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Lower exponential strong well-posedness of hyperbolic boundary value problems in a strip

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Abstract

In this article we are interested in the lower exponential strong well-posedness of hyperbolic boundary value problems set in the strip $\mathbb{R}^{d-1} \times [0, 1]$. We assume that each component of the boundary of the strip satisfies the so-called uniform Kreiss-Lopatinskii condition (which is the condition ensuring the lower exponential strong well-posedness of the boundary value problem in the half space) and we show that due to selfinteraction phenomenon, an extra condition has to be made to ensure the lower exponential strong well-posedness of the boundary value problem in the strip. This condition imposes that a selfinteracting wave in the strip can not be amplified by repetitive reflections against each component of the boundary. This condition is very analogous to the one imposed in [Osher, 1973] to study the problem in the quarter space. However, due to the simplicity of the phases generation process for the strip geometry (less phases are generated) and this condition involves matrices and not Fourier integral operators.

AMS subject classification :35L50

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1 Introduction

In this article we are interested in the study of hyperbolic boundary value problems set in the strip $\mathbb{R}^{d-1} \times [0, 1]$. Such problems read:

$$\begin{cases} L(\partial)u := \partial_t u + \sum_{j=1}^d A_j \partial_j u = f & \text{for } (t, x', x_d) \in \mathbb{R}_t \times \mathbb{R}^{d-1} \times]0, 1[, \\ B_0 u|_{x_d=0} = g_0 & \text{for } (t, x') \in \mathbb{R}_t \times \mathbb{R}^{d-1}, \\ B_1 u|_{x_d=1} = g_1 & \text{for } (t, x') \in \mathbb{R}_t \times \mathbb{R}^{d-1}, \\ u|_{t \leq 0} = 0 & \text{for } (x', x_d) \in \mathbb{R}^{d-1} \times [0, 1], \end{cases} \quad (1)$$

where the coefficients A_j are given matrices in $\mathbf{M}_{N \times N}(\mathbb{R})$, the source terms f , g_0 and g_1 are given functions and where the solution u has its values in \mathbb{R}^N .

The boundary matrices in (1) namely B_0 and B_1 are generic matrices encoding the good number of boundary conditions to make sure that the problem (1) is not underdetermined or at the contrary overdetermined. More precisely $B_0 \in \mathbf{M}_{p \times N}(\mathbb{R})$ and $B_1 \in \mathbf{M}_{(N-p) \times N}(\mathbb{R})$, the value of p will be made precise in Assumption 2.2.

Our aim in this article is to establish a full characterization of the lower exponential strong well-posedness of (1) in terms of the boundary conditions B_0 and B_1 . Firstly, let us give more details about what we mean by lower exponential strong well-posedness by considering the analogous of (1) in the half space geometry, namely

$$\begin{cases} L(\partial)u = f & \text{for } (t, x', x_d) \in \mathbb{R}_t \times \mathbb{R}^{d-1} \times \mathbb{R}_+, \\ B_0 u|_{x_d=0} = g & \text{for } (t, x') \in \mathbb{R}_t \times \mathbb{R}^{d-1}, \\ u|_{t \leq 0} = 0 & \text{for } (x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R}_+. \end{cases} \quad (2)$$

The boundary value problem (2) has been deeply studied since the seminal work of [Kreiss, 1970] and in particular we know that the strong well-posedness of (2) is equivalent to the fulfillment of the so called uniform Kreiss-Lopatinskii condition. This condition states that in the normal mode analysis no stable mode is solution of the homogeneous boundary value problem (we refer to [Kreiss, 1970] for more details).

By strong well-posedness we mean that for all source terms $f \in L_\gamma^2(\mathbb{R}_t \times \mathbb{R}^{d-1} \times \mathbb{R}_+)$ and $g \in L_\gamma^2(\mathbb{R}_t \times \mathbb{R}^{d-1})$ the boundary value problem (2) admits a unique solution $u \in L_\gamma^2(\mathbb{R}_t \times \mathbb{R}^{d-1} \times \mathbb{R}_+)$ with trace $u|_{x_d=0} \in L_\gamma^2(\mathbb{R}_t \times \mathbb{R}^{d-1})$ satisfying the following energy estimate: there exists $C > 0$ and $\gamma_0 \geq 0$ such that for all $\gamma > \gamma_0$ we have the inequality:

$$\gamma \|u\|_{L_\gamma^2(\mathbb{R}_t \times \mathbb{R}^{d-1} \times \mathbb{R}_+)}^2 + \|u|_{x_d=0}\|_{L_\gamma^2(\mathbb{R}_t \times \mathbb{R}^{d-1})}^2 \leq C \left(\frac{1}{\gamma} \|f\|_{L_\gamma^2(\mathbb{R}_t \times \mathbb{R}^{d-1} \times \mathbb{R}_+)}^2 + \|g\|_{L_\gamma^2(\mathbb{R}_t \times \mathbb{R}^{d-1})}^2 \right) \quad (3)$$

where for some Banach space X we defined $L_\gamma^2(X) := \{u \in \mathcal{D}'(X) \setminus ue^{-\gamma t} \in L^2(X)\}$ (equipped with the obvious norm).

The fact of interest here is that the estimate (3) permits a solution with exponential growth in time (bounded by $e^{\gamma_0 t}$). However it has also been shown in [Kreiss, 1970] that in fact one can take γ_0 to be zero and consequently the solution u does not have exponential growth in time. It is what we mean by lower exponential strong well-posedness.

We now go back to the problem in the strip namely (1) and derive some consequences of the result of [Kreiss, 1970]. As the operator $L(\partial)$ is assumed to be hyperbolic we have finite speed of propagation so that it is clear (by localization) that impose the uniform Kreiss-Lopatinskii condition on each side of the boundary is a necessary condition for the strong well-posedness of (1) (the definition is the same as for the half space problem except that we now want to control the two traces, namely $u|_{x_d=0}$ and $u|_{x_d=1}$). Moreover the fact that $L(\partial)$ is stable by zero order perturbations directly implies that impose the uniform Kreiss-Lopatinskii condition on each side of the boundary define a strongly well-posed problem.

Let us briefly sketch the proof of the *a priori* energy estimate in such a situation. Let $\chi_0, \chi_1 \in \mathcal{D}([0, 1])$ be such that:

$$\chi_0(x_d) = \begin{cases} 1 & \text{for } x_d \in [0, \frac{1}{4}] \\ 0 & \text{for } x_d \geq \frac{1}{2} \end{cases} \quad \text{and} \quad \chi_1(x_d) = \begin{cases} 1 & \text{for } x_d \in [\frac{3}{4}, 1] \\ 0 & \text{for } x_d \leq \frac{1}{2} \end{cases}$$

and define $\chi_{\frac{1}{2}} = 1 - \chi_0 - \chi_1$. Let u be a regular solution of (1) and define $u_0 := \chi_0 u$, $u_{\frac{1}{2}} := \chi_{\frac{1}{2}} u$ and $u_1 := \chi_1 u$. Clearly, $u = u_0 + u_{\frac{1}{2}} + u_1$, $u|_{x_d=j} = u_j|_{x_d=j}$ for $j = 0, 1$. By hyperbolicity the problem in the full space is strongly well-posed (and even lower exponentially strongly well-posed) so that there exists $C > 0$ such that $\gamma > 0$

$$\begin{aligned} \gamma \|u_{\frac{1}{2}}\|_{L_\gamma^2(\mathbb{R}_t \times \mathbb{R}^{d-1} \times [0, 1])}^2 &\leq \frac{C}{\gamma} \|L(\partial)u_{\frac{1}{2}}\|_{L_\gamma^2(\mathbb{R}_T \times \mathbb{R}^{d-1} \times [0, 1])}^2 \\ &\leq \frac{C}{\gamma} \|L(\partial)u\|_{L_\gamma^2(\mathbb{R}_T \times \mathbb{R}^{d-1} \times [0, 1])}^2 + \frac{C_{A_d}}{\gamma} \|u\|_{L_\gamma^2(\mathbb{R}_T \times \mathbb{R}^{d-1} \times [0, 1])}^2, \end{aligned}$$

by standard estimate on the commutator $[L(\partial), \varphi_{\frac{1}{2}}]$. Consequently, taking γ large enough, we obtain

$$\gamma \|u_{\frac{1}{2}}\|_{L_\gamma^2(\mathbb{R}_t \times \mathbb{R}^{d-1} \times [0, 1])}^2 \leq \frac{C}{\gamma} \|f\|_{L_\gamma^2(\mathbb{R}_t \times \mathbb{R}^{d-1} \times [0, 1])}^2. \quad (4)$$

By the Kreiss-Lopatinskii condition on each side, the same reasoning gives (for γ large enough)

$$\gamma \|u_0\|_{L_\gamma^2(\mathbb{R}_t \times \mathbb{R}^{d-1} \times [0, 1])}^2 + \|u_0|_{x_d=0}\|_{L_\gamma^2(\mathbb{R}_t \times \mathbb{R}^{d-1})}^2 \leq C \left(\frac{1}{\gamma} \|f\|_{L_\gamma^2(\mathbb{R}_t \times \mathbb{R}^{d-1} \times [0, 1])}^2 + \|g_0\|_{L_\gamma^2(\mathbb{R}_t \times \mathbb{R}^{d-1})}^2 \right), \quad (5)$$

and

$$\gamma \|u_1\|_{L_\gamma^2(\mathbb{R}_t \times \mathbb{R}^{d-1} \times [0, 1])}^2 + \|u_1|_{x_d=1}\|_{L_\gamma^2(\mathbb{R}_t \times \mathbb{R}^{d-1})}^2 \leq C \left(\frac{1}{\gamma} \|f\|_{L_\gamma^2(\mathbb{R}_t \times \mathbb{R}^{d-1} \times [0, 1])}^2 + \|g_1\|_{L_\gamma^2(\mathbb{R}_t \times \mathbb{R}^{d-1})}^2 \right). \quad (6)$$

Sum (4)-(5) and (6) gives the desired result that is to say: that there exist $C > 0$ and $\gamma_0 \geq 0$ such that for all $\gamma > \gamma_0$ we have the following *a priori* energy estimate for (1):

$$\begin{aligned} \gamma \|u\|_{L_\gamma^2(\mathbb{R}_t \times \mathbb{R}^{d-1} \times [0, 1])}^2 + \|u|_{x_d=0}\|_{L_\gamma^2(\mathbb{R}_t \times \mathbb{R}^{d-1})}^2 + \|u|_{x_d=1}\|_{L_\gamma^2(\mathbb{R}_t \times \mathbb{R}^{d-1})}^2 \\ \leq C \left(\frac{1}{\gamma} \|f\|_{L_\gamma^2(\mathbb{R}_t \times \mathbb{R}^{d-1} \times [0, 1])}^2 + \|g_0\|_{L_\gamma^2(\mathbb{R}_t \times \mathbb{R}^{d-1})}^2 + \|g_1\|_{L_\gamma^2(\mathbb{R}_t \times \mathbb{R}^{d-1})}^2 \right), \end{aligned} \quad (7)$$

so that the problem in the strip (1) is strongly well-posed. The rest of this paper address the question of the lower exponential strong well-posedness of (1) that is, crudely speaking, "does (7) holds with $\gamma_0 = 0$?"

Before we turn to the description of our main result let us motivate a bit this study. Firstly hyperbolic boundary value problems in the strip are, in the author's knowledge, very little considered in the litterature. The reason of that is probably that, as we have just shown, they are easily solvable by localization. However the localization technique is far to be entirely satisfactory because the maximal time exponential growth given in (7) is a crude one (especially if there is no time exponential growth at all). So a full characterization of the lower exponential strong well-posedness is a first substancial refinement.

More generally, hyperbolic boundary value problems with more than one boundary are also not a lot considered in the litterature. For example, in the quarter space geometry, we refere to the seminal work of [Osher, 1973]-[Osher, 1974b] (see also [Benoit, 2015] and the references therein), the strong well-posedness of the problem is far to be clearly characterized. As the reader will notice, many of the ideas described in this paper are adapted for the ones use in the quarter space geometry. So the author believes that a good understanding of the hyperbolic boundary value problem in the (simpler) geometry of the strip will improve the understanding of the one in the quarter space.

An other interest of this study is linked to the study of the stability of finite difference scheme approximations. Indeed when one implements a numerical scheme to approximate the solution of the Cauchy problem associated to (1) then due to the impossibility to modelized the full space \mathbb{R}^d artificial boundary conditions have to be imposed numerically. Consequently the scheme will be defined on some (possibly big) rectangle.

Let us for simplicity only truncate the space \mathbb{R}^d with respect to one variable, to fix the ideas x_d , then the associated scheme is defined in a strip and it is interesting to determine the boundary conditions leading to the stability of the scheme (without entering into details, let us say that some discretized version of (7) holds (we refer to [Coulombel, 2011b] or [B. Gustafsson and Sundstrom., 1972] for precise estimates)).

The seminal result of [B. Gustafsson and Sundstrom., 1972] ensures that if each boundary condition satisfies some discretized version of the uniform Kreiss-Lopatinskii condition, then the associated scheme is strongly stable (that is that the analogous of (7) holds with some threshold for γ_0). The question of an estimate like (7) with $\gamma_0 = 0$, referred as the P -stability (P stands for practical), is also considered in [Trefethen, 1985] where the author shows that two stable boundary conditions are not sufficient to ensure the P -stability of the associated scheme defined in the strip. In the author's opinion a full characterization of the numerical boundary condition leading to P -stability is an interesting question. Indeed for numerical schemes, compared to the analogous continuous problem, all the components of the traces have to be specified so that some extra (and arbitrary, in the sense that they are not prescribed by the physics of the problem) boundary conditions have to be imposed. Having a characterization allowing to determine if these extra boundary conditions induce an exponential growth of the solution, or not, is thus a natural question that we expect to clarify in a forthcoming contribution.

Unsurprisingly impose the uniform Kreiss-Lopatinskii condition on each side of the boundary will not be sufficient to ensure lower exponential strong well-posedness. Intuitively and in a formal way the reason is the following. Consider the strip problem (1) with $f \equiv 0$ and $g_1 \equiv 0$ and assume that the source term g_0 induces some information traveling to the side $\{x_d = 1\}$ (think, for example, to a transport phenomenon) then this information will hit the side $\{x_d = 1\}$ after a finite time and be reflected back (by the uniform Kreiss-Lopatinskii condition) with some amplification reflection coefficient α_1 . The reflected information travels to the side $\{x_d = 0\}$, it hits it in finite time, and is reflected again with an amplification reflection coefficient α_0 . Now imagine that this reflected information is (or contains) the initial one (this kind of phenomenon was referred as selfinteraction in [Benoit, 2016]), then this process will be repeated periodically in time and one expects that the solution u behaves more or less like $(\alpha_0\alpha_1)^t$. Consequently if $|\alpha_0\alpha_1| > 1$ (that is to say the information is amplified during the two reflections) then u is expected to have a non trivial exponential growth in time and (1) can not be lower exponentially strongly well-posed. As the uniform Kreiss-Lopatinskii condition implies the existence of reflected rays but does not say anything about the associated reflection coefficient we deduce that impose this condition on each side of the boundary is not sufficient for lower exponential strong well-posedness.

Consequently two new invisibility conditions are necessary and (also) sufficient to ensure the lower exponential strong well-posedness. These conditions have to be understood as a non amplification condition for selfinteracting modes and thus have to be compared with the condition described in [Osher, 1973] for the hyperbolic corner problem. The first condition ensures the existence of a weak solution while the second one is used to establish the *a priori* energy estimate so that it gives the uniqueness of the solution.

However the results of Proposition 3.6 and 3.13 indicate that these two conditions seem to be equivalent (we are not so far for the proof of such a result see Proposition 3.6 and 3.13) so that in the rest of the introduction we will only describe the first invisibility condition.

More precisely this new condition in this article involves two operators $T_{0 \rightarrow 1}(\zeta)$ (resp. $T_{1 \rightarrow 0}(\zeta)$), where ζ stands for a frequency parameter, that to the value of the trace of the solution on the side $\{x_d = 0\}$ (resp. $\{x_d = 1\}$) gives in output the value of the trace of the solution on the side $\{x_d = 1\}$ (resp. $\{x_d = 0\}$). The new necessary and sufficient condition for lower exponential strong well-posedness of the hyperbolic boundary value problem in the strip then means that the operator $(I - T_{1 \rightarrow 0}(\zeta)T_{0 \rightarrow 1}(\zeta))$ is uniformly invertible, on some subspace, in terms of the frequency parameter ζ . The operator $T_{0 \rightarrow 1}(\zeta)$ (resp. $T_{1 \rightarrow 0}(\zeta)$) encodes the amplification factor during the reflection of an incoming wave for the side $\{x_d = 1\}$ (resp. $\{x_d = 0\}$) against this side. Consequently to ensure that $(I - T_{1 \rightarrow 0}(\zeta)T_{0 \rightarrow 1}(\zeta))$ it is sufficient to impose that $|T_{1 \rightarrow 0}(\zeta)T_{0 \rightarrow 1}(\zeta)| < 1$, which means the natural condition that to ensure the lower exponential strong well-posedness it is necessary, to prevent some kind of blow-up in time, to impose that the energy amplification of a wave decreases when it comes from an information on the side $\{x_d = 0\}$, hits $\{x_d = 1\}$ and finally goes back to $\{x_d = 0\}$ as described before.

This phenomenon was already present in the geometry of the corner for which some phases can regenerate themselves after a suitable number of reflections against the sides of the corner. However in the geometry of the corner, this phenomenon is a bit hidden in some really specific and probably not so common geometries of the characteristic variety (see [Benoit, 2016] for more details).

Whereas for the problem in the strip, this phenomenon becomes generic. Indeed as soon as the side $\{x_d = 0\}$ generates incoming waves, they will travel to $\{x_d = 1\}$ and be reflected back to $\{x_d = 0\}$ and consequently the initial considered wave will be regenerated after two rebounds. The author expects to give more details about this assertion in a forthcoming contribution dealing with WKB expansions for the strip problem.

The paper is organized as follows: in Section 2 we introduce some notations and definitions and state the main result, that is to say the full characterization of lower exponentially strong well-posed problems.

Section 3 is devoted to the proof of this result. Firstly we give a proof in the more favorable framework of strictly dissipative boundary conditions (see Paragraph 3.1) and then we use this result to deal with the general framework in Paragraphs 3.2-3.3.

The proof of the *a priori* energy estimate that we choose to follow here is in the spirit of the one given by [Osher, 1973] to treat the case of the corner problem and uses symmetrizers as for the half space [Kreiss, 1970]. However, as the reader will notice the geometry of the characteristic variety induced by the strip problem will lead to really simple operators $T_{0 \rightarrow 1}(\zeta)$ (resp. $T_{1 \rightarrow 0}(\zeta)$). These operators involve matrices and not integral Fourier operators as the operators used in [Osher, 1973]. As a matter of fact, we are able to complete the discussion and obtain an *a priori* energy estimate without loss of derivatives which is not the case in [Osher, 1973].

Existence of the solution can be obtained by two distinct ways, the first one is an explicit construction under a stronger version of the invisibility condition and the second one uses a standard duality argument. Finally uniqueness follows from a "weak=strong" lemma which is essentially trivial for the strip geometry. At the end of Section 3, more precisely in Paragraph 3.4, we discuss and illustrate the necessary and sufficient condition for lower exponential strong well-posedness on some examples.

At last Section 4 contains a conclusion, a summary and gives some conjectures that seem to be reasonable.

2 Notations and main results

2.1 Notations and assumptions

Throughout this paper, the strip will be denoted by $\Gamma := \mathbb{R}^{d-1} \times [0, 1]$, the two sides of its boundary will be denoted by $\partial\Gamma_0 := \mathbb{R}^{d-1} \times \{0\}$ and $\partial\Gamma_1 := \mathbb{R}^{d-1} \times \{1\}$.

For $\chi, s \in \mathbb{R}$ and for some Banach space X including the time variable t , we introduce the weighted H^s -weighted (in time) Sobolev spaces spaces by:

$$H_\chi^s(X) := \{u \in \mathcal{D}'(X) \mid ue^{-\chi t} \in H^s(X)\},$$

equipped with the norm $\|\cdot\|_{H_\chi^s(X)} := \|\cdot e^{-\chi t}\|_{H^s(X)}$. We have $(H_{-\chi}^{-s}(X))^* = H_\chi^s(X)$.

We will make the following assumptions about the system (1). The first one is a hyperbolicity assumption while the second one restricts the analysis to non-characteristic boundaries. More precisely:

Assumption 2.1 (Constantly hyperbolic operator) *The system (1) is constantly hyperbolic in the sense that there exists $q \geq 1$, real valued analytic functions $\lambda_1, \dots, \lambda_q$ on $\mathbb{R}^d \setminus \{0\}$ and positive integers μ_1, \dots, μ_q such that:*

$$\forall \xi \in \mathbb{S}^{d-1}, \det \left(\tau + \sum_{j=1}^d \xi_j A_j \right) = \prod_{j=1}^q (\tau + \lambda_j(\xi))^{\mu_j},$$

with $\lambda_1(\xi) < \dots < \lambda_q(\xi)$ and the eigenvalues $\lambda_j(\xi)$ of $\sum_{j=1}^d \xi_j A_j$ are semi-simple.

Assumption 2.2 (Non-characteristic boundaries) *The matrix A_d is invertible. We denote by p its number of strictly positive eigenvalues.*

The matrices $B_0 \in \mathbf{M}_{p \times N}(\mathbb{R})$ and $B_1 \in \mathbf{M}_{(N-p) \times N}(\mathbb{R})$ are of maximal rank.

For technical reasons that will be made precise in the core of the proofs we will in addition to the hyperbolicity Assumption 2.1 assume that the coefficients of (1) are symmetric

Assumption 2.3 (Symmetric operator) *The matrices A_j , $j = 1, \dots, d$ are symmetric.*

As well as the following assumption

Assumption 2.4 *We assume that*

$$\ker B_0 \cap \ker B_1 = \{0\}.$$

Combined with Assumption 2.2 it immediately implies that we have the following decomposition:

$$\ker B_0 \oplus \ker B_1 = \mathbb{R}^N.$$

We introduce the following partition of the frequency space:

$$\Xi := \{\zeta := (\sigma = \gamma + i\tau, \eta) \in \mathbb{C} \times \mathbb{R}^{d-1} \setminus \{\gamma \geq 0\}\} \text{ and } \Xi_0 := \Xi \cap \{\gamma = 0\}.$$

After Laplace transform in time ($t \leftrightarrow \sigma$) and Fourier transform in the tangential space ($x' \leftrightarrow \eta$), denoted by $\widehat{\cdot}$, in the hyperbolic boundary value problem (1) we obtain the following ordinary differential equation in the normal variable x_d :

$$\begin{cases} \frac{d}{dx_d} \widehat{u}(\zeta, x_d) = \mathcal{A}(\zeta) \widehat{u}(\zeta, x_d) + A_d^{-1} \widehat{f}(\zeta, x_d), & \text{for } x_d \in]0, 1[, \\ B_0 \widehat{u}|_{x_d=0}(\zeta) = \widehat{g}_0(\zeta), \\ B_1 \widehat{u}|_{x_d=1}(\zeta) = \widehat{g}_1(\zeta), \end{cases} \quad (8)$$

where $\mathcal{A}(\zeta)$ is the so-called resolvent matrix defined by:

$$\mathcal{A}(\zeta) := -A_d^{-1} \left(\sigma I + i \sum_{j=1}^{d-1} \eta_j A_j \right), \quad (9)$$

and we introduce

$$M(\zeta) := \left(\sigma I + i \sum_{j=1}^{d-1} \eta_j A_j \right). \quad (10)$$

A direct consequence of Assumption 2.1 is the following lemma due to [Hersh, 1963]:

Lemma 2.1 [Hersh, 1963] *For all frequency parameter $\zeta \in \Xi \setminus \Xi_0$, the resolvent matrix $\mathcal{A}(\zeta)$ only admits eigenvalues with non-zero real part, and thus do not has purely imaginary eigenvalues. We denote by $E^s(\zeta)$ (resp. $E^u(\zeta)$), the stable (resp. unstable) space of $\mathcal{A}(\zeta)$ that is the eigenspace associated with the negative (resp. positive) real part eigenvalues. Then independently of ζ , $\dim E^s(\zeta) = p$ and $\dim E^u(\zeta) = N - p$ and we have the following decomposition:*

$$\mathbb{C}^N = E^s(\zeta) \oplus E^u(\zeta). \quad (11)$$

We denote by $\Pi^s(\zeta)$ (resp. $\Pi^u(\zeta)$) the projection upon $E^s(\zeta)$ (resp. $E^u(\zeta)$) with respect to the decomposition (11).

For $\zeta \in \Xi_0$, the decomposition (11) does not hold anymore. However it has been shown by [Kreiss, 1970] for strictly hyperbolic operators and then generalized by [Métivier, 2000] to constanly hyperbolic operators that the stable subspace $E^s(\zeta)$ and the unstable subspace $E^u(\zeta)$ admit continuous extension up to Ξ_0 . It is a consequence of the following Theorem:

Theorem 2.1 (Block structure) *Under Assumptions 2.1-2.2, for all $\zeta \in \Xi$, there exists a neighborhood \mathcal{V} of ζ in Ξ , an integer $L \geq 1$, a partition $N = \mu_1 + \dots + \mu_L$, with $\mu_1, \dots, \mu_L \geq 1$ and an invertible matrix T , regular on \mathcal{V} such that:*

$$\forall \zeta \in \mathcal{V}, T^{-1}(\zeta) \mathcal{A}(\zeta) T(\zeta) = \text{diag}(\mathcal{A}_1(\zeta), \dots, \mathcal{A}_L(\zeta))$$

where the blocks $\mathcal{A}_j(\zeta) \in \mathbf{M}_{\mu_j \times \mu_j}(\mathbb{C})$ satisfy one of the following alternatives:

- i) all the elements in the spectrum of $\mathcal{A}_j(\zeta)$ have positive real part.
- ii) All the elements in the spectrum of $\mathcal{A}_j(\zeta)$ have negative real part.
- iii) $\mu_j = 1$, $\mathcal{A}_j(\zeta) \in i\mathbb{R}$, $\partial_\gamma \mathcal{A}_j(\zeta) \in \mathbb{R} \setminus \{0\}$ and $\mathcal{A}_j(\zeta) \in i\mathbb{R}$ for all $\zeta \in \Xi_0 \cap \mathcal{V}$.
- iv) $\mu_j > 1$ and there exists $k_j \in i\mathbb{R}$ such that

$$\mathcal{A}_j(\zeta) = \begin{bmatrix} k_j & i & 0 \\ & \ddots & i \\ 0 & & k_j \end{bmatrix},$$

the coefficient in the lower left corner of $\partial_\gamma \mathcal{A}_j(\zeta) \in \mathbb{R} \setminus \{0\}$ and for all $\zeta \in \Xi_0 \cap \mathcal{V}$, $\mathcal{A}_j(\zeta) \in i\mathbf{M}_{\mu_j \times \mu_j}(\mathbb{R})$.

An important remark for what follows is this nice property induced by the geometry of the strip. On the one hand, if one considers the boundary value problem in the half space:

$$\begin{cases} \frac{d}{dx_d} \widehat{u}(\zeta, x_d) = \mathcal{A}(\zeta) \widehat{u}(\zeta, x_d) + A_d^{-1} \widehat{f}(\zeta, x_d) & \text{for } x_d \in]0, +\infty[, \\ B_0 \widehat{u}|_{x_d=0}(\zeta) = \widehat{g}_0(\zeta), \end{cases} \quad (12)$$

then the subspaces $E_0^s(\zeta)$ and $E_0^u(\zeta)$ (extended by continuity to Ξ_0) associated to the problem (12) are given by $E_0^s(\zeta) := E^s(\zeta)$ and $E_0^u(\zeta) := E^u(\zeta)$. But on the other hand if one considers the hyperbolic problem in the half space:

$$\begin{cases} \frac{d}{dx_d} \widehat{u}(\zeta, x_d) = \mathcal{A}(\zeta) \widehat{u}(\zeta, x_d) + A_d^{-1} \widehat{f}(\zeta, x_d) & \text{for } x_d \in]-\infty, 1[, \\ B_1 \widehat{u}|_{x_d=1}(\zeta) = \widehat{g}_1(\zeta), \end{cases} \quad (13)$$

and makes the change of variable $\tilde{x}_d = 1 - x_d$, then he recovers the problem:

$$\begin{cases} \frac{d}{d\tilde{x}_d} \widehat{u}(\zeta, \tilde{x}_d) = -\mathcal{A}(\zeta) \widehat{u}(\zeta, \tilde{x}_d) - A_d^{-1} \widehat{f}(\zeta, \tilde{x}_d) & \text{for } \tilde{x}_d \in]0, +\infty[, \\ B_1 \widehat{u}|_{\tilde{x}_d=0}(\zeta) = \widehat{g}_1(\zeta). \end{cases} \quad (14)$$

So from (14) Lemma 2.1 and Theorem 2.1 it is easy to see that the stable subspace $E_1^s(\zeta)$ and the unstable subspace $E_1^u(\zeta)$ associated for (13) (both extended by continuity to Ξ_0) are respectively given by $E_1^s(\zeta) = E^u(\zeta)$ and $E_1^u(\zeta) = E^s(\zeta)$.

In other words, a stable or (outgoing) information for the side $\partial\Gamma_0$ is unstable or (incoming) for the side $\partial\Gamma_1$ and *vice versa*.

With these remarks in hand we can state the definition of the uniform Kreiss-Lopatinskii condition for each side of the strip:

Assumption-Definition 2.1 (Uniform Kreiss-Lopatinskii condition) *For all frequency parameter, $\zeta \in \Xi$, we still denote by $E^s(\zeta)$ and $E^u(\zeta)$ the extension up to Ξ_0 of the stable/unstable space defined in Lemma 2.1, we have:*

$$\ker B_0 \cap E^s(\zeta) = \{0\} \text{ and } \ker B_1 \cap E^u(\zeta) = \{0\}$$

so that we have the decompositions

$$\ker B_0 \oplus E^s(\zeta) = \mathbb{C}^N, \quad (15)$$

$$\ker B_1 \oplus E^u(\zeta) = \mathbb{C}^N. \quad (16)$$

Equivalently the restriction of B_0 (resp. B_1) to the stable subspace $E^s(\zeta)$ (resp. $E^u(\zeta)$) is uniformly invertible in terms of ζ . More precisely there exists $C_0, C_1 > 0$ such that for all $\zeta \in \Xi$:

$$\forall v \in E^s(\zeta), |v| \leq C_0 |B_0 v| \text{ and } \forall w \in E^u(\zeta), |w| \leq C_1 |B_1 w|,$$

where we stress that the constants C_0 and C_1 do not depend on $\zeta \in \Xi$.

We introduce the notations

$$\phi_0(\zeta) := B_0^{-1}|_{E^s(\zeta)} \text{ and } \phi_1(\zeta) := B_1^{-1}|_{E^u(\zeta)}. \quad (17)$$

The classical work of [Kreiss, 1970] states that the boundary value problem in the half space (12) (resp. (13)) is strongly well-posed if and only if the uniform Kreiss-Lopatinskii condition (15) (resp. (16)) is satisfied.

Some parts of the proofs in the article are a bit technical, especially the ones based on several decomposition of the space \mathbb{C}^N . That is why, for the reader's convenience, we summarize some of the projectors used in this article in the following definition

Definition 2.1 (Projectors) *From Lemma 2.1, for all $\zeta \in \Xi \setminus \Xi_0$, we have the decomposition:*

$$\mathbb{C}^N = E^s(\zeta) \oplus E^u(\zeta),$$

and we recall that $\Pi^s(\zeta)$ (resp. $\Pi^u(\zeta)$) is the projector upon $E^s(\zeta)$ (resp. $E^u(\zeta)$) with respect to the decomposition $\mathbb{C}^N = E^s(\zeta) \oplus E^u(\zeta)$.

From Assumption-Definition 2.1, for all $\zeta \in \Xi$, we have the decomposition

$$\mathbb{C}^N = \ker B_0 \oplus E^s(\zeta),$$

and we define by $\bar{\Pi}^s(\zeta)$ the projector upon $E^s(\zeta)$, $\bar{\Pi}_{\ker B_0}(\zeta)$ the projector upon $\ker B_0$ with respect to the decomposition $\mathbb{C}^N = \ker B_0 \oplus E^s(\zeta)$. Similarly we have the decomposition

$$\mathbb{C}^N = \ker B_1 \oplus E^u(\zeta),$$

so that we define by $\bar{\Pi}^u(\zeta)$ the projector upon $E^u(\zeta)$, $\bar{\Pi}_{\ker B_1}(\zeta)$ the projector upon $\ker B_1$ with respect to the decomposition $\mathbb{C}^N = \ker B_1 \oplus E^u(\zeta)$.

Finally from Assumption 2.4, we have the decomposition

$$\mathbb{R}^N = \ker B_0 \oplus \ker B_1,$$

and we define $\Pi_{\ker B_0}$ (resp. $\Pi_{\ker B_1}$) the projector upon $\ker B_0$ (resp. $\ker B_1$) with respect to the decomposition $\mathbb{R}^N = \ker B_0 \oplus \ker B_1$.

To conclude this paragraph we make clear the concepts of well-posedness that we will use in the following definition. Both of them are natural generalizations of the ones introduced in the half space geometry.

Definition 2.2 *i) The hyperbolic boundary value problem in the strip (1) is said to be strongly well-posed if and only if, for all source terms $f \in L^2_\gamma(\Omega)$, $g_0 \in L^2_\gamma(\partial\Omega_0)$ and $g_1 \in L^2_\gamma(\partial\Omega_1)$, the system (1) admits a unique solution $u \in L^2_\gamma(\Omega)$, with traces in $L^2_\gamma(\partial\Omega_0)$ and $L^2_\gamma(\partial\Omega_1)$ satisfying the energy estimate: there exists $C > 0$ and $\gamma_0 \geq 0$ such that for all $\gamma > \gamma_0$:*

$$\gamma \|u\|_{L^2_\gamma(\Omega)}^2 + \|u|_{x_d=0}\|_{L^2_\gamma(\partial\Omega_0)}^2 + \|u|_{x_d=1}\|_{L^2_\gamma(\partial\Omega_1)}^2 \leq C \left(\frac{1}{\gamma} \|f\|_{L^2_\gamma(\Omega)}^2 + \|g_0\|_{L^2_\gamma(\partial\Omega_0)}^2 + \|g_1\|_{L^2_\gamma(\partial\Omega_1)}^2 \right). \quad (18)$$

ii) If the hyperbolic boundary value problem in the strip (1) is strongly well-posed with $\gamma_0 = 0$ then it is said to be lower exponentially strongly well-posed.

We recall that if (18) holds for all $\gamma > \gamma_0$ and $\gamma_0 \neq 0$ then the solution may admit a exponential growth in time whereas when the problem is lower exponentially strongly well-posed such a growth is impossible.

2.2 Main result

The main result of this article is the following. It gives a necessary and sufficient condition for lower exponential strong well-posedness.

Theorem 2.2 *Under Assumptions 2.1-2.2-2.3 and 2.4, the hyperbolic boundary value problem in the strip (1) is lower exponentially strongly well-posed in the sense of Definition 2.2 if and only if each boundary condition satisfies the uniform Kreiss-Lopatinskii condition (see Assumption-Definition 2.1) and if moreover*

the matrix $(I - T_{0 \rightarrow 0}(\zeta)) := (I - T_{1 \rightarrow 0}(\zeta)T_{0 \rightarrow 1}(\zeta))$ (resp. $(I - \bar{T}_{0 \rightarrow 0}(\zeta)) := (I - \bar{T}_{1 \rightarrow 0}(\zeta)\bar{T}_{0 \rightarrow 1}(\zeta))$) where the matrices $T_{0 \rightarrow 1}(\zeta)$ and $T_{1 \rightarrow 0}(\zeta)$ (resp. $\bar{T}_{0 \rightarrow 1}(\zeta)$ and $\bar{T}_{1 \rightarrow 0}(\zeta)$) are respectively given by:

$$T_{0 \rightarrow 1}(\zeta) := \phi_1(\zeta)B_1e^{\mathcal{A}(\zeta)} \text{ and } T_{1 \rightarrow 0}(\zeta) = \phi_0(\zeta)B_0e^{-\mathcal{A}(\zeta)},$$

$$\left(\text{resp. } \bar{T}_{0 \rightarrow 1}(\zeta) := (I - \phi_1(\zeta)B_1)e^{\mathcal{A}(\zeta)} \text{ and } \bar{T}_{1 \rightarrow 0}(\zeta) = (1 - \phi_0(\zeta)B_0)e^{-\mathcal{A}(\zeta)} \right)$$

is uniformly invertible over $E^s(\zeta)$ (resp. $\ker B_0$) compared to the frequency parameter $\zeta \in \Xi \setminus \Xi_0$. That is that there exist $C, \bar{C} > 0$ such that for all $\zeta \in \Xi \setminus \Xi_0$ we have:

$$\forall v \in E^s(\zeta), |v| \leq C|(I - T_{0 \rightarrow 0}(\zeta))v|.$$

$$\forall w \in \ker B_0, |w| \leq \bar{C}|(I - \bar{T}_{0 \rightarrow 0}(\zeta))w|.$$

This theorem has to be compared with the case of a hyperbolic system of equations in a corner domain studied in [Osher, 1973]-[Osher, 1974b]. Indeed, in this setting under a new invertibility condition involving Fourier integral operators and reading under the form $(I - T(\zeta))$, the author manages to construct symmetrizers leading to an *a priori* energy estimate. However, in this analysis the number of losses of derivatives in the energy estimate is non-explicit. This is due to the difficulty to study, with enough precision, the structure of the integral Fourier operator $T(\zeta)$.

In [Benoit, 2015], the author precises the results of [Osher, 1973] – [Osher, 1974b] by showing that if the operator $T(\zeta)$ is uniformly bounded in terms of the parameter ζ then the obtained *a priori* energy estimate is without loss. Moreover, the boundedness (unfortunately not the uniform part of it) of this operator $T(\zeta)$ is also establishes.

For the strip problem the operator $T(\zeta)$, namely $T_{0 \rightarrow 0}(\zeta)$, is not a Fourier integral operator anymore but is just a matrix (see Theorem 2.2). Consequently this boundedness question is much more easy to deal with and permits us to show the lower exponential strong well-posedness of (1) because the *a priori* energy estimate that we are able to show is lossless.

Another difference compared to the quarter space geometry is that for the strip problem we do not ask the invertibility of the operator $(I - T(\zeta))$ on the all space but only on the stable subspace $E^s(\zeta)$. This restriction will be crucial especially to construct a weak solution for (1).

3 Lower exponential strong well-posedness of the problem in a strip

The first step in the study of strong well-posedness for the strip problem (1) is to reduce the analysis to the question of the lower exponential strong well-posedness of the time-space resolvent problem (8). In order to do so we introduce the following concepts of strong well-posedness for the problem (8):

Definition 3.1 *i) Under Assumptions 2.1 and 2.2 we say that the resolvent strip problem (8) is strongly well-posed if for all sources terms $f \in L^2([0, 1], \mathbb{C}^N)$, $g_0, g_1 \in \mathbb{C}^N$, the system (8) admits a unique solution $u \in L^2([0, 1], \mathbb{C}^N)$, with well-defined traces on $\partial\Gamma_0$ and $\partial\Gamma_1$ satisfying the energy estimate: there exists $C, \gamma_0 > 0$, such that for all $\gamma > \gamma_0$ we have:*

$$\gamma \int_0^1 |u(x_d)|^2 dx_d + |u|_{x_d=0}|^2 + |u|_{x_d=1}|^2 \leq C \left(\frac{1}{\gamma} \int_0^1 |f(x_d)|^2 dx_d + |g_0|^2 + |g_1|^2 \right). \quad (19)$$

ii) If the resolvent strip problem is (8) strongly well-posed in the sense of i) and if moreover one can choose $\gamma_0 = 0$ then we say that this problem is lower exponentially strongly well-posed.

The following proposition states the equivalence between the two definitions of strong well-posedness introduced so far. The proof of this proposition is a straightforward generalization of the ones in the half space geometry or in the quarter space geometry and will be omitted here.

Proposition 3.1 *The hyperbolic strip boundary value problem (1) is strongly well-posed (resp. lower exponentially strongly well-posed) in the sense of Definition 2.2 if and only if the time-space resolvent problem is strongly well-posed (resp. lower exponentially strongly well-posed) in the sense of Definition 3.1.*

Consequently (19) written for $f \equiv 0$, $g_1 \equiv 0$ and $f \equiv 0$, $g_0 \equiv 0$ imply that impose the uniform Kreiss-Lopatinskii condition (see Definition 2.1) on each component of the boundary is a necessary condition for the strong well-posedness of (1).

3.1 The case of strictly dissipative boundary conditions

There exists particular boundary conditions for which the proof of the lower exponential strong well-posedness of the strip problem (1) is much more simple. Namely these boundary conditions are the strictly dissipative ones, that is to say the boundary conditions that impose that the energy decreases during a reflection.

In this setting the proof of the *a priori* energy estimate is classical and relies on a standard integration by parts argument. Then the existence of a weak solution comes from a slight adaptation of the duality argument introduced in [Lax and Phillips, 1960] to deal with the geometry of the half space. Finally the uniqueness of the solution is provided by the fact that from a "weak=strong" lemma, the weak solution satisfies the energy estimate (18).

In this paragraph we describe this proof. This is firstly motivated by the fact that this proof in this simplified setting is a good preliminary work to study the proof in the general one. But also because the proof in the general setting is based upon the lower exponential strong well-posedness of the problem in a strip for strictly dissipative boundary conditions (see [Osher, 1973] or [Benoit, 2015] for more details in the quarter space geometry).

Moreover for a fixed symmetric hyperbolic operator $L(\partial)$ strictly dissipative boundary conditions always exist and are algebraically characterized. Consequently these conditions are rather easy to determine compared to the boundary conditions that only satisfy the uniform Kreiss-Lopatinskii condition. Thus the strictly dissipative framework is an interesting one because it gives a non empty and easily determinable set of suitable boundary conditions ensuring lower exponential strong well-posedness.

Before to state the lower exponential strong well-posedness result we give some more details about what we mean by strict dissipativity as well as some classical properties of these particular boundary conditions.

Definition-Proposition 3.1 *The boundary condition on the side $\partial\Gamma_0$ (resp. $\partial\Gamma_1$) is said to be strictly dissipative is the following inequality holds:*

$$\forall v \in \ker B_0 \setminus \{0\}, \langle A_d v, v \rangle < 0 \text{ (resp. } \ker B_1 \setminus \{0\}), \langle A_d v, v \rangle > 0). \quad (20)$$

Or equivalently (see for example [Benzoni-Gavage and Serre, 2007]), the boundary condition on the side $\partial\Gamma_0$ (resp. $\partial\Gamma_1$) is strictly dissipative if and only if there exists $\varepsilon_0, C_0 > 0$ (resp. $\varepsilon_1, C_1 > 0$) such that:

$$\forall v \in \mathbb{R}^N, \varepsilon_0 |v|^2 + \langle A_d v, v \rangle - C_0 |B_0 v|^2 < 0, \text{ (resp. } \varepsilon_1 |v|^2 + \langle A_d v, v \rangle - C_1 |B_1 v|^2 > 0). \quad (21)$$

Under Assumption 2.2 for all fixed operator $L(\partial)$ with symmetric coefficients there exists boundary matrices B_0 and B_1 such that the boundary conditions in (1) are strictly dissipative.

Finally under Assumptions 2.1-2.2 if the boundary condition on the side $\partial\Gamma_0$ (resp. $\partial\Gamma_1$) is strictly dissipative then it satisfies the uniform Kreiss-Lopatinskii condition on the side $\partial\Gamma_0$ (resp. $\partial\Gamma_1$) (we refer, for example, to [Benzoni-Gavage and Serre, 2007]-Paragraph 4.3.4 for a proof).

In particular, we remark that if the boundary conditions in (1) are strictly dissipative then the necessary Assumption 2.4 is trivially satisfied because of (20).

The lower exponential strong well-posedness result for strictly dissipative boundary condition is the following:

Theorem 3.1 *Under Assumptions 2.2-2.3, assume that each boundary condition in (1) is strictly dissipative in the sense of Definition-Proposition 3.1. Then the strip problem (1) is lower exponentially strongly well posed in the sense of Definition 2.2.*

The proof of Theorem 3.1 is given in the following paragraphs.

3.1.1 The *a priori* energy estimate

The main point in the proof of Theorem (3.1) is the following *a priori* energy estimate, based upon an easy integration by parts argument.

Proposition 3.2 (A priori energy estimate) *Under Assumption 2.3, assume that each boundary condition in (1) is strictly dissipative in the sense of Definition-Proposition 3.1, then let u be a regular (for example $H^1(\Omega)$ with decay at infinity in space) solution of (1), then u satisfies the a priori energy estimate that is there exists $C > 0$ such that for all $\gamma > 0$ we have*

$$\gamma \|u\|_{L_\gamma^2(\Omega)}^2 + \|u|_{x_d=0}\|_{L_\gamma^2(\partial\Omega_0)}^2 + \|u|_{x_d=1}\|_{L_\gamma^2(\partial\Omega_1)}^2 \leq C \left(\frac{1}{\gamma} \|f\|_{L_\gamma^2(\Omega)}^2 + \|g_0\|_{L_\gamma^2(\partial\Omega_0)}^2 + \|g_1\|_{L_\gamma^2(\partial\Omega_1)}^2 \right), \quad (22)$$

where we stress that C does not depend on γ .

Proof : Let \widehat{u} be the Laplace in time, Fourier in the x' -space transform of u the regular solution of (1). Then \widehat{u} solves the resolvent strip problem (8).

Multiply the evolution equation of (8) by \widehat{u} and integrate over $[0, 1]$ with respect to the variable x_d gives:

$$\int_0^1 \left\langle A_d \frac{d}{dx_d} \widehat{u}, \widehat{u} \right\rangle dx_d + \int_0^1 \langle M(\zeta) \widehat{u}, \widehat{u} \rangle dx_d = \int_0^1 \langle \widehat{f}, \widehat{u} \rangle dx_d, \quad (23)$$

where we recall that $M(\zeta)$ is defined in (10). By symmetry of A_d we derive the relation:

$$\frac{d}{dx_d} \langle A_d \widehat{u}, \widehat{u} \rangle = 2\text{Re} \left\langle A_d \frac{d}{dx_d} \widehat{u}, \widehat{u} \right\rangle,$$

and from the symmetry of the A_j , $j = 1, \dots, d-1$ and the special form of M we obtain by taking the real part of (23):

$$\langle A_d \widehat{u}|_{x_d=1}, \widehat{u}|_{x_d=1} \rangle - \langle A_d \widehat{u}|_{x_d=0}, \widehat{u}|_{x_d=0} \rangle + 2\gamma \int_0^1 |\widehat{u}|^2 dx_d = 2\text{Re} \int_0^1 \langle \widehat{f}, \widehat{u} \rangle dx_d.$$

Then by strict dissipativity (see Definition-Proposition 3.1) and Cauchy-Schwarz/Young inequalities applied to the right hand side it follows:

$$\gamma \int_0^1 |\widehat{u}|^2 dx_d + |u|_{x_d=0}|^2 + |u|_{x_d=1}|^2 \leq C \left(\frac{1}{\gamma} \int_0^1 |\widehat{f}|^2 dx_d + |\widehat{g}_0|^2 + |\widehat{g}_1|^2 \right).$$

Finally we integrate over \mathbb{R}^{d-1} with respect to the Fourier variable and over \mathbb{R} with respect with the imaginary part of the Laplace variable. By Plancherel formula we obtain that u satisfies (22). □

3.1.2 Construction of a weak solution by duality method

To show the existence of a weak solution of (1) we use a duality method. This method is made possible due to the fact that strictly dissipative boundary conditions are suitable for the dual problem. More precisely we mean that if the primal problem (1) admits strictly dissipative boundary conditions then we are free to define a problem associated to (1) the so-called dual problem which also admits strictly dissipative boundary conditions.

Note that as for the half space geometry, as long as the boundary conditions are strictly dissipative, an other proof based on Hile-Yosida theorem [Yosida, 1995] (we refer to [Benzoni-Gavage and Serre, 2007] Chapter 3 for a precise proof) should also operate and be more straightfull. However as we will see in the general framework (that is to say when boundary conditions are not strictly dissipative anymore), see Paragraph 3.3.3, this duality method will be more relevant, that is why we decided to describ it in the setting of strictly dissipative boundary conditions.

To use this duality method we introduce the dual problem:

Definition 3.2 *The strip problem:*

$$\begin{cases} L^*(\partial)v = f & \text{on } \Omega, \\ C_0 v|_{x_d=0} = g_0 & \text{on } \partial\Omega_0, \\ C_1 v|_{x_d=1} = g_1 & \text{on } \partial\Omega_1, \end{cases} \quad (24)$$

is said to be dual for the strip problem (1) if for all regular u and v (for example $H^1(\Omega)$) we have the duality formula:

$$\langle L(\partial)u, v \rangle - \langle u, L^*(\partial)v \rangle = \sum_{j=0}^1 (-1)^{j+1} (\langle N_j u|_{x_d=j}, C_j v|_{x_d=j} \rangle + \langle B_j u|_{x_d=j}, M_j v|_{x_d=j} \rangle), \quad (25)$$

where the matrices $N_0, C_0, M_1 \in \mathbf{M}_{N-p \times N}(\mathbb{R})$, $N_1, C_1, M_0 \in \mathbf{M}_{p \times N}(\mathbb{R})$, appearing in the right hand side of (25) are characterized by the relations:

$$j = 0, 1, \quad A_d = C_j^T N_j + M_j^T B_j. \quad (26)$$

The following proposition, whose proof is classical (see for example [Benzoni-Gavage and Serre, 2007]) ensures the existence of a dual problem to (1).

Proposition 3.3 *Under Assumptions 2.2 and 2.3 let $L^*(\partial)$ be given by:*

$$L^*(\partial) := -L(\partial),$$

then there exists matrices $N_0, C_0, M_1 \in \mathbf{M}_{N-p \times N}(\mathbb{R})$, $N_1, C_1, M_0 \in \mathbf{M}_{p \times N}(\mathbb{R})$ such that (24) is a dual problem for (1). The matrices C_j are characterized by the relations:

$$\ker C_j = (A_d \ker B_j)^\perp, \quad (27)$$

the matrices N_j are onto and satisfy:

$$\ker B_j \cap \ker N_j = \{0\}.$$

The following result which is a slight modification of the arguments of [Lax and Phillips, 1960] shows that if the strip problem (1) admits strictly dissipative boundary conditions then so do for the associated dual problem (24). More precisely:

Lemma 3.1 *If the strip problem (8) satisfies Assumptions 2.2-2.3 and admits strictly dissipative boundary conditions in the sense of Definition-Proposition 3.1 then for all choices of $C_0 \in \mathbf{M}_{N-p \times N}(\mathbb{R})$, $C_1 \in \mathbf{M}_{p \times N}(\mathbb{R})$ satisfying (27) the dual strip problem (24) satisfies Assumptions 2.2-2.3 and the associated boundary condition are strictly dissipative.*

Proof : The fact that (24) also satisfies Assumptions 2.2-2.3 is clear from the definition of $L^*(\partial)$ and from the characterization (27). We now turn to the strict dissipativity of the boundary conditions.

As $L^*(\partial) = -L(\partial)$ we shall show that:

$$\forall v \in \ker C_0, \langle A_d v, v \rangle > 0, \text{ and } \forall w \in \ker C_1, \langle A_d w, w \rangle < 0.$$

We only show the first inequality, the proof of the second one is totally similar (except for the sign).

By contradiction we assume that there exists $\underline{v} \in \ker C_0$ such that

$$\langle A_d \underline{v}, \underline{v} \rangle \leq 0.$$

Then for all $u \in \ker B_0$ and for all $\lambda \in \mathbb{R}$ we have:

$$\langle A_d(u + \lambda \underline{v}), u + \lambda \underline{v} \rangle = \langle A_d u, u \rangle + 2\lambda \langle A_d u, \underline{v} \rangle + \lambda^2 \langle A_d \underline{v}, \underline{v} \rangle < 0.$$

Indeed from (27) the second term in the right hand side is zero, by strict dissipativity of the boundary condition on $\partial\Gamma_0$ the first one is strictly negative and the last one is negative by construction of \underline{v} .

As we assumed $\ker B_0$ to be of maximal rank for the property $\langle A_d \cdot, \cdot \rangle < 0$ it implies that $u + \lambda \underline{v} \in \ker B_0$ and consequently $\underline{v} \in \ker B_0$. As a consequence we have, by (26), that $A_d \underline{v} = C_0^T N_0 \underline{v}$ and thus

$$\langle A_d \underline{v}, \underline{v} \rangle = \langle C_0 \underline{v}, N_0 \underline{v} \rangle = 0,$$

because $\underline{v} \in \ker C_0$. This contradicts the strict dissipativity of the boundary condition $\partial\Gamma_0$.

□

Lemma 3.1 also admits an other version in the general setting that is the one of boundary conditions that only satisfy the uniform Kreiss-Lopatinskii condition. More precisely, we refer for example to [[Benzoni-Gavage and Serre, 2007]-Paragraph 4.4] that we have the following result:

Lemma 3.2 *Under Assumptions 2.1-2.2 the primal problem satisfies the uniform Kreiss-Lopatinskii condition for the side $\partial\Gamma_0$ (resp. the side $\partial\Gamma_1$) if and only if the dual problem defined in (24) satisfies the backward uniform Kreiss-Lopatinskii condition for the side $\partial\Gamma_0$ (resp. the side $\partial\Gamma_1$).*

More precisely it means that we have the equivalence

$$\ker B_0 \cap E^s(\zeta) = \{0\} \Leftrightarrow \ker C_0 \cap \tilde{E}^s(\zeta) = \{0\},$$

and

$$\ker B_1 \cap E^u(\zeta) = \{0\} \Leftrightarrow \ker C_1 \cap \tilde{E}^u(\zeta) = \{0\},$$

where $\tilde{E}^s(\zeta)$ and $\tilde{E}^u(\zeta)$ stand for the stable/unstable subspaces associated to $\tilde{\mathcal{A}}(\sigma, \eta) := \mathcal{A}(-\sigma, \eta)$ the resolvent matrix of (24).

Consequently we define for the dual problem (24) we define the inverses $\tilde{\phi}_0(\zeta) := C_{0|\tilde{E}^s(\zeta)}$ and $\tilde{\phi}_1(\zeta) := C_{1|\tilde{E}^u(\zeta)}$

This result will be the keystone in the construction of the symmetrizer in the general setting and also to construct a weak solution by duality method.

We now go back to the particular case of strictly dissipative boundary conditions and we show that Lemma 3.1 implies the existence of a weak solution $u \in L^2_\gamma(\Omega)$ of (1) to conclude this paragraph.

For simplicity, for $\chi \in \mathbb{R}$, we introduce the functional space \mathcal{H}_χ by

$$\mathcal{H}_\chi := L^2_\chi(\Omega) \times L^2_\chi(\partial\Omega_0) \times L^2_\chi(\partial\Omega_1),$$

which is a Hilbert space with the product norm:

$$\|\cdot\|_{\mathcal{H}_\chi} := \sqrt{\|\cdot\|_{L^2_\chi(\Omega)}^2 + \|\cdot\|_{L^2_\chi(\partial\Omega_0)}^2 + \|\cdot\|_{L^2_\chi(\partial\Omega_1)}^2}.$$

Proposition 3.4 *Under Assumptions 2.2-2.3 if the boundary conditions of the strip problem (1) are strictly dissipative in the sense of Definition-Proposition 3.1, then the strip problem (1) admits a weak solution (in the distributional sense) $u \in L^2_\gamma(\Omega)$.*

The proof of Proposition 3.4 is rather classical. Indeed it is a straightforward adaptation of the one for the boundary value problem in the half space given, for example, in [[Benzoni-Gavage and Serre, 2007]-Paragraph 4.5.3, see also [Sarason, 1962] for a slight modification of this proof]. However we recall this proof for a sake of completeness.

Proof : We define

$$X_{-\gamma} := \{L^*(\partial)v \text{ for } v \in H^1_{-\gamma}(\Omega) \text{ satisfying } C_0 v|_{x_d=0} = 0, C_1 v|_{x_d=1} = 0\}.$$

Using Lemma 3.1, the dual problem (24) satisfies the assumptions of Proposition 3.2. Consequently, as $v \in H^1_{-\gamma}(\Omega)$ satisfies $C_0 v|_{x_d=0} = C_1 v|_{x_d=1} = 0$ we deduce that we have the energy estimate:

$$\gamma \|v\|_{L^2_{-\gamma}(\Omega)}^2 + \|v|_{x_d=0}\|_{L^2_{-\gamma}(\partial\Omega_0)}^2 + \|v|_{x_d=1}\|_{L^2_{-\gamma}(\partial\Omega_1)}^2 \leq \frac{C}{\gamma} \|L^*(\partial)v\|_{L^2_{-\gamma}(\Omega)}^2,$$

that is to say

$$\|v\|_{\mathcal{H}_{-\gamma}} \leq C_\gamma \|L^*(\partial)v\|_{L^2_{-\gamma}(\Omega)}. \quad (28)$$

Let $(f, g_0, g_1) \in \mathcal{H}_\gamma$ we define a linear form ℓ on $X_{-\gamma}$ by:

$$\ell(L^*(\partial)v) := \langle f, L^*(\partial)v \rangle - \langle g_1, M_1 v|_{x_d=1} \rangle + \langle g_0, M_0 v|_{x_d=1} \rangle. \quad (29)$$

Then combining Cauchy-Schwarz inequality and (28) we deduce that:

$$\begin{aligned} |\ell(L^*(\partial)v)| &\leq C \left(\|f\|_{L_\gamma^2(\Omega)} \|v\|_{L_{-\gamma}^2(\Omega)} + \|g_0\|_{L_\gamma^2(\partial\Omega_0)} \|v|_{x_d=0}\|_{L_{-\gamma}^2(\partial\Omega_0)} + \|g_1\|_{L_\gamma^2(\partial\Omega_1)} \|v|_{x_d=1}\|_{L_{-\gamma}^2(\partial\Omega_1)} \right) \\ &\leq C_\gamma \|L^*(\partial)v\|_{L_{-\gamma}^2(\Omega)}. \end{aligned}$$

Consequently, from Hahn-Banach theorem, ℓ can be extended by continuity up to $L_{-\gamma}^2(\Omega)$ and by Riesz representation theorem, there exists $\underline{u} \in L_\gamma^2(\Omega)$ such that for all $L^*(\partial)v \in L_{-\gamma}^2(\Omega)$ we have:

$$\ell(L^*(\partial)v) = \langle \underline{u}, L^*(\partial)v \rangle,$$

which by definition of ℓ is equivalent to:

$$\langle f, L^*(\partial)v \rangle - \langle \underline{u}, L^*(\partial)v \rangle = \langle g_1, M_1 v|_{x_d=1} \rangle - \langle g_0, M_0 v|_{x_d=1} \rangle.$$

For $u \in L_\gamma^2(\Omega)$, the traces $u|_{x_d=0} \in H_\gamma^{-\frac{1}{2}}(\partial\Omega_0)$ and $u|_{x_d=1} \in H_\gamma^{-\frac{1}{2}}(\partial\Omega_1)$ are well-defined and by Green formula we have that for all $v \in H_{-\gamma}^1(\Omega)$:

$$\begin{aligned} \langle L(\partial)u, v \rangle_{H_\gamma^{-1}(\Omega); H_{-\gamma}^1(\Omega)} - \langle u, L^*(\partial)v \rangle &= \sum_{j=0}^1 (-1)^{j+1} \left(\langle B_j u|_{x_d=j}, M_j v|_{x_d=j} \rangle_{H_\gamma^{-\frac{1}{2}}(\partial\Omega_j); H_\gamma^{\frac{1}{2}}(\partial\Omega_j)} \right. \\ &\quad \left. + \langle N_j u|_{x_d=j}, C_j v|_{x_d=j} \rangle_{H_\gamma^{-\frac{1}{2}}(\partial\Omega_j); H_\gamma^{\frac{1}{2}}(\partial\Omega_j)} \right). \quad (30) \end{aligned}$$

Testing (30) written for $u = \underline{u}$, against functions $L^*(\partial)v$ where $v \in \mathcal{D}(\Omega)$ gives that $L(\partial)\underline{u} = f$ in the distributional sense and that $L(\partial)\underline{u} \in L_\gamma^2(\Omega)$. Next test (30) against functions in $L^*(\partial)v$ where $v \in \mathcal{D}(\mathbb{R} \times \mathbb{R}^{d-1} \times [0, 1])$ satisfies $v|_{x_d=0} \in \ker C_0$ (resp. $v \in \mathcal{D}(\mathbb{R} \times \mathbb{R}^{d-1} \times]0, 1])$ satisfies $v|_{x_d=1} \in \ker C_1$) gives

$$\left\langle g_0 - B_0 \underline{u}|_{x_d=0}, M_0 v|_{x_d=0} \right\rangle_{H_\gamma^{-\frac{1}{2}}(\partial\Omega_0); H_\gamma^{\frac{1}{2}}(\partial\Omega_0)} = 0, \left(\text{resp. } \left\langle B_1 \underline{u}|_{x_d=1} - g_1, M_1 v|_{x_d=1} \right\rangle_{H_\gamma^{-\frac{1}{2}}(\partial\Omega_1); H_\gamma^{\frac{1}{2}}(\partial\Omega_1)} = 0 \right). \quad (31)$$

Using the characterization $\ker C_j = (A_d \ker B_j)^\perp$, the fact that B_j is onto we deduce that $M_j|_{\ker C_j}$ is onto so that the test function in (31) can be replaced by any test function. Consequently $B_0 \underline{u}|_{x_d=0} = g_0$ and $B_1 \underline{u}|_{x_d=1} = g_1$ in the distributional sense and $B_0 \underline{u}|_{x_d=0} \in L_\gamma^2(\partial\Omega_0)$, $B_1 \underline{u}|_{x_d=1} \in L_\gamma^2(\partial\Omega_1)$, which concludes the existence of a weak solution for (1). \square

3.1.3 The "weak=strong" lemma

To conclude the proof of Theorem 3.1 it remains to show that the weak solution constructed in Proposition 3.4 admits the extra trace regularity (that is $u|_{x_d=j} \in L_\gamma^2(\partial\Omega_j)$) and that it satisfies the energy estimate (22), which immediately implies uniqueness by linearity.

The interesting point here is that, due to the fact that the strip geometry "looks like" the half space geometry, the classical proof for the half space (see, for example [[Benzoni-Gavage and Serre, 2007], Lemma 4.7] or [Chazarain and Piriou, 1981]) can be used to show these points in the strip geometry¹. More precisely, we have:

Lemma 3.3 ("Weak=strong") *Let $u \in L_\gamma^2(\Omega)$ be a solution of (1) such that $f := L(\partial)u \in L_\gamma^2(\Omega)$, $g_0 = (B_0 u)|_{x_d=0} \in L^2(\partial\Omega_0)$ and $g_1 = (B_1 u)|_{x_d=1} \in L^2(\partial\Omega_1)$ then:*

¹Note that this was not the case for the quarter space geometry in which it is not possible to regularized the solution without modify the value of (at least) one of its traces. However in the quarter space geometry this analysis can also be done by modifying the notion of weak solution and by using the "weak=strong" lemma of [Sarason, 1962]. We refer to [Benoit, 2015]-Chapter 4 for a precise construction.

- $u|_{x_d=0} \in L^2(\partial\Omega_0)$ and $u|_{x_d=1} \in L^2(\partial\Omega_1)$.
- u satisfies (22).

Note that this "weak=strong" lemma does not depend on the kind of considered boundary conditions. As a consequence, Lemma 3.3 can also be used in the general framework of non strictly dissipative boundary conditions.

Proof : Following [Benzoni-Gavage and Serre, 2007], it is sufficient to consider the case where $u \in L^2_\gamma(\Omega)$ has a compact support. Let ρ^ε be a mollifier in the tangential variables that is (t, x') then $u^\varepsilon := \rho^\varepsilon *_{(t, x')} u$ is a compactly supported function which is smooth in (t, x') .

The same holds for $L(\partial)u^\varepsilon = \rho^\varepsilon *_{(t, x')} L(\partial)u \in L^2_\gamma(\Omega)$ and also for $(B_0u^\varepsilon)|_{x_d=0} = \rho^\varepsilon *_{(t, x')} (B_0u)|_{x_d=0} \in L^2_\gamma(\partial\Omega_0)$, $(B_1u^\varepsilon)|_{x_d=1} = \rho^\varepsilon *_{(t, x')} (B_1u)|_{x_d=1} \in L^2_\gamma(\partial\Omega_1)$ because ρ does not depend on x_d .

Let $j = 1, \dots, d-1$ then we have $\partial_j u^\varepsilon = (\partial_j \rho) *_{(t, x')} u$ so that $\partial_j u^\varepsilon \in L^2_\gamma(\Omega)$. The same holds for $\partial_t u^\varepsilon$. Consequently the trivial identity

$$\partial_d u^\varepsilon = A_d^{-1} \left[L(\partial)u^\varepsilon - \partial_t u^\varepsilon - \sum_{j=1}^{d-1} A_j \partial_j u^\varepsilon \right],$$

implies that $\partial_d u^\varepsilon \in L^2_\gamma(\Omega)$ so that $u^\varepsilon \in H^1_\gamma(\Omega)$. We deduce that u^ε satisfies (22) and using the classical properties of mollifiers we obtain, by passing to the limit in (22), that u satisfies (22) and that $u|_{x_d=0} \in L^2_\gamma(\partial\Omega_0)$, $u|_{x_d=1} \in L^2_\gamma(\partial\Omega_1)$. □

This Lemma concludes the proof of Theorem 3.1.

3.2 Necessary conditions for strong well-posedness in the general setting

To derive necessary conditions for strong well-posedness we consider the problem:

$$\begin{cases} \frac{d}{dx_d} u = \mathcal{A}(\zeta)u & \text{for } x_d \in]0, 1[, \\ B_0 u|_{x_d=0} = g, \\ B_1 u|_{x_d=1} = 0, \end{cases} \quad (32)$$

which is assumed to be strongly well-posed and we extend the solution by zero for $x_d \geq 1$. The extension, denoted by v , satisfies:

$$\begin{cases} \frac{d}{dx_d} v = \mathcal{A}(\zeta)v - \delta_{x_d=1} v|_{x_d=1}, & \text{for } x_d \in]0, +\infty[, \\ B_0 v|_{x_d=0} = g. \end{cases}$$

Then we apply Duhamel's formula to obtain:

$$\Pi^s(\zeta)v = e^{x_d \mathcal{A}(\zeta)} \Pi^s(\zeta)v|_{x_d=0} - \int_0^{x_d} \delta_{|s=1} e^{(x_d-s)\mathcal{A}(\zeta)} ds \Pi^s(\zeta)v|_{x_d=1}, \quad (33)$$

$$\Pi^u(\zeta)v = \int_{x_d}^{\infty} \delta_{|s=1} e^{(x_d-s)\mathcal{A}(\zeta)} ds \Pi^u(\zeta)v|_{x_d=1}. \quad (34)$$

Thanks to the uniform Kreiss-Lopatinskii condition on the side $\partial\Gamma_0$ we have:

$$\Pi^s(\zeta)v|_{x_d=0} = \phi_0(\zeta)g - \phi_0(\zeta)B_0\Pi^u(\zeta)v|_{x_d=0},$$

which combined with (33)-(34) gives:

$$\begin{aligned} u|_{x_d=0} = v|_{x_d=0} &= \phi_0(\zeta)g - (I - \phi_0(\zeta)B_0) \int_0^{+\infty} \delta_{|s=1} e^{-s\mathcal{A}(\zeta)} ds \Pi^u(\zeta)u|_{x_d=1}, \\ &= \phi_0(\zeta)g + (\phi_0(\zeta)B_0 - I) e^{-\mathcal{A}(\zeta)} \Pi^u(\zeta)u|_{x_d=1}. \end{aligned} \quad (35)$$

Then we extend the solution u by zero for $x_d \leq 0$ to derive a relation between $u|_{x_d=1}$ and $u|_{x_d=0}$. More precisely, by repeating exactly the same kind of computations, we obtain:

$$u|_{x_d=1} = (\phi_1(\zeta)B_1 - I)e^{-\mathcal{A}(\zeta)}\Pi^s(\zeta)u|_{x_d=0}. \quad (36)$$

Combine (35) and (36) then leads to the compatibility condition:

$$\begin{aligned} u|_{x_d=0} &= \phi_0(\zeta)g + (\phi_0(\zeta)B_0 - I)e^{-\mathcal{A}(\zeta)}\Pi^u(\zeta)(\phi_1(\zeta)B_1 - I)e^{-\mathcal{A}(\zeta)}\Pi^s(\zeta)u|_{x_d=0}, \\ &= \phi_0(\zeta)g + (\phi_0(\zeta)B_0 - I)e^{-\mathcal{A}(\zeta)}\Pi^u(\zeta)\phi_1(\zeta)B_1e^{-\mathcal{A}(\zeta)}\Pi^s(\zeta)u|_{x_d=0}, \\ &\quad - (\phi_0(\zeta)B_0 - I)e^{-\mathcal{A}(\zeta)}\underbrace{\Pi^u(\zeta)e^{\mathcal{A}(\zeta)}\Pi^s(\zeta)}_{=0}u|_{x_d=0} \\ &= \phi_0(\zeta)g - (I - \phi_0(\zeta)B_0)e^{-\mathcal{A}(\zeta)}\phi_1(\zeta)B_1e^{-\mathcal{A}(\zeta)}\Pi^s(\zeta)u|_{x_d=0}, \end{aligned} \quad (37)$$

because for all $\zeta \in \Xi$, $\text{Ran}(\phi_1(\zeta)) = E^u(\zeta)$.

For simplicity, let:

$$T_{0 \rightarrow 1}(\zeta) = \phi_1(\zeta)B_1e^{-\mathcal{A}(\zeta)} \text{ and } T_{1 \rightarrow 0}(\zeta) = \phi_0(\zeta)B_0e^{-\mathcal{A}(\zeta)}, \quad (38)$$

in such a way that (37) becomes:

$$(I - T_{1 \rightarrow 0}(\zeta)T_{0 \rightarrow 1}(\zeta))\Pi^s(\zeta)u|_{x_d=0} + \Pi^u(\zeta)u|_{x_d=0} + e^{-\mathcal{A}(\zeta)}\phi_1(\zeta)B_1e^{-\mathcal{A}(\zeta)}\Pi^s(\zeta)u|_{x_d=0} = \phi_0(\zeta)g.$$

On the one hand, by identification on $E^s(\zeta)$ and $E^u(\zeta)$, it follows (by using the fact that $E^u(\zeta)$ is a stable space of $e^{-\mathcal{A}(\zeta)}$):

$$\begin{cases} (I - T_{1 \rightarrow 0}(\zeta)T_{0 \rightarrow 1}(\zeta))\Pi^s(\zeta)u|_{x_d=0} = \phi_0(\zeta)g, \\ \Pi^u(\zeta)u|_{x_d=0} = -e^{-\mathcal{A}(\zeta)}\phi_1(\zeta)B_1e^{-\mathcal{A}(\zeta)}\Pi^s(\zeta)u|_{x_d=0}, \end{cases} \quad (39)$$

from which we deduce that the unstable component of the trace of the solution u is uniquely determined in terms of the stable component.

Then the first equation of (39) immediately implies that for all $\zeta \in \Xi \setminus \Xi_0$, the operator $I - T_{1 \rightarrow 0}(\zeta)T_{0 \rightarrow 1}(\zeta) : E^s(\zeta) \rightarrow E^s(\zeta)$ is invertible and thus determines in a unique way the value of $\Pi^s(\zeta)u|_{x_d=0}$. Consequently, the unstable part of the trace of the solution is uniquely determined in terms of g by:

$$\Pi^u(\zeta)u|_{x_d=0} = -e^{-\mathcal{A}(\zeta)}\phi_1(\zeta)B_1e^{-\mathcal{A}(\zeta)}(I - T_{1 \rightarrow 0}(\zeta)T_{0 \rightarrow 1}(\zeta))_{|E^s(\zeta)}^{-1}\phi_0(\zeta)g,$$

from which we deduce that

$$u|_{x_d=0} = \left(I - e^{-\mathcal{A}(\zeta)}\phi_1(\zeta)B_1e^{-\mathcal{A}(\zeta)} \right) (I - T_{1 \rightarrow 0}(\zeta)T_{0 \rightarrow 1}(\zeta))_{|E^s(\zeta)}^{-1}\phi_0(\zeta)g. \quad (40)$$

If one reiterates exactly the same reasoning but this time with a non zero source term on the side $\{x_d = 1\}$ instead of the side $\{x_d = 0\}$, then he will recover the following necessary condition for the lower exponential strong well-posedness: for all $\zeta \in \Xi \setminus \Xi_0$, the operator $I - T_{0 \rightarrow 1}(\zeta)T_{1 \rightarrow 0}(\zeta) : E^u(\zeta) \rightarrow E^u(\zeta)$ is invertible.

However, on the other hand, by identification on $E^s(\zeta)$ and $\ker B_0$ (we recall that the associated projectors are defined in Definition 2.1) (37) reads:

$$\begin{cases} \bar{\Pi}^s(\zeta)u|_{x_d=0} = \phi_0(\zeta)g, \\ \bar{\Pi}_{\ker B_0}(\zeta)u|_{x_d=0} = (I - \phi_0(\zeta)B_0)e^{-\mathcal{A}(\zeta)}\phi_1(\zeta)B_1e^{-\mathcal{A}(\zeta)}\Pi^s(\zeta)u|_{x_d=0}, \end{cases} \quad (41)$$

because $\text{Ran}(I - \phi_0(\zeta)B_0) \subset \ker B_0$. The first equation of (41) uniquely determines the stable part of the trace on $\partial\Omega_0$ and we shall make the second one more explicit in terms of $\bar{\Pi}_{\ker B_0}(\zeta)u|_{x_d=0}(\zeta)$. To this aim we decompose:

$$\Pi^s(\zeta)u|_{x_d=0}(\zeta) = \underbrace{\Pi^s(\zeta)\bar{\Pi}^s(\zeta)}_{=\bar{\Pi}^s(\zeta)}u|_{x_d=0}(\zeta) + \Pi^s(\zeta)\bar{\Pi}_{\ker B_0}(\zeta)u|_{x_d=0}(\zeta),$$

in the right hand side of the second equation of (41). However for all $X \in \mathbb{C}^N$ we have the identity:

$$e^{-\mathcal{A}(\zeta)}(I - \phi_1(\zeta)B_1)e^{\mathcal{A}(\zeta)}X = e^{-\mathcal{A}(\zeta)} \underbrace{(I - \phi_1(\zeta)B_1)e^{\mathcal{A}(\zeta)}\Pi^u(\zeta)}_{=0}X + e^{-\mathcal{A}(\zeta)}(I - \phi_1(\zeta)B_1)e^{\mathcal{A}(\zeta)}\Pi^s(\zeta)X,$$

because both spaces $E^s(\zeta)$ and $E^u(\zeta)$ are stable subspaces for $e^{\mathcal{A}(\zeta)}$. So we deduce that the contribution of $\Pi^s(\zeta)\bar{\Pi}_{\ker B_0}(\zeta)u|_{x_d=0}(\zeta)$ in the second equation of (41) is:

$$\begin{aligned} & (I - \phi_0(\zeta)B_0)e^{-\mathcal{A}(\zeta)}\phi_1(\zeta)B_1e^{\mathcal{A}(\zeta)}\Pi^s(\zeta)\bar{\Pi}_{\ker B_0}(\zeta)u|_{x_d=0}(\zeta) \\ &= (I - \phi_0(\zeta)B_0)e^{-\mathcal{A}(\zeta)}(I - \phi_1(\zeta)B_1)e^{\mathcal{A}(\zeta)}\bar{\Pi}_{\ker B_0}(\zeta)u|_{x_d=0}(\zeta) - \underbrace{(I - \phi_0(\zeta)B_0)\Pi^s(\zeta)}_{=0}\bar{\Pi}_{\ker B_0}(\zeta)u|_{x_d=0}. \end{aligned}$$

Consequently the second equation of (41) reads:

$$(I - \bar{T}_{1 \rightarrow 0}(\zeta)\bar{T}_{0 \rightarrow 1}(\zeta))\bar{\Pi}_{\ker B_0}(\zeta)u|_{x_d=0}(\zeta) = (I - \phi_0(\zeta)B_0)e^{-\mathcal{A}(\zeta)}\phi_1(\zeta)B_1e^{\mathcal{A}(\zeta)}\phi_0(\zeta)g, \quad (42)$$

where we set:

$$\bar{T}_{1 \rightarrow 0}(\zeta) := (I - \phi_0(\zeta)B_0)e^{-\mathcal{A}(\zeta)} \text{ and } \bar{T}_{0 \rightarrow 1}(\zeta) := (I - \phi_1(\zeta)B_1)e^{\mathcal{A}(\zeta)}, \quad (43)$$

and where we used the first equation of (41) to express the right hand side of (42) in terms of g . In view of (42) (recall that $(I - \phi_0(\zeta)B_0)E^u(\zeta) = (I - \phi_0(\zeta)B_0)\mathbb{C}^N = \ker B_0$, we assume that the operator $(I - \bar{T}_{1 \rightarrow 0}(\zeta)\bar{T}_{0 \rightarrow 1}(\zeta))$ is invertible on $\ker B_0$ with values in $\ker B_0$ uniformly in terms of $\zeta \in \Xi \setminus \Xi_0$.

Once again, if one reiterates exactly the same reasoning but this time with a non zero source term on the side $\{x_d = 1\}$ instead of the side $\{x_d = 0\}$, then he will recover the following necessary condition for the strong well-posedness: for all $\zeta \in \Xi \setminus \Xi_0$, the operator $I - \bar{T}_{0 \rightarrow 1}(\zeta)\bar{T}_{1 \rightarrow 0}(\zeta) : \ker B_1 \rightarrow \ker B_1$ is invertible, uniformly in terms of ζ .

For simplicity let us define:

$$T_{0 \rightarrow 0}(\zeta) := T_{1 \rightarrow 0}(\zeta)T_{0 \rightarrow 1}(\zeta) \text{ and } T_{1 \rightarrow 1}(\zeta) := T_{0 \rightarrow 1}(\zeta)T_{1 \rightarrow 0}(\zeta),$$

where $T_{0 \rightarrow 1}(\zeta)$ and $T_{1 \rightarrow 0}(\zeta)$ are defined in (38) and

$$\bar{T}_{0 \rightarrow 0}(\zeta) := \bar{T}_{1 \rightarrow 0}(\zeta)\bar{T}_{0 \rightarrow 1}(\zeta) \text{ and } \bar{T}_{1 \rightarrow 1}(\zeta) := \bar{T}_{0 \rightarrow 1}(\zeta)\bar{T}_{1 \rightarrow 0}(\zeta),$$

where $\bar{T}_{0 \rightarrow 1}(\zeta)$ and $\bar{T}_{1 \rightarrow 0}(\zeta)$ are defined in (43).

We summarize this paragraph in the following proposition:

Proposition 3.5 *If the strip problem (32) is strongly well posed then for all $\zeta \in \Xi \setminus \Xi_0$, the operator $(I - T_{0 \rightarrow 0}(\zeta))$ (resp. $(I - T_{1 \rightarrow 1}(\zeta))$) is an isomorphism from $E^s(\zeta)$ to $E^s(\zeta)$ (resp. $E^u(\zeta)$ to $E^u(\zeta)$). Moreover we assume the bounds: for all $\zeta \in \Xi \setminus \Xi_0$;*

$$\forall v \in E^s(\zeta), |v| \leq C_0|(I - T_{0 \rightarrow 0}(\zeta))v|,$$

and

$$\forall w \in E^u(\zeta), |w| \leq C_1|(I - T_{1 \rightarrow 1}(\zeta))w|,$$

where $C_0, C_1 > 0$ do not depend on ζ .

If the strip problem (32) is strongly well posed then for all $\zeta \in \Xi \setminus \Xi_0$, the operator $(I - \bar{T}_{0 \rightarrow 0}(\zeta))$ (resp. $(I - \bar{T}_{1 \rightarrow 1}(\zeta))$) is an isomorphism from $\ker B_0$ to $\ker B_0$ (resp. $\ker B_1$ to $\ker B_1$). Moreover we assume the bounds: for all $\zeta \in \Xi \setminus \Xi_0$;

$$\forall v \in \ker B_0, |v| \leq \bar{C}_0|(I - \bar{T}_{0 \rightarrow 0}(\zeta))v|,$$

and

$$\forall w \in \ker B_1, |w| \leq \bar{C}_1|(I - \bar{T}_{1 \rightarrow 1}(\zeta))w|,$$

where $\bar{C}_0, \bar{C}_1 > 0$ do not depend on ζ .

The end of this section is devoted to the proof of the fact that these necessary conditions are also sufficient conditions for the strip problem (32) to be lower exponentially strongly well posed.

3.3 Sufficient conditions for lower exponential strong well-posedness via symmetrizers

We first make the following assumptions:

Assumption 3.1 For all $\zeta \in \Xi \setminus \Xi_0$ we assume that the operator $(I - T_{0 \rightarrow 0}(\zeta))$ (resp. $(I - T_{1 \rightarrow 1}(\zeta))$) is uniformly invertible (in terms of ζ) from $E^s(\zeta)$ to $E^s(\zeta)$ (resp. $E^u(\zeta)$ to $E^u(\zeta)$).

Assumption 3.2 For all $\zeta \in \Xi \setminus \Xi_0$ we assume that the operator $(I - \bar{T}_{0 \rightarrow 0}(\zeta))$ (resp. $(I - \bar{T}_{1 \rightarrow 1}(\zeta))$) is uniformly invertible (in terms of ζ) from $\ker B_0$ to $\ker B_0$ (resp. $\ker B_1$ to $\ker B_1$).

These assumptions has been shown to be necessary for the lower exponential strong well-posedness of (1) in Paragraph 3.2.

Before to turn to the proof of their sufficiency, let us describe a link between these two assumptions. More precisely we show in the following proposition that Assumption 3.1 implies Assumption 3.2²

Proposition 3.6 If the operator $(I - T_{0 \rightarrow 0}(\zeta))$ is invertible over $E^s(\zeta)$ then the operator $(I - \bar{T}_{0 \rightarrow 0}(\zeta))$ is invertible over $\ker B_0$.

Proof : Consider $v \in \mathbb{C}^N$ such that $v = \bar{T}_{0 \rightarrow 0}(\zeta)v$, then $v \in \ker B_0$ and we decompose $v = \Pi^s(\zeta)v + \Pi^u(\zeta)v$. We compute:

$$\begin{aligned} \bar{T}_{0 \rightarrow 0}(\zeta)v &= (I - \phi_0(\zeta)B_0)e^{-\mathcal{A}(\zeta)}(I - \phi_1(\zeta)B_1)e^{\mathcal{A}(\zeta)}\Pi^s(\zeta)v, \\ &= \underbrace{(I - \phi_0(\zeta)B_0)\Pi^s(\zeta)}_{=0}v - (I - \phi_0(\zeta)B_0)e^{-\mathcal{A}(\zeta)}\phi_1(\zeta)B_1e^{\mathcal{A}(\zeta)}\Pi^s(\zeta)v, \\ &= -e^{-\mathcal{A}(\zeta)}\phi_1(\zeta)B_1e^{\mathcal{A}(\zeta)}\Pi^s(\zeta)v + T_{0 \rightarrow 0}(\zeta)\Pi^s(\zeta)v. \end{aligned}$$

Identifying on $E^s(\zeta)$ and $E^u(\zeta)$ we derive:

$$\begin{cases} (I - T_{0 \rightarrow 0}(\zeta))\Pi^s(\zeta)v = 0, \\ \Pi^u(\zeta)v = -e^{-\mathcal{A}(\zeta)}\phi_1(\zeta)B_1e^{\mathcal{A}(\zeta)}\Pi^s(\zeta)v. \end{cases}$$

By assumption we have $\Pi^s(\zeta)v = \Pi^u(\zeta)v = 0$ and consequently $(I - \bar{T}_{0 \rightarrow 0}(\zeta))$ is invertible over $\ker B_0$. □

Consequently Proposition 3.6 is a strong argument in favor of the fact that Assumption 3.1 implies 3.2. To show such an implication only the uniform bound of the inverse has to be considered and it is left for future studies.

3.3.1 Outline of the proof, and comments about Assumptions 3.1-3.2

As in the proof of the lower exponential strong well-posedness in the strictly dissipative setting the sketch of proof that we follow is a classical proof in three steps. More precisely, the first step is to obtain an *a priori* energy estimate for all regular enough solution of (1). This first step uses Assumption 3.2 but not Assumption 3.1.

Then in a second step we will construct a weak solution of (1). This can be done by two distinct ways: the first one is to construct this weak solution "by hand" and will be done under a stronger assumption than Assumption 3.1, while the second one will use the dual problem (as it has been done in Paragraph 3.1.2). More precisely, we will introduce a dual problem for (1), show the uniqueness of its solution from which we immediately deduce the existence of a weak solution for the primal problem.

The simplest way to show that the dual problem admits a unique solution is of course to show that it satisfies the *a priori* energy estimate and, from the first step of the proof, it will be sufficient that Assumption 3.2 holds for the dual problem. The main point in this proof is the quite surprising fact that Assumption

²In the core of the proof of Theorem 2.2, more precisely in Paragraph 3.3.3, we will obtain that in fact Assumption 3.2 and Assumption 3.1 are equivalent. However, at present time, we do not have a direct proof of the reverse implication.

3.2 for the dual problem is nothing but Assumption 3.1 for the primal problem.

Consequently assume that the primal problem satisfies both Assumptions 3.1 and 3.2 then we have the *a priori* energy estimate and the existence of a weak solution. The "weak=strong" lemma used for the problem with strictly dissipative boundary conditions ends the proof of Theorem 2.2.

3.3.2 The *a priori* energy estimate via symmetrizers

The proof relies on several substeps and follow essentially the ideas developed by [Osher, 1973] to establish *a priori* energy estimate for the corner problem. Let us list the main arguments :

i) The first substep consists in a reduction to a homogeneous problem in the interior and used the fact that the symmetric problem with strictly dissipative boundary conditions is lower exponentially strongly well-posed (see Theorem 3.1).

ii) Then we show *a priori* energy estimate for the reduced problem thanks to symmetrizers. This adopt was already used by [Kreiss, 1970] to treat the problem in the half space.

iii) Finally the last substep is to show that such symmetrizers exist. This point of the proof of the *a priori* energy estimate is the only one that uses Assumption 3.2.

Step 1: Reduction to a homogeneous problem in the interior. The purpose of this paragraph is to show the following:

Proposition 3.7 (Reduction to an homogeneous problem in the interior) *Under Assumptions 2.1, 2.2 and 2.3 consider the strip problem with homogeneous forcing term in the interior, that is*

$$\begin{cases} \frac{d}{dx_d}v = \mathcal{A}(\zeta)v & \text{for } x_d \in]0, 1[, \\ B_0v|_{x_d=0} = g_0(\zeta), \\ B_1v|_{x_d=1} = g_1(\zeta), \end{cases} \quad (44)$$

then the resolvent strip problem (8) is lower exponentially strongly well-posed if and only if the resolvent strip corner problem with homogeneous interior forcing term (44) is lower exponentially strongly well-posed in the sense that it admits a unique solution $v \in L^2([0, 1])$, satisfying the energy estimate:

$$|v|_{x_d=0}|^2 + |v|_{x_d=1}|^2 \leq C (|g_0|^2 + |g_1|^2), \quad (45)$$

where we stress that C does not depend on ζ .

Proof : We only have to show that the lower exponential strong well-posedness of (44) implies the one of (8). To this aim we introduce two strictly dissipative boundary conditions \tilde{B}_0 and \tilde{B}_1 (note that such boundary conditions exist from Definition-Proposition 3.1) and we consider the auxiliary problem:

$$\begin{cases} \frac{d}{dx_d}w(\zeta, x_d) = \mathcal{A}(\zeta)w + A_d^{-1}f(\zeta, x_d) & \text{for } x_d \in]0, 1[, \\ \tilde{B}_0w|_{x_d=0}(\zeta) = 0, \\ \tilde{B}_1w|_{x_d=1}(\zeta) = 0, \end{cases} \quad (46)$$

as the boundary conditions are strictly dissipative Theorem 3.1 applies that is there exists a unique solution $w \in L^2([0, 1])$ of (46). Moreover w satisfies the energy estimate: there exists $C > 0$ such that for all $\gamma > 0$,

$$\gamma \|w\|_{L^2([0,1])}^2 + |w|_{x_d=0}|^2 + |w|_{x_d=1}|^2 \leq \frac{C}{\gamma} \|f\|_{L^2([0,1])}^2. \quad (47)$$

Let v be the unique solution of (44), as we assume lower exponential strong well-posedness, we have, on the one hand (see (45)):

$$|v|_{x_d=0}|^2 + |v|_{x_d=1}|^2 \leq C (|g_0|^2 + |g_1|^2). \quad (48)$$

But on the other hand, test the evolution equation of (44) against v , integrate over $[0, 1]$ and finally taking the real part gives the estimate (see the proof of Proposition 3.2):

$$\gamma \|v\|_{L^2([0,1])}^2 \leq |v|_{x_d=0}|^2 + |v|_{x_d=1}|^2. \quad (49)$$

Let u be defined by $u := w + v$ then $u \in L^2([0, 1])$ is a solution of the strip problem:

$$\begin{cases} \frac{d}{dx_d} u = \mathcal{A}(\zeta)u + A_d^{-1}f(\zeta, x_d) & \text{for } x_d \in]0, 1[, \\ B_0 u|_{x_d=0} = \tilde{g}_0(\zeta), \\ B_1 u|_{x_d=1} = \tilde{g}_1(\zeta), \end{cases}$$

where the source terms on the boundaries are in \mathbb{C}^N and are defined by $\tilde{g}_j := g_j + B_j w|_{x_d=j}$, $j = 0, 1$. Finally, from the energy estimates (47), (48), (49) and the triangle inequality we deduce that:

$$\begin{aligned} \gamma \|u\|_{L^2([0,1])}^2 + |u|_{x_d=0}|^2 + |u|_{x_d=1}|^2 &\leq \gamma \|v\|_{L^2([0,1])}^2 + |v|_{x_d=0}|^2 + |v|_{x_d=1}|^2 \\ &\quad + \gamma \|w\|_{L^2([0,1])}^2 + |w|_{x_d=0}|^2 + |w|_{x_d=1}|^2, \\ &\leq C \left(\frac{1}{\gamma} \|f\|_{L^2([0,1])}^2 + |g_0|^2 + |g_1|^2 \right), \end{aligned}$$

which concluded the proof of the strong well-posedness of (8) (the uniqueness follows from the energy estimate).

□

Step 2: Definition of the symmetrizer and last reduction. In this second substep we show that the energy estimate (19) holds for any regular (let us say $H^1([0, 1])$) solution of (8). To show this result we will need the following object: the so-called symmetrizer³.

Definition 3.3 (Symmetrizer) *We say that $\mathcal{S} : \Xi \setminus \Xi_0 \times [0, 1] \rightarrow \mathbf{M}_{N \times N}(\mathbb{C})$ is a symmetrizer for the resolvent strip problem (8) if it satisfies the following conditions:*

- i) For all $\zeta \in \Xi \setminus \Xi_0$ and for all $x_d \in [0, 1]$, the matrix $\Sigma(\zeta, x_d) := \mathcal{S}(\zeta, x_d)A_d$ is hermitian.*
- ii) For all $\zeta \in \Xi \setminus \Xi_0$, $\Sigma(\zeta, 0)$ (resp. $\Sigma(\zeta, 1)$) is bounded, uniformly in terms of ζ on $E^s(\zeta)$ (resp. $E^u(\zeta)$). In other words there exists $c_0, c_1 > 0$ such that for all $\zeta \in \Xi \setminus \Xi_0$ the following bounds holds:*

$$\forall v \in E^s(\zeta), |\Sigma(\zeta, 0)v| \leq c_0|v| \text{ and } \forall w \in E^u(\zeta), |\Sigma(\zeta, 1)w| \leq c_0|w|.$$

- iii) For all $\zeta \in \Xi \setminus \Xi_0$, $\mathcal{S}(\zeta, \cdot)$ solves the differential equation:*

$$\frac{1}{2} \frac{d}{dx_d} (\mathcal{S}(\zeta, x_d)A_d) + \text{Re}(\mathcal{S}(\zeta, x_d)M(\zeta)) = 0, \text{ for } x_d \in]0, 1[,$$

where we recall that $M(\zeta) = [\sigma + iA'(\eta)]$.

- iv) For $x_d = 0$, for all $v \in \mathbb{C}^N$ and for all $\zeta \in \Xi \setminus \Xi_0$ we have the inequality:*

$$\langle v, \mathcal{S}(\zeta, 0)v \rangle \leq C_0|B_0v|^2 - \varepsilon_0|v|^2,$$

for some constants $\varepsilon_0, C_0 > 0$ that do not depend on ζ .

- v) For $x_d = 1$, for all $w \in \mathbb{C}^N$ and for all $\zeta \in (\Xi \setminus \Xi_0)$ we have the inequality:*

$$-\langle w, \mathcal{S}(\zeta, 1)w \rangle \leq C_1|B_1w|^2 - \varepsilon_1|w|^2,$$

for some constants $\varepsilon_1, C_1 > 0$ that do not depend on ζ .

³The terminology of symmetrizer is possibly not the best one to use. Indeed compared to [Kreiss, 1970], the matrix defined in Definition 3.3 does not symmetrize the operator in the sense that we assumed that the operator satisfies Assumption 2.3. This is due to the fact that *iii)* of Definition 3.3 is based on a equality and not on an inequality allowing to compensate the lack of symmetry of the operator as in [Kreiss, 1970]. That is why the term of "positizer" may be more relevant.

In a first time we assume that there exists a symmetrizer \mathcal{S} for (8) and we conclude this substep by showing that the *a priori* energy estimate (18) is satisfied. The construction of a symmetrizer is postponed to the following substep.

Proposition 3.8 (A priori energy estimate) *Under Assumptions 2.1, and 2.2, and 2.3 if the resolvent strip problem (8) admits a symmetrizer in the sense of Definition 3.3 then the a priori energy estimate (18) is satisfied by all regular enough solution u of (8).*

Proof : From Proposition 3.7 we can assume that $f \equiv 0$, so that we only have to show (18) without the forcing term f in the right hand side and the L^2 -norm of the solution in the left hand side that is:

$$|u|_{x_d=0}|^2 + |u|_{x_d=1}|^2 \leq C (|g_0|^2 + |g_1|^2),$$

where u is a regular solution of (44) or equivalently of:

$$\begin{cases} A_d \frac{d}{dx_d} u + M(\zeta)u = 0 & \text{for } x_d \in]0, 1[, \\ B_0 u|_{x_d=0} = g_0(\zeta), \\ B_1 u|_{x_d=1} = g_1(\zeta). \end{cases}$$

We multiply the evolution equation of this system by $\mathcal{S}(\zeta, x_d)$ and test against u to derive that:

$$\left\langle \Sigma(\zeta, x_d) \frac{d}{dx_d} u, u \right\rangle + \langle \mathcal{S}(\zeta, x_d) M(\zeta) u, u \rangle = 0. \quad (50)$$

Then we use the relation

$$\frac{d}{dx_d} \langle \Sigma(\zeta, x_d) u, u \rangle = \left\langle \frac{d}{dx_d} (\Sigma(\zeta, x_d) u), u \right\rangle + 2 \left\langle \Sigma(\zeta, x_d) \frac{d}{dx_d} u, u \right\rangle,$$

which holds because of point *i*) of Definition 3.3 in (50) and we obtain:

$$\frac{1}{2} \frac{d}{dx_d} \langle \Sigma(\zeta, x_d) u, u \rangle - \frac{1}{2} \left\langle \frac{d}{dx_d} (\Sigma(\zeta, x_d) u), u \right\rangle + \langle \mathcal{S}(\zeta, x_d) M(\zeta) u, u \rangle = 0. \quad (51)$$

From point *iii*) of Definition 3.3, this equation is equivalent to:

$$\frac{d}{dx_d} \langle \Sigma(\zeta, x_d) u, u \rangle = 0.$$

Integrate over $[0, 1]$ gives:

$$\langle \Sigma(\zeta, 1) u|_{x_d=1}, u|_{x_d=1} \rangle = \langle \Sigma(\zeta, 0) u|_{x_d=0}, u|_{x_d=0} \rangle.$$

Finally, using points *iv*) and *v*) of Definition 3.3 we obtain as desired:

$$\varepsilon_0 |u|_{x_d=0}|^2 + \varepsilon_1 |u|_{x_d=1}|^2 \leq C_0 |g_0|^2 + C_1 |g_1|^2.$$

□

Let us remark that in this proof we do not use point *ii*) of Definition 3.3. Indeed, *ii*) of Definition 3.3 is in fact assumed to introduce a new notion of symmetrizer for which the control on the trace on $\partial\Gamma_0$ (resp. $\partial\Gamma_1$) is only imposed on $\ker B_0$ (resp. $\ker B_1$). More precisely we define the following notion of symmetrizer, the reduced symmetrizer:

Definition 3.4 [Reduced symmetrizer] *We say that $\mathcal{S} : (\Xi \setminus \Xi_0) \times [0, 1] \rightarrow \mathbf{M}(\mathbb{C})$ is a reduced symmetrizer for the resolvent strip problem (8) if it satisfies the conditions *i*) – *iii*) of Definition 3.3 and if in addition it satisfies:*

iv') For $x_d = 0$, for all $v \in \ker B_0$ and for all $\zeta \in \Xi \setminus \Xi_0$ we have the inequality:

$$\langle v, \mathcal{S}(\zeta, 0)v \rangle \leq -\varepsilon_0 |v|^2,$$

for some constant $\varepsilon_0 > 0$ that does not depend on ζ .

v') For $x_d = 1$, for all $w \in \ker B_1$ and for all $\zeta \in \Xi \setminus \Xi_0$ we have the inequality:

$$-\langle w, \mathcal{S}(\zeta, 1)w \rangle \leq -\varepsilon_1 |w|^2,$$

for some constant $\varepsilon_1 > 0$ that does not depend on ζ .

The main interest of this reduced symmetrizer is that thanks to point *ii)* of Definition 3.3 the control on the traces can be relaxed to the kernel of the boundary conditions while the reduced symmetrizer still enable to obtain *a priori* energy estimate. This result is given in the following Proposition

Proposition 3.9 *Under Assumptions 2.1, 2.2 and 2.3 if the strip problem (8) admits a reduced symmetrizer in the sense of Definition 3.4 then the a priori energy estimate (45) (and consequently (18)) is satisfied.*

Proof : We reiterate the proof of Proposition 3.8 (note that this part of the proof only require *i)* and *iii)* in Definition 3.3) to derive the equality

$$\langle \Sigma(\zeta, 1)u|_{x_d=1}, u|_{x_d=1} \rangle = \langle \Sigma(\zeta, 0)u|_{x_d=0}, u|_{x_d=0} \rangle.$$

Using point *iv')* of Definition 3.4 for all $u^b \in \ker B_0$, we have $\langle u^b, \mathcal{S}(0, \zeta)u^b \rangle \leq -\varepsilon_0 |u^b|^2$. By the uniform Kreiss-Lopatinskii condition we decompose $u|_{x_d=0} = u^b + u^\sharp$, where $u^b \in \ker B_0$ and $u^\sharp \in E^s(\zeta)$, $B_0 u^\sharp = g_0$.

Then because of point *i)* in Definition 3.3 we have:

$$\begin{aligned} \langle \Sigma(\zeta, 0)u|_{x_d=0}, u|_{x_d=0} \rangle &= \langle \Sigma(\zeta, 0)u^b, u^b \rangle + 2\operatorname{Re} \langle \Sigma(\zeta, 0)u^b, u^\sharp \rangle + \langle u^\sharp, \Sigma(\zeta, 0)u^\sharp \rangle, \\ &\leq -\frac{\varepsilon_0}{2} |u^b|^2 + C_{\varepsilon_0} |\Sigma(\zeta, 0)u^\sharp|^2 + |u^\sharp|^2, \end{aligned}$$

where we used point *iv')* of Definition 3.4 in the first term of the right hand side, Cauchy-Schwarz and Young inequalities in the second and third terms. From point *ii)* in Definition 3.3 we obtain:

$$\langle \Sigma(\zeta, 0)u|_{x_d=0}, u|_{x_d=0} \rangle \leq -\frac{\varepsilon_0}{2} |u^b|^2 + C |u^\sharp|^2,$$

where we stress that the constant C in the right hand side may depend on ε_0 but does not depend on ζ .

Finally by the uniform Kreiss-Lopatinskii condition for the side $\partial\Gamma_0$ we obtain:

$$\langle \Sigma(\zeta, 0)u|_{x_d=0}, u|_{x_d=0} \rangle \leq -\frac{\varepsilon_0}{2} |u^b|^2 + C |B_0 u^\sharp|^2,$$

which is nothing but the inequality *iv)* of Definition 3.3. The derivation of the inequality *v)* of Definition 3.3 follows exactly the same lines and will be omitted here.

We then conclude exactly as in the end of the proof of Proposition 3.8. □

With Proposition 3.9 in hand, to show the *a priori* energy estimate for the strip problem (1) it only remains to construct a reduced symmetrizer in the sense of Definition 3.4. This construction occupies the end of this subsection.

Step 3: Construction of a reduced symmetrizer.

The construction of a reduced symmetrizer for the strip problem has to be compared with the one given by [Osher, 1973] to treat corner problems. In the author opinion, this fact is interesting in its own because it shows that the symmetrizer is not linked to the geometry of the space in which the problem lies but that it is constructed to deal with autointeraction phenomena which are proper to boundary value problems with

several boundaries.

We are looking for a reduced symmetrizer, in the sense of Definition 3.4, under the form:

$$\mathcal{S}(\zeta, x_d) := A_d J(\zeta, x_d) N(\zeta) J^*(\zeta, x_d). \quad (52)$$

A symmetrizer under this special form will of course satisfies the point *i*) of Definition 3.3 provided that

$$N^*(\zeta) = N(\zeta). \quad (53)$$

The first substep of this construction is to choose J in (52) in such a way that point *iii*) of Definition 3.3 is achieved. As we will see, a way to determine such a J involves the operators $\overline{T}_{0 \rightarrow 1}$ and $\overline{T}_{1 \rightarrow 0}$ defined in (43).

Then the second substep is to choose N in (52) as a suitable weight to ensure points *iv'*) and *v'*) of Definition 3.4. An important remark for what follows is that because N will not depend on the variable x_d , it will not destroy the fact that a symmetrizer reading under the form (52) will still satisfy point *iii*) in Definition 3.3. In some sense, N will depend on J (and consequently N will depend on $\overline{T}_{0 \rightarrow 1}$ and $\overline{T}_{1 \rightarrow 0}$) but J will not depend on N , so that we are have enough liberty to choose N after having fixed J .

At last, we check that the constructed \mathcal{S} satisfies point *ii*) of Definition 3.3.

Substep 3.1: Construction of J

A good candidate for J is a solution of the equation: for all $\zeta \in \Xi \setminus \Xi_0$, $x_d \in]0, 1[$

$$\frac{d}{dx_d} A_d J(\zeta, x_d) + M^*(\zeta) J(\zeta, x_d) = 0, \quad (54)$$

or equivalently (recall that A_d is assumed to be symmetric)

$$\frac{d}{dx_d} J^*(\zeta, x_d) A_d + J^*(\zeta, x_d) M(\zeta) = 0. \quad (55)$$

Such a J ensures that \mathcal{S} reading under the form (52) satisfies point *iii*) of Definition 3.3. Indeed we have:

$$\begin{aligned} \frac{d}{dx_d} \mathcal{S}(\zeta, x_d) A_d &= \frac{d}{dx_d} A_d J(\zeta, x_d) N(\zeta) J^*(\zeta, x_d) A_d, \\ &= \frac{d}{dx_d} (A_d J(\zeta, x_d) N(\zeta) J^*(\zeta, x_d) A_d + A_d J(\zeta, x_d) N(\zeta) \frac{d}{dx_d} (J^*(\zeta, x_d) A_d)), \\ &= -M^*(\zeta) J(\zeta, x_d) N(\zeta) J^*(\zeta, x_d) A_d - A_d J(\zeta, x_d) N(\zeta) J^*(\zeta, x_d) M(\zeta), \\ &= -2\text{Re}(\mathcal{S}(\zeta, x_d) M(\zeta)) \end{aligned}$$

that is to say exactly *iii*) of Definition 3.3.

So it remains to construct J as a solution of (54). Note that this equation is equivalent to

$$\frac{d}{dx_d} J(\zeta, x_d) = \widetilde{\mathcal{A}}(\zeta) J(\zeta, x_d),$$

where $\widetilde{\mathcal{A}}(\zeta)$ is the resolvent matrix of the dual problem of (12) given by $\widetilde{\mathcal{A}}(\sigma, \eta) = \mathcal{A}(-\sigma, \eta)$ (see Paragraph 3.1.2). So that $J(\zeta, x_d)$ is a solution of the dual problem.

We choose J under the form $J(\zeta, x_d) := K_0(\zeta, x_d) + K_1(\zeta, x_d)$ where K_0 and K_1 are solutions of:

$$\forall \zeta \in \Xi \setminus \Xi_0, \begin{cases} \frac{d}{dx_d} K_0(\zeta, x_d) = \widetilde{\mathcal{A}}(\zeta) K_0(\zeta, x_d) & \text{for } x_d \geq 0, \\ K_0|_{x_d=0} = \widetilde{\phi}_0(\zeta) C_0 v, \end{cases} \quad (56)$$

and

$$\forall \zeta \in \Xi \setminus \Xi_0, \begin{cases} \frac{d}{dx_d} K_1(\zeta, x_d) = \widetilde{\mathcal{A}}(\zeta) K_1(\zeta, x_d) & \text{for } x_d \leq 1, \\ K_1|_{x_d=1} = \widetilde{\phi}_1(\zeta) C_1 v, \end{cases} \quad (57)$$

where v is a given element of \mathbb{C}^N so that K_0 and K_1 are matrices. In (56)-(57) we recall that the terms $\widetilde{\phi}_0(\zeta)$ and $\widetilde{\phi}_1(\zeta)$ are the inverse given by the uniform Kreiss-Lopatinskii condition applied to the dual problem.

Such a choice of J is perfectly possible because the sum $K_0 + K_1$ is well-defined on $[0, 1]$ and because no boundary conditions are imposed for J in Definition 3.3.

Using Duhamel's formula we define \widetilde{K}_0 a solution of (56) and \widetilde{K}_1 a solution (57) by:

$$\widetilde{K}_0(\zeta, x_d) v = e^{x_d \widetilde{\mathcal{A}}(\zeta)} \widetilde{\phi}_0(\zeta) C_0 v, \quad (58)$$

and

$$\widetilde{K}_1(\zeta, x_d) v = e^{(x_d-1) \widetilde{\mathcal{A}}(\zeta)} \widetilde{\phi}_1(\zeta) C_1 v, \quad (59)$$

So $\widetilde{J} := \widetilde{K}_0 + \widetilde{K}_1$ solves (54). However, in order to make the dependency of $\widetilde{J}(\zeta, 0)$ and $\widetilde{J}(\zeta, 1)$ explicit in terms of $\widetilde{T}_{0 \rightarrow 1}(\zeta)$ and $\widetilde{T}_{1 \rightarrow 0}(\zeta)$ we shall modify a bit the definition of \widetilde{K}_0 and \widetilde{K}_1 . This is made in the following substep after a study of the boundary conditions (iv') and v').

Substep 3.2: The adjoint operator \widetilde{J}^*

With a reduced symmetrizer \mathcal{S} under the form (52), points (iv') and v') of Definition 3.4 read:

$$\begin{cases} \forall v \in \ker B_0, \langle v, A_d \widetilde{J}(\zeta, 0) N(\zeta) \widetilde{J}^*(\zeta, 0) A_d v \rangle \leq -\varepsilon_0 |v|^2, \\ \forall w \in \ker B_1, -\langle w, A_d \widetilde{J}(\zeta, 1) N(\zeta) \widetilde{J}^*(\zeta, 1) A_d w \rangle \leq -\varepsilon_1 |w|^2, \end{cases}$$

or equivalently

$$\begin{cases} \forall v \in \ker B_0, \langle \widetilde{J}^*(\zeta, 0) A_d v, N(\zeta) \widetilde{J}^*(\zeta, 0) A_d v \rangle \leq -\varepsilon_0 |v|^2, \\ \forall w \in \ker B_1, -\langle \widetilde{J}^*(\zeta, 1) A_d w, N(\zeta) \widetilde{J}^*(\zeta, 1) A_d w \rangle \leq -\varepsilon_1 |w|^2. \end{cases} \quad (60)$$

So points (iv') and v') of Definition 3.4 do not involve the operator \widetilde{J} , but only \widetilde{J}^* .

We have (recall the definitions of \widetilde{K}_0 and \widetilde{K}_1 (see (58) and (59)):

$$\widetilde{J}^*(\zeta, 0) = (\widetilde{\phi}_0(\zeta) C_0)^* + \left(e^{-\widetilde{\mathcal{A}}(\zeta)} \widetilde{\phi}_1(\zeta) C_1 \right)^* \quad \text{and} \quad \widetilde{J}^*(\zeta, 1) = \left(e^{\widetilde{\mathcal{A}}(\zeta)} \widetilde{\phi}_0(\zeta) C_0 \right)^* + (\widetilde{\phi}_1(\zeta) C_1)^*.$$

Next we use the following lemmas:

Lemma 3.4 *Let $\widetilde{E}^s(\zeta)$ (resp. $\widetilde{E}^u(\zeta)$) be the stable (resp. unstable) subspace of $\widetilde{\mathcal{A}}(\zeta)$ then for all $\zeta \in \Xi$ we have the equalities:*

$$\widetilde{E}^s(\zeta) = (A_d E^s(\zeta))^\perp \quad \text{and} \quad \widetilde{E}^u(\zeta) = (A_d E^u(\zeta))^\perp.$$

Proof : We refer, for example, to [[Benzoni-Gavage and Serre, 2007]-Paragraph 4.4] for a proof.

Lemma 3.5 *For all $\zeta \in \Xi$ we have the equalities:*

$$(\widetilde{\phi}_0(\zeta) C_0)^* = A_d (I - \phi_0(\zeta) B_0) A_d^{-1} \quad \text{and} \quad (\widetilde{\phi}_1(\zeta) C_0)^* = A_d (I - \phi_1(\zeta) B_1) A_d^{-1}.$$

Proof : We give the poof of the first equality, the proof of the second one follows exactly the same lines.

Firstly we show that:

$$(\widetilde{\phi}_0(\zeta) C_0)^* = A_d \overline{\Pi}_{\ker B_0}(\zeta) A_d^{-1},$$

where we recall that $\overline{\Pi}_{\ker B_0}(\zeta)$ is the projector upon $\ker B_0$ with respect with the decomposition given by the uniform Kreiss-Lopatinskii condition on $\partial\Gamma_0$, that is $\mathbb{C}^N = \ker B_0 \oplus E^s(\zeta)$. By definition of $\widetilde{\phi}_0(\zeta)$, $\widetilde{\phi}_0(\zeta) C_0$

is a projector so it is uniquely determined by its the kernel and its range of $(\widetilde{\phi}_0(\zeta)C_0)^*$. We have by Lemma 3.4:

$$\begin{aligned}\ker(\widetilde{\phi}_0(\zeta)C_0)^* &= (\text{Ran}\widetilde{\phi}_0(\zeta)C_0)^\perp = \widetilde{E}^s(\zeta)^\perp = A_d E^s(\zeta), \\ \text{Ran}(\widetilde{\phi}_0(\zeta)C_0)^* &= (\ker\widetilde{\phi}_0(\zeta)C_0)^\perp = \ker C_0^\perp = A_d \ker B_0,\end{aligned}$$

where we used Lemma 3.4 to derive the first equality and (27) to derive the second one. Consequently we have:

$$(\widetilde{\phi}_0(\zeta)C_0)^* = A_d \overline{\Pi}_{\ker B_0}(\zeta) A_d^{-1}.$$

Then we remark that for all $X \in \mathbb{C}^N$ we have:

$$(I - \phi_0(\zeta)B_0)X = \underbrace{(I - \phi_0(\zeta)B_0)\overline{\Pi}^s(\zeta)X}_{=0} + (I - \phi_0(\zeta)B_0)\overline{\Pi}_{\ker B_0}(\zeta)X = \overline{\Pi}_{\ker B_0}(\zeta)X,$$

from which we immediately deduce that

$$(\widetilde{\phi}_0(\zeta)C_0)^* = A_d(I - \phi_0(\zeta)B_0)A_d^{-1}.$$

□

From Lemma 3.5 we deduce that

$$\begin{aligned}\widetilde{J}^*(\zeta, 0)A_d &= A_d [(I - \phi_0(\zeta)B_0) + \overline{T}_{0 \rightarrow 1}(\zeta)], \\ \widetilde{J}^*(\zeta, 1)A_d &= A_d [\overline{T}_{1 \rightarrow 0}(\zeta) + (I - \phi_1(\zeta)B_1)],\end{aligned}$$

where we recall that $\overline{T}_{0 \rightarrow 1}(\zeta)$ and $\overline{T}_{1 \rightarrow 0}(\zeta)$ are defined in (43). Consequently to conclude the construction of J we define J such that $J^*(\zeta, x_d) = A_d^{-1}\widetilde{J}^*(\zeta, x_d)$, which satisfies (54). Moreover we have the following traces values:

$$J^*(\zeta, 0)A_d = (I - \phi_0(\zeta)B_0) + \overline{T}_{0 \rightarrow 1}(\zeta), \quad (61)$$

$$J^*(\zeta, 1)A_d = \overline{T}_{1 \rightarrow 0}(\zeta) + (I - \phi_1(\zeta)B_1). \quad (62)$$

Once J is fixed, it remains to choose the operator N to ensure (60) (written with J instead of \widetilde{J}). Before to give a precise construction for N , let us be a little more precise about these boundary conditions. Indeed, we remark that the first equation of (60) is assumed to hold for $v \in \ker B_0$, the second one for $w \in \ker B_1$ so that we in fact have:

$$J^*(\zeta, 0)A_d v = (I + \overline{T}_{0 \rightarrow 1}(\zeta))v, \text{ and } J^*(\zeta, 1)A_d w = (\overline{T}_{1 \rightarrow 0}(\zeta) + I)w. \quad (63)$$

The next substep of the proof is devoted to the construction of N to make sure that (60) holds.

Substep 3.3: Construction of N .

We are looking for N under the form:

$$N(\zeta) := \Pi_{\ker B_0}^* N_{00}(\zeta) \Pi_{\ker B_0} + \Pi_{\ker B_1}^* N_{11}(\zeta) \Pi_{\ker B_1} + 2\text{Re}(\Pi_{\ker B_0}^* N_{01}(\zeta) \Pi_{\ker B_1}), \quad (64)$$

where we recall that $\Pi_{\ker B_0}$ (resp. $\Pi_{\ker B_1}$) is the projector upon $\ker B_0$ (resp. $\ker B_1$) with respect with the decomposition⁴ $\mathbb{R}^N = \ker B_0 \oplus \ker B_1$. Note that such a N satisfies (53) independently of N_{00} , N_{11} and N_{01} so that \mathcal{S} defined in (52) satisfies point *i*) of Definition 3.3.

To make the first equation of (60) more explicit we shall compute $N(\zeta)J(\zeta, 0)^*A_d v$ for $v \in \ker B_0$. Because of (63) and (64) (also recall that for all $\zeta \in \Xi$, $\text{Ran}(\overline{T}_{0 \rightarrow 1}(\zeta)) \subset \ker B_1$) we obtain that for all $v \in \ker B_0$

$$\begin{aligned}N(\zeta)J(\zeta, 0)^*A_d v &= \Pi_{\ker B_0}^* N_{00}(\zeta)v + \Pi_{\ker B_1}^* N_{11}(\zeta)\overline{T}_{0 \rightarrow 1}(\zeta)v \\ &\quad - \Pi_{\ker B_0}^* N_{01}(\zeta)\overline{T}_{0 \rightarrow 1}(\zeta)v + \Pi_{\ker B_1}^* N_{01}(\zeta)^* \Pi_{\ker B_0} v,\end{aligned}$$

⁴It is the only point in the proof where we use Assumption 2.4

so that the first equation of (60) becomes:

$$\forall v \in \ker B_0, \quad \left\langle v, \left(N_{00}(\zeta) + \bar{T}_{0 \rightarrow 1}^*(\zeta) N_{11}(\zeta) T_{0 \rightarrow 1}(\zeta) + N_{01}(\zeta) \bar{T}_{0 \rightarrow 1}(\zeta) + \bar{T}_{0 \rightarrow 1}^*(\zeta) N_{01}^*(\zeta) \right) v \right\rangle \leq -\varepsilon_0 |v|^2.$$

The same reasoning applied to the second equation of (60) gives:

$$\forall w \in \ker B_1, \quad - \left\langle w, \left(\bar{T}_{1 \rightarrow 0}^*(\zeta) N_{00}(\zeta) \bar{T}_{1 \rightarrow 0}(\zeta) + N_{11}(\zeta) + \bar{T}_{1 \rightarrow 0}^*(\zeta) N_{01}(\zeta) + N_{01}^*(\zeta) \bar{T}_{1 \rightarrow 0}(\zeta) \right) w \right\rangle \leq -\varepsilon_1 |w|^2.$$

In terms of operators these equations are equivalent to:

$$\begin{cases} N_{00}(\zeta) + \bar{T}_{0 \rightarrow 1}^*(\zeta) N_{11}(\zeta) \bar{T}_{0 \rightarrow 1}(\zeta) + 2\operatorname{Re}(N_{01}(\zeta) \bar{T}_{0 \rightarrow 1}(\zeta)) \leq -I, \\ -\bar{T}_{1 \rightarrow 0}^*(\zeta) N_{00}(\zeta) \bar{T}_{1 \rightarrow 0}(\zeta) - N_{11}(\zeta) - 2\operatorname{Re}(\bar{T}_{1 \rightarrow 0}^*(\zeta) N_{01}(\zeta)) \leq -I, \end{cases} \quad (65)$$

where without loss of generality we assumed that $\varepsilon_0 = \varepsilon_1 = 1$.

To solve (65) we essentially follow the proof given in [Osher, 1973]. Note that in (65) we have three degrees of freedom to solve two equations that is why the choice of the coefficients of $N(\zeta)$ is not unique. To determine good candidates we multiply the second equation of (65) by $\bar{T}_{0 \rightarrow 1}^*(\zeta)$ in the left hand side and by $\bar{T}_{0 \rightarrow 1}(\zeta)$ in the right hand side and then sum to the first equation of (65). We obtain

$$N_{00}(\zeta) - \bar{T}_{0 \rightarrow 0}^*(\zeta) N_{00}(\zeta) \bar{T}_{0 \rightarrow 0}(\zeta) + 2\operatorname{Re}((I - \bar{T}_{0 \rightarrow 0}^*(\zeta)) N_{01}(\zeta) \bar{T}_{0 \rightarrow 1}(\zeta)) \leq -(I + \bar{T}_{0 \rightarrow 1}^*(\zeta) \bar{T}_{0 \rightarrow 1}(\zeta)), \quad (66)$$

where we recall that $\bar{T}_{0 \rightarrow 0}(\zeta) = \bar{T}_{1 \rightarrow 0}(\zeta) \bar{T}_{0 \rightarrow 1}(\zeta)$ and where we used the identity:

$$\begin{aligned} 2\operatorname{Re}(N_{01}(\zeta) \bar{T}_{0 \rightarrow 1}(\zeta)) - 2\bar{T}_{0 \rightarrow 1}^*(\zeta) \operatorname{Re}(\bar{T}_{1 \rightarrow 0}^*(\zeta) N_{01}(\zeta)) \bar{T}_{0 \rightarrow 1}(\zeta) \\ = 2\operatorname{Re}(N_{01}(\zeta) \bar{T}_{0 \rightarrow 1}(\zeta)) - 2\operatorname{Re}(\bar{T}_{0 \rightarrow 1}^*(\zeta) \bar{T}_{1 \rightarrow 0}^*(\zeta) N_{01}(\zeta) \bar{T}_{0 \rightarrow 1}(\zeta)) \\ = 2\operatorname{Re}((I - \bar{T}_{0 \rightarrow 0}(\zeta))^* N_{01}(\zeta) \bar{T}_{0 \rightarrow 1}(\zeta)). \end{aligned}$$

Let formally⁵ $N_{00}(\zeta) := -I$ and $N_{01}(\zeta) := (I - \bar{T}_{0 \rightarrow 0}(\zeta))^{-*} V(\zeta)$ and we choose $V(\zeta)$ in order to ensure the equality in (66). So we shall solve:

$$V(\zeta) \bar{T}_{0 \rightarrow 1}(\zeta) + \bar{T}_{0 \rightarrow 1}^*(\zeta) V^*(\zeta) + \bar{T}_{0 \rightarrow 0}^*(\zeta) \bar{T}_{0 \rightarrow 0}(\zeta) = -\bar{T}_{0 \rightarrow 1}^*(\zeta) \bar{T}_{0 \rightarrow 1}(\zeta),$$

and we verify that $V(\zeta) := -\frac{1}{2} \bar{T}_{0 \rightarrow 1}^*(\zeta) \left(\bar{T}_{1 \rightarrow 0}^*(\zeta) \bar{T}_{1 \rightarrow 0}(\zeta) + I \right)$ is suitable. Under this choice of $N_{00}(\zeta)$ and $N_{01}(\zeta)$ we now determine N_{11} in such a way that (65) is satisfied.

Firstly, we consider the second equation of (65) in which we choose $N_{11}(\zeta)$ in order to impose the equality. It implies that:

$$N_{11}(\zeta) := I - \bar{T}_{1 \rightarrow 0}^*(\zeta) \bar{T}_{1 \rightarrow 0}(\zeta) - 2\operatorname{Re}(\bar{T}_{1 \rightarrow 0}^*(\zeta) N_{01}(\zeta)).$$

Secondly it remains to show that the first equation of (65) is satisfied under this choice of $N_{00}(\zeta)$, $N_{01}(\zeta)$ and $N_{11}(\zeta)$. This equation is equivalent to

$$\bar{T}_{0 \rightarrow 1}^*(\zeta) \left(I - \bar{T}_{1 \rightarrow 0}^*(\zeta) \bar{T}_{1 \rightarrow 0}(\zeta) - 2\operatorname{Re}(\bar{T}_{1 \rightarrow 0}^*(\zeta) N_{01}(\zeta)) \right) \bar{T}_{0 \rightarrow 1}(\zeta) + 2\operatorname{Re}(N_{01}(\zeta) \bar{T}_{0 \rightarrow 1}^*(\zeta)) \leq 0,$$

which can be developed as

$$\begin{aligned} -\bar{T}_{0 \rightarrow 1}^*(\zeta) \bar{T}_{0 \rightarrow 1}(\zeta) + \bar{T}_{0 \rightarrow 0}^*(\zeta) \bar{T}_{0 \rightarrow 0}(\zeta) - \bar{T}_{0 \rightarrow 0}^*(\zeta) N_{01}(\zeta) \bar{T}_{0 \rightarrow 1}(\zeta) - \bar{T}_{0 \rightarrow 1}^*(\zeta) N_{01}^*(\zeta) \bar{T}_{0 \rightarrow 0}(\zeta) \\ + N_{01}(\zeta) \bar{T}_{0 \rightarrow 1}(\zeta) + \bar{T}_{0 \rightarrow 1}^*(\zeta) N_{01}^*(\zeta) \leq 0, \end{aligned}$$

that is

$$-\bar{T}_{0 \rightarrow 1}^*(\zeta) \bar{T}_{0 \rightarrow 1}(\zeta) + \bar{T}_{0 \rightarrow 0}^*(\zeta) \bar{T}_{0 \rightarrow 0}(\zeta) + 2\operatorname{Re}((I - \bar{T}_{0 \rightarrow 0}(\zeta))^* N_{01}(\zeta) \bar{T}_{0 \rightarrow 1}(\zeta)) \leq 0, \quad (67)$$

⁵Note that it is not clear at first glance that such N_{01} is well-defined. It will depend on $\operatorname{Ran} V$, because Assumption 3.2 only ensures that $(I - \bar{T}_{0 \rightarrow 0}(\zeta))^{-1}$ is well-defined on $\ker B_0$

From the definition of $N_{01}(\zeta)$ equation (67) becomes:

$$\begin{aligned} & -\bar{T}_{0 \rightarrow 1}^*(\zeta)\bar{T}_{0 \rightarrow 1}(\zeta) + \bar{T}_{0 \rightarrow 0}^*(\zeta)\bar{T}_{0 \rightarrow 0}(\zeta) + V(\zeta)\bar{T}_{0 \rightarrow 1}(\zeta) + \bar{T}_{0 \rightarrow 1}^*(\zeta)V^*(\zeta) \leq 0, \\ \Leftrightarrow & -2\bar{T}_{0 \rightarrow 1}^*(\zeta)\bar{T}_{0 \rightarrow 1}(\zeta) \leq 0. \end{aligned}$$

Consequently with

$$\begin{aligned} N_{00} & := -I, \quad N_{01}(\zeta) := -\frac{1}{2}(I - \bar{T}_{0 \rightarrow 0}(\zeta))^{-*}\bar{T}_{0 \rightarrow 1}^*(\zeta)(\bar{T}_{1 \rightarrow 0}^*(\zeta)\bar{T}_{1 \rightarrow 0}(\zeta) + I) \\ & \text{and } N_{11}(\zeta) := I - \bar{T}_{1 \rightarrow 0}^*(\zeta)\bar{T}_{1 \rightarrow 0}(\zeta) - 2\text{Re}(\bar{T}_{1 \rightarrow 0}^*(\zeta)N_{01}(\zeta)), \end{aligned} \quad (68)$$

the symmetrizer \mathcal{S} defined in (52), with J characterized as a solution of (54) and N under the form (64) with coefficients defined in (68) solves, at the formal level, points *i*) and *iii*) of Definition 3.3 as well as *iv'*) and *v'*) of Definition 3.4.

Substep 3.4: Justification and proof of point *ii*) in Definition 3.3

To show that \mathcal{S} is a reduced symmetrizer it remains to show that it is well-defined, which amounts to show that N is well-defined, and that it also satisfies *ii*) of Definition 3.3.

Firstly we justify that with the coefficients defined in (68), N is well-defined under Assumption 3.2. By definition, we have:

$$\begin{aligned} N(\zeta) & = -\Pi_{\ker B_0}^* \Pi_{\ker B_0} + \Pi_{\ker B_1}^* \left(I - \bar{T}_{1 \rightarrow 0}^*(\zeta)\bar{T}_{1 \rightarrow 0}(\zeta) - \bar{T}_{1 \rightarrow 0}^*(\zeta)N_{01}(\zeta) - N_{01}^*(\zeta)\bar{T}_{1 \rightarrow 0}(\zeta) \right) \Pi_{\ker B_1} \\ & \quad + \Pi_{\ker B_0}^* N_{01}(\zeta)\Pi_{\ker B_1} + \Pi_{\ker B_1}^* N_{01}^*(\zeta)\Pi_{\ker B_0}, \end{aligned} \quad (69)$$

so that to ensure that $N(\zeta)$ is well-defined we have to make sure that each terms involving $N_{01}(\zeta)$ or $N_{01}^*(\zeta)$ in (69) are well-defined under Assumption 3.2.

Concerning the terms involving $N_{01}^*(\zeta)$, using the fact that $\text{Ran } \bar{T}_{1 \rightarrow 0}(\zeta) \subset \ker B_0$, we only have to justify that $N_{01}^*(\zeta)v$ makes sense for all $v \in \ker B_0$. It is trivially the case under Assumption 3.2 because we have:

$$N_{01}^*(\zeta)v = V^*(\zeta)(I - \bar{T}_{0 \rightarrow 0}(\zeta))^{-1}v.$$

Then we remark that the only terms involving $N_{01}(\zeta)$ in (69) are the terms $\bar{T}_{1 \rightarrow 0}^*(\zeta)N_{01}(\zeta)$ or $\Pi_{\ker B_0}^* N_{01}(\zeta)$. These terms can be rewritten under the form:

$$\bar{T}_{1 \rightarrow 0}^*(\zeta)N_{01}(\zeta) = ((I - \bar{T}_{0 \rightarrow 0}(\zeta))^{-1}\bar{T}_{1 \rightarrow 0}(\zeta))^*V \text{ and } \Pi_{\ker B_0}^* N_{01}(\zeta) = ((I - \bar{T}_{0 \rightarrow 0}(\zeta))^{-1}\Pi_{\ker B_0})^*V,$$

and consequently they are well-defined under Assumption 3.2.

So to show that the symmetrizer \mathcal{S} defined in (52) is a reduced symmetrizer in the sense of Definition 3.4 it remains to show that it satisfies point *ii*) of Definition 3.3. We recall that point *ii*) of Definition 3.3 means that there exists $c_0, c_1 > 0$ such that for all $\zeta \in \Xi \setminus \Xi_0$ we have the bounds:

$$\forall v \in E^s(\zeta), |\mathcal{S}(\zeta, 0)A_d v| \leq c_0|v| \text{ and } \forall w \in E^u(\zeta), |\mathcal{S}(\zeta, 1)A_d w| \leq c_1|w|.$$

From (61) we have that for all $v \in E^s(\zeta)$:

$$J^*(\zeta, 0)A_d v = \underbrace{(I - \phi_0(\zeta)B_0)}_{=0} v + \bar{T}_{0 \rightarrow 1}(\zeta)v,$$

so that, because $\text{Ran } \bar{T}_{0 \rightarrow 1}(\zeta) \subset \ker B_1$, we have:

$$N(\zeta)J^*(\zeta, 0)A_d v = (\Pi_{\ker B_1}^* N_{11}(\zeta) + \Pi_{\ker B_0}^* N_{01}(\zeta))\bar{T}_{0 \rightarrow 1}(\zeta)v,$$

from which we deduce that for all $v \in E^s(\zeta)$

$$\begin{aligned} \mathcal{S}(\zeta, 0)A_d v & = A_d J(\zeta, 0)N(\zeta)J^*(\zeta, 0)A_d v \\ & = A_d \left(\tilde{\phi}_0(\zeta)C_0 + e^{-\tilde{\mathcal{A}}(\zeta)}\tilde{\phi}_1(\zeta)C_1 \right) A_d^{-1} (\Pi_{\ker B_1}^* N_{11}(\zeta) + \Pi_{\ker B_0}^* N_{01}(\zeta))\bar{T}_{0 \rightarrow 1}(\zeta)v. \end{aligned} \quad (70)$$

In order to simplify as much as possible (70) we use the following lemma:

Lemma 3.6 *Let C_0 and C_1 denote the boundary conditions for the dual problem characterized by (27) then we have the decomposition:*

$$\mathbb{R}^N = \ker C_0 \oplus \ker C_1,$$

let $\Pi_{\ker C_1}$ (resp. $\Pi_{\ker C_0}$) be the projector upon $\ker C_1$ (resp. $\ker C_0$) with respect with this decomposition. Then we have the equalities

$$\Pi_{\ker B_1}^* = A_d \Pi_{\ker C_0} A_d^{-1} \text{ and } \Pi_{\ker B_0}^* = A_d \Pi_{\ker C_1} A_d^{-1}.$$

Proof : The proof of this lemma is essentially the same as the proof of Lemma 3.5. However let us give the proof of the first equality for a sake of completeness. Considering that $\Pi_{\ker B_1}^*$ is a projector we just have to determine its kernel and its range. We have:

$$\begin{aligned} \ker \Pi_{\ker B_1}^* &= \ker B_1^\perp = A_d \ker C_1, \\ \text{Ran} \Pi_{\ker B_1}^* &= \ker B_0^\perp = A_d \ker C_0, \end{aligned}$$

where we used the characterization (27). Consequently, we have $\Pi_{\ker B_1}^* = A_d \Pi_{\ker C_0} A_d^{-1}$

□

From Lemma 3.6, (70) becomes for $v \in E^s(\zeta)$

$$\mathcal{S}(\zeta, 0) A_d v = A_d \left(\tilde{\phi}_0(\zeta) C_0 \Pi_{\ker C_1} A_d^{-1} N_{01}(\zeta) + e^{-\tilde{\mathcal{A}}(\zeta)} \tilde{\phi}_1(\zeta) C_1 \Pi_{\ker C_0} A_d^{-1} N_{11}(\zeta) \right) \bar{T}_{0 \rightarrow 1}(\zeta) v. \quad (71)$$

The same kind of computations leads to the fact for all $w \in E^u(\zeta)$

$$\mathcal{S}(\zeta, 1) A_d w = A_d \left(\tilde{\phi}_1(\zeta) C_1 \Pi_{\ker C_0} A_d^{-1} N_{01}^*(\zeta) + e^{\tilde{\mathcal{A}}(\zeta)} \tilde{\phi}_0(\zeta) C_0 \Pi_{\ker C_1} A_d^{-1} \right) \bar{T}_{1 \rightarrow 0}(\zeta) v. \quad (72)$$

So to show that point *ii*) of Definition 3.3 is satisfied, from (71) and (72), the uniform Kreiss-Lopatinskii for the dual problem it is sufficient to show that the terms

$$\Pi_{\ker C_1} A_d^{-1} N_{01}(\zeta) \bar{T}_{0 \rightarrow 1}(\zeta) v, \quad N_{11}(\zeta) \bar{T}_{0 \rightarrow 1}(\zeta) v \text{ for } v \in E^s(\zeta) \quad (73)$$

and

$$N_{01}^*(\zeta) \bar{T}_{1 \rightarrow 0}(\zeta) w \text{ for } w \in E^u(\zeta) \quad (74)$$

are uniformly controlable in terms of v and w respectively.

To treat the first term in (73), from Lemma 3.6 and (68) we write:

$$\Pi_{\ker C_1} A_d^{-1} N_{01}(\zeta) \bar{T}_{0 \rightarrow 1}(\zeta) v = -\frac{1}{2} A_d^{-1} [(I - \bar{T}_{0 \rightarrow 0}(\zeta))^{-1} \Pi_{\ker B_0}^* \bar{T}_{0 \rightarrow 1}(\zeta) (\bar{T}_{1 \rightarrow 0}^*(\zeta) \bar{T}_{1 \rightarrow 0}(\zeta) + I) \bar{T}_{0 \rightarrow 1}(\zeta) v].$$

From Assumption 3.2 the fact that the operators $\bar{T}_{0 \rightarrow 1}^*(\zeta)$ and $\bar{T}_{1 \rightarrow 0}^*(\zeta)$ are respectively given by:

$$\bar{T}_{0 \rightarrow 1}^*(\zeta) = A_d e^{-\tilde{\mathcal{A}}(\zeta)} \tilde{\phi}_1(\zeta) C_1 A_d^{-1} \text{ and } \bar{T}_{1 \rightarrow 0}^*(\zeta) = A_d e^{\tilde{\mathcal{A}}(\zeta)} \tilde{\phi}_0(\zeta) C_0 A_d^{-1},$$

so that (from the fact that $\text{Ran } \tilde{\phi}_0(\zeta) \subset \tilde{E}^s(\zeta)$ and $\text{Ran } \tilde{\phi}_1(\zeta) \subset \tilde{E}^u(\zeta)$) they are uniformly bounded with respect to the parameter ζ (we refer to substep 3.2 or to Paragraph 3.3.3 for explicit computations). We only have to justify that for all $v \in E^s(\zeta)$

$$|\bar{T}_{0 \rightarrow 1}(\zeta) v| \leq C |v| \text{ and } |\bar{T}_{1 \rightarrow 0} \bar{T}_{0 \rightarrow 1}(\zeta) v| \leq C |v|.$$

The first inequality is clear from the definition of $\bar{T}_{0 \rightarrow 1}(\zeta)$ (see (43)) and from the fact that $v \in E^s(\zeta)$, while the second one is not clear at first glance because $\text{Ran } \bar{T}_{0 \rightarrow 1}(\zeta) \subset \ker B_1$. However from the uniform Kreiss-Lopatinskii condition on the side $\partial\Gamma_0$ we have:

$$\begin{aligned} |\bar{T}_{0 \rightarrow 0}(\zeta) v| &\leq C |e^{-\mathcal{A}(\zeta)} (I - \phi_1(\zeta) B_1) e^{\mathcal{A}(\zeta)} v| \leq C \left(|v| + |e^{-\mathcal{A}(\zeta)} \phi_1(\zeta) B_1 e^{\mathcal{A}(\zeta)} v| \right) \\ &\leq C |v|, \end{aligned} \quad (75)$$

because $v \in E^s(\zeta)$ and $\text{Ran } \phi_1(\zeta) \subset E^u(\zeta)$.

Consequently we obtain that:

$$|\Pi_{\ker C_1} A_d^{-1} N_{01}(\zeta) \bar{T}_{0 \rightarrow 1}(\zeta) v| \leq C|v|,$$

which conclude the discussion for the first term in (73). We then turn to the second term in (73). From the definition of $N_{11}(\zeta)$ (see (68)) it reads:

$$N_{11}(\zeta) \bar{T}_{0 \rightarrow 1}(\zeta) v = \left(I - \bar{T}_{1 \rightarrow 0}^*(\zeta) \bar{T}_{1 \rightarrow 0}(\zeta) - \bar{T}_{1 \rightarrow 0}^*(\zeta) N_{01}(\zeta) - N_{01}^*(\zeta) \bar{T}_{1 \rightarrow 0}(\zeta) \right) \bar{T}_{0 \rightarrow 1}(\zeta) v, \quad (76)$$

and we treat each term in (76) separately. The first term in (76) is bounded from the fact that $v \in E^s(\zeta)$, the second one is bounded from (75) and the proof for the third one follows the one of the first term in (73) (with $\bar{T}_{1 \rightarrow 0}^*(\zeta)$ instead of $\Pi_{\ker C_1} A_d^{-1}$) so we essentially only have to treat the last term in (76). From (68) we write:

$$\begin{aligned} N_{01}^*(\zeta) \bar{T}_{1 \rightarrow 0}(\zeta) &= \left(\bar{T}_{1 \rightarrow 0}(\zeta)^* N_{01}(\zeta) \right)^* \\ &= -\frac{1}{2} \left(\left[(I - \bar{T}_{0 \rightarrow 0}(\zeta))^{-1} \bar{T}_{1 \rightarrow 0}(\zeta) \right]^* \bar{T}_{0 \rightarrow 1}^*(\zeta) (I + \bar{T}_{1 \rightarrow 0}^*(\zeta) \bar{T}_{1 \rightarrow 0}(\zeta)) \right)^*, \end{aligned}$$

and we conclude as for the proof of the first term in (73). The proof of the uniform boundedness of (74) follows exactly the same reasoning.

This concludes the proof of the fact that \mathcal{S} defined in (52) satisfies *ii*) of Definition 3.3.

Consequently we have shown the following proposition:

Proposition 3.10 *Under Assumptions 2.1, 2.2, 2.3, 2.4 and 3.2. Also assume that each boundary condition satisfies the uniform Kreiss-Lopatinskii condition then the strip problem (8) admits a reduced symmetrizer in the sense of Definition 3.4.*

From which we immediately deduce the following corollary by using Proposition 3.9.

Corollary 3.1 *Under Assumptions 2.1, 2.2, 2.3, 2.4 and 3.2. Also assume that each boundary condition satisfies the uniform Kreiss-Lopatinskii condition then the strip problem (1) satisfies the a priori energy estimate, that is to say that (18) is satisfied by all regular enough solution of (1).*

3.3.3 Existence of a weak solution

In this paragraph we give two constructions of a weak solution of (8). The first one uses a stronger version of Assumption 3.1 but it has the advantage to give an explicit formula for the solution, whereas the second one only uses Assumption 3.1 and is an adaptation of the construction of the weak solution in the strictly dissipative framework (see Paragraph 3.1).

Construction of a weak solution "by hand" In this paragraph we assume the following:

Assumption 3.3 *The restriction of $T_{0 \rightarrow 0}(\zeta)$ (resp. $T_{1 \rightarrow 1}(\zeta)$) to $E^s(\zeta)$ (resp. $E^u(\zeta)$) is a uniform (in terms of ζ) contraction. More precisely, this means that there exists $C_0 \in]0, 1[$ (resp. $C_1 \in]0, 1[$) such that for all $\zeta \in \Xi \setminus \Xi_0$:*

$$\forall v \in E^s(\zeta) \text{ (resp. } \forall w \in E^u(\zeta)), |T_{0 \rightarrow 0}(\zeta)v| \leq C_0|v| \text{ (resp. } |T_{1 \rightarrow 1}(\zeta)w| < C_1|w|),$$

where we stress that C_0 (resp. C_1) does not depend on ζ .

Proposition 3.11 *Under Assumptions 2.1-3.3, also assume that each boundary condition satisfies the uniform Kreiss-Lopatinskii condition then the problem (1) admits a weak solution u .*

Proof : By linearity we can assume that $g_1 \equiv 0$ and we thus have to consider:

$$\begin{cases} \frac{d}{dx_d} u(\zeta, x_d) = \mathcal{A}(\zeta)u(\zeta, x_d), \text{ for } x_d \in]0, 1[\\ B_0 u|_{x_d=0}(\zeta) = g_0(\zeta), \\ B_1 u|_{x_d=1}(\zeta) = 0. \end{cases} \quad (77)$$

We define u_0 the solution of

$$\begin{cases} \frac{d}{dx_d} u_0(\zeta, x_d) = \mathcal{A}(\zeta)u_0(\zeta, x_d), \text{ for } x_d \geq 0 \\ u_0|_{x_d=0}(\zeta) = \phi_0(\zeta)g(\zeta), \end{cases} \quad (78)$$

Duhamel's formula gives:

$$u_0(\zeta, x_d) = e^{x_d \mathcal{A}(\zeta)} \phi_0(\zeta)g(\zeta), \text{ and consequently } u_0|_{x_d=1}(\zeta) = e^{\mathcal{A}(\zeta)} \phi_0(\zeta)g(\zeta).$$

Then define u_1 the solution of the system:

$$\begin{cases} \frac{d}{dx_d} u_1(\zeta, x_d) = \mathcal{A}(\zeta)u_1(\zeta, x_d), \text{ for } x_d \leq 1 \\ u_1|_{x_d=1}(\zeta) = -\phi_1(\zeta)B_1 u_0|_{x_d=1}(\zeta), \end{cases} \quad (79)$$

once again Duhamel's formula gives:

$$u_1(\zeta, x_d) = -e^{(x_d-1)\mathcal{A}(\zeta)} \phi_1(\zeta)B_1 u_0|_{x_d=1}(\zeta) = -e^{(x_d-1)\mathcal{A}(\zeta)} \phi_1(\zeta)B_1 e^{\mathcal{A}(\zeta)} \phi_0(\zeta)g(\zeta),$$

from which we deduce:

$$u_1|_{x_d=0}(\zeta) = -e^{-\mathcal{A}(\zeta)} \phi_1(\zeta)B_1 e^{\mathcal{A}(\zeta)} \phi_0(\zeta)g(\zeta).$$

Note that by construction we have:

$$u_0|_{x_d=0} + u_1|_{x_d=0} = \left(I - e^{-\mathcal{A}(\zeta)} \phi_1(\zeta)B_1 e^{\mathcal{A}(\zeta)} \right) \phi_0(\zeta)g(\zeta),$$

which has to be compared with (40) (which we recall is expected to give the "true value" of the trace of the solution on $\partial\Gamma_0$) and

$$u_0|_{x_d=1} + u_1|_{x_d=1} = (I - \phi_1(\zeta)B_1) u_0|_{x_d=1},$$

which immediately implies that $B_1(u_0|_{x_d=1} + u_1|_{x_d=1}) = 0$. We then define recursively the sequence $(u_n)_{n \geq 2}$ where u_n is the solution of:

$$\begin{cases} \frac{d}{dx_d} u_n(\zeta, x_d) = \mathcal{A}(\zeta)u_n(\zeta, x_d), \text{ for } x_d \geq 0 \\ u_n|_{x_d=0}(\zeta) = -\phi_0(\zeta)B_0 u_{n-1}|_{x_d=0}(\zeta), \end{cases} \quad \text{if } n \text{ is even,}$$

and

$$\begin{cases} \frac{d}{dx_d} u_n(\zeta, x_d) = \mathcal{A}(\zeta)u_n(\zeta, x_d), \text{ for } x_d \leq 1 \\ u_n|_{x_d=1}(\zeta) = -\phi_1(\zeta)B_1 u_{n-1}|_{x_d=1}(\zeta), \end{cases} \quad \text{if } n \text{ is odd.}$$

Duhamel's formula (combined with a recursive argument) immediately gives the following expressions: for all $p \in \mathbb{N}$

$$\begin{aligned} u_{2p}(\zeta, x_d) &= e^{x_d \mathcal{A}(\zeta)} T_{0 \rightarrow 0}(\zeta)^p \phi_0(\zeta)g(\zeta), \\ u_{2p+1}(\zeta, x_d) &= -e^{(x_d-1)\mathcal{A}(\zeta)} \phi_1(\zeta)B_1 e^{\mathcal{A}(\zeta)} T_{0 \rightarrow 0}(\zeta)^p \phi_0(\zeta)g(\zeta). \end{aligned}$$

Finally, let $S_p(\zeta, x_d) := \sum_{q=0}^p (u_{2q} + u_{2q+1})(\zeta, x_d)$. In view of its definition it is clear that for all $p \in \mathbb{N}$, S_p solves the evolution equation of (77). Moreover for all $p \in \mathbb{N}$, we have:

$$B_1 S_p|_{x_d=1}(\zeta) = \sum_{q=0}^p \underbrace{B_1 (I - \phi_1(\zeta)B_1)}_{=0} u_{2q}|_{x_d=1}(\zeta) = 0,$$

and

$$S_{p|x_d=0}(\zeta) = \left(I - e^{-\mathcal{A}(\zeta)} \phi_1(\zeta) B_1 e^{\mathcal{A}(\zeta)} \right) \sum_{q=0}^p T_{0 \rightarrow 0}(\zeta)^q \phi_0(\zeta) g,$$

and once again this equation has to be compared with (40) where $(I - T_{0 \rightarrow 0}(\zeta))^{-1}$ has been replaced by the truncated Neumann expansion $\sum_{q=0}^p T_{0 \rightarrow 0}(\zeta)^q$.

From Assumption 3.3, $\underline{u}(\zeta, x_d) := \lim_{p \rightarrow \infty} S_p(\zeta, x_d)$ is well-defined and $\underline{u}(\zeta, \cdot) \in L^2([0, 1])$ solves the evolution equation of (77) as well as the boundary condition on $x_d = 1$. Finally:

$$\begin{aligned} B_0 \underline{u}|_{x_d=0}(\zeta) &= B_0 \left(I - e^{-\mathcal{A}(\zeta)} \phi_1(\zeta) B_1 e^{\mathcal{A}(\zeta)} \right) \sum_{q=0}^{\infty} T_{0 \rightarrow 0}(\zeta)^q \phi_0(\zeta) g, \\ &= g(\zeta) + B_0 T_{0 \rightarrow 0}(\zeta) \sum_{q=0}^{\infty} T_{0 \rightarrow 0}(\zeta)^q \phi_0(\zeta) g - B_0 e^{-\mathcal{A}(\zeta)} \phi_1(\zeta) B_1 e^{\mathcal{A}(\zeta)} \sum_{q=0}^{\infty} T_{0 \rightarrow 0}(\zeta)^q \phi_0(\zeta) g \\ &= g(\zeta), \end{aligned}$$

where we used the relation:

$$B_0 T_{0 \rightarrow 0}(\zeta) = \underbrace{B_0 \phi_0(\zeta)}_{=I} B_0 e^{-\mathcal{A}(\zeta)} \phi_1(\zeta) B_1 e^{\mathcal{A}(\zeta)}.$$

Consequently \underline{u} is a weak solution of (77). Then to treat the case $g_1 \neq 0$, we repeat essentially the same construction. □

Before to end this paragraph let us make some comments about the constructed weak solution. This weak solution is constructed as the superposition of wave reflected wave packets. Each wave packet reads under the form $T_{0 \rightarrow 0}(\zeta)^p$ so that it corresponds to the initial boundary trace $\phi_0(\zeta)g$ which has been reflected on $\partial\Gamma_1$ and on $\partial\Gamma_0$, p times.

Consequently it seems natural to impose that $T_{0 \rightarrow 0}(\zeta)$ is a contraction (that is the energy of reflected rays decreases) because if it is not then we expect the solution to blow up in finite time.

Existence of a weak solution by duality To show that (1) admits a solution we use the duality method. We essentially just have to show that Assumption 3.2 is satisfied by the dual problem of (1) (if it is satisfied by the primal problem (1)) and we can then conclude that there exists a weak solution of (1) exactly as it has been done in Paragraph (3.1.2).

For convenience we recall that the dual problem reads:

$$\begin{cases} L^*(\partial)v = f & \text{on } \Omega, \\ C_0 v|_{x_d=0} = g_0 & \text{on } \partial\Omega_0, \\ C_1 v|_{x_d=1} = g_1 & \text{on } \partial\Omega_1, \end{cases} \quad (80)$$

and that the matrices $C_0 \in \mathbf{M}_{(N-p) \times N}(\mathbb{R})$ and $C_1 \in \mathbf{M}_{p \times N}(\mathbb{R})$ are characterized by:

$$\ker C_j = (A_d \ker B_j)^\perp, \quad (81)$$

and also recall $\tilde{\phi}_0(\zeta)$ and $\tilde{\phi}_1(\zeta)$ stand for the inverse given by the uniform Kreiss-Lopatinskii condition for the dual problem (80).

In fact we will show that if the primal problem (1) satisfies Assumption 3.1 then the operator $(I - \tilde{\tilde{T}}_{0 \rightarrow 0}(\zeta))$ (where $\tilde{\tilde{T}}_{0 \rightarrow 0}(\zeta)$ denotes the operator $\tilde{T}_{0 \rightarrow 0}(\zeta)$ for the dual problem) is uniformly invertible over $\ker C_0$. This result will however be sufficient to conclude because to perform the remaining existence part of the proof in

Paragraph (3.1.2) we just use the fact that the solution of the dual problem (24) is unique, thanks to the fact that this problem satisfies the *a priori* energy estimate.

Recall that in the proof of the *a priori* energy estimate in Paragraph 3.3.2 we used the fact that $(I - \widetilde{T}_{0 \rightarrow 0}(\zeta))$ is uniformly invertible over $\ker B_0$ (that is Assumption 3.2) and not that $(I - T_{0 \rightarrow 0}(\zeta))$ is uniformly invertible on $E^s(\zeta)$ (that is Assumption 3.1).

We introduce the operators $\widetilde{T}_{0 \rightarrow 1}(\zeta)$ and $\widetilde{T}_{1 \rightarrow 0}(\zeta)$ associated to the dual problem (80). They are given by:

$$\widetilde{T}_{0 \rightarrow 1}(\zeta) := (I - \widetilde{\phi}_1(\zeta)C_1)e^{\widetilde{\mathcal{A}}(\zeta)} \text{ and } \widetilde{T}_{1 \rightarrow 0}(\zeta) := (I - \widetilde{\phi}_0(\zeta)C_0)e^{-\widetilde{\mathcal{A}}(\zeta)}.$$

The main result of this paragraph is the following :

Proposition 3.12 *The primal problem (1) satisfies Assumption 3.1 if and only if the dual problem (80) satisfies Assumption 3.2.*

Proof : We want to show that for all $\zeta \in \Xi \setminus \Xi_0$ the operator $(I - \widetilde{T}_{0 \rightarrow 0}(\zeta))$ is uniformly invertible over $\ker C_0$. Noting that for all $v \in \ker C_0$ we have $(I - \widetilde{T}_{0 \rightarrow 0}(\zeta))(I - \widetilde{\phi}_0(\zeta)C_0)v = (I - \widetilde{T}_{0 \rightarrow 0}(\zeta))v$ it is sufficient to show that $(I - \widetilde{T}_{0 \rightarrow 0}(\zeta))(I - \widetilde{\phi}_0(\zeta)C_0)$ is uniformly invertible over $\ker C_0$.

To do this we use the following lemma:

Lemma 3.7 *Let $A, B \subset \mathbb{C}^N$ such that $E = A \oplus B$ and let $T : A \rightarrow A$. Then $T : A \rightarrow A$ is invertible if and only if $T^* : B^\perp \rightarrow B^\perp$ is invertible. Moreover $T : A \rightarrow A$ is uniformly invertible in terms of some parameter ζ if and only if $T^* : B^\perp \rightarrow B^\perp$ is uniformly invertible.*

From Lemma 3.7 applied to the decomposition $\mathbb{C}^N = \widetilde{E}^s(\zeta) \oplus \ker C_0$ is it equivalent to show that $(I - \widetilde{\phi}_0(\zeta)C_0)^*(I - \widetilde{T}_{0 \rightarrow 0}(\zeta))^*$ is uniformly invertible over $\widetilde{E}^s(\zeta)^\perp = A_d E^s(\zeta)$ (by Lemma 3.4). By Lemma 3.5 we have $(I - \widetilde{\phi}_0(\zeta)C_0)^* = A_d \phi_0(\zeta) B_0 A_d^{-1}$ and we compute:

$$\begin{aligned} \widetilde{T}_{0 \rightarrow 1}^*(\zeta) &= \left(e^{\widetilde{\mathcal{A}}(\zeta)} \right)^* (I - \widetilde{\phi}_1(\zeta)C_1)^* = A_d e^{-\widetilde{\mathcal{A}}(\zeta)} \phi_1(\zeta) B_1 A_d^{-1}, \\ \widetilde{T}_{1 \rightarrow 0}^*(\zeta) &= \left(e^{-\widetilde{\mathcal{A}}(\zeta)} \right)^* (I - \widetilde{\phi}_0(\zeta)C_0)^* = A_d e^{\widetilde{\mathcal{A}}(\zeta)} \phi_0(\zeta) B_0 A_d^{-1}, \end{aligned} \quad (82)$$

where we refer to Paragraph 3.3.2 for the details about $\left(e^{\widetilde{\mathcal{A}}(\zeta)} \right)^*$. Consequently; for $X \in A_d E^s(\zeta)$, $X = A_d Y$ with $Y \in E^s(\zeta)$ we have:

$$\begin{aligned} (I - \widetilde{\phi}_0(\zeta)C_0)^*(I - \widetilde{T}_{0 \rightarrow 0}(\zeta))^* X &= A_d \phi_0(\zeta) B_0 A_d^{-1} \left(I - A_d e^{-\widetilde{\mathcal{A}}(\zeta)} \phi_1(\zeta) B_1 e^{+\widetilde{\mathcal{A}}(\zeta)} \phi_0(\zeta) B_0 A_d^{-1} \right) X \\ &= A_d \phi_0(\zeta) B_0 \left(I - e^{-\widetilde{\mathcal{A}}(\zeta)} \phi_1(\zeta) B_1 e^{+\widetilde{\mathcal{A}}(\zeta)} \right) Y, \\ &= A_d (\phi_0(\zeta) B_0 - T_{0 \rightarrow 0}(\zeta)) Y, \\ &= A_d (I - T_{0 \rightarrow 0}(\zeta)) Y, \end{aligned}$$

because $Y \in E^s(\zeta)$. Consequently Assumption 3.1 for the primal problem (8) is equivalent to Assumption 3.2 for the dual problem (80) which concludes the proof. \square

With Proposition 3.12 we can reiterate exactly the same reasoning as for the problem with strictly dissipative boundary conditions (see Paragraph 3.1.2). We immediately obtain the existence of a weak solution $u \in L_\gamma^2(\Omega)$ for (1).

Reiterating exactly the same kind of computations as those described in the proof of Proposition 3.12 but for the operator $(I - \widetilde{T}_{0 \rightarrow 0}(\zeta))$ instead of $(I - \widetilde{T}_{0 \rightarrow 0}(\zeta))$ immediately gives the following result :

Proposition 3.13 *The primal problem (1) satisfies Assumption 3.2 if and only if the dual problem (80) satisfies Assumption 3.1.*

3.3.4 Summary of the proof of Theorem 2.2.

Assume that the strip problem (1) satisfies Assumptions 2.1-2.2-2.3-2.3 and that each boundary condition satisfies the uniform Kreiss-Lopatinskii condition.

Assume that Assumption 3.2 holds, so that (1) admits a reduced symmetrizer and Corollary 3.1 applies and (1) satisfies the *a priori* energy estimate.

Assume that Assumption 3.1 (or its stronger version that is Assumption 3.3) holds then one of the constructions of a weak solution given in Paragraph 3.3.3 applies and we deduce that 1 admits a weak solution $u \in L^2_\gamma(\Omega)$.

Reiterating exactly the same reasoning as the one described for strictly dissipative boundary conditions (see Lemma 3.3) we show that this weak solution is a strong solution so that it satisfies the *a priori* energy estimate and by linearity it is unique. This concludes the proof of Theorem 2.2.

Before to study some examples let us give more details about the links between Assumptions 3.1 and 3.2. As already mentioned Proposition 3.6 shows that if $(I - T_{0 \rightarrow 0}(\zeta))$ is invertible over $E^s(\zeta)$ then $(I - \bar{T}_{0 \rightarrow 0}(\zeta))$ is invertible over $\ker B_0$.

Assume that we can show the following stronger version of Proposition 3.6

Proposition 3.14 *If the strip problem (1) satisfies Assumption 3.1 then it satisfies Assumption 3.2.*

Then we can also show the following version of Theorem 2.2 (that is sharp, in the author's opinion)

Theorem 3.2 *Under Assumptions 2.1-2.2-2.3 and 2.4, assume that Proposition 3.13 holds then Assumption 3.1 and Assumption 3.2 are equivalent.*

Consequently the hyperbolic boundary value problem in the strip (1) is lower exponentially strongly well-posed in the sense of Definition 2.2 if and only if each boundary condition satisfies the uniform Kreiss-Lopatinskii condition (see Assumption-Definition 2.1) and if moreover it satisfies Assumption 3.1.

Proof : We just have to justify the equivalence between Assumption 3.1 and Assumption 3.2. From Proposition 3.13 Assumption 3.1 implies Assumption 3.2.

The fact that Assumption 3.2 implies Assumption 3.1 is a consequence of the duality method. Assume that Assumption 3.2 holds for the primal problem (1), then by Proposition 3.13, Assumption 3.1 holds for the dual problem (80). By Proposition 3.14, the dual problem satisfies Assumption 3.2. This implies, by Proposition 3.12 that the primal problem (1) satisfies Assumption 3.1. □

3.4 Examples

3.4.1 An example in the spirit of [Osher, 1974a]

In this paragraph we consider the following hyperbolic boundary value problem inspired from [Osher, 1974a]:

$$\begin{cases} \partial_t u + A_1 \partial_x u = \partial_t u + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \partial_x u = f & \text{for } (t, x) \in \mathbb{R} \times]0, 1[, \\ B_0 u|_{x=0} := \begin{bmatrix} 1 & -\alpha_0 \end{bmatrix} u|_{x=0} = g_0 & \text{for } t \in \mathbb{R}, \\ B_1 u|_{x=1} := \begin{bmatrix} -\alpha_1 & 1 \end{bmatrix} u|_{x=1} = g_1 & \text{for } t \in \mathbb{R}, \\ u|_{t \leq 0} = 0 & \text{for } x \in [0, 1], \end{cases} \quad (83)$$

where α_0 and α_1 are fixed real numbers and where the source terms f, g_0, g_1 are zero for negative times. Equation (83) is nothing but two transport equations (a left going one and a right going one) coupled by the boundary conditions. To ensure that Assumption 2.4 is satisfied we impose that $(\alpha_0, \alpha_1) \neq (1, 1)$. It is

also easy to check that Assumptions 2.1-2.2 and 2.3 are satisfied, moreover we can show that the boundary conditions are strictly dissipative if and only if

$$|\alpha_0| < 1 \text{ and } |\alpha_1| < 1, \quad (84)$$

consequently if α_0 and α_1 satisfy (84), Theorem 3.1 applies and (83) is strongly well-posed.

Then we compute $\mathcal{A}(\sigma) = -\sigma A_1$ which immediately implies that $E^s(\sigma) = E^s = \text{vect} \{(1, 0)\}$ and $E^u(\sigma) = E^u = \text{vect} \{(0, 1)\}$. So the uniform Kreiss-Lopatinskii condition is satisfied on the side $\partial\Gamma_0$ (resp. $\partial\Gamma_1$) for all α_0 (resp. α_1).

We then study Assumptions 3.1 and 3.2, we compute:

$$(I - T_{0 \rightarrow 0}(\sigma)) \begin{bmatrix} 1 \\ 0 \end{bmatrix} = (I - \phi_0(\sigma)B_0 e^{-\mathcal{A}(\sigma)} \phi_1(\sigma)B_1 e^{\mathcal{A}(\sigma)}) \begin{bmatrix} 1 \\ 0 \end{bmatrix} = (1 - \alpha_0 \alpha_1 e^{-2\sigma}) \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

from which we deduce that Assumption 3.1 is satisfied if and only if $|\alpha_0 \alpha_1| < 1$. We now study Assumption 3.2 by a direct computation, we have:

$$(I - \bar{T}_{0 \rightarrow 0}(\sigma)) \begin{bmatrix} \alpha_0 \\ 1 \end{bmatrix} = (I - (I - \phi_0(\sigma)B_0) e^{-\mathcal{A}(\sigma)} (1 - \phi_1(\sigma)B_1) e^{\mathcal{A}(\sigma)}) \begin{bmatrix} \alpha_0 \\ 1 \end{bmatrix} = (\alpha_0 \alpha_1 e^{-2\sigma} - 1) \begin{bmatrix} \alpha_0 \\ 1 \end{bmatrix},$$

from which we deduce that Assumption 3.2 is satisfied if and only if $|\alpha_0 \alpha_1| < 1$. In particular we recall that Assumption 3.2 holds if and only if Assumption 3.1 holds.

Consequently if $|\alpha_0 \alpha_1| < 1$ we can apply Theorem 2.2 and therefore (83) is lower exponentially strongly well-posed.

So we have shown on this example that the strip problem (83) can be lower exponentially strongly well-posed for boundary conditions which are not strictly dissipative and that Assumptions 3.1-3.2 are satisfied for more boundary conditions than only the strictly dissipative ones.

3.4.2 The wave equation

In this paragraph we consider the two dimensional wave equation:

$$\begin{cases} \partial_t u + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \partial_1 u + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \partial_2 u = 0 & \text{for } (t, x) \in \mathbb{R} \times]0, 1[\times \mathbb{R}, \\ B_0 u|_{x_1=0} := \begin{bmatrix} 1 & -\alpha_0 \end{bmatrix} u|_{x_1=0} = g & \text{for } (t, x_2) \in \mathbb{R} \times \mathbb{R}, \\ B_1 u|_{x_1=1} := \begin{bmatrix} 1 & -\alpha_1 \end{bmatrix} u|_{x_1=1} = 0 & \text{for } (t, x_2) \in \mathbb{R} \times \mathbb{R}, \\ u|_{t \leq 0} = 0 & \text{for } x \in [0, 1] \times \mathbb{R}, \end{cases} \quad (85)$$

where α_0 and α_1 are fixed real numbers satisfying $\alpha_0 \neq \alpha_1$ to ensure Assumption 2.4. We can verify that the boundary conditions are strictly dissipative if and only if

$$|\alpha_0| < 1 \text{ and } |\alpha_1| > 1. \quad (86)$$

As the system (85) only has two equations we can apply the result of [Strang, 1969] establishing that the boundary conditions of (85) satisfy the uniform Kreiss-Lopatinskii condition if and only if they are strictly dissipative. Consequently the strip problem (85) is strongly well-posed if and only if (86) holds.

Our aim is here to verify that under (86), Assumption 3.1 is satisfied. To this end, we compute:

$$\mathcal{A}(\zeta) = \begin{bmatrix} -\sigma & -i\eta \\ i\eta & \sigma \end{bmatrix},$$

and let $X(\zeta) \in \mathbb{C}$ be a solution of $\det(\mathcal{A}(\zeta) - X(\zeta)) = 0$. We have

$$X(\zeta)^2 = \sigma^2 + \eta^2, \quad (87)$$

and we define $X^s := X^s(\zeta) = X_R^s + iX_I^s$ the solution of (87) with strictly negative real part and $X^u := X^u(\zeta)$ the one with strictly positive real part. We have $X^s = -X^u$.

Thus we can parametrized $E^s(\zeta) = \text{vect} \{(-i\eta, \sigma + X^s)\}$ and $E^u(\zeta) = \text{vect} \{(-i\eta, \sigma - X^s)\}$ from which we deduce the inverses given by the uniform Kreiss-Lopatinskii condition, we have:

$$\phi_0(\zeta)x := -\frac{x}{i\eta + \alpha_0(\sigma + X^s)} \begin{bmatrix} -i\eta \\ \sigma + X^s \end{bmatrix} \text{ and } \phi_1(\zeta)y := -\frac{y}{i\eta + \alpha_1(\sigma - X^s)} \begin{bmatrix} -i\eta \\ \sigma - X^s \end{bmatrix}.$$

Some computations give:

$$(I - \phi_0(\zeta)B_0e^{-\mathcal{A}(\zeta)}\phi_1(\zeta)B_1e^{\mathcal{A}(\zeta)}) \begin{bmatrix} -i\eta \\ \sigma + X^s \end{bmatrix} = \left(1 - e^{2X^s} \frac{i\eta + \alpha_1(\sigma + X^s)}{i\eta + \alpha_1(\sigma - X^s)} \frac{i\eta + \alpha_0(\sigma - X^s)}{i\eta + \alpha_0(\sigma + X^s)}\right) \begin{bmatrix} -i\eta \\ \sigma + X^s \end{bmatrix},$$

from which it follows that for Assumption 3.1 (or equivalently Assumption 3.1) to hold it is sufficient that

$$\left| \frac{i\eta + \alpha_1(\sigma + X^s)}{i\eta + \alpha_1(\sigma - X^s)} \frac{i\eta + \alpha_0(\sigma - X^s)}{i\eta + \alpha_0(\sigma + X^s)} \right| < 1, \quad (88)$$

because $|e^{2X^s}| \leq 1$.

Some tedious (but not difficult) computations show that (88) reads under the form $-\lambda < \lambda$, where

$$\lambda := (\alpha_0 - \alpha_1)(1 + \alpha_0\alpha_1)\eta X_I^s + (\alpha_0^2 - \alpha_1^2)(\tau X_I^s + \gamma X_R^s). \quad (89)$$

So (88) holds if and only if $\lambda > 0$. By definition we have $\gamma X_R^s < 0$ and from (87) (in which we take the imaginary part) we obtain that $X_R^s X_I^s = \gamma\tau$ so that $\tau X_I^s \leq 0$ and the second term in (89) is strictly positive if $|\alpha_0| < |\alpha_1|$ (which holds in particular for all strictly dissipative boundary conditions).

The first term in (89) is unfortunately much more difficult to deal with because it is not clear that the product ηX_I^s has a constant sign. To deal with we choose to cancel it by choosing $(1 + \alpha_0\alpha_1) = 0$. An consequently we show that for $(1 + \alpha_0\alpha_1) = 0$ and $|\alpha_0| < |\alpha_1|$, the operator $T_{0 \rightarrow 0}(\zeta)$ is a contraction with respect to ζ and we can conclude in a (really) particular case.

A more clever analysis to treat (at least) the strictly dissipative boundary conditions is to remark that for all strictly dissipative boundary conditions we have $(\alpha_0 - \alpha_1)(1 + \alpha_0\alpha_1) > (\alpha_0^2 - \alpha_1^2)$ so that:

$$\lambda > (\alpha_0^2 - \alpha_1^2)(\eta X_I^s + \tau X_I^s + \gamma X_R^s),$$

and to determine the sign of $X_I^s(\eta + \tau)$.

4 Conclusion and conjectures

In this article we gave two theorems establishing the lower exponential strong well-posedness of hyperbolic boundary value problems in a strip. The first one (see Theorem 3.1) applies in the particular framework of strictly dissipative boundary conditions. In particular it gives easily checkable sufficient conditions for lower exponential strong well-posedness. The second one, that is Theorem 2.2 is stated, in terms of the boundary conditions, in the more general possible setting. It states that the lower exponential strong well-posedness is equivalent to the fulfillment of two new invertibility condition. These conditions asks that the operators $(I - T_{0 \rightarrow 0}(\zeta))$ and $(I - \bar{T}_{0 \rightarrow 0}(\zeta))$ are uniformly, in terms of the frequency parameter ζ , invertible on the stable subspace $E^s(\zeta)$.

Of course, when Theorem 3.1 applies then Theorem 2.2 should also applies and consequently for the consistency of the theory it should be interesting to show that when the boundary conditions are strictly dissipative then the operators $(I - T_{0 \rightarrow 0}(\zeta))$ and $(I - \bar{T}_{0 \rightarrow 0}(\zeta))$ are uniformly invertible.

About this question, our main conjecture is that when the boundary condition on the side $partial\Gamma_0$ (resp. $\partial\Gamma_1$) is strictly dissipative then the operator $T_{1 \rightarrow 0}(\zeta) : E^u(\zeta) \rightarrow E^s(\zeta)$ (resp. $T_{0 \rightarrow 1}(\zeta) : E^s(\zeta) \rightarrow E^u(\zeta)$) is a contraction so that the same holds for $T_{0 \rightarrow 0}(\zeta) : E^s(\zeta) \rightarrow E^s(\zeta)$. The same conjecture holds for the operators $\bar{T}_{1 \rightarrow 0}(\zeta) : \ker B_1 \rightarrow \ker B_0$ (resp. $\bar{T}_{0 \rightarrow 1}(\zeta) : \ker B_0 \rightarrow \ker B_1$).

In Paragraph 3.2 we have shown that both conditions on the operators $(I - T_{0 \rightarrow 0}(\zeta))$ and $(I - \bar{T}_{0 \rightarrow 0}(\zeta))$ are necessary for the lower exponential strong well-posedness of the boundary value problem in the strip. Moreover the result of Proposition 3.6 combined with the result of Proposition 3.13 seems to indicate that in fact Assumptions 3.1 and 3.2 are equivalent. We believe that establish such an equivalent should be an interesting for the all theory (because it permits to give a version of Theorem 2.2 with Assumption 3.1 (or 3.2) only, and this question is left for future studies. The only point that we should show is a stronger version of Proposition 3.6 including the uniform bound for the inverse.

However as described in paragraph 3.4.2, this condition (specifically the uniform part of the inversibility) seems to be really difficult to establish for a given hyperbolic boundary value problem in the strip with general boundary conditions.

In the author's opinion, this difficulty is a reminiscence of the same one in the half space geometry. Let us briefly explain what we mean by this assertion.

In this article we used a version of the uniform Kreiss-Lopatinskii condition (see Assumption-Definition 2.1) stated on the all space Ξ . This condition is equivalent to the following condition: for all $\zeta \in \Xi \setminus \Xi_0$ the restriction of B_0 (resp. B_1) to $E^s(\zeta)$ (resp. $E^u(\zeta)$) is uniformly invertible. However when one wants to check that a given boundary condition satisfies (or not) the uniform Kreiss-Lopatinskii condition the first version of the uniform Kreiss-Lopatinskii condition (see see Assumption-Definition 2.1) is much more easy to handle with because it gives for free the uniform part of the bound of the inverse. Indeed by homogeneity we can restrict the study to the unit sphere of Ξ which is compact and consequently the uniform part of the bound comes for free.

Our new inversibility condition is stated on the open space $\Xi \setminus \Xi_0$ so that, by analogy, with the uniform Kreiss-Lopatinskii condition, we believe that a simple way (and possibly the simplest one) to show that the inversibility condition holds uniformly on the space $\Xi \setminus \Xi_0$ is to consider the boundary frequencies, that is the $\zeta \in \Xi_0$. Extend the operator $(I - T_{0 \rightarrow 0}(\zeta))$ on $E^s(\zeta)$ up to the boundary Ξ_0 is not difficult because as noticed by [Kreiss, 1970] and [Métivier, 2000] the stable subspace $E^s(\zeta)$ admits a continuous extension up to Ξ_0 . Moreover, instead of showing that $(I - T_{0 \rightarrow 0}(\zeta))$ is uniformly invertible on $E^s(\zeta)$, it may be easier to show that $T_{0 \rightarrow 0}(\zeta)$ is a uniform contraction on $E^s(\zeta)$.

However the difficulty by considering $(I - T_{0 \rightarrow 0}(\zeta))$ for $\zeta \in \Xi_0$ is that this operator will generically not be invertible on $E^s(\zeta)$ as soon as ζ is a glancing frequency⁶.

Indeed, for simplicity, let us consider the wave equation (85) for which the glancing area (we recall that glancing modes are the ones such that Theorem 2.1 is satisfied with a block of type iv) is parametrized by $\gamma = 0, |\tau| = |\eta|$. For such a ζ we have: $X^s(\zeta) = X^u(\zeta) = 0$ and $E^s(\zeta) = E^u(\zeta)$. So that if v_s stands for a generator of $E^s(\zeta)$ we obtain:

$$T_{0 \rightarrow 0}(\zeta)v_s = e^{2X^s(\zeta)}\phi_0(\zeta)B_0\phi_1(\zeta)B_1v_s = v_s,$$

and $(I - T_{0 \rightarrow 0}(\zeta))$ can not be invertible on $E^s(\zeta)$. Moreover it can also not be a contraction.

In more generality we obtain essentially

$$T_{0 \rightarrow 0}(\zeta) = e^{2X^s(\zeta)},$$

on $E^s(\zeta)$ which does not prevent $(I - T_{0 \rightarrow 0}(\zeta))$ to be invertible but which prevent $T_{0 \rightarrow 0}(\zeta)$ to be a contraction (because $X^s(\zeta) \in i\mathbb{R}$ for glancing modes).

The fact that $(I - T_{0 \rightarrow 0}(\zeta))$ is sometimes clearly not invertible for the glancing frequencies ζ may sound alarming. However we believe that it should not be.

Indeed, if we see the inversibility condition on $(I - T_{0 \rightarrow 0}(\zeta))$ as a non amplification condition on self-interacting modes, then this condition should not be imposed on glancing modes because these modes are linked to boundary layers and so they do not propagate the information (see [Williams, 1996]) from one side to the other. So we believe, even if at present time some technical issues remain, that the construction of

⁶Recall that glancing modes are generic for hyperbolic boundary value problems, except for the degenerated example (83) because from [Coulombel, 2011a].

the symmetrizer, described in paragraph 3.3.2, can be performed for $\zeta \in \Xi_0$ under an invertibility condition which does not involve glancing modes.

In order to do so we believe that a complete geometric optics expansion of the problem (1) should help to understand the action of $T_{0 \rightarrow 0}(\zeta)$ near glancing modes which is probably the keystone of the construction. The author aims to give such a WKB expansion in a forthcoming contribution.

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