(v(x, s)) - \Pi_i(u(x, t))) \cdot \\ \nabla \varphi_i(v(x, s)) \xi(x, t, s) \end{array} \right] dx dt ds \geq 0.
\end{aligned} \tag{43}$$

Adding (42) and (43), and using (38), we get:

$$\begin{aligned}
& \int_0^T \sum_{i=1}^N \int_{\Omega_i} \left(\int_0^{u_0(x)} S_n^\pm(\Pi_i(a) - \Pi_i(v(x, s))) da \right) \xi(x, 0, s) dx ds \\
& + \sum_{i=1}^N \int_0^T \int_0^T \int_{\Omega_i} \left(\int_0^{u(x, t)} S_n^\pm(\Pi_i(a) - \Pi_i(v(x, s))) da \right) \partial_t \xi(x, t, s) dx dt ds \\
& + \sum_{i=1}^N \int_0^T \int_0^T \int_{\Omega_i} \left(\int_0^{v(x, s)} S_n^\mp(\Pi_i(a) - \Pi_i(u(x, t))) da \right) \partial_s \xi(x, t, s) dx dt ds \\
& \quad - \sum_{i=1}^N \int_0^T \int_0^T \int_{\Omega_i} \left[\begin{array}{c} S_n^\pm(\Pi_i(u(x, t)) - \Pi_i(v(x, s))) \\ \nabla(\varphi_i(u(x, t)) - \varphi_i(v(x, s))) \end{array} \right] \cdot \nabla \xi(x, t, s) dx dt ds \\
& \quad - \sum_{i=1}^N \int_0^T \int_0^T \int_{\Omega_i} \left[\begin{array}{c} S_n^{\pm'}(\Pi_i(u(x, t)) - \Pi_i(v(x, s))) \\ \nabla(\varphi_i(u(x, t)) - \varphi_i(v(x, s))) \cdot \\ \nabla(\Pi_i(v(x, s)) - \Pi_i(u(x, t))) \end{array} \right] \xi(x, t, s) dx dt ds \geq 0.
\end{aligned}$$

Let us rewrite it $A_1^n + A_2^n + A_3^n + A_4^n \geq 0$, with:

$$\begin{aligned}
A_1^n &= \int_0^T \sum_{i=1}^N \int_{\Omega_i} \left(\int_0^{u_0(x)} S_n^\pm(\Pi_i(a) - \Pi_i(v(x, s))) da \right) \xi(x, 0, s) dx ds, \\
A_2^n &= \sum_{i=1}^N \int_0^T \int_0^T \int_{\Omega_i} \left(\int_0^{u(x, t)} S_n^\pm(\Pi_i(a) - \Pi_i(v(x, s))) da \right) \partial_t \xi(x, t, s) dx dt ds \\
&\quad + \sum_{i=1}^N \int_0^T \int_0^T \int_{\Omega_i} \left(\int_0^{v(x, s)} S_n^\mp(\Pi_i(a) - \Pi_i(u(x, t))) da \right) \partial_s \xi(x, t, s) dx dt ds, \\
A_3^n &= - \sum_{i=1}^N \int_0^T \int_0^T \int_{\Omega_i} \left[\frac{S_n^\pm(\Pi_i(u(x, t)) - \Pi_i(v(x, s)))}{\nabla(\varphi_i(u(x, t)) - \varphi_i(v(x, s)))} \right] \cdot \nabla \xi(x, t, s) dx dt ds, \\
A_4^n &= - \sum_{i=1}^N \int_0^T \int_0^T \int_{\Omega_i} \left[\frac{S_n^{\pm'}(\Pi_i(u(x, t)) - \Pi_i(v(x, s)))}{\nabla(\varphi_i(u(x, t)) - \varphi_i(v(x, s))) \cdot \nabla(\Pi_i(v(x, s)) - \Pi_i(u(x, t)))} \right] \xi(x, t, s) dx dt ds.
\end{aligned}$$

Now, we let n tend to $+\infty$, then, using the dominated convergence theorem and the fact that Π_i is strictly increasing,

$$\lim_{n \rightarrow +\infty} A_1^n = \int_0^T \sum_{i=1}^N \int_{\Omega_i} |u_0(x) - v(x, s)|^\pm \xi(x, 0, s) dx ds. \quad (44)$$

The same way, remarking that

$$\text{Sign}^+(\Pi_i(u) - \Pi_i(v)) = \text{Sign}^+(u - v) = -\text{Sign}^-(v - u),$$

$$\lim_{n \rightarrow +\infty} A_2^n = \int_0^T \int_0^T \sum_{i=1}^N \int_{\Omega_i} |u(x, t) - v(x, s)|^\pm (\partial_t \xi(x, t, s) + \partial_s \xi(x, t, s)) dx ds, \quad (45)$$

$$\lim_{n \rightarrow +\infty} A_3^n = - \sum_{i=1}^N \int_0^T \int_0^T \int_{\Omega_i} |\nabla(\varphi_i(u(x, t)) - \varphi_i(v(x, s)))|^\pm \cdot \nabla \xi(x, t, s) dx dt ds. \quad (46)$$

$$\begin{aligned}
A_4^n &= -\sum_{i=1}^N \int_0^T \int_0^T \int_{\Omega_i} \left[\begin{array}{c} S_n^{\pm'}(\Pi_i(u(x,t)) - \Pi_i(v(x,s))) \\ \nabla(\varphi_i \circ \Pi_i^{-1}(\Pi_i(u(x,t))) - \varphi_i \circ \Pi_i^{-1}(\Pi_i(v(x,s)))) \cdot \\ \nabla(\Pi_i(v(x,s)) - \Pi_i(u(x,t)))\xi(x,t,s) \end{array} \right] dxdt ds \\
&= -\sum_{i=1}^N \int_0^T \int_0^T \int_{\Omega_i} \left[\begin{array}{c} S_n^{\pm'}(\Pi_i(u(x,t)) - \Pi_i(v(x,s))) \\ (\varphi_i \circ \Pi_i^{-1})'(\Pi_i(u(x,t))) \\ |\nabla(\Pi_i(v(x,s)) - \Pi_i(u(x,t)))|^2 \xi(x,t,s) \end{array} \right] dxdt ds \\
&\quad + \sum_{i=1}^N \int_0^T \int_0^T \int_{\Omega_i} \left[\begin{array}{c} S_n^{\pm'}(\Pi_i(u(x,t)) - \Pi_i(v(x,s))) \\ \nabla(\Pi_i(u(x,t)) - \Pi_i(v(x,s))) \cdot \\ \nabla \Pi_i(v(x,s))\xi(x,t,s) \\ (\varphi_i \circ \Pi_i^{-1})'(\Pi_i(v(x,s))) \\ -(\varphi_i \circ \Pi_i^{-1})'(\Pi_i(u(x,t))) \end{array} \right] dxdt ds \\
&\leq \sum_{i=1}^N \int_0^T \int_0^T \int_{\Omega_i} \left[\begin{array}{c} S_n^{\pm'}(\Pi_i(u(x,t)) - \Pi_i(v(x,s))) \\ ((\varphi_i \circ \Pi_i^{-1})'(\Pi_i(v(x,s))) - (\varphi_i \circ \Pi_i^{-1})'(\Pi_i(u(x,t)))) \\ \nabla(\Pi_i(u(x,t)) - \Pi_i(v(x,s))) \cdot \nabla \Pi_i(v(x,s))\xi(x,t,s) \end{array} \right] dxdt ds \\
&\leq \sum_{i=1}^N \int_0^T \int_0^T \int_{\Omega_i} \left[\begin{array}{c} S_n^{\pm'}(\Pi_i(u(x,t)) - \Pi_i(v(x,s))) \\ M_i |\Pi_i(u(x,t)) - \Pi_i(v(x,s))| |\nabla \Pi_i(v(x,s))| \\ |\nabla(\Pi_i(u(x,t)) - \Pi_i(v(x,s)))| \xi(x,t,s) \end{array} \right] dxdt ds,
\end{aligned}$$

where M_i denotes a Lipschitz constant for the function $(\varphi_i \circ \Pi_i^{-1})'$. Such a constant exists thanks to Assumption 2. Let us now define a partition of $\Omega_i \times (0, T) \times (0, T)$:

- $E1 = \{(x, t, s) \in \Omega \times (0, T) \times (0, T), \Pi_i(u)(x, t) = \Pi_i(v)(x, s)\}$,
- $E2 = \{(x, t, s) \in \Omega \times (0, T) \times (0, T), \Pi_i(u)(x, t) \neq \Pi_i(v)(x, s)\}$.

Then

$$\begin{aligned}
&\text{for almost every } (x, t, s) \in E1, \quad \nabla(\Pi_i(u) - \Pi_i(v)) = 0, \\
&\text{for all } (x, t, s) \in E2, \quad \lim_{n \rightarrow \infty} S_n^{\pm'}(\Pi_i(u) - \Pi_i(v)) = 0.
\end{aligned}$$

Thus for almost every $(x, t, s) \in \Omega \times (0, T) \times (0, T)$:

$$\lim_{n \rightarrow \infty} S_n^{\pm'}(\Pi_i(u) - \Pi_i(v)) \nabla(\Pi_i(u) - \Pi_i(v)) = 0.$$

Furthermore, $(x, t, s) \mapsto S_n^{\pm'}(\Pi_i(u) - \Pi_i(v)) M_i |\Pi_i(u) - \Pi_i(v)| |\nabla \Pi_i(v)| |\nabla(\Pi_i(u) - \Pi_i(v))|$ is integrable on $\Omega_i \times (0, T) \times (0, T)$, thus, using the dominated convergence theorem, we can claim that:

$$\liminf_{n \rightarrow \infty} A_4^n \leq 0. \tag{47}$$

So (44)-(45)-(46)-(47) implies:

$$\begin{aligned}
&\int_0^T \sum_{i=1}^N \int_{\Omega_i} |u_0(x) - v(x, s)|^{\pm} \xi(x, 0, s) dx ds \\
&+ \int_0^T \int_0^T \sum_{i=1}^N \int_{\Omega_i} |u(x, t) - v(x, s)|^{\pm} (\partial_t \xi(x, t, s) + \partial_s \xi(x, t, s)) dx ds \tag{48} \\
&- \sum_{i=1}^N \int_0^T \int_0^T \int_{\Omega_i} \nabla |\varphi_i(u(x, t)) - \varphi_i(v(x, s))|^{\pm} \cdot \nabla \xi(x, t, s) dx dt ds \geq 0.
\end{aligned}$$

Let $\psi \in \mathcal{D}^+(\overline{\Omega} \times [0, T])$, let ρ belong to $\mathcal{D}^+(\mathbb{R})$, with $\text{supp}(\rho) \subset [-1, 0]$ and $\int_{\mathbb{R}} \rho(s) ds = 1$. For $n \geq 1$, one sets $\rho_n(s) = n\rho(ns)$. One sets $\xi(x, t, s) = \psi(x, t)\rho_n(t - s)$, so that ξ belongs to $\mathcal{D}^+(\overline{\Omega} \times [0, T] \times (0, T))$. One has:

$$\partial_t \xi(x, t, s) + \partial_s \xi(x, t, s) = \partial_t \psi(x, t)\rho_n(t - s),$$

and then inequality (48) can be rewritten:

$$\begin{aligned} & \int_0^T \sum_{i=1}^N \int_{\Omega_i} |u_0(x) - v(x, s)|^\pm \psi(x, 0)\rho_n(-s) dx ds \\ & + \int_0^T \int_0^T \sum_{i=1}^N \int_{\Omega_i} |u(x, t) - v(x, s)|^\pm \partial_t \psi(x, t)\rho_n(t - s) dx ds \\ & - \sum_{i=1}^N \int_0^T \int_0^T \int_{\Omega_i} \nabla |\varphi_i(u(x, t)) - \varphi_i(v(x, s))|^\pm \cdot \nabla \psi(x, t)\rho_n(t - s) dx dt ds \geq 0. \end{aligned} \quad (49)$$

The weak solution v has been chosen in $C([0, T], L^1(\Omega))$, (such a solution exists for regular enough initial data v_0 , i.e. $v_0 \in W_{pw}^{1,\infty}(\Omega)$, as exposed in Proposition 2.15). We can apply the theorem of continuity in mean to let tend n to $+\infty$ in inequality (49), thus we get, for all $\psi \in \mathcal{D}^+(\overline{\Omega} \times [0, T])$,

$$\begin{aligned} & \int_0^T \int_{\Omega} |u(x, t) - v(x, t)|^\pm \partial_t \psi(x, t) dx dt + \int_{\Omega} |u_0(x) - v_0(x)|^\pm \psi(x, 0) dx \\ & - \sum_{i=1}^N \int_0^T \int_{\Omega_i} \nabla |\varphi_i(u(x, t)) - \varphi_i(v(x, t))|^\pm \nabla \psi(x, t) dx dt \geq 0. \end{aligned} \quad (50)$$

The inequality (50) still holds for any $\psi \in W^{1,1}(0, T; L^1(\Omega)) \cap L^2(0, T; H^1(\Omega))$ with $\psi(\cdot, T) = 0$, and so for $\psi(x, t) = (T - t)$. In this case, we get the following comparison principle:

$$\int_0^T \int_{\Omega} |u(x, t) - v(x, t)|^\pm dx dt \leq T \int_{\Omega} |u_0(x) - v_0(x)|^\pm dx. \quad (51)$$

This particularly insures the uniqueness of the solution, and its time-continuity, if $\varphi_i(u_0)$ belongs to $W^{1,\infty}(\Omega_i)$. Moreover, we can state the following L^1 -contraction principle in this case: $\forall t \in [0, T]$,

$$\int_{\Omega} |u(x, t) - v(x, t)|^\pm dx dt \leq \int_{\Omega} |u_0(x) - v_0(x)|^\pm dx. \quad (52)$$

If u_0 belongs to $L^\infty(\Omega)$, $0 \leq u_0 \leq 1$, there exists a sequence $(u_{0,n})_{n \geq 1}$ of approximated initial data fulfilling:

- $\forall n \geq 1, \varphi_i(u_{0,n}) \in W^{1,\infty}(\Omega_i)$,

- $\|u_{0,n} - u_0\|_{L^1(\Omega)} \rightarrow 0$ as $n \rightarrow +\infty$.

Let u_n be the unique solution associated to initial data $u_{0,n}$. Then for all $t \in [0, T]$, the sequence $(u_n(\cdot, t))_n$ is a Cauchy sequence thanks to (52), and so it converges to $u(\cdot, t)$ thanks to (51).

Let $v_0 \in L^\infty(\Omega)$. Let $(v_{0,n})_n$ such that, for all $n \in \mathbb{N}$, $\varphi_i(v_{0,n}) \in W^{1,\infty}(\Omega_i)$ and $v_{0,n} \rightarrow v_0$ in the $L^1(\Omega)$ -topology. Then, thanks to (52),

$$\int_{\Omega} |u_n(x, t) - v_n(x, t)|^{\pm} dx dt \leq \int_{\Omega} |u_{0,n}(x) - v_{0,n}(x)|^{\pm} dx \quad (53)$$

where v_n is the unique weak solution associated to the initial data $v_{0,n}$. We can now let tend n to $+\infty$. Thanks to the short discussion stated above, for all $t \in [0, T]$, $v_n(\cdot, t)$ tends to $v(\cdot, t)$ in the $L^1(\Omega)$ -topology. Then we deduce from (53) that for any $u_0, v_0 \in L^\infty(\Omega)$, $0 \leq u_0, v_0 \leq 1$, for any $t \in [0, T]$, one has the following L^1 -contraction principle:

$$\int_{\Omega} |u(x, t) - v(x, t)|^{\pm} dx dt \leq \int_{\Omega} |u_0(x) - v_0(x)|^{\pm} dx. \quad (54)$$

Let $t \in [0, T]$, $\tau \in]0, T - t[$. For all $n \geq 1$, one has, using (54)

$$\begin{aligned} \int_{\Omega} |u(x, t + \tau) - u(x, t)|^{\pm} dx &\leq \int_{\Omega} \left[\begin{array}{l} |u(x, t + \tau) - u_n(x, t + \tau)|^{\pm} \\ + |u_n(x, t + \tau) - u_n(x, t)|^{\pm} \\ + |u(x, t) - u_n(x, t)|^{\pm} \end{array} \right] dx \\ &\leq \int_{\Omega} \left[\begin{array}{l} 2|u_0(x) - u_{0,n}(x)|^{\pm} \\ + |u_n(x, t + \tau) - u_n(x, t)|^{\pm} \end{array} \right] dx \end{aligned}$$

Since $u_n \in C([0, T]; L^1(\Omega))$, one gets, for all $n \geq 1$,

$$\lim_{\tau \rightarrow 0} \int_{\Omega} |u(x, t + \tau) - u(x, t)|^{\pm} dx \leq 2 \int_{\Omega} |u_0(x) - u_{0,n}(x)|^{\pm} dx.$$

Letting n tend to $+\infty$ gives the time-continuity of u . ■

The proof of Theorem 3.1 completes the proof of Theorem 1.1.

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