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# Affine Versus Multi-Affine Models for S-Variable LMI Conditions \*

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## Abstract

This paper discusses and compares LMI results built using the S-variable approach starting from equivalent yet different representations of uncertain systems. Using the fact that S-variable results are well suited to handle descriptor systems and that descriptor system modeling is versatile, we compare results in terms of the impact of modeling on the computational burden and on conservatism. Multi-affine representations allow reduced numerical burden while affine representations lead to less conservative results. Numerical examples show that conservatism reduction is not systematic, but is in some cases quite significant without major increase of the numerical burden.

*Keywords:* LMIs, Polytopic uncertainties, S-variables, Robustness, Descriptor systems

## 1 Introduction

During the past twenty years an efficient method for handling affine polytopic uncertainties which involves additional S-variables has been intensively used for robustness purpose. The method originated in [8, 4, 14] and has had many derivations. The book [6] discusses many of these, including analysis/control design problems, and conservatism/numerical complexity issues. Among the nice features of the approach which is noticed as soon as [20, 3, 21, 5] is that it is well suited for systems in descriptor form, and this descriptor form is appropriate for manipulating systems rational in the uncertainties (or non-linearities), as if affine [10, 3, 21]. Any rational in the uncertainties linear system can be converted to a descriptor linear system with affine dependency in the parameters for which S-variable results apply readily. Moreover, it is shown in [15] that S-variable results apply as well to models which are multi-affine in the parameters, a modeling inspired from multi-simplex models from [12, 11], which may as well be represented using tensors as exploited in [2] that provide related LMI results for Takagi-Sugeno fuzzy systems.

One conclusion drawn from [15] is that S-variable results apply readily to affine and to multi-affine descriptor models. But, in case a system has a multi-affine representation it can always be converted to an affine representation. This is usually at the expense of increasing the size of the descriptor model and hence increases the numerical burden for the solvers in the end. If we employ a multi-affine representation, we obtain an LMI that is more conservative but computationally less demanding, whereas if we employ an affine representation, we obtain an LMI that is less conservative but computationally more demanding. The question addressed in this paper is whether the increase of the numerical burden worths the effort in terms of reduced conservatism. We provide notations for handling the affine and multi-affine models. We show the intrinsic source of conservatism in the multi-affine results and we show on examples that it is indeed the case, but not always.

Results are compared numerically to Pólya based results for robust stability analysis ([17, 18, 13]). The comparison reveals the relevance of the proposed methodology in terms of the tradeoff conservatism/complexity. For space limitation reasons we did not perform comparisons with sum-of-squares type results that may be applied as well. The reason why we concentrated the comparison with the Pólya approach is that [16] shows that in the considered situation S-variable results include those issued from sum-of-squares methodology.

The outline of the paper is as follows. The second section is devoted to multi-affine modeling using multi-simplex representation of uncertainties. The third section recalls the S-variable results for descriptor systems (for the simplest

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case when the  $E$  matrix is full column rank, for more advanced results the reader is invited to consult [6]). The fourth section is then dedicated to numerical examples which illustrate the impact of modeling on the conservatism of LMI results.

*Notation:*

$I_n$  stands for the identity matrix of size  $n$ .  $0_{n,m}$  is the zero matrix with  $n$  rows and  $m$  columns. When the size can be deduced from the context the subscripts are avoided.  $A^T$  is the transpose of the matrix  $A$ .  $A \succ B$  is the matrix inequality stating that  $A - B$  is symmetric positive definite. The terminology “congruence operation of  $A$  on  $B$ ” is used to denote  $A^T B A$ . If  $A$  is full column rank, and  $B \succ 0$ , the congruence operation of  $A$  on  $B$  gives a positive definite matrix:  $A^T B A \succ 0$ . A matrix inequality of the type  $N(X) \succ 0$  is said to be a linear matrix inequality (LMI for short), if  $N(X)$  is affine in the decision variables  $X$ . In the following, decision variables are highlighted using the blue color.  $\Xi_{\bar{v}} = \{\xi_{v=1 \dots \bar{v}} \geq 0, \sum_{v=1}^{\bar{v}} \xi_v = 1\}$  is the unitary simplex in  $\mathbb{R}^{\bar{v}}$ . The elements  $\xi$  of unitary simplexes are used to describe polytopic type uncertainties. In the following, uncertainties are highlighted using the red color. The vertices of  $\Xi_{\bar{v}}$  are the  $\bar{v}$  vectors  $\xi^{[v]}$  with all zeros coefficients except one equal to 1.

## 2 Polytopes and Multi-affine representations

We shall consider uncertain representations depending on parameters  $\theta$  assumed to lie in a set  $\Theta$  defined as the cross product of  $\bar{p}$  polytopes:

$$\theta \in \Theta = \{(\theta_1, \dots, \theta_{\bar{p}}) \in \Theta_1 \times \dots \times \Theta_{\bar{p}}\}. \quad (1)$$

The  $\bar{p}$  components of  $\theta$  are independent vectors  $\theta_p \in \mathbb{R}^{m_p}$ . Each set  $\Theta_p$  is assumed to be a polytope with  $\bar{v}_p$  vertices from the set  $\mathcal{V}_p = \{\theta_p^{[1]}, \dots, \theta_p^{[\bar{v}_p]}\}$ .  $\Theta_p$  is the convex hull of the vertices, or equivalently, each  $\theta_p$  writes as the weighted sum of vertices with weight from unitary simplexes:

$$\Theta_p = Co(\mathcal{V}_p) = \left\{ \theta_p = \sum_{v=1}^{\bar{v}_p} \xi_{p,v} \theta_p^{[v]} : \xi_p \in \Xi_{\bar{v}_p} \right\}. \quad (2)$$

In the following,  $\mathcal{V} = \mathcal{V}_1 \times \dots \times \mathcal{V}_{\bar{p}}$  is the finite set of all extremal values of the parameters gathered in  $\theta$ . A generic element of  $\mathcal{V}$  will be denoted  $\theta^{[v]}$  with  $v = (v_1, \dots, v_{\bar{p}})$  the vector of indices of vertices for each component.  $\mathcal{I}$  is the set of all vectors of indices  $v$ .  $\theta^{[v]}$  is the one to one mapping from  $\mathcal{I}$  to  $\mathcal{V}$ . The cardinality of  $\mathcal{V}$  is  $\bar{v} = \prod_{p=1}^{\bar{p}} \bar{v}_p$ . Choosing any ordering of the  $\bar{v}$  components of  $\mathcal{V}$  the set shall also be described using the notation  $\mathcal{V} = \{\theta^{[1]}, \dots, \theta^{[\bar{v}]}, \dots, \theta^{[\bar{v}]}\}$  where  $\bar{v}$  is a scalar indexing of the  $\bar{v}$  elements.

A matrix  $M(\theta)$  is said to be multi-affine in the parameters if it is affine in each  $\theta_p$  taken independently. A multi-affine matrix can be written as the multi-sum of weighted vertices from  $\mathcal{V}$  denoted as

$$M(\theta) = \sum_{v \in \mathcal{I}} \xi_{1,v_1} \dots \xi_{\bar{p},v_{\bar{p}}} M(\theta^{[v]}) : \xi_p \in \Xi_{\bar{v}_p} \quad (3)$$

As we will see on an example, it is quite trivial to notice that any such matrix belongs as well to the polytopic set defined by

$$M(\tilde{\theta}) = \sum_{\bar{v}=1}^{\bar{v}} \xi_{\bar{v}} M(\theta^{[\bar{v}]}) : \xi \in \Xi_{\bar{v}} \quad (4)$$

The simplest case is when the  $\theta_p$  elements are scalars ( $m_p = 1$ ) defined in intervals  $\theta_p \in [\theta_p^{[1]}, \theta_p^{[2]}]$ , which are polytopes of  $\bar{v}_p = 2$  vertices. For the case when all elements are scalar, the cardinality of  $\mathcal{V}$  is  $\bar{v} = 2^{\bar{p}}$ . In case of two scalar parameters, the multi-sum reads as

$$\begin{aligned} M(\theta) &= \sum_{v \in \{(1,1), (1,2), (2,1), (2,2)\}} \xi_{1,v_1} \xi_{2,v_2} M(\theta^{[v]}) \\ &= \xi_{1,1} \xi_{2,1} M(\theta^{[(1,1)]}) + \xi_{1,1} \xi_{2,2} M(\theta^{[(1,2)]}) + \xi_{1,2} \xi_{2,1} M(\theta^{[(2,1)]}) + \xi_{1,2} \xi_{2,2} M(\theta^{[(2,2)]}). \end{aligned}$$

The following fact

$$\xi_{1,1} \xi_{2,1} + \xi_{1,1} \xi_{2,2} + \xi_{1,2} \xi_{2,1} + \xi_{1,2} \xi_{2,2} = \xi_{1,1} + \xi_{1,2} = 1$$

allows to conclude that the multi-affine matrix  $M(\theta)$  is included in the polytope of four vertices  $M(\theta^{[(1,1)]})$ ,  $M(\theta^{[(1,2)]})$ ,  $M(\theta^{[(2,1)]})$ ,  $M(\theta^{[(2,2)]})$ . The converse is not true in general. There are potentially elements in the convex hull of these four vertices that are not in the multi-affine model. An example of this is the matrix  $M(\theta) = \begin{bmatrix} \theta_1 & \theta_1 \theta_2 & \theta_2 \end{bmatrix}$  with  $\theta_1 \in [1, 2]$  and  $\theta_2 \in [1, 2]$ . The middle of vertices  $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$  and  $\begin{bmatrix} 2 & 4 & 2 \end{bmatrix}$  is equal to  $\begin{bmatrix} \frac{3}{2} & \frac{5}{2} & \frac{3}{2} \end{bmatrix}$ . It is inside polytope (4), but since  $\frac{5}{2} \neq (\frac{3}{2})^2$  it is not a realization (3) of  $M(\theta)$ .

### 3 LMI S-Variable conditions

The previous section allows to conclude about the fact that multi-affine polytopes can be considered as included in polytopes of larger size. This might bring some conservatism. To evaluate this fact we study the LMI conditions for robust analysis of polytopic and multi-polytopic systems. To do so we consider one type of LMI conditions issued from the S-Variable approach.

Let an uncertain descriptor systems described by the following equation:

$$E_{xx}(\theta)\dot{x}(t) + E_{\pi x}(\theta)\pi(t) = A(\theta)x(t) \quad (5)$$

where  $x \in \mathbb{R}^{n_x}$  is the state and  $\pi \in \mathbb{R}^{n_\pi}$  is some auxiliary signal linked to the state via algebraic constraints. Neither  $E_{xx}(\theta)$  nor  $A(\theta)$  are supposed to be square and have  $n$  rows. For simplicity of the presentation the matrix  $\begin{bmatrix} E_{xx}(\theta) & E_{\pi x}(\theta) \end{bmatrix}$  is assumed to be full column rank for all  $\theta \in \Theta$ . More general cases are described in [6].

For conciseness of notation denote

$$M(\theta) = \begin{bmatrix} E_{xx}(\theta) & E_{\pi x}(\theta) & -A(\theta) \end{bmatrix}$$

the uncertain matrix which fully describes the dynamics of the system.

**Proposition 1** *If the system (6) is defined using a multi-affine matrix  $M(\theta)$  then there exists another augmented representation*

$$\hat{E}_{xx}(\theta)\dot{x}(t) + \hat{E}_{\pi x}(\theta)\hat{\pi}(t) = \hat{A}(\theta)x(t) \quad (6)$$

in which the matrix of size  $\hat{n}$ -by- $(2n_x + \hat{n}_\pi)$ , where  $\hat{n} \geq n$  and  $\hat{n}_\pi \geq n_\pi$ ,

$$\hat{M}(\theta) = \begin{bmatrix} \hat{E}_{xx}(\theta) & \hat{E}_{\pi x}(\theta) & -\hat{A}(\theta) \end{bmatrix}$$

is affine in the parameters.

Rather than proving the proposition for a general representation we shall consider an illustrative example. The proof of the general case follows the same lines. Let the system described by

$$\begin{bmatrix} 1 & \theta_1 \\ 0 & 1 \end{bmatrix} \dot{x}(t) = \begin{bmatrix} -\theta_2 & 0 \\ \theta_1\theta_2 & -1 \end{bmatrix} x(t). \quad (7)$$

This multi-affine representation admits the following equivalent affine representation

$$\begin{bmatrix} 1 & \theta_1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \dot{x}(t) + \begin{bmatrix} 0 \\ -\theta_1 \\ 1 \end{bmatrix} \hat{\pi}(t) = \begin{bmatrix} -\theta_2 & 0 \\ 0 & -1 \\ \theta_2 & 0 \end{bmatrix} x(t). \quad (8)$$

The proof comes from the fact the last row of the affine representation gives  $\hat{\pi} = \theta_2 x_1$ , which when included in the second row, gives exactly the multi-affine representation.

**Theorem 1** *The uncertain system (6) defined by a multi-affine matrix  $M(\theta)$  is robustly stable if there exist  $\bar{v}$   $n_x$ -by- $n_x$  symmetric matrices  $P^{[v]} \succ 0$  and a common to all inequalities  $(2n_x + n_\pi)$ -by- $n$  matrix  $S$  such that for all vertices  $v \in \mathcal{I}$  the following LMIs hold*

$$Q(P^{[v]}) \prec SM(\theta^{[v]}) + M^T(\theta^{[v]})S^T \quad (9)$$

where

$$Q(P^{[v]}) = \begin{bmatrix} 0_{n_x, n_x} & 0 & P^{[v]} \\ 0 & 0_{n_\pi, n_\pi} & 0 \\ P^{[v]} & 0 & 0_{n_x, n_x} \end{bmatrix}.$$

*Proof* If the inequalities of Theorem 1 hold for all vertices  $v \in \mathcal{I}$ , these hold as well for all reordered vertices  $\theta^{[\tilde{v}]}$  with  $\tilde{v} = 1 \dots \bar{v}$ . By convexity of matrix inequalities this implies that the inequalities hold for the convex hull of the vertices, that is for all  $\xi_{\tilde{v}} \in \Xi_{\bar{v}}$  one has:

$$Q(P(\tilde{\theta})) \prec SM(\tilde{\theta}) + M^T(\tilde{\theta})S^T \quad (10)$$

for the following parameter-dependent Lyapunov matrix

$$P(\tilde{\theta}) = \sum_{\bar{v}=1}^{\bar{v}} \xi_{\bar{v}} P^{[\bar{v}]}$$

Congruence of  $\eta = \begin{pmatrix} \dot{x}^T & \pi^T & x^T \end{pmatrix}^T$  on inequality (10) gives that

$$\dot{x}^T(t) P(\tilde{\theta}) x(t) + x^T(t) P(\tilde{\theta}) \dot{x}(t) < 0$$

holds along the trajectories of  $M(\tilde{\theta})\eta(t) = 0$ . Hence robust stability of the embedding polytopic uncertain system is proved by the parameter-dependent Lyapunov function  $x^T P(\tilde{\theta}) x$ . The multi-affine representation  $M(\theta)$  being included inside the polytopic embedding  $M(\tilde{\theta})$ , stability of the multi-affine model is proved. ■

The proof is well known but is recalled here because it clarifies that these conditions for proving robust stability of multi-affine models are equivalent to proving robust stability of a larger embedding polytopic set of models.

An alternative method to prove robust stability of the multi-affine models is to take advantage of the equivalent augmented representation proposed in Proposition 1.

**Corollary 1** *The uncertain system (6) defined by a multi-affine matrix  $M(\theta)$  is robustly stable if there exist  $\bar{v}$   $n_x$ -by- $n_x$  symmetric matrices  $\hat{P}^{[v]} \succ 0$  and a common to all inequalities  $(2n_x + \hat{n}_\pi)$ -by- $\hat{n}$  matrix  $\hat{S}$  such that for all vertices  $v \in \mathcal{I}$  the following LMIs hold*

$$\hat{Q}(\hat{P}^{[v]}) \prec \hat{S} \hat{M}(\theta^{[v]}) + \hat{M}^T(\theta^{[v]}) \hat{S}^T \quad (11)$$

where

$$\hat{Q}(\hat{P}^{[v]}) = \begin{bmatrix} 0_{n_x, n_x} & 0 & \hat{P}^{[v]} \\ 0 & 0_{\hat{n}_\pi, \hat{n}_\pi} & 0 \\ \hat{P}^{[v]} & 0 & 0_{n_x, n_x} \end{bmatrix}.$$

At this stage here are the characteristics of the two results.

- Theorem 1 is such that:  
The decision variables are the  $\bar{v}$  matrices  $P^{[v]}$  of size  $n_x$ -by- $n_x$  and a  $(2n_x + n_\pi)$ -by- $n$  matrix  $S$ ;  
The  $\bar{v}$  constraints (9) have  $(2n_x + n_\pi)$  rows (and as many columns);  
Stability also holds for a polytopic model in which the multi-affine representation is embedded.
- Corollary 1 is such that:  
The decision variables are the  $\bar{v}$  matrices  $\hat{P}^{[v]}$  of size  $n_x$ -by- $n_x$  and a  $(2n_x + \hat{n}_\pi)$ -by- $\hat{n}$  matrix  $\hat{S}$ ;  
The  $\bar{v}$  constraints (11) have  $(2n_x + \hat{n}_\pi)$  rows (and as many columns).

Theorem 1 is hence potentially more conservative than Corollary 1 but is of smaller size (both in number of decision variables and in size of the constraints). The increase in size of LMIs of Corollary 1 compared to those of Theorem 1 is related to the increase from  $n_\pi$  to  $\hat{n}_\pi$ . This increase is at most equal to the number of products between parameters in the original multi-affine representation.

Since the goal of the paper is to compare results in terms of conservatism and numerical complexity, we shall also compare these to another LMI-based approach. The comparison is done with respect to the Pólya based method [17, 18, 13] when applied to inequality

$$Q(P(\theta)) \prec S(\theta)M(\theta) + M^T(\theta)S(\theta)^T \quad (12)$$

with parameter dependent S-variables  $S(\theta)$ . The choice of a multi-affine  $P(\theta)$  and a common to all  $S(\theta) = S$  gives exactly condition of Theorem 1. Choosing polynomial  $P(\theta)$  of degree higher than one and polynomial  $S(\theta)$  leads to LMIs of decreasing conservatism. Conservatism is known to be vanishing as the order grows.

## 4 Numerical examples

### 4.1 Example 1

The LMI conditions are tested on the example (7) with  $\theta_1 \in [-10\delta, 10\delta]$  and  $\theta_2 \in [1, 1 - \delta]$ . The LMI conditions are tested for different values of  $\delta$  and are summarized in Table 1. OK indicates that the LMIs are feasible for this value of  $\delta$ . Corollary 1 is feasible for larger values of  $\delta$  thus illustrating that it is less conservative than Theorem 1. The

Table 1: Example 1

$\delta$	0.556	0.557	0.999	1	nb vars	nb rows
Theorem 1	OK	-	-	-	20	24
Corollary 1	OK	OK	OK	-	27	28

last columns indicate the number of decision variables and the total number or rows of the LMI constraints (including constraints  $P^{[v]} \succ 0$ ).

For  $\theta_1 = 0$  and  $\theta_2 = 0$ , which is a possible realization when  $\delta = 1$ , the system is not asymptotically stable. For this example Theorem 1 is highly conservative while Corollary 1 is not. Conservatism reduction thanks to Corollary 1 is not at the expense of a major increase of the computation burden.

These results are now compared to those obtained using Pólya methodology. Table 2 gives the results for different choices of parameter-dependent S-variables. In all cases  $P(\theta)$  is chosen to be multi-affine in the uncertainties. Four choices are tested:

- (1a)  $S(\theta) = S$  is parameter-independent;
- (1b)  $S(\theta) = S(\theta_1)$  is affine in the first uncertainty;
- (1c)  $S(\theta) = S(\theta_2)$  is affine in the second uncertainty;
- (1d)  $S(\theta)$  is multi-affine in the two uncertainties.

Table 2: Example 1 - Pólya method

$\delta$	0.556	0.557	0.999	1	nb vars	nb rows
(1a)	OK	-	-	-	20	24
(1b)	OK	-	-	-	28	32
(1c)	OK	OK	OK	-	28	32
(1d)	OK	OK	OK	-	44	44

Results of Table 2 illustrate that it is indeed possible to get non-conservative results starting from the multi-affine representation by applying the Pólya methodology with parameter-dependent S-variables. But, in comparison with our proposed strategy consisting in converting the multi-affine problem into an affine representation, the results using parameter-dependent S-variables lead to numerically more demanding LMIs. Our interpretation of this fact is that our proposed methodology exploits more accurately the structure of the problem while systematic methods for handling polynomially parameter-dependent LMIs tend to over parameterize the LMIs.

## 4.2 Example 2

We consider now a more complex example with three uncertain parameters:

$$\begin{bmatrix} 1 & \theta_1 & 0 & 0 \\ 0 & 1 & \theta_{3,1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \theta_1\theta_2 & \theta_2\theta_{3,1} & 1 \end{bmatrix} \dot{x} = \begin{bmatrix} -\theta_2 & 0 & 0 & 0 \\ \theta_1\theta_2 & -1 & 0 & 0 \\ \theta_1\theta_{3,2} & 0 & -1 & 0 \\ \theta_1\theta_2\theta_{3,2} & 0 & 0 & -1 \end{bmatrix} x \quad (13)$$

with  $\theta_1 \in Co(-10\delta, +10\delta)$ ,  $\theta_2 \in Co(1, 1 - \delta)$  and

$$\theta_3 = \begin{pmatrix} \theta_{3,1} \\ \theta_{3,2} \end{pmatrix} \in Co\left(\begin{pmatrix} \delta \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \delta \end{pmatrix}, \begin{pmatrix} -\delta \\ -\delta \end{pmatrix}\right).$$

Choose  $\pi_1 = \begin{bmatrix} \theta_1 \theta_{3,2} & 0 & 0 & 0 \end{bmatrix} x - \begin{bmatrix} 0 & \theta_1 & \theta_{3,1} & 0 \end{bmatrix} \dot{x}$  to get this other multi-affine model

$$\begin{bmatrix} 1 & \theta_1 & 0 & 0 \\ 0 & 1 & \theta_{3,1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \theta_1 & \theta_{3,1} & 0 \end{bmatrix} \dot{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \theta_2 \\ 1 \end{bmatrix} \pi_1 = \begin{bmatrix} -\theta_2 & 0 & 0 & 0 \\ \theta_1 \theta_2 & -1 & 0 & 0 \\ \theta_1 \theta_{3,2} & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ \theta_1 \theta_{3,2} & 0 & 0 & 0 \end{bmatrix} x. \quad (14)$$

Choose  $\pi_2 = \begin{bmatrix} \theta_1 & 0 & 0 & 0 \end{bmatrix} x$  to get this third multi-affine model

$$\begin{bmatrix} 1 & \theta_1 & 0 & 0 \\ 0 & 1 & \theta_{3,1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \theta_1 \theta_2 & \theta_2 \theta_{3,1} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \dot{x} + \begin{bmatrix} 0 \\ -\theta_2 \\ -\theta_{3,2} \\ -\theta_2 \theta_{3,2} \\ 1 \end{bmatrix} \pi_2 = \begin{bmatrix} -\theta_2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ \theta_1 & 0 & 0 & 0 \end{bmatrix} x. \quad (15)$$

Combining  $\pi_1$  and  $\pi_2$ , the system has the following affine representation

$$\begin{bmatrix} 1 & \theta_1 & 0 & 0 \\ 0 & 1 & \theta_{3,1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \theta_1 & \theta_{3,1} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \dot{x} + \begin{bmatrix} 0 & 0 \\ 0 & -\theta_2 \\ 0 & -\theta_{3,2} \\ \theta_2 & 0 \\ 1 & -\theta_{3,2} \\ 0 & 1 \end{bmatrix} \pi = \begin{bmatrix} -\theta_2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ \theta_1 & 0 & 0 & 0 \end{bmatrix} x. \quad (16)$$

Models (13), (14) and (15) are multi-affine, and (16) is affine. The LMI results are tested for each of these models and results are summarized in Table 3.

Table 3: Example 2

$\delta$	0.469	0.470	0.602	0.603	nb vars	nb rows
(13)	OK	-	-	-	152	144
(14)	OK	-	-	-	165	156
(15)	OK	OK	OK	-	165	156
(16)	OK	OK	OK	-	180	168

In this example the LMIs built based on models (13) and (14) give exactly the same results in terms of conservatism but for an increased numerical burden in case of model (14). The same comment holds when comparing the LMIs built based on models (15) and (16). This illustrates the fact that results for multi-affine representations are not necessarily more conservative. But the results for affine representations are, as guaranteed by the theory, the less conservative ones, and the more demanding in terms of numerical complexity.

In order to reduce further the conservatism [7] suggests to augment the model representation by including higher derivatives of the states. See also [6] for discussions about this method. For the treated example, starting from model (14) and including the second derivative of the state, it amounts to considering the augmented system:

$$\begin{bmatrix} E_{x(14)}(\theta) & 0 \\ 0 & E_{x(14)}(\theta) \\ 0 & I \end{bmatrix} \begin{pmatrix} \ddot{x} \\ \dot{x} \end{pmatrix} + \begin{bmatrix} E_{\pi(14)}(\theta) & 0 \\ 0 & E_{\pi(14)}(\theta) \\ 0 & 0 \end{bmatrix} \tilde{\pi}_1 = \begin{bmatrix} A_{(14)}(\theta) & 0 \\ 0 & A_{(14)}(\theta) \\ I & 0 \end{bmatrix} \begin{pmatrix} \dot{x} \\ x \end{pmatrix}$$

where

$$E_{x(14)}(\theta) = \begin{bmatrix} 1 & \theta_1 & 0 & 0 \\ 0 & 1 & \theta_{3,1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \theta_1 & \theta_{3,1} & 0 \end{bmatrix} \quad E_{\pi(14)}(\theta) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \theta_2 \\ 1 \end{bmatrix} \quad A_{(14)}(\theta) = \begin{bmatrix} -\theta_2 & 0 & 0 & 0 \\ \theta_1 \theta_2 & -1 & 0 & 0 \\ \theta_1 \theta_{3,2} & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ \theta_1 \theta_{3,2} & 0 & 0 & 0 \end{bmatrix}$$

We do not provide all the formulas, but the augmentation procedure applies the same way to all four models (13), (14), (15), (16) and we shall denote (13-a), (14-a), (15-a), (16-a) the augmented versions of these. In the augmented

models the last rows correspond to the fact that  $\dot{x}$  and  $\ddot{x}$  are the same vectors on both sides of the equality constraint. As discussed in [6] the method hence includes the information that the first derivatives of the uncertain parameters are zero. The rows involved to include this key information are parameter independent and this fact can be used to reduce partly, and without conservatism, the numerical burden (see Chapter 3 in [6]). Table 4 provides the results for the augmented systems. The size of the LMIs and the number of decision variables are while employing this size reduction technique.

Table 4: Example 2 with augmentation

$\delta$	0.541	0.542	0.643	0.644	nb vars	nb rows
(13-a)	OK	-	-	-	528	240
(14-a)	OK	-	-	-	572	264
(15-a)	OK	OK	OK	-	572	264
(16-a)	OK	OK	OK	-	624	288

Same conclusions apply as for Table 3. Moreover, as expected, the LMIs tested for Table 4 are less conservative than those for Table 3 when comparing results based augmented and non-augmented models. What is more surprising is the conservatism of results based on model (13-a) compared to those based on model (15). With less decision variables, and without introducing the knowledge about derivatives of the parameters, the results based on (15) are less conservative.

For comparison, we apply the Pólya methodology to the parameter-dependent inequality (12) for different choices of parameter-dependent variables. Two cases are tested, in both of these  $P(\theta)$  is taken multi-affine, and

(2a)  $S(\theta) = S$  is common to all uncertainties (result coincides with conditions of Theorem 1);

(2b) multi-affine  $S(\theta)$ .

Results are given in Table 5 and illustrate once again that our proposed strategy has advantages in terms of the numerical burden versus conservatism.

Table 5: Example 2 - Pólya method

$\delta$	0.469	0.470	0.643	0.644	nb vars	nb rows
(2a)	OK	-	-	-	152	144
(2b)	OK	OK	OK	-	504	480

To finalize the study of this example it should be said that for  $\delta = 0.644$  the vertex defined by the following values

$$\theta_1 = 6.44, \quad \theta_2 = 0.3560, \quad \theta_3 = \begin{pmatrix} -0.644 \\ -0.644 \end{pmatrix}$$

gives an unstable system. LMI results based on models (15-a) and (16-a) are hence non-conservative.

### 4.3 Example 3

Now we modify slightly Example 2 and replace the set where the  $\theta_3$  parameter lies to the following polytope of four vertices

$$\theta_3 = \begin{pmatrix} \theta_{3,1} \\ \theta_{3,2} \end{pmatrix} \in Co \left( \begin{pmatrix} \delta \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \delta \end{pmatrix}, \begin{pmatrix} -\delta \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -\delta \end{pmatrix} \right).$$

Results are given in Table 6 where (13-0) indicates that the model used is (13) and no augmentation is applied; (13-1) indicates that a first augmentation including  $\ddot{x}$  in the equations is applied as in Example 2; (13-2) indicates that an other augmentation including  $x^{(3)}$  in the equations is applied. We do not give results based on equations (14) and (16) because these are once again strictly the same as for LMIs built based on equations (13) and (15) respectively. Only the maximal values of  $\delta$  for which the LMI results are feasible are given.

For the third augmentation that involves derivatives of the state up to  $x^{(4)}$  the LMI results provide no measurable improvement compared to the second augmentation (at least with the precision of thee digits that we used). But,



Table 6: Example 3 with augmentation

	$\delta$	nb vars	nb rows
(13-0)	0.502	192	192
(15-0)	0.659	205	208
(13-1)	0.587	672	320
(15-1)	0.841	716	352
(13-2)	0.587	1440	448
(15-2)	<b>0.849</b>	1533	496
(13-3)	0.587	2496	576
(15-3)	<b>0.849</b>	2656	640

applying results from [17, 18] on the dual variables of the LMIs obtained for (16-3) with  $\delta = 0.850$  we were able to find a de-stabilizing worst case:

$$\theta_1 = 8.4995, \quad \theta_2 = 0.15, \quad \theta_3 = \begin{pmatrix} 0.3947 \\ 0.4552 \end{pmatrix}$$

for which the poles of the system are

$$0.0030 \pm 3.4839i, \quad -0.0124, \quad -1.0000.$$

There is clearly an unstable pair of poles which indicates that the upper bound  $\delta = 0.849$  obtained at the second relaxation is tight (at the precision of three digits that we have chosen). It should be noticed that without the results of [17, 18] which were implemented using the methodology described in [6], it would have been quite complex to find this worst case that is not at one of the  $\bar{v} = 16$  vertices.

We now compare these results with Pólya methodology. 5 cases are tested:

- (3a)  $P(\tilde{\theta})$  is multi-affine and  $S(\tilde{\theta}) = S$  is common to all uncertainties (identical results to (13-0));
- (3b) both  $P(\tilde{\theta})$  and  $S(\tilde{\theta})$  are multi-affine;
- (3c)  $P(\tilde{\theta})$  is polynomial or degree 2 in all the parameters and  $S(\tilde{\theta})$  is multi-affine;
- (3d)  $P(\tilde{\theta})$  and  $S(\tilde{\theta})$  are polynomial or degree 2 in all the parameters;
- (3e)  $P(\tilde{\theta})$  is polynomial of degree 3 in all the parameters and  $S(\tilde{\theta})$  is polynomial of degree 2.

Results are given in Table 7 and illustrate once again that Pólya based results tend to be numerically more demanding compared to our approach, especially in terms of the size of LMI constraints.

Table 7: Example 3 - Pólya method

	$\delta$	nb vars	nb rows
(3a)	0.502	192	192
(3b)	0.752	672	784
(3c)	0.809	1412	1080
(3d)	0.847	3780	2920
(3e)	<b>0.849</b>	6080	3840

In terms of computation, all tests have been made on a MacBook Pro 2.9 GHz Intel Core i5 with Matlab2016b. LMIs were coded using YALMIP (R20141030 release) [9] and solved using SDPT3 (version 4.0) [19]. Solver time for the 4 LMI problems built for models (13-3), (14-3), (15-3) and (16-3) for  $\delta = 0.849$  is about 70 seconds. The LMIs for the comparison with Pólya based results were build using the ROLMIP toolbox [1].

The code of the examples is available on the web at:

<http://homepages.laas.fr/peaucell/papers/rocond18.m>

## 5 Conclusions

The goal of this paper is to discuss the conservatism of S-variable LMI results when these are applied to multi-affine representations. We have shown that at the expense of a minor augmentation of the numerical burden it is preferable to convert the models into affine representations. Doing so does not increase much the numerical burden, but reduces the conservatism quite significantly on examples. Yet it is not always the case, and we provide examples of the two situations. Methods to detect beforehand if the removal of some of the products between parameters will indeed reduce the conservatism is left as an open question.

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