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On Bernstein processes generated by hierarchies of linear parabolic systems in \mathbb{R}^d

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Abstract

In this article we investigate the properties of Bernstein processes generated by infinite hierarchies of forward-backward systems of decoupled linear deterministic parabolic partial differential equations defined in \mathbb{R}^d , where d is arbitrary. An important feature of those systems is that the elliptic part of the parabolic operators may be realized as an unbounded Schrödinger operator with compact resolvent in standard L^2 -space. The Bernstein processes we are interested in are in general non-Markovian, may be stationary or non-stationary and are generated by weighted averages of measures naturally associated with the pure point spectrum of the operator. We also introduce time-dependent trace-class operators which possess most of the attributes of density operators in Quantum Statistical Mechanics, and prove that the statistical averages of certain bounded self-adjoint observables usually evaluated by means of such operators coincide with the expectation values of suitable functions of the underlying processes. In the particular case where the given parabolic equations involve the Hamiltonian of an isotropic system of quantum harmonic oscillators, we show that one of the associated processes is identical in law with the periodic Ornstein-Uhlenbeck process.

1 Introduction and outline

Bernstein (or reciprocal) processes constitute a generalization of Markov processes and have played an increasingly important rôle in various areas of mathematics and mathematical physics over the years, particularly in view of the recent advances in the Monge-Kantorovitch formulation of Optimal Transport Theory and Stochastic Geometric Mechanics (see, e.g., [1], [6]-[9], [16], [20]-[22], [27], [32]-[34] and the many references therein for a history and other works on the subject, which trace things back to the pioneering works [5] and [28]). As such they may be intrinsically defined without any reference to partial differential equations, and may take values in any topological space countable at infinity as was shown in [16]. However, in this article we restrict ourselves to the consideration of Bernstein processes generated by certain systems of parabolic partial

differential equations, whose state space is the Euclidean space \mathbb{R}^d endowed with its Borel σ -algebra \mathcal{B}_d . We begin with the following:

Definition 1. Let $d \in \mathbb{N}^+$ and $T \in (0, +\infty)$ be arbitrary. We say the \mathbb{R}^d -valued process $Z_{\tau \in [0, T]}$ defined on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a Bernstein process if

$$\mathbb{E}(b(Z_r) | \mathcal{F}_s^+ \vee \mathcal{F}_t^-) = \mathbb{E}(b(Z_r) | Z_s, Z_t) \quad (1)$$

\mathbb{P} -almost everywhere for every bounded Borel measurable function $b : \mathbb{R}^d \mapsto \mathbb{C}$, and for all r, s, t satisfying $r \in (s, t) \subset [0, T]$. In (1), the σ -algebras are

$$\mathcal{F}_s^+ = \sigma \{ Z_\tau^{-1}(F) : \tau \leq s, F \in \mathcal{B}_d \} \quad (2)$$

and

$$\mathcal{F}_t^- = \sigma \{ Z_\tau^{-1}(F) : \tau \geq t, F \in \mathcal{B}_d \}, \quad (3)$$

where $\mathbb{E}(\cdot | \cdot)$ denotes the conditional expectation on $(\Omega, \mathcal{F}, \mathbb{P})$.

This definition obviously extends that of a Markov process in the sense of a complete independence of the dynamics of $Z_{\tau \in [0, T]}$ within the interval (s, t) once Z_s and Z_t are known, no matter what the behavior of the process is prior to instant s and after instant t . This last property also shows that there are two time directions coming into play from the outset, since \mathcal{F}_s^+ may be interpreted as the σ -algebra gathering all available information before time s and \mathcal{F}_t^- as that collecting all available information after time t . It is therefore no surprise that any system of parabolic partial differential equations susceptible of generating Bernstein processes should exhibit two time directions, one pointing toward the future and one toward the past. Accordingly, we introduce below hierarchies of partial differential equations which we shall define from adjoint parabolic Cauchy problems of the form

$$\begin{aligned} \partial_t u(\mathbf{x}, t) &= \frac{1}{2} \Delta_{\mathbf{x}} u(\mathbf{x}, t) - V(\mathbf{x}) u(\mathbf{x}, t), \quad (\mathbf{x}, t) \in \mathbb{R}^d \times (0, T], \\ u(\mathbf{x}, 0) &= \varphi_0(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d \end{aligned} \quad (4)$$

and

$$\begin{aligned} -\partial_t v(\mathbf{x}, t) &= \frac{1}{2} \Delta_{\mathbf{x}} v(\mathbf{x}, t) - V(\mathbf{x}) v(\mathbf{x}, t), \quad (\mathbf{x}, t) \in \mathbb{R}^d \times [0, T), \\ v(\mathbf{x}, T) &= \psi_T(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d \end{aligned} \quad (5)$$

where $\Delta_{\mathbf{x}}$ denotes Laplace's operator with respect to the spatial variable, and where φ_0 and ψ_T are real-valued functions or measures to be specified below. In the sequel we write $L^2(\mathbb{R}^d)$ and $L^\infty(\mathbb{R}^d)$ for the usual Lebesgue spaces of all square integrable and essentially bounded real- or complex-valued functions on \mathbb{R}^d , respectively, and $L_{\text{loc}}^\infty(\mathbb{R}^d)$ for the local version of $L^\infty(\mathbb{R}^d)$, without ever distinguishing notationally between the real and the complex case. It will

indeed be clear from the context which case we are referring to, or else further specifications will be made. Finally we shall denote by $(\cdot, \cdot)_2$ the inner product in $L^2(\mathbb{R}^d)$ which we assume to be linear in the first argument, and by $\|\cdot\|_2$ the corresponding norm.

Throughout this article we impose the following hypothesis regarding V , where $|\cdot|$ stands for the usual Euclidean norm:

(H) The real-valued function V is bounded from below and satisfies $V \in L_{\text{loc}}^\infty(\mathbb{R}^d)$ with $V(x) \rightarrow +\infty$ as $|x| \rightarrow +\infty$.

It is well known that Hypothesis (H) allows the self-adjoint realization of the elliptic operator on the right-hand side of (4)-(5), which is up to a sign the operator associated with the closure of the quadratic form

$$\mathbf{Q}(f) = \frac{1}{2} \sum_{j=1}^d \int_{\mathbb{R}^d} dx \left| \frac{\partial f(x)}{\partial x_j} \right|^2 + \int_{\mathbb{R}^d} dx V(x) |f(x)|^2 \quad (6)$$

first defined for every complex-valued, compactly supported and smooth function f on \mathbb{R}^d (see, e.g., Section 2 in Chapter VI of [17]). Moreover the self-adjoint realization of the operator associated with (6), henceforth denoted by

$$H = -\frac{1}{2} \Delta_x + V, \quad (7)$$

generates a symmetric semigroup $\exp[-tH]$ on $L^2(\mathbb{R}^d)$ whose integral kernel satisfies

$$\begin{cases} g(x, t, y) = g(y, t, x), \\ c_1 t^{-\frac{d}{2}} \exp\left[-c_1^* \frac{|x-y|^2}{t}\right] \leq g(x, t, y) \leq c_2 t^{-\frac{d}{2}} \exp\left[-c_2^* \frac{|x-y|^2}{t}\right] \end{cases} \quad (8)$$

for all $x, y \in \mathbb{R}^d$ and every $t \in (0, T]$, where $c_{1,2}$ and $c_{1,2}^*$ are positive constants (see, e.g., Theorem 1 in [3] and its complete proof in [4]). At the same time Hypothesis (H) also implies that the resolvent of the self-adjoint realization of (7) is compact in $L^2(\mathbb{R}^d)$. As a Schrödinger operator this means that its spectrum $(E_m)_{m \in \mathbb{N}^d}$ is entirely discrete with $E_m \rightarrow +\infty$ as $|m| \rightarrow +\infty$, and that there exists an orthonormal basis $(f_m)_{m \in \mathbb{N}^d}$ consisting entirely of its eigenfunctions which we shall assume to be real (see, e.g., Section XIII.14 in [25], which allows for more general conditions on V). In the sequel we shall refer to the function g in (8) as the (parabolic) Green function associated with (4)-(5), also called fundamental solution to (4) in references [3] and [4].

In the context of this article we also assume that

$$\mathcal{Z}(t) := \sum_{m \in \mathbb{N}^d} \exp[-tE_m] < +\infty \quad (9)$$

for every $t \in (0, T]$, so that the strong convergence of

$$g(x, t, y) = \sum_{m \in \mathbb{N}^d} \exp[-tE_m] f_m(x) f_m(y) \quad (10)$$

holds in $L^2(\mathbb{R}^d \times \mathbb{R}^d)$. Then the construction of Bernstein processes associated with (4)-(5) rests on the availability of Green's function (8) and on the existence of probability measures on $\mathcal{B}_d \times \mathcal{B}_d$ whose joint densities μ satisfy the normalization condition

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} dx dy \mu(x, y) = 1. \quad (11)$$

Given these facts we organize the remaining part of this article in the following way: in Section 2 we use the knowledge of g and μ to state a general proposition about the existence of a probability space which supports a Bernstein process $Z_{\tau \in [0, T]}$ characterized by its finite-dimensional distributions, the joint distribution of Z_0 and Z_T and the probability of finding Z_t at any time $t \in [0, T]$ in a given region of space. In Section 3 we proceed with the construction of specific families of probability measures by introducing the hierarchies of equations we alluded to above. That is, with each level m of the spectrum of (7) we associate a pair of adjoint Cauchy problems of the form

$$\begin{aligned} \partial_t u(x, t) &= \frac{1}{2} \Delta_x u(x, t) - V(x)u(x, t), \quad (x, t) \in \mathbb{R}^d \times (0, T], \\ u(x, 0) &= \varphi_{m,0}(x), \quad x \in \mathbb{R}^d \end{aligned} \quad (12)$$

and

$$\begin{aligned} -\partial_t v(x, t) &= \frac{1}{2} \Delta_x v(x, t) - V(x)v(x, t), \quad (x, t) \in \mathbb{R}^d \times [0, T], \\ v(x, T) &= \psi_{m,T}(x), \quad x \in \mathbb{R}^d. \end{aligned} \quad (13)$$

To wit, we are considering as many pairs of such systems as is necessary to take into account the whole pure point spectrum of (7), and then focus our attention on the sequence of probability measures μ_m given by the joint densities

$$\mu_m(x, y) = \varphi_{m,0}(x)g(x, T, y)\psi_{m,T}(y) \quad (14)$$

where

$$\begin{cases} \varphi_{m,0}(x) = \frac{\delta(x - \mathbf{a}_m)}{g^{\frac{1}{2}}(\mathbf{a}_m, T, \mathbf{b}_m)}, \\ \psi_{m,T}(x) = \frac{\delta(x - \mathbf{b}_m)}{g^{\frac{1}{2}}(\mathbf{a}_m, T, \mathbf{b}_m)}, \end{cases} \quad (15)$$

thus having

$$\mu_m(G) = \int_G dx dy \varphi_{m,0}(x)g(x, T, y)\psi_{m,T}(y) \quad (16)$$

for every $G \in \mathcal{B}_d \times \mathcal{B}_d$. In the preceding expressions the points $\mathbf{a}_m, \mathbf{b}_m \in \mathbb{R}^d$ are arbitrarily chosen for every $m \in \mathbb{N}^d$ and δ stands for the Dirac measure so that

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} dx dy \mu_m(x, y) = 1, \quad (17)$$

in agreement with (11). In this manner and by applying the general proposition of Section 2 we obtain a sequence of Markovian bridges $Z_{\tau \in [0, T]}^m$ whose properties we analyze thoroughly. With each level of the spectrum we then associate a weight p_m and consider probability measures of the form

$$\bar{\mu} = \sum_{m \in \mathbb{N}^d} p_m \mu_m, \quad p_m > 0, \quad \sum_{m \in \mathbb{N}^d} p_m = 1, \quad (18)$$

that is, statistical mixtures of the measures μ_m . Yet another application of the proposition of Section 2 then allows us to generate a non-stationary and non-Markovian process $\bar{Z}_{\tau \in [0, T]}$ associated with $\bar{\mu}$. We also introduce a linear, time-dependent trace-class operator which plays the same rôle as the so-called density operator in Quantum Statistical Mechanics (see, e.g., [30]), and prove that the statistical averages of certain bounded self-adjoint observables evaluated by means of that operator coincide with the expectations of suitable functions of $\bar{Z}_{\tau \in [0, T]}$. In Section 4, keeping the same notation as in Section 3 for the initial-final data in (12) and (13), we carry out a similar construction but this time with $\varphi_{m,0}$ and $\exp[-TH] \psi_{m,T}$ forming a complete biorthonormal system in $L^2(\mathbb{R}^d)$, thus satisfying in particular

$$(\varphi_{m,0}, \exp[-TH] \psi_{n,T})_2 = \delta_{m,n} \quad (19)$$

for all $m, n \in \mathbb{N}^d$ where $\exp[-TH]$ stands for the Schrödinger semigroup generated by (7) evaluated at the terminal time $t = T$. The simplest system of this kind is

$$\begin{cases} \varphi_{m,0}(x) = f_m(x), \\ \psi_{m,T}(x) = \exp[TE_m] f_m(x) \end{cases} \quad (20)$$

where E_m and f_m are the eigenvalues and the eigenfunctions introduced above, respectively, but generally speaking a pair of initial-final data satisfying (19) always exists provided $\exp[-TH] \psi_{m,T}$ is sufficiently close to f_m for every $m \in \mathbb{N}^d$ in some sense. This statement essentially comes from an adaptation of a result by Paley and Wiener according to Theorem XXXVII of Chapter VII in [23], but then the corresponding measures (16) are signed since we impose no requirement about the positivity of $\varphi_{m,0}$ and $\exp[-TH] \psi_{m,T}$. In particular, regarding (20) the eigenfunctions f_m are typically not positive on \mathbb{R}^d with the possible exception of f_0 , so that it becomes intrinsically impossible to construct a Bernstein process from each μ_m individually in contrast to the method of Section 3. Nevertheless, the averaging procedure defined by (18) still allows us to generate genuine probability measures on $\mathcal{B}_d \times \mathcal{B}_d$ and hence other non-Markovian processes, which turns out to be particularly simple to do in the case of (20) when

$$p_m = \mathcal{Z}^{-1}(T) \exp[-TE_m] \quad (21)$$

where $\mathcal{Z}(T)$ is given by (9). In Section 4 we also define a linear, time-dependent trace-class operator from a pair of suitably chosen Riesz bases and prove again

that the corresponding statistical averages of certain bounded self-adjoint observables coincide with the expectations of suitable functions of the processes, along with many other properties. We devote Section 5 to the application of the results of Sections 3 and 4 to the case where the operator on the right-hand side of (12)-(13) is that of an isotropic system of quantum harmonic oscillators, up to a sign. That is, we consider hierarchies of the form

$$\begin{aligned} \partial_t u(\mathbf{x}, t) &= \frac{1}{2} \Delta_{\mathbf{x}} u(\mathbf{x}, t) - \frac{\lambda^2}{2} |\mathbf{x}|^2 u(\mathbf{x}, t), \quad (\mathbf{x}, t) \in \mathbb{R}^d \times (0, T], \\ u(\mathbf{x}, 0) &= \varphi_{\mathbf{m}, 0, \lambda}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d \end{aligned} \quad (22)$$

and

$$\begin{aligned} -\partial_t v(\mathbf{x}, t) &= \frac{1}{2} \Delta_{\mathbf{x}} v(\mathbf{x}, t) - \frac{\lambda^2}{2} |\mathbf{x}|^2 v(\mathbf{x}, t), \quad (\mathbf{x}, t) \in \mathbb{R}^d \times [0, T], \\ v(\mathbf{x}, T) &= \psi_{\mathbf{m}, T, \lambda}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d \end{aligned} \quad (23)$$

with $\lambda > 0$ and suitable choices of $\varphi_{\mathbf{m}, 0, \lambda}$ and $\psi_{\mathbf{m}, T, \lambda}$, and prove that the processes constructed there are intimately tied up with various types of conditioned Ornstein-Uhlenbeck processes. In particular, we show that one of these is identical in law with the periodic Ornstein-Uhlenbeck process, which was also analyzed by means of completely different techniques by various authors in different contexts (see, e.g., [19], [24] and [27]). To this end we carry out explicit computations of the laws and of the covariances based on the fact that in this situation Green's function identifies with Mehler's d -dimensional kernel, namely,

$$\begin{aligned} &g_{\lambda}(\mathbf{x}, t, \mathbf{y}) \\ &= \left(\frac{2\pi \sinh(\lambda t)}{\lambda} \right)^{-\frac{d}{2}} \exp \left[-\frac{\lambda \left(\cosh(\lambda t) \left(|\mathbf{x}|^2 + |\mathbf{y}|^2 \right) - 2(\mathbf{x}, \mathbf{y})_{\mathbb{R}^d} \right)}{2 \sinh(\lambda t)} \right] \end{aligned} \quad (24)$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$ and every $t \in (0, T]$, where $(\cdot, \cdot)_{\mathbb{R}^d}$ stands for the usual inner product in \mathbb{R}^d . Finally, we point out that the periodic Ornstein-Uhlenbeck process we just alluded to has the same law as one of the processes used in [14] to discuss properties of certain quantum systems in equilibrium with a thermal bath, and that it also identifies with the process "indexed by the circle" and possessing the "two-sided Markov property on the circle" investigated in [18]. Our work indeed shows that many of the processes investigated in those references may be viewed as belonging to a very special class of non-Markovian and stationary Bernstein processes.

2 On the existence of Bernstein processes in \mathbb{R}^d

Aside from a probability measure μ on $\mathcal{B}_d \times \mathcal{B}_d$ that satisfies (11), the typical construction of a Bernstein process requires a transition function as is the case

for Markov processes. Since there are two time directions provided by (4)-(5) we shall see that the natural choice is

$$Q(x, t; F, r; y, s) := \int_F dz q(x, t; z, r; y, s) \quad (25)$$

for every $F \in \mathcal{B}_d$, where

$$q(x, t; z, r; y, s) := \frac{g(x, t - r, z)g(z, r - s, y)}{g(x, t - s, y)}. \quad (26)$$

Both functions are well defined and positive for all $x, y, z \in \mathbb{R}^d$ and all r, s, t satisfying $r \in (s, t) \subset [0, T]$ by virtue of (8), and moreover the normalization condition

$$Q(x, t; \mathbb{R}^d, r; y, s) = 1$$

holds as a consequence of the semigroup composition law for g . It is the knowledge of both μ and Q that makes it possible to associate a Bernstein process with (4)-(5) in the following way:

Proposition 1. *Let μ satisfy (11) and let Q be given by (25). Then there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P}_\mu)$ supporting an \mathbb{R}^d -valued Bernstein process $Z_{\tau \in [0, T]}$ such that the following properties are valid:*

(a) *The function Q is the two-sided transition function of $Z_{\tau \in [0, T]}$ in the sense that*

$$\mathbb{P}_\mu(Z_r \in F | Z_s, Z_t) = Q(Z_t, t; F, r; Z_s, s) \quad (27)$$

for each $F \in \mathcal{B}_d$ and all r, s, t satisfying $r \in (s, t) \subset [0, T]$. Moreover,

$$\mathbb{P}_\mu(Z_0 \in F_0, Z_T \in F_T) = \mu(F_0 \times F_T) \quad (28)$$

for all $F_0, F_T \in \mathcal{B}_d$, that is, μ is the joint probability distribution of Z_0 and Z_T . In particular we have

$$\mathbb{P}_\mu(Z_0 \in F) = \mu(F \times \mathbb{R}^d) \quad (29)$$

and

$$\mathbb{P}_\mu(Z_T \in F) = \mu(\mathbb{R}^d \times F) \quad (30)$$

for each $F \in \mathcal{B}_d$.

(b) *For every $n \in \mathbb{N}^+$ the finite-dimensional distributions of the process are given by*

$$\begin{aligned} & \mathbb{P}_\mu(Z_{t_1} \in F_1, \dots, Z_{t_n} \in F_n) \\ &= \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{d\mu(x, y)}{g(x, T, y)} \int_{F_1} dx_1 \dots \int_{F_n} dx_n \\ & \quad \times \prod_{k=1}^n g(x_k, t_k - t_{k-1}, x_{k-1}) \times g(y, T - t_n, x_n) \end{aligned} \quad (31)$$

for all $F_1, \dots, F_n \in \mathcal{B}_d$ and all $t_0 = 0 < t_1 < \dots < t_n < T$, where $\mathbf{x}_0 = \mathbf{x}$. In particular we have

$$\begin{aligned} & \mathbb{P}_\mu(Z_t \in F) \\ &= \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{d\mu(\mathbf{x}, y)}{g(\mathbf{x}, T, y)} \int_F dz g(\mathbf{x}, t, z) g(z, T - t, y) \end{aligned} \quad (32)$$

for each $F \in \mathcal{B}_d$ and every $t \in (0, T)$.

(c) \mathbb{P}_μ is the only probability measure leading to the above properties.

There already exists a proof of an abstract version of a related statement in [16] as well as a more analytic version of it in [32], so that we limit ourselves here to showing how the basic quantities of interest can be expressed in terms of Green's function (8):

Proof. The existence of $(\Omega, \mathcal{F}, \mathbb{P}_\mu)$ and of $Z_{\tau \in [0, T]}$ is through Kolmogorov's extension theorem, with the probability \mathbb{P}_μ defined on cylindrical sets by

$$\begin{aligned} & \mathbb{P}_\mu(Z_0 \in F_0, Z_{t_1} \in F_1, \dots, Z_{t_n} \in F_n, Z_T \in F_T) \\ &= \int_{F_0 \times F_T} d\mu(\mathbf{x}, y) \int_{F_1} d\mathbf{x}_1 \dots \int_{F_n} d\mathbf{x}_n \prod_{k=1}^n q(y, T; x_k, t_k; x_{k-1}, t_{k-1}) \end{aligned}$$

for all $F_0, \dots, F_T \in \mathcal{B}_d$ and all $t_0 = 0 < t_1 < \dots < t_n < T$, where $\mathbf{x}_0 = \mathbf{x}$ and q is given by (26). Since

$$\begin{aligned} & \prod_{k=1}^n q(y, T; x_k, t_k; x_{k-1}, t_{k-1}) \\ &= \prod_{k=1}^n \frac{g(y, T - t_k, \mathbf{x}_k) g(\mathbf{x}_k, t_k - t_{k-1}, \mathbf{x}_{k-1})}{g(y, T - t_{k-1}, \mathbf{x}_{k-1})} \\ &= \frac{1}{g(\mathbf{x}, T, y)} \prod_{k=1}^n g(\mathbf{x}_k, t_k - t_{k-1}, \mathbf{x}_{k-1}) \times g(y, T - t_n, \mathbf{x}_n) \end{aligned}$$

after $n - 1$ cancellations in the products, we therefore obtain

$$\begin{aligned} & \mathbb{P}_\mu(Z_0 \in F_0, Z_{t_1} \in F_1, \dots, Z_{t_n} \in F_n, Z_T \in F_T) \\ &= \int_{F_0 \times F_T} \frac{d\mu(\mathbf{x}, y)}{g(\mathbf{x}, T, y)} \int_{F_1} d\mathbf{x}_1 \dots \int_{F_n} d\mathbf{x}_n \\ & \quad \times \prod_{k=1}^n g(\mathbf{x}_k, t_k - t_{k-1}, \mathbf{x}_{k-1}) \times g(y, T - t_n, \mathbf{x}_n), \end{aligned} \quad (33)$$

which is (31) when $F_0 = F_T = \mathbb{R}^d$. We now prove (28) by using the symmetry property in (8) along with the semigroup composition law for g to get

$$\int_{\mathbb{R}^d} d\mathbf{x}_1 \dots \int_{\mathbb{R}^d} d\mathbf{x}_n \prod_{k=1}^n g(\mathbf{x}_k, t_k - t_{k-1}, \mathbf{x}_{k-1}) \times g(y, T - t_n, \mathbf{x}_n) = g(\mathbf{x}, T, y) \quad (34)$$

by means of an easy induction argument on n . The substitution of (34) into (33) with the choice $F_1 = \dots = F_n = \mathbb{R}^d$ then leads to the desired relation

$$\mathbb{P}_\mu(Z_0 \in F_0, Z_T \in F_T) = \int_{F_0 \times F_T} d\mu(x, y),$$

of which (29) and (30) are obvious particular cases. Finally, (32) is (31) with $n = 1$. ■

REMARK. It is plain that the only relevant conditions in the proof of the proposition are the symmetry and the positivity of Green's function (8), aside from the data of a probability measure μ . Furthermore, as we shall see below Bernstein processes may be stationary and Markovian but in general they are neither one nor the other, as these properties are intimately tied up with the structure of μ . More specifically, according to Theorem 3.1 in [16] adapted to the present situation, a Bernstein process is Markovian if, and only if, there exist positive measures ν_0 and ν_T on \mathcal{B}_d such that μ be of the form

$$\mu(G) = \int_G d(\nu_0 \otimes \nu_T)(x, y) g(x, T, y) \quad (35)$$

for every $G \in \mathcal{B}_d \times \mathcal{B}_d$, with

$$\mu(\mathbb{R}^d \times \mathbb{R}^d) = 1.$$

We refer the reader for instance to [31], [32] and some of their references for an analysis of the Markovian case in various situations. Finally, in various different forms Bernstein processes have also recently appeared in applications of Optimal Transport Theory as testified for instance in [21] and in the monographs [11] and [29], and in the developments of Stochastic Geometric Mechanics as in [34].

In the next section we carry out the program described in Section 1 starting with (12), (13) and (14) when the initial-final data are given by (15).

3 On mixing Bernstein bridges in \mathbb{R}^d

Relation (15) implies that measures (16) are already probability measures so that we may apply Proposition 1 directly and in this manner associate a Bernstein process $Z_{\tau \in [0, T]}^m$ with each level of the spectrum. This leads to the following result where u_m and v_m denote the solutions to (12) and (13), respectively, that is,

$$u_m(x, t) = \int_{\mathbb{R}^d} dy g(x, t, y) \varphi_{m,0}(y) > 0 \quad (36)$$

and

$$v_m(x, t) = \int_{\mathbb{R}^d} dy g(x, T - t, y) \psi_{m,T}(y) > 0. \quad (37)$$

Theorem 1. Assume that Hypothesis (H) holds. Then for every $\mathbf{m} \in \mathbb{N}^d$ there exists a non-stationary Bernstein process $Z_{\tau \in [0, T]}^{\mathbf{m}}$ in \mathbb{R}^d such that the following statements are valid:

(a) The process $Z_{\tau \in [0, T]}^{\mathbf{m}}$ is a forward Markov process whose finite-dimensional distributions are

$$\begin{aligned} & \mathbb{P}_{\mu_{\mathbf{m}}} (Z_{t_1}^{\mathbf{m}} \in F_1, \dots, Z_{t_n}^{\mathbf{m}} \in F_n) \\ &= \int_{\mathbb{R}^d} d\mathbf{x} \rho_{\mathbf{m}, 0}(\mathbf{x}) \int_{F_1} d\mathbf{x}_1 \dots \int_{F_n} d\mathbf{x}_n \prod_{k=1}^n w_{\mathbf{m}}^*(\mathbf{x}_{k-1}, t_{k-1}; \mathbf{x}_k, t_k) \end{aligned} \quad (38)$$

for every $n \in \mathbb{N}^+$, all $F_1, \dots, F_n \in \mathcal{B}_d$ and all $0 = t_0 < t_1 < \dots < t_n < T$, with $\mathbf{x}_0 = \mathbf{x}$. In the preceding expression the density of the forward Markov transition function is

$$w_{\mathbf{m}}^*(\mathbf{x}, s; \mathbf{y}, t) = g(\mathbf{x}, t - s, \mathbf{y}) \frac{v_{\mathbf{m}}(\mathbf{y}, t)}{v_{\mathbf{m}}(\mathbf{x}, s)} \quad (39)$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$ and all $s, t \in [0, T]$ with $t > s$, while the initial distribution of the process reads

$$\rho_{\mathbf{m}, 0}(\mathbf{x}) = \varphi_{\mathbf{m}, 0}(\mathbf{x}) v_{\mathbf{m}}(\mathbf{x}, 0). \quad (40)$$

(b) The process $Z_{\tau \in [0, T]}^{\mathbf{m}}$ may also be viewed as a backward Markov process since the finite-dimensional distributions (38) may also be written as

$$\begin{aligned} & \mathbb{P}_{\mu_{\mathbf{m}}} (Z_{t_1}^{\mathbf{m}} \in F_1, \dots, Z_{t_n}^{\mathbf{m}} \in F_n) \\ &= \int_{\mathbb{R}^d} d\mathbf{x} \rho_{\mathbf{m}, T}(\mathbf{x}) \int_{F_1} d\mathbf{x}_1 \dots \int_{F_n} d\mathbf{x}_n \prod_{k=1}^n w_{\mathbf{m}}(\mathbf{x}_{k+1}, t_{k+1}; \mathbf{x}_k, t_k) \end{aligned} \quad (41)$$

for every $n \in \mathbb{N}^+$, all $F_1, \dots, F_n \in \mathcal{B}_d$ and all $0 < t_1 < \dots < t_n < t_{n+1} = T$, with $\mathbf{x}_{n+1} = \mathbf{x}$. In the preceding expression the density of the backward Markov transition function is

$$w_{\mathbf{m}}(\mathbf{x}, t; \mathbf{y}, s) = g(\mathbf{x}, t - s, \mathbf{y}) \frac{u_{\mathbf{m}}(\mathbf{y}, s)}{u_{\mathbf{m}}(\mathbf{x}, t)} \quad (42)$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$ and all $s, t \in [0, T]$ with $t > s$, while the final distribution of the process reads

$$\rho_{\mathbf{m}, T}(\mathbf{x}) = \psi_{\mathbf{m}, T}(\mathbf{x}) u_{\mathbf{m}}(\mathbf{x}, T).$$

(c) We have

$$\mathbb{P}_{\mu_{\mathbf{m}}} (Z_0^{\mathbf{m}} = \mathbf{a}_{\mathbf{m}}) = \mathbb{P}_{\mu_{\mathbf{m}}} (Z_T^{\mathbf{m}} = \mathbf{b}_{\mathbf{m}}) = 1 \quad (43)$$

and

$$\mathbb{P}_{\mu_{\mathbf{m}}} (Z_t^{\mathbf{m}} \in F) = \int_F d\mathbf{x} u_{\mathbf{m}}(\mathbf{x}, t) v_{\mathbf{m}}(\mathbf{x}, t) \quad (44)$$

for each $t \in (0, T)$ and every $F \in \mathcal{B}_d$.

(d) Finally,

$$\mathbb{E}_{\mu_{\mathbf{m}}} (b(Z_t^{\mathbf{m}})) = \int_{\mathbb{R}^d} d\mathbf{x} b(\mathbf{x}) u_{\mathbf{m}}(\mathbf{x}, t) v_{\mathbf{m}}(\mathbf{x}, t) \quad (45)$$

for each bounded Borel measurable function $b : \mathbb{R}^d \mapsto \mathbb{C}$ and every $t \in (0, T)$.

Proof. From (39) and the semigroup composition law for g we get

$$w_m^*(\mathbf{x}, s; \mathbf{y}, t) = \int_{\mathbb{R}^d} d\mathbf{z} w_m^*(\mathbf{x}, s; \mathbf{z}, r) w_m^*(\mathbf{z}, r; \mathbf{y}, t)$$

for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^d$ and every $r \in (s, t) \subset [0, T]$, so that the Chapman-Kolmogorov relation

$$W_m^*(\mathbf{x}, s; F, t) = \int_{\mathbb{R}^d} d\mathbf{y} w_m^*(\mathbf{x}, s; \mathbf{y}, r) W_m^*(\mathbf{y}, r; F, t)$$

holds for every $F \in \mathcal{B}_d$, where

$$W_m^*(\mathbf{x}, s; F, t) := \int_F d\mathbf{y} w_m^*(\mathbf{x}, s; \mathbf{y}, t).$$

Therefore W_m^* is the transition function of a forward Markov process with density w_m^* . In order to prove (38) we start with (31) into which we substitute (14) to obtain

$$\begin{aligned} & \mathbb{P}_{\mu_m} (Z_{t_1}^m \in F_1, \dots, Z_{t_n}^m \in F_n) \\ &= \int_{\mathbb{R}^d} d\mathbf{x} \varphi_{m,0}(\mathbf{x}) \int_{F_1} d\mathbf{x}_1 \dots \int_{F_n} d\mathbf{x}_n \prod_{k=1}^n g(\mathbf{x}_k, t_k - t_{k-1}, \mathbf{x}_{k-1}) \times v_m(\mathbf{x}_n, t_n) \end{aligned} \quad (46)$$

where $\mathbf{x}_0 = \mathbf{x}$ and $t_0 = 0$. Furthermore, using (39) we may rewrite the product in the preceding expression as

$$\begin{aligned} & \prod_{k=1}^n g(\mathbf{x}_k, t_k - t_{k-1}, \mathbf{x}_{k-1}) \times v_m(\mathbf{x}_n, t_n) \\ &= \prod_{k=1}^n w_m^*(\mathbf{x}_{k-1}, t_{k-1}; \mathbf{x}_k, t_k) \times v_m(\mathbf{x}, 0) \end{aligned}$$

after $n-1$ cancellations, which eventually leads to Statement (a) by taking (40) into account. The proof of Statement (b) is similar and thereby omitted. Now, from (14) and (29) we have

$$\mathbb{P}_{\mu_m} (Z_0^m \in F) = \int_F d\mathbf{x} \varphi_{m,0}(\mathbf{x}) v_m(\mathbf{x}, 0) = \begin{cases} 0 & \text{if } \mathbf{a}_m \notin F \\ 1 & \text{if } \mathbf{a}_m \in F \end{cases} \quad (47)$$

by using the first relation in (15), and similarly from (30) we get

$$\mathbb{P}_{\mu_m} (Z_T^m \in F) = \int_F d\mathbf{x} u_m(\mathbf{x}, T) \psi_{m,T}(\mathbf{x}) = \begin{cases} 0 & \text{if } \mathbf{b}_m \notin F \\ 1 & \text{if } \mathbf{b}_m \in F \end{cases} \quad (48)$$

so that (43) holds. Moreover, (44) is an immediate consequence of (14), (32) and (36), (37), which proves Statement (c) and thereby Statement (d). Finally,

a glance at (39) shows that (38) lacks translation invariance in time so that $Z_{\tau \in [0, T]}^m$ is indeed non-stationary. ■

REMARKS. (1) The fact that $Z_{\tau \in [0, T]}^m$ is both a forward and a backward Markov process is a manifestation of its reversibility in the sense of Definition 2 in [32], which is also readily apparent in (44) since the probability density of finding the process in a given region of space at a given time is expressed as the product of the forward solution (36) times the backward solution (37). As a matter of fact we can also obtain (44) either from (38) or from (41) when $n = 1$, and we have

$$\mathbb{P}_{\mu_m}(Z_t^m \in \mathbb{R}^d) = \int_{\mathbb{R}^d} dx u_m(x, t) v_m(x, t) = 1 \quad (49)$$

for every $t \in [0, T]$ as it should be. Indeed, substituting (15) into (36)-(37) and the resulting expression into (44) we get

$$\mathbb{P}_{\mu_m}(Z_t^m \in F) = \frac{1}{g(\mathbf{a}_m, T, \mathbf{b}_m)} \int_F dx g(\mathbf{a}_m, t, x) g(x, T - t, \mathbf{b}_m),$$

which implies (49) thanks to the semigroup composition law for g . Finally, we stress the fact that the forward density (39) is defined from the backward solution (37), while the backward density (42) is defined from the forward solution (36), and not the other way around.

(2) We note that (46) may also be written as

$$\begin{aligned} & \mathbb{P}_{\mu_m}(Z_{t_1}^m \in F_1, \dots, Z_{t_n}^m \in F_n) \\ &= \int_{F_1} dx_1 \dots \int_{F_n} dx_n \prod_{k=2}^n g(x_k, t_k - t_{k-1}, x_{k-1}) \times u_m(x_1, t_1) v_m(x_n, t_n) \end{aligned} \quad (50)$$

by carrying out the integral over x and by taking (36) into account. Relation (50) will play an important rôle in Section 5 since the integrand determines the density of the law of the random vector $(Z_{t_1}^m, \dots, Z_{t_n}^m) \in \mathbb{R}^{nd}$.

(3) Relation (43) shows that the process $Z_{\tau \in [0, T]}^m$ is pinned down at \mathbf{a}_m when $t = 0$ and at \mathbf{b}_m when $t = T$. We have therefore obtained a sequence of Markovian bridges associated with the discrete spectrum of the operator on the right-hand side of (12)-(13), which we shall call Bernstein bridges in the sequel. In particular, each process $Z_{\tau \in [0, T]}^m$ reduces to a Markovian loop in \mathbb{R}^d when $\mathbf{a}_m = \mathbf{b}_m$ for every m .

It turns out that Theorem 1 is the stepping stone toward the construction of a non-Markovian process we alluded to at the beginning of this article, which we shall carry out through the averaging procedure briefly sketched in the introduction. Accordingly, by mixing the Bernstein bridges constructed above we obtain the following result:

Theorem 2. *Assume that Hypothesis (H) holds, and for every $m \in \mathbb{N}^d$ let $Z_{\tau \in [0, T]}^m$ be the process of Theorem 1. Let $\bar{Z}_{\tau \in [0, T]}$ be the Bernstein process in the*

sense of Proposition 1 where the probability measure is (18) with the initial-final conditions given by (15). Then the following statements are valid:

(a) The process $\bar{Z}_{\tau \in [0, T]}$ is non-stationary, non-Markovian and its finite-dimensional distributions are

$$\begin{aligned} & \mathbb{P}_{\bar{\mu}}(\bar{Z}_{t_1} \in F_1, \dots, \bar{Z}_{t_n} \in F_n) \\ &= \sum_{\mathbf{m} \in \mathbb{N}^d} p_{\mathbf{m}} \mathbb{P}_{\mu_{\mathbf{m}}}(Z_{t_1}^{\mathbf{m}} \in F_1, \dots, Z_{t_n}^{\mathbf{m}} \in F_n) \end{aligned} \quad (51)$$

for every $n \in \mathbb{N}^+$ and all $F_1, \dots, F_n \in \mathcal{B}_d$, where $\mathbb{P}_{\mu_{\mathbf{m}}}(Z_{t_1}^{\mathbf{m}} \in F_1, \dots, Z_{t_n}^{\mathbf{m}} \in F_n)$ is given either by (38) or (41).

(b) We have

$$\mathbb{P}_{\bar{\mu}}(\bar{Z}_t \in F) = \sum_{\mathbf{m} \in \mathbb{N}^d} p_{\mathbf{m}} \mathbb{P}_{\mu_{\mathbf{m}}}(Z_t^{\mathbf{m}} \in F) \quad (52)$$

for each $t \in [0, T]$ and every $F \in \mathcal{B}_d$, where $\mathbb{P}_{\mu_{\mathbf{m}}}(Z_t^{\mathbf{m}} \in F)$ is given by (44), (47) and (48).

(c) We have

$$\mathbb{E}_{\bar{\mu}}(b(\bar{Z}_t)) = \sum_{\mathbf{m} \in \mathbb{N}^d} p_{\mathbf{m}} \mathbb{E}_{\mu_{\mathbf{m}}}(b(Z_t^{\mathbf{m}})) \quad (53)$$

for each bounded Borel measurable function $b : \mathbb{R}^d \mapsto \mathbb{C}$ and every $t \in [0, T]$, where $\mathbb{E}_{\mu_{\mathbf{m}}}(b(Z_t^{\mathbf{m}}))$ is given by (45).

Proof. It is sufficient to substitute the joint density

$$\bar{\mu}(\mathbf{x}, \mathbf{y}) = g(\mathbf{x}, T, \mathbf{y}) \sum_{\mathbf{m} \in \mathbb{N}^d} p_{\mathbf{m}} \varphi_{\mathbf{m}, 0}(\mathbf{x}) \psi_{\mathbf{m}, T}(\mathbf{y})$$

with $\varphi_{\mathbf{m}, 0}$ and $\psi_{\mathbf{m}, T}$ given by (15) into (31) and (32) to obtain (51) and (52), respectively, from which (53) follows. Owing to the lack of translation invariance in time of (41), it is then clear that the process $\bar{Z}_{\tau \in [0, T]}$ is also non-stationary. Finally, we note that $\bar{\mu}$ is not of the form (35) so that $\bar{Z}_{\tau \in [0, T]}$ is indeed non-Markovian. ■

Having associated an arbitrary weight $p_{\mathbf{m}}$ with each level of the spectrum of (7), it is now natural to ask whether there exists a linear bounded operator $\mathcal{R}(t)$ acting in $L^2(\mathbb{R}^d)$ for every $t \in (0, T)$ possessing most of the attributes of a so-called density operator in Quantum Statistical Mechanics. If so, an interesting question is to know whether the averages of certain bounded self-adjoint observables computed by means of such a density operator are in one way or another related to some expectation values of the process $\bar{Z}_{\tau \in [0, T]}$. We shall see that the answer is affirmative if we define

$$\mathcal{R}(t) f := \sum_{\mathbf{m} \in \mathbb{N}^d} p_{\mathbf{m}} (f, u_{\mathbf{m}}(\cdot, t))_2 v_{\mathbf{m}}(\cdot, t) \quad (54)$$

for each complex-valued $f \in L^2(\mathbb{R}^d)$ and every $t \in (0, T)$, where $u_{\mathbf{m}}(\cdot, t)$ and $v_{\mathbf{m}}(\cdot, t)$ are given by

$$u_{\mathbf{m}}(\mathbf{x}, t) = \frac{g(\mathbf{x}, t, \mathbf{a}_{\mathbf{m}})}{g^{\frac{1}{2}}(\mathbf{a}_{\mathbf{m}}, T, \mathbf{b}_{\mathbf{m}})} \quad (55)$$

and

$$v_{\mathbf{m}}(\mathbf{x}, t) = \frac{g(\mathbf{x}, T-t, \mathbf{b}_{\mathbf{m}})}{g^{\frac{1}{2}}(\mathbf{a}_{\mathbf{m}}, T, \mathbf{b}_{\mathbf{m}})}, \quad (56)$$

respectively, after substitution of (15) into (36) and (37). We begin with the following result in whose proof we write c for all the irrelevant positive constants depending only on the universal constants $c_{1,2}$ and $c_{1,2}^*$ in (8):

Theorem 3. *Let us assume that the sequences of points $\mathbf{a}_{\mathbf{m}}, \mathbf{b}_{\mathbf{m}}$ in (15) satisfy*

$$\sup_{\mathbf{m} \in \mathbb{N}^d} |\mathbf{a}_{\mathbf{m}} - \mathbf{b}_{\mathbf{m}}| < +\infty. \quad (57)$$

Then the following statements hold:

(a) *Formula (54) defines a linear trace-class operator in $L^2(\mathbb{R}^d)$ for every $t \in (0, T)$ and we have*

$$\mathrm{Tr} \mathcal{R}(t) = \sum_{\mathbf{m} \in \mathbb{N}^d} p_{\mathbf{m}} = 1.$$

(b) *Let us consider the linear bounded self-adjoint multiplication operator on $L^2(\mathbb{R}^d)$ given by $Bf = bf$ for every complex-valued $f \in L^2(\mathbb{R}^d)$, where $b \in L^\infty(\mathbb{R}^d)$ is real-valued. If $\bar{Z}_{\tau \in [0, T]}$ denotes the Bernstein process of Theorem 2 then we have*

$$\mathrm{Tr}(\mathcal{R}(t)B) = \mathbb{E}_{\bar{\mu}}(b(\bar{Z}_t)) \quad (58)$$

for every $t \in (0, T)$, where the right-hand side of (58) is given by (53).

Proof. We first prove that $u_{\mathbf{m}}(\cdot, t), v_{\mathbf{m}}(\cdot, t) \in L^2(\mathbb{R}^d)$ and that there exists a constant $c_* > 0$ independent of \mathbf{m} and depending only on t, T and on the constants in (8) such that

$$\|u_{\mathbf{m}}(\cdot, t)\|_2 \leq c_* < +\infty, \quad (59)$$

$$\|v_{\mathbf{m}}(\cdot, t)\|_2 \leq c_* < +\infty. \quad (60)$$

Indeed, from the right-hand side inequality (8) we have

$$\int_{\mathbb{R}^d} dx g^2(\mathbf{x}, t, \mathbf{a}_{\mathbf{m}}) \leq ct^{-d} \int_{\mathbb{R}^d} dx \exp\left[-c \frac{|\mathbf{x}|^2}{t}\right] = ct^{-\frac{d}{2}} < +\infty$$

for every $t \in (0, T)$ independently of \mathbf{m} by translation invariance of the integral, and similarly

$$\int_{\mathbb{R}^d} dx g^2(\mathbf{x}, T-t, \mathbf{b}_{\mathbf{m}}) \leq c(T-t)^{-\frac{d}{2}} < +\infty.$$

On the other hand, from the left-hand side inequality (8) we obtain

$$\frac{1}{g(\mathbf{a}_m, T, \mathbf{b}_m)} \leq cT^{\frac{d}{2}} \exp \left[c \frac{|\mathbf{a}_m - \mathbf{b}_m|^2}{T} \right]$$

so that we eventually get

$$\|u_m(\cdot, t)\|_2^2 \leq c \left(\frac{T}{t} \right)^{\frac{d}{2}} \exp \left[c \frac{|\mathbf{a}_m - \mathbf{b}_m|^2}{T} \right] \leq c \left(\frac{T}{t} \right)^{\frac{d}{2}} \exp \left[\frac{c}{T} \right] := c_*^2 < +\infty$$

by virtue of (55) and (57). In a completely similar way we have

$$\|v_m(\cdot, t)\|_2^2 \leq c \left(\frac{T}{T-t} \right)^{\frac{d}{2}} \exp \left[c \frac{|\mathbf{a}_m - \mathbf{b}_m|^2}{T-t} \right] \leq c_*^2$$

by changing the value of c_* if necessary, so that (59) and (60) hold. Therefore, series (54) converges strongly in $L^2(\mathbb{R}^d)$ and defines there a linear bounded operator since

$$\sum_{\mathbf{m} \in \mathbb{N}^d} p_m |(f, u_m(\cdot, t))_2| \|v_m(\cdot, t)\|_2 \leq c_*^2 \|f\|_2 < +\infty$$

for each $f \in L^2(\mathbb{R}^d)$ and every $t \in (0, T)$. In order to prove that $\mathcal{R}(t)$ is trace-class, it is then necessary and sufficient to show that

$$\sum_{\mathbf{n} \in \mathbb{N}^d} (\mathcal{R}(t) \mathbf{h}_n, \mathbf{h}_n)_2 < +\infty \quad (61)$$

for any orthonormal basis $(\mathbf{h}_n)_{\mathbf{n} \in \mathbb{N}^d}$ in $L^2(\mathbb{R}^d)$, in which case (61) will not depend on the choice of the basis (see, e.g., Theorem 8.1 in Chapter III of [13]). To this end let us introduce momentarily the auxiliary function

$$A(\mathbf{m}, \mathbf{n}, t) := p_m (\mathbf{h}_n, u_m(\cdot, t))_2 (v_m(\cdot, t), \mathbf{h}_n)_2$$

so that

$$\sum_{\mathbf{m} \in \mathbb{N}^d} A(\mathbf{m}, \mathbf{n}, t) = (\mathcal{R}(t) \mathbf{h}_n, \mathbf{h}_n)_2 \quad (62)$$

for any fixed \mathbf{n} . Moreover, for any fixed \mathbf{m} we have

$$\sum_{\mathbf{n} \in \mathbb{N}^d} A(\mathbf{m}, \mathbf{n}, t) = p_m (u_m(\cdot, t), v_m(\cdot, t))_2 = p_m \quad (63)$$

by virtue of (49). In addition, the preceding series converges absolutely as a consequence of the Cauchy-Schwarz inequality and estimates (59), (60) since

for any positive integers N_1, \dots, N_d we have successively

$$\begin{aligned}
& \sum_{\mathbf{n}: 0 \leq n_j \leq N_j} |A(\mathbf{m}, \mathbf{n}, t)| \\
& \leq p_{\mathbf{m}} \left(\sum_{\mathbf{n} \in \mathbb{N}^d} |(u_{\mathbf{m}}(\cdot, t), \mathbf{h}_{\mathbf{n}})_2|^2 \right)^{\frac{1}{2}} \left(\sum_{\mathbf{n} \in \mathbb{N}^d} |(v_{\mathbf{m}}(\cdot, t), \mathbf{h}_{\mathbf{n}})_2|^2 \right)^{\frac{1}{2}} \\
& = p_{\mathbf{m}} \|u_{\mathbf{m}}(\cdot, t)\|_2 \|v_{\mathbf{m}}(\cdot, t)\|_2 \leq c_*^2 p_{\mathbf{m}}
\end{aligned} \tag{64}$$

for any fixed \mathbf{m} so that

$$\sum_{\mathbf{n} \in \mathbb{N}^d} |A(\mathbf{m}, \mathbf{n}, t)| < +\infty$$

since the partial sums of this series remain bounded. Finally, (64) still implies

$$\sum_{\mathbf{m} \in \mathbb{N}^d} \sum_{\mathbf{n} \in \mathbb{N}^d} |A(\mathbf{m}, \mathbf{n}, t)| \leq c_*^2 \sum_{\mathbf{m} \in \mathbb{N}^d} p_{\mathbf{m}} = c_*^2 < +\infty.$$

Therefore the corresponding iterated series are equal (see, e.g., Theorem 8.43 in Chapter 8 of [2]), that is,

$$\sum_{\mathbf{n} \in \mathbb{N}^d} \sum_{\mathbf{m} \in \mathbb{N}^d} A(\mathbf{m}, \mathbf{n}, t) = \sum_{\mathbf{m} \in \mathbb{N}^d} \sum_{\mathbf{n} \in \mathbb{N}^d} A(\mathbf{m}, \mathbf{n}, t)$$

or, equivalently,

$$\mathrm{Tr} \mathcal{R}(t) := \sum_{\mathbf{n} \in \mathbb{N}^d} (\mathcal{R}(t) \mathbf{h}_{\mathbf{n}}, \mathbf{h}_{\mathbf{n}})_2 = \sum_{\mathbf{m} \in \mathbb{N}^d} p_{\mathbf{m}} = 1$$

according to (62) and (63), which is (a). As for the proof of (b), arguing as above for the computation of the trace we have

$$\begin{aligned}
& \mathrm{Tr}(\mathcal{R}(t) B) \\
& = \sum_{\mathbf{n} \in \mathbb{N}^d} \sum_{\mathbf{m} \in \mathbb{N}^d} p_{\mathbf{m}} (\mathbf{h}_{\mathbf{n}}, b u_{\mathbf{m}}(\cdot, t))_2 (v_{\mathbf{m}}(\cdot, t), \mathbf{h}_{\mathbf{n}})_2 \\
& = \sum_{\mathbf{m} \in \mathbb{N}^d} p_{\mathbf{m}} (b u_{\mathbf{m}}(\cdot, t), v_{\mathbf{m}}(\cdot, t))_2 = \mathbb{E}_{\bar{\mu}}(b(\bar{Z}_t))
\end{aligned}$$

where the last equality follows from (45) and (53) (note that $u_{\mathbf{m}}(\cdot, t)$ and $v_{\mathbf{m}}(\cdot, t)$ are also real-valued). ■

REMARKS. (1) The preceding considerations show that $\mathcal{R}(t)$ is not self-adjoint in general for it is easily seen that its adjoint is obtained by swapping the rôle of (55) and (56), that is,

$$\mathcal{R}^*(t) f = \sum_{\mathbf{m} \in \mathbb{N}^d} p_{\mathbf{m}} (f, v_{\mathbf{m}}(\cdot, t))_2 u_{\mathbf{m}}(\cdot, t).$$

Aside from that and in addition to the conclusion of Theorem 3, (54) possesses most of the properties of a density operator. For instance, every operator $\mathcal{P}^m(t) : L^2(\mathbb{R}^d) \mapsto L^2(\mathbb{R}^d)$ defined by

$$\mathcal{P}^m(t)f := (f, u_m(\cdot, t))_2 v_m(\cdot, t)$$

satisfies $(\mathcal{P}^m(t))^2 = \mathcal{P}^m(t)$ as a consequence of (49) and thus represents an oblique projection rather than an orthogonal projection, but (54) is still a statistical mixture of the $\mathcal{P}^m(t)$ obtained by sweeping over the whole spectrum of (7). Moreover, we remark that (54) involves both the forward and the backward solutions to (12) and (13), again in agreement with the fact that there are two time directions in the theory from the outset.

(2) It is tempting to believe that for *every* linear bounded selfadjoint operator there exists a real-valued $b \in L^\infty(\mathbb{R}^d)$ such that (58) holds, since such an operator is unitarily equivalent to a multiplication operator by the spectral theorem. We defer the general analysis of this question to a separate publication.

In the next section we carry out the program described in the introduction when the initial-final data satisfy suitable biorthogonality properties, and where we keep the same notation $\varphi_{m,0}$ and $\psi_{m,T}$ for them.

4 On generating Bernstein processes in \mathbb{R}^d by mixing signed measures

What we first need lies in the following adaptation of a result by Paley and Wiener (see the abstract form given in Section 86 of Chapter V in [26] of Theorem XXXVII of Chapter VII in [23]). We omit the proof as it is essentially available therein modulo trivial changes and up to the observation that the equality

$$(\exp[-tH] \varphi_{m,0}, \exp[-(T-t)H] \psi_{n,T})_2 = (\varphi_{m,0}, \exp[-TH] \psi_{n,T})_2$$

holds for every $t \in [0, T]$ as a consequence of the symmetry of the semigroup $\exp[-tH]$. In the following statement all functions are supposed to be real-valued with $(\mathbf{f}_m)_{m \in \mathbb{N}^d}$ the orthonormal basis of Section 1:

Proposition 2. *Let $(\psi_{m,T})_{m \in \mathbb{N}^d}$ be an arbitrary sequence in $L^2(\mathbb{R}^d)$ and let us assume that there exists $\theta \in [0, 1)$ such that the estimate*

$$\left\| \sum_{m \in I} \gamma_m (\mathbf{f}_m - \exp[-TH] \psi_{m,T}) \right\|_2 \leq \theta \left(\sum_{m \in I} |\gamma_m|^2 \right)^{\frac{1}{2}} \quad (65)$$

holds for every sequence $(\gamma_m)_{m \in \mathbb{N}^d}$ of real numbers, where the sums in (65) run over the same finite set $I \subset \mathbb{N}^d$ which may be chosen arbitrarily. Then there

exists a sequence $(\varphi_{\mathbf{m},0})_{\mathbf{m} \in \mathbb{N}^d} \subset L^2(\mathbb{R}^d)$ such that the following statements are valid:

(a) We have

$$(\exp[-tH] \varphi_{\mathbf{m},0}, \exp[-(T-t)H] \psi_{\mathbf{n},T})_2 = \delta_{\mathbf{m},\mathbf{n}} \quad (66)$$

for every $t \in [0, T]$ and the strongly convergent expansions

$$\begin{cases} f = \sum_{\mathbf{m} \in \mathbb{N}^d} (f, \varphi_{\mathbf{m},0})_2 \exp[-TH] \psi_{\mathbf{m},T}, \\ f = \sum_{\mathbf{m} \in \mathbb{N}^d} (f, \exp[-TH] \psi_{\mathbf{m},T})_2 \varphi_{\mathbf{m},0} \end{cases} \quad (67)$$

hold for every $f \in L^2(\mathbb{R}^d)$.

(b) The coefficients in (67) satisfy the estimates

$$(1 + \theta)^{-1} \|f\|_2 \leq \left(\sum_{\mathbf{m} \in \mathbb{N}^d} |(f, \varphi_{\mathbf{m},0})_2|^2 \right)^{\frac{1}{2}} \leq (1 - \theta)^{-1} \|f\|_2, \quad (68)$$

$$(1 - \theta) \|f\|_2 \leq \left(\sum_{\mathbf{m} \in \mathbb{N}^d} |(f, \exp[-TH] \psi_{\mathbf{m},T})_2|^2 \right)^{\frac{1}{2}} \leq (1 + \theta) \|f\|_2. \quad (69)$$

Thus the sequences $(\exp[-TH] \psi_{\mathbf{m},T})_{\mathbf{m} \in \mathbb{N}^d}$ and $(\varphi_{\mathbf{m},0})_{\mathbf{m} \in \mathbb{N}^d}$ constitute Riesz bases of $L^2(\mathbb{R}^d)$ in the terminology of [13] and it is plain that (20) corresponds to $\theta = 0$ in Proposition 2, in which case (67) reduces to the usual Fourier expansion of f along the orthonormal basis $(\mathbf{f}_{\mathbf{m}})_{\mathbf{m} \in \mathbb{N}^d}$ and (68), (69) to Parseval's equality. The reason why we have to consider $\exp[-TH] \psi_{\mathbf{m},T}$ rather than just $\psi_{\mathbf{m},T}$ lies in Relation (70) of the following result:

Lemma 1. Let $\varphi_{\mathbf{m},0}$ and $\exp[-TH] \psi_{\mathbf{m},T}$ be as in Proposition 2 and let us again define the density $\mu_{\mathbf{m}}$ by

$$\mu_{\mathbf{m}}(x, y) = \varphi_{\mathbf{m},0}(x) g(x, T, y) \psi_{\mathbf{m},T}(y).$$

Then the induced measures $\mu_{\mathbf{m}}$ on $\mathcal{B}_d \times \mathcal{B}_d$ are signed and we have

$$\mu_{\mathbf{m}}(\mathbb{R}^d \times \mathbb{R}^d) = 1 \quad (70)$$

for every $\mathbf{m} \in \mathbb{N}^d$.

Proof. The measures are signed since there is no requirement about the positivity of $\varphi_{\mathbf{m},0}$ and $\exp[-TH] \psi_{\mathbf{m},T}$. In particular, regarding (20) the eigenfunctions $\mathbf{f}_{\mathbf{m}}$ are typically not positive on \mathbb{R}^d with the possible exception of \mathbf{f}_0 . Moreover, expanding $\psi_{\mathbf{m},T}$ along the orthonormal basis $(\mathbf{f}_{\mathbf{k}})_{\mathbf{k} \in \mathbb{N}^d}$ we get

$$\sum_{\mathbf{k} \in \mathbb{N}^d} \exp[-TE_{\mathbf{k}}] (\varphi_{\mathbf{m},0}, \mathbf{f}_{\mathbf{k}})_2 (\psi_{\mathbf{n},T}, \mathbf{f}_{\mathbf{k}})_2 = \delta_{\mathbf{m},\mathbf{n}}$$

from (19), and therefore

$$\mu_{\mathbf{m}}(\mathbb{R}^d \times \mathbb{R}^d) = \sum_{\mathbf{k} \in \mathbb{N}^d} \exp[-TE_{\mathbf{k}}] (\varphi_{\mathbf{m},0}, \mathbf{f}_{\mathbf{k}})_2 (\psi_{\mathbf{m},T}, \mathbf{f}_{\mathbf{k}})_2 = 1$$

by substituting (10) into (16). ■

The fact that (18) may define a probability measure in the case under consideration is then ensured by the following result:

Lemma 2. *Let the initial-final conditions form a complete biorthonormal system in the sense of Proposition 2, and let $\bar{\mu}$ be the measure determined by*

$$\bar{\mu}(G) = \sum_{\mathbf{m} \in \mathbb{N}^d} p_{\mathbf{m}} \mu_{\mathbf{m}}(G) \quad (71)$$

for every $G \in \mathcal{B}_d \times \mathcal{B}_d$. If

$$(\mathbf{x}, \mathbf{y}) \mapsto \sum_{\mathbf{m} \in \mathbb{N}^d} p_{\mathbf{m}} \varphi_{\mathbf{m},0}(\mathbf{x}) \psi_{\mathbf{m},T}(\mathbf{y}) \quad (72)$$

is a positive measure on $\mathbb{R}^d \times \mathbb{R}^d$ then $\bar{\mu}$ is a probability measure on $\mathcal{B}_d \times \mathcal{B}_d$.

Proof. We have $\bar{\mu}(\mathbb{R}^d \times \mathbb{R}^d) = 1$ because of Lemma 1 and the fact that $\sum_{\mathbf{m} \in \mathbb{N}^d} p_{\mathbf{m}} = 1$. On the other hand, the joint density associated with (71) reads

$$\bar{\mu}(\mathbf{x}, \mathbf{y}) = g(\mathbf{x}, T, \mathbf{y}) \sum_{\mathbf{m} \in \mathbb{N}^d} p_{\mathbf{m}} \varphi_{\mathbf{m},0}(\mathbf{x}) \psi_{\mathbf{m},T}(\mathbf{y}) \quad (73)$$

where $g(\mathbf{x}, T, \mathbf{y}) > 0$ according to (8). ■

REMARK. It may seem abrupt to assume off-hand that (72) is positive as a measure. However, an important example illustrating this situation comes about when the initial-final data are given by (20) and the weights associated with the spectrum by (21). Indeed, in this case we have

$$\mu_{\mathbf{m}}(\mathbf{x}, \mathbf{y}) = \exp[TE_{\mathbf{m}}] g(\mathbf{x}, T, \mathbf{y}) \mathbf{f}_{\mathbf{m}}(\mathbf{x}) \mathbf{f}_{\mathbf{m}}(\mathbf{y})$$

and therefore

$$\begin{aligned} \bar{\mu}(\mathbf{x}, \mathbf{y}) &= \mathcal{Z}^{-1}(T) g(\mathbf{x}, T, \mathbf{y}) \sum_{\mathbf{m} \in \mathbb{N}^d} \mathbf{f}_{\mathbf{m}}(\mathbf{x}) \mathbf{f}_{\mathbf{m}}(\mathbf{y}) \\ &= \mathcal{Z}^{-1}(T) g(\mathbf{x}, T, \mathbf{y}) \delta(\mathbf{x} - \mathbf{y}) \end{aligned} \quad (74)$$

where the last equality is a consequence of the completeness of the orthogonal system $(\mathbf{f}_{\mathbf{m}})_{\mathbf{m} \in \mathbb{N}^d}$. It is (74) that will allow us to relate the above considerations to the periodic Ornstein-Uhlenbeck process in the next section.

Since the solutions u_m and v_m to (12) and (13) may now be written in terms of the Schrödinger semigroup defined in Section 1, namely,

$$u_m(\cdot, t) = \exp[-tH] \varphi_{m,0} \quad (75)$$

and

$$v_m(\cdot, t) = \exp[-(T-t)H] \psi_{m,T}, \quad (76)$$

respectively, then by mixing the measures μ_m as in Lemma 2 we obtain:

Theorem 4. *Assume that Hypothesis (H) holds, and let $\bar{Z}_{\tau \in [0, T]}$ be the Bernstein process in the sense of Proposition 1 with $\bar{\mu}$ given by (73), the particular case (74) being excluded. Then the following statements are valid:*

(a) *The process $\bar{Z}_{\tau \in [0, T]}$ is non-stationary, non-Markovian and for every $n \in \mathbb{N}^+$ with $n \geq 2$ its finite-dimensional distributions are*

$$\begin{aligned} & \mathbb{P}_{\bar{\mu}}(\bar{Z}_{t_1} \in F_1, \dots, \bar{Z}_{t_n} \in F_n) \\ &= \sum_{m \in \mathbb{N}^d} p_m \int_{F_1} dx_1 \dots \int_{F_n} dx_n \prod_{k=2}^n g(x_k, t_k - t_{k-1}, x_{k-1}) \\ & \quad \times (\exp[-t_1 H] \varphi_{m,0})(x_1) (\exp[-(T-t_n)H] \psi_{m,T})(x_n) \end{aligned}$$

for all $F_1, \dots, F_n \in \mathcal{B}_d$ and all $0 < t_1 < \dots < t_n < T$.

(b) *We have*

$$\begin{aligned} & \mathbb{P}_{\bar{\mu}}(\bar{Z}_t \in F) \\ &= \sum_{m \in \mathbb{N}^d} p_m \int_F dx (\exp[-tH] \varphi_{m,0})(x) (\exp[-(T-t)H] \psi_{m,T})(x) \end{aligned}$$

for each $F \in \mathcal{B}_d$ and every $t \in [0, T]$.

(c) *We have*

$$\begin{aligned} & \mathbb{E}_{\bar{\mu}}(b(\bar{Z}_t)) \quad (77) \\ &= \sum_{m \in \mathbb{N}^d} p_m \int_{\mathbb{R}^d} dx b(x) (\exp[-tH] \varphi_{m,0})(x) (\exp[-(T-t)H] \psi_{m,T})(x) \end{aligned}$$

for each bounded Borel measurable function $b: \mathbb{R}^d \mapsto \mathbb{C}$ and every $t \in [0, T]$.

Proof. The substitution of (73) into (31) gives

$$\begin{aligned} & \mathbb{P}_{\mu}(Z_{t_1} \in F_1, \dots, Z_{t_n} \in F_n) \\ &= \sum_{m \in \mathbb{N}^d} p_m \int_{F_1} dx_1 \dots \int_{F_n} dx_n \prod_{k=2}^n g(x_k, t_k - t_{k-1}, x_{k-1}) \\ & \quad \times \int_{\mathbb{R}^d \times \mathbb{R}^d} dx dy \varphi_{m,0}(x) \psi_{m,T}(y) g(x_1, t_1, x) g(y, T - t_n, x_n) \end{aligned}$$

for all $F_1, \dots, F_n \in \mathcal{B}_d$ and all $0 < t_1 < \dots < t_n < T$, where we have used the fact that $t_0 = 0$ and $\mathbf{x}_0 = \mathbf{x}$. This proves (a) since

$$(\exp[-t_1 H] \varphi_{\mathbf{m},0})(\mathbf{x}_1) = \int_{\mathbb{R}^d} d\mathbf{x} g(\mathbf{x}_1, t_1, \mathbf{x}) \varphi_{\mathbf{m},0}(\mathbf{x})$$

and

$$(\exp[-(T-t_n) H] \psi_{\mathbf{m},T})(\mathbf{x}_n) = \int_{\mathbb{R}^d} d\mathbf{x} g(\mathbf{x}_n, T-t_n, \mathbf{x}) \psi_{\mathbf{m},T}(\mathbf{x}).$$

The proof of (b) is similar by using (73) in (32). It is also plain that (c) follows from (b) and that $\bar{Z}_{\tau \in [0, T]}$ is non-stationary and non-Markovian for the same reasons as those given in the proof of Theorem 2 of the preceding section. ■

When $\bar{\mu}$ is given by (74) the associated process remains stationary and it is useful to discuss its properties separately by writing out the various quantities of interest in view of the applications discussed in the next section:

Corollary 1. *Assume that Hypothesis (H) holds, and let $\bar{Z}_{\tau \in [0, T]}$ be the Bernstein process in the sense of Proposition 1 with $\bar{\mu}$ given by (74). Then the following statements are valid:*

(a) *The process $\bar{Z}_{\tau \in [0, T]}$ is stationary, non-Markovian and for every $n \in \mathbb{N}^+$ with $n \geq 2$ its finite-dimensional distributions are*

$$\begin{aligned} & \mathbb{P}_{\bar{\mu}}(\bar{Z}_{t_1} \in F_1, \dots, \bar{Z}_{t_n} \in F_n) \\ &= \mathcal{Z}^{-1}(T) \int_{F_1} d\mathbf{x}_1 \dots \int_{F_n} d\mathbf{x}_n \\ & \quad \times \prod_{k=2}^n g(\mathbf{x}_k, t_k - t_{k-1}, \mathbf{x}_{k-1}) \times g(\mathbf{x}_1, T - (t_n - t_1), \mathbf{x}_n) \end{aligned} \quad (78)$$

for all $F_1, \dots, F_n \in \mathcal{B}_d$ and all $0 < t_1 < \dots < t_n < T$.

(b) *We have*

$$\mathbb{P}_{\bar{\mu}}(\bar{Z}_t \in F) = \mathcal{Z}^{-1}(T) \int_F d\mathbf{x} g(\mathbf{x}, T, \mathbf{x}) \quad (79)$$

for each $F \in \mathcal{B}_d$ and every $t \in [0, T]$.

(c) *We have*

$$\mathbb{E}_{\bar{\mu}}(b(\bar{Z}_t)) = \mathcal{Z}^{-1}(T) \int_{\mathbb{R}^d} d\mathbf{x} b(\mathbf{x}) g(\mathbf{x}, T, \mathbf{x}) \quad (80)$$

for each bounded Borel measurable function $b: \mathbb{R}^d \mapsto \mathbb{C}$ and every $t \in [0, T]$.

Proof. Relation (78) follows from the substitution of (74) into (31) and from the semigroup composition law for g , while (79) is a consequence of (74) into (32) and (80) a consequence of (79) since the density of the law of the

process is $\mathbf{x} \mapsto \mathcal{Z}^{-1}(T)g(\mathbf{x}, T, \mathbf{x})$. Now for any $\tau > 0$ sufficiently small such that $0 < t_1 + \tau < \dots < t_n + \tau < T$ we have

$$\mathbb{P}_{\bar{\mu}}(\bar{Z}_{t_1+\tau} \in F_1, \dots, \bar{Z}_{t_n+\tau} \in F_n) = \mathbb{P}_{\bar{\mu}}(\bar{Z}_{t_1} \in F_1, \dots, \bar{Z}_{t_n} \in F_n)$$

from (78) and therefore $\bar{Z}_{\tau \in [0, T]}$ is stationary, which entails the fact that both (79) and (80) are independent of t . Finally the process is non-Markovian since (74) is not of the form (35). ■

As in the preceding section we can now define a linear transformation in $L^2(\mathbb{R}^d)$ which will play the rôle of a density operator. Let us set

$$\mathcal{R}(t)f := \sum_{\mathbf{m} \in \mathbb{N}^d} p_{\mathbf{m}}(f, u_{\mathbf{m}}(\cdot, t))_2 v_{\mathbf{m}}(\cdot, t) \quad (81)$$

for each $f \in L^2(\mathbb{R}^d)$ and every $t \in [0, T]$, where $u_{\mathbf{m}}$ and $v_{\mathbf{m}}$ are given by (75) and (76), respectively. We begin with the following:

Lemma 3. *Assume that $(\psi_{\mathbf{m}, T})_{\mathbf{m} \in \mathbb{N}^d}$ is an arbitrary bounded sequence in $L^2(\mathbb{R}^d)$, and that $(\varphi_{\mathbf{m}, 0})_{\mathbf{m} \in \mathbb{N}^d}$ is the sequence associated with $(\psi_{\mathbf{m}, T})_{\mathbf{m} \in \mathbb{N}^d}$ in the sense of Proposition 2. Then (81) defines a linear bounded operator in $L^2(\mathbb{R}^d)$.*

Proof. Since the function V is bounded from below according to Hypothesis (H) we first have

$$\|u_{\mathbf{m}}(\cdot, t)\|_2 \leq c_T \|\varphi_{\mathbf{m}, 0}\|_2, \quad (82)$$

$$\|v_{\mathbf{m}}(\cdot, t)\|_2 \leq c_T \|\psi_{\mathbf{m}, T}\|_2 \quad (83)$$

for some finite constant $c_T > 0$ depending only on T (and on the lower bound in question). Moreover, by choosing $f = \varphi_{\mathbf{n}, 0}$ in the first inequality in (69) and by using the biorthogonality relation (19), we see that for each $\theta \in [0, 1)$ there exists a finite constant $c_{\theta} > 0$ such that $\|\varphi_{\mathbf{m}, 0}\|_2 \leq c_{\theta}$ for every $\mathbf{m} \in \mathbb{N}^d$. Combining this with the boundedness of $(\psi_{\mathbf{m}, T})_{\mathbf{m} \in \mathbb{N}^d}$ and with (82), (83) we obtain

$$\begin{aligned} \|u_{\mathbf{m}}(\cdot, t)\|_2 &\leq c_* < +\infty, \\ \|v_{\mathbf{m}}(\cdot, t)\|_2 &\leq c_* < +\infty \end{aligned}$$

where c_* depends only on T , the lower bound of V and θ . Therefore we have

$$\sum_{\mathbf{m} \in \mathbb{N}^d} p_{\mathbf{m}} |(f, u_{\mathbf{m}}(\cdot, t))_2| \|v_{\mathbf{m}}(\cdot, t)\|_2 \leq c_*^2 \|f\|_2 < +\infty$$

for each $f \in L^2(\mathbb{R}^d)$ and every $t \in [0, T]$, so that (81) converges strongly in $L^2(\mathbb{R}^d)$ with the desired properties. ■

REMARK. It is essential that the sequence $(\psi_{\mathbf{m}, T})_{\mathbf{m} \in \mathbb{N}^d}$ be bounded for the above argument to hold, but this does not follow from the first inequality in

(68) as the boundedness of $(\varphi_{\mathbf{m},0})_{\mathbf{m} \in \mathbb{N}^d}$ followed from the first inequality in (69). Indeed, the first inequality in (68) along with the biorthogonality relation (19) only shows that there exists a finite constant $c_\theta > 0$ such that $\|\exp[-TH] \psi_{\mathbf{m},T}\|_2 \leq c_\theta$ for every $\mathbf{m} \in \mathbb{N}^d$, but that does not entail the boundedness of $(\psi_{\mathbf{m},T})_{\mathbf{m} \in \mathbb{N}^d}$.

In fact we have much more than the conclusion of Lemma 3:

Theorem 5. *The hypothesis is the same as in Lemma 3. Then the following statements hold:*

(a) *Expression (81) defines a linear trace-class operator in $L^2(\mathbb{R}^d)$ with*

$$\mathrm{Tr} \mathcal{R}(t) = \sum_{\mathbf{m} \in \mathbb{N}^d} p_{\mathbf{m}} = 1, \quad (84)$$

$$\mathrm{Tr} \mathcal{R}^2(t) = \sum_{\mathbf{m} \in \mathbb{N}^d} p_{\mathbf{m}}^2 \leq 1 \quad (85)$$

for every $t \in [0, T]$. In particular we have $\mathrm{Tr} \mathcal{R}^2(t) = 1$ if, and only if, $p_{\mathbf{m}^*} = 1$ for exactly one \mathbf{m}^* , thus having $p_{\mathbf{m}} = 0$ for every $\mathbf{m} \neq \mathbf{m}^*$.

(b) *The eigenvalue equations*

$$\mathcal{R}(t)v_{\mathbf{m}}(\cdot, t) = p_{\mathbf{m}}v_{\mathbf{m}}(\cdot, t), \quad (86)$$

$$\mathcal{R}^*(t)u_{\mathbf{m}}(\cdot, t) = p_{\mathbf{m}}u_{\mathbf{m}}(\cdot, t) \quad (87)$$

hold for every $\mathbf{m} \in \mathbb{N}^d$ and every $t \in [0, T]$, where

$$\mathcal{R}^*(t)f = \sum_{\mathbf{m} \in \mathbb{N}^d} p_{\mathbf{m}} (f, v_{\mathbf{m}}(\cdot, t))_2 u_{\mathbf{m}}(\cdot, t) \quad (88)$$

is the adjoint of $\mathcal{R}(t)$.

(c) *Let us consider the linear bounded self-adjoint multiplication operator on $L^2(\mathbb{R}^d)$ given by $Bf = bf$ for every $f \in L^2(\mathbb{R}^d)$, where $b \in L^\infty(\mathbb{R}^d)$ is real-valued. If $\bar{Z}_\tau \in [0, T]$ denotes the Bernstein process of Theorem 4 then we have*

$$\mathrm{Tr}(\mathcal{R}(t)B) = \mathbb{E}_\mu(b(\bar{Z}_t)) \quad (89)$$

for every $t \in [0, T]$, where the right-hand side of (89) is given by (77).

Proof. The proof of (84) is quite similar to that of Statement (a) in Theorem 3 and is thereby omitted, while that of (85) follows from the biorthogonality of $u_{\mathbf{m}}(\cdot, t)$ and $v_{\mathbf{m}}(\cdot, t)$. Equations (86), (87) are an immediate consequence of (81), (88) and of the biorthogonality relation (66), while the proof of (c) is identical to that of the last statement of Theorem 3. ■

REMARK. It follows directly from (66) and (86) that

$$\sum_{\mathbf{m} \in \mathbb{N}^d} (\mathcal{R}(t)v_{\mathbf{m}}(\cdot, t), u_{\mathbf{m}}(\cdot, t))_2 = \sum_{\mathbf{m} \in \mathbb{N}^d} p_{\mathbf{m}} = 1.$$

Nevertheless, the fact that the preceding expression holds true is not specific to the problem at hand, but is a general property of trace-class operators whose trace may be computed by means of Lidskii's theorem using biorthogonal systems generated by Riesz bases (see, e.g., Theorems 5 and 6 in Section 2, Chapter I, of [12]). Finally, at the end of this article we will dwell a bit more on the meaning of (85).

If the initial-final conditions are given by (20) we note that (81) reduces to the self-adjoint, positive, trace-class time-independent operator

$$\mathcal{R}f = \sum_{\mathbf{m} \in \mathbb{N}^d} p_{\mathbf{m}} (f, \mathbf{f}_{\mathbf{m}})_2 \mathbf{f}_{\mathbf{m}}. \quad (90)$$

In this case we have the following:

Corollary 2. *Let $B : L^2(\mathbb{R}^d) \mapsto L^2(\mathbb{R}^d)$ be the same operator as in (c) of Theorem 5, and let $\bar{Z}_{\tau \in [0, T]}$ be the Bernstein process of Corollary 1. Then we have*

$$\text{Tr}(\mathcal{R}B) = \mathbb{E}_{\bar{\mu}}(b(\bar{Z}_t)) = \mathcal{Z}^{-1}(T) \sum_{\mathbf{m} \in \mathbb{N}^d} \exp[-TE_{\mathbf{m}}] \int_{\mathbb{R}^d} dx b(x) |\mathbf{f}_{\mathbf{m}}(x)|^2 \quad (91)$$

for every $t \in [0, T]$.

Proof. It is easily verified that

$$\text{Tr}(\mathcal{R}B) = \sum_{\mathbf{m} \in \mathbb{N}^d} p_{\mathbf{m}} \int_{\mathbb{R}^d} dx b(x) |\mathbf{f}_{\mathbf{m}}(x)|^2$$

whenever $p_{\mathbf{m}} > 0$ satisfies the normalization condition in (18), while if the probabilities associated with the spectrum are given by (21) we have

$$\begin{aligned} \mathbb{E}_{\bar{\mu}}(b(\bar{Z}_t)) &= \mathcal{Z}^{-1}(T) \int_{\mathbb{R}^d} dx b(x) g(x, T, x) \\ &= \mathcal{Z}^{-1}(T) \sum_{\mathbf{m} \in \mathbb{N}^d} \exp[-TE_{\mathbf{m}}] \int_{\mathbb{R}^d} dx b(x) |\mathbf{f}_{\mathbf{m}}(x)|^2 \end{aligned}$$

for every $t \in [0, T]$ according to (10) and (80). ■

In the final section of this article we apply some of the above results to the class of Bernstein processes generated by (22) and (23).

5 On the periodic Ornstein-Uhlenbeck process and related processes

We begin by recalling that the eigenvalue equation

$$\left(-\frac{1}{2} \Delta_x + \frac{\lambda^2}{2} |x|^2 \right) h_{\mathbf{m}, \lambda}(x) = E_{\mathbf{m}, \lambda} h_{\mathbf{m}, \lambda}(x)$$

holds for every $\mathbf{m} \in \mathbb{N}^d$, with

$$E_{\mathbf{m},\lambda} := \left(\sum_{j=1}^d m_j + \frac{d}{2} \right) \lambda \quad (92)$$

and

$$\mathbf{h}_{\mathbf{m},\lambda}(\mathbf{x}) := \prod_{j=1}^d h_{m_j,\lambda}(x_j).$$

In these expressions m_j is the j^{th} component of \mathbf{m} , x_j the j^{th} component of \mathbf{x} and $h_{m,\lambda}$ denotes the one-dimensional, suitably scaled Hermite function

$$h_{m,\lambda}(x) := \sqrt[4]{\lambda} h_m(\sqrt{\lambda}x)$$

where

$$h_m(x) = (-1)^m \left(\pi^{\frac{1}{2}} 2^m m! \right)^{-\frac{1}{2}} e^{\frac{x^2}{2}} \frac{d^m}{dx^m} e^{-x^2}.$$

Furthermore we have

$$\mathcal{Z}_\lambda(T) := \sum_{\mathbf{m} \in \mathbb{N}^d} \exp[-TE_{\mathbf{m},\lambda}] = (2(\cosh(\lambda T) - 1))^{-\frac{d}{2}} \quad (93)$$

by summing the series explicitly, so that Mehler's kernel (24) may be expanded as

$$g_\lambda(\mathbf{x}, t, \mathbf{y}) = \sum_{\mathbf{m} \in \mathbb{N}^d} \exp[-tE_{\mathbf{m},\lambda}] \mathbf{h}_{\mathbf{m},\lambda}(\mathbf{x}) \mathbf{h}_{\mathbf{m},\lambda}(\mathbf{y})$$

according to the considerations of Section 1, where the series is now convergent for every $t \in (0, T]$ uniformly in all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$. This last property is a consequence of the Cramér-Charlier inequality

$$|\mathbf{h}_{\mathbf{m},\lambda}(\mathbf{x}) \mathbf{h}_{\mathbf{m},\lambda}(\mathbf{y})| \leq \left(\frac{\lambda}{\pi^2} \right)^{\frac{d}{4}} k^{2d}$$

valid with $k \leq 1.086435$ uniformly in all \mathbf{x}, \mathbf{y} and \mathbf{m} (see, e.g., Section 10.18 in [10] for the one-dimensional case). We first illustrate some of the consequences of Theorem 1 by considering the initial-final data

$$\begin{cases} \varphi_{\mathbf{m},0}(\mathbf{x}) = \mathcal{N}_{\mathbf{m},\lambda} \delta(\mathbf{x}), \\ \psi_{\mathbf{m},T}(\mathbf{x}) = \mathcal{N}_{\mathbf{m},\lambda} \delta(\mathbf{x} - \mathbf{b}_{\mathbf{m}}) \end{cases} \quad (94)$$

where $(\mathbf{b}_{\mathbf{m}})_{\mathbf{m} \in \mathbb{N}^d} \subset \mathbb{R}^d$ is an arbitrary sequence of points associated with (92), and where

$$\mathcal{N}_{\mathbf{m},\lambda} := \left(\frac{2\pi \sinh(\lambda T)}{\lambda} \right)^{\frac{d}{4}} \exp \left[\frac{\lambda \coth(\lambda T) |\mathbf{b}_{\mathbf{m}}|^2}{4} \right].$$

A glance at (24) shows that (94) is a particular case of (15) when $\mathbf{a}_m = 0$ for every m . The corresponding solutions to (22) and (23) given by (55) and (56) then read

$$u_{m,\lambda}(\mathbf{x}, t) = \mathcal{N}_{m,\lambda}^* \sinh^{-\frac{d}{2}}(\lambda t) \exp \left[-\frac{\alpha_\lambda(t) |\mathbf{x}|^2}{2} \right] \quad (95)$$

and

$$\begin{aligned} v_{m,\lambda}(\mathbf{x}, t) &= \mathcal{N}_{m,\lambda}^* \exp \left[-\frac{\alpha_\lambda(T-t) |\mathbf{b}_m|^2}{2} \right] \sinh^{-\frac{d}{2}}(\lambda(T-t)) \\ &\quad \times \exp \left[-\frac{1}{2} \left(\alpha_\lambda(T-t) |\mathbf{x}|^2 - \frac{2\lambda(\mathbf{b}_m, \mathbf{x})_{\mathbb{R}^d}}{\sinh(\lambda(T-t))} \right) \right], \end{aligned} \quad (96)$$

respectively, where we have defined

$$\alpha_\lambda(t) := \lambda \coth(\lambda t) \quad (97)$$

for every $t \in (0, T]$ and

$$\mathcal{N}_{m,\lambda}^* := \left(\frac{\lambda \sinh(\lambda T)}{2\pi} \right)^{\frac{d}{4}} \exp \left[\frac{\alpha_\lambda(T) |\mathbf{b}_m|^2}{4} \right].$$

Then the following result holds:

Corollary 3. *The Bernstein process $Z_{\tau \in [0, T]}^{m, \lambda}$ associated with (22), (23) and (94) in the sense of Theorem 1 is a non-stationary Gaussian and Markovian process such that the following properties are valid:*

(a) *We have*

$$\mathbb{P}_{\mu_m} \left(Z_t^{m, \lambda} \in F \right) = (2\pi\sigma_\lambda(t))^{-\frac{d}{2}} \int_F d\mathbf{x} \exp \left[-\frac{|\mathbf{x} - \mathbf{b}_{m,\lambda}(t)|^2}{2\sigma_\lambda(t)} \right] \quad (98)$$

for each $t \in (0, T)$ and every $F \in \mathcal{B}_d$, where

$$\mathbf{b}_{m,\lambda}(t) = \frac{\sinh(\lambda t)}{\sinh(\lambda T)} \mathbf{b}_m \quad (99)$$

and

$$\sigma_\lambda(t) = \frac{\sinh(\lambda(T-t)) \sinh(\lambda t)}{\lambda \sinh(\lambda T)}. \quad (100)$$

Furthermore we have

$$\mathbb{P}_{\mu_m} \left(Z_0^{m, \lambda} = \mathbf{o} \right) = \mathbb{P}_{\mu_m} \left(Z_T^{m, \lambda} = \mathbf{b}_m \right) = 1 \quad (101)$$

for every $m \in \mathbb{N}^d$.

(b) We have

$$\mathbb{E}_{\mu_m} \left((Z_s^{m,\lambda,i} - b_{m,\lambda}^i(s))(Z_t^{m,\lambda,j} - b_{m,\lambda}^j(t)) \right) = \begin{cases} \frac{\sinh(\lambda(T-t)) \sinh(\lambda s)}{\lambda \sinh(\lambda T)} \delta_{i,j} & \text{for } t \geq s, \\ \frac{\sinh(\lambda(T-s)) \sinh(\lambda t)}{\lambda \sinh(\lambda T)} \delta_{i,j} & \text{for } t \leq s \end{cases} \quad (102)$$

for all $s, t \in [0, T]$ and all $i, j \in \{1, \dots, d\}$, where $b_{m,\lambda}^i$ denotes the i^{th} component of $\mathbf{b}_{m,\lambda}$.

(c) We have

$$\mathbb{E}_{\mu_m} \left(b(Z_t^{m,\lambda}) \right) = (2\pi\sigma_\lambda(t))^{-\frac{d}{2}} \int_{\mathbb{R}^d} dx b(x) \exp \left[-\frac{|x - \mathbf{b}_{m,\lambda}(t)|^2}{2\sigma_\lambda(t)} \right] \quad (103)$$

for each bounded Borel measurable function $b : \mathbb{R}^d \mapsto \mathbb{C}$ and every $t \in (0, T)$.

Proof. We begin by proving (98). Using (95) and (96) we first have

$$\begin{aligned} & u_{m,\lambda}(\mathbf{x}, t) v_{m,\lambda}(\mathbf{x}, t) \\ &= \left(\frac{\lambda \sinh(\lambda T)}{2\pi \sinh(\lambda(T-t)) \sinh(\lambda t)} \right)^{\frac{d}{2}} \exp \left[\frac{(\alpha_\lambda(T) - \alpha_\lambda(T-t)) |\mathbf{b}_m|^2}{2} \right] \\ & \quad \times \exp \left[-\frac{1}{2} \left((\alpha_\lambda(t) + \alpha_\lambda(T-t)) |\mathbf{x}|^2 - \frac{2\lambda (\mathbf{b}_m, \mathbf{x})_{\mathbb{R}^d}}{\sinh(\lambda(T-t))} \right) \right] \end{aligned} \quad (104)$$

after regrouping terms, and furthermore

$$\alpha_\lambda(T) - \alpha_\lambda(T-t) = -\frac{\lambda \sinh(\lambda t)}{\sinh(\lambda(T-t)) \sinh(\lambda T)} \quad (105)$$

$$\alpha_\lambda(t) + \alpha_\lambda(T-t) = \frac{\lambda \sinh(\lambda T)}{\sinh(\lambda(T-t)) \sinh(\lambda t)} \quad (106)$$

from (97). The substitution of (105) and (106) into (104) then leads to

$$\begin{aligned} & u_{m,\lambda}(\mathbf{x}, t) v_{m,\lambda}(\mathbf{x}, t) \\ &= \left(\frac{\lambda \sinh(\lambda T)}{2\pi \sinh(\lambda(T-t)) \sinh(\lambda t)} \right)^{\frac{d}{2}} \exp \left[-\frac{\lambda \sinh(\lambda t) |\mathbf{b}_m|^2}{2 \sinh(\lambda(T-t)) \sinh(\lambda T)} \right] \\ & \quad \times \exp \left[-\frac{\lambda}{2} \left(\frac{\sinh(\lambda T) |\mathbf{x}|^2 - 2 \sinh(\lambda t) (\mathbf{b}_m, \mathbf{x})_{\mathbb{R}^d}}{\sinh(\lambda(T-t)) \sinh(\lambda t)} \right) \right]. \end{aligned} \quad (107)$$

Now, for the numerator of the argument in the second exponential of the preceding expression we have

$$\begin{aligned} & \sinh(\lambda T) |\mathbf{x}|^2 - 2 \sinh(\lambda t) (\mathbf{b}_m, \mathbf{x})_{\mathbb{R}^d} \\ &= \sinh(\lambda T) |\mathbf{x} - \mathbf{b}_{m,\lambda}(t)|^2 - \frac{\sinh^2(\lambda t) |\mathbf{b}_m|^2}{\sinh(\lambda T)} \end{aligned} \quad (108)$$

by virtue of (99). Therefore, taking (100) and (108) into account in (107) we get

$$\begin{aligned} & u_{\mathbf{m},\lambda}(\mathbf{x}, t)v_{\mathbf{m},\lambda}(\mathbf{x}, t) \\ &= (2\pi\sigma_\lambda(t))^{-\frac{d}{2}} \exp\left[-\frac{|\mathbf{x} - \mathbf{b}_{\mathbf{m},\lambda}(t)|^2}{2\sigma_\lambda(t)}\right] \end{aligned}$$

following the cancellation of two exponential factors, which proves Statement (a) according to (44). We also remark that (101) is a particular case of (43), and that (103) holds according to (45).

We now turn to the proof of (102). According to (50) we note that the density of the law of $(Z_{t_1}^{\mathbf{m},\lambda}, \dots, Z_{t_n}^{\mathbf{m},\lambda}) \in \mathbb{R}^{nd}$ is

$$\begin{aligned} & \prod_{k=2}^n g_\lambda(\mathbf{x}_k, t_k - t_{k-1}, \mathbf{x}_{k-1}) \times u_{\mathbf{m},\lambda}(\mathbf{x}_1, t_1)v_{\mathbf{m},\lambda}(\mathbf{x}_n, t_n) \\ &= (2\pi)^{-\frac{nd}{2}} \left(\frac{\lambda^n \sinh(\lambda T)}{\sinh(\lambda(T - t_n)) \sinh(\lambda t_1)} \right)^{\frac{d}{2}} \left(\prod_{k=2}^n \sinh(\lambda(t_k - t_{k-1})) \right)^{-\frac{d}{2}} \\ & \quad \times \exp\left[\frac{1}{2}(\alpha_\lambda(T) - \alpha_\lambda(T - t_n)) |\mathbf{b}_{\mathbf{m}}|^2\right] \\ & \quad \times \exp\left[-\frac{\lambda}{2} \sum_{k=2}^n \frac{\cosh(\lambda(t_k - t_{k-1})) (|\mathbf{x}_k|^2 + |\mathbf{x}_{k-1}|^2) - 2(\mathbf{x}_k, \mathbf{x}_{k-1})_{\mathbb{R}^d}}{\sinh(\lambda(t_k - t_{k-1}))}\right] \\ & \quad \times \exp\left[-\frac{1}{2}(\alpha_\lambda(t_1) |\mathbf{x}_1|^2 + \alpha_\lambda(T - t_n) |\mathbf{x}_n|^2)\right] \times \exp\left[\frac{\lambda(\mathbf{b}_{\mathbf{m}}, \mathbf{x}_n)_{\mathbb{R}^d}}{\sinh(\lambda(T - t_n))}\right] \end{aligned}$$

for every $n \geq 2$. Therefore, the tridiagonal matrix C_λ^{-1} corresponding to the quadratic part when $d = 1$ is identified as

$$C_{\lambda,k,k}^{-1} = \begin{cases} \frac{\lambda \sinh(\lambda t_2)}{\sinh(\lambda(t_2 - t_1)) \sinh(\lambda t_1)} & \text{for } k = 1, \\ \frac{\lambda \sinh(\lambda(t_{k+1} - t_{k-1}))}{\sinh(\lambda(t_{k+1} - t_k)) \sinh(\lambda(t_k - t_{k-1}))} & \text{for } k = 2, \dots, n-1, \\ \frac{\lambda \sinh(\lambda(T - t_{n-1}))}{\sinh(\lambda(T - t_n)) \sinh(\lambda(t_n - t_{n-1}))} & \text{for } k = n \end{cases}$$

(the second line not being there if $n = 2$), and

$$C_{\lambda,k,k-1}^{-1} = C_{\lambda,k-1,k}^{-1} = -\frac{\lambda}{\sinh(\lambda|t_k - t_{k-1}|)} \quad \text{for } k = 2, \dots, n.$$

Consequently, inverting the matrix and using numerous identities among hyperbolic functions we eventually get

$$C_{\lambda,k,l} = \begin{cases} \frac{\sinh(\lambda(T - t_k)) \sinh(\lambda t_l)}{\lambda \sinh(\lambda T)} & \text{for } k \geq l, \\ \frac{\sinh(\lambda(T - t_l)) \sinh(\lambda t_k)}{\lambda \sinh(\lambda T)} & \text{for } k \leq l, \end{cases}$$

which leads to (102) by standard arguments. ■

REMARKS. (1) Corollary 3 thus describes a sequence of random curves all pinned down at the origin when $t = 0$ and at \mathbf{b}_m when $t = T$, with probability one. We also remark that the Gaussian law is not centered unless $\mathbf{b}_m = \mathbf{o}$, and that the process is clearly non-stationary and Markovian since (98) depends explicitly on time and (102) factorizes as the product of a function of s times a function of t . Moreover, we note that the curve $\sigma_\lambda : [0, T] \mapsto \mathbb{R}_0^+$ given by (100) is concave aside from satisfying $\sigma_\lambda(0) = \sigma_\lambda(T) = 0$, and that it takes on the maximal value at the midpoint of the time interval, namely,

$$\sigma_\lambda \left(\frac{T}{2} \right) = \frac{\sinh^2 \left(\frac{\lambda T}{2} \right)}{\lambda \sinh(\lambda T)},$$

thereby retaining the main features of a Brownian bridge. In fact, $Z_{\tau \in [0, T]}^{m, \lambda}$ does reduce to a Brownian bridge in the limit $\lambda \rightarrow 0_+$ since

$$\lim_{\lambda \rightarrow 0_+} \mathbb{E}_{\mu_m} \left((Z_s^{m, \lambda, i} - b_{m, \lambda}^i(s))(Z_t^{m, \lambda, j} - b_{m, \lambda}^j(t)) \right) = \begin{cases} \frac{(T-t)s}{T} \delta_{i,j} & \text{for } t \geq s, \\ \frac{(T-s)t}{T} \delta_{i,j} & \text{for } t \leq s \end{cases}$$

according to (102).

(2) Relations (94) represent a very degenerate case of Gaussian data. It would have been possible to replace (94) by choosing genuine Gaussian curves for both $\varphi_{m,0}$ and $\psi_{m,T}$, or by

$$\begin{cases} \varphi_{m,0}(\mathbf{x}) = \exp \left[\frac{\lambda T d}{4} \right] \delta(\mathbf{x}), \\ \psi_{m,T}(\mathbf{x}) = \exp \left[\frac{\lambda T d}{4} \right] \exp \left[-\frac{\lambda |\mathbf{x}|^2}{2} \right] \end{cases}$$

for every m . In this case the corresponding Bernstein process would have been a non-stationary, Markovian centered process $Z_{\tau \in [0, T]}^\lambda$ satisfying

$$\mathbb{P}_\mu (Z_0^\lambda = \mathbf{o}) = 1,$$

whose variance and covariance are given by

$$\begin{cases} \sigma_\lambda(t) = \frac{\sinh(\lambda t) \exp[-\lambda t]}{\lambda}, \\ \mathbb{E} \left(Z_s^{\lambda, i} Z_t^{\lambda, j} \right) = \frac{\exp[-\lambda(s+t)]}{2\lambda} (\exp[2\lambda(s \wedge t)] - 1) \delta_{i,j} \end{cases}$$

respectively, in other words a process identical in law with a d -dimensional Ornstein-Uhlenbeck process conditioned to start at the origin. We omit the details of the computations that led to the above formulae, which are quite similar to those carried out above.

Finally, we still have the following consequence of Theorem 3, where the density operator is defined by

$$\mathcal{R}_\lambda(t) f := \sum_{\mathbf{m} \in \mathbb{N}^d} p_{\mathbf{m}}(f, u_{\mathbf{m}, \lambda}(\cdot, t))_2 v_{\mathbf{m}, \lambda}(\cdot, t)$$

for each complex-valued $f \in L^2(\mathbb{R}^d)$ and every $t \in (0, T)$, where $u_{\mathbf{m}, \lambda}(\cdot, t)$ and $v_{\mathbf{m}, \lambda}(\cdot, t)$ are given by (95) and (96), respectively:

Corollary 4. *Let $\bar{Z}_{\tau \in [0, T]}^\lambda$ be the Bernstein process in the sense of Proposition 1 corresponding to the joint probability density*

$$\bar{\mu}_\lambda(\mathbf{x}, \mathbf{y}) = g_\lambda(\mathbf{x}, T, \mathbf{y}) \delta(\mathbf{x}) \sum_{\mathbf{m} \in \mathbb{N}^d} p_{\mathbf{m}} \mathcal{N}_{\mathbf{m}, \lambda}^2 \delta(\mathbf{y} - \mathbf{b}_{\mathbf{m}})$$

generated from (94), where $(\mathbf{b}_{\mathbf{m}})_{\mathbf{m} \in \mathbb{N}^d} \subset \mathbb{R}^d$ is an arbitrary sequence such that

$$\sup_{\mathbf{m} \in \mathbb{N}^d} |\mathbf{b}_{\mathbf{m}}| < +\infty.$$

If B is the multiplication operator of Theorem 3, then we have

$$\text{Tr}(\mathcal{R}_\lambda(t) B) = (2\pi\sigma_\lambda(t))^{-\frac{d}{2}} \sum_{\mathbf{m} \in \mathbb{N}^d} p_{\mathbf{m}} \int_{\mathbb{R}^d} \mathbf{d}\mathbf{x} b(\mathbf{x}) \exp\left[-\frac{|\mathbf{x} - \mathbf{b}_{\mathbf{m}, \lambda}(t)|^2}{2\sigma_\lambda(t)}\right]$$

for each $t \in (0, T)$ and every $p_{\mathbf{m}} > 0$ satisfying the normalization condition in (18).

The situation is quite different from that we just described if we consider the hierarchy (22), (23) with the initial-final data

$$\begin{aligned} \varphi_{\mathbf{m}, 0, \lambda}(\mathbf{x}) &= \mathbf{h}_{\mathbf{m}, \lambda}(\mathbf{x}), \\ \psi_{\mathbf{m}, T, \lambda}(\mathbf{x}) &= \exp[TE_{\mathbf{m}, \lambda}] \mathbf{h}_{\mathbf{m}, \lambda}(\mathbf{x}) \end{aligned} \quad (109)$$

and with (21) for the probabilities associated with each level of the spectrum, thus having

$$\mathcal{R}_\lambda f := \mathcal{Z}_\lambda^{-1}(T) \sum_{\mathbf{m} \in \mathbb{N}^d} \exp[-TE_{\mathbf{m}, \lambda}] (f, \mathbf{h}_{\mathbf{m}, \lambda})_2 \mathbf{h}_{\mathbf{m}, \lambda} \quad (110)$$

for the density operator (90). Then we have:

Theorem 6. *For every $\lambda > 0$ the Bernstein process $\bar{Z}_{\tau \in [0, T]}^\lambda$ associated with the infinite hierarchy (22)-(23) in the sense of Corollary 1 is a stationary, non-Markovian Gaussian process such that the following statements are valid:*

(a) *We have*

$$\mathbb{P}_{\bar{\mu}}(\bar{Z}_t^\lambda \in F) = (2\pi\sigma_\lambda)^{-\frac{d}{2}} \int_F \mathbf{d}\mathbf{x} \exp\left[-\frac{|\mathbf{x}|^2}{2\sigma_\lambda}\right] \quad (111)$$

for each $t \in [0, T]$ and every $F \in \mathcal{B}_d$, where

$$\sigma_\lambda = \frac{\sinh(\lambda T)}{2\lambda(\cosh(\lambda T) - 1)}. \quad (112)$$

(b) The components of $\bar{Z}_{\tau \in [0, T]}^\lambda$ satisfy the relation

$$\mathbb{E}_{\bar{\mu}} \left(\bar{Z}_s^{\lambda, i} \bar{Z}_t^{\lambda, j} \right) = \frac{\cosh \left(\lambda \left(|t - s| - \frac{T}{2} \right) \right)}{2\lambda \sinh \left(\frac{\lambda T}{2} \right)} \delta_{i, j} \quad (113)$$

for all $s, t \in [0, T]$ and all $i, j \in \{1, \dots, d\}$.

(c) For every linear bounded self-adjoint multiplication operator B on $L^2(\mathbb{R}^d)$ as defined in Theorem 5 we have

$$\text{Tr}(\mathcal{R}_\lambda B) = \mathbb{E}_{\bar{\mu}}(b(\bar{Z}_t^\lambda)) = (2\pi\sigma_\lambda)^{-\frac{d}{2}} \int_{\mathbb{R}^d} dx b(x) \exp \left[-\frac{|x|^2}{2\sigma_\lambda} \right]$$

for every $t \in [0, T]$.

Proof. The process $\bar{Z}_{\tau \in [0, T]}^\lambda$ is Gaussian by virtue of (78) with Green's function (24). Furthermore we have

$$g_\lambda(x, T, x) = \left(\frac{\lambda}{2\pi \sinh(\lambda T)} \right)^{\frac{d}{2}} \exp \left[-\frac{\lambda(\cosh(\lambda T) - 1)|x|^2}{\sinh(\lambda T)} \right]$$

so that (111) with (112) follows immediately from (79) and (93). We now turn to the proof of (113) by determining the Gaussian density of $(\bar{Z}_{t_1}^\lambda, \dots, \bar{Z}_{t_n}^\lambda)$ in \mathbb{R}^{nd} for any $n \in \mathbb{N}^+$ by substituting (24) and (93) into (78). We obtain

$$\begin{aligned} & (2(\cosh(\lambda T) - 1))^{\frac{d}{2}} \\ & \times \prod_{k=2}^n g_\lambda(x_k, t_k - t_{k-1}, x_{k-1}) \times g_\lambda(x_1, T - (t_n - t_1), x_n) \\ = & (2\pi)^{-\frac{nd}{2}} \left(\frac{2\lambda^n (\cosh(\lambda T) - 1)}{\sinh(\lambda(T - (t_n - t_1)))} \right)^{\frac{d}{2}} \prod_{k=2}^n (\sinh(\lambda(t_k - t_{k-1})))^{-\frac{d}{2}} \\ & \times \exp \left[-\frac{\lambda}{2} \sum_{k=2}^n \frac{\cosh(\lambda(t_k - t_{k-1})) \left(|x_k|^2 + |x_{k-1}|^2 \right) - 2(x_k, x_{k-1})_{\mathbb{R}^d}}{\sinh(\lambda(t_k - t_{k-1}))} \right] \\ & \times \exp \left[-\frac{\lambda \cosh(\lambda(T - (t_n - t_1))) \left(|x_1|^2 + |x_n|^2 \right) - 2(x_1, x_n)_{\mathbb{R}^d}}{2 \sinh(\lambda(T - (t_n - t_1)))} \right]. \end{aligned}$$

For the sake of clarity we identify the inverse of the covariance matrix C_λ by considering the case $n = 2$ separately from the case $n \geq 3$. For $n = 2$ we obtain

$$C_{\lambda, k, k}^{-1} = \frac{\lambda \sinh(\lambda T)}{\sinh(\lambda(t_2 - t_1)) \sinh(\lambda(T - (t_2 - t_1)))} \quad \text{for } k = 1, 2$$

and

$$C_{\lambda,2,1}^{-1} = C_{\lambda,1,2}^{-1} = -\frac{\lambda}{\sinh(\lambda|t_2 - t_1|)} - \frac{\lambda}{\sinh(\lambda(T - |t_2 - t_1|))},$$

while for $n \geq 3$ we get

$$C_{\lambda,k,k}^{-1} = \begin{cases} \frac{\lambda \sinh(\lambda(T - (t_n - t_2)))}{\sinh(\lambda(t_2 - t_1)) \sinh(\lambda(T - (t_n - t_1)))} & \text{for } k = 1, \\ \frac{\lambda \sinh(\lambda(t_{k+1} - t_{k-1}))}{\sinh(\lambda(t_{k+1} - t_k)) \sinh(\lambda(t_k - t_{k-1}))} & \text{for } k = 2, \dots, n-1, \\ \frac{\lambda \sinh(\lambda(T - (t_{n-1} - t_1)))}{\sinh(\lambda(t_n - t_{n-1})) \sinh(\lambda(T - (t_n - t_1)))} & \text{for } k = n, \end{cases}$$

$$C_{\lambda,k,k-1}^{-1} = C_{\lambda,k-1,k}^{-1} = -\frac{\lambda}{\sinh(\lambda|t_k - t_{k-1}|)} \quad \text{for } k = 2, \dots, n,$$

and

$$C_{\lambda,n,1}^{-1} = C_{\lambda,1,n}^{-1} = -\frac{\lambda}{\sinh(\lambda(T - |t_n - t_1|))},$$

all the remaining matrix elements being zero. In both cases we then obtain by inversion

$$C_{\lambda,k,l} = \frac{\sinh(\lambda|t_k - t_l|) - \sinh(\lambda(|t_k - t_l| - T))}{2\lambda(\cosh(\lambda T) - 1)}$$

for all $k, l \in \{1, \dots, n\}$ or, equivalently,

$$C_{\lambda,k,l} = \frac{\cosh(\lambda(|t_k - t_l| - \frac{T}{2}))}{2\lambda \sinh(\frac{\lambda T}{2})},$$

so that (113) eventually follows. Finally, Statement (c) follows from Corollary 2 by taking (111) into account. Note that independently of the considerations of the preceding section a glance at (111) and (113) shows directly that $\bar{Z}_{\tau \in [0, T]}^\lambda$ is stationary, as well as non-Markovian since (113) does not factorize as the product of a function of s times a function of t . ■

REMARKS. (1) It turns out that the process of Theorem 6 identifies in law with the d -dimensional periodic Ornstein-Uhlenbeck process. In order to see this we define

$$X_t := \frac{e^{-\lambda t}}{1 - e^{-\lambda T}} \int_0^T e^{-\lambda(T-\tau)} dW_\tau + \int_0^t e^{-\lambda(t-\tau)} dW_\tau, \quad t \in [0, T] \quad (114)$$

where $W_{\tau \in [0, T]}$ is a given Wiener process in \mathbb{R}^d , and where the integrals in (114) are both forward Itô integrals. It is known from a particular case of Theorem 2.1 in [19], or from a direct computation using the rules of Itô calculus (see also Section 5 in [27] for the case $d = 1$), that (114) may be viewed as a way of

writing the forward Ornstein-Uhlenbeck integral equation with random periodic boundary conditions

$$\begin{aligned} X_t &= e^{-\lambda t} X_0 + \int_0^t e^{-\lambda(t-\tau)} dW_\tau, \quad t \in [0, T], \\ X_0 &= X_T \end{aligned} \tag{115}$$

when $\mathbb{E}(X_0) = 0$, whose covariance is precisely (113). Therefore, our analysis shows that the periodic Ornstein-Uhlenbeck process may be viewed as a very special example of a stationary and non-Markovian Bernstein process. Incidentally, that process happens to be quite relevant to the mathematical investigation of certain quantum systems in equilibrium with a thermal bath when the inverse temperature is interpreted as the period. This is indeed a consequence of the fact that it also identifies in law with the Gaussian process of mean zero used in Theorem 2.1 of [14] when the positive matrix therein is chosen as $A = \lambda \mathbb{I}_d$ with \mathbb{I}_d the identity in \mathbb{R}^d . This, in turn, follows immediately from (113) which may also be written as

$$\mathbb{E}_{\bar{\mu}} \left(\bar{Z}_s^{\lambda, i} \bar{Z}_t^{\lambda, j} \right) = \frac{\exp[-\lambda|t-s|] + \exp[-\lambda(T-|t-s|)]}{2\lambda(1 - \exp[-\lambda T])} \delta_{i,j}$$

by virtue of the identity

$$\frac{\cosh\left(\lambda\left(t - \frac{T}{2}\right)\right)}{\sinh\left(\frac{\lambda T}{2}\right)} = \frac{\exp[-\lambda t] + \exp[-\lambda(T-t)]}{1 - \exp[-\lambda T]}$$

valid for every $t \in [0, T]$. Finally, we observe that the definition of a periodic process "indexed by the circle" that satisfies the "two-sided Markov property on the circle" given in Section 4 of [18] is a very special case of our definition of a Bernstein process given at the beginning of this paper. Indeed, a standard argument shows that Relation (1) is equivalent to the statement that $\mathcal{F}_s^+ \vee \mathcal{F}_t^-$ is conditionally independent of the σ -algebra

$$\mathcal{F}_{[s,t]} := \sigma \left\{ Z_\tau^{-1}(F) : \tau \in [s, t], F \in \mathcal{B}_d \right\}$$

when

$$\mathcal{F}_{\{s,t\}} := \sigma \left\{ Z_s^{-1}(F), Z_t^{-1}(F) : F \in \mathcal{B}_d \right\}$$

is given. In this respect we also refer the reader to [8] and [15] for the stationary Gaussian case when $d = 1$. More generally, we remark that Problem (115) falls into the realm of a much more general class of periodic linear stochastic differential equations which were investigated by several authors, including [19] where some of the multidimensional time-periodic processes considered there were useful regarding the resolution of filtering, smoothing and prediction problems.

(2) We complete this article with an observation concerning the interpretation of (85). Following the analogy with Quantum Statistical Mechanics, we may say that the operator $\mathcal{R}(t)$ represents a so-called pure state when $\text{Tr } \mathcal{R}^2(t) = 1$

and a mixed state when $\text{Tr } \mathcal{R}^2(t) < 1$ (see, e.g., [30] for explanations regarding this terminology). In view of the first part of Theorem 5, it is therefore legitimate to say that the non-Markovian Bernstein processes that we constructed from the method of Section 4 correspond to mixed states in the above sense. Similar considerations hold for operator (110), which satisfies the inequalities

$$0 \leq \mathcal{R}_\lambda^2 \leq \mathcal{R}_\lambda \leq \mathbb{I}$$

in the sense of quadratic forms since \mathcal{R}_λ is self-adjoint, where \mathbb{I} stands for the identity in $L^2(\mathbb{R}^d)$. In this case we always have $\text{Tr } \mathcal{R}_\lambda^2(t) < 1$ by virtue of (21), and the only process that would correspond to a pure state in this context is the Markovian process generated by the measure

$$\mu_{0,\lambda}(G) = \int_G dx dy \varphi_{0,0,\lambda}(x) g_\lambda(x, T, y) \psi_{0,T,\lambda}(y)$$

which is of the form (35), where g_λ is Mehler's kernel (24) and $\varphi_{0,0,\lambda}$, $\psi_{0,T,\lambda}$ are given by (109) with $\mathbf{m} = 0$. There are, however, many other interesting Markovian Bernstein processes associated with (22)-(23) (see, e.g., [34]).

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