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► **To cite this version:**

Matthieu Romagny, Dajano Tossici. Affine smooth group schemes over the dual numbers. 2018. hal-01712886v2

**HAL Id: hal-01712886**

**<https://hal.science/hal-01712886v2>**

Preprint submitted on 23 Feb 2018 (v2), last revised 25 Jun 2019 (v4)

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# Affine smooth group schemes over the dual numbers

Matthieu Romagny, Dajano Tossici

February 23, 2018

## Abstract

We provide an equivalence between the category of affine, smooth group schemes over the ring of generalized dual numbers  $k[I]$ , and the category of extensions of the form  $1 \rightarrow \mathrm{Lie}(G, I) \rightarrow E \rightarrow G \rightarrow 1$  where  $G$  is an affine, smooth group scheme over  $k$ . Here  $k$  is an arbitrary commutative ring and  $k[I] = k \oplus I$  with  $I^2 = 0$ . The equivalence is given by Weil restriction, and we provide a quasi-inverse which we call *Weil extension*. It is compatible with the exact structures and the  $\mathbb{O}_k$ -module stack structures on both categories. Our constructions rely on the use of the group algebra scheme of an affine group scheme; we introduce this object and establish its main properties. As an application, we establish a Dieudonné classification for smooth, commutative, unipotent group schemes over  $k[I]$ .

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# 1 Introduction

Throughout this article, we fix a commutative ring  $k$  and a free  $k$ -module  $I$  of finite rank  $r \geq 1$ . We consider the ring of (generalized) dual numbers  $k[I] := k \oplus I$  with  $I^2 = 0$ . We write  $h : \text{Spec}(k[I]) \rightarrow \text{Spec}(k)$  the structure map and  $i : \text{Spec}(k) \rightarrow \text{Spec}(k[I])$  the closed immersion. Also we denote by  $\mathbb{O}_k$  the  $\text{Spec}(k)$ -ring scheme such that if  $R$  is a  $k$ -algebra then  $\mathbb{O}_k(R) = R$  with its ring structure.

## 1.1 Motivation, results, plan of the article

**1.1.1 Motivation.** The starting point of our work is a relation between deformations and group extensions. To explain the idea, let  $G$  be an affine, flat, finitely presented algebraic  $k$ -group scheme. It is shown in Illusie’s book [Ill72] that the set of isomorphism classes of deformations of  $G$  to  $k[I]$  is in bijection with the cohomology group  $H^2(BG, \underline{\mathcal{L}}_G^\vee \otimes I)$ , see chap. VII, thm 3.2.1 in *loc. cit.* Here, the coefficients of cohomology are the derived dual of the equivariant cotangent complex  $\underline{\mathcal{L}}_G \in D(BG)$ , tensored (in the derived sense) by  $I$  viewed as the coherent sheaf it defines on the fpqc site of  $BG$ , also equal to the vector bundle  $\mathbb{V}(I^\vee)$ . If we assume moreover that  $G$  is smooth, then the augmentation  $\underline{\mathcal{L}}_G \rightarrow \omega_G^1$  to the sheaf of invariant differential 1-forms is a quasi-isomorphism and it follows that  $\underline{\mathcal{L}}_G^\vee \simeq \text{Lie } G$ . Since coherent cohomology of  $BG$  is isomorphic to group cohomology of  $G$ , the cohomology group of interest ends up being  $H^2(G, \text{Lie}(G, I))$  where  $\text{Lie}(G, I) := \text{Lie } G \otimes \mathbb{V}(I^\vee)$ . The latter cohomology group is meaningful also in the theory of group extensions, where it is known to classify isomorphism classes of extensions of  $G$  by  $\text{Lie}(G, I)$  viewed as a  $G$ -module via the adjoint representation, see [DG70], chap. II, § 3, no 3.3 and III, § 6, no 2.1.

In this paper, our aim is to give a direct algebro-geometric construction of this correspondence between deformations and group extensions of affine, smooth group schemes. Our main result is that the Weil restriction functor  $h_*$  provides such a construction. Thereby, we obtain a categorification of a link that has been available only as a bijection between sets of isomorphism classes. This improvement is crucial for a better understanding of  $k[I]$ -group schemes, since in applications most groups occur as kernels of quotients of morphisms. We illustrate this by giving a Dieudonné-type theory for unipotent group schemes. Natural extensions of our result to more general (higher-order) thickenings of the base, or to non-smooth group schemes, would have further interesting applications. We were in fact driven here by potential improvements of our previous work on models of finite flat group schemes, and we hope to come back to this in the future.

**1.1.2 Main results.** For reasons that are discussed below, it is convenient for us to work with group schemes slightly more general than smooth ones. We say that a morphism of schemes  $X \rightarrow S$  is *differentially flat* if both  $X$  and  $\Omega_{X/S}^1$  are flat over  $S$ . Examples of differentially flat group schemes include smooth group schemes, pullbacks from the spectrum of a field, Tate-Oort group schemes with parameter  $a = 0$  in characteristic  $p$ , flat vector bundles i.e.  $\mathbb{V}(\mathcal{F})$  with  $\mathcal{F}$  flat over the base, and truncated Barsotti-Tate groups of level  $n$  over a base where  $p^n = 0$  ([Ill85], 2.2.1). If  $\mathcal{G}$  is a  $k[I]$ -group scheme, we call *rigidification* of  $\mathcal{G}$  an isomorphism of  $k[I]$ -schemes  $\sigma : h^*\mathcal{G}_k \xrightarrow{\sim} \mathcal{G}$  that lifts the identity of the scheme  $\mathcal{G}_k := i^*\mathcal{G}$ . Such a map need not be a morphism of group schemes. We say that  $\mathcal{G}$  is *rigid* if it admits a rigidification. Examples of rigid group schemes include smooth group schemes, pullbacks from  $\text{Spec}(k)$ , and groups of multiplicative type.

Let  $\text{Gr}/k[I]$  be the category of affine, differentially flat, rigid  $k[I]$ -group schemes (this includes all affine, smooth  $k[I]$ -group schemes). The morphisms in this category are the morphisms of  $k[I]$ -group schemes. Let  $\text{Ext}(I)/k$  be the category of extensions of the form  $1 \rightarrow \text{Lie}(G, I) \rightarrow E \rightarrow G \rightarrow 1$  where  $G$  is an affine differentially flat  $k$ -group scheme, and "extension" means that  $E \rightarrow G$  is an fpqc torsor under  $\text{Lie}(G, I)$ . The morphisms in this category are the commutative diagrams:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \text{Lie}(G, I) & \longrightarrow & E & \longrightarrow & G & \longrightarrow & 1 \\ & & \downarrow \text{d}\psi & & \downarrow \varphi & & \downarrow \psi & & \\ 1 & \longrightarrow & \text{Lie}(G', I) & \longrightarrow & E' & \longrightarrow & G' & \longrightarrow & 1 \end{array}$$

where  $\text{d}\psi = \text{Lie}(\psi)$  is the differential of  $\psi$ . Usually such a morphism will be denoted simply  $\varphi : E \rightarrow E'$ .

The categories  $\text{Gr}/k[I]$  and  $\text{Ext}(I)/k$  are exact categories. They are also fpqc stacks over  $\text{Spec}(k)$  equipped with the structure of  $\mathbb{O}_k$ -module stacks fibred in groupoids over  $\text{Gr}/k$ . This means that there exist notions of *sum* and *scalar multiple* for objects of  $\text{Gr}/k[I]$  and  $\text{Ext}(I)/k$  (for extensions, the sum is the Baer sum). The precise notion is reviewed in 1.2 and Appendix A. We can now state our main result.

**Theorem A** (see Theorem 6.1.1) (1) *The Weil restriction functor provides an equivalence:*

$$h_* : \text{Gr}/k[I] \xrightarrow{\sim} \text{Ext}(I)/k.$$

*This equivalence commutes with base changes on  $\text{Spec}(k)$ .*

(2) *If  $1 \rightarrow \mathcal{G}' \rightarrow \mathcal{G} \rightarrow \mathcal{G}'' \rightarrow 1$  is an exact sequence in  $\text{Gr}/k[I]$ , then  $1 \rightarrow h_*\mathcal{G}' \rightarrow h_*\mathcal{G} \rightarrow h_*\mathcal{G}'' \rightarrow 1$  is exact in  $\text{Ext}(I)/k$ . If moreover  $\mathcal{G}'$  is smooth then  $1 \rightarrow h_*\mathcal{G}' \rightarrow h_*\mathcal{G} \rightarrow h_*\mathcal{G}'' \rightarrow 1$  is exact. In particular,  $h_*$  is an exact equivalence between the subcategories of smooth objects endowed with their natural exact structure.*

(3) *The equivalence  $h_*$  is a morphism of  $\mathbb{O}_k$ -module stacks fibred over  $\text{Gr}/k$ , i.e. it transforms the addition and scalar multiplication of deformations of a fixed  $G \in \text{Gr}/k$  into the Baer sum and scalar multiplication of extensions.*

(4) *Let  $P$  be one of the properties of group schemes over a field: of finite type, smooth, connected, unipotent, split unipotent, solvable, commutative. Say that a group scheme over an arbitrary ring has property  $P$  if it is flat and its fibres have  $P$ . Then  $\mathcal{G} \in \text{Gr}/k[I]$  has property  $P$  if and only if the  $k$ -group scheme  $E = h_*\mathcal{G}$  has  $P$ .*

In order to show that  $h_*$  is an equivalence, we build a quasi-inverse  $h^+$  which we call *Weil extension*. The construction and study of this functor is in fact the hardest part of the proof.

As an application of the Theorem, we prove a Dieudonné classification for smooth, commutative, unipotent group schemes over the generalized dual numbers of a perfect field  $k$ . This takes the form of an exact equivalence of categories with a category of extensions of smooth, erasable Dieudonné modules. Here is the precise statement (we refer to Section 7 for the definition of all undefined terms).

**Theorem B** (see Theorem 7.2.7) *Let  $\text{SCU}/k[I]$  be the category of smooth, commutative, unipotent  $k[I]$ -group schemes. Let  $\mathbb{D}\text{-I-Mod}$  be the category of  $I$ -extensions of smooth erasable Dieudonné modules. Then the Dieudonné functor  $M : \text{CU}/k \rightarrow \mathbb{D}\text{-Mod}$  induces a contravariant equivalence of categories:*

$$\mathcal{M} : \text{SCU}/k[I] \rightarrow \mathbb{D}\text{-I-Mod}$$

*that sends  $\mathcal{U}$  to the extension  $0 \rightarrow M(\mathcal{U}_k) \rightarrow M(h_*\mathcal{U}) \rightarrow M(\text{Lie}(\mathcal{U}_k, I)) \rightarrow 0$ . A quasi-inverse functor is obtained by sending an extension  $0 \rightarrow M \rightarrow M' \rightarrow \text{Lie}(M, I) \rightarrow 0$  to the Weil extension  $h^+(U(M'))$  of the extension  $0 \rightarrow \text{Lie}(U(M), I) \rightarrow U(M') \rightarrow U(M) \rightarrow 0$ , where  $U$  is a quasi-inverse for  $M$ .*

**1.1.3 Comments.** An important tool in many of our arguments is the *group algebra scheme*. It provides a common framework to conduct computations in the groups and their tangent bundles simultaneously. It allows us to describe conveniently the Weil restriction of a group scheme, and is essential in the proof of the Main Theorem. Since we are not aware of any treatment of the group algebra scheme in the literature, we give a detailed treatment in Section 2. We illustrate the usefulness of this concept by showing how it allows to work out easily the deformation theory of affine smooth group schemes (Subsection 3.3).

Let us say a word on the assumptions. The choice to work with differentially flat group schemes instead of simply smooth ones is not just motivated by the search for maximal generality or aesthetic reasons. It is also extremely useful because when working with an affine, smooth group scheme  $G$ , we use our results also for the group algebra  $\mathbb{O}[G]$  in the course of proving the main theorem; and the group algebra  $\mathbb{O}[G]$  is differentially flat and rigid, but usually infinite-dimensional and hence not smooth.

There are at least two advantages to work over generalized dual numbers  $k[I]$  rather than simply the usual ring  $k[\varepsilon]$  with  $\varepsilon^2 = 0$ . The first is that in order to prove that our equivalence of categories respects the  $\mathbb{O}_k$ -module stack structure, we have to introduce the ring  $k[I]$  with the two-dimensional  $k$ -module  $I = k\varepsilon + k\varepsilon'$ . Indeed, this is needed to describe the sum of deformations and the Baer sum of extensions. The other advantage is that since arbitrary local Artin  $k$ -algebras are filtered by Artin  $k$ -algebras whose maximal ideal has square zero, our results may be useful in handling deformations along more general thickenings.

**1.1.4 Plan of the article.** The present Section 1 ends with material of preliminary nature on the explanation of the  $\mathbb{O}_k$ -module stack structure of the categories  $\text{Gr}/k[I]$  and  $\text{Ext}(I)/k$  (Subsection 1.2) and on the notations of Weil restriction (Subsection 1.3). Then in Section 2 are introduced the group algebra schemes. In Section 3 are introduced notions from differential calculus: tangent bundle, Lie algebra and exponentials, in the level of generality we need. In Section 4 we describe the functor  $h_*$  of our Main Theorem, while in the next Section 5 we construct a quasi-inverse. In Section 6 we prove the Main Theorem, namely, that these functors are indeed quasi-inverse. Finally, in Section 7 we deduce a classification of Dieudonné type for smooth, commutative, unipotent group schemes over the dual numbers. In the Appendix, we provide a careful definition of  $\mathbb{O}_k$ -module stacks in groupoids and we highlight some of its features.

**1.1.5 Acknowledgements.** For various discussions and remarks, we thank Sylvain Brochard, Xavier Caruso, Brian Conrad, Bernard Le Stum, Brian Osserman, and Tobias Schmidt. We are also grateful to the CIRM in Luminy where part of this research was done. Finally, the first author would like to thank the executive and administrative staff of IRMAR and of the Centre Henri Lebesgue ANR-11-LABX-0020-01 for creating an attractive mathematical environment.

## 1.2 The $\mathbb{O}_k$ -module stack structure of $\text{Gr}/k[I]$ and $\text{Ext}(I)/k$

Both categories  $\text{Gr}/k[I]$  and  $\text{Ext}(I)/k$  are endowed with the structure of  *$\mathbb{O}_k$ -module stacks in groupoids over  $\text{Gr}/k$* . We rely on Appendix A where a full-fledged definition of this notion is given.

In rough terms, this means that in the fibre categories over a fixed  $k$ -group scheme  $G$ , there is an addition law by which one can add pairs of objects, and an external law by which one can multiply objects by scalars of the ring scheme  $\mathbb{O}_k$ . In this subsection, we give a description of these structures. The proofs that we indeed obtain  $\mathbb{O}_k$ -module stack structures, apart from being tedious and not especially enlightening, are present in the literature (sometimes hidden beneath the surface). For these reasons, we just give hints.

**1.2.1 The  $\mathbb{O}_k$ -module stack  $\text{Gr}/k[I] \rightarrow \text{Gr}/k$ .** Let  $G \in \text{Gr}/k$  be fixed. Let  $\mathcal{G}_1, \mathcal{G}_2 \in \text{Gr}/k[I]$  with identifications  $i^*\mathcal{G}_1 \simeq G \simeq i^*\mathcal{G}_2$ . The addition is obtained by a two-step process. First we glue these

group schemes along their common closed subscheme  $G$ :

$$\mathcal{G}' := \mathcal{G}_1 \amalg_G \mathcal{G}_2.$$

This lives over the scheme  $\mathrm{Spec}(k[I]) \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k[I]) = \mathrm{Spec}(k[I \oplus I])$ . The properties of gluing along infinitesimal thickenings are studied in the Stacks Project [SP]. We point out some statements relevant to our situation: existence of the coproduct in Tag 07RV, a list of properties preserved by gluing in Tag 07RX, gluing of modules in Tag 08KU and preservation of flatness of modules in Tag 07RW. It follows from these results that  $\mathcal{G}'$  is an object of  $\mathrm{Gr}/k[I \oplus I]$ . Then by pullback along the closed immersion

$$j : \mathrm{Spec}(k[I]) \hookrightarrow \mathrm{Spec}(k[I \oplus I])$$

induced by the addition morphism  $I \oplus I \rightarrow I$ ,  $i_1 \oplus i_2 \mapsto i_1 + i_2$  we form the desired sum:

$$\mathcal{G}_1 + \mathcal{G}_2 := j^*(\mathcal{G}') = j^*(\mathcal{G}_1 \amalg_G \mathcal{G}_2).$$

The neutral element for this addition is the group scheme  $h^*G$ . The scalar multiplication by  $\lambda \in k$  is given by rescaling using the scheme map  $s_\lambda : \mathrm{Spec}(k[I]) \rightarrow \mathrm{Spec}(k[I])$  induced by the  $k$ -algebra map  $k[I] \rightarrow k[I]$  which is multiplication by  $\lambda$  in  $I$ :

$$\lambda \mathcal{G} := s_\lambda^* \mathcal{G}.$$

The axioms of  $\mathbb{O}_k$ -module stacks can be checked but we leave the details to the courageous reader.

**1.2.2 The  $\mathbb{O}_k$ -module stack  $\mathrm{Ext}(I)/k \rightarrow \mathrm{Gr}/k$ .** Again let  $G \in \mathrm{Gr}/k$  be fixed and set  $L := \mathrm{Lie}(G, I)$ . Let  $E_1, E_2 \in \mathcal{E}xt(G, L)$  be two extensions. Their addition is given by Baer sum; here again this is a two-step process. Namely, we first form the fibre product  $E' = E_1 \times_G E_2$  which is an extension of  $G$  by  $L \times L$ . Then the Baer sum is the pushforward of this extension along the addition map  $+ : L \times L \rightarrow L$ . All in all, we have the following diagram which serves as a definition of  $E_1 + E_2$ :

$$\begin{array}{ccccccc} 1 & \longrightarrow & L \times L & \longrightarrow & E' & \longrightarrow & G \longrightarrow 1 \\ & & \downarrow + & \lrcorner & \downarrow & & \parallel \\ 1 & \longrightarrow & L & \longrightarrow & E_1 + E_2 & \longrightarrow & G \longrightarrow 1. \end{array}$$

Explicitly, the underlying group scheme of the Baer sum is given by  $E_1 + E_2 = E'/M$  where  $M = \ker(L \times L \rightarrow L) = \{(x, -x), x \in L\}$ . The neutral element for this addition is the trivial extension  $E = L \rtimes G$ . Even though this is not emphasized in the literature, the usual proofs of the fact that the set of extensions endowed with the Baer sum operation is an abelian group provide explicit associativity and commutativity constraints proving that  $\mathcal{E}xt(G, L)$  is a Picard category. The associativity constraint is obtained by expressing  $(E_1 + E_2) + E_3$  and  $E_1 + (E_2 + E_3)$  as isomorphic quotients of  $E_1 \times_G E_2 \times_G E_3$ , and the commutativity constraint is obtained from the flipping morphism in  $E_1 \times_G E_2$ . The scalar multiplication by  $\lambda \in k$  is given by pushforward along the multiplication-by- $\lambda$  morphism in the module scheme  $L = \mathrm{Lie}(G, I)$ . All in all, we have the following diagram which serves as a definition of  $\lambda E$ :

$$\begin{array}{ccccccc} 1 & \longrightarrow & L & \longrightarrow & E & \longrightarrow & G \longrightarrow 1 \\ & & \downarrow \lambda & \lrcorner & \downarrow & & \parallel \\ 1 & \longrightarrow & L & \longrightarrow & \lambda E & \longrightarrow & G \longrightarrow 1. \end{array}$$

Again, the verification of the axioms of an  $\mathbb{O}_k$ -module stack is tedious but not difficult.

### 1.3 Weil restriction generalities

Weil restriction plays a prominent role in our work; we briefly give the main definitions and notations related to this formalism. We refer to [BLR90], § 7.6 for more details; we will occasionally make use of some well-known results found in that reference without explicitly saying so. Let  $h : \mathrm{Spec}(k') \rightarrow \mathrm{Spec}(k)$  be a finite, locally free morphism of affine schemes. Let  $(\mathrm{Sch}/k)$  be the category of  $k$ -schemes and  $(\mathrm{Fun}/k)$  the category of functors  $(\mathrm{Sch}/k)^\circ \rightarrow (\mathrm{Sets})$ . The Yoneda functor embeds the former category into the latter. We define  $(\mathrm{Sch}/k')$  and  $(\mathrm{Fun}/k')$  similarly. By sending a morphism of functors  $f : X' \rightarrow \mathrm{Spec}(k')$  to the morphism  $h \circ f : X' \rightarrow \mathrm{Spec}(k)$  we obtain a functor  $h_! : (\mathrm{Fun}/k') \rightarrow (\mathrm{Fun}/k)$ . Sometimes we will refer to  $h_!X'$  as *the  $k'$ -functor  $X'$  viewed as a  $k$ -functor* and the notation  $h_!$  will be omitted. The pullback functor  $h^* : (\mathrm{Fun}/k) \rightarrow (\mathrm{Fun}/k')$  is right adjoint to  $h_!$ , in particular  $(h^*X)(S') = X(h_!S')$  for all  $k'$ -schemes  $S'$ . The Weil restriction functor  $h_* : (\mathrm{Fun}/k') \rightarrow (\mathrm{Fun}/k)$  is right adjoint to  $h^*$ , in particular we have  $(h_*X')(S) = X'(h^*S')$  for all  $k$ -schemes  $S$ . Thus we have a sequence of adjoint functors:

$$h_!, h^*, h_*.$$

The functors  $h_!$  and  $h^*$  preserve the subcategories of schemes. The same is true for  $h_*$  if  $h$  is radicial, a case which covers our needs (and see [BLR90], § 7.6 for refined representability results). We write

$$\alpha : 1 \longrightarrow h_*h^* \quad \text{and} \quad \beta : h^*h_* \longrightarrow 1$$

the unit and counit of the  $(h^*, h_*)$ -adjunction. If  $X$  is a  $k$ -scheme then  $\alpha_X : X \rightarrow h_*h^*X$  is a closed immersion. If  $X'$  is a  $k'$ -group (resp. algebra) functor (resp. scheme), then also  $h_*X'$  is a  $k$ -group (resp. algebra) functor (resp. scheme). If moreover  $X' \rightarrow \mathrm{Spec}(k')$  is smooth of relative dimension  $n$ , then  $h_*X' \rightarrow \mathrm{Spec}(k)$  is smooth of relative dimension  $n[k' : k]$  where  $[k' : k]$  is the locally constant rank of  $h$ .

Quite often, it is simpler to consider functors defined on the subcategory of affine schemes, in other words functors of the form

$$(k\text{-Alg}) \xrightarrow{\mathrm{Spec}} (\mathrm{Sch}/k)^\circ \longrightarrow (\mathrm{Sets}).$$

The functors  $h_!, h^*, h_*$  are defined similarly in this context.

## 2 Group algebras of group schemes

Let  $k$  be a (commutative, unitary) ring. Let  $\mathbb{O}_k$  be the  $\mathrm{Spec}(k)$ -ring scheme such that if  $R$  is a  $k$ -algebra then  $\mathbb{O}_k(R) = R$  with its ring structure. If the base ring is clear from context, we write  $\mathbb{O}$  instead of  $\mathbb{O}_k$ . Let  $G$  be an affine  $k$ -group scheme. In this subsection, we explain the construction of the group algebra  $\mathbb{O}[G]$ , which is the analogue in the setting of algebraic geometry of the usual group algebra of abstract discrete groups. Note that for a finite constant group scheme, the set  $\mathbb{O}[G](k)$  of  $k$ -rational points of the group algebra and the usual group algebra  $k[G]$  are isomorphic, but for other groups they do not have much in common in general; this will be emphasized below.

Since we are not aware of any treatment of the concept of group algebra of an affine group scheme in the existing literature, we give a somewhat detailed treatment, including examples (2.2.4–2.2.6), basic properties (2.3.1) and a proof of the universal property (2.3.2).

### 2.1 Linear algebra schemes

**2.1.1 Vector bundles.** Let  $S = \mathrm{Spec}(k)$ . As in [EGA1<sub>new</sub>], we call *vector bundle* an  $\mathbb{O}_S$ -module scheme of the form  $\mathbb{V}(\mathcal{F}) = \mathrm{Spec} S(\mathcal{F})$  where  $\mathcal{F}$  is a quasi-coherent  $\mathcal{O}_S$ -module and  $S(\mathcal{F})$  is its symmetric algebra. We say *smooth vector bundle* if  $\mathcal{F}$  is locally free of finite rank. For all  $S$ -schemes  $T$ , we have:

$$\mathbb{V}(\mathcal{F})(T) = \mathrm{Hom}_{\mathcal{O}_T\text{-Mod}}(\mathcal{F}_T, \mathcal{O}_T)$$

where  $\mathcal{F}_T$  is the pullback of  $\mathcal{F}$  along  $T \rightarrow S$ ; see [EGA1<sub>new</sub>], 9.4.9. If  $f : X \rightarrow S$  is a quasi-compact, quasi-separated  $S$ -scheme, the sheaf  $f_*\mathcal{O}_X$  is quasi-coherent and we set  $\mathbb{V}(X/S) := \mathbb{V}(f_*\mathcal{O}_X)$ . There is a canonical  $S$ -morphism

$$\nu_X : X \longrightarrow \mathbb{V}(X/S)$$

which is initial among all  $S$ -morphisms  $X \rightarrow \mathbb{V}(\mathcal{F})$  from  $X$  to a vector bundle. We call it the *vector bundle envelope* of  $X/S$ . If  $X$  is affine over  $S$ , the map  $\nu_X$  is a closed immersion because it is induced by the morphism of algebras  $S(f_*\mathcal{O}_X) \rightarrow f_*\mathcal{O}_X$  induced by the identity  $f_*\mathcal{O}_X \rightarrow f_*\mathcal{O}_X$ , hence surjective.

**2.1.2 The ring scheme  $\mathbb{O}_k$ .** The  $k$ -ring scheme  $\mathbb{O}_k$  is defined as a functor by the requirement that  $\mathbb{O}_k(R)$  is the ring  $R$  itself, for all  $k$ -algebras  $R$ . If  $h : \mathrm{Spec}(k') \rightarrow \mathrm{Spec}(k)$  is a morphism of affine schemes, there is a morphism of  $k'$ -ring schemes  $h^*\mathbb{O}_k \rightarrow \mathbb{O}_{k'}$  which is the identity on points, hence an isomorphism of ring schemes. However, whereas the target has a natural structure of  $\mathbb{O}_{k'}$ -algebra, the source does not. For this reason, the pullback of module functors or module schemes along  $h$  is defined as  $M \mapsto h^*M \otimes_{h^*\mathbb{O}_k} \mathbb{O}_{k'}$ , as is familiar for the pullback of modules on ringed spaces. Usually we will write simply  $h^*M$ .

For use in Section 3 and later sections, we give some complements on the case where  $k' = k[I]$  is the ring of generalized dual numbers, for some finite free  $k$ -module  $I$ . We will identify  $I$  and its dual  $I^\vee = \mathrm{Hom}_k(I, k)$  with the coherent  $\mathcal{O}_{\mathrm{Spec}(k)}$ -modules they define, thus we have the vector bundle  $\mathbb{V}(I^\vee)$ . For each  $k$ -algebra  $R$ , we have  $R[I] = R \oplus I \otimes_k R$  where  $IR \simeq I \otimes_k R$  has square 0. This decomposition functorial in  $R$  gives rise to a direct sum decomposition of  $\mathbb{O}_k$ -module schemes:

$$h_*\mathbb{O}_{k[I]} = \mathbb{O}_k \oplus \mathbb{V}(I^\vee).$$

It is natural to use the notation  $\mathbb{O}_k[\mathbb{V}(I^\vee)]$  for the  $\mathbb{O}_k$ -algebra scheme on the right-hand side, however we will write more simply  $\mathbb{O}_k[I]$ . Now we move up on  $\mathrm{Spec}(k[I])$ , where there is a morphism of  $\mathbb{O}_{k[I]}$ -module schemes  $h^*\mathbb{V}(I^\vee) \rightarrow \mathbb{O}_{k[I]}$  defined for all  $k[I]$ -algebras  $R$  as the morphism  $I \otimes_k R \rightarrow R$ ,  $i \otimes x \mapsto i_R x$  where  $i_R := i \cdot 1_R$ . According to what we said before, the pullback  $h^*\mathbb{V}(I^\vee)$  has the module structure such that a section  $a \in \mathbb{O}_{k[I]}$  acts by  $a \cdot i \otimes x := i \otimes ax$ . The image of  $h^*\mathbb{V}(I^\vee) \rightarrow \mathbb{O}_{k[I]}$  is the ideal  $I \cdot \mathbb{O}_{k[I]}$  defined by  $(I \cdot \mathbb{O}_{k[I]})(R) = IR$  for all  $k[I]$ -algebras  $R$ . There is an exact sequence of  $\mathbb{O}_{k[I]}$ -module schemes:

$$0 \longrightarrow I \cdot \mathbb{O}_{k[I]} \longrightarrow \mathbb{O}_{k[I]} \longrightarrow i_*\mathbb{O}_k \longrightarrow 0.$$

**2.1.3  $\mathbb{O}$ -Algebra schemes.** Here we define a category ( $\mathbb{O}$ -Alg) of *linear  $\mathbb{O}$ -algebra schemes* and give a summary of elementary properties. For us an  *$\mathbb{O}$ -algebra scheme* is a  $k$ -scheme  $D$  endowed with two internal composition laws  $+, \times : D \times D \rightarrow D$  called addition and multiplication possessing neutral sections  $0, 1 : \mathrm{Spec}(k) \rightarrow D$ , and an external law  $\cdot : \mathbb{O} \times D \rightarrow D$ , such that for each  $k$ -algebra  $R$  the tuple  $(D(R), +, \times, 0, 1, \cdot)$  is an associative unitary  $R$ -algebra. In particular  $(D, +, 0)$  is a commutative group scheme, and  $(D(R), \times, 1)$  is a (possibly noncommutative) monoid. We say that  $D$  is a *linear  $\mathbb{O}$ -algebra scheme* if its underlying  $\mathbb{O}$ -module scheme is a vector bundle. In this case  $\mathcal{F}$  can be recovered as the "dual bundle" sheaf  $\mathcal{F} = \mathrm{Hom}_{\mathbb{O}\text{-Mod}}(D, \mathbb{O})$ , the Zariski sheaf over  $S$  whose sections over an open  $U$  are the morphisms of  $\mathbb{O}|_U$ -modules  $D|_U \rightarrow \mathbb{O}|_U$  (read [EGA1<sub>new</sub>], 9.4.9 between the lines). For an affine  $\mathbb{O}$ -algebra scheme in our sense, the comultiplication is a map  $\Delta : S(\mathcal{F}) \rightarrow S(\mathcal{F}) \otimes S(\mathcal{F})$  and the bilinearity of  $m$  implies that this map is induced from a map  $\Delta_0 : \mathcal{F} \rightarrow \mathcal{F} \otimes \mathcal{F}$ . Finally we point out two constructions on linear  $\mathbb{O}$ -algebra schemes. The first is the tensor product  $D \otimes_{\mathbb{O}} D'$ , which as a functor is defined as  $R \mapsto D(R) \otimes_R D'(R)$ . If  $D = \mathbb{V}(\mathcal{F})$  and  $D' = \mathbb{V}(\mathcal{F}')$  with  $\mathcal{F}, \mathcal{F}'$  locally free of finite ranks, then  $D \otimes_{\mathbb{O}} D' \simeq \mathbb{V}(\mathcal{F} \otimes \mathcal{F}')$ . The second construction is the group of units. We observe that for any linear  $\mathbb{O}$ -algebra scheme  $D$ , the subfunctor  $D^\times \subset D$  of (multiplicative) units is the preimage under the multiplication  $m : D \times D \rightarrow D$  of the unit section  $1 : \mathrm{Spec}(k) \rightarrow D$  and is therefore representable by an affine scheme. This gives rise to the *group of units* functor  $(\mathbb{O}\text{-Alg}) \rightarrow (k\text{-Gr})$ ,  $D \mapsto D^\times$  where  $(k\text{-Gr})$  is the category of affine  $k$ -group schemes.



**2.1.4 Remark.** We do not know if an  $\mathbb{O}$ -algebra scheme whose underlying scheme is affine over  $S$  is always of the form  $\mathbb{V}(\mathcal{F})$ .

## 2.2 Group algebra: construction and examples

Let  $G = \text{Spec}(A)$  be an affine  $k$ -group scheme. We write  $(u, v) \mapsto u \star v$  or sometimes simply  $(u, v) \mapsto uv$  the multiplication of  $G$ . This operation extends to the vector bundle envelope  $\mathbb{V}(G/k)$ , as follows. Let  $\Delta : A \rightarrow A \otimes_k A$  be the comultiplication. For each  $k$ -algebra  $R$ , we have  $\mathbb{V}(G/k)(R) = \text{Hom}_{k\text{-Mod}}(A, R)$ . If  $u, v : A \rightarrow R$  are morphisms of  $k$ -modules, we consider the composition  $u \star v := (u \otimes v) \circ \Delta$ :

$$u \star v : A \xrightarrow{\Delta} A \otimes_k A \xrightarrow{u \otimes v} R.$$

Here the map  $u \otimes v : A \otimes_k A \rightarrow R$  is  $a \otimes b \mapsto u(a)v(b)$ .

**2.2.1 Definition.** The *group algebra of the  $k$ -group scheme  $G$* :

$$\mathbb{O}[G] := (\mathbb{V}(G/k), +, \star),$$

is the vector bundle  $\mathbb{V}(G/k)$  endowed with the product just defined. We write  $\nu_G : G \hookrightarrow \mathbb{O}[G]$  the closed immersion as in paragraph 2.1.1.

We check below that  $\mathbb{O}[G]$  is a linear  $\mathbb{O}$ -algebra scheme. Apart from  $G(R)$ , there is another noteworthy subset inside  $\mathbb{O}[G](R)$ , namely the set  $\text{Der}_G(R)$  of  $k$ -module maps  $d : A \rightarrow R$  which are  $u$ -derivations for some  $k$ -algebra map  $u : A \rightarrow R$  (which need not be determined by  $d$ ); a more accurate notation would be  $\text{Der}(\mathcal{O}_G, \mathbb{O})(R)$  but we favour lightness. Here are the first basic properties coming out of the construction.

**2.2.2 Proposition.** *Let  $G$  be an affine  $k$ -group scheme. Let  $\mathbb{O}[G]$  and  $\text{Der}_G$  be as described above.*

- (1) *The tuple  $\mathbb{O}[G] := (\mathbb{V}(G/k), +, \star)$  is a linear  $\mathbb{O}$ -algebra scheme.*
- (2) *As a  $k$ -scheme,  $\mathbb{O}[G]$  is  $k$ -flat (resp. has  $k$ -projective function ring) iff  $G$  has the same property.*
- (3) *The composition  $G \hookrightarrow \mathbb{O}[G]^\times \hookrightarrow \mathbb{O}[G]$  is a closed embedding, hence also  $G \hookrightarrow \mathbb{O}[G]^\times$ .*
- (4) *The subfunctor  $\text{Der}_G \subset \mathbb{O}[G]$  is stable by multiplication by  $G$  on the left and on the right, so it acquires left and right  $G$ -actions. More precisely, if  $u, v : A \rightarrow R$  are maps of algebras,  $d : A \rightarrow R$  is a  $u$ -derivation and  $d' : A \rightarrow R$  is a  $v$ -derivation, then  $d \star v$  and  $u \star d'$  are  $(u \star v)$ -derivations.*

**Proof :** (1) We must check that  $\star$  is associative, unitary and bilinear. For associativity:

$$\begin{aligned} (u \star v) \star w &= [((u \otimes v) \circ \Delta) \otimes w] \circ \Delta \\ &= (u \otimes v \otimes w) \circ (\Delta \otimes \text{id}) \circ \Delta \\ &= (u \otimes v \otimes w) \circ (\text{id} \otimes \Delta) \circ \Delta \text{ by coassociativity of } \Delta, \\ &= [u \otimes ((v \otimes w) \circ \Delta)] \circ \Delta \\ &= u \star (v \star w). \end{aligned}$$

Now we prove that if  $s : k \rightarrow R$  is the structure map then  $e_R := s \circ e : A \rightarrow k \rightarrow R$  is a two-sided unit for  $\star$ . This follows from the fact that because  $e_R$  factors through  $k$ , for any  $k$ -linear map  $u : A \rightarrow R$ , the map  $u \otimes e_R : A \otimes_k A \rightarrow R$  is the composition of  $\text{id} \otimes e : A \otimes_k A \rightarrow A$  with  $u$ . Thus:

$$\begin{aligned} u \star e_R &= (u \otimes e_R) \circ \Delta \\ &= u \circ (\text{id} \otimes e) \circ \Delta \\ &= u \circ \text{id} \text{ by the property of the counit,} \\ &= u. \end{aligned}$$

The fact that  $e_R \star u = u$  is proved in a similar way. The fact that  $\star$  is bilinear follows from the bilinearity of  $\otimes$ . This finishes the proof of (1).

(2) This is because the symmetric algebra of  $A$  is flat (resp. projective) if and only if  $A$  is.

(3) From the definitions, the morphism  $\nu_G : G \hookrightarrow \mathbb{O}[G]$  factors through  $\mathbb{O}[G]^\times$ . Since  $\nu_G$  is a closed embedding as noticed in paragraph 2.1.1, then  $G \hookrightarrow \mathbb{O}[G]^\times$  is also a closed embedding.

(4) Using the fact that multiplication in  $A \otimes_k A$  is given by linear extension of the rule  $a_1 \otimes a_2 \cdot b_1 \otimes b_2 = a_1 b_1 \otimes a_2 b_2$ , one computes that  $d \otimes v$  is a  $(u \otimes v)$ -derivation. The claim for  $d \star v$  follows by precomposition with  $\Delta$ . The claim for  $u \star d'$  is proven similarly.  $\square$

**2.2.3 Remark on Hopf algebra structure.** If  $G$  is a finite, locally free  $k$ -group scheme, then the multiplication of its function ring induces a comultiplication on  $\mathbb{O}[G]$  making it a Hopf  $\mathbb{O}$ -algebra scheme. Moreover the  $k$ -algebra  $\mathbb{O}[G](k)$  is the ring of functions of the Cartier dual  $G^\vee$ . Even though we will not make much use of this Hopf algebra structure, it highlights Examples 2.2.5 and 2.2.6 below.

**2.2.4 Example 1: finite locally free groups.** If  $G$  is a finite locally free  $k$ -group scheme, we shall prove the following things. The algebra  $\mathbb{O}[G]$  is a smooth vector bundle of rank  $[G : \text{Spec}(k)]$  and its group of units is the complement in  $\mathbb{O}[G]$  of the Cartier divisor equal to the zero locus of the determinant of the left regular representation:

$$\mathbb{O}[G] \xrightarrow{\mathbb{L}} \text{End}_{\mathbb{O}\text{-Mod}}(\mathbb{O}[G]) \xrightarrow{\det} \mathbb{A}^1.$$

If moreover  $G$  is the finite constant  $k$ -group scheme defined by a finite abstract group  $\Gamma$ , then  $\mathbb{O}[G]$  is isomorphic to the algebra scheme defined by the abstract group algebra  $k[\Gamma]$ , that is to say  $\mathbb{O}[G](R) \simeq R[\Gamma]$  functorially in  $R$ .

**Proof :** That  $\mathbb{O}[G]$  is a smooth vector bundle of rank  $n$  is clear. For the description of the group of units, since the left regular representation  $\mathbb{L}$  is a morphism of algebra schemes, any element  $u \in \mathbb{O}[G]^\times(R)$  maps to an invertible element of  $\text{End}_{\mathbb{O}\text{-Mod}}(\mathbb{O}[G])(R)$  and its determinant is thus invertible. Conversely if  $\det(\mathbb{L}(u)) \in R^\times$  then  $\mathbb{L}(u)$  is invertible in the endomorphism algebra, and the Cayley-Hamilton theorem says that its inverse is a polynomial in  $\mathbb{L}(u)$ , hence an element of  $\mathbb{L}(\mathbb{O}[G]^\times)(R)$ .

Now we assume that  $G$  is defined by the finite group  $\Gamma$ . Then  $A$  is isomorphic to the algebra of functions  $\Gamma \rightarrow k$ , a basis of which is given by the indicator functions  $\delta_\gamma$  for  $\gamma \in \Gamma$ . The comultiplication is given by  $\Delta(\delta_\gamma) = \sum_{\alpha\beta=\gamma} \delta_\alpha \otimes \delta_\beta$ . Let  $\{\delta_\gamma^\vee\}$  be the dual basis of  $\text{Hom}_{k\text{-Mod}}(A, k)$ . We have:

$$\mathbb{O}[G](R) = \text{Hom}_{k\text{-Mod}}(A, R) = \bigoplus_{\gamma \in \Gamma} R \delta_\gamma^\vee$$

and the product is given by  $\delta_\alpha^\vee \star \delta_\beta^\vee = (\delta_\alpha^\vee \otimes \delta_\beta^\vee) \circ \Delta = \delta_{\alpha\beta}^\vee$ . This is the group algebra  $R[G]$  as defined in algebra textbooks, hence the claim.  $\square$

**2.2.5 Example 2: the additive group.** Let  $G = \mathbb{G}_a = \text{Spec}(k[x])$ . For a  $k$ -algebra  $R$ , let

$$R\langle\langle t \rangle\rangle = R[[t, t^{[2]}, t^{[3]}, \dots]]$$

be the  $R$ -algebra of divided power formal power series in one variable  $t$ , with  $t^{[i]}t^{[j]} = \binom{i+j}{i} t^{[i+j]}$  for all  $i, j \geq 0$ . Then we claim that we have an isomorphism  $\mathbb{O}[G] \xrightarrow{\sim} \mathbb{O}\langle\langle t \rangle\rangle$  given by:

$$\mathbb{O}[G](R) \xrightarrow{\sim} R\langle\langle t \rangle\rangle \quad , \quad (k[x] \xrightarrow{u} R) \longmapsto \sum_{i \geq 0} u(x^i) t^i.$$

Indeed, the product  $u \star v$  of two maps  $u, v : k[x] \rightarrow R$  inside  $\mathbb{O}[G](R)$  is determined by its values on  $x^i$  for all  $i$ . We compute  $(u \star v)(x^i)$ :

$$x^i \xrightarrow{\Delta} (x_1 + x_2)^i = \sum_{j+k=i} \binom{i}{j} x_1^j x_2^k \xrightarrow{u \otimes v} \sum_{j+k=i} \binom{i}{j} u(x_1^j) v(x_2^k).$$

This is the  $i$ -th coefficient of the product of  $\sum_{j \geq 0} u(x_1^j) t^{[j]}$  and  $\sum_{k \geq 0} v(x_2^k) t^{[k]}$  in the ring  $R\langle\langle t \rangle\rangle$ , as claimed. Under this identification,  $G(R)$  is the set of power series of the form  $e^{\alpha t} := \sum_{i \geq 0} \alpha^i t^{[i]}$  for some  $\alpha \in R$ ,  $\mathbb{O}[G]^\times(R)$  is the set of power series with invertible constant term, and  $\text{Der}_G(R)$  is the set of power series of the form  $\beta t e^{\alpha t}$  for some  $\alpha, \beta \in R$ .

Let us assume moreover that  $k$  has characteristic  $p > 0$ . Then we have the kernel of Frobenius  $H = \alpha_p \hookrightarrow G$ . The algebra  $\mathbb{O}[H](R)$  is identified with the sub- $R$ -algebra of  $R\langle\langle t \rangle\rangle$  generated by  $t$ , which is isomorphic to  $R[t]/(t^p)$  because  $t^p = pt^{[p]} = 0$ .

**2.2.6 Example 3: the multiplicative group.** Let  $G = \mathbb{G}_m = \text{Spec}(k[x, 1/x])$ . Let  $R^{\mathbb{Z}}$  be the product algebra, whose elements are sequences with componentwise addition and multiplication. We have an isomorphism  $\mathbb{O}[G] \xrightarrow{\sim} \prod_{i \in \mathbb{Z}} \mathbb{O}$  given by:

$$\mathbb{O}[G](R) \xrightarrow{\sim} R^{\mathbb{Z}} \quad , \quad (k[x] \xrightarrow{u} R) \longmapsto \{u(x^i)\}_{i \in \mathbb{Z}}.$$

Under this identification  $G(R)$  is the set of geometric sequences  $\{\alpha^i\}_{i \in \mathbb{Z}}$  for some  $\alpha \in R^\times$ ,  $\mathbb{O}[G]^\times(R)$  is the set of sequences of invertible elements, and  $\text{Der}_G(R)$  is the set of sequences of the form  $\{i\beta\alpha^{i-1}\}_{i \in \mathbb{Z}}$  for some  $\alpha \in R^\times$  and  $\beta \in R$ . One proves more generally that for any torus  $G = T$  with character group  $X(T)$ , we have  $\mathbb{O}[T] \xrightarrow{\sim} \prod_{i \in X(T)} \mathbb{O}$ .

Let us now fix an integer  $n \geq 1$ . Then we have the subgroup of  $n$ -th roots of unity  $H = \mu_n \hookrightarrow G$ . The algebra  $\mathbb{O}[H](R)$  is identified with the sub- $R$ -algebra of  $\mathbb{Z}/n\mathbb{Z}$ -invariants  $\mathbb{O}[G](R)^{\mathbb{Z}/n\mathbb{Z}}$  composed of sequences  $\{r_i\}_{i \in \mathbb{Z}}$  such that  $r_{i+nj} = r_i$  for all  $i, j \in \mathbb{Z}$ .

## 2.3 Properties: functoriality and adjointness

Here are some functoriality properties of the group algebra.

**2.3.1 Proposition.** *Let  $G$  be an affine  $k$ -group scheme. The formation of the group algebra  $\mathbb{O}[G]$ :*

- (1) *is functorial in  $G$  and faithful;*
- (2) *commutes with base change  $k'/k$ ;*
- (3) *is compatible with products: there is a canonical isomorphism  $\mathbb{O}[G] \otimes_{\mathbb{O}} \mathbb{O}[H] \xrightarrow{\sim} \mathbb{O}[G \times H]$ ;*
- (4) *is compatible with Weil restriction: if  $h : \text{Spec}(k') \rightarrow \text{Spec}(k)$  is a finite locally free morphism of schemes, there is an isomorphism of  $\mathbb{O}_k$ -algebra schemes  $\mathbb{O}_k[G] \otimes_{\mathbb{O}_k} h_* \mathbb{O}_{k'} \xrightarrow{\sim} h_* h^* \mathbb{O}_k[G]$ .*

**Proof :** (1) Any map of affine  $k$ -group schemes  $G = \text{Spec}(A) \rightarrow H = \text{Spec}(B)$  gives rise to a map of  $k$ -Hopf algebras  $B \rightarrow A$  and then by precomposition to a map of  $R$ -algebras  $\mathbb{O}[G](R) \rightarrow \mathbb{O}[H](R)$  which is functorial in  $R$ . Faithfulness follows from the fact that  $G \hookrightarrow \mathbb{O}[G]$  is a closed immersion.

(2) The isomorphism  $\text{Hom}_{k'-\text{Mod}}(A \otimes_k k', R') \xrightarrow{\sim} \text{Hom}_{k-\text{Mod}}(A, R')$ , functorial in the  $k'$ -algebra  $R'$ , gives an isomorphism  $\mathbb{O}_{k'}[G_{k'}] \xrightarrow{\sim} \mathbb{O}[G] \times_{\text{Spec}(k)} \text{Spec}(k')$ .

(3) Write  $G = \text{Spec}(A)$  and  $H = \text{Spec}(B)$ . To a pair of  $k$ -module maps  $u : A \rightarrow R$  and  $v : B \rightarrow R$  we attach the map  $u \otimes v : A \otimes_k B \rightarrow R$ ,  $a \otimes b \mapsto u(a)v(b)$ . This defines an isomorphism

$$\text{Hom}_{k-\text{Mod}}(A, R) \otimes_R \text{Hom}_{k-\text{Mod}}(B, R) \xrightarrow{\sim} \text{Hom}_{k-\text{Mod}}(A \otimes_k B, R)$$

which is functorial in  $R$ . The result follows.

(4) If  $D$  is an  $\mathbb{O}_k$ -algebra scheme, the functorial  $R$ -algebra maps

$$\begin{aligned} D(R) \otimes_R (R \otimes_k k') &\longrightarrow D(R \otimes_k k') \\ d \otimes r' &\longmapsto r'd \end{aligned}$$

fit together to give a morphism  $D \otimes_{\mathbb{O}_k} h_* \mathbb{O}_{k'} \rightarrow h_* h^* D$ . In case  $D = \mathbb{O}_k[G]$  and  $h$  finite locally free, this is none other than the isomorphism  $\mathrm{Hom}_{k\text{-Mod}}(A, R) \otimes_k k' \xrightarrow{\simeq} \mathrm{Hom}_{k\text{-Mod}}(A, R \otimes_k k')$ .  $\square$

Finally we prove the adjointness property. We recall from paragraph 2.1.3 (see also Remark 2.1.4) that  $(\mathbb{O}\text{-Alg})$  is the category of  $\mathbb{O}$ -algebra schemes whose underlying  $\mathbb{O}$ -module scheme is of the form  $\mathbb{V}(\mathcal{F}) = \mathrm{Spec} S(\mathcal{F})$  for some quasi-coherent  $\mathcal{O}_{\mathrm{Spec}(k)}$ -module  $\mathcal{F}$ , and that  $(k\text{-Gr})$  is the category of affine  $k$ -group schemes.

**2.3.2 Theorem (Adjointness property of the group algebra).** *The group algebra functor is left adjoint to the group of units functor. In other words, for all affine  $k$ -group schemes  $G$  and all linear  $\mathbb{O}$ -algebra schemes  $D$ , the map that sends a morphism of algebra schemes  $\mathbb{O}[G] \rightarrow D$  to the composition  $G \subset \mathbb{O}[G]^\times \rightarrow D^\times$  gives a bifunctorial bijection:*

$$\mathrm{Hom}_{\mathbb{O}\text{-Alg}}(\mathbb{O}[G], D) \xrightarrow{\simeq} \mathrm{Hom}_{k\text{-Gr}}(G, D^\times).$$

**Proof :** We describe a map in the other direction and we leave to the reader the proof that it is an inverse. Let  $f : G \rightarrow D^\times$  be a morphism of  $k$ -group schemes. We will construct a map of functors  $f' : \mathbb{O}[G] \rightarrow D$ . We know from paragraph 2.1.3 that  $D = \mathbb{V}(F)$  where  $F$  is a  $k$ -module, and that the comultiplication  $\Delta_D : S(F) \rightarrow S(F) \otimes S(F)$  is induced by a morphism  $\Delta_0 : F \rightarrow F \otimes F$ . Consider the composition  $G \rightarrow D^\times \subset D$  and let  $g : S(F) \rightarrow A$  be the corresponding map of algebras. For each  $k$ -algebra  $R$  we have  $\mathbb{O}[G](R) = \mathrm{Hom}_{k\text{-Mod}}(A, R)$  and  $D(R) = \mathrm{Hom}_{k\text{-Mod}}(F, R)$ . We define  $f'$  as follows:

$$\begin{aligned} \mathbb{O}[G](R) &\longrightarrow D(R) \\ (u : A \rightarrow R) &\longmapsto (u \circ g \circ i : F \rightarrow R) \end{aligned}$$

where  $i : F \hookrightarrow S(F)$  is the inclusion as the degree 1 piece in the symmetric algebra. The map  $f'$  is a map of modules, and we only have to check that it respects the multiplication. Let  $u, v : A \rightarrow R$  be module homomorphisms. We have the following commutative diagram:

$$\begin{array}{ccccc} A & \xrightarrow{\Delta_G} & A \otimes A & \xrightarrow{u \otimes v} & R \\ \uparrow g & & \uparrow g \otimes g & & \\ S(F) & \xrightarrow{\Delta_D} & S(F) \otimes S(F) & & \\ \uparrow i & & \uparrow i \otimes i & & \\ F & \xrightarrow{\Delta_0} & F \otimes F & & \end{array}$$

With the  $\star$  notation as in Subsection 2.2, we compute:

$$\begin{aligned} (u \star_G v) \circ g \circ i &= (u \otimes v) \circ \Delta_G \circ g \circ i \\ &= (ugi \otimes vgi) \circ \Delta_0 \\ &= ugi \star_D vgi. \end{aligned}$$

Thus  $f' : \mathbb{O}[G] \rightarrow D$  is a map of algebra schemes and this ends the construction.  $\square$

**2.3.3 Remark.** It follows from this result that a smooth vector bundle with action of  $G$  is the same as a (smooth)  $\mathbb{O}[G]$ -module scheme. Indeed, the endomorphism algebra of such a vector bundle is representable by a linear  $\mathbb{O}$ -algebra scheme  $D$ . For example, if  $G$  is an affine, finite type, differentially flat  $k$ -group scheme, then the adjoint action on  $\mathrm{Lie} G$  makes it an  $\mathbb{O}[G]$ -module scheme.

### 3 Differential calculus on group schemes

We review the notions of tangent bundle and Lie algebra in the required generality. We introduce the exponential morphism of a  $k$ -group scheme and we establish its main properties, including some special properties needed in the sequel of the paper. Finally in 3.3 we show how these notions, together with the group algebra, allow to recover easily the deformation theory of affine smooth group schemes.

#### 3.1 Tangent bundle and Lie algebra

Let  $k$  be a ring and  $I$  a free  $k$ -module of finite rank  $r \geq 1$  with dual  $I^\vee = \mathrm{Hom}_k(I, k)$ . Let  $k[I]$  be the algebra of dual numbers, i.e.  $k[I] := k \oplus I$  with multiplication determined by the condition  $I^2 = 0$ . Let  $h : \mathrm{Spec}(k[I]) \rightarrow \mathrm{Spec}(k)$  be the structure map and  $i : \mathrm{Spec}(k) \rightarrow \mathrm{Spec}(k[I])$  the closed immersion. Basic structure facts on the ring schemes  $\mathbb{O}_k$  and  $\mathbb{O}_{k[I]}$  are recalled in Paragraph 2.1.2.

Let  $G$  be a  $k$ -group scheme with unit section  $e : \mathrm{Spec}(k) \rightarrow G$ . The *tangent bundle of  $G/k$  relative to  $I$*  is the  $k$ -group scheme defined by:

$$\mathrm{T}(G, I) := h_* h^* G.$$

The  $(h^*, h_*)$  adjunction (see Subsection 1.3) provides two morphisms of group schemes:

$$\begin{aligned} \alpha_G : G &\longrightarrow \mathrm{T}(G, I), \\ \beta_{h^*G} : h^* \mathrm{T}(G, I) &= h^* h_* h^* G \longrightarrow h^* G. \end{aligned}$$

From these we derive  $\pi_G := i^* \beta_{h^*G}$  and its kernel, the *Lie algebra of  $G/k$  relative to  $I$* :

$$\begin{aligned} \pi_G &:= i^* \beta_{h^*G} : \mathrm{T}(G, I) \longrightarrow G, \\ \mathrm{Lie}(G, I) &:= \ker(\pi_G). \end{aligned}$$

The map  $\mathrm{T}(G, I) \rightarrow G$  is a vector bundle which can be described in terms of derivations. If we identify the  $k$ -module  $I$  with the corresponding locally free sheaf on  $\mathrm{Spec}(k)$ , then for all points  $f : S \rightarrow G$  with values in a  $k$ -scheme  $S$  we have ([SGA3.1], Exposé II, Prop. 3.3):

$$\mathrm{Hom}_G(S, \mathrm{T}(G, I)) \xrightarrow{\sim} \mathrm{Der}^f(\mathcal{O}_G, f_* \mathcal{O}_S \otimes_k I).$$

In particular  $\mathrm{Lie}(G, I) \rightarrow \mathrm{Spec}(k)$  is an  $\mathbb{O}_k$ -Lie algebra scheme such that for all  $S/k$  we have

$$\mathrm{Hom}_k(S, \mathrm{Lie}(G, I)) \xrightarrow{\sim} \mathrm{Der}^e(\mathcal{O}_G, e_* \mathcal{O}_S \otimes_k I).$$

It supports the *adjoint representation*, i.e. the action of  $G$  by conjugation inside  $\mathrm{T}(G, I)$ :

$$\mathrm{Ad} : G \rightarrow \mathrm{GL}(\mathrm{Lie}(G, I)) \quad , \quad \mathrm{Ad}(g)(x) = \alpha_G(g)x\alpha_G(g)^{-1}$$

for all points  $g$  of  $G$  and  $x$  of  $\mathrm{Lie}(G, I)$ . Applying the functor  $\mathrm{Lie}$ , that is differentiating at the unit section of  $G$ , we obtain the infinitesimal adjoint representation of the Lie algebra:

$$\mathrm{ad} : \mathrm{Lie} G \rightarrow \mathrm{End}(\mathrm{Lie}(G, I)).$$

When  $I = k\varepsilon$ , from  $\text{ad}$  we deduce the bilinear form called *bracket*  $[-, -] : \text{Lie } G \times \text{Lie } G \rightarrow \text{Lie } G$ . That is, we have  $[x, y] := (\text{ad } x)(y)$  for all points  $x, y$  of  $\text{Lie } G$ .

The tangent bundle  $\text{T}(G, I)$  carries a structure of extension which we now explain. By the triangle identity of adjunction,  $h^*\alpha_G$  is a section of  $\beta_{h^*G}$ , hence  $\alpha_G$  is a section of  $\pi_G$ . Letting  $\gamma_G$  be the inclusion  $\text{Lie}(G, I) \hookrightarrow \text{T}(G, I)$ , we have a split exact sequence:

$$1 \longrightarrow \text{Lie}(G, I) \xrightarrow{\gamma_G} \text{T}(G, I) \begin{array}{c} \xleftarrow{\pi_G} \\ \xrightarrow{\alpha_G} \end{array} G \longrightarrow 1.$$

This is an exact sequence of functors, hence also an exact sequence of group schemes (i.e. of fppf sheaves). Let  $m$  be the multiplication of  $\text{T}(G, I)$ . The splitting gives rise to an isomorphism of  $k$ -schemes:

$$\varrho_G : \text{Lie}(G, I) \times G \xrightarrow{\gamma_G \times \alpha_G} \text{T}(G, I) \times \text{T}(G, I) \xrightarrow{m} \text{T}(G, I),$$

that is, any point of  $\text{T}(G, I)$  may be written uniquely as a product  $\gamma_G(x) \cdot \alpha_G(g)$  for some points  $x \in \text{Lie}(G, I)$  and  $g \in G$ . We will sometimes write briefly  $(x, g) = \gamma_G(x) \cdot \alpha_G(g)$  to denote this point of  $\text{T}(G, I)$ . The conjugation action of  $G$  on  $\text{Lie}(G, I)$  related to the extension structure is given by the adjoint action, thus the group structure of  $\text{T}(G, I)$  can be described by:

$$(x, g) \cdot (x', g') = (x + \text{Ad}(g)x', gg').$$

The dependence of  $\text{T}(G, I)$  and  $\text{Lie}(G, I)$  on  $I$  can be described further. If  $I = k\varepsilon$  so that  $k[I] = k[\varepsilon]$  with  $\varepsilon^2 = 0$ , we write simply  $\text{T}G = \text{T}(G, k\varepsilon)$  and  $\text{Lie } G = \text{Lie}(G, k\varepsilon)$  and we call them the *tangent bundle* and the *Lie algebra* of  $G$ . For general  $I$ , the isomorphisms  $\text{Der}^f(\mathcal{O}_G, f_*\mathcal{O}_S) \otimes_k I \xrightarrow{\sim} \text{Der}^f(\mathcal{O}_G, f_*\mathcal{O}_S \otimes_k I)$  functorial in  $S/G$  induce an isomorphism of vector bundles:

$$\text{T}G \otimes_{\mathbb{O}_G} u^*\mathbb{V}(I^\vee) \xrightarrow{\sim} \text{T}(G, I),$$

with  $u : G \rightarrow \text{Spec}(k)$  the structure map and  $u^*\mathbb{V}(I^\vee)$  the pullback vector bundle. Similarly, the isomorphisms  $\text{Der}(\mathcal{O}_G, f_*\mathcal{O}_S) \otimes_k I \xrightarrow{\sim} \text{Der}(\mathcal{O}_G, f_*\mathcal{O}_S \otimes_k I)$  functorial in  $S/k$  induce an isomorphism of  $\mathbb{O}_k$ -Lie algebra schemes:

$$\text{Lie } G \otimes \mathbb{V}(I^\vee) \xrightarrow{\sim} \text{Lie}(G, I).$$

With  $G$  acting trivially on  $\mathbb{V}(I^\vee)$ , this isomorphism is  $G$ -equivariant.

**3.1.1 Idempotence of Lie.** An important property for us will be the idempotence of the Lie functor, namely the existence of an isomorphism  $d : \text{Lie}(G, I) \rightarrow \text{Lie}(\text{Lie}(G, I))$  as in [SGA3.1], Exposé II, 4.3.2. We give the explicit construction of  $d$  here since it will be needed later on. For this, let  $I, J$  be two finite free  $k$ -modules and note that  $R[I][J] = R \oplus IR \oplus JR \oplus IJR$ . If  $G = \text{Spec}(A)$  is affine, for a  $k$ -algebra  $R$ , the elements of the set  $G(R[I][J])$  are the maps  $f : A \rightarrow R[I][J]$  written  $f(x) = r(x) + s(x) + t(x) + u(x)$  where  $r : A \rightarrow R$  is an algebra map,  $s : A \rightarrow IR$  and  $t : A \rightarrow JR$  are  $r$ -derivations, and  $u : A \rightarrow IJR$  satisfies the identity:

$$u(xy) = r(x)u(y) + r(y)u(x) + s(x)t(y) + s(y)t(x).$$

Thus  $u$  is an  $r$ -derivation if  $s = 0$  or  $t = 0$ . Consider the maps  $p : R[I][J] \rightarrow R[I]$ ,  $J \mapsto 0$  and  $q : R[I][J] \rightarrow R[J]$ ,  $I \mapsto 0$ . Unwinding the definition we see that:

$$(\text{Lie}(\text{Lie}(G, I), J)(R) = \ker (G(R[I][J]) \xrightarrow{(p,q)} G(R[I]) \times G(R[J])).$$

For varying  $R$ , the maps  $R[I \otimes_k J] \rightarrow R[I][J]$ ,  $i \otimes j \mapsto ij$  induce a morphism  $d : \text{Lie}(G, I \otimes_k J) \rightarrow \text{Lie}(\text{Lie}(G, I), J)$ .

**3.1.2 Lemma.** *The morphism  $d : \mathrm{Lie}(G, I \otimes_k J) \rightarrow \mathrm{Lie}(\mathrm{Lie}(G, I), J)$  is an isomorphism.*

If  $I = J = k\varepsilon$  this means simply that  $\mathrm{Lie} G = \mathrm{Lie}(\mathrm{Lie} G)$ . For simplicity of notation, after this lemma and its proof we will most often write  $d$  as an equality, in other words  $\mathrm{Lie}(G, I \otimes_k J) = \mathrm{Lie}(\mathrm{Lie}(G, I), J)$ . This will not cause any ambiguity.

**Proof :** For a  $k$ -algebra  $R$ , the map  $R[I \otimes_k J] \rightarrow R[I][J]$  induces  $G(R[I \otimes_k J]) \rightarrow G(R[I][J])$  which takes  $\mathrm{Lie}(G, I \otimes_k J)(R)$  into  $\mathrm{Lie}(\mathrm{Lie}(G, I), J)(R)$ . We want to construct an inverse for this map. Start from an element  $x \in (\mathrm{Lie}(\mathrm{Lie}(G, I), J))(R)$  i.e.  $x : \mathrm{Spec}(R[I][J]) \rightarrow G$ . Choosing open affine coverings  $G = \cup U_i$  and  $\mathrm{Spec}(R) = \cup V_{i,j}$  such that  $x(V_{i,j}) \subset U_i$  for all  $i, j$ , we reduce to the case where  $G$  is affine equal to  $\mathrm{Spec}(A)$ . Then  $x$  corresponds to a morphism of  $k$ -algebras  $A \rightarrow R[I][J]$ ,  $a \mapsto r(a) + s(a) + t(a) + u(a)$  such that  $r = e$ ,  $s = t = 0$ , in other words  $a \mapsto e(a) + u(a)$ . Since  $I \otimes_k J \simeq IJ$ , we have  $R[I \otimes_k J] \simeq R \oplus IJR$ . We see that our morphism factors uniquely through a morphism  $A \rightarrow R[I \otimes_k J]$  and we are done.  $\square$

## 3.2 Exponential and infinitesimal translation

Demazure and Gabriel in [DG70] use an exponential notation which is flexible enough to coincide in some places with the morphism  $\exp_G$  as we define it below (*loc. cit.* chap. II, § 4, 3.7) and in other places with the morphism  $\gamma_G$  (*loc. cit.* chap. II, § 4, 4.2). The drawback of flexibility is a little loss of precision. We introduce the exponential in a somehow more formal way, as an actual morphism between functors.

**3.2.1 Definition.** The *exponential* of a  $k$ -group scheme  $G$  is the composition:

$$\exp_{G,I} : h^* \mathrm{Lie}(G, I) \xrightarrow{h^* \gamma_G} h^* \mathrm{T}G \xrightarrow{\beta_{h^*G}} h^* G.$$

When  $I$  is clear from context, and also when  $I = k\varepsilon$ , we write  $\exp_G$  instead of  $\exp_{G,I}$ . The following proposition collects some elementary properties of the exponential.

**3.2.2 Proposition.** *The exponential  $\exp_{G,I}$  of a  $k$ -group scheme  $G$  has the following properties.*

(1. Functoriality) *For all morphisms of group functors  $f : G \rightarrow G'$  we have a commutative square:*

$$\begin{array}{ccc} h^* \mathrm{Lie}(G, I) & \xrightarrow{\exp_{G,I}} & h^* G \\ h^* \mathrm{Lie}(f) \downarrow & & \downarrow h^* f \\ h^* \mathrm{Lie}(G', I) & \xrightarrow{\exp_{G',I}} & h^* G'. \end{array}$$

(2. Equivariance) *The map  $\exp_{G,I}$  is  $h^*G$ -equivariant for the adjoint action on  $h^* \mathrm{Lie}(G, I)$  and the conjugation action of  $h^*G$  on itself.*

(3. Infinitesimal translation) *Let  $x$  be a point of  $h^* \mathrm{Lie}(G, I)$ ,  $g$  a point of  $h^*G$  and  $(x, g) = h^* \gamma_G(x) \cdot h^* \alpha_G(g)$  the corresponding point of  $h^* \mathrm{T}(G, I)$ . Then we have  $\beta_{h^*G}(x, g) = \exp_{G,I}(x)g$ .*

(4. Adjoint action of exponentials) *Using the description  $\mathrm{Lie}(G, I) = \mathrm{Lie} G \otimes \mathbb{V}(I^\vee)$ , the morphism*

$$h^* \mathrm{Ad} \circ \exp_{G,I} : h^* \mathrm{Lie}(G, I) \rightarrow h^* \mathrm{GL}(\mathrm{Lie}(G, I))$$

*is equal to  $x \otimes i \mapsto \mathrm{Id} + i \mathrm{ad}(x)$ , that is,  $\mathrm{Ad}(\exp_{G,I}(x \otimes i))x' = x' + i[x, x']$ .*

(5. Exponential of a Lie algebra) *Let  $J$  be another finite free  $k$ -module. Using the isomorphism*

$$\mathrm{Lie}(G, J) \otimes \mathbb{V}(I^\vee) \simeq \mathrm{Lie}(G, J \otimes_k I) \xrightarrow{d} \mathrm{Lie}(\mathrm{Lie}(G, J), I)$$

from Lemma 3.1.2, the morphism of  $k[I]$ -group schemes

$$h^* \mathrm{Lie}(G, J) \otimes_{\mathbb{O}_{k[I]}} h^* \mathbb{V}(I^\vee) = h^* \mathrm{Lie}(\mathrm{Lie}(G, J), I) \xrightarrow{\exp_{\mathrm{Lie}(G, J), I}} h^* \mathrm{Lie}(G, J)$$

is given by the external law of the  $\mathbb{O}_{k[I]}$ -module scheme  $h^* \mathrm{Lie}(G, J)$ . In particular, its image is equal to  $I \cdot h^* \mathrm{Lie}(G, J)$  and its kernel contains  $I \cdot h^* \mathrm{Lie}(\mathrm{Lie}(G, J), I)$ . Besides, if  $I = k\varepsilon$  the map  $\exp_{\mathrm{Lie}(G, J)} : h^* \mathrm{Lie}(G, J) \rightarrow h^* \mathrm{Lie}(G, J)$  is multiplication-by- $\varepsilon$  in the module scheme  $h^* \mathrm{Lie}(G, J)$ .

(6. Kernel) *The two maps:*

$$\exp_{G, I} : h^* \mathrm{Lie}(G, I) \longrightarrow h^* G, \quad \exp_{\mathrm{Lie}(G), I} : h^* \mathrm{Lie}(G, I) \xrightarrow{d} h^* \mathrm{Lie}(\mathrm{Lie}(G), I) \longrightarrow h^* \mathrm{Lie}(G)$$

have equal kernels, thus  $I \cdot h^* \mathrm{Lie}(G, I) \subset \ker(\exp_{G, I})$ . In particular, in case  $I = k\varepsilon$ , the kernel of  $\exp_G : h^* \mathrm{Lie} G \rightarrow h^* G$  is equal to the kernel of the multiplication-by- $\varepsilon$  map in  $h^* \mathrm{Lie} G$ .

**Proof :** (1) This follows from the fact that the morphisms  $\gamma$  and  $\beta$  are functorial.

(2) This follows from the definition of  $\mathrm{Ad}$ .

(3) This is because  $\beta_{h^*G}$  is a morphism of group schemes, so that

$$\beta_{h^*G}(x, g) = \beta_{h^*G}(h^* \gamma_G(x) \cdot h^* \alpha_G(g)) = \beta_{h^*G}(h^* \gamma_G(x)) \cdot \beta_{h^*G}(h^* \alpha_G(g)) = \exp_G(x)g.$$

(4) We have the following diagram with commutative square:

$$\begin{array}{ccccc} h^* \mathrm{Lie} G & \xrightarrow{h^* \gamma_G} & h^* h_* h^* G & \xrightarrow{\beta_{h^*G}} & h^* G \\ & & \downarrow h^* h_* h^* \mathrm{Ad} & & \downarrow h^* \mathrm{Ad} \\ & & h^* h_* h^*(\mathrm{GL}(\mathrm{Lie} G)) & \xrightarrow{\beta_{h^* \mathrm{GL}(\mathrm{Lie} G)}} & h^*(\mathrm{GL}(\mathrm{Lie} G)). \end{array}$$

The composition in the first line is  $\exp_G$  so the commutativity of the diagram says that

$$h^* \mathrm{Ad} \circ \exp_G = \beta_{h^* \mathrm{GL}(\mathrm{Lie} G)} \circ h^* h_* h^* \mathrm{Ad} \circ h^* \gamma_G.$$

Now let  $x \otimes i$  be a point of the group  $\mathrm{Lie}(G, I) = \mathrm{Lie} G \otimes \mathbb{V}(I^\vee)$  (which is generated by such elementary tensors). By [DG70], chap. II, § 4, 4.2, formula (5) with square-zero element  $\varepsilon := i$  we have  $h_* h^* \mathrm{Ad} \circ \gamma_G = \mathrm{Id} + i \mathrm{ad}$ , whence the result.

(5) Let  $R$  be a  $k[I]$ -algebra and  $i_R := i1_R$ , for each  $i \in I$ . On the  $R$ -valued points, the maps in the statement are obtained by application of the functor  $G$  to some morphisms of algebras, as follows:

- $h^* d$  is deduced from the morphism  $R[I \otimes_k J] \rightarrow R[I][J]$ ,  $i \otimes j \mapsto ij$ .
- $\exp_{\mathrm{Lie}(G, I)}$  is deduced from the morphism  $R[I][J] \rightarrow R[J]$ ,  $i \mapsto i_R$ ,  $j \mapsto j$ .

The composition is  $R[I \otimes_k J] \rightarrow R[J]$ ,  $i \otimes j \mapsto i_R j$ . This is the external multiplication by scalars of  $\mathbb{O}_{k[I]}$  in  $(h^* \mathrm{Lie}(G, I))(R)$ , see [DG70], chap. II, § 4, 1.2. The result follows.



(6) Covering  $G$  by open affines, we reduce to the case where  $G = \text{Spec}(A)$ . Let  $R$  be a  $k[I]$ -algebra. An  $R$ -valued point of the kernel of  $\exp_{G,I}$  is a morphism of  $k$ -algebras  $f : A \rightarrow R[I]$ ,  $a \mapsto e(a) + v(a)$  such that  $e(a) + v_R(a) = e(a)$  for all  $a \in A$  where  $v_R : A \rightarrow R$  is the composition

$$A \xrightarrow{v} I \otimes_k R \xrightarrow{i \otimes r \mapsto ir} R.$$

As we saw in (5), the  $e$ -derivation  $v_R$  is the image of  $v$  under  $\exp_{\text{Lie}(G,I)}$ , hence the claim.  $\square$

We finish this subsection with a corollary of the computation of the exponential of a Lie algebra. This will be very useful in the sequel.

**3.2.3 Lemma.** *Let  $G, H$  be two group schemes over  $k$ . Let  $\phi : G \rightarrow \text{Lie}(H, I)$  be a morphism of pointed schemes. Let  $i$  be a section of the ideal  $I \cdot \mathbb{O}_{k[I]} \subset \mathbb{O}_{k[I]}$ . Then the following compositions are both equal to the trivial morphism:*

$$(1) \quad h^* \text{Lie}(G, I) \xrightarrow{\exp_{G,I}} h^* G \xrightarrow{h^* \phi} h^* \text{Lie}(H, I) \xrightarrow{i} h^* \text{Lie}(H, I) .$$

$$(2) \quad h^* \text{Lie}(G, I) \xrightarrow{\exp_{G,I}} h^* G \xrightarrow{h^* \phi} h^* \text{Lie}(H, I) \xrightarrow{\exp_{H,I}} h^* H .$$

**Proof :** By functoriality of  $\exp$  we have a commutative diagram:

$$\begin{array}{ccc} h^* \text{Lie}(G, I) & \xrightarrow{\exp_{G,I}} & h^* G \\ h^* d\phi \downarrow & & \downarrow h^* \phi \\ h^* \text{Lie}(\text{Lie}(H, I), I) & \xrightarrow{\exp_{\text{Lie}(H,I),I}} & h^* \text{Lie}(H, I). \end{array}$$

According to Proposition 3.2.2(5), the image of  $\exp_{\text{Lie}(H,I),I}$  is equal to the subfunctor  $I \cdot h^* \text{Lie}(G, I)$ . Since  $I^2 = 0$ , from this (1) follows. Moreover, by 3.2.2(6) the map  $\exp_{H,I}$  has the same kernel as  $\exp_{\text{Lie}(H),I}$  which by 3.2.2(5) contains  $I \cdot h^* \text{Lie}(H, I)$ . Again, since  $I^2 = 0$ , point (2) follows.  $\square$

### 3.3 Deformations of affine group schemes

In this section, we assume that the  $k$ -group scheme  $G$  is *affine*. In this situation, we want to illustrate the usefulness of the tools presented so far, including the tangent bundle and the group algebra. In particular, we give a clean and short proof of the fact that if  $G$  is affine and smooth, its isomorphism classes of deformations to  $k[I]$  are classified by the second cohomology group  $H^2(G, \text{Lie}(G, I))$ , see Proposition 3.3.3.

The first proposition below shows that many concepts related to differential calculus in  $G$  can be handled very conveniently using the group algebra. One key point is that the identification  $\text{T}(\mathbb{O}[G], I) \simeq \mathbb{O}[G][I]$  in 3.3.1 (1) below allows to transport the elements  $i \in I$  from the world of rings of functions to the world of points. Many differential calculus computations are thereby simplified; we include the examples of the adjoint action and the Lie bracket, but other results can be derived painlessly in a similar fashion (e.g. the results of Proposition 3.2.2).

We note that no care is needed (regarding topologies or flatness assumptions) in order to handle the notion of *split exact sequence* of group schemes; indeed the notion of *H-extension* used in Subsection 5.2 formalizes this.

**3.3.1 Proposition.** *Let  $G$  be an affine  $k$ -group scheme and  $(\mathbb{O}[G], +, \star)$  its group algebra.*

(1) The tangent bundle  $T(\mathbb{O}[G], I)$  of the group algebra is canonically isomorphic to the algebra scheme  $\mathbb{O}[G][I] = \mathbb{O}[G] \oplus \mathbb{O}[G] \cdot I$ . We have a commutative diagram of affine monoid schemes:

$$\begin{array}{ccccccc}
1 & \longrightarrow & \text{Lie } G \otimes \mathbb{V}(I^\vee) & \longrightarrow & T(G, I) & \longrightarrow & G \longrightarrow 1 \\
& & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & (\mathbb{O}[G] \cdot I, +) & \longrightarrow & (\mathbb{O}[G][I]^\times, \star) & \longrightarrow & (\mathbb{O}[G]^\times, \star) \longrightarrow 1 \\
& & \parallel & & \downarrow & & \downarrow \\
& & & & (\mathbb{O}[G] \cdot I, +) & \xrightarrow{a_2 \mapsto 1+a_2} & (\mathbb{O}[G][I], \star) & \xrightarrow{a_1+a_2 \mapsto a_1} & (\mathbb{O}[G], \star)
\end{array}$$

The first two rows are split exact sequences of group schemes. In the last row we have written the points of  $\mathbb{O}[G][I]$  as  $a = a_1 + a_2$  with  $a_1 \in \mathbb{O}[G]$  and  $a_2 \in \mathbb{O}[G] \cdot I$ . The map  $\text{Lie } G \otimes \mathbb{V}(I^\vee) \hookrightarrow (\mathbb{O}[G] \cdot I, +)$  induces an isomorphism between  $\text{Lie } G \otimes \mathbb{V}(I^\vee)$  and the subscheme  $\text{Der}_G^1 \subset \mathbb{O}[G]$  of  $e$ -derivations where  $e$  is the neutral element of  $G$ .

(2) The adjoint action  $\text{Ad} : G \rightarrow \text{GL}(\text{Lie}(G, I))$  can be expressed as a conjugation inside  $\mathbb{O}[G][I]$ :

$$1 + \text{Ad}(g)x = g(1 + x)g^{-1} = 1 + gxg^{-1},$$

for all  $x \in \text{Lie}(G, I) = \text{Lie } G \otimes \mathbb{V}(I^\vee)$  and  $g \in G$ .

(3) The Lie bracket  $[-, -] : \text{Lie } G \times \text{Lie } G \rightarrow \text{Lie } G$  can be expressed as the bracket defined by the associative multiplication of  $\mathbb{O}[G]$ :

$$[x, y] = xy - yx.$$

**3.3.2 Remark.** More generally, for all finite free  $k$ -modules  $I, J$  we have an infinitesimal adjoint action

$$\text{ad} := \text{Lie}(\text{Ad}, J) : \text{Lie}(G, J) \rightarrow \text{Hom}(\text{Lie}(G, I), \text{Lie}(G, I \otimes J))$$

which can be expressed as the bracket defined by the associative multiplication of  $\mathbb{O}[G][I][J]$ , that is,  $[x, y] = xy - yx$  for  $x \in \mathbb{O}[G] \cdot J$  and  $y \in \mathbb{O}[G] \cdot I$ . The expression in the particular case  $I = J = k\varepsilon$  is obtained by writing  $1 + \varepsilon x$  instead of  $1 + x$ .

**Proof :** (1) By Proposition 2.3.1(4), we have an isomorphism of  $\mathbb{O}$ -algebra schemes:

$$T(\mathbb{O}[G], I) = h_* h^* \mathbb{O}[G] \xrightarrow{\simeq} \mathbb{O}[G] \otimes_{\mathbb{O}} h_* \mathbb{O}_{k[I]} \simeq \mathbb{O}[G][I].$$

Under this identification, the map  $T\nu_G : T(G, I) \rightarrow T(\mathbb{O}[G], I) = \mathbb{O}[G][I]$  is given as follows: for a  $k$ -algebra  $R$ , a point  $f \in T(G, I)(R)$  is a morphism  $f : A \rightarrow R[I]$ ,  $f(x) = u(x) + v(x)$  for some unique  $k$ -module homomorphisms  $u : A \rightarrow R$  and  $v : A \rightarrow IR$ , and we have

$$T\nu_G(f) = u + v \in \mathbb{O}[G][I](R).$$

This is the central vertical map in the pictured diagram. The rest is clear.

(2) Using the inclusions of multiplicative monoids  $\alpha_G : G \hookrightarrow \text{T}G$  and  $T\nu_G : \text{T}G \hookrightarrow \mathbb{O}[G][\varepsilon]$ , we can view the conjugation action by  $G$  inside the tangent bundle or inside the tangent group algebra, as we wish. The result follows.

(3) In order to compute  $\text{ad}$  we differentiate and hence work in  $\mathbb{O}[G][\varepsilon, \varepsilon']$ . That is, the Lie algebra embedded by  $y \mapsto 1 + \varepsilon y$  is acted upon by the Lie algebra embedded by  $x \mapsto 1 + \varepsilon' x$ , via conjugation in

the ambient  $\mathbb{O}[G][\varepsilon, \varepsilon']$ . With these notations, the identification  $\text{End}(\text{Lie } G) \xrightarrow{\sim} \text{Lie}(\text{GL}(\text{Lie } G))$  goes by  $f \mapsto 1 + \varepsilon' f$ . All in all, the outcome is that  $\text{ad}(x)$  is determined by the condition that for all  $y$  we have:

$$(1 + \varepsilon' x)(1 + \varepsilon y)(1 - \varepsilon' x) = 1 + \varepsilon((\text{id} + \varepsilon' \text{ad}(x))(y)).$$

Since the left-hand side is equal to  $1 + \varepsilon y + \varepsilon \varepsilon'(xy - yx)$ , this proves our claim.  $\square$

In the next proposition, we say that a deformation  $\mathcal{G}$  of  $G$  is *rigid* if there exists an isomorphism of  $k[\varepsilon]$ -schemes  $\mathcal{G} \xrightarrow{\sim} h^*G$ . Note that if  $G$  is formally smooth then all deformations are rigid, as follows from Lemma 3.3.4 below which is well-known and included for the convenience of the reader.

**3.3.3 Proposition.** *Let  $k$  be a base ring and let  $G$  be an affine  $k$ -group scheme.*

(1) *The set of  $k[I]$ -group scheme structures on the scheme  $h^*G$  that lift the  $k$ -group scheme structure of  $G$  is in bijection with the set of 2-cocycles  $c : G \times G \rightarrow \text{Lie } G$ .*

(2) *The set of isomorphism classes of rigid deformations of  $G$  to  $k[\varepsilon]$  is in bijection with  $H^2(G, \text{Lie}(G, I))$ , the second group cohomology of  $G$  with coefficients in the adjoint representation  $\text{Lie}(G, I) \simeq \text{Lie } G \otimes \mathbb{V}(I^\vee)$ .*

**Proof :** (1) We want to deform the multiplication  $m : G \times G \rightarrow G$ ,  $(u, v) \mapsto uv$  into a multiplication  $\tilde{m} : h^*G \times h^*G \rightarrow h^*G$  which by adjunction we can view as a map:

$$G \times G \rightarrow h_*h^*G = T(G, I), \quad (u, v) \mapsto u \odot v.$$

We use embeddings inside the group algebra as in Proposition 3.3.1; thus both targets of  $m$  and  $\tilde{m}$  are embedded in  $\mathbb{O}[G][I]$ . The condition that  $\tilde{m}$  equals  $m$  modulo  $I$  is that these morphisms agree after composition with the projection  $\pi : \mathbb{O}[G][I] \rightarrow \mathbb{O}[G]$ . In other words the condition is that for all points  $u, v \in G$ , the element  $(u \odot v)(uv)^{-1}$  equals 1 modulo  $I$ . Since this is also a point of  $T(G, I)$ , it belongs to  $\text{Lie}(G, I)$ . Hence we can write

$$u \odot v = (1 + c(u, v))uv$$

for some  $c : G \times G \rightarrow \text{Lie}(G, I)$ . The associativity constraint for  $\odot$  gives equality between the following two expressions:

$$\begin{aligned} (u \odot v) \odot w &= (1 + \varepsilon c(uv + c(u, v)uv, w))(1 + c(u, v))uvw \\ &= (1 + c(uv, w))(1 + c(u, v))uvw \quad \text{using Lemma 3.2.3,} \\ &= (1 + (c(uv, w) + c(u, v)))uvw \end{aligned}$$

and

$$\begin{aligned} u \odot (v \odot w) &= (1 + c(u, vw + c(v, w)vw))u(1 + c(v, w))vw \\ &= (1 + c(u, vw))u(1 + c(v, w))vw \quad \text{using Lemma 3.2.3,} \\ &= (1 + (c(u, vw) + uc(v, w)u^{-1}))uvw. \end{aligned}$$

Simplifying by  $uvw$  which is invertible in the group algebra, and equating, gives:

$$c(uv, w) + c(u, v) = c(u, vw) + uc(v, w)u^{-1}.$$

This is the cocycle relation. Conversely, if  $c : G \times G \rightarrow \text{Lie}(G, I)$  is a 2-cocycle, we define

$$u \odot v := (1 + c(u, v))uv.$$

The cocycle identity gives the associativity of this map. Moreover, inverses for this law exist and are given by the formula  $u^{\odot -1} = u^{-1}(1 - c(u, u^{-1}))$ . So we obtain a  $k[I]$ -group scheme  $\mathcal{G}_c := (\text{Spec}(A[I]), \odot)$  which is a deformation of  $G$ .

(2) Let  $\mathcal{G}$  be a rigid deformation of  $G$ . Choosing an isomorphism of schemes  $\varphi_1 : \mathcal{G} \xrightarrow{\sim} h^*G$ , the induced group scheme structure on  $h^*G$  gives rise to a cocycle  $c$  as explained in (1). Choosing another isomorphism  $\varphi_2 : \mathcal{G} \xrightarrow{\sim} h^*G$ , we have an automorphism  $\xi = \varphi_2 \circ \varphi_1^{-1} : h^*G \rightarrow h^*G$  which restricts to the identity on  $G$ . The map  $G \rightarrow h_*h^* = \text{T}(G, I)$  obtained by adjunction is of the form  $u \mapsto (1 + \psi(u))u$  for some morphism  $\psi : G \rightarrow \text{Lie}(G, I)$ . This means that  $\xi(u) = (1 + \psi(u))u$ . We want to see how the multiplication is transformed by the change of isomorphism. If  $u' = (1 + \psi(u))u$  then  $u = (1 - \psi(u'))u'$ , and we have:

$$\begin{aligned} (u', v') &\xrightarrow{\xi^{-1}} ((1 - \psi(u'))u', (1 - \psi(v'))v') \\ &\xrightarrow{\odot} (1 + c(u', v'))(1 - \psi(u'))(1 - \text{Ad}(u')\psi(v'))u'v' \\ &= (1 + (c(u', v') - \psi(u') - \text{Ad}(u')\psi(v')))u'v' \\ &\xrightarrow{\xi} [1 + (\psi(u'v') + c(u', v') - \psi(u') - \text{Ad}(u')\psi(v'))]u'v' \end{aligned}$$

We see that  $c$  changes by the coboundary  $\partial\psi$  and the class  $[c] \in H^2(G, \text{Lie}(G, I))$  does not depend on the choice of isomorphism  $\mathcal{G} \xrightarrow{\sim} h^*G$ . To obtain the inverse bijection, one chooses a cocycle  $c$  and attaches the deformation  $\mathcal{G}_c$  as in (1). The map is well-defined because another choice of  $c$  in the same cohomology class gives an isomorphic deformation.  $\square$

**3.3.4 Lemma.** *Let  $R$  be a ring. Let  $X, Y$  be  $R$ -schemes with  $X$  affine and  $Y$  formally smooth. Then the functor  $\text{Clim}_R(X, Y)$  of closed immersions is formally smooth. If moreover  $X$  is flat then the functor  $\text{Isom}_R(X, Y)$  of isomorphisms is formally smooth.*

**Proof :** Let  $C \rightarrow C_0 = C/N$  be an extension with square zero ideal. Notationally we may replace  $X, Y$  by  $X \otimes_R C, Y \otimes_R C$  and hence assume that  $R = C$ . Let  $f_0 : X_0 \rightarrow Y_0$  be a closed immersion over  $R_0 = R/N$  with  $X_0 = X \otimes R_0, Y_0 = Y \otimes R_0$ . Since  $Y$  is formally smooth, the composition  $X_0 \rightarrow Y_0 \hookrightarrow Y$  lifts to a morphism  $f : X \rightarrow Y$ . Topologically  $f$  is the same as  $f_0$  which is a closed immersion, hence  $f$  is affine by [SP], Tag 04DE. It remains to prove that  $f^\sharp : \mathcal{O}_Y \rightarrow f_*\mathcal{O}_X$  is surjective, and an isomorphism when  $X$  is flat. Since  $f$  is affine, we reduce to the case  $X = \text{Spec}(A), Y = \text{Spec}(B)$  and  $u := f^\sharp : B \rightarrow A$ . Let  $M = \text{coker}(u)$  and  $N = \text{ker}(u)$ . Since the formation of cokernels commutes with base change and  $f_0$  is an isomorphism, we have  $M \otimes C_0 = 0$  i.e.  $M = NM$ . Since  $N$  is nilpotent it follows that  $M = 0$ , hence  $u$  is surjective. In the case where moreover  $A$  is flat, the formation of  $N$  commutes with base change. Again using that  $f_0$  is an isomorphism we deduce that  $N = 0$  and  $f$  is an isomorphism.  $\square$

## 4 Weil restriction

We keep the notations from the previous sections. In this section, we describe the Weil restriction  $E = h_*\mathcal{G}$  of a  $k[I]$ -group scheme  $\mathcal{G} \in \text{Gr}/k[I]$  and show how it carries the structure of an object of the category of extensions  $\mathcal{E}xt(\mathcal{G}_k, \text{Lie}(\mathcal{G}_k, I))$ .

### 4.1 Weil restriction of the group algebra

If  $\mathcal{G}$  is an affine  $k[I]$ -group scheme, its Weil restriction  $h_*\mathcal{G}$  embeds in the pushforward algebra  $h_*\mathbb{O}[\mathcal{G}]$ , with  $\mathbb{O}[\mathcal{G}] = \mathbb{O}_{k[I]}[\mathcal{G}]$  the group algebra of  $\mathcal{G}$ . (It also embeds in the algebra  $\mathbb{O}_k[h_*\mathcal{G}]$  which however is

less interesting in that it does not reflect the Weil restriction structure.) Our aim in this subsection is to give a description of  $h_*\mathbb{O}[\mathcal{G}]$  suited to the computation of the adjunction map  $\beta_{\mathcal{G}}$ .

The starting point is the following definition and lemma. Let  $A$  be a  $k[I]$ -algebra and  $R$  a  $k$ -algebra. Let  $v : A \rightarrow I \otimes_k R$  be a  $k$ -linear map such that there is a  $k$ -linear map  $v^* : A \rightarrow R$  satisfying  $v(ix) = iv^*(x)$  for all  $i \in I$  and  $x \in A$ . Then  $v^*$  is uniquely determined by  $v$ ; in fact it is already determined by the identity  $v(ix) = iv^*(x)$  for any fixed  $i$  belonging to a basis of  $I$  as a  $k$ -module.

**4.1.1 Definition.** Let  $A$  be a  $k[I]$ -algebra and  $R$  a  $k$ -algebra. We say that a  $k$ -linear map  $v : A \rightarrow I \otimes_k R$  is *suitable* if there is a  $k$ -linear map  $v^* : A \rightarrow R$  such that  $v(ix) = iv^*(x)$  for all  $i \in I$  and  $x \in A$ . We denote by  $\text{Homs}_k(A, I \otimes_k R)$  the  $R$ -module of suitable maps and  $\text{Homs}_k(A, I \otimes_k R) \rightarrow \text{Hom}_k(A, R)$ ,  $v \mapsto v^*$  the  $R$ -module map that sends  $v$  to the unique map  $v^*$  with the properties above.

Note that since  $I^2 = 0$  in  $R[I]$ , if  $v$  is suitable then  $v^*$  vanishes on  $IA$ . In other words, the map  $v \mapsto v^*$  factors through  $\text{Hom}_k(A/IA, R)$ .

**4.1.2 Lemma.** *Let  $A$  be a  $k[I]$ -algebra and  $R$  a  $k$ -algebra.*

(1) *Each morphism of  $k[I]$ -modules  $f : A \rightarrow R[I]$  is of the form  $f = v^* + v$  for a unique suitable  $k$ -linear map  $v : A \rightarrow I \otimes_k R$ , and conversely.*

(2) *Each morphism of  $k[I]$ -algebras  $f : A \rightarrow R[I]$  is of the form  $f = v^* + v$  as above with  $v$  satisfying moreover  $v(i) = i$  for all  $i \in I$  and  $v(xy) = v^*(x)v(y) + v^*(y)v(x)$  for all  $x, y \in A$ , and conversely. In particular  $v^* : A \rightarrow R$  is a  $k$ -algebra homomorphism and  $v : A \rightarrow I \otimes_k R$  is a  $v^*$ -derivation.*

**Proof :** (1) Using the decomposition  $R[I] = R \oplus I \otimes_k R$ , we can write  $f(x) = u(x) + v(x)$  for some unique  $k$ -linear maps  $u : A \rightarrow R$  and  $v : A \rightarrow I \otimes_k R$ . Then  $f$  is  $k[I]$ -linear if and only if  $f(ix) = if(x)$  for all  $i \in I$  and  $x \in A$ . Taking into account that  $I^2 = 0$ , this means that  $v$  is suitable and  $u = v^*$ .

(2) The condition  $f(1) = 1$  means that  $v^*(1) = 1$ , that is  $v(i) = i$  for all  $i \in I$ , and  $v(1) = 0$ . The condition of multiplicativity of  $f$  means that  $v^*$  is multiplicative and  $v$  is a  $v^*$ -derivation, i.e.  $v(xy) = v^*(x)v(y) + v^*(y)v(x)$ . In the presence of the derivation property, the multiplicativity of  $v^*$  is automatic (computing  $v(ixy)$  in two different ways) as well as the condition  $v(1) = 0$  (setting  $x = y = 1$ ). Conversely if  $v$  is suitable with  $v(i) = i$  and  $v(xy) = v^*(x)v(y) + v^*(y)v(x)$ , one sees that  $v^*$  is a morphism of rings and  $f = v^* + v$  is a morphism of  $k[I]$ -algebras.  $\square$

Now let  $\mathcal{G}$  be an affine  $k[I]$ -group scheme. Lemma 4.1.2 shows that the Weil restriction  $h_*\mathbb{V}(\mathcal{G}/k[I])$  can be described in terms of the scheme of suitable maps defined as a functor on  $k$ -algebras by:

$$\mathbb{O}_s(\mathcal{G})(R) := \text{Homs}_k(A, I \otimes_k R).$$

We will now identify the multiplication induced on  $\mathbb{O}_s(\mathcal{G})$  by means of this isomorphism.

**4.1.3 Proposition.** *Let  $\mathcal{G} = \text{Spec}(A)$  be an affine  $k[I]$ -group scheme with comultiplication  $\Delta : A \rightarrow A \otimes_{k[I]} A$  and counit  $e : A \rightarrow k[I]$ , with  $e = d^* + d$  for a unique suitable  $k$ -linear map  $d : A \rightarrow I$ .*

(1) *Let  $R$  be a  $k$ -algebra and let  $v, w : A \rightarrow I \otimes_k R$  be suitable  $k$ -linear maps. Then the morphism  $v^* \otimes_k w + v \otimes_k w^* : A \otimes_k A \rightarrow R$  factors through a well-defined  $k$ -linear morphism*

$$v^* \otimes_k w + v \otimes_k w^* : A \otimes_{k[I]} A \rightarrow R.$$

(2) For  $v, w$  as before let:

$$v \diamond w := (v^* \otimes_k w + v \otimes_k w^*) \circ \Delta.$$

Then  $(\mathbb{O}_s(\mathcal{G}), +, \diamond)$  is an associative unitary  $\mathbb{O}$ -algebra with multiplicative unit  $d$ , and the map

$$\theta_{\mathcal{G}} : \mathbb{O}_s(\mathcal{G}) \xrightarrow{\sim} h_*\mathbb{O}[\mathcal{G}], \quad v \mapsto v^* + v$$

is an isomorphism of associative unitary  $\mathbb{O}$ -algebras.

**Proof :** (1) The  $k$ -linear morphism  $v^* \otimes_k w + v \otimes_k w^*$  takes the same value  $iv^*(a)w^*(b)$  on the tensors  $ia \otimes b$  and  $a \otimes ib$  for all  $i \in I$ ,  $a, b \in A$ . Therefore it vanishes on tensors of the form  $(a \otimes b)(i \otimes 1 - 1 \otimes i)$ . Since these tensors generate the kernel of the ring map  $A \otimes_k A \rightarrow A \otimes_{k[I]} A$ , we obtain an induced morphism  $v^* \otimes_k w + v \otimes_k w^* : A \otimes_{k[I]} A \rightarrow R$ .

(2) According to (1) the definition of  $v \diamond w$  makes sense. For the rest it is enough to prove that  $\theta_{\mathcal{G}}(v \diamond w) = \theta_{\mathcal{G}}(v) \star \theta_{\mathcal{G}}(w)$  because if this is the case then all the known properties of the product  $\star$  in  $h_*\mathbb{O}[\mathcal{G}]$  are transferred to  $\diamond$  by the isomorphism  $\theta_{\mathcal{G}}$ . On one hand, using the expression  $v \diamond w = (v^* \otimes_k w + v \otimes_k w^*) \circ \Delta$  and the fact that  $\Delta$  is  $k[I]$ -linear, we find  $(v \diamond w)^* = (v^* \otimes_k w^*) \circ \Delta$  hence:

$$\begin{aligned} \theta_{\mathcal{G}}(v \diamond w) &= (v^* \otimes_k w^*) \circ \Delta + (v^* \otimes_k w + v \otimes_k w^*) \circ \Delta \\ &= [v^* \otimes_k w^* + (v^* \otimes_k w + v \otimes_k w^*)] \circ \Delta. \end{aligned}$$

On the other hand, we have:

$$\theta_{\mathcal{G}}(v) \star \theta_{\mathcal{G}}(w) = [(v^* + v) \otimes_k (w^* + w)] \circ \Delta.$$

The maps in the brackets are equal, whence  $\theta_{\mathcal{G}}(v \diamond w) = \theta_{\mathcal{G}}(v) \star \theta_{\mathcal{G}}(w)$  as desired.  $\square$

**4.1.4 Remark.** If  $(\varepsilon_1, \dots, \varepsilon_r)$  is a basis of  $I$  we have a concrete description as follows. A  $k$ -linear map  $v : A \rightarrow I \otimes_k R$  can be written  $v = \varepsilon_1 v_1 + \dots + \varepsilon_r v_r$  for some maps  $v_j : A \rightarrow R$ . Then  $v$  is suitable if and only if  $v_j \varepsilon_i = \delta_{i,j} v^*$  for all  $i, j$ . If this is the case,  $v_j$  induces a  $k$ -linear morphism  $A \otimes_{k[I]} k[\varepsilon_j] \rightarrow R$  and  $v^* = v_j \varepsilon_j$  for each  $j$ . Now write  $\mathcal{G}_j = \mathcal{G} \otimes_{k[I]} k[\varepsilon_j]$  and  $h_j : \text{Spec}(k[\varepsilon_j]) \rightarrow \text{Spec}(k)$  the structure map. Also let  $\mathcal{G}_k = \mathcal{G} \otimes_{k[I]} k$  so we have maps  $\mathbb{V}(h_{j,!}\mathcal{G}_j/k) \rightarrow \mathbb{V}(\mathcal{G}_k/k)$ ,  $v_j \mapsto v^* := v_j \varepsilon_j$ . Then we have an isomorphism:

$$\mathbb{V}(h_{1,!}\mathcal{G}_1/k) \times_{\mathbb{V}(\mathcal{G}_k/k)} \dots \times_{\mathbb{V}(\mathcal{G}_k/k)} \mathbb{V}(h_{r,!}\mathcal{G}_r/k) \xrightarrow{\sim} \mathbb{O}_s(\mathcal{G})$$

given by  $(v_1, \dots, v_r) \mapsto v = \varepsilon_1 v_1 + \dots + \varepsilon_r v_r$ .

## 4.2 Kernel of the adjunction $\beta : h^*h_*\mathcal{G} \rightarrow \mathcal{G}$

Again let  $\mathcal{G} = \text{Spec}(A)$  be an affine  $k[I]$ -group scheme. Denote by  $\Delta : A \rightarrow A \otimes_{k[I]} A$  the comultiplication and  $e : A \rightarrow k[I]$  the counit, of the form  $e = d^* + d$  for a unique suitable  $k$ -linear map  $d : A \rightarrow I$ . The purpose of this subsection is to prove the following proposition.

**4.2.1 Proposition.** *Let  $\mathcal{G}$  be an affine  $k[I]$ -group scheme and  $\mathcal{G}_k = i^*\mathcal{G}$ . Let  $\beta = \beta_{\mathcal{G}} : h^*h_*\mathcal{G} \rightarrow \mathcal{G}$  be the adjunction, and  $L(\mathcal{G}) := \ker(\beta)$ .*

(1) *Functoriality. The formation of  $L(\mathcal{G})$  is functorial for morphisms of pointed  $k[I]$ -schemes, and for morphisms of  $k[I]$ -group schemes.*

(2) Explicit description. We have embeddings of monoids  $L(\mathcal{G}) \subset h^*h_*\mathcal{G} \subset (h^*\mathbb{O}_s(\mathcal{G}), \diamond)$  under which, functorially in the  $k[I]$ -algebra  $R$ :

$$(h^*h_*\mathcal{G})(R) = \left\{ v \in (h^*\mathbb{O}_s(\mathcal{G}))(R) \mid \begin{array}{l} v(i) = i \text{ for all } i \in I \\ v(xy) = v^*(x)v(y) + v^*(y)v(x) \text{ for all } x, y \in A \end{array} \right\}$$

and if  $v_R : A \rightarrow R$  denotes the composition  $A \xrightarrow{v} I \otimes_k R \xrightarrow{i \otimes r \mapsto ir} R$ :

$$L(\mathcal{G})(R) = \{v \in (h^*h_*\mathcal{G})(R) : v^* + v_R = e_R\}.$$

(3) Special fibre. There is a functorial isomorphism  $i^*L(\mathcal{G}) \xrightarrow{\sim} \text{Lie}(\mathcal{G}_k, I)$ ,  $v \mapsto v - d$ . Under this isomorphism, the action by conjugation of  $h_*\mathcal{G} = i^*h^*h_*\mathcal{G}$  on  $i^*L(\mathcal{G})$  is given by the morphism:

$$h_*\mathcal{G} \xrightarrow{i^*\beta} \mathcal{G}_k \xrightarrow{\text{Ad}} \text{GL}(\text{Lie}(\mathcal{G}_k, I)).$$

(4) Case of trivial deformation groups. If  $\mathcal{G} = h^*G$  for some affine  $k$ -group scheme  $G$ , there is a canonical and functorial isomorphism  $L(\mathcal{G}) \xrightarrow{\sim} h^*\text{Lie}(G, I)$ . More precisely, under the isomorphism (see 3.1)

$$\varrho_G : \text{Lie}(G, I) \times G \xrightarrow{\sim} \text{T}(G, I),$$

the subgroup  $L(\mathcal{G}) \subset h^*\text{T}(G, I)$  has for points the pairs  $(x, g) \in h^*\text{Lie}(G, I) \times h^*G$  such that  $g = \exp(-x)$ , and the isomorphism is given by  $(x, g) \mapsto x$ .

**Proof :** (1) If  $\varphi : \mathcal{G} \rightarrow \mathcal{G}'$  is a morphism of pointed schemes, then by functoriality of  $\beta$  the morphism  $h_*h^*\varphi$  takes the kernel of  $\beta_{\mathcal{G}}$  into the kernel of  $\beta_{\mathcal{G}'}$ . If moreover  $\varphi$  is a map of group schemes then the restriction of  $h_*h^*\varphi$  to  $L(\mathcal{G})$  also.

(2) In the rest of the proof we use the possibility to compute in the group algebra  $(\mathbb{O}_s(\mathcal{G}), +, \diamond)$ , see Proposition 4.1.3. The description of  $h^*h_*\mathcal{G}$  as a submonoid of the multiplicative monoid of  $\mathbb{O}_s(\mathcal{G})$  is copied from Lemma 4.1.2. The description of  $L(\mathcal{G})$  follows from the fact that for  $f \in \mathcal{G}(R[I])$ ,  $f = v^* + v : A \rightarrow R[I]$ , the image  $\beta(f) \in \mathcal{G}(R)$ ,  $\beta(f) : A \rightarrow R$  is the map  $v^* + v_R$ .

(3) – first claim. The pullback  $i^*$  is the restriction to the category of those  $k[I]$ -algebras  $R$  such that  $IR = 0$ . For such an  $R$ , an element  $v \in (i^*L(\mathcal{G}))(R)$  is a  $k$ -linear map  $v : A \rightarrow R$  such that  $v(i) = i$ ,  $v(xy) = v^*(x)v(y) + v^*(y)v(x)$  and  $v^* = e_R$ . In particular we see that  $v$  is an  $e_R$ -derivation. Since also  $d_R$  is an  $e_R$ -derivation with  $d_R(i) = i$  for all  $i \in I$ , the difference  $\delta := v - d_R$  induces an  $e_R$ -derivation  $\delta : A \rightarrow I \otimes_k R$  vanishing on  $IA$ , i.e. an  $R$ -point of  $\text{Lie}(\mathcal{G}_k, I)$ . Conversely, any  $e_R$ -derivation  $\delta : A \rightarrow I \otimes_k R$  vanishing on  $IA$  gives rise to a  $k$ -linear map  $v : A \rightarrow I \otimes_k R$  defined by  $v := d_R + \delta$  and satisfying the properties required to be a point of  $(i^*L(\mathcal{G}))(R)$ . Finally let  $\delta_1, \delta_2 \in \text{Lie}(\mathcal{G}_k, I)(R)$ . Since  $\delta_1, \delta_2$  vanish on  $IA$ , we have  $d_R^* + \delta_1^* = d_R^* + \delta_2^* = e_R$  and then:

$$\begin{aligned} (d_R + \delta_1) \diamond (d_R + \delta_2) &= [e_R \otimes (d_R + \delta_2) + (d_R + \delta_1) \otimes e_R] \circ \Delta \\ &= [(e_R \otimes d_R + d_R \otimes e_R) + (e_R \otimes \delta_2 + \delta_1 \otimes e_R)] \circ \Delta. \end{aligned}$$

All three morphisms  $e_R \otimes d_R + d_R \otimes e_R$ ,  $e_R \otimes \delta_2$  and  $\delta_1 \otimes e_R$  factor through  $A \otimes_{k[I]} A$ , so the precomposition with  $\Delta$  is distributive for them. Since also  $d_R$  is the neutral element for the law  $\diamond$  and  $e_R$  is the neutral element for the law  $\star$ , we obtain:

$$\begin{aligned} (d_R + \delta_1) \diamond (d_R + \delta_2) &= [e_R \otimes d_R + d_R \otimes e_R] \circ \Delta + [e_R \otimes \delta_2] \circ \Delta + [\delta_1 \otimes e_R] \circ \Delta \\ &= d_R + \delta_1 + \delta_2. \end{aligned}$$

This shows that the isomorphism  $i^*L(\mathcal{G}) \xrightarrow{\sim} \text{Lie}(\mathcal{G}_k)$ ,  $v \mapsto v - d_R$  is a morphism of groups.

(3) – second claim. Let  $R$  be a  $k[I]$ -algebra such that  $IR = 0$ . The action of an element  $u \in (h^*h_*\mathcal{G})(R)$  on an element  $v = d_R + \delta \in L(\mathcal{G})(R)$  by conjugation can be computed in the algebra  $h^*\mathbb{O}_s(\mathcal{G})$ :

$$u \diamond v \diamond u^{-1} = u \diamond (d_R + \delta) \diamond u^{-1} = u \diamond d_R \diamond u^{-1} + u \diamond \delta \diamond u^{-1} = d_R + u \diamond \delta \diamond u^{-1}.$$

We see that  $u$  is acting on  $\delta$  by conjugation in the group algebra. This is the adjoint action, as explained in Proposition 3.3.1.

(4) When  $\mathcal{G} = h^*G$ , the adjunction map is the infinitesimal translation. The  $R$ -points of  $L(\mathcal{G})$  are the pairs  $(x, g)$  such that  $\exp(x)g = 1$  in  $G(R)$ . This amounts to  $g = \exp(-x)$  which proves (4).  $\square$

### 4.3 Extension structure of the Weil restriction

**4.3.1 Definition.** Let  $X$  be a  $k[I]$ -group scheme and  $X_k := i^*X = X \times_{\mathrm{Spec}(k[I])} \mathrm{Spec}(k)$ . A *rigidification* of  $X$  is an isomorphism of  $k[I]$ -schemes  $\sigma : h^*X_k \xrightarrow{\sim} X$  that restricts to the identity on  $X_k$ . We say that  $X$  is *rigid* if it admits a rigidification. If  $X = \mathcal{G}$  is a group scheme, a rigidification is not required to be a group scheme homomorphism.

**4.3.2 Remarks.** (1) If  $\sigma : h^*\mathcal{G}_k \xrightarrow{\sim} \mathcal{G}$  is a rigidification, then  $\sigma(1)^{-1} \cdot \sigma$  is another. Therefore if there exists a rigidification  $\sigma$ , we may always assume moreover that  $\sigma(1) = 1$ .

(2) By the infinitesimal lifting criterion, all affine smooth  $k[I]$ -group schemes are rigid. By Cartier duality,  $k[I]$ -group schemes of multiplicative type are rigid.

(3) If  $I = k\varepsilon$ , an example of a flat  $k[\varepsilon]$ -group scheme that does not admit a rigidification is the kernel of the endomorphism  $\mathbb{G}_a \rightarrow \mathbb{G}_a$ ,  $x \mapsto x^p - \varepsilon x$ , see Example 4.3.5 below.

(4) If  $\alpha : \mathcal{G} \rightarrow \mathcal{G}'$  is a morphism between rigid  $k[I]$ -group schemes, it is not always possible to choose rigidifications for  $\mathcal{G}$  and  $\mathcal{G}'$  that are compatible in the sense that  $\sigma' \circ h^*\alpha = \alpha \circ \sigma$ . For instance let  $I = k\varepsilon$  and let  $\alpha : \mathbb{G}_a \rightarrow \mathbb{G}_a$  be the morphism defined by  $\alpha(x) = \varepsilon x$ . Then  $\mathcal{G}$  and  $\mathcal{G}'$  are rigid but since  $h^*\alpha = 0$ , there do not exist compatible rigidifications.

**4.3.3 Lemma.** *Let  $\mathcal{G}$  be a  $k[I]$ -group scheme such that the restriction  $\mathcal{G}_k = i^*\mathcal{G}$  is  $k$ -flat. Let  $\pi := i^*\beta_{\mathcal{G}} : h_*\mathcal{G} \rightarrow \mathcal{G}_k$ . Then the adjunction  $\mathrm{Hom}_{\mathrm{Sch}/k[I]}(h^*\mathcal{G}_k, \mathcal{G}) = \mathrm{Hom}_{\mathrm{Sch}/k}(\mathcal{G}_k, h_*\mathcal{G})$  induces a bijection between rigidifications of  $\mathcal{G}$  and sections of  $\pi$ .*

**Proof :** Let  $\sigma : h^*\mathcal{G}_k \xrightarrow{\sim} \mathcal{G}$  be a rigidification and  $s := h_*\sigma \circ \alpha_{\mathcal{G}_k}$ . Then we have  $\sigma = \beta_{\mathcal{G}} \circ h^*s$  and by applying  $i^*$  we find  $\mathrm{id}_{\mathcal{G}_k} = i^*\sigma = \pi \circ s$  hence  $s$  is a section of  $\pi$ . Conversely let  $s$  be a section of  $\pi$  and  $\sigma := \beta_{\mathcal{G}} \circ h^*s$ . Then  $i^*\sigma = i^*\beta_{\mathcal{G}} \circ i^*h^*s = \pi \circ s = \mathrm{id}_{\mathcal{G}_k}$  hence  $\sigma$  lifts the identity. In particular  $\sigma$  is an affine morphism. Since moreover  $h^*\mathcal{G}_k$  is flat, we conclude as in the proof of Lemma 3.3.4 that  $\sigma$  is an isomorphism, hence a rigidification.  $\square$

**4.3.4 Lemma.** *Let  $\mathcal{G}$  be an affine, differentially flat and rigid  $k[I]$ -group scheme. Then  $\beta : h^*h_*\mathcal{G} \rightarrow \mathcal{G}$  is faithfully flat and we have an exact sequence:*

$$1 \longrightarrow L(\mathcal{G}) \longrightarrow h^*h_*\mathcal{G} \xrightarrow{\beta} \mathcal{G} \longrightarrow 1.$$

*If  $\mathcal{G}$  is of finite type over  $k[I]$ , then  $\beta$  is smooth.*



**Proof :** Again we put  $\pi = i^*\beta : h_*\mathcal{G} \rightarrow \mathcal{G}_k$ . Let  $\sigma : h^*\mathcal{G}_k \xrightarrow{\sim} \mathcal{G}$  be a rigidification and  $s : \mathcal{G}_k \rightarrow h_*\mathcal{G}$  the corresponding section of  $\pi$ , see Lemma 4.3.3. By Proposition 4.2.1(3) the kernel of  $\pi$  is  $\text{Lie}(\mathcal{G}_k)$  which is  $k$ -flat. The section  $s$  provides an isomorphism of  $k$ -schemes  $\text{Lie}(\mathcal{G}_k) \times \mathcal{G}_k \simeq h_*\mathcal{G}$  which shows that  $h_*\mathcal{G}$  is  $k$ -flat so that  $h^*h_*\mathcal{G}$  is  $k[I]$ -flat. It follows also that  $\pi : h_*\mathcal{G} \rightarrow \mathcal{G}_k$  is faithfully flat and by the "critère de platitude par fibres" in the nilpotent case ([SP], Tag 06A5) we deduce that the morphism  $\beta$  is faithfully flat. Finally if  $\mathcal{G}$  is of finite type over  $k[I]$ , then the special fibre of  $L(\mathcal{G})$  is the smooth vector group  $\text{Lie}(\mathcal{G}_k)$ , hence  $L(\mathcal{G})$  is smooth and so is  $\beta$ .  $\square$

**4.3.5 Example.** Here is an example where the result above fails, for a group that does not admit a rigidification. Assume that  $k$  is a field of characteristic  $p > 0$ . Let  $I = k\varepsilon$  be free of rank 1. Let  $\mathcal{G}$  be the kernel of the endomorphism  $\mathbb{G}_a \rightarrow \mathbb{G}_a$ ,  $x \mapsto x^p - \varepsilon x$ . Then  $(h^*h_*\mathcal{G})(R)$  is the set of elements  $a \oplus b\varepsilon \in R \oplus R\varepsilon$  such that  $(a \oplus b\varepsilon)^p = \varepsilon(a \oplus b\varepsilon)$ . This equation is equivalent to  $a^p = \varepsilon a$ , hence  $a = 0$ . Thus  $(a, b) \mapsto b$  is an isomorphism  $h^*h_*\mathcal{G} \xrightarrow{\sim} \mathbb{G}_{a, k[\varepsilon]}$ . The map  $(h^*h_*\mathcal{G})(R) \rightarrow \mathcal{G}(R)$  sends  $b$  to  $b\varepsilon_R$ . In other words, if we let  $K_\varepsilon \simeq \text{Spec}(k[\varepsilon][x]/(\varepsilon x))$  denote the kernel of  $\varepsilon : \mathbb{G}_{a, k[\varepsilon]} \rightarrow \mathbb{G}_{a, k[\varepsilon]}$ , then the sequence of the lemma is  $1 \rightarrow K_\varepsilon \rightarrow \mathbb{G}_{a, k[\varepsilon]} \xrightarrow{\varepsilon} \mathcal{G} \rightarrow 1$ . Here, the map  $\varepsilon : \mathbb{G}_{a, k[\varepsilon]} \rightarrow \mathcal{G}$  is not flat so this is not an exact sequence of flat group schemes.  $\square$

**4.3.6 Proposition.** *For each  $\mathcal{G} \in \text{Gr}/k[I]$ , the restriction via  $i^*$  of the exact sequence of Lemma 4.3.4 gives  $E := h_*\mathcal{G}$  the structure of an object of  $\text{Ext}/k$ . As a result, Weil restriction gives a functor:*

$$h_* : \text{Gr}/k[I] \rightarrow \text{Ext}/k.$$

**Proof :** From Lemma 4.3.4 we have an exact sequence  $1 \rightarrow L(\mathcal{G}) \rightarrow h^*h_*\mathcal{G} \rightarrow \mathcal{G} \rightarrow 1$ . It follows from point (2) in Proposition 4.2.1 that when we restrict to the closed fibre, we obtain an exact sequence:

$$1 \longrightarrow \text{Lie}(\mathcal{G}_k, I) \longrightarrow E \longrightarrow \mathcal{G}_k \longrightarrow 1$$

where the  $\mathcal{G}_k$ -action on  $\text{Lie}(\mathcal{G}_k)$  induced by the extension is the adjoint representation. The same reference proves that this extension is functorial in  $\mathcal{G}$ . More precisely, if  $u : \mathcal{G} \rightarrow \mathcal{G}'$  is a morphism of affine, differentially flat, rigid  $k[I]$ -group schemes, then we obtain a morphism between the extensions  $E = h_*\mathcal{G}$  and  $E' = h_*\mathcal{G}'$  as follows:

$$\begin{array}{ccccccc} 1 & \longrightarrow & \text{Lie}(\mathcal{G}_k, I) & \longrightarrow & E & \longrightarrow & \mathcal{G}_k \longrightarrow 1 \\ & & \downarrow d\psi & & \downarrow \varphi & & \downarrow \psi \\ 1 & \longrightarrow & \text{Lie}(\mathcal{G}'_k, I) & \longrightarrow & E' & \longrightarrow & \mathcal{G}'_k \longrightarrow 1 \end{array}$$

where  $\varphi = h_*u$  and  $\psi = u_k = i^*u$ , the restriction of  $u$  along  $i : \text{Spec}(k) \hookrightarrow \text{Spec}(k[I])$ .  $\square$

We draw a corollary that will be useful in Section 6.

**4.3.7 Corollary.** *Let  $Y$  be an affine, flat, rigid  $k[I]$ -scheme and  $\mathbb{V}(Y) = \mathbb{V}(Y/k[I])$ ,  $\mathbb{V}(Y_k) = \mathbb{V}(Y_k/k)$  the vector bundle envelopes. Then we have a split exact sequence of flat  $k$ -group schemes:*

$$0 \longrightarrow \mathbb{V}(Y_k) \otimes_{\mathbb{O}_k} \mathbb{V}(I^\vee) \longrightarrow h_*\mathbb{V}(Y) \xrightarrow{\pi} \mathbb{V}(Y_k) \longrightarrow 0.$$

Moreover  $h_*\mathbb{V}(Y)$  is flat as an  $\mathbb{O}_k[I]$ -module scheme and the surjection  $\pi$  is isomorphic to the reduction-modulo- $I$  map.

**Proof :** Recall from paragraph 2.1.2 that  $\mathbb{O}_k[I] := h_*\mathbb{O}_{k[I]}$ . By the assumptions on  $Y$ , the vector bundle  $\mathbb{V}(Y)$  is affine, differentially flat and rigid over  $k[I]$ . Thus Proposition 4.3.6 yields the displayed exact sequence. Using a rigidification for  $Y$  and base change for the functor  $\mathbb{V}$ , we have  $h_*\mathbb{V}(Y) \simeq h_*\mathbb{V}(h^*Y_k) \simeq h_*h^*\mathbb{V}(Y_k)$ . Like in Proposition 2.3.1(4), we have an isomorphism of  $\mathbb{O}_k$ -modules  $\mathbb{V}(Y_k) \otimes_{\mathbb{O}_k} \mathbb{O}_k[I] \xrightarrow{\sim} h_*h^*\mathbb{V}(Y_k)$ ,  $x \otimes a \mapsto ax$ . Given that  $\mathbb{V}(Y_k)$  is  $\mathbb{O}_k$ -flat, this proves that  $h_*\mathbb{V}(Y)$  is  $\mathbb{O}_k[I]$ -flat. Even more, we have an isomorphism  $h_*\mathbb{V}(Y) \simeq \mathbb{V}(Y_k) \oplus (\mathbb{V}(Y_k) \otimes_{\mathbb{O}_k} \mathbb{V}(I^V))$  and  $\pi$  is the projection onto the first factor, i.e. the reduction-modulo- $I$  map.  $\square$

## 5 Weil extension

In this section, we construct a functor  $h^+$  called *Weil extension* which is a quasi-inverse to the functor  $h_*$  of Weil restriction described in the previous section. The idea behind the construction is that one can recover a  $k[I]$ -group scheme  $\mathcal{G}$  from the extension  $E = h_*\mathcal{G}$  by looking at the target of the adjunction  $\beta_{\mathcal{G}} : h^*E = h^*h_*\mathcal{G} \rightarrow \mathcal{G}$ . In turn, in order to reconstruct the faithfully flat morphism  $\beta_{\mathcal{G}}$  it is enough to know its kernel  $K$ . In the case where  $\mathcal{G}$  is a constant group  $h^*G$ , which in other words is the case where  $E$  is a tangent bundle  $\mathbb{T}(G, I)$ , Proposition 4.2.1(4) hints the correct expression  $K = \{(x, g) \in h^*E; g = \exp(-x)\}$ . The definition of  $K$  for general extensions  $1 \rightarrow \text{Lie}(G, I) \rightarrow E \rightarrow G \rightarrow 1$  where  $G$  is an affine, differentially flat  $k$ -group scheme, builds on this intuition.

### 5.1 Hochschild extensions

The construction of an extension from a 2-cocycle is well-known; we recall it to set up the notations. Recall from [DG70], chap. II, § 3, no 2 that if  $G$  is a  $k$ -group functor and  $M$  is a  $k$ - $G$ -module functor, then a *Hochschild extension* or simply *H-extension* of  $G$  by  $M$  is an exact sequence of group functors

$$1 \longrightarrow M \xrightarrow{i} E \xrightarrow{\pi} G$$

such that  $\pi$  has a section (which is not required to be a morphism of groups). From a given section  $s : G \rightarrow E$ , we can produce a unique morphism  $c : G \times G \rightarrow M$  such that  $i(c(g, g')) := s(g)s(g')s(gg')^{-1}$ . This is a 2-cocycle, i.e. it satisfies the identity

$$c(g, g') + c(gg', g'') = g \cdot c(g', g'') + c(g, g'g'').$$

Note that we may always replace  $s$  by the section  $G \rightarrow E$ ,  $g \mapsto s(1)^{-1}s(g)$  to obtain a section such that  $s(1) = 1$ . When this is the case, we have  $c(g, 1) = c(1, g') = 0$  for all  $g, g'$  and we say that  $c$  is *normalized*. Conversely, starting from a cocycle  $c$ , the functor  $E_c = M \times G$  with multiplication defined by

$$(x, g) \cdot (x', g') := (x + g \cdot x' + c(g, g'), gg')$$

is an *H-extension*. The map  $s : G \rightarrow E_c$ ,  $g \mapsto (0, g)$  is a possible choice of section for  $\pi$ . It follows from the previous comments that we may always change the cocycle into a normalized cocycle.

### 5.2 Kernel of the adjunction, reprise

In this subsection, we prepare the construction of the kernel of the adjunction map  $\beta_{h^+E}$  of the (yet to be produced) Weil extension  $h^+E$ . The end result is in Proposition 5.3.1 of the next subsection. Note that in spite of the similarity of titles, the viewpoint is different from that of Subsection 4.2.

Let  $G$  be an affine  $k$ -group scheme, and  $\text{Lie}(G, I)$  its Lie algebra relative to  $I$ , viewed as an affine  $k$ -group scheme with the adjoint action of  $G$ . To any 2-cocycle  $c : G \times G \rightarrow \text{Lie}(G, I)$  we attach as before an  $H$ -extension  $E_c = \text{Lie}(G, I) \times G$  with multiplication:

$$(x, g) \cdot (x', g') := (x + \text{Ad}(g)x' + c(g, g'), gg').$$

Our group  $E_c$  has a structure of  $H$ -extension:

$$1 \longrightarrow \text{Lie}(G, I) \xrightarrow{x \mapsto (x, 1)} E_c \xrightarrow{(x, g) \mapsto g} G \longrightarrow 1.$$

The following result is the heart of the construction of the Weil extension functor  $h^+$ . We point out that among the groups  $K_\lambda(E_c)$  introduced here, it is especially  $K_{-1}(E_c)$  that will be relevant in the sequel, as Proposition 4.2.1(4) shows. However, we include the whole family  $K_\lambda(E_c)$  since it comes with no extra cost and brings interesting insight, see Remark 6.1.2.

**5.2.1 Proposition.** *Let  $h : \text{Spec}(k[I]) \rightarrow \text{Spec}(k)$  be the  $k$ -scheme of dual numbers. Fix  $\lambda \in k$ .*

(1) *Let  $G$  be an affine  $k$ -group scheme and  $E_c$  the  $H$ -extension constructed out of a normalized 2-cocycle  $c : G \times G \rightarrow \text{Lie}(G, I)$ . Let  $K_\lambda(E_c) \subset h^*E_c$  be the subfunctor defined by:*

$$K_\lambda(E_c) = \{(x, g) \in h^*E_c ; g = \exp(\lambda x)\}.$$

*Then  $K_\lambda(E_c)$  is a closed normal sub- $k[I]$ -group scheme of  $h^*E_c$ .*

(2) *Let  $G, G'$  be affine  $k$ -group schemes and  $E_c, E_{c'}$  be the  $H$ -extensions constructed out of some chosen normalized 2-cocycles  $c, c'$ . Let  $f : E_c \rightarrow E_{c'}$  be a morphism of extensions:*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \text{Lie}(G, I) & \longrightarrow & E_c & \longrightarrow & G & \longrightarrow & 1 \\ & & \downarrow \text{Lie}(\alpha) & & \downarrow f & & \downarrow \alpha & & \\ 1 & \longrightarrow & \text{Lie}(G', I) & \longrightarrow & E_{c'} & \longrightarrow & G' & \longrightarrow & 1 \end{array}$$

*Then  $(h^*f)(K_\lambda(E_c)) \subset K_\lambda(E_{c'})$ , with equality if  $f$  is an isomorphism.*

When the extension  $E_c$  is clear from context, we write  $K_\lambda$  instead of  $K_\lambda(E_c)$ . We will prove the proposition after a few preliminaries. First of all, for the convenience of the reader, we recall the description of morphisms of extensions, in the abstract group setting for simplicity.

**5.2.2 Lemma.** *Let  $\alpha : G \rightarrow G'$  be a morphism of groups and  $\delta : L \rightarrow L'$  be a morphism from a  $G$ -module to a  $G'$ -module which is  $\alpha$ -equivariant. Let  $E \in \mathcal{E}xt(G, L)$  and  $E' \in \mathcal{E}xt(G', L')$  be two extensions.*

(1) *There exists a morphism of extensions  $f : E \rightarrow E'$ , i.e. a diagram*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & L & \longrightarrow & E & \longrightarrow & G & \longrightarrow & 1 \\ & & \downarrow \delta & & \downarrow f & & \downarrow \alpha & & \\ 1 & \longrightarrow & L' & \longrightarrow & E' & \longrightarrow & G' & \longrightarrow & 1, \end{array}$$

*if and only if  $\alpha^*[E'] = \delta_*[E]$  in  $H^2(G, L')$ , and if this condition holds then the set of morphisms is a principal homogeneous space under the set of 1-cocycles  $Z^1(G, L')$ . More precisely, assume that we describe  $E$  with a normalized cocycle  $c : G \times G \rightarrow L$  so that  $E \simeq L \times G$  with multiplication  $(x, g) \cdot (x', g') = (x + g \cdot x' + c(g, g'), gg')$ , and we describe  $E'$  similarly with a normalized cocycle  $c'$ . Then all morphisms  $f : E \rightarrow E'$  are of the form  $f(x, g) = (\delta(x) + \varphi(g), \alpha(g))$  for a unique 1-cochain  $\varphi : G \rightarrow L'$  such that  $\partial\varphi = c' \circ \alpha - \delta \circ c$ .*

(2) If  $E, E'$  are two extensions of  $G$  by  $L$ , then the set of morphisms of extensions  $E \rightarrow E'$  is a principal homogeneous space under the group  $Z^1(G, L)$ , more precisely all morphisms are of the form  $f(x, g) = (x + \varphi(g), g)$  for a unique  $\varphi \in Z^1(G, L)$ . All of them are isomorphisms.

(3) Assume that the extension is trivial, so that  $[E] = 0 \in H^2(G, L)$ . Then all group sections  $G \rightarrow E$  of the extension are of the form  $s(x, g) = (\varphi(g), g)$  for a unique  $\varphi \in C^1(G, L)$  such that  $\partial\varphi = c$ .

**Proof :** We can write a morphism of extensions as  $f : L \times G \rightarrow L' \times G'$ ,  $(x, g) \mapsto (u(x, g), \alpha(g))$  with  $u(x, 1) = \delta(x)$ . The property that  $f$  is a morphism of groups translates into the identity:

$$u(x_1 + g_1 \cdot x_2 + c(g_1, g_2), g_1 g_2) = u(x_1, g_1) + \alpha(g_1) \cdot u(x_2, g_2) + c'(\alpha g_1, \alpha g_2).$$

Setting  $x_1 = x$ ,  $x_2 = 0$ ,  $g_1 = 1$ ,  $g_2 = g$ , and  $\varphi(g) := u(0, g)$  we find  $u(x, g) = \delta(x) + \varphi(g)$  for all  $x, g$ . The above identity implies  $\varphi(g_1 g_2) - \varphi(g_1) - \alpha(g_1) \cdot \varphi(g_2) = c'(\alpha g_1, \alpha g_2) - \delta(c(g_1, g_2))$ . This means that  $\partial\varphi = c' \circ \alpha - \delta \circ c$  as claimed in (1). Considering the particular case of morphisms

$$\begin{array}{ccccccccc} 1 & \longrightarrow & L & \longrightarrow & E & \longrightarrow & G & \longrightarrow & 1 \\ & & \downarrow \text{id} & & \downarrow & & \downarrow \text{id} & & \\ 1 & \longrightarrow & L & \longrightarrow & E' & \longrightarrow & G & \longrightarrow & 1 \end{array}$$

we get (2), and considering the case of morphisms

$$\begin{array}{ccccccccc} 1 & \longrightarrow & 0 & \longrightarrow & G & \xrightarrow{\text{id}} & G & \longrightarrow & 1 \\ & & \downarrow 0 & & \downarrow & & \downarrow \text{id} & & \\ 1 & \longrightarrow & L & \longrightarrow & E & \longrightarrow & G & \longrightarrow & 1 \end{array}$$

we get (3). □

We come back to the extension  $E_c$ . The lemma tells us that the group  $\text{Aut}_{\text{ext}}(E_c)$  of automorphisms of  $E_c$  as an extension is isomorphic to the group of 1-cocycles  $Z^1(G, \text{Lie}(G, I))$ . Item (2) of Proposition 5.2.1 says in particular that  $K_\lambda(E_c)$  is stable under these particular automorphisms.

Now we record a few technical properties concerning the exponential and the cocycles. For simplicity we write  $\exp$  instead of  $\exp_G$ .

**5.2.3 Lemma.** *Let  $G$  be an affine  $k$ -group scheme and  $c : G \times G \rightarrow \text{Lie}(G, I)$  a normalized 2-cocycle. Let  $\exp : h^* \text{Lie}(G, I) \rightarrow h^* G$  be the exponential morphism as defined in Subsection 3.2. Let  $R$  be a  $k[I]$ -algebra and  $g, g', g'' \in G(R)$ . Assume that  $g$  is an exponential i.e. an element in the image of  $\exp$ . Then, we have:*

- (1)  $\exp(c(g, g')) = \exp(c(g', g)) = 1$ ,
- (2)  $\exp(c(gg', g'')) = \exp(c(g'g, g'')) = \exp(c(g', gg'')) = \exp(c(g', g''g)) = \exp(c(g', g''))$ .

The same statements hold with  $c$  replaced by  $\lambda c$ , for each  $\lambda \in k$ .

**Proof :** (1) Applying Lemma 3.2.3 to the morphism of pointed schemes  $\phi = c(-, g') : G \otimes_k R \rightarrow (\text{Lie}(G, I)) \otimes_k R$ , we obtain  $\exp(c(g, g')) = 1$ . Similarly  $\exp(c(g', g)) = 1$ .

(2) Write  $g = \exp(x)$ . Since  $\text{Lie}(G, I) = \text{Lie}(G) \otimes \mathbb{V}(I^\vee)$  we can write  $x$  as a sum of tensors  $y \otimes i$ . Working inductively on the number of tensors in the sum, we can assume that  $x = y \otimes i$ . We prove successively that each of the first four terms equals  $\exp(c(g', g''))$ .

- a. The cocycle identity  $c(g, g') + c(gg', g'') = \text{Ad}(g)c(g', g'') + c(g, g'g'')$  together with (1) imply  $\exp(c(gg', g'')) = \exp(\text{Ad}(g)c(g', g''))$ . Since  $g$  is an exponential, according to Proposition 3.2.2(4) its adjoint action is given by  $\text{Ad}(g)c(g', g'') = c(g', g'') + i[x, c(g', g'')]$ . Since multiples of  $i$  lie in the kernel of  $\exp$ , see Proposition 3.2.2(6), we deduce  $\exp(\text{Ad}(g)c(g', g'')) = \exp(c(g', g''))$ .
- b. Since  $\gamma := g'g(g')^{-1}$  is an exponential,  $\exp(c(g'g, g'')) = \exp(c(\gamma g', g'')) = \exp(c(g', g''))$  by a.
- c. The cocycle identity with  $g$  and  $g'$  exchanged reads  $c(g', g) + c(g'g, g'') = \text{Ad}(g')c(g, g'') + c(g', gg'')$ . We deduce  $\exp(c(g'g, g'')) = \exp(c(g', gg''))$ . We conclude with b.
- d. Again this follows from the fact that  $g''g(g'')^{-1}$  is an exponential.

The final claim of the lemma holds because  $\lambda c$  is again a normalized cocycle. □

**Proof of Proposition 5.2.1 :** Let us write  $K = K_\lambda(E_c)$  for simplicity. Obviously  $K$  is a closed subfunctor of  $h^*E_c$  which is isomorphic to  $h^*\text{Lie}(G, I)$  as a  $k[I]$ -scheme. For the verification of points (1) and (2) we let  $R$  be an arbitrary  $k[I]$ -algebra.

(1) First, let us prove that  $K$  is a subgroup scheme. Let  $(x, g)$  and  $(x', g')$  be two  $R$ -valued points of  $K$  so we have  $g = \exp(\lambda x)$  and  $g' = \exp(\lambda x')$ . On one hand, by Proposition 3.2.2(4) we see that  $\text{Ad}(g)x' = x' + \varepsilon[x, x']$  and by Proposition 3.2.2(6) we deduce  $\exp(\lambda \text{Ad}(g)x') = \exp(\lambda x')$ . On the other hand, by Lemma 5.2.3(1) we have  $\exp(\lambda c(g, g')) = 1$ . Putting all this together we get:

$$\begin{aligned} \exp(\lambda x + \lambda \text{Ad}(g)x' + \lambda c(g, g')) &= \exp(\lambda x) \exp(\lambda \text{Ad}(g)x') \exp(\lambda c(g, g')) \\ &= \exp(\lambda x) \exp(\lambda x') \\ &= gg'. \end{aligned}$$

This proves that the product  $(x, g) \cdot (x', g')$  is a point of  $K$ . Using the same arguments we prove that the inverse  $(x, g)^{-1} = (-\text{Ad}(g^{-1})x - c(g^{-1}, g), g^{-1})$  is a point of  $K$ . Hence  $K$  is a subgroup scheme.

Second, let us prove that  $K$  is stable by inner automorphisms. Let  $(x, g)$  and  $(x', g')$  be  $R$ -valued points of  $h^*E_c$  and  $K$  respectively. We must prove that  $(x'', g'') := (x, g) \cdot (x', g') \cdot (x, g)^{-1}$  lies in  $K$ . Writing  $x'$  as a sum of tensors  $x_s = y_s \otimes i_s$  and setting  $g_s = \exp(\lambda x_s)$ , we have  $(x_s, g_s) \in K(R)$  and  $K$  is a subgroup scheme, it is enough to prove that  $(x, g) \cdot (x_s, g_s) \cdot (x, g)^{-1}$  lies in  $K$ . In other words, we may and do assume in the sequel that  $x' = y \otimes i$ . We first consider  $(x_1, g_1) := (x', g') \cdot (x, g)^{-1}$ . Using the fact that  $g' = \exp(\lambda x')$  and Proposition 3.2.2(4), we find

$$\text{Ad}(g')(-\text{Ad}(g^{-1})x - c(g^{-1}, g)) = -\text{Ad}(g^{-1})x - c(g^{-1}, g) + b$$

where  $b \in I \cdot \text{Lie}(G, I)(R)$  is a certain bracket, and hence:

$$\begin{aligned} (x_1, g_1) &= (x', g') \cdot (-\text{Ad}(g^{-1})x - c(g^{-1}, g), g^{-1}) \\ &= (x' - \text{Ad}(g^{-1})x - c(g^{-1}, g) + b + c(g', g^{-1}), g'g^{-1}). \end{aligned}$$

Now  $(x'', g'') = (x, g) \cdot (x_1, g_1) = (x + \text{Ad}(g)x_1 + c(g, g_1), gg_1)$  and our task is to check that

$$\exp(\lambda x + \lambda \text{Ad}(g)x_1 + \lambda c(g, g_1)) = gg_1.$$

We note the following:

- a. We have:  $\lambda x + \lambda \operatorname{Ad}(g)x_1 = \lambda \operatorname{Ad}(g)x' - \lambda \operatorname{Ad}(g)c(g^{-1}, g) + b + \lambda \operatorname{Ad}(g)c(g', g^{-1})$ . By Proposition 3.2.2(6), the term  $b$  will disappear upon taking exponentials, so we may disregard it. Similarly, by Lemma 5.2.3(1) the exponential of  $\lambda \operatorname{Ad}(g)c(g', g^{-1})$  equals 1. Also, using the cocycle relation we see that  $\operatorname{Ad}(g)c(g^{-1}, g) = c(g, g^{-1})$ . Hence:

$$\exp(\lambda x + \lambda \operatorname{Ad}(g)x_1) = \exp(\lambda \operatorname{Ad}(g)x' - \lambda c(g, g^{-1})).$$

- b. By Lemma 5.2.3(2) we have  $\exp(\lambda c(g, g_1)) = \exp(\lambda c(g, g'g^{-1})) = \exp(\lambda c(g, g^{-1}))$ .

- c. Using Proposition 3.2.2(3) we have  $gg_1 = gg'g^{-1} = g \exp(\lambda x')g^{-1} = \exp(\lambda \operatorname{Ad}(g)x')$ .

Putting a-b-c together we get  $\exp(\lambda x + \lambda \operatorname{Ad}(g)x_1 + \lambda c(g, g_1)) = \exp(\lambda \operatorname{Ad}(g)x') = gg_1$  as desired.

(2) Let us write  $K = K_\lambda(E_c)$  and  $K' = K_\lambda(E_{c'})$  for simplicity. According to Lemma 5.2.2, any morphism of extensions  $f : E_c \rightarrow E_{c'}$  is of the form  $f(x, g) = (\operatorname{Lie}(\alpha)(x) + \varphi(g), \alpha(g))$  for a unique  $\varphi : G \rightarrow \operatorname{Lie} G'$  satisfying  $\varphi(gg') - \varphi(g) - \operatorname{Ad}(g)\varphi(g') = c'(\alpha g, \alpha g') - \operatorname{Lie}(\alpha)(c(g, g'))$ . Setting  $g = g' = 1$  we see that such a  $\varphi$ , hence also  $\lambda\varphi$ , is a map of pointed schemes. This being said, if  $(x, g)$  is an  $R$ -valued point of  $K$ , the following computation shows that  $f(x, g)$  is a point of  $K'$ :

$$\begin{aligned} \exp_{G'}(\lambda \operatorname{Lie}(\alpha)(x) + \lambda\varphi(g)) &= \exp_{G'}(\lambda \operatorname{Lie}(\alpha)(x)) \exp_{G'}(\lambda\varphi(g)) \\ &= \exp_{G'}(\lambda \operatorname{Lie}(\alpha)(x)) \text{ by Lemma 3.2.3,} \\ &= \exp_{G'}(\operatorname{Lie}(\alpha)(\lambda x)) \\ &= \alpha(\exp_G(\lambda x)) \text{ by functoriality of exp,} \\ &= \alpha(g) \text{ because } g = \exp_G(\lambda x). \end{aligned}$$

When  $f$  is an isomorphism, applying the statement to  $f^{-1}$ , we find  $(h^*f)(K) = K'$ .  $\square$

### 5.3 Weil extension functor

Now let  $G$  be an affine and differentially flat  $k$ -group scheme. Thus  $G$  as well as the adjoint representation  $\operatorname{Lie}(G, I)$  are  $k$ -flat. We consider an arbitrary extension:

$$1 \longrightarrow \operatorname{Lie}(G, I) \xrightarrow{i} E \xrightarrow{\pi} G \longrightarrow 1.$$

Then  $E \rightarrow G$  is an fpqc torsor under  $\operatorname{Lie}(G, I)$ . It has a cohomology class in  $H^1(G, \operatorname{Lie}(G, I))$  which vanishes, being quasi-coherent cohomology of an affine scheme. It follows that  $\pi$  has a section  $s : G \rightarrow E$ , and the extension becomes an  $H$ -extension. We may and do replace  $s$  by  $s(1)^{-1} \cdot s$  in order to ensure that  $s(1) = 1$ . From  $s$  we build a normalized cocycle  $c : G \times G \rightarrow \operatorname{Lie}(G, I)$  as follows:

$$i(c(g, g')) := s(g)s(g')s(gg')^{-1}.$$

These data give rise to the group  $E_c$  as defined in Subsection 5.2.

**5.3.1 Proposition.** *Let  $1 \rightarrow \operatorname{Lie}(G, I) \rightarrow E \rightarrow G \rightarrow 1$  be an object of the category  $\operatorname{Ext}(I)/k$ . Let  $s : G \rightarrow E$ , with  $s(1) = 1$ , be as chosen above, and  $c$  the normalized cocycle derived from it. Let  $\lambda \in k$ .*

- (1) *The map  $\tau_s : E_c \rightarrow E$ ,  $(x, g) \mapsto i(x)s(g)$  is an isomorphism of extensions.*
- (2) *The closed normal subgroup scheme  $K_\lambda(E) := (h^*\tau_s)(K_\lambda(E_c)) \subset h^*E$  does not depend on the choice of  $s$ .*

(3) For all morphisms  $f : E \rightarrow E'$  in  $\text{Ext}(I)/k$  we have  $(h^*f)(K_\lambda(E)) \subset K_\lambda(E')$  with equality if  $f$  is an isomorphism.

If the extension  $E$  is clear from context, we write  $K_\lambda$  instead of  $K_\lambda(E)$ . Note that if  $E$  is the trivial extension and  $s = \alpha$ , the map  $\tau_\alpha$  is the map  $\varrho_G$  defined in paragraph 3.1.

**Proof :** (1) follows from the constructions of  $c$  and  $E_c$ .

For the proof of (2) and (3) we will rely on the following basic remark. Let  $f : E \rightarrow E'$  be a morphism in  $\text{Ext}(I)/k$ . Let  $\tau_s : E_c \rightarrow E$  and  $\tau_{s'} : E_{c'} \rightarrow E'$  be the isomorphisms associated to choices of sections  $s, s'$  preserving 1 and corresponding normalized cocycles  $c, c'$ . Let  $K_{\lambda,s}(E) := (h^*\tau_s)(K_\lambda(E_c)) \subset h^*E$  and similarly  $K_{\lambda,s'}(E') := (h^*\tau_{s'})(K_\lambda(E_{c'})) \subset h^*E'$ . We have a morphism of extensions:

$$\rho = \tau_{s'}^{-1} \circ f \circ \tau_s : E_c \longrightarrow E_{c'}.$$

According to Proposition 5.2.1(2) we have  $(h^*\rho)(K_\lambda(E_c)) \subset K_\lambda(E_{c'})$ . It follows that:

$$(h^*f)(K_{\lambda,s}(E)) = (h^*f)((h^*\tau_s)(K_\lambda(E_c))) = (h^*\tau_{s'})((h^*\rho)(K_\lambda(E_c))) \subset (h^*\tau_{s'})(K_\lambda(E_{c'})) = K_{\lambda,s'}(E').$$

When  $f$  is an isomorphism, applying the statement to  $f^{-1}$  gives equality.

(2) Applying the basic remark to  $E = E'$  and  $f = \text{id} : E \rightarrow E$  proves that  $K_{\lambda,s}(E) = K_{\lambda,s'}(E)$ , that is, the subgroup  $K_{\lambda,s}(E)$  does not depend on the choice of  $s$ . Since  $\tau_s$  is an isomorphism of groups, the fact that  $K_\lambda(E)$  is a closed normal subgroup scheme follows from Proposition 5.2.1(1).

(3) Applying the basic remark to a general  $f$  gives the statement.  $\square$

For an extension  $1 \rightarrow \text{Lie}(G) \rightarrow E \rightarrow G \rightarrow 1$ , we let  $K_\lambda := K_\lambda(E)$  be the subgroup defined in the proposition. Point (4) in Proposition 4.2.1 gives motivation to consider  $K_{-1}$ . The fpqc quotient sheaf  $h^+E := h^*E/K_{-1}$  is representable by an affine flat  $k[I]$ -scheme (see Perrin [Per76]).

**5.3.2 Definition.** We call *Weil extension* the quotient  $h^+E := h^*E/K_{-1}$ .

**5.3.3 Lemma.** *Weil extension is a functor*  $\text{Ext}(I)/k \rightarrow \text{Gr}/k[I]$ .

**Proof :** The  $k[I]$ -group scheme  $G := h^*E/K_{-1}$  is affine and flat. Let  $s : G \rightarrow E$  be a section of  $E \rightarrow G$  such that  $s(1) = 1$ . By pullback, this induces a morphism  $h^*G \rightarrow h^*E \rightarrow G$  which is the identity on the special fibre, hence an isomorphism, hence a rigidification. This proves that the functor of the statement is well-defined on objects. Proposition 5.3.1 proves that the functor is well-defined on morphisms.  $\square$

## 6 The equivalence of categories

### 6.1 Roadmap for the proof of the Main Theorem

This section is devoted to the proof of the theorem which we now recall. Let  $\text{Gr}/k[I]$  be the category of affine, differentially flat, rigid  $k[I]$ -group schemes. Let  $\text{Ext}(I)/k$  be the category of extensions  $1 \rightarrow \text{Lie}(G, I) \rightarrow E \rightarrow G \rightarrow 1$  where  $G$  is an affine differentially flat  $k$ -group scheme. By descent theory, both categories are fpqc stacks over  $\text{Spec}(k)$ . Moreover, they have structures of  $\mathbb{O}_k$ -module fpqc stacks fibred over  $\text{Gr}/k$ , see 1.2. Finally, recall that the relative notions such as smooth, connected, unipotent... are as defined in Theorem A in the Introduction.

**6.1.1 Theorem.** (1) *The Weil restriction/extension functors provide quasi-inverse equivalences:*

$$\mathrm{Gr}/k[I] \begin{array}{c} \xrightarrow{h_*} \\ \xleftarrow{h^+} \end{array} \mathrm{Ext}(I)/k.$$

*These equivalences commute with base changes  $\mathrm{Spec}(k') \rightarrow \mathrm{Spec}(k)$ .*

(2) *If  $1 \rightarrow \mathcal{G}' \rightarrow \mathcal{G} \rightarrow \mathcal{G}'' \rightarrow 1$  is an exact sequence in  $\mathrm{Gr}/k[I]$ , then  $1 \rightarrow h_*\mathcal{G}' \rightarrow h_*\mathcal{G} \rightarrow h_*\mathcal{G}''$  is exact in  $\mathrm{Ext}(I)/k$ . If moreover  $\mathcal{G}'$  is smooth then  $1 \rightarrow h_*\mathcal{G}' \rightarrow h_*\mathcal{G} \rightarrow h_*\mathcal{G}'' \rightarrow 1$  is exact. In particular,  $h_*$  is an exact equivalence between the subcategories of smooth objects endowed with their natural exact structure.*

(3) *The equivalence  $h_*$  is a morphism of  $\mathbb{O}_k$ -module stacks fibred over  $\mathrm{Gr}/k$ , i.e. it transforms the addition and scalar multiplication of deformations of a fixed  $G \in \mathrm{Gr}/k$  into the Baer sum and scalar multiplication of extensions of  $G$  by  $\mathrm{Lie}(G, I)$ .*

(4) *Let  $P$  be one of the properties: of finite type, smooth, connected, unipotent, split unipotent, solvable, commutative. Then  $\mathcal{G} \in \mathrm{Gr}/k[I]$  has the property  $P$  if and only if the  $k$ -group scheme  $E = h_*\mathcal{G}$  has  $P$ .*

The rest of Section 6 is organized as follows. In Subsection 6.2 we prove a preliminary result used in the proof of (1). In Subsection 6.3 we prove (1), (2), (4). Finally in Subsection 6.4 we prove (3).

**6.1.2 Remark.** Let  $\mathcal{G} \in \mathrm{Gr}/k[I]$  be an affine, differentially flat, rigid  $k[\varepsilon]$ -group scheme. Let  $E = h_*\mathcal{G}$  be the Weil restriction. In Proposition 5.3.1 is constructed a family of subgroup schemes

$$K_\lambda := K_\lambda(E) \subset h^*E$$

parametrized by the line  $\mathbb{A}^1 = \mathrm{Spec}(k[\lambda])$ . For  $\lambda = 0$  we have  $K_0 = h^* \mathrm{Lie} \mathcal{G}_k \hookrightarrow h^*E$  and  $h^*E/K_0 \simeq h^*\mathcal{G}_k$  while for  $\lambda = -1$  we have  $h^*E/K_{-1} = h^+E \simeq \mathcal{G}$  by the Theorem. Therefore the family  $h^*E/K_\lambda \rightarrow \mathbb{A}^1$  is an algebraic deformation from  $h^*\mathcal{G}_k$  to  $\mathcal{G}$  in the category  $\mathrm{Gr}/k[I]$ . Since the category of deformations of  $\mathcal{G}_k$  is a module category, the existence of such a line is not suprising; here we produce it explicitly.

## 6.2 Equivariance of rigidifications under Lie algebra translation

Let  $\mathcal{G}$  be an affine, differentially flat, rigid  $k[I]$ -group scheme. Let  $\sigma : h^*\mathcal{G}_k \xrightarrow{\sim} \mathcal{G}$  be a rigidification such that  $\sigma(1) = 1$ . We consider the morphism of  $k$ -schemes:

$$h_*\sigma : h_*h^*\mathcal{G}_k \longrightarrow h_*\mathcal{G}.$$

This is not a morphism of group schemes, because source and target are not isomorphic groups in general. However, it satisfies an important equivariance property. To state it, note that source and target are extensions of  $\mathcal{G}_k$  by  $\mathrm{Lie}(\mathcal{G}_k, I)$ ; in particular both carry an action of  $\mathrm{Lie}(\mathcal{G}_k, I)$  by left translation.

**6.2.1 Proposition.** *With notation as above, the morphism of  $k$ -schemes*

$$h_*\sigma : h_*h^*\mathcal{G}_k \longrightarrow h_*\mathcal{G}$$

*is  $\mathrm{Lie}(\mathcal{G}_k, I)$ -equivariant.*

**Proof :** Consider the extension of  $\sigma$  to the group algebras:

$$\sigma' := h_*\mathbb{O}[\sigma] : h_*\mathbb{O}[h^*\mathcal{G}_k] \rightarrow h_*\mathbb{O}[\mathcal{G}].$$



Note that by compatibility of  $\mathbb{O}[-]$  with base change and Weil restriction (see Proposition 2.3.1, (2)-(4)), we have  $h_*\mathbb{O}[h^*\mathcal{G}_k] \xrightarrow{\sim} h_*h^*\mathbb{O}[\mathcal{G}_k] \xrightarrow{\sim} \mathbb{O}[\mathcal{G}_k][I]$ . We obtain a commutative diagram:

$$\begin{array}{ccc} h_*h^*\mathcal{G}_k & \xrightarrow{h_*\sigma} & h_*\mathcal{G} \\ \downarrow & & \downarrow \\ \mathbb{O}[\mathcal{G}_k][I] & \xrightarrow{\sigma'} & h_*\mathbb{O}[\mathcal{G}]. \end{array}$$

We identify  $I \cdot \mathbb{O}[\mathcal{G}_k] := \mathbb{O}[\mathcal{G}_k] \otimes_{\mathbb{O}_k} \mathbb{V}(I^\vee)$  as the ideal generated by  $I$  inside both algebras  $\mathbb{O}[\mathcal{G}_k][I]$  and  $h_*\mathbb{O}[\mathcal{G}]$ , see Corollary 4.3.7. With this convention we formulate:

**Claim.** *We have  $\sigma'(x + y) = \sigma'(x) + y$  for all points  $x \in \mathbb{O}[\mathcal{G}_k][I]$  and  $y \in I \cdot \mathbb{O}[\mathcal{G}_k]$ .*

To prove this, we introduce another copy  $J = I$  of our square-zero ideal as follows:

$$\begin{array}{ccc} h_*h^*h_*h^*\mathcal{G}_k & \xrightarrow{h_*h^*h_*\sigma} & h_*h^*h_*\mathcal{G} \\ \downarrow & & \downarrow \\ \mathbb{O}[\mathcal{G}_k][I][J] & \xrightarrow{\sigma''} & h_*\mathbb{O}[\mathcal{G}][J] \end{array}$$

where we've set  $\sigma'' = h_*h^*\sigma'$  for brevity. Let  $s, t : \mathbb{O}[\mathcal{G}_k][I] \times J \cdot \mathbb{O}[\mathcal{G}_k][I][J] \rightarrow h_*\mathbb{O}[\mathcal{G}][J]$  be defined by

$$s(x, y) = \sigma''(x + y) \quad \text{and} \quad t(x, y) = \sigma''(x) + y.$$

Then  $s$  and  $t$  are equal modulo  $I$  because of the fact that  $\sigma$  is the identity modulo  $I$ . Clearly they are also equal modulo  $J$ . Since  $h_*\mathbb{O}[\mathcal{G}][J]$  is flat as an  $\mathbb{O}_k[I][J]$ -module, we deduce that  $s - t$  takes its values in the ideal  $IJ \cdot h_*\mathbb{O}[\mathcal{G}][I]$ . Setting  $J = I$ , that is composing with the morphism  $h_*\mathbb{O}[\mathcal{G}][J] \rightarrow h_*\mathbb{O}[\mathcal{G}]$  that takes a section of  $J$  to the corresponding section of  $I$ , we obtain the Claim.

We now conclude the proof of the proposition. We write  $\star_1$  and  $\star_2$  the multiplications of  $h_*h^*\mathcal{G}_k$  and  $h_*\mathcal{G}$  respectively, extended to  $h_*\mathbb{O}[h^*\mathcal{G}_k]$  and  $h_*\mathbb{O}[\mathcal{G}]$ . It is enough to show more generally that

$$h_*\sigma : h_*\mathbb{O}[h^*\mathcal{G}_k] \longrightarrow h_*\mathbb{O}[\mathcal{G}]$$

is  $(1 + I \cdot \mathbb{O}[\mathcal{G}_k])$ -equivariant. We take points  $x \in I \cdot \mathbb{O}[\mathcal{G}_k]$  and  $y \in h_*\mathbb{O}[h^*\mathcal{G}_k]$  and compute:

$$\begin{aligned} (h_*\sigma)((1 + x) \star_1 y) &= (h_*\sigma)(y + x \star_1 y) \\ &= (h_*\sigma)(y) + x \star_1 y \quad \text{by the Claim above,} \\ &= (h_*\sigma)(y) + x \star_2 y \quad \text{by Corollary 4.3.7 since } \star_1 = \star_2 \text{ modulo } I, \\ &= (h_*\sigma)(y) + x \star_2 (h_*\sigma)(y) \quad \text{by Corollary 4.3.7 since } \sigma = \text{id modulo } I, \\ &= (1 + x) \star_2 (h_*\sigma)(y). \end{aligned}$$

This proves that  $h_*\sigma$  is  $\text{Lie}(\mathcal{G}_k, I)$ -equivariant. □

### 6.3 Proof of the main theorem: equivalence and exactness

**6.3.1 Proof of 6.1.1(1).** Firstly, we prove that  $h^+ \circ h_*$  is isomorphic to the identity. Let  $\mathcal{G} \rightarrow \text{Spec}(k[I])$  be an affine, differentially flat, rigid  $k[I]$ -group scheme. Let  $E = h_*\mathcal{G}$  be the associated extension:

$$1 \longrightarrow \text{Lie}(\mathcal{G}_k, I) \longrightarrow E \longrightarrow \mathcal{G}_k \longrightarrow 1.$$

We fix a rigidification  $\sigma : h^*\mathcal{G}_k \xrightarrow{\sim} \mathcal{G}$  such that  $\sigma(1) = 1$ . We know from Proposition 6.2.1 that  $h_*\sigma : h_*h^*\mathcal{G}_k \rightarrow h_*\mathcal{G}$  is  $\text{Lie}(\mathcal{G}_k, I)$ -equivariant. If we use the letter  $\gamma$  to denote the inclusions of  $\text{Lie}(\mathcal{G}_k, I)$  into the relevant extensions, this can be written:

$$(h_*\sigma)(\gamma_{h_*h^*\mathcal{G}_k}(x) \cdot y) = \gamma_{h_*\mathcal{G}}(x) \cdot (h_*\sigma)(y), \quad \text{all } x \in \text{Lie}(\mathcal{G}_k, I), y \in h_*h^*\mathcal{G}_k.$$

Restricting to  $y$  in the image of  $\alpha = \alpha_{\mathcal{G}_k} : \mathcal{G}_k \hookrightarrow h^*h^*\mathcal{G}_k$ , so  $\tau_\alpha = \varrho_{\mathcal{G}_k}$ , we obtain:

$$h_*\sigma \circ \tau_\alpha = \tau_s.$$

Using functoriality of  $\beta$  and the fact that  $\sigma(1) = 1$ , we build a commutative diagram:

$$\begin{array}{ccccccc} h^*E_0 = h^*(\text{Lie}(\mathcal{G}_k, I) \times \mathcal{G}_k) & \xrightarrow{h^*\tau_\alpha} & h^*h_*h^*\mathcal{G}_k & \xrightarrow{\beta_{h^*\mathcal{G}_k}} & h^*\mathcal{G}_k & \longrightarrow & 1 \\ \text{id} \downarrow & & h_*h_*\sigma \downarrow & & \sigma \downarrow & & \\ h^*E_c = h^*(\text{Lie}(\mathcal{G}_k, I) \times \mathcal{G}_k) & \xrightarrow{h^*\tau_s} & h^*h_*\mathcal{G} & \xrightarrow{\beta_{\mathcal{G}}} & \mathcal{G} & \longrightarrow & 1 \end{array}$$

Here the horizontal maps are morphisms of groups and the vertical maps are *not* morphisms of groups (not even the leftmost map  $\text{id} : h^*E_0 \rightarrow h^*E_c$ ). Note also that  $E_0 = T(\mathcal{G}_k, I)$ , that is, the extension  $E_c$  with the zero cocycle  $c = 0$ . Now we consider  $K_{-1}(E)$  as defined in Proposition 5.3.1. According to Proposition 4.2.1(4), the group  $K_{-1}(E_0) \subset E_0$  is the kernel of  $\beta_{h^*\mathcal{G}_k} \circ \tau_\alpha$ . On one hand the identity takes  $K_{-1}(E_0)$  to  $K_{-1}(E_c)$ , and on the other hand the map  $h_*\sigma$  takes  $\ker(\beta_{h^*\mathcal{G}_k})$  onto  $\ker(\beta_{\mathcal{G}})$  since it takes 1 to 1. By commutativity of the left-hand square, we find  $K_{-1}(E) = \ker(\beta_{\mathcal{G}})$  and therefore  $\beta_{\mathcal{G}}$  induces an isomorphism  $h^+E = h^*E/K_{-1}(E) \simeq \mathcal{G}$  which is visibly functorial.

Secondly, we prove that  $h_* \circ h^+$  is isomorphic to the identity. Let  $1 \rightarrow \text{Lie}(G, I) \rightarrow E \rightarrow G \rightarrow 1$  be an extension. We fix a section  $s : G \rightarrow E$  such that  $s(1) = 1$  and we let  $c$  be the normalized cocycle defined by  $s$ . Let  $K_{-1} = K_{-1}(E) \subset h^*E$  be the closed normal subgroup defined in Proposition 5.3.1, and let  $\mathcal{G} := h^+E = h^*E/K_{-1}$  with quotient map  $\pi : h^*E \rightarrow \mathcal{G}$ . Define  $\sigma = \pi \circ h^*s : h^*G \rightarrow \mathcal{G}$ . Since  $i^*K_{-1} = \text{Lie}(G, I)$  as a subgroup of  $E$ , we see that  $i^*\mathcal{G} \simeq G$  and  $i^*\sigma$  is the identity of  $G$ . Since  $G$  is  $k$ -flat, it follows as in the proof of Lemma 3.3.4 (see also Lemma 4.3.3) that  $\sigma$  is an isomorphism. From the construction of  $K_{-1}$ , we see that after we compose with the isomorphisms

$$h^*\tau_s : h^*h_*h^*G \xrightarrow{\sim} h^*E \quad \text{and} \quad \sigma : h^*G \xrightarrow{\sim} \mathcal{G},$$

the flat surjection  $\pi : h^*E \rightarrow \mathcal{G}$  is identified with the counit of the adjunction:

$$\beta_{h^*G} : h^*h_*h^*G \rightarrow h^*G.$$

We apply  $h_*$  and obtain the commutative diagram:

$$\begin{array}{ccccc} h_*h^*G & \xrightarrow{\alpha_{h_*h^*G}} & h_*h^*h_*h^*G & \xrightarrow{h_*\beta_{h^*G}} & h_*h^*G \\ \tau_s \downarrow \wr & & h_*h^*\tau_s \downarrow \wr & & h_*\sigma \downarrow \wr \\ E & \xrightarrow{\alpha_E} & h_*h^*E & \xrightarrow{h_*\pi} & h_*\mathcal{G} \end{array}$$

Since the top row is the identity, we see that the bottom row is an isomorphism, i.e.  $E \xrightarrow{\sim} h_*\mathcal{G}$ . Again, it is clear that this isomorphism is functorial.

Finally, we consider the commutation with base changes. For Weil restriction, this is a standard fact. For Weil extension, this follows from base change commutation for pullbacks and for quotients by flat subgroups.

**6.3.2 Proof of 6.1.1(2).** Let  $1 \rightarrow \mathcal{G}' \rightarrow \mathcal{G} \rightarrow \mathcal{G}'' \rightarrow 1$  be an exact sequence in  $\text{Gr}/k[I]$ . Then the exact sequences with solid arrows are exact:

$$\begin{aligned} 1 &\longrightarrow \mathcal{G}'_k \longrightarrow \mathcal{G}_k \longrightarrow \mathcal{G}''_k \longrightarrow 1 \\ 1 &\longrightarrow \text{Lie}(\mathcal{G}'_k, I) \longrightarrow \text{Lie}(\mathcal{G}_k, I) \longrightarrow \text{Lie}(\mathcal{G}''_k, I) \cdots \longrightarrow 1 \end{aligned}$$

Moreover, if  $\mathcal{G}'$  is smooth then the second sequence is exact also if we include the dotted arrow. By an easy diagram chase, we find that the commutative diagram below has exact rows (again including dotted arrows if  $\mathcal{G}'$  is smooth):

$$\begin{array}{ccccccc} & & 1 & & 1 & & 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \text{Lie}(\mathcal{G}'_k, I) & \longrightarrow & \text{Lie}(\mathcal{G}_k, I) & \longrightarrow & \text{Lie}(\mathcal{G}''_k, I) \cdots \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & h_*\mathcal{G}'_k & \longrightarrow & h_*\mathcal{G}_k & \longrightarrow & h_*\mathcal{G}''_k \cdots \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \mathcal{G}'_k & \longrightarrow & \mathcal{G}_k & \longrightarrow & \mathcal{G}''_k \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 1 & & 1 & & 1 \end{array}$$

This proves the claim.

**6.3.3 Proof of 6.1.1(4).** If  $\mathcal{G}$  is of finite type, or smooth, or connected, or unipotent, or split unipotent, or solvable, then  $G = i^*\mathcal{G}$  as well as  $\text{Lie}(G, I)$  have the same property. It follows that  $E = h_*\mathcal{G}$  has the property. Moreover, if  $\mathcal{G}$  is commutative then  $E$  also.

Conversely, if  $E$  is of finite type, or smooth, or connected, or unipotent, or split unipotent, or solvable, or commutative, then  $h^*E$  has the same property. Therefore the quotient  $h^+E := h^*E/K_{-1}$  has the same property.

## 6.4 Proof of the main theorem: isomorphism of $\mathbb{O}_k$ -module stacks

In this paragraph, we prove 6.1.1(3), i.e. that the Weil restriction functor  $h_* : \text{Gr}/k[I] \rightarrow \text{Ext}(I)/k$  exchanges the addition and the scalar multiplication on both sides. Before we start, we point out that these properties will imply that the image of a trivial deformation group scheme  $\mathcal{G} = h^*G$  under Weil restriction is the tangent bundle (i.e. trivial) extension  $T(G, I)$ , a fact which can be shown directly using Proposition 4.2.1(4).

We work in the fibre category over a fixed  $G \in \text{Gr}/k$  and we set  $L := \text{Lie}(G, I)$ . Let  $\mathcal{G}_1, \mathcal{G}_2 \in \text{Gr}/k[I]$  with identifications  $i^*\mathcal{G}_1 \simeq G \simeq i^*\mathcal{G}_2$ . For clarity, we introduce three copies  $I_1 = I_2 = I$  of the same finite free  $k$ -module. For  $c = 1, 2$  we have obvious maps:

$$\begin{array}{ccc} \text{Spec}(k[I_c]) & \xrightarrow{j_c} & \text{Spec}(k[I_1 \oplus I_2]) \\ & \searrow h_c \quad \swarrow \ell & \\ & \text{Spec}(k) & \end{array}$$

Set  $\mathcal{G}' = \mathcal{G}_1 \amalg_G \mathcal{G}_2 \in \text{Gr}/k[I_1 \oplus I_2]$ , so  $\mathcal{G}_1 + \mathcal{G}_2 = j^*\mathcal{G}'$  where  $j : \text{Spec}(k[I]) \hookrightarrow \text{Spec}(k[I \oplus I])$  is the closed immersion induced by the addition morphism  $I \oplus I \rightarrow I$ . We have a morphism

$$\xi : \ell_*\mathcal{G}' \longrightarrow h_{1,*}\mathcal{G}_1 \times_G h_{2,*}\mathcal{G}_2$$

whose component  $\xi_c : \ell_* \mathcal{G}' \rightarrow h_{c,*} \mathcal{G}_c$  is the  $\ell_*$ -pushforward of the adjunction  $\mathcal{G}' \rightarrow j_{c,*} j_c^* \mathcal{G}' = j_{c,*} \mathcal{G}_c$ . Besides, we have a morphism

$$\omega : \ell_* \mathcal{G}' \longrightarrow h_*(\mathcal{G}_1 + \mathcal{G}_2)$$

obtained as the  $\ell_*$ -pushforward of the adjunction  $\mathcal{G}' \rightarrow j_* j^* \mathcal{G}' = j_*(\mathcal{G}_1 + \mathcal{G}_2)$ .

**6.4.1 Lemma.** *The morphism  $\xi$  is an isomorphism and it induces an isomorphism of extensions on the bottom row of the following commutative square:*

$$\begin{array}{ccc} \ell_* \mathcal{G}' & \xrightarrow{\xi} & h_{1,*} \mathcal{G}_1 \times_G h_{2,*} \mathcal{G}_2 \\ \omega \downarrow & & \downarrow \\ h_*(\mathcal{G}_1 + \mathcal{G}_2) & \xrightarrow{\sim} & h_{1,*} \mathcal{G}_1 + h_{2,*} \mathcal{G}_2 \end{array}$$

**Proof :** Write  $G = \text{Spec}(A)$  and  $\mathcal{G}_c = \text{Spec}(\mathcal{A}_c)$  so  $\mathcal{G}' = \text{Spec}(\mathcal{A}_1 \times_A \mathcal{A}_2)$ . There is a morphism of algebra schemes  $\xi' : \ell_* \mathbb{O}[\mathcal{G}'] \rightarrow h_{1,*} \mathbb{O}[\mathcal{G}_1] \times_{\mathbb{O}[G]} h_{2,*} \mathbb{O}[\mathcal{G}_2]$  constructed in the same way as  $\xi$ . In order to describe  $\xi'$  we can express the Weil restrictions in terms of “suitable maps” as in Lemma 4.1.2. For a  $k$ -algebra  $R$ , we have:

$$\begin{array}{ccc} \text{Homs}_k(\mathcal{A}_1 \times_A \mathcal{A}_2, I_1 R \oplus I_2 R) & \xrightarrow{\xi'(R)} & \text{Homs}_k(\mathcal{A}_1, I_1 R) \times_{\text{Hom}_k(A,R)} \text{Homs}_k(\mathcal{A}_2, I_2 R) \\ v & \longmapsto & (v_1, v_2) \end{array}$$

where  $v_1$  (resp.  $v_2$ ) is  $v$  modulo  $I_2$  (resp.  $I_1$ ). This is a bijection whose inverse sends a pair  $(v_1, v_2)$  with  $v_1^* = v_2^* : A \rightarrow R$  to the map  $v : \mathcal{A}_1 \times_A \mathcal{A}_2 \rightarrow I_1 R \oplus I_2 R$ ,  $(a_1, a_2) \mapsto v_1(a_1) + v_2(a_2)$ . The morphism  $\xi$  is the bijection obtained by restriction of  $\xi'$  to the subsets of algebra maps as in Lemma 4.1.2(2). Namely, an algebra map is of the form  $f = v^* + v$  where  $v$  is suitable, and  $\omega(R)$  sends  $f$  to  $(f_1, f_2)$  while  $\omega(R)^{-1}$  sends  $(f_1 = v^* + v_1, f_2 = v^* + v_2)$  to  $f = v^* + v_1 + v_2$ .

In order to describe  $\omega$  note that  $(h_*(\mathcal{G}_1 + \mathcal{G}_2))(R) = \text{Hom}_{k[I_1 \oplus I_2]\text{-Alg}}(\mathcal{A}_1 \times_A \mathcal{A}_2, R[I])$  where  $R[I]$  is a  $k[I_1 \oplus I_2]$ -algebra via the map  $k[I_1 \oplus I_2] \rightarrow k[I]$  induced by addition  $+ : I \oplus I \rightarrow I$ . Then  $\omega(R)$  sends  $f$  to the composition

$$\mathcal{A}_1 \times_A \mathcal{A}_2 \xrightarrow{f} R[I_1 \oplus I_2] \xrightarrow{+} R[I].$$

Thus  $\omega(R)$  is surjective because  $R[I_1 \oplus I_2] \rightarrow R[I]$  has  $R$ -algebra sections, i.e.  $\omega$  is a surjection of functors. Its kernel is the set of maps  $f = v^* + v_1 + v_2$  such that  $v_1 + v_2 = d_1 + d_2 : \mathcal{A}_1 \times_A \mathcal{A}_2 \rightarrow R[I]$ , with  $e_c = d_c^* + d_c : \mathcal{A}_c \rightarrow k[I_c]$  the counits of the Hopf algebras. After translation by the derivations as indicated by Proposition 4.2.1(3), on the side of extensions the kernel is  $\ker(+ : L \times L \rightarrow L)$ , giving rise to a quotient isomorphic to the Baer sum extension  $h_{1,*} \mathcal{G}_1 + h_{2,*} \mathcal{G}_2$ .  $\square$

It remains to prove that  $h_* : \text{Gr}/k[I] \rightarrow \text{Ext}(I)/k$  exchanges the scalar multiplication on both sides. Let  $\mathcal{G} \in \text{Gr}/k[I]$  with an identification  $i^* \mathcal{G} \simeq G$ . We will reduce to a situation similar as that of Lemma 6.4.1 thanks to the following trick.

**6.4.2 Lemma.** *Let  $j_\lambda : \text{Spec}(k[I]) \hookrightarrow \text{Spec}(k[I_1 \oplus I_2])$  be the closed immersion defined by the surjective  $k$ -algebra map  $I_1 \oplus I_2 \rightarrow I$ ,  $i_1 \oplus i_2 \mapsto \lambda i_1 + i_2$ . Then we have  $s_\lambda^* \mathcal{G} \xrightarrow{\sim} j_\lambda^*(\mathcal{G} \amalg_G h^* G)$  canonically.*

**Proof :** If we think of  $\text{Spec } k[I_1 \oplus I_2]$  as the coproduct  $\text{Spec } k[I_1] \amalg_{\text{Spec } k} \text{Spec } k[I_2]$ , the map  $j_\lambda$  is the composition  $(s_\lambda \amalg \text{id}) \circ j$  as follows:

$$\text{Spec}(k[I]) \xrightarrow{j} \text{Spec}(k[I_1 \oplus I_2]) \xrightarrow{s_\lambda \amalg \text{id}} \text{Spec}(k[I_1 \oplus I_2]).$$

It follows that:

$$j_\lambda^*(\mathcal{G} \amalg_G h^*G) = j^*((s_\lambda \amalg \text{id})^*(\mathcal{G} \amalg_G h^*G)) = j^*(s_\lambda^*\mathcal{G} \amalg h^*G) = s_\lambda^*\mathcal{G} + h^*G = s_\lambda^*\mathcal{G},$$

because  $h^*G$  is the neutral element for the sum in the fibre category of  $\text{Gr}/k[I] \rightarrow \text{Gr}/k$  at  $G$ .  $\square$

Set  $\mathcal{G}_1 = \mathcal{G}$  and  $\mathcal{G}_2 = h^*G$ . Recall that  $L = \text{Lie}(G, I)$ . The Weil restrictions are  $E := E_1 = h_*\mathcal{G}$  and the trivial extension  $E_2 = h_*h^*G = L \times G$ . As in Lemma 6.4.1, there are morphisms  $\xi : \ell_*\mathcal{G}' \rightarrow E_1 \times_G E_2 = E \times L$ .

**6.4.3 Lemma.** *The morphism  $\xi$  is an isomorphism which induces an isomorphism of extensions on the bottom row of the following commutative square:*

$$\begin{array}{ccc} \ell_*\mathcal{G}' & \xrightarrow{\xi} & h_*\mathcal{G} \times L \\ \omega \downarrow & & \downarrow \\ h_*(\lambda\mathcal{G}) & \xrightarrow{\sim} & \lambda h_*\mathcal{G}. \end{array}$$

**Proof :** The proof is the same as that of Lemma 6.4.1 except that in the final step we use the map  $I \oplus I \rightarrow I$ ,  $i_1 \oplus i_2 \mapsto \lambda i_1 + i_2$ . Again this morphism is surjective and on the side of extensions, the kernel corresponds to the kernel of  $L \times L \rightarrow L$ ,  $(v_1, v_2) \mapsto \lambda v_1 + v_2$ . The quotient of  $E \times L$  by this kernel is exactly the extension  $\lambda E$ , the pushout of the diagram:

$$\begin{array}{ccc} L & \longrightarrow & E \\ \lambda \downarrow & \lrcorner & \downarrow \\ L & \longrightarrow & \lambda E. \end{array}$$

This finishes the proof.  $\square$

## 7 Dieudonné theory for unipotent groups over the dual numbers

In this section, as an application of the Main Theorem, we give a classification of smooth, unipotent group schemes over the dual numbers of a perfect field  $k$ , in terms of (extensions of) Dieudonné modules. So throughout the section, the ring  $k$  is a perfect field of characteristic  $p > 0$ .

### 7.1 Reminder on Dieudonné theory

We denote by  $W$  the Witt ring scheme over  $k$  and  $F, V$  its Frobenius and Verschiebung endomorphisms. For all  $n \geq 1$ , we write  $W_n := W/V^n W$  the ring scheme of Witt vectors of length  $n$ . We use the same notation also for these operators over the  $R$ -points, with  $R$  a  $k$ -algebra. We also define  $\tilde{V} : W_n \rightarrow W_{n+1}$  as the morphism induced on  $W_n$  by the composition

$$W \xrightarrow{V} W \xrightarrow{\pi_{n+1}} W_{n+1}$$

where  $\pi_{n+1}$  is the natural projection.

The *Dieudonné ring*  $\mathbb{D}$  is the  $W(k)$ -algebra generated by two variables  $\mathbb{F}$  and  $\mathbb{V}$  with the relations:

$$\begin{aligned}\mathbb{F}x &= \mathbb{F}(x)\mathbb{F} \\ x\mathbb{V} &= \mathbb{V}\mathbb{F}(x) \\ \mathbb{F}\mathbb{V} &= \mathbb{V}\mathbb{F} = p,\end{aligned}$$

for varying  $x \in W(k)$ . A *Dieudonné module* is a left  $\mathbb{D}$ -module. A Dieudonné module  $M$  is called *erasable* if for any  $m \in M$  there exists a positive integer  $n$  such that  $\mathbb{V}^n m = 0$ .

Let  $R$  be a  $k$ -algebra. Then, for any  $n \geq 1$ , we make  $W_n(R)$  a left  $\mathbb{D}$ -module with the rules:

$$\begin{aligned}\mathbb{F} \cdot u &= \mathbb{F}(u) \\ \mathbb{V} \cdot u &= \mathbb{V}(u) \\ x \cdot u &= \mathbb{F}^{1-n}(x)u\end{aligned}$$

for all  $u \in W_n(R)$  and  $x \in W(k)$ . The twist in the latter definition is designed to make  $\tilde{V} : W_n(R) \rightarrow W_{n+1}(R)$  a morphism of  $\mathbb{D}$ -modules, see Demazure and Gabriel [DG70], chap. V, § 1, no 3.3. All of this is functorial in  $R$  and gives  $W_n$  a structure of  $\mathbb{D}$ -module scheme. In particular,  $\text{End}_k(W_n)$  is a  $\mathbb{D}$ -module. According to [DG70], chap. V, § 1, no 3.4 the morphism  $\mathbb{D} \rightarrow \text{End}_k(W_n)$  induces an isomorphism of  $\mathbb{D}$ -modules:

$$\mathbb{D}/\mathbb{D}\mathbb{V}^n \xrightarrow{\sim} \text{End}_k(W_n).$$

If  $U$  is a commutative, unipotent  $k$ -group scheme, the set  $\text{Hom}_k(U, W_n)$  is a Dieudonné module with its structure given by postcomposition, i.e. for any  $f : U \rightarrow W_n$ :

$$\begin{aligned}\mathbb{F} \cdot f &= \mathbb{F} \circ f \\ \mathbb{V} \cdot f &= \mathbb{V} \circ f \\ x \cdot f &= \mathbb{F}^{1-n}(x)f,\end{aligned}$$

all  $x \in W(k)$ . We define the *Dieudonné module of  $U$*  as:

$$M(U) := \varinjlim_n \text{Hom}_k(U, W_n)$$

where the transition maps of the inductive system are induced by  $\tilde{V} : W_n \rightarrow W_{n+1}$ . Since  $\text{Hom}_k(U, W_n)$  is killed by  $\mathbb{V}^n$  and  $M(U)$  is a union of these subgroups, we see that  $M(U)$  is erasable. If  $M$  is a Dieudonné module, we define its *Frobenius twist*  $M^{(p)}$  as the module with underlying group  $M^{(p)} = M$  and  $\mathbb{D}$ -module structure given by:

$$\mathbb{F}_{M^{(p)}} = \mathbb{F}_M, \quad \mathbb{V}_{M^{(p)}} = \mathbb{V}_M, \quad x_{M^{(p)}} = \mathbb{F}^{-1}(x)_M \quad \text{for all } x \in W(k).$$

Then the maps  $\mathbb{F}_M : M^{(p)} \rightarrow M$  and  $\mathbb{V}_M : M \rightarrow M^{(p)}$  are visibly  $\mathbb{D}$ -linear. Moreover, according to [DG70], chap. V, § 1, 4.5 we have a canonical isomorphism  $M(U)^{(p)} \xrightarrow{\sim} M(U^{(p)})$  for all  $U \in \text{CU}$  with Frobenius twist  $U^{(p)}$ .

**7.1.1 Theorem.** *Let  $\text{CU}/k$  be the category of commutative unipotent  $k$ -group schemes, and  $\mathbb{D}\text{-Mod}$  the category of Dieudonné modules. The contravariant functor*

$$M : \text{CU}/k \longrightarrow \mathbb{D}\text{-Mod}$$

*is exact, fully faithful with essential image the full subcategory of erasable Dieudonné modules. It transforms the Frobenius (resp. Verschiebung) of  $U$  into the Frobenius (resp. Verschiebung) of  $M(U)$ . Moreover a unipotent group scheme  $U$  is of finite type if and only if  $M(U)$  is of finite type.*

**Proof :** See [DG70], chap. V, § 1, 4.3, 4.4, 4.5. □

## 7.2 Dieudonné theory over the dual numbers

Before stating our Dieudonné classification, we need to define the notions of *Lie algebra* and *smoothness* of Dieudonné modules.

**7.2.1 Definition.** Let  $M \in \mathbb{D}\text{-Mod}$  be a Dieudonné module. We define the *Lie algebra of  $M$*  by:

$$\text{Lie } M := (M/\mathbb{F}M) \otimes_k k[\mathbb{F}].$$

If  $I$  a finite dimensional  $k$ -vector space, the  *$I$ -Lie algebra of  $M$*  is  $\text{Lie}(M, I) := \text{Lie } M \otimes_k I^\vee$ . This gives rise to endofunctors  $\text{Lie}(-)$  and  $\text{Lie}(-, I)$  of the category  $\mathbb{D}\text{-Mod}^{\text{eff}}$ .

**7.2.2 Proposition.** *We have an isomorphism, functorial in  $U \in \text{CU}/k$ :*

$$\text{Lie}(M(U), I) \xrightarrow{\sim} M(\text{Lie}(U, I)).$$

**Proof :** We start with the case of dimension one  $I = k$ , so  $\text{Lie}(U, k\varepsilon) = \text{Lie}(U)$ . Let  $U'$  be the kernel of Frobenius in  $U$  and  $M = M(U)$ ,  $M' = M(U')$ . We have an exact sequence:

$$0 \longrightarrow U' \longrightarrow U \xrightarrow{\mathbb{F}_U} U(p).$$

We deduce isomorphisms  $\text{Lie } U \xrightarrow{\sim} \text{Lie } U'$  and  $M' = M/\mathbb{F}M$ . In the sequel set  $L_U := (\text{Lie } U)(k)$ , a  $k$ -vector space. Since  $U'$  is a finite commutative group scheme, according to Fontaine [Fo77], chap. III, 4.2 there is a canonical isomorphism:

$$\eta_{U'} : M'/\mathbb{F}M' \xrightarrow{\sim} L_{U'}^\vee.$$

We deduce a composed isomorphism  $\eta_U$  as follows:

$$M/\mathbb{F}M \xrightarrow{\sim} M'/\mathbb{F}M' \xrightarrow{\eta_{U'}} L_{U'}^\vee \xrightarrow{\sim} L_U^\vee.$$

From the isomorphism of  $k$ -group schemes  $\text{Lie } U \simeq \mathbb{V}(L_U^\vee)$ , we deduce

$$M(\text{Lie } U) = \text{Hom}_k(\mathbb{V}(L_U^\vee), \mathbb{G}_a) \xrightarrow{\sim} L_U^\vee \otimes k[\mathbb{F}].$$

By tensoring  $\eta_U$  with  $k[\mathbb{F}]$ , we find

$$\text{Lie } M = (M/\mathbb{F}M) \otimes_k k[\mathbb{F}] \xrightarrow{\sim} L_U^\vee \otimes k[\mathbb{F}] \xrightarrow{\sim} M(\text{Lie } U).$$

The result for general  $I$  follows since  $\text{Lie}(U, I) \simeq \text{Lie } U \otimes_{\mathbb{O}_k} \mathbb{V}(I^\vee)$  and  $\text{Lie}(M, I) = \text{Lie } M \otimes_k I^\vee$ .  $\square$

We can characterize the functor  $\text{Lie}$  on Dieudonné modules by its values on the modules  $\mathbb{D}/\mathbb{D}\mathbb{V}^n$ .

**7.2.3 Proposition.** *There exists a unique covariant functor  $\mathcal{L} : \mathbb{D}\text{-Mod}^{\text{eff}} \rightarrow \mathbb{D}\text{-Mod}^{\text{eff}}$  with the following properties:*

- (1)  $\mathcal{L}$  is right exact and commutes with filtering inductive limits;
- (2)  $\mathcal{L}(\mathbb{D}/\mathbb{D}\mathbb{V}^n) = k[\mathbb{F}]^n$  for all  $n \geq 1$ ;
- (3)  $\mathcal{L} : \text{End}(\mathbb{D}/\mathbb{D}\mathbb{V}^n) \rightarrow \text{End}(k[\mathbb{F}]^n)$  sends  $\mathbb{F}$  to 0;  $\mathbb{V}$  to the endomorphism  $(a_0, a_1, \dots, a_{n-1}) \mapsto (0, a_0, a_1, \dots, a_{n-2})$ ; and multiplication by  $x = (x_0, x_1, x_2, \dots) \in W(k)$  to the diagonal endomorphism with diagonal entries  $(x_0, x_0^p, \dots, x_0^{p^{n-1}})$ .

**Proof :** Uniqueness. The key is the fact that  $\mathbb{D}/\mathbb{D}\mathbb{V}^n$  is a projective generator of the subcategory  $C_n := (\mathbb{D}\text{-Mod}^{\text{eff}})^{\mathbb{V}^n=0}$  of objects killed by  $\mathbb{V}^n$ . More precisely, since any erasable  $\mathbb{D}$ -module is a filtering union of its submodules of finite type, property (a) implies that  $\mathcal{L}$  is determined by its restriction to the subcategory of modules of finite type. Any finite type module  $M$  is killed by  $\mathbb{V}^n$  for some  $n \geq 1$ . Since  $\mathbb{D}/\mathbb{D}\mathbb{V}^n$  is noetherian, for any  $M \in C_n$  there exist  $r, s$  and an exact sequence:

$$(\mathbb{D}/\mathbb{D}\mathbb{V}^n)^s \longrightarrow (\mathbb{D}/\mathbb{D}\mathbb{V}^n)^r \longrightarrow M \longrightarrow 0,$$

and for any morphism  $f : M \rightarrow M'$  in  $C_n$  there is a commutative diagram:

$$\begin{array}{ccccccc} (\mathbb{D}/\mathbb{D}\mathbb{V}^n)^s & \longrightarrow & (\mathbb{D}/\mathbb{D}\mathbb{V}^n)^r & \longrightarrow & M & \longrightarrow & 0 \\ \downarrow h & & \downarrow g & & \downarrow f & & \\ (\mathbb{D}/\mathbb{D}\mathbb{V}^n)^{s'} & \longrightarrow & (\mathbb{D}/\mathbb{D}\mathbb{V}^n)^{r'} & \longrightarrow & M' & \longrightarrow & 0. \end{array}$$

Since  $\mathcal{L}$  is right exact, this implies that  $\mathcal{L}(M)$  is determined by the values of  $\mathcal{L}((\mathbb{D}/\mathbb{D}\mathbb{V}^n)^r)$  for variable  $r$ , and  $\mathcal{L}(f)$  is determined by the values of  $\mathcal{L}(g)$  for variable  $g$  as above. Again since  $\mathcal{L}$  is right exact, it is additive. Hence  $\mathcal{L}((\mathbb{D}/\mathbb{D}\mathbb{V}^n)^r)$  is determined by  $\mathcal{L}(\mathbb{D}/\mathbb{D}\mathbb{V}^n)$  which is prescribed in (b). Similarly  $\mathcal{L}(g)$  is determined by the values of  $\mathcal{L}$  on the various maps  $\mathbb{D}/\mathbb{D}\mathbb{V}^n \hookrightarrow (\mathbb{D}/\mathbb{D}\mathbb{V}^n)^r \twoheadrightarrow \mathbb{D}/\mathbb{D}\mathbb{V}^n$ . Since  $\text{End}(\mathbb{D}/\mathbb{D}\mathbb{V}^n) = \mathbb{D}/\mathbb{D}\mathbb{V}^n$  is generated by  $W(k)$ ,  $F$  and  $V$ , the assignment in (b) ensures uniqueness of  $\mathcal{L}$ . Existence. Because of Proposition 7.2.2, it is enough to check that the functors  $\mathcal{L}(M(-))$  and  $M(\text{Lie}(-))$  take the same values on the Witt groups  $U = W_n$  and the endomorphisms of these groups. This is a simple computation which is left to the reader.  $\square$

We come to the notion of smoothness. It is known that a  $k$ -group scheme of finite type  $U$  is smooth if and only if its relative Frobenius  $F_{U/k} : U \rightarrow U^{(p)}$  is an epimorphism of  $k$ -group schemes. This motivates the following definition.

**7.2.4 Definition.** An erasable Dieudonné module  $M$  is called *smooth* if it is of finite type and its Frobenius morphism  $F_M : M^{(p)} \rightarrow M$  is a monomorphism.

**7.2.5 Lemma.** An erasable Dieudonné module  $M$  is smooth if and only if there exists a sequence of epimorphisms  $M = M_d \rightarrow M_{d-1} \rightarrow \cdots \rightarrow M_1 \rightarrow M_0 = 0$  whose kernels are all isomorphic to  $k[\mathbb{F}]$ .

The integer  $d$  is uniquely determined; it is called the *dimension* of  $M$ .

**Proof :** Since  $M(\mathbb{G}_a) \simeq k[\mathbb{F}]$ , this follows from Theorem 7.1.1, and [DG70], chap. IV, §2, Cor 3.9.  $\square$

**7.2.6 Definition.** An *I-extension* in  $\mathbb{D}\text{-Mod}^{\text{eff}}$  is an extension of smooth erasable Dieudonné modules of the form

$$0 \longrightarrow M \longrightarrow M' \longrightarrow \text{Lie}(M, I) \longrightarrow 0.$$

A *morphism of I-extensions* is a morphism of extensions in the usual sense, that is, a morphism of short exact sequences of  $\mathbb{D}$ -modules.

**7.2.7 Theorem.** Let  $\text{SCU}/k[I]$  be the category of smooth, commutative, unipotent (i.e. with unipotent special fibre)  $k[I]$ -group schemes. Let  $\mathbb{D}\text{-I-Mod}$  be the category of *I-extensions* of smooth erasable Dieudonné modules. Then the Dieudonné functor  $M$  induces a contravariant equivalence of categories:

$$\mathcal{M} : \text{SCU}/k[I] \longrightarrow \mathbb{D}\text{-I-Mod}$$



that sends  $\mathcal{U}$  to the extension  $0 \rightarrow M(\mathcal{U}_k) \rightarrow M(h_*\mathcal{U}) \rightarrow M(\mathrm{Lie}(\mathcal{U}_k, I)) \rightarrow 0$ . A quasi-inverse functor is obtained by sending an extension  $0 \rightarrow M \rightarrow M' \rightarrow \mathrm{Lie}(M, I) \rightarrow 0$  to the Weil extension  $h^+(U(M'))$  of the extension  $0 \rightarrow \mathrm{Lie}(U(M), I) \rightarrow U(M') \rightarrow U(M) \rightarrow 0$ , where  $U$  is a quasi-inverse for  $M$ .

**Proof :** It suffices to put together Theorem 6.1.1 and Theorem 7.1.1. In little more detail, let  $\mathcal{U}$  be a smooth, commutative, unipotent group scheme over the ring of dual numbers  $k[I]$ , and let  $U = \mathcal{U}_k$  be its special fibre. By Theorem 6.1.1 this datum is equivalent to an extension

$$0 \longrightarrow \mathrm{Lie}(U, I) \longrightarrow E \longrightarrow U \longrightarrow 0$$

with  $E = h_*\mathcal{U}$  smooth, commutative, unipotent. By Theorem 7.1.1 this is equivalent to an extension

$$0 \longrightarrow M(U) \longrightarrow M' \longrightarrow M(\mathrm{Lie}(U, I)) \longrightarrow 0,$$

where  $M' = M(E)$ . Since  $\mathrm{Lie}(M(U), I) \xrightarrow{\sim} M(\mathrm{Lie}(U, I))$  by Proposition 7.2.2, we obtain an  $I$ -extension as desired.  $\square$

## A Appendix: Module stacks in groupoids

Both categories  $\mathrm{Gr}/k[I]$  and  $\mathrm{Ext}(I)/k$  are endowed with the structure of  $\mathbb{O}_k$ -module stacks in groupoids over  $\mathrm{Gr}/k$  and the purpose of this Appendix is to explain what this means. In the two cases this seems to be a well-known fact, but we were able to locate only very few discussions of this topic in the literature: the reader can check Osserman [Os10] subsections 2.3, 2.4, 2.5 or Weissman [We11] Proposition 1.5 which is hardly more than a hint. In fact, the additive part of the structure, which goes by the name of a ‘‘Picard category’’, is well documented, a landmark being Deligne’s expos e in [SGA4.3], Exp. XVIII, § 1.4. However, the linear part of the structure, that is the  $\mathbb{O}_k$ -scalar multiplication and its interplay with the additive structure, are almost absent. Subsections 2.4 and 2.5 of Osserman [Os10] are a first step, but the author writes: *Although it is possible to [define scalar multiplication maps] on a categorical level as we did with addition, expressing the proper conditions for associativity and distributivity isomorphisms becomes substantially more complicated.* Here we tackle the question of the definition, and we establish a few basic results in order to highlight the nontrivial features of the theory.

We start with the definition of Picard categories. The alternative phrase *commutative group groupoids* is a more accurate name to refer to them, but we stick with the traditional name. In order to make the axioms reader-friendly, we adopt a simplified description for the multifunctors involved, e.g. the associativity isomorphism  $a : T_1 \rightarrow T_2$  between the trifunctors  $T_1, T_2 : P \times P \times P \rightarrow P$  given by  $T_1 = + \circ (+ \times \mathrm{id})$  and  $T_2 = + \circ (\mathrm{id} \times +)$  is given in the form of isomorphisms  $a_{x,y,z} : (x+y)+z \rightarrow x+(y+z)$  functorial in  $x, y, z \in P$ .

**A.1 Definition.** Let  $P$  be a category and  $+ : P \times P \rightarrow P$  a bifunctor.

- (1) An *associativity constraint* for  $+$  is an isomorphism of functors  $a_{x,y,z} : (x+y)+z \xrightarrow{\sim} x+(y+z)$  such that the pentagon axiom ([SGA4.3], Exp. XVIII, 1.4.1) is satisfied. It is called *trivial* or *strict* if  $a_{x,y,z} = \mathrm{id}$  for all  $x, y, z \in P$ .
- (2) A *commutativity constraint* for  $+$  is an isomorphism of functors  $c_{x,y} : x+y \xrightarrow{\sim} y+x$  which satisfies  $c_{y,x} \circ c_{x,y} = \mathrm{id}_{x+y}$  for all  $x, y \in P$ . It is called *trivial* or *strict* if  $c_{x,y} = \mathrm{id}$  for all  $x, y \in P$ .
- (3) The associativity and commutativity constraints  $a$  and  $c$  are *compatible* if the hexagon axiom ([SGA4.3], Exp. XVIII, 1.4.1) is satisfied.
- (4) A *neutral element* for  $+$  is an object  $0 \in P$  with an isomorphism  $\varphi : 0+0 \xrightarrow{\sim} 0$ .

In other mathematical contexts, associativity constraints are also called *associators* and commutativity constraints are also called *symmetric braidings*.

**A.2 Definition.** Let  $(P_1, +)$  and  $(P_2, +)$  be categories endowed with bifunctors. Let  $F : P_1 \rightarrow P_2$  be a functor and  $\varphi_{F,x,y} : F(x + y) \xrightarrow{\sim} F(x) + F(y)$  an isomorphism of functors.

(1) Let  $a_1, a_2$  be associativity constraints on  $(P_1, +)$  and  $(P_2, +)$ . We say that  $(F, \varphi_F)$  is *compatible with*  $a_1, a_2$  if the following diagram commutes:

$$\begin{array}{ccccc} F((x + y) + z) & \xrightarrow{\varphi_F} & F(x + y) + F(z) & \xrightarrow{\varphi_F} & (F(x) + F(y)) + F(z) \\ F(a_1) \downarrow & & & & \downarrow a_2 \\ F(x + (y + z)) & \xrightarrow{\varphi_F} & F(x) + F(y + z) & \xrightarrow{\varphi_F} & F(x) + (F(y) + F(z)). \end{array}$$

(2) Let  $c_1, c_2$  be commutativity constraints on  $(P_1, +)$  and  $(P_2, +)$ . We say that  $(F, \varphi_F)$  is *compatible with*  $c_1, c_2$  if the following diagram commutes:

$$\begin{array}{ccc} F(x + y) & \xrightarrow{\varphi_F} & F(x) + F(y) \\ F(c_1) \downarrow & & \downarrow c_2 \\ F(y + x) & \xrightarrow{\varphi_F} & F(y) + F(x). \end{array}$$

**A.3 Definition.** A *Picard category* is a quadruple  $(P, +, a, c)$  composed of a nonempty groupoid  $P$ , a bifunctor  $+: P \times P \rightarrow P$  with compatible associativity and commutativity constraints  $a$  and  $c$ , such that for each  $x \in P$  the functor  $P \rightarrow P, y \mapsto x + y$  is an equivalence.

Any Picard category  $P$  has a neutral element  $0$  which is unique up to a unique isomorphism ([SGA4.3], Exp. XVIII, 1.4.4). Moreover, for each  $x, y \in P$  the set of morphisms  $\text{Hom}(x, y)$  is either empty or a torsor under the group  $G := \text{Aut}(0)$ . More precisely, the functors  $+x : P \rightarrow P$  and  $x+ : P \rightarrow P$  induce the same bijection  $G \rightarrow \text{Aut}(x), \varphi \mapsto \varphi + \text{id}_x$ . Viewing this bijection as an identification, the set  $\text{Hom}(x, y)$  with its right  $\text{Aut}(x)$ -action and left  $\text{Aut}(y)$ -action becomes a pseudo- $G$ -bitorsor, i.e. it is either empty or a  $G$ -bitorsor.

**A.4 Lemma.** *The left and right actions of  $G$  on the sets  $\text{Hom}(x, y)$  coincide, that is,  $ga = ag$  for all  $g \in G$  and  $a \in \text{Hom}(x, y)$ . In particular, the group  $G$  is abelian.*

**Proof :** The proof is a variation on the Eckmann-Hilton argument. In order to avoid carrying the cumbersome isomorphism  $\varphi$  attached to the neutral element  $0$  (see Definition A.1(4)), we assume notationally that it is the identity. That is, we consider that the functors  $0+$  and  $+0$  are identities. Let  $1 = \text{id}_0$ . For all  $a \in \text{Hom}(x, y)$  we have:

- (i)  $a1 = a$  and  $1a = a$ ,
- (ii)  $a + 1 = 1 + a = a$ .

Moreover, since  $+: P \times P \rightarrow P$  is a functor, it respects composition. Hence for all  $a, b \in \text{Hom}(x, y)$  and  $g, h \in G$  we have:

- (iii)  $ga + hb = (g + h)(a + b)$ ,
- (iv)  $ag + bh = (a + b)(g + h)$ .

From (iii), we deduce  $a + h = (1 \circ a) + (h \circ 1) = (1 + h) \circ (a + 1) = h \circ a$ . From (iv), we deduce  $a + h = (a \circ 1) + (1 \circ h) = (a + 1) \circ (1 + h) = a \circ h$ . It follows that  $h \circ a = a \circ h$ , that is, the left and right actions of  $G$  on  $\text{Hom}(x, y)$  coincide. For  $x = y = 0$  this proves that  $G$  is abelian.  $\square$

**A.5 Definition.** Let  $P_1, P_2$  be Picard categories.

(2) An *additive functor* is a pair  $(F, \varphi_F)$  where  $F : P_1 \rightarrow P_2$  is a functor and  $\varphi_{F,x,y} : F(x + y) \xrightarrow{\sim} F(x) + F(y)$  is an isomorphism of functors that is compatible with associativity and commutativity constraints.

(3) Let  $F, G : P_1 \rightarrow P_2$  be additive functors. A *morphism of additive functors* is a morphism of functors  $u : F \rightarrow G$  such that the following diagram is commutative:

$$\begin{array}{ccc} F(x + y) & \xrightarrow{u_{x+y}} & G(x + y) \\ \varphi_F \downarrow & & \downarrow \varphi_G \\ F(x) + F(y) & \xrightarrow{u_x + u_y} & G(x) + G(y). \end{array}$$

We emphasize that since a Picard category is a groupoid (that is, all its morphisms are isomorphisms), all morphisms of additive functors  $u : F \rightarrow G$  are isomorphisms.

The *category of additive functors*  $\text{Hom}(P_1, P_2)$  is itself a Picard category ([SGA4.3], Exp. XVIII, 1.4.7). Additive functors can be composed and the identity functors behave as neutral elements. In the particular case where  $P_1 = P_2 = P$ , along with its addition law, the Picard category  $\text{End}(P) = \text{Hom}(P, P)$  enjoys an internal multiplication given by composition. Note that in this case multiplication is strictly associative, because so is composition of functors in categories.

**A.6 Remark.** In  $\text{End}(P)$ , the neutral element  $0_E$  is the functor  $0_E : P \rightarrow P$ ,  $x \mapsto 0_P$ ,  $(u : x \rightarrow y) \mapsto \text{id}_{0_P}$ . Let us describe its automorphism group in the case where  $P$  is a small category. An automorphism is a natural transformation  $\gamma : 0_E \rightarrow 0_E$ , that is, for each  $x \in P$  we have a morphism  $\gamma_x : 0_P \rightarrow 0_P$  such that for all morphisms  $u : x \rightarrow y$  the following square is commutative:

$$\begin{array}{ccc} 0_P & \xrightarrow{\gamma_x} & 0_P \\ \text{id}_{0_P} \downarrow & & \downarrow \text{id}_{0_P} \\ 0_P & \xrightarrow{\gamma_y} & 0_P. \end{array}$$

This means that  $\gamma_x = \gamma_y$  whenever  $x$  and  $y$  are isomorphic. It follows that  $\text{Aut}(0_E)$  is the group of functions  $\pi_0(P) \rightarrow G$  where  $\pi_0(P)$  is the set of isomorphism classes of objects of  $P$ .

In fact  $\text{End}(P)$  is a *ring category*, but in order to introduce module groupoids, we do not actually need to define what is such a thing.

**A.7 Definition.** Let  $\Lambda$  be a commutative ring. A  $\Lambda$ -*module groupoid* is a Picard category  $P$  endowed with a functor  $F = (F, \varphi_F, \psi_F) : \Lambda \rightarrow \text{End}(P)$  called *scalar multiplication* such that:

- (1)  $(F, \varphi_F)$  is an additive functor.

For each  $\lambda \in \Lambda$ , for simplicity we write  $(\lambda, \varphi_\lambda)$  for  $(F_\lambda, \varphi_{F_\lambda}) : P \rightarrow P$ . Moreover:

- (2)  $(F, \psi_F)$  is multiplicative, i.e.  $F(1) = \text{id}_P$  and  $F$  is compatible with the associativity constraints of multiplication.

(3)  $F$  is compatible with the distributivity of multiplication over addition:

$$\begin{array}{ccccc} (\lambda(\mu + \nu))x & \xrightarrow{\psi_F} & \lambda((\mu + \nu)x) & \xrightarrow{\varphi_F} & \lambda(\mu x + \nu x) \\ \parallel & & & & \downarrow \varphi_\lambda \\ (\lambda\mu + \lambda\nu)x & \xrightarrow{\varphi_F} & (\lambda\mu)x + (\lambda\nu)x & \xrightarrow{\psi_F + \psi_F} & \lambda(\mu x) + \lambda(\nu x). \end{array}$$

**A.8 Remarks.** (1) The associativity constraints of multiplication are trivial in both  $\Lambda$  and  $\text{End}(P)$ , but this does not mean that the condition of compatibility in (2) is empty. Rather, it means that the following diagram commutes:

$$\begin{array}{ccccc} ((\lambda\mu)\nu)x & \xrightarrow{\psi_F} & (\lambda\mu)(\nu x) & \xrightarrow{\psi_F} & \lambda(\mu(\nu x)) \\ \parallel & & & & \parallel \\ (\lambda(\mu\nu))x & \xrightarrow{\psi_F} & \lambda((\mu\nu)x) & \xrightarrow{\psi_F} & \lambda(\mu(\nu x)) \end{array}$$

The same remark applies for distributivity of multiplication over addition (and the relevant commutative diagram is spelled out in A.7(3) above.)

(2) Since  $\text{End}(P)$  is not endowed with a multiplicative commutativity constraint (for the simple reason that such a constraint does not exist in general), no corresponding requirement is imposed on  $(F, \psi_F)$ .

In usual set-valued module theory, from the fact that  $\Lambda$  is commutative it would follow that the functor  $F : \Lambda \rightarrow \text{End}(P)$  in Definition A.7 takes values in a commutative subring of  $\text{End}(P)$ . However, in our context commutativity is not a property but an *extra datum*. Correspondingly, the correct statement is the following.

**A.9 Lemma.** *Let  $P$  be a  $\Lambda$ -module groupoid with structure map  $F : \Lambda \rightarrow \text{End}(P)$ . Then the essential image of  $F$  has a natural commutativity constraint such that for all  $\lambda, \mu \in \Lambda$  the commutativity isomorphisms  $c_{\lambda, \mu}$  is the composition:*

$$c_{\lambda, \mu} : \lambda(\mu x) \xrightarrow{\psi_{F, \lambda, \mu}^{-1}} (\lambda\mu)x = (\mu\lambda)x \xrightarrow{\psi_{F, \mu, \lambda}} \mu(\lambda x).$$

**Proof :** Let  $G, H$  be in the essential image of  $F : \Lambda \rightarrow \text{End}(P)$ . Thus there exist  $\lambda, \mu \in \Lambda$  and isomorphisms  $u : G \xrightarrow{\sim} F_\lambda$  and  $v : H \xrightarrow{\sim} F_\mu$ . There is an induced isomorphism  $u \star v : GH \xrightarrow{\sim} F_\lambda F_\mu$ , described as follows. Since  $u$  is a functor, for each  $x \in P$  we have a commutative square:

$$\begin{array}{ccc} GHx & \xrightarrow{u_{Hx}} & F_\lambda Hx \\ Gv_x \downarrow & & \downarrow F_\lambda v_x \\ GF_\mu x & \xrightarrow{u_{F_\mu x}} & F_\lambda F_\mu x. \end{array}$$

Either way defines  $(u \star v)_x : GHx \rightarrow F_\lambda F_\mu x$ . Now we define:

$$c_{G, H} = GH \xrightarrow{u \star v} F_\lambda F_\mu \xrightarrow{c_{\lambda, \mu}} F_\mu F_\lambda \xrightarrow{v^{-1} \star u^{-1}} HG.$$

We must prove that this does not depend on the choices of  $u$  and  $v$ . Let  $\Gamma$  be the automorphism group of the neutral functor  $0_{\text{End}(P)}$ . Another choice of isomorphism  $u'$  is of the form  $u' = u\gamma$  for some  $\gamma \in \Gamma$ . Then  $u'^{-1} = \gamma^{-1}u^{-1}$  and  $c'_{G, H} = (v^{-1} \star \gamma^{-1}u^{-1}) \circ c_{\lambda, \mu} \circ (u\gamma \star v)$ . But according to Lemma A.4, the

element  $\gamma$  commutes with all morphisms, hence  $u\gamma \star v = \gamma(u \star v)$  and  $(v^{-1} \star \gamma^{-1} u^{-1}) = (v^{-1} \star u^{-1})\gamma^{-1}$ . It follows that  $c'_{G,H} = c_{G,H}$ . Similarly we prove that another choice  $v' = v\gamma$  does not change  $c_{G,H}$ . Finally we check that the symmetry condition of a commutativity constraint holds. The computation for  $c_{\lambda,\mu}$  follows from its definition, and for  $c_{G,H}$  follows from the fact that  $(u \star v)(u^{-1} \star v^{-1}) = \text{id}$ .  $\square$

**A.10 Definition.** Let  $\mathcal{S}$  be a site. Let  $\Lambda$  be a sheaf of commutative rings on  $\mathcal{S}$ . A  $\Lambda$ -module stack (in groupoids) over  $\mathcal{S}$  is a stack in groupoids  $P$  over  $\mathcal{S}$  endowed with

- (1) a functor  $+$  :  $P \times P \rightarrow P$ ,
- (2) isomorphisms of functors  $a_{x,y,z} : (x + y) + z \xrightarrow{\sim} x + (y + z)$  and  $c_{x,y} : x + y \xrightarrow{\sim} y + x$ ,
- (3) a functor  $F = (F, \varphi_F, \psi_F) : \Lambda \rightarrow \text{End}(P)$ ,

such that for each  $U \in \mathcal{S}$  the fibre category  $P(U)$  is a  $\Lambda(U)$ -module groupoid.

There is an obvious corresponding relative notion of  $\Lambda$ -module stack (in groupoids) over a given  $\mathcal{S}$ -stack  $Q$ , namely, it is a morphism of stacks  $P \rightarrow Q$  that makes  $P$  a stack fibred in groupoids over  $Q$ , with an addition functor  $+$  :  $P \times_Q P \rightarrow P$  etc. It is the relative notion that is useful in the paper.

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