



HAL
open science

Opinion diffusion and influence: A logical approach

Laurence Cholvy

► **To cite this version:**

Laurence Cholvy. Opinion diffusion and influence: A logical approach. International Journal of Approximate Reasoning, 2018, 93, pp.24-39. 10.1016/j.ijar.2017.10.014 . hal-01712001

HAL Id: hal-01712001

<https://hal.science/hal-01712001>

Submitted on 21 Feb 2018

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Opinion Diffusion and Influence: A logical Approach

Laurence Cholvy

ONERA, Toulouse, France

Abstract

This paper aims at modelling opinion diffusion among agents with a logical approach by assuming that opinions are propositional formulas. In a first part, we present a model in which an agent changes its opinion by merging the opinions of some influential agents, its influencers. More precisely, it merges these opinions, which may be contradictory, by taking into account an order of importance among its influencers. In a second part, we generalize this model so that influencers are ordered according to several orders of influence which depend on the topics of opinions.

1. Introduction

Understanding the dynamics of opinion among agents is an important question which has recently received considerable attention in the community of autonomous agents and multi-agent systems [6, 1, 2, 17, 5, 11, 19, 16]. This question depends on several parameters.

The first important parameter is the population of agents. This population may be unstructured, in such a case, agents interact randomly [13, 6]. But generally, some relations exist between agents. The population of agents may be divided into communities modelling neighborhood relations between agents [12, 1, 5]. The population may also be structured via an influence relation which relates two agents, the opinion of one of these agents being influenced by the opinion of the other [17]. Graphs are widely used to model the structured population: nodes are agents and links are the relations between agents. Links are symmetrical or not, depending on the type of relations and they may also be labelled with probabilities [18].

The second parameter is the model of opinion. Here again, several options exist. Most of the works previously cited consider only one opinion and model it as a real number in $[0, 1]$. For instance, if the question is to evaluate the opinion of people about the fact that Canada will host the Winter Olympics in 2026, then an opinion which is close to 1 means that the agent is quite confident in Canada candidature or that according to this agent, the probability that Canada will host the Olympics is high. An opinion which is close to 0 means that the agent thinks that Canada candidature will be rejected or that according to this agent, the probability that Canada will host the Olympics is low. Some other works are based on formal logic and model opinions as

propositional formulas or, more precisely, as the sets of their models. In [11], an opinion is a single interpretation, called a ballot. For instance, an agent whose opinion is $CAN \wedge acroski$, thinks that Canada will organize the Olympics in 2026 and that there will be acroski trials. Another agent whose opinion is $\neg CAN \wedge acroski$, thinks that 2026 Games will not be hosted by Canada but there will be acroski trials. More generally, [16] considers that an opinion is any propositional formula, thus modelled by a set of interpretations which is not necessarily a singleton. For instance, an agent opinion is $(acroski \vee ski.joering) \wedge CAN$ when it thinks that the 2026 Winter Olympics will be hosted by Canada and that there will be *acroski* trials or *ski.joering* trials.

The last parameter is the model of opinion dynamics. Many works in the field of opinion dynamics in multi-agent systems are based on a theory introduced in the field of Social Psychology called *Social Judgment Theory (SJT)*. The basic idea of *SJT* is that individual opinion changing is a judgmental process: if an agent considers that a presented opinion is close to its current opinion, then it is likely to shift in the direction of this opinion (*assimilation*); if it considers that the presented opinion is distant to its current opinion, then it is likely to shift away from this opinion (*contrast*); otherwise, the agent does not change its opinion (*non-commitment*). This general idea has led to different formal models [13, 6, 2] in which the thresholds agents use to characterize what are close and distant opinions are identical or not, universal or agent dependent. Some other works, like [17], are based on the theory of *motivated cognition*, defined in Cognitive Psychology, and which also says that agents are skeptical of another agent when their opinions diverge, but are more receptive to persuasion when their opinions better align. Some other works in the field of diffusion in multi agent systems claim to be based on models provided by the Network Science community. For instance, [5] is based on the *SIR model* which says that the value of an agent's feature evolves according to the values of its neighbors feature values. For instance an agent is infected if one of its neighbor is. Or an agent may say that it believes something if its neighbors said that they also do.

Simulating opinion diffusion may be applied in many contexts. In particular, simulation of opinion diffusion is important to prepare Psychological Actions (PSYOPS) in the military context. These actions aim at changing the perception and the behavior of some people. They consist in elaborating and spreading out a message that must reach these individuals, directly or indirectly via their social networks [10, 9].

In the present paper, we extend a work recently presented in [4]¹. We model opinions by any kind of propositional formulas. As a consequence, opinions may be in disjunctive form and thus be incomplete. We also assume that the population of agents is structured by a binary relation of influence which relates two agents when one influences the other. In a first part, we assume that any agent orders its influencers (i.e. agents which influence it) according to the strength of the influence relation. Then, any agent changes its opinion by merging the opinions of its influencers from the most

¹In this present paper, we prove many more properties on IODS and TIODS that we did in the previous paper. But the main difference between the two papers is that here, we changed the definition for TIODS so that *Unanimity Preservation* is now satisfied while it was not in the previous paper. It is also proved that TIODS extend IODS, which was not the case in the previous paper. Moreover, proofs are given in extension

influential one to the least. In a second part, we consider that agents order their influencers according to the topics of opinions. For instance, you may be more influenced by your friend *Paul* than by your friend *Mary* about *winter sport events* while being more influenced by *Mary* than by *Paul* about *litterature*. In this case, any agent changes its opinion by merging the opinions of its influencers, topic by topic.

This paper is organized as follows. Section 2 presents the notion of Importance-Based Merging Operators and provides some original properties of these operators. Section 3 presents the notion of Influence-Based Opinion Diffusion Structures (IODS) to model opinion diffusion. Section 4 presents some properties of IODS. Section 5 extends IODS so that influencers may be ordered according to several orders of influence, depending on the topics of opinions. It also studies some of their properties. Finally, Section 6 presents some conclusions and discussions.

2. Importance-Based Merging Operators

In this section, we aim at defining a merging operator which takes into account the relative importance of the formulas to be merged for building the result. For doing this, we adopt the same kind of approach introduced in [14]: we adopt a semantical approach and we characterize the models of the result from the models of the initial formulas; we also consider a special formula, called *integrity constraint*, which expresses some law of nature and which restricts the possible models.

We consider a finite propositional language L i.e. a finite set of propositional letters. A literals is a propositional letter or the negation of a propositional letter. By convention, an interpretation of L is represented by a set of literals so that a propositional letter is positive iff it is satisfied in the interpretation, negative iff it is not satisfied in the interpretation. If φ is a formula of L , $Mod(\varphi)$ denotes the set of models of φ i.e., the set of interpretations in which φ is true. A multi-set of formulas $\{\varphi_1, \dots, \varphi_n\}$ equipped with a total order \prec s.t. $\varphi_i \prec \varphi_{i+1}$ ($i = 1..n - 1$) is called *an ordered multi-set of formulas* and denoted $\varphi_1 \prec \dots \prec \varphi_n$.

Given a consistent formula μ and an ordered multi-set of formulas $\varphi_1 \prec \dots \prec \varphi_n$, an Importance-Based Merging Operator characterizes a formula $\Delta_\mu(\varphi_1 \prec \dots \prec \varphi_n)$ whose models are selected among the models of μ by taking into account the relative importance of the formulas φ_i . More precisely, the operator selects the models of μ which first are the closest to the models of φ_1 , then the closest to the models of φ_2 etc. For doing this, we assume a pseudo-distance d between interpretations of L . Notice that $\Delta_\mu(\varphi_1 \prec \dots \prec \varphi_n)$ should be indexed by d but we write $\Delta_\mu(\varphi_1 \prec \dots \prec \varphi_n)$ to simplify the notation. The formal definition of Δ is the following:

Definition 1 (Importance-Based Merging Operator). *An Importance-Based Merging Operator is a function Δ which associates a formula μ and a non empty ordered multi-set of consistent formulas $\varphi_1 \prec \dots \prec \varphi_n$ with a formula denoted $\Delta_\mu(\varphi_1 \prec \dots \prec \varphi_n)$ so that: $Mod(\Delta_\mu(\varphi_1 \prec \dots \prec \varphi_n)) = Min_{\leq_{d, \varphi_1 \prec \dots \prec \varphi_n}} Mod(\mu)$ with:*

- $w \leq_{d, \varphi_1 \prec \dots \prec \varphi_n} w'$ iff
 $[D(w, \varphi_1), \dots, D(w, \varphi_n)] \leq_{lex} [D(w', \varphi_1), \dots, D(w', \varphi_n)]$
- $[D(w, \varphi_1), \dots, D(w, \varphi_n)]$ is a vector whose k^{th} element is $D(w, \varphi_k)$

- $D(w, \varphi) = \min_{w' \in \text{Mod}(\varphi)} d(w, w')$, w and w' being two interpretations and d is a pseudo-distance² between interpretations.
- \leq_{lex} is a lexicographic comparison of vectors of reals defined by: $[v_1, \dots, v_n] \leq_{lex} [v'_1, \dots, v'_n]$ iff (i) $\forall k \ v_k = v'_k$ or (ii) $\exists k \ v_k < v'_k$ and $\forall j < k \ v_j = v'_j$

Some simple distances d which can be used for instantiating the previous definition are: d_D , the drastic distance, ($d_D(w, w') = 0$ iff $w = w'$, 1 otherwise); d_H , the Hamming distance ($d_H(w, w') = m$ iff w and w' differ on m variables).

Example 1. Let μ be a tautology. $\varphi_1 = a \vee b$, $\varphi_2 = \neg a$, $\varphi_3 = \neg b \wedge c$. The eight interpretations are $w_1 = \{a, b, c\}$, $w_2 = \{a, b, \neg c\}$, $w_3 = \{a, \neg b, c\}$, $w_4 = \{a, \neg b, \neg c\}$, $w_5 = \{\neg a, b, c\}$, $w_6 = \{\neg a, b, \neg c\}$, $w_7 = \{\neg a, \neg b, c\}$, $w_8 = \{\neg a, \neg b, \neg c\}$ and $\text{Mod}(\Delta_\mu(\varphi_1 \prec \varphi_2 \prec \varphi_3)) = \text{Min}_{\leq_{lex}}(\{w_1, \dots, w_8\})$. With $d = d_H$ we get: $\text{Mod}(\Delta_\mu(\varphi_1 \prec \varphi_2 \prec \varphi_3)) = \{w_5\}$, thus $\Delta_\mu(\varphi_1 \prec \varphi_2 \prec \varphi_3) = \neg a \wedge b \wedge c$. With $d = d_D$ we get: $\text{Mod}(\Delta_\mu(\varphi_1 \prec \varphi_2 \prec \varphi_3)) = \{w_5, w_6\}$, thus $\Delta_\mu(\varphi_1 \prec \varphi_2 \prec \varphi_3) = \neg a \wedge b$.

Suppose now that $\mu = a \wedge b$ i.e., $\text{Mod}(\mu) = \{w_1, w_2\}$. With $d = d_H$ we get: $\text{Mod}(\Delta_\mu(\varphi_1 \prec \varphi_2 \prec \varphi_3)) = \{w_1\}$, thus $\Delta_\mu(\varphi_1 \prec \varphi_2 \prec \varphi_3) = a \wedge b \wedge c$. With $d = d_D$ we get: $\text{Mod}(\Delta_\mu(\varphi_1 \prec \varphi_2 \prec \varphi_3)) = \{w_1, w_2\}$, thus $\Delta_\mu(\varphi_1 \prec \varphi_2 \prec \varphi_3) = a \wedge b$.

Let's now examine some properties of Importance-Based Merging Operators. First we consider the nine postulates that merging operators should satisfy according to [14]. We reformulate them within our context since here, formulas to be merged are ordered. Results are given in the following proposition³.

Proposition 1.

- $\Delta_\mu(\varphi_1 \prec \dots \prec \varphi_n) \models \mu$
- If μ is consistent then $\Delta_\mu(\varphi_1 \prec \dots \prec \varphi_n)$ is consistent
- If $\bigwedge_{i=1}^n \varphi_i \wedge \mu$ is consistent then $\models \Delta_\mu(\varphi_1 \prec \dots \prec \varphi_n) \leftrightarrow \bigwedge_{i=1}^n \varphi_i \wedge \mu$
- It is not the case that if $\models \mu \leftrightarrow \mu'$ and if there is a permutation f over $\{1 \dots n\}$ such that $\forall i = 1 \dots n, \models \varphi_i \rightarrow \varphi'_{f(i)}$ then $\models \Delta_\mu(\varphi_1 \prec \dots \prec \varphi_n) \leftrightarrow \Delta_{\mu'}(\varphi'_1 \prec \dots \prec \varphi'_n)$. We only have: If $\models \mu \leftrightarrow \mu'$ and if $\forall i = 1 \dots n, \models \varphi_i \leftrightarrow \varphi'_i$ then $\models \Delta_\mu(\varphi_1 \prec \dots \prec \varphi_n) \leftrightarrow \Delta_{\mu'}(\varphi'_1 \prec \dots \prec \varphi'_n)$
- It is not necessarily the case that if $\varphi_1 \models \mu$ and $\varphi_2 \models \mu$ then $\Delta_\mu(\varphi_1 \prec \varphi_2) \wedge \varphi_1$ is consistent iff $\Delta_\mu(\varphi_1 \prec \varphi_2) \wedge \varphi_2$ is consistent
- $\Delta_\mu(\varphi_1 \prec \dots \prec \varphi_n) \wedge \Delta_\mu(\varphi_{n+1} \prec \dots \prec \varphi_m) \models \Delta_\mu(\varphi_1 \prec \dots \prec \varphi_m)$

² $\forall w \forall w' \ d(w, w') = d(w', w)$ and $d(w, w') = 0 \implies w = w'$

³Proofs are given in Appendix.

- If $\Delta_\mu(\varphi_1 \prec \dots \prec \varphi_n) \wedge \Delta_\mu(\varphi_{n+1} \prec \dots \prec \varphi_m)$ is consistent then $\Delta_\mu(\varphi_1 \prec \dots \prec \varphi_m) \models \Delta_\mu(\varphi_1 \prec \dots \prec \varphi_n) \wedge \Delta_\mu(\varphi_{n+1} \prec \dots \prec \varphi_m)$
- $\Delta_{\mu_1}(\varphi_1 \prec \dots \prec \varphi_n) \wedge \mu_2 \models \Delta_{\mu_1 \wedge \mu_2}(\varphi_1 \prec \dots \prec \varphi_n)$
- If $\Delta_{\mu_1}(\varphi_1 \prec \dots \prec \varphi_n) \wedge \mu_2$ is consistent then $\Delta_{\mu_1 \wedge \mu_2}(\varphi_1 \prec \dots \prec \varphi_n) \models \Delta_{\mu_1}(\varphi_1 \prec \dots \prec \varphi_n)$

Furthermore we can also prove the following propositions.

Proposition 2. *If $\varphi_1 \models \mu$ then*

$$\text{Mod}(\Delta_\mu(\varphi_1 \prec \dots \prec \varphi_n)) = \text{Mod}(\Delta_{\varphi_1}(\varphi_1 \prec \dots \prec \varphi_n))$$

I.e., if the most important formula to be merged satisfies μ then replacing the integrity constraint μ by that formula does not change the result of merging⁴. As a consequence we have:

Proposition 3. *If $\varphi_1 \models \mu$ then $\Delta_\mu(\varphi_1 \prec \dots \prec \varphi_n) \models \varphi_1$.*

Proposition 4. *Consider n formulas $\varphi_1, \dots, \varphi_n$ and \prec an order on these formulas. Let $k \in \{1 \dots n\}$ and let \prec^k be the order obtained from \prec by replacing φ_k by $\Delta_\mu(\prec)$. Then, $\text{Mod}(\Delta_\mu(\prec)) = \text{Mod}(\Delta_\mu(\prec^k))$.*

I.e., the result of merging different formulas with Δ_μ does not change when one formula, whatever its importance, is replaced by the merged formula. This shows that Δ_μ satisfies the property called “ballot-monotonicity” in [11] and which was there restricted to ballots.

3. Influence-Based Opinion Diffusion Structure (IODS)

In the following, we present the notions of *Influence-Based Opinion Diffusion Structure* and *Influence-Based Opinion Sequence* which are inspired by the notions of *Belief Revision Games* and *Belief Sequences* introduced in [16].

Definition 2 (Preliminary definition and notation).

- A totally-ordered set is a set S equipped with a total order \preceq . It is usually denoted (S, \preceq) .
- If $S = \{a_1, \dots, a_n\}$ is a finite set and \preceq is a total order on S so that $\forall i = 1 \dots (n-1) \ a_i \preceq a_{i+1}$, then by convention, we will write $S = \{a_1 \prec \dots \prec a_n\}$ to compactly denote the totally ordered set (S, \preceq) . As usual, we will write $e \in S$ iff e is one of the a_i s.

⁴We can prove the equality if $\varphi_1 \wedge \mu$ is consistent, which is a less strong condition. But we need the stronger condition in the rest of the paper

Definition 3 (Influence-Based Opinion Diffusion Structure). An Influence-Based Opinion Diffusion Structure (IODS) is a quadruplet $S = (A, \mu, B, Inf)$ where:

- $A = \{1, \dots, n\}$ is a finite set of agents.
- μ is a consistent formula of L .
- B is a function which associates any agent i of A with a consistent formula of L denoted for short B_i such that $B_i \models \mu$.
- Inf is a function which associates any agent i of A with a totally ordered set $Inf(i) = \{i_1 \prec_i \dots \prec_i i_{n_i}\}$, $n_i \geq 1$.

A is the finite set of agents. The formula μ represents the information which is true in the world. It is called *integrity constraint*. For any agent i , the formula B_i represents its *initial opinion*. We assume that agents are rational and thus that B_i is consistent and satisfies the integrity constraint μ . For any agent i , agents i_1, \dots, i_{n_i} are the influencers of i . With the total order \prec_i , i ranks them according to their degree of influence: for any agents j and k in $Inf(i)$, $j \prec_i k$ means that, according to i , its own opinion is (strictly) more influenced by j 's opinion than by k 's opinion. If $Inf(i) = \{i_1 \prec_i \dots \prec_i i_{n_i}\}$, i_1 is called *the main influencer* of i .

Notice that i may belong to $Inf(i)$ i.e., we do not require that i 's opinion is only influenced by other agents opinion: i may take its own opinion into account in the process of opinion changing. Consequently, i has to rank itself in $Inf(i)$. For instance, $i \prec_i j$, for any $j \in Inf(i)$ and $j \neq i$ when i considers that, even if it is influenced by other influential agents, its own opinion will only be strengthened by their opinions. At the opposite, $j \prec_i i$, for any $j \in Inf(i)$ and $j \neq i$ when i is not confident in its own opinion and wants to take opinions of its influencers (but itself) first. As a particular case, $Inf(i) = \{i\}$ is allowed and represents the fact that agent i is not influenced by some other agent but itself.

Notice also that i may not belong to $Inf(i)$. This means that i is only influenced by other people and is not confident at all in its own opinion.

Definition 4 (Influence-Based Opinion Sequence). Let $S = (A, \mu, B, Inf)$ be an IODS and $i \in A$ with $Inf(i) = \{i_1 \prec_i \dots \prec_i i_{n_i}\}$. The Influence-Based Opinion Sequence of i , denoted $(B_i^s)_{s \in \mathbb{N}}$, is defined by:

- (i) $B_i^0 = B_i$
- (ii) $\forall s > 0, B_i^s = \Delta_\mu(B_{i_1}^{s-1} \prec \dots \prec B_{i_{n_i}}^{s-1})$

The Influence-Based Opinion Sequence (or Opinion Sequence for short) of agent i , $(B_i^s)_{s \in \mathbb{N}}$, represents the history of i 's opinion evolution. This evolution is done according to the Importance-Based Merging Operator Δ_μ : i 's opinion at step s is the result of Δ_μ applied to the ordered multi-set of opinions: $B_{i_1}^{s-1} \prec \dots \prec B_{i_{n_i}}^{s-1}$.

As already written, the definition of IODS is inspired by the definition of Belief Revision Games (BRG), given in [16] according to which a BRG G is a 5-uple

$G = (A, Ac, L, \mathcal{B}, \mathcal{R})^5$ where:

- A is a finite set of agents,
- Ac is a binary irreflexive relation on A . $\{j : (j, i) \in Ac\}$ is the set of *acquaintances* of i
- L is a finite propositional language,
- \mathcal{B} is a mapping from A to L .
- $\mathcal{R} = \{R_i : i \in A\}$. Each R_i (called *the revision policy of agent i*) is a mapping from $L \times L^{in(i)}$ to L with $in(i) = |\{j : (j, i) \in Ac\}|$, such that,
 1. if $\varphi_0^1 \equiv \varphi_0^2$ and $\langle \varphi_1^1, \dots, \varphi_{in(i)}^1 \rangle \succ \langle \varphi_1^2, \dots, \varphi_{in(i)}^2 \rangle$ ⁶ then

$$R_i(\varphi_0^1, \varphi_1^1, \dots, \varphi_{in(i)}^1) \equiv R_i(\varphi_0^2, \varphi_1^2, \dots, \varphi_{in(i)}^2)$$
 2. if $in(i) = 0$ then R_i is the identity function

At first glance, IODS could be seen as a particular case of BRG in which all the agents have the same revision policy: Δ_μ . But this is not true because Δ_μ is not a revision policy as defined previously. Indeed, the first constraint imposed on R_i in BRG definition implies that acquaintances cannot be ordered: i 's opinion will evolve the same, whatever a permutation of its acquaintance. This is obviously not satisfied by Δ_μ since, as proved by the fourth item of proposition 1, inputs of the Importance-Based Merging Operator Δ_μ cannot be permuted.

However, an interesting result proved in proposition 1 of [16] is still valid for IODS. This result shows that in a BRG, the opinion sequence of any agent is cyclic i.e., is characterized by an initial segment $B_i^0 \dots B_i^{b-1}$ and a cycle $B_i^b \dots B_i^e$ which will be repeated ad infinitum. The proof is based on the fact that since the language is finite, the number of possible opinions which are distinct up to logical equivalence is finite. Moreover, there is a finite number of agents and for each agent, the set of its acquaintances is fixed. So the number of possible n-uples $BRG^s = (B_1^s, \dots, B_n^s)$ (vectors of agent opinions at step s) is finite, up to equivalence. Consequently, the belief sequence of each agent s is cyclic. The proof does not use the fact that R_i is irrelevant to permutation of its arguments. So, we can re-use this proof for proving the following:

Proposition 5. *In an IODS, the opinion sequence of any agent is cyclic i.e., the opinion sequence of any agent i is characterized by an initial segment $B_i^0 \dots B_i^{b-1}$ and a cycle $B_i^b \dots B_i^e$ which will be repeated ad infinitum.*

If the cycle of the opinion sequence of i is $B_i^b \dots B_i^e$, then the size of the cycle is defined by: $|Cyc(B_i)| = e - b + 1$.

⁵We change the initial notation in order to easily compare IODS and BRG

⁶ $\langle \varphi_1^1, \dots, \varphi_{in(i)}^1 \rangle \succ \langle \varphi_1^2, \dots, \varphi_{in(i)}^2 \rangle$ iff there is a permutation f over $\{1 \dots in(i)\}$ so that $\forall j = 1 \dots in(i) \varphi_j^1 \equiv \varphi_{f(j)}^2$

Example 2. Let $S = (A, \mu, B, Inf)$ be an IODS with $A = \{1, 2, 3\}$ (I.e., there are three agents named 1, 2 and 3), μ is a tautology, $B_1 = a \vee b$, $B_2 = \neg a \wedge \neg b$, $B_3 = a \vee \neg a$, $Inf(1) = \{1\}$, $Inf(2) = \{2 \prec_2 1\}$, $Inf(3) = \{1 \prec_3 2\}$. I.e., agent 1 is its own and only influential agent. Agent 2 is influenced by 1 and by itself, but more influenced by itself than by 1. Agent 3 is influenced by 1 and by 2, 1 being the agent who influences it the most. 1 initially believes $a \vee b$, 2 initially believes $\neg a \wedge \neg b$, and 3 has initially no opinion. Table 1 shows the evolution of agent opinions when the distance used is d_H . Agent 1 is not influenced by anyone except itself, so its opinion remains $a \vee b$. Agent 2 is keen to change its opinion by integrating opinion of 1 if possible. But here, 2 has a complete opinion which is inconsistent with 1's opinion. Consequently, 2's opinion remains $\neg a \wedge \neg b$. Finally, 3 who had initially no opinion, is keen to form its own opinion by integrating 1's opinion and 2's opinion by giving preference to the first one. Here it gets: $(a \wedge \neg b) \vee (b \wedge \neg a)$ Here, $|Cyc(B_1)| = |Cyc(B_2)| = |Cyc(B_3)| = 1$.

	$s = 0$	$s \geq 1$
$i = 1$	$a \vee b$	$a \vee b$
$i = 2$	$\neg a \wedge \neg b$	$\neg a \wedge \neg b$
$i = 3$	$a \vee \neg a$	$(a \vee b) \wedge (\neg a \vee \neg b)$

Table 1: Opinion evolution in example 2

The following example shows a case where lengths of cycles are greater than 1.

Example 3. Let $S = (A, \mu, B, Inf)$ be an IODS with: $A = \{1, 2\}$, μ being a tautology, $Inf(1) = \{2\}$, $Inf(2) = \{1\}$, $B_1 = a$, $B_2 = \neg a$. S represents a network of two agents each one being influenced by the other one only. Agent 1 initially believes a and agent 2 initially believes $\neg a$. Assume that μ is a tautology. Table 2 shows the evolution of opinions (for distance d_H and d_D as well). By definition, agent 1 adopts 2's opinion and agent 2 adopts 1's opinion at the same time. Since their initial opinions contradict, the agents change opinion recursively. Here, $|Cyc(B_1)| = |Cyc(B_2)| = 2$

	$s = 0$	$s = 1$	$s=0 \bmod 2$	$s=1 \bmod 2$
$i = 1$	a	$\neg a$	a	$\neg a$
$i = 2$	$\neg a$	a	$\neg a$	a

Table 2: Opinion evolution in example 3

Finally, let us introduce some more definitions which will be useful for the next section. The following adapts the definition provided in [11] and defines some type of IODS in which only some particular loops are permitted in the relation of influence.

Definition 5 (DAG with self-loops). From $S = (A, \mu, B, Inf)$, we can build a graph whose nodes are agents of A and edges are $i \rightarrow j$ iff $i \in Inf(j)$. We say that S is a DAG with self-loops if this graph is a directed graph where the only permitted cycles are of type $i \rightarrow i$.

The following introduces the notion of *sphere of influence* of an agent.

Definition 6 (Sphere of Influence of an agent). Let $S = (A, \mu, B, Inf)$ and $i \in A$. The *sphere of influence* of i is defined by: $Sphere(i) = \bigcup_{k \geq 1} Sphere^k(i)$ with

$$Sphere^1(i) = \{j_1 : Inf(j_1) = \{i \prec \dots\}\}$$

$$Sphere^k(i) = \{j_k : Inf(j_k) = \{j_{k-1} \prec \dots\} \text{ and } j_{k-1} \in Sphere^{k-1}(i)\}$$

The *sphere of influence* of an agent is thus the set of agents which are directly or indirectly mostly influenced by i . Let's now introduce some different types of agents.

Definition 7 (Dogmatic agent). Let $S = (A, \mu, B, Inf)$ and $i \in A$. i is a *dogmatic agent* iff $Inf(i) = \{i\}$.

Definition 8 (Self-confident agent). Let $S = (A, \mu, B, Inf)$ and $i \in A$. i is *self-confident* iff i is its main influencer.

Definition 9 (Diffident agent). Let $S = (A, \mu, B, Inf)$ and $i \in A$. i is *diffident* iff $i \notin Inf(i)$ or $Inf(i) = \{i_1 \prec_i \dots \prec_i i\}$ with $i_1 \neq i$.

An agent is *dogmatic* when it is not influenced by other agents. As a consequence, a dogmatic agent i will never change its opinion i.e., $\forall s \geq 0 B_i^s = B_i^0$. An agent is *self-confident* when it is its main influencer. Notice that dogmatic agents are self-confident. Moreover, if i is self-confident then $i \in Sphere(i)$. An agent is *diffident* when it is influenced only by others or it has at least two influencers and it is its least influential influencer. Finally, notice that if S is a DAG with self-loops then there is at least one dogmatic agent.

Definition 10 (Leader of a group, leader). Let $S = (A, \mu, B, Inf)$ and $i \in A$.

- Let $I \subseteq A$. i is the *leader of I* iff i is dogmatic and $I \subseteq Sphere(i)$.
- i is a *leader* iff there is a subset I of A st i is the leader of I .

In other words, for being a *leader*, an agent must be the leader of a group. And, for being the *leader of a group of agents I* , agent i must not be influenced by no other agent and any agent of I must be directly or indirectly mostly influenced by i .

4. Properties of IODS

In this section, we consider some properties, many of them being introduced in [16], and we check whether Importance-Based Opinion Structures satisfy them or not.

Proposition 6. Let $S = (A, \mu, B, Inf)$ be an IODS. Then, $\forall i \in A \forall s \in \mathbb{N} B_i^s$ is consistent.

This property corresponds to the property of BRG called *Consistency Preservation (CP)* defined in [16] as: $\forall i \in A$, if B_i is consistent then $\forall s \in \mathbb{N} B_i^s$ is consistent. In the case of an IODS, the premiss is omitted because the initial opinions are consistent.

Proposition 7. Let $S = (A, \mu, B, Inf)$ be an IODS. Then, $\forall i \in A \forall s \in \mathbb{N} B_i^s \models \mu$.

This proposition shows that agents take integrity constraints into account to revise their opinion.

Proposition 8. Let $S = (A, \mu, B, Inf)$ be an IODS and φ be a consistent formula of L . Then, if $\forall i \in A, \varphi \models B_i$ then $\forall i \in A \forall s \in \mathbb{N}, \varphi \models B_i^s$.

This proves that IODS satisfy the property called *Agreement Preservation (AP)* defined in [16]: if all agents initially agree on some alternatives then they will not change their mind about them.

Proposition 9. Let $S = (A, \mu, B, Inf)$ be an IODS and let φ be a formula of L . Then, if $\forall i \in A, B_i \models \varphi$ then $\forall i \in A, \forall s \in \mathbb{N}, B_i^s \models \varphi$

This proposition proves that any Importance-Based Opinion Structures satisfy the property called *Unanimity Preservation (UP)* defined in [16]: every formula which is a logical consequence of the initial opinions remains so after opinion diffusion. I.e., any opinion initially shared by the agents remains so after opinion diffusion.

The property of *Responsiveness (Resp)* introduced in [16] states that an agent should take into account the opinions of the agents who influence it whenever (i) its opinions are inconsistent with the opinions of its acquaintances (but itself) and (ii) its acquaintances (but itself) agree on some alternatives. It is adapted here as follows:

Definition 11 (Resp). $S = (A, \mu, B, Inf)$ satisfies **(Resp)** iff $\forall i \in A \forall s \in \mathbb{N}$, if $\forall j \in Inf(i) \setminus \{i\} B_j^s \wedge B_i^s$ is inconsistent and if $\bigwedge_{j \in Inf(i) \setminus \{i\}} B_j^s$ is consistent, then $B_i^{s+1} \not\models B_i^s$.

(Resp) is not necessarily satisfied here. As a counterexample, consider an IODS in which $A = \{1, 2, 3\}$, $Inf(1) = \{1\}$, $Inf(2) = \{2\}$, $Inf(3) = \{3 \prec 1 \prec 2\}$, $B_1 = a \wedge b$, $B_2 = a$, $B_3 = \neg a$ and μ is a tautology. Notice that agents 1 and 2 influence agent 3 and that 1 and 2 agree on a . However, if the underlying distance is Hamming, we have $\forall s \geq 1, B_3^s = \neg a \wedge b$. Consequently, $B_3^s \models B_3$.

Let us now focus on the property of *Convergence (Conv)* introduced in [16] and in [11] as well, which states that there is a step in the evolution process when all opinions stop evolving.

Definition 12 (Conv). An IODS $S = (A, \mu, B, Inf)$ satisfies **(Conv)** iff $\forall i \in A |Cyc(B_i)| = 1$.

Some IODS satisfy **(Conv)** and some don't. In example 2, S converges: the stable opinions of 1, 2 and 3 are respectively $a \vee b$, $\neg a \wedge \neg b$ and $(a \vee b) \wedge (\neg a \vee \neg b)$. In example 3, S does not converge.

In the case when opinions are single logical interpretations (i.e., ballots), [11] proves that, if the aggregation operator satisfies the property of ballot-monotonicity,

then a DAG with self-loops converges. In the present work, assumptions are different since opinions are sets of logical interpretations (i.e., sets of ballots). Fortunately, proposition 4 proves that Δ_μ satisfies ballot-monotonicity and the proof given in [11] does not use the fact that opinions are supposed to be ballots. Thus we can re-use the proof of [11] and claim:

Proposition 10. *IODS which are DAG with self-loops satisfy (Conv).*

Let us finally consider the property of *Acceptability* given in [16]. We extend it for IODS as follows:

Definition 13 (Acceptability). *Let $S = (A, \mu, B, Inf)$ be an IODS and φ a formula of L . φ is accepted by agent i of A iff for all $B_i^s \in Cyc(B_i)$, we have $B_i^s \models \varphi$. φ is unanimously accepted in S iff φ is accepted by all i in A . φ is majoritary accepted if the number of agents who accept it is strictly greater than the number of agents who do not.*

In example 2, for instance, $a \vee b$ is majoritary accepted, as well as $\neg a \vee \neg b$.

The following proposition identifies cases in which an IODS satisfies Acceptability.

Proposition 11. *Let S be an IODS and i the leader of a set of agents I . Then B_i is accepted by any agent in I .*

This means that the opinion of the leader of a group is accepted by any agent in this group.

Proposition 12. *Let S be an IODS and i a dogmatic agent. Then:*

- (i) *If $Sphere(i) = A$ then B_i is unanimously accepted.*
- (ii) *If $|Sphere(i)| > \frac{|A|}{2}$ then B_i is majoritary accepted.*

Obviously, if a dogmatic agent is the main influencer of all agents in the population, then its opinion is unanimously accepted. If a dogmatic agent is the main influencer of more than a half population, then its opinion is majoritary accepted.

Proposition 13. *Let S be an IODS and i a self-confident agent. Then $\forall s \ B_i^s \models B_i^{s-1}$*

This shows that the opinion of a self-confident agent becomes stronger and stronger. This is illustrated below.

Example 4. *Assume μ is a tautology. Consider three agents 1, 2, 3 so that: $Inf(1) = \{1\}$, $Inf(2) = \{1 \prec 2\}$, $Inf(3) = \{3 \prec 2 \prec 1\}$ and $B_1 = a$, $B_2 = b$, $B_3 = a \vee \neg a$. Agent 3 is a self confident who initially has no opinion. At step 1, its opinion becomes b , which is stronger. Finally, after step 2, its opinion is $a \wedge b$, which is even stronger.*

Proposition 14. *Let S be an IODS. Let i be a dogmatic agent st $|Mod(B_i)| = 1$. Then $\forall j \in Sphere(i) \ \exists k \ \forall s > k \ |Mod(B_j^s)| = 1$*

We can consider that an opinion with only one model is a strong opinion since, the agent which has this opinion has a firm position (yes or no) towards each propositional letter. This proposition shows that any agent in the sphere of influence of a dogmatic agent whose opinion is strong converges to an opinion which is strong.

5. Taking topics into account

In this section, we extend IODS in order to take topics of opinions into account. Our motivation is to model the fact that an agent may influence another agent for opinions which belong to a given topic only and not for every opinion. As a consequence, topics will parametrize the ranking of influencers, i.e., for each agent, there will be as many rankings of influencers as topics.

5.1. Topics

Definition 14 (Topics). *Topics T_1, \dots, T_m of L are some sets of propositional literals of L so that: (i) any literal of L belongs to a topic; (ii) for any proposition letter p of L , for any topic T_i in $\{T_1, \dots, T_m\}$, we have: $p \in T_i \iff \neg p \in T_i$.*

Example 5. *Suppose that the letters of the language are a, b, c . We can define: $T_1 = \{a, \neg a\}$ and $T_2 = \{b, \neg b, c, \neg c\}$. In this case these two topics are disjoint. We can also define: $T_1 = \{a, \neg a, b, \neg b\}$, $T_2 = \{b, \neg b, c, \neg c\}$, $T_3 = \{b, \neg b\}$. There, $T_3 = T_1 \cap T_2$.*

As written, each agent will rank its influencers according to several rankings, each of them depending on a topic. So the question of compatibility between the topics and the rankings they parametrize must be addressed: in particular, given a set of influencers, is there a relation between their ranking parametrized by the topic T and their ranking parametrized by a topic $T' \subseteq T$? More generally, is there a relation between their ranking parametrized by the topic T and their ranking parametrized by a topic T' when $T \cap T' \neq \emptyset$? We answer this question by considering that if an agent i is strictly more influential than another agent j for opinions related to the topic T , then j cannot be strictly more influential than i for opinions related to topic T' when $T \cap T' \neq \emptyset$. Indeed, if it was not the case, then, for any opinion belonging to $T \cap T'$, i would be considered as strictly more influential than j because $T \cap T' \subseteq T$ and j would be considered as strictly more influential than i because $T \cap T' \subseteq T'$. Which is not possible. For instance, consider the two topics *skiing* and *ballets* (i.e choreographic disciplines). They are not disjoint because *acroski* is a kind of skiing with a choreography. Consider now two agents *Paul* and *Mary* providing information related to these topics. Assume that *Paul* influences you strictly more than *Mary* regarding *skiing* and that *Mary* influences you strictly more than *Paul* regarding *ballets*. On a first hand, you can conclude that *Paul* influences you strictly more than *Mary* regarding *acroski* since *acroski* is a kind of *skiing*. On the other hand, you can conclude that *Mary* influences you strictly more than *Paul* regarding *acroski* since *acroski* is a kind of *ballets*. But concluding both is impossible.

The following definition characterize rankings which are compatible with topics.

Definition 15 (Topic compatible orders). *Consider m topics $T_1 \dots T_m$. Let $\prec_{T_1} \dots \prec_{T_m}$ be m total orders on a given set of formulas. We say that $\prec_{T_1} \dots \prec_{T_m}$ are topic-compatible iff for any i and j , if $T_i \cap T_j \neq \emptyset$ then \prec_{T_i} and \prec_{T_j} are identical.*

Notice that we could have followed a different solution and adopted a default reasoning by stating that: if an agent i is strictly more influential than another agent j for opinions related to the topic T , and if $T \cap T' \neq \emptyset$ then, typically, j cannot be strictly

more influent than i for opinions related to topic T' . This rule could be modeled in Default Logic [15]. In the example, this would lead to two extensions: one in which Paul is considered as strictly more influent than Mary for opinions related to acroski; the other in which Mary is considered as strictly more influent than Paul for opinions related to acroski. This is not the solution we followed in the present paper.

5.2. Topic-dependent Importance-Based Merging Operators

We now extend the notion of Importance-Based Merging Operators for taking topics into account. Before, we introduce the following definition:

Definition 16. Let φ a formula and T_i a topic. $\prod_i \varphi$ is the formula defined by: $Mod(\prod_i \varphi) = \{w \cap T_i : w \in Mod(\varphi)\}$

Definition 17 (Topic-dependent Importance-Based Merging Operator). A Topic-dependent Importance-Based Merging Operator, is a function Θ which, given a formula μ , given a multi-set of consistent formulas of L and m topic-compatible orders on this multi-set denoted $\prec_{T_1} \dots \prec_{T_m}$, produces a formula denoted $\Theta_\mu(\prec_{T_1} \dots \prec_{T_m})$ so that:

$$Mod(\Theta_\mu(\prec_{T_1} \dots \prec_{T_m})) = \bigoplus_{i=1 \dots m} Mod(\Delta_{\prod_i \mu}(\prec_{T_i}))$$

with

- Δ an importance-based merging operator as defined in 1.
- $\bigoplus_{i=1 \dots m} M_i = \{w : w \in Mod(\mu) \text{ and } \forall i \ w \cap T_i \in M_i\}$ if not empty;
 $\bigoplus_{i=1 \dots m} M_i = Mod(\mu)$ else.

First, Θ merges formulas topic by topic and computes $Mod(\Delta_{\prod_i \mu}(\prec_{T_i}))$. Then it aggregates the results with operator \bigoplus . This agregation operator aims at selecting models of μ whose projections on all topics are results of independent mergings (i.e., belong to $Mod(\Delta_{\prod_i \mu}(\prec_{T_i}))$). But it may happen that no model of μ satisfies this condition, even if each initial formula satisfies μ as shown in example 6. In such a case, the operator selects all the models of μ i.e., the merging is vacuous and the resulting formula is nothing else than μ . Notice that, when μ is consistent, $Mod(\Delta_\mu(\prec_{T_1} \dots \prec_{T_m}))$ is not empty.

Example 6. Consider a language with two letters a, b and two topics: $T_1 = \{a, \neg a\}$, $T_2 = \{b, \neg b\}$. Consider $\varphi_1 = a \wedge b$, $\varphi_2 = \neg a \wedge \neg b$, $\mu = a \rightarrow b$. Notice that $\varphi_1 \models \mu$ and $\varphi_2 \models \mu$. Consider two orders, $\{\varphi_1 \prec_{T_1} \varphi_2, \varphi_2 \prec_{T_2} \varphi_1\}$. If we consider the first topic only we get: $Mod(\Delta_{\prod_{T_1} \mu}(\prec_{T_1})) = \{\{a\}\}$. If we consider the second topic only we get: $Mod((\Delta_{\prod_{T_2} \mu}(\prec_{T_2})) = \{\{-b\}\}$. By definition 17 we get $\{\{a\}\} \oplus \{\{-b\}\} = Mod(\mu)$ because $\{a, \neg b\} \notin Mod(\mu)$. Finally, we get $Mod(\Theta_\mu(\prec_{T_1} \prec_{T_2})) = Mod(\mu)$ i.e., $\Theta_\mu(\prec_{T_1} \prec_{T_2}) = a \rightarrow b$.

Notice that definition 1 is a special case of definition 17 i.e., if there is only one topic and only one order on a multi-set of formulas, then $\Theta_\mu(\prec)$ as defined in definition 17 is identical to $\Delta_\mu(\prec)$ as defined in definition 1.

Again, we consider the different postulates that merging operators should satisfy according to [14]. Here, we focus on the only ones which will be necessary for proving properties about TIODS. We reformulate them within the context of several orders.

Proposition 15.

- $\Theta_\mu(\prec_{T_1} \dots \prec_{T_m}) \models \mu$
- If μ is consistent then $\Theta_\mu(\prec_{T_1} \dots \prec_{T_m})$ is consistent
- If $\bigwedge_{i=1}^n \varphi_i \wedge \mu$ is consistent then $\models \Theta_\mu(\prec_{T_1} \dots \prec_{T_m}) \leftrightarrow \bigwedge_{i=1}^n \varphi_i \wedge \mu$

Proposition 16. Consider n formulas $\varphi_1, \dots, \varphi_n$ and m orders on these formulas $\prec_{T_1} \dots \prec_{T_m}$. Let $k \in \{1, \dots, n\}$ and $\prec_{T_1}^k, \dots, \prec_{T_m}^k$ be the orders obtained from $\prec_{T_1} \dots \prec_{T_m}$ by replacing φ_k by $\Theta_\mu(\prec_{T_1} \dots \prec_{T_m})$. Then,

$$\text{Mod}(\Theta_\mu(\prec_{T_1} \dots \prec_{T_m})) = \text{Mod}(\Theta_\mu(\prec_{T_1}^k \prec \dots \prec_{T_m}^k))$$

This shows that the result of merging different formulas ordered by several topic-dependant orders does not change when one formula, whatever its importance in any order, is replaced by the merged formula. This is an extension of proposition 4.

5.3. *Topic-dependent Influence-Based Opinion Diffusion Structure (TIODS)*

We now extend the notion of Influence-Based Opinion Diffusion Structure to take topics into account.

Definition 18 (Topic-dependent Influence-Based Opinion Diffusion Structure). A *Topic-dependent Influence-Based Opinion Diffusion Structure (TIODS)* is a quadruplet $S = (A, \mu, B, \text{Inf})$ where:

- $A = \{1, \dots, n\}$ is a finite set of agents.
- μ is a consistent formula of L .
- B is a function which associates any agent i of A with a consistent formula of L denoted B_i such that $B_i \models \mu$.
- For any agent i of A , $\text{Inf}(i) = \{\prec_{T_1}, \dots, \prec_{T_m}\}$ where $\prec_{T_1}, \dots, \prec_{T_m}$ are m total topic-compatible orders on a single set of agents $\{i_{i_1} \dots i_{n_i}\}$.

A , μ , B_i are defined as before. $\text{Inf}(i)$ is here defined by m orders on the set of i 's influencers. Each order corresponds to a topic. Given a topic T_k and $\prec_{T_k} \in \text{Inf}(i)$, $i_1 \prec_{T_k} i_2$ means that i 's opinion is more influenced by i_1 's opinion than by i_2 's opinion regarding the topic T_k . Obviously, this definition extends definition 3.

Definition 19 (Topic-dependent Influence-Based Opinion Sequence). Consider a TIODS $S = (A, \mu, B, Inf)$. Let $i \in A$ with $Inf(i) = \{\prec_{T_1}, \dots, \prec_{T_m}\}$ m orders on the set $\{i_{i_1} \dots i_{i_m}\}$. The Topic-dependent Influence-Based Opinion Sequence of i , denoted $(B_i^s)_{s \in \mathbb{N}}$, is defined as follows:

- $B_i^0 = B_i$
- $\forall s \in \mathbb{N}, B_i^{s+1} = \Theta_{B_{i_1}^s \vee \dots \vee B_{i_m}^s}(\prec_{T_1} \dots \prec_{T_m})$ with $B_{i_1}^s \prec_{T_k} B_{i_2}^s$ iff $i_1 \prec_{T_k} i_2$ for any $k \in \{1 \dots m\}$.

As before, the Opinion Sequence of agent i , $(B_i^s)_{s \in \mathbb{N}}$, represents the history of i 's opinion evolution. Here, i 's opinion evolves according to the Topic-Dependant Importance-Based Merging Operator Θ . More precisely, i 's opinion at step s is the result of $\Theta_{B_{i_1}^s \vee \dots \vee B_{i_m}^s}$ applied to the multi-set of opinions: $B_{i_1}^s \dots B_{i_m}^s$ ordered topic by topic by $\prec_{T_1} \dots \prec_{T_m}$. The following proposition shows that definition 19 extends definition 4.

Proposition 17. When there is only one topic and only one order on a multi-set of formulas, then the Topic-dependent Influence-Based Opinion Diffusion Sequence as defined in definition 19 is identical to the Influence-Based Opinion Diffusion Sequence as defined in definition 4.

Proposition 18. In a TIODS, the opinion sequence of any agent is cyclic i.e., the opinion sequence of any agent i is characterized by an initial segment $B_i^0 \dots B_i^{b-1}$ and a cycle $B_i^b \dots B_i^e$ which will be repeated ad infinitum.

Example 7. Consider a language whose topics are $T_1 = \{a, \neg a\}$ and $T_2 = \{b, \neg b\}$ and a TIODS $S = (A, \mu, B, Inf)$ with: $A = \{1, 2\}$, μ is a tautology, $Inf(1) = \{1\}$, $Inf(2) = \{1 \prec_{T_1} 2, 2 \prec_{T_2} 1\}$, $B_1 = a$, $B_2 = \neg a \wedge b$. In other words, we consider two agents 1 and 2. 1 is its own and only influencer. Regarding topic T_1 , 2 is influenced by 1 and by itself, but more by 1 than by itself; regarding topic T_2 , 2 is influenced by 1 and by itself, but more by itself than by 1. Initially, 1's opinion is a and 2's opinion is $\neg a \wedge b$. Table 3 shows the evolution of opinions (for distance d_H and d_D as well).

	$s = 0$	$s \geq 1$
$i = 1$	a	a
$i = 2$	$\neg a \wedge b$	$a \wedge b$

Table 3: Opinion evolution in example 7

5.4. Some properties of TIODS

We present here some results about TIODS. First we prove that *Consistency preservation*, *Agreement preservation* and *Unanimity preservation* are still satisfied.

Proposition 19. Let $S = (A, \mu, B, Inf)$ be a TIODS. Then $\forall i \in A \forall s \in \mathbb{N} B_i^s \models \mu$.

Proposition 20. *Let $S = (A, \mu, B, Inf)$ be a TIODS. Then $\forall i \in A \forall s \in \mathbb{N} B_i^s$ is consistent.*

Proposition 21. *Let $S = (A, \mu, B, Inf)$ be a TIODS and φ a consistent formula of L . If $\forall i \in A, \varphi \models B_i$ then $\forall i \in A \forall s \in \mathbb{N}, \varphi \models B_i^s$.*

Proposition 22. *Let $S = (A, \mu, B, Inf)$ be a TIODS and φ a consistent formula of L . If $\forall i \in A, B_i \models \varphi$ then $\forall i \in A, \forall s \in \mathbb{N}, B_i^s \models \varphi$.*

Responsiveness is obviously not satisfied (see counter example given in section 4). Moreover, the definition of DAG with self-loops can obviously be extended to TIODS, as well as the definition of convergence. Then we can prove the following:

Proposition 23. *TIODS which are DAG with self-loops converge.*

6. Conclusion and perspectives

The present paper addressed opinion diffusion in the case when opinions are modeled by propositional formulas. In a first part, we have presented a formal model which assumes that each agent changes its opinion by merging the opinions of its influencers, from the most influential to the least one. In a second part, we have extended this model in order to take the topics of opinions into account. This work is original since such models had never been studied before and properties that have been proved are rather encouraging.

However, this work is based on several assumptions which can be discussed and changing them opens up new perspectives. Let's present some of them.

The first main assumption on IODS is that each agent has a fixed set of influencers and the way it orders them is fixed also. This means that all their life long, agents are influenced by the same people, in the same way. This is of course very restrictive and a model in which, for each agent, the ordered set $Inf(i)$ may change would be much more general than the one presented here. For sure, convergence will not be guaranteed. Notice that $Inf(i)$ could change because the preorder \prec changes or because the agents in $Inf(i)$ change (for instance, i choses its influencers among the ones whose opinions are close to its own).

The second assumption made here is that agents order their influencers according to a strict order (or according to several strict orders if many topics). Thus, two agents cannot be considered as equally influential for a given agent. Changing this assumption will lead us to consider preorders instead of orders. Defining a merging operator which takes into account preorders is an open question.

Moreover, in TIODS, an agent has a single set of influencers, which are ordered differently depending on the topics. Again, this is restrictive and it will be interesting to consider that the sets of influencers are themselves topic-dependent.

A generalization of this work will be to consider that there is no global integrity constraint as it is the case here, but only some local ones, shared by the agents who, in some way, belong the same community. Studying the acceptability property in such a context is challenging.

Another perspective we plan to work on in the near future is to inject in our model the Bounded-Confidence assumption. According to this assumption, agents are only influenced by people whose opinions are not absolutely different from their own. In the present model, there is no such constraint on influencers but we plan to take it into account and to examine what become the properties of IODS then.

We would also like to extend our model in order to take the communication durations into account. Indeed, the present models assume that agent opinions change regularly at the same time. This comes to assume that communication between agents have the same duration. But this is a too strong assumption because in real applications, some interactions may be very short (electronic communications, emails, telephone..) and some others may be longer (mails, physical meeting ...). Taking this into account will lead us to redefine the notion of Opinion Sequence and to check if the properties are still satisfied.

Finally, we would like to revisit our work in order to represent uncertain agents i.e., agents whose opinions are weighted propositional formulas, weights intending to represent the degrees of uncertainty of agents. Following [8, 3, 7] we think that the Belief Functions theory is a good candidate for doing so. But redefining IODS in such formal tool and studying their properties is an open question.

Acknowledgements: This work was supported by ONERA under grant number 25348.01F (ROSARIO project).

References

- [1] Sreerupa Chatterjee, Feyza Merve Hafizoglu, and Sandip Sen. Predicting migration and opinion adoption patterns in agent communities. In *International conference on Autonomous Agents and Multi-Agent Systems, AAMAS '13, Saint Paul, MN, USA, May 6-10, 2013*, pages 1287–1288, 2013.
- [2] H.F. Chau, C.Y. Wong, F.K. Xhow, and C.F Fung. Social judgment theory based model on opinion formation, polarization and evolution. *Physica A*, 415:133–140, 2014.
- [3] Jin-Hee Cho and Ananthram Swami. Dynamics of uncertain opinions in social networks. In *2014 IEEE Military Communications Conference*, 2014.
- [4] Laurence Cholvy. Diffusion of opinion and influence. In *Scalable Uncertainty Management - 10th International Conference, SUM 2016, Nice, France, September 21-23, 2016, Proceedings*, pages 112–125, 2016.
- [5] Zoé Christoff and Jens Ulrik Hansen. A logic for diffusion in social networks. *J. Applied Logic*, 13(1):48–77, 2015.
- [6] Chad Crawford, Logan Brooks, and Sandip Sen. Opposites repel: the effect of incorporating repulsion on opinion dynamics in the bounded confidence model. In *International conference on Autonomous Agents and Multi-Agent Systems, AAMAS '13, Saint Paul, MN, USA, May 6-10, 2013*, pages 1225–1226, 2013.

- [7] Michael Crosscombe and Jonathan Lawry. A model of multi-agent consensus for vague and uncertain beliefs. *Adaptive Behaviour*, 24(4):249–260, 2016.
- [8] Ranga Dabarera, Rafael C. Nunez, Kamal Premaratne, and Manohar N. Murthi. Dynamics of belief theoretic agent opinions under bounded confidence. In *17th International Conference on Information Fusion, FUSION 2014, Salamanca, Spain, July 7-10, 2014*, pages 1–8, 2014.
- [9] Colette Faucher. Propagation of the effects of certain types of military psychological operations in a networked population. In *Proceedings of the 2015 AISB Convention, April 20-22, Canterbury, England, 2015*.
- [10] Colette Faucher, Gregory Zacharewicz, Maâmar El-Amine Hamri, and Claudia S. Frydman. PSYOPS and CIMIC operations: from concepts to G-DEVS models. In *2012 Spring Simulation Multiconference, SpringSim '12, Orlando, FL, USA, March 26-29, 2012, Proceedings of the 2012 Symposium on Theory of Modeling and Simulation - DEVS Integrative M&S Symposium*, page 42, 2012.
- [11] Umberto Grandi, Emiliano Lorini, and Laurent Perrussel. Propositional opinion diffusion. In *Proceedings of the 2015 International Conference on Autonomous Agents and Multiagent Systems, AAMAS 2015, Istanbul, Turkey, May 4-8, 2015*, pages 989–997, 2015.
- [12] Feyza Merve Hafizoglu and Sandip Sen. Analysis of opinion spread through migration and adoption in agent communities. In *PRIMA 2012: Principles and Practice of Multi-Agent Systems - 15th International Conference, Kuching, Sarawak, Malaysia, September 3-7, 2012. Proceedings*, pages 153–167, 2012.
- [13] Wander Jager and Frédéric Amblard. A dynamical perspective on attitude change. In *Proceedings of NAACSOS (North American Association for Computational Social and Organizational Science) Conference, Pittsburgh, USA, June 22-25, 2004*.
- [14] Sébastien Konieczny and Ramón Pino Pérez. Merging information under constraints: A logical framework. *J. Log. Comput.*, 12(5):773–808, 2002.
- [15] Raymond Reiter. A logic for default reasoning. *Artif. Intell.*, 13(1-2):81–132, 1980.
- [16] Nicolas Schwind, Katsumi Inoue, Gauvain Bourgne, Sébastien Konieczny, and Pierre Marquis. Belief revision games. In *Proceedings of the Twenty-Ninth AAAI Conference on Artificial Intelligence, January 25-30, 2015, Austin, Texas, USA.*, pages 1590–1596, 2015.
- [17] Alan Tsang and Kate Larson. Opinion dynamics of skeptical agents. In *International conference on Autonomous Agents and Multi-Agent Systems, AAMAS '14, Paris, France, May 5-9, 2014*, pages 277–284, 2014.

- [18] Zhefeng Wang, Enhong Chen, Qi Liu, Yu Yang, Yong Ge, and Biao Chang. Maximizing the coverage of information propagation in social networks. In *Proceedings of the Twenty-Fourth International Joint Conference on Artificial Intelligence, IJCAI 2015, Buenos Aires, Argentina, July 25-31, 2015*, pages 2104–2110, 2015.
- [19] Fuhan Yan, Zhaofeng Li, and Yichuan Jiang. Noised diffusion dynamics with individual biased opinion. In *ECAI 2014 - 21st European Conference on Artificial Intelligence, 18-22 August 2014, Prague, Czech Republic - Including Prestigious Applications of Intelligent Systems (PAIS 2014)*, pages 1129–1130, 2014.

Appendix: Proofs

Proposition 1.

Proof 1.

- *The first item is obviously satisfied.*
- *The second item is satisfied. Indeed, if μ is consistent then $\text{Mod}(\mu) \neq \emptyset$, thus $\text{Mod}(\Delta_\mu(\varphi_1 \prec \dots \prec \varphi_n)) \neq \emptyset$, i.e, $\Delta_\mu(\varphi_1 \prec \dots \prec \varphi_n)$ is consistent.*
- *The third item is satisfied. Indeed, assume that $\bigwedge_{i=1..n} \varphi_i \wedge \mu$ is consistent.*
 - *Consider $w \models \Delta_\mu(\varphi_1 \prec \dots \prec \varphi_n)$. This implies that $[D(w, \varphi_1), \dots, D(w, \varphi_n)]$ is minimal in $\text{Mod}(\mu)$. But, since $\bigwedge_{i=1..n} \varphi_i \wedge \mu$ is consistent, there exists w_0 st $w_0 \models \bigwedge_{i=1..n} \varphi_i \wedge \mu$. Thus $[D(w_0, \varphi_1), \dots, D(w_0, \varphi_n)] = [0, \dots, 0]$, i.e, $[D(w_0, \varphi_1), \dots, D(w_0, \varphi_n)]$ is minimal. Consequently, $\forall i \ D(w, \varphi_i) = 0$ i.e, $\forall i \ w \models \varphi_i$. Thus $w \models \bigwedge_{i=1..n} \varphi_i \wedge \mu$.*
 - *Now consider $w \models \bigwedge_{i=1..n} \varphi_i \wedge \mu$. (such a model exists since $\bigwedge_{i=1..n} \varphi_i \wedge \mu$ is consistent). Thus by definition, $w \models \mu$ and $\forall i = 1..n \ D(w, \varphi_i) = 0$. Thus $[D(w, \varphi_1), \dots, D(w, \varphi_n)]$ is minimal in $\text{Mod}(\mu)$. Thus $w \models \Delta_\mu(\varphi_1 \prec \dots \prec \varphi_n)$.*
- *If $\models \mu \leftrightarrow \mu'$ and $\forall i = 1..n, \models \varphi_i \leftrightarrow \varphi'_i$ then $\text{Mod}(\mu) = \text{Mod}(\mu')$. Thus $\text{Mod}(\Delta_\mu(\varphi_1 \prec \dots \prec \varphi_n)) = \text{Mod}(\Delta_{\mu'}(\varphi'_1 \prec \dots \prec \varphi'_n))$ i.e., $\models \Delta_\mu(\varphi_1 \prec \dots \prec \varphi_n) \leftrightarrow \Delta_{\mu'}(\varphi'_1 \prec \dots \prec \varphi'_n)$*
- *Consider μ a tautology, $\varphi_1 = a \wedge b$ and $\varphi_2 = a \wedge \neg b$. We have $\varphi_1 \models \mu$, $\varphi_2 \models \mu$ and $\Delta_\mu(\varphi_1 \prec \varphi_2) = a \wedge b$. $\Delta_\mu(\varphi_1 \prec \varphi_2) \wedge \varphi_1$ is consistent but $\Delta(\varphi_1 \prec \varphi_2) \wedge \varphi_2$ is inconsistent.*
- *Consider w and w' two models of μ . We can show that: If $[D(w, \varphi_1), \dots, D(w, \varphi_n)] \leq_{lex} [D(w', \varphi_1), \dots, D(w', \varphi_n)]$ and $[D(w, \varphi_{n+1}), \dots, D(w, \varphi_m)] \leq_{lex} [D(w', \varphi_{n+1}), \dots, D(w', \varphi_m)]$ then $[D(w, \varphi_1), \dots, D(w, \varphi_m)] \leq_{lex} [D(w', \varphi_1), \dots, D(w', \varphi_m)]$. Thus, if $w \models \Delta(\varphi_1 \prec \dots \prec \varphi_n)$ and $w \models \Delta(\varphi_{n+1} \prec \dots \prec \varphi_m)$ then $w \models \Delta(\varphi_1 \prec \dots \prec \varphi_m)$*

- Assume $\Delta(\varphi_1 \prec \dots \prec \varphi_n) \wedge \Delta(\varphi_{n+1} \prec \dots \prec \varphi_m)$ consistent. Thus $\exists w_0$ st $[D(w_0, \varphi_1), \dots, D(w_0, \varphi_n)]$ and $[D(w_0, \varphi_{n+1}), \dots, D(w_0, \varphi_m)]$ are minimal in $\text{Mod}(\mu)$. Thus $[D(w_0, \varphi_1), \dots, D(w_0, \varphi_m)]$ is minimal in $\text{Mod}(\mu)$.
Consider $w \models \Delta(\varphi_1 \prec \dots \prec \varphi_m)$. I.e., $[D(w, \varphi_1), \dots, D(w, \varphi_m)]$ is minimal in $\text{Mod}(\mu)$. Thus $\forall i, D(w, \varphi_i) = D(w_0, \varphi_i)$. Thus both $[D(w, \varphi_1), \dots, D(w, \varphi_n)]$ and $[D(w, \varphi_{n+1}), \dots, D(w, \varphi_m)]$ are minimal in $\text{Mod}(\mu)$. Thus, $w \models \Delta(\varphi_1 \prec \dots \prec \varphi_n)$ and $w \models \Delta(\varphi_{n+1} \prec \dots \prec \varphi_m)$, i.e., $\Delta(\varphi_1 \prec \dots \prec \varphi_m) \models \Delta(\varphi_1 \prec \dots \prec \varphi_n) \wedge \Delta(\varphi_{n+1} \prec \dots \prec \varphi_m)$
- If $w \models \Delta_\mu(\varphi_1 \prec \dots \prec \varphi_n) \wedge \mu'$ then $w \models \mu'$ and $w \models \Delta_\mu(\varphi_1 \prec \dots \prec \varphi_n)$. Thus $w \in \text{Mod}(\mu \wedge \mu')$ and $\forall w' \in \text{Mod}(\mu) [D(w, \varphi_1), \dots, D(w, \varphi_n)] \leq_{\text{lex}} [D(w', \varphi_1), \dots, D(w', \varphi_n)]$.
Let $w'' \in \text{Mod}(\mu \wedge \mu')$. Thus $w'' \in \text{Mod}(\mu)$ thus $[D(w, \varphi_1), \dots, D(w, \varphi_n)] \leq_{\text{lex}} [D(w'', \varphi_1), \dots, D(w'', \varphi_n)]$. Thus $w \models \Delta_{\mu \wedge \mu'}(\varphi_1 \prec \dots \prec \varphi_n)$.
- Assume $\Delta_{\mu_1}(\varphi_1 \prec \dots \prec \varphi_n) \wedge \mu_2$ is consistent. Then $\exists w_0 \in \text{Mod}(\mu_1 \wedge \mu_2)$ st $[D(w_0, \varphi_1), \dots, D(w_0, \varphi_n)]$ is minimal in $\text{Mod}(\mu_1)$. I.e., $\forall w' \in \text{Mod}(\mu_1) [D(w_0, \varphi_1), \dots, D(w_0, \varphi_n)] \leq_{\text{lex}} [D(w', \varphi_1), \dots, D(w', \varphi_n)]$.
Consider $w \models \Delta_{\mu_1 \wedge \mu_2}(\varphi_1 \prec \dots \prec \varphi_n)$. Then $w \in \text{Mod}(\mu_1)$ and $w \in \text{Mod}(\mu_2)$ and $[D(w, \varphi_1), \dots, D(w, \varphi_n)]$ is minimal in $\text{Mod}(\mu_1 \wedge \mu_2)$. I.e., $\forall w' \in \text{Mod}(\mu_1 \wedge \mu_2) [D(w, \varphi_1), \dots, D(w, \varphi_n)] \leq_{\text{lex}} [D(w', \varphi_1), \dots, D(w', \varphi_n)]$. In particular $[D(w, \varphi_1), \dots, D(w, \varphi_n)] \leq_{\text{lex}} [D(w_0, \varphi_1), \dots, D(w_0, \varphi_n)]$. Thus $\forall w' \in \text{Mod}(\mu_1), [D(w, \varphi_1), \dots, D(w, \varphi_n)] \leq_{\text{lex}} [D(w', \varphi_1), \dots, D(w', \varphi_n)]$. Consequently, $[D(w, \varphi_1), \dots, D(w, \varphi_n)]$ is minimal in $\text{Mod}(\mu_1)$. I.e., $w \models \Delta_{\mu_1}(\varphi_1 \prec \dots \prec \varphi_n)$.

Proposition 2. If $\varphi_1 \models \mu$ then

$$\text{Mod}(\Delta_\mu(\varphi_1 \prec \dots \prec \varphi_n)) = \text{Mod}(\Delta_{\varphi_1}(\varphi_1 \prec \dots \prec \varphi_n))$$

Proof 2.

- Assume $\text{Mod}(\Delta_\mu(\varphi_1 \prec \dots \prec \varphi_n)) \not\subseteq \text{Mod}(\Delta_{\varphi_1}(\varphi_1 \prec \dots \prec \varphi_n))$. Then $\exists w_0 \in \text{Mod}(\Delta_\mu(\varphi_1 \prec \dots \prec \varphi_n))$ st $w_0 \notin \text{Mod}(\Delta_{\varphi_1}(\varphi_1 \prec \dots \prec \varphi_n))$.
On the first hand, $w_0 \in \text{Mod}(\Delta_\mu(\varphi_1 \prec \dots \prec \varphi_n))$ implies $\forall w \in \text{Mod}(\mu), [D(w_0, \varphi_1), \dots, D(w_0, \varphi_n)] \leq [D(w, \varphi_1), \dots, D(w, \varphi_n)]$
On the other hand, $w_0 \notin \text{Mod}(\Delta_{\varphi_1}(\varphi_1 \prec \dots \prec \varphi_n))$ implies $\exists w_1 \in \text{Mod}(\varphi_1)$ st $[D(w_1, \varphi_1), \dots, D(w_1, \varphi_n)] < [D(w_0, \varphi_1), \dots, D(w_0, \varphi_n)]$.
Since, $\varphi_1 \models \mu$, we have $\text{Mod}(\varphi_1) \subseteq \text{Mod}(\mu)$. Thus the two previous assertions are contradictory.
- Assume $\text{Mod}(\Delta_{\varphi_1}(\varphi_1 \prec \dots \prec \varphi_n)) \not\subseteq \text{Mod}(\Delta_\mu(\varphi_1 \prec \dots \prec \varphi_n))$. Then $\exists w_1 \in \text{Mod}(\Delta_{\varphi_1}(\varphi_1 \prec \dots \prec \varphi_n))$ st $w_1 \notin \text{Mod}(\Delta_\mu(\varphi_1 \prec \dots \prec \varphi_n))$. I.e., $\exists w_2 \in \text{Mod}(\mu)$ st $[D(w_2, \varphi_1), \dots, D(w_2, \varphi_n)] < [D(w_1, \varphi_1), \dots, D(w_1, \varphi_n)]$.
But, $w_1 \in \text{Mod}(\Delta_{\varphi_1}(\varphi_1 \prec \dots \prec \varphi_n))$ implies $D(w_1, \varphi_1) = 0$ (because of the first point of proposition 1). Thus $[D(w_2, \varphi_1), \dots] < [D(w_1, \varphi_1), \dots]$ is impossible.

Proposition 3. If $\varphi_1 \models \mu$ then $\Delta_\mu(\varphi_1 \prec \dots \prec \varphi_n) \models \varphi_1$.

Proof 3. This is a consequence of proposition 1 and proposition 2.

Proposition 4. Consider n formulas $\varphi_1, \dots, \varphi_n$ and \prec an order on these formulas. Let $k \in \{1..n\}$ and let \prec^k be the order obtained from \prec by replacing φ_k by $\Delta_\mu(\prec)$. Then, $\text{Mod}(\Delta_\mu(\prec)) = \text{Mod}(\Delta_\mu(\prec^k))$.

Proof 4.

Let us assume that \prec is defined by $\varphi_1 \prec \dots \prec \varphi_n$. Let $k \in \{1..n\}$.

(\subseteq) Let $w \models \Delta_\mu(\prec)$. Thus (1) $w \in \text{Mod}(\mu)$, (2) $[D(w, \varphi_1), \dots, D(w, \varphi_n)]$ is minimal in $\text{Mod}(\mu)$ and (3) $D(w, \Delta_\mu(\prec)) = 0$.

(2) implies, $\forall w' \in \text{Mod}(\mu) [D(w, \varphi_1), \dots, D(w, \varphi_n)] \leq_{lex} [D(w', \varphi_1), \dots, D(w', \varphi_n)]$.
I.e., $\forall j = 1..n D(w', \varphi_j) = D(w, \varphi_j)$ or $\exists k' \in \{1, \dots, n\}$ st $\forall j < k' D(w, \varphi_j) = D(w', \varphi_j)$ and $D(w, \varphi_{k'}) < D(w', \varphi_{k'})$

- Suppose $\forall j = 1..n D(w', \varphi_j) = D(w, \varphi_j)$. Then $w' \in \Delta_\mu(\prec)$. Thus $[D(w', \varphi_1), \dots, D(w', \Delta_\mu(\prec)), \dots, D(w', \varphi_n)] = [D(w', \varphi_1), \dots, 0, \dots, D(w', \varphi_n)] = [D(w, \varphi_1), \dots, D(w, \Delta_\mu(\prec)), \dots, D(w, \varphi_n)]$. Finally, $[D(w, \varphi_1), \dots, D(w, \Delta_\mu(\prec)), \dots, D(w, \varphi_n)] \leq_{lex} [D(w', \varphi_1), \dots, D(w', \Delta_\mu(\prec)), \dots, D(w', \varphi_n)]$.
- Suppose $\exists k' \in \{1, \dots, n\}$ st $\forall j < k' D(w, \varphi_j) = D(w', \varphi_j)$ and $D(w, \varphi_{k'}) < D(w', \varphi_{k'})$

- If $k' \geq k$ then $[D(w, \varphi_1), \dots, D(w, \varphi_{k'})] = [D(w', \varphi_1), \dots, D(w', \varphi_{k'})]$. Thus $[D(w, \varphi_1), \dots, 0] \leq_{lex} [D(w', \varphi_1), \dots, D(w', \Delta_\mu(\prec))]$. Finally, $[D(w, \varphi_1), \dots, D(w, \Delta_\mu(\prec)), \dots, D(w, \varphi_n)] \leq_{lex} [D(w', \varphi_1), \dots, D(w', \Delta_\mu(\prec)), \dots, D(w', \varphi_n)]$.
- If $k' < k$, since $[D(w, \varphi_1), \dots, D(w, \varphi_{k'})] < [D(w', \varphi_1), \dots, D(w', \varphi_{k'})]$, we have $[D(w, \varphi_1), \dots, D(w, \varphi_{k'}), \dots, D(w, \Delta_\mu(\prec)), \dots, D(w, \varphi_n)] < [D(w', \varphi_1), \dots, D(w', \varphi_{k'}), \dots, D(w', \Delta_\mu(\prec)), \dots, D(w', \varphi_n)]$.

Thus finally, in any case we have:

$$[D(w, \varphi_1), \dots, D(w, \Delta_\mu(\prec)), \dots, D(w, \varphi_n)] \leq_{lex} [D(w', \varphi_1), \dots, D(w', \Delta_\mu(\prec)), \dots, D(w', \varphi_n)].$$

I.e., $w \models \Delta_\mu(\prec^k)$

(\supseteq) Let $w \models \Delta_\mu(\prec^k)$. Then (4) $[D(w, \varphi_1), \dots, D(w, \Delta_\mu(\prec)), \dots, D(w, \varphi_n)]$ is minimal in $\text{Mod}(\mu)$. Suppose that $w \not\models \Delta_\mu(\prec)$ and consider $w' \models \Delta_\mu(\prec)$. Then

(5) $[D(w', \varphi_1), \dots, D(w', \varphi_k), \dots, D(w', \varphi_n)]$ is minimal; (6) $D(w', \Delta_\mu(\prec)) = 0$ and (7) $D(w, \Delta_\mu(\prec)) \neq 0$.

(4) and (5) implies $[D(w', \varphi_1), \dots, D(w', \varphi_k), \dots, D(w', \varphi_n)] < [D(w, \varphi_1), \dots, D(w, \varphi_k), \dots, D(w, \varphi_n)]$

Thus, (6) $\exists k' = 1..n \forall j < k' D(w', \varphi_j) = D(w, \varphi_j)$ and $D(w', \varphi_{k'}) < D(w, \varphi_{k'})$.

- Suppose $k' \leq k$
 (6) implies $[D(w', \varphi_1), \dots, D(w', \varphi_{k'})] < [D(w, \varphi_1) \dots D(w, \varphi_{k'})]$. Thus
 $[D(w', \varphi_1), \dots, D(w', \varphi_{k'}), \dots, D(w', \Delta_\mu(\prec), \dots, D(w', \varphi_n))] <$
 $[D(w, \varphi_1), \dots, D(w, \varphi_{k'}), \dots, D(w, \Delta_\mu(\prec), \dots, D(w, \varphi_n)]$. This contradicts (4).
- Suppose $k < k'$. Then (6) implies $[D(w', \varphi_1), \dots, D(w', \varphi_{k-1})] = [D(w, \varphi_1), \dots, D(w, \varphi_{k-1})]$.
 Besides, (5) and (7) implies $D(w', \Delta_\mu(\prec)) < D(w, \Delta_\mu(\prec))$.
 Finally, $[D(w', \varphi_1), \dots, D(w', \Delta_\mu(\prec)), \dots, D(w', \varphi_n)] <_{lex} [D(w, \varphi_1), \dots, D(w, \Delta_\mu(\prec))$
 $, \dots, D(w, \varphi_n)]$. Thus, $[D(w, \varphi_1), \dots, D(w, \Delta_\mu(\prec)), \dots, D(w, \varphi_n)]$ is not minimal in $Mod(\mu)$. This contradicts (4).

Proposition 5. In an IODS, the opinion sequence of any agent is cyclic i.e., the opinion sequence of any agent i is characterized by an initial segment $B_i^0 \dots B_i^{b-1}$ and a cycle $B_i^0 \dots B_i^e$ which will be repeated up as infinitum.

Proof 5. This result has been proved in proposition 1 of [16]. The proof is based on the fact that since the language is finite, the number of possible opinions which are distinct up to logical equivalence is finite. Moreover, there is a finite number of agents and for each agent, the set of its acquaintances is fixed. So the number of possible n -uples $BRG^s = (B_1^s, \dots, B_n^s)$ (vectors of agent opinions at step s) is finite, up to logical equivalence. Consequently, the belief sequence of each agent s is cyclic. The proof does not use the fact that revision policies have to be irrelevant to permutation of their arguments. So we can transpose this proof for IODS.

Proposition 6. Let $S = (A, \mu, B, Inf)$ be an IODS. Then, $\forall i \in A \forall s \in \mathbb{N} B_i^s$ is consistent.

Proof 6. By definition, the initial opinion of agent i is consistent and μ is consistent. Moreover, Importance-based Merging Operators satisfy proposition 1 (second item). So i 's opinion remains consistent at any step of the sequence.

Proposition 7. Let $S = (A, \mu, B, Inf)$ be an IODS. Then, $\forall i \in A \forall s \in \mathbb{N} B_i^s \models \mu$.

Proof 7. By definition, the initial opinion of agent i satisfies μ . Moreover Importance-based Merging Operators satisfy 1 (first item). So i ' opinion satisfies μ at any step of the sequence.

Proposition 8. Let $S = (A, \mu, B, Inf)$ be an IODS and a consistent formula φ of L . Then, if $\forall i \in A, \varphi \models B_i$ then $\forall i \in A \forall s \in \mathbb{N}, \varphi \models B_i^s$.

Proof 8. This is proved by induction First, $\forall i \in A \varphi \models B_i^0$. Then if $\forall j \in Inf(i) \varphi \models B_j^{s-1}$ then $\varphi \models \bigwedge_{j \in Inf(i)} B_j^{s-1} \wedge \mu$. Consequently, $\bigwedge_{j \in Inf(i)} B_j^{s-1} \wedge \mu$ is consistent. Because Importance-based Merging Operators satisfy proposition 1 (third item), we conclude that $B_i^s = \bigwedge_{j \in Inf(i)} B_j^{s-1} \wedge \mu$ thus $\varphi \models B_i^s$.

Proposition 9. Let $S = (A, \mu, B, Inf)$ be an IODS and let φ be a formula of L . Then, if $\forall i \in A, B_i \models \varphi$ then $\forall i \in A, \forall s \in \mathbb{N}, B_i^s \models \varphi$

Proof 9. We prove it by induction. Assume $\forall i \in A, B_i \models \varphi$.

- For $s = 0$. If $\forall i \in A, B_i \models \varphi$, then obviously $\forall i \in A, B_i^0 \models \varphi$
- Suppose the property true for any $t \leq s-1$ and let's prove it for s . By hypothesis we have $\forall i \in A B_i^{s-1} \models \varphi$. Consider an agent i so that $\text{Inf}(i) = \{i_1 \prec \dots i_n\}$. We have in particular, $B_{i_1}^{s-1} \models \varphi$. By proposition 3, we have $B_i^s \models B_{i_1}^{s-1}$. Thus $B_i^s \models \varphi$

Proposition 10. IODS which are DAG with self-loops satisfy **(Conv)**.

Proof 10. In the case when opinions are single logical interpretations (i.e., ballots), [11] proves that, if the aggregation operator satisfies the property of “ballot-monotonicity” then a DAG with self-loops converges. Here, assumptions are different since opinions are sets of logical interpretations (i.e., sets of ballots). But fortunately, proposition 4 proves that Δ_μ satisfies ballot-monotonicity. Moreover, the proof given in [11] does not use the fact that opinions are supposed to be ballots. Thus we can re-use the proof of [11] in our case.

Proposition 11. Let S be an IODS and i the leader of a set of agents I . Then B_i is accepted by any agent in I .

Proof 11. We first show that for i being dogmatic, $\forall j \in \text{Sphere}(i), \exists s_j \forall k \geq s_j B_j^k \models B_i$. I.e. B_i is accepted by any agent in $\text{Sphere}(i)$.

First, notice that since i is dogmatic, $\forall k \geq 0 B_i^k = B_i$.

Now, consider $j \in \text{Sphere}(i)$.

- Suppose that $\text{Inf}(j) = \{i \prec \dots\}$. Then $\forall k \geq 1, B_j^k = \Delta_\mu(B_i \prec \dots)$. Thus, because $B_i \models \mu$, we have $B_j^k \models B_i$, i.e. B_i is accepted by j .
- Suppose that there exists $j_1 \dots j_{k-1}$ so that $\text{Inf}(j_1) = \{i \prec \dots\}$, $\text{Inf}(j_2) = \{j_1 \prec \dots\}$, $\text{Inf}(j_{k-1}) = \{j_{k-2} \prec \dots\}$, $\text{Inf}(j) = \{j_{k-1} \prec \dots\}$. Thus $\forall r \geq 1 B_{j_1}^r = B_i, \forall r \geq 2 B_{j_2}^r = B_{j_1}, \dots, \forall r \geq k B_j^r = B_i$. I.e. B_i is accepted by j .

As a consequence, if i is the leader of a group I , then $I \subseteq \text{Sphere}(i)$ thus B_i is accepted by any agent in I .

Proposition 12. Let S be an IODS and i a dogmatic agent. Then:

- If $\text{Sphere}(i) = A$ then B_i is unanimously accepted.
- If $|\text{Sphere}(i)| > \frac{|A|}{2}$ then B_i is majoritary accepted.

Proof 12. This is a corollary of proposition 11

Proposition 13. Let S be an IODS and i a self-confident agent. Then

$$\forall s B_i^s \models B_i^{s-1}$$

Proof 13. i being self-confident, we can write $\text{Inf}(i) = \{i \prec i_1 \prec \dots i_{n_i}\}$ (with $n_i \geq 0$). Because $B_i \models \mu$, we can apply proposition 2 and conclude that $\forall s \Delta_\mu(B_i^{s-1} \prec \dots B_{i_{n_i}}^{s-1}) \models B_i^{s-1}$. I.e., $B_i^s \models B_i^{s-1}$. Thus $\text{Mod}(B_i^s) \subseteq \text{Mod}(B_i^{s-1})$, i.e., $B_i^s \models B_i^{s-1}$.

Proposition 14. *Let S be an IODS. Let i be a dogmatic agent st $| \text{Mod}(B_i) | = 1$. Then $\forall j \in \text{Sphere}(i) \exists k \forall s > k \ | \text{Mod}(B_j^s) | = 1$*

Proof 14. *This is a consequence of propositions 11 and 13.*

Proposition 15.

- $\Theta_\mu(\prec_{T_1} \dots \prec_{T_m}) \models \mu$
- If μ is consistent then $\Theta_\mu(\prec_{T_1} \dots \prec_{T_m})$ is consistent
- If $\bigwedge_{i=1}^n \varphi_i \wedge \mu$ is consistent then $\models \Theta_\mu(\prec_{T_1} \dots \prec_{T_m}) \leftrightarrow \bigwedge_{i=1}^n \varphi_i \wedge \mu$

Proof 15.

- The first item is proved by definition of \oplus .
- The second item is satisfied because $\text{Mod}(\mu) \neq \emptyset$ implies $\text{Mod}(\Theta_\mu(\prec_{T_1} \dots \prec_{T_m})) \neq \emptyset$.
- The third item is satisfied. Indeed, if $\bigwedge_i \varphi_i \wedge \mu$ is consistent, then $\prod_k (\bigwedge_i \varphi_i \wedge \mu)$ is consistent i.e., $\prod_k \bigwedge_i \varphi_i \wedge \prod_k \mu$ is consistent. Since Δ satisfies proposition 1 (third item), we get: $\text{Mod}(\Delta_{\prod_k \mu}(\prec_k)) = \text{Mod}(\prod_k \bigwedge_i \varphi_i \wedge \prod_k \mu)$. Thus $\bigoplus_{k=1 \dots m} \text{Mod}(\Delta_{\prod_k \mu}(\prec_k)) = \bigoplus_{k=1 \dots m} \text{Mod}(\prod_k \bigwedge_i \varphi_i \wedge \prod_k \mu) = \text{Mod}(\bigwedge_i \varphi_i \wedge \mu)$.

Proposition 16. *Consider n formulas $\varphi_1, \dots, \varphi_n$ and m orders on these formulas $\prec_{T_1} \dots \prec_{T_m}$. Let $k \in \{1, \dots, n\}$ and $\prec_{T_1}^k, \dots, \prec_{T_m}^k$ be the orders obtained from $\prec_{T_1} \dots \prec_{T_m}$ by replacing φ_k by $\Theta_\mu(\prec_{T_1} \dots \prec_{T_m})$. Then,*

$$\text{Mod}(\Theta_\mu(\prec_{T_1} \dots \prec_{T_m})) = \text{Mod}(\Theta_\mu(\prec_{T_1}^k \dots \prec_{T_m}^k))$$

Proof 16. *Let $k \in \{1 \dots n\}$. We can write $\{w : w \in \text{Mod}(\mu) \text{ and } \forall i \in \{1 \dots m\} \prod_i w \in \text{Mod}(\Delta_{\prod_i \mu}(\prec_{T_i}))\} = \text{Mod}(\mu) \cap \{w : \prod_1 w \in \text{Mod}(\Delta_{\prod_1 \mu}(\prec_{T_1}))\} \cap \dots \cap \{w : \prod_m w \in \text{Mod}(\Delta_{\prod_m \mu}(\prec_{T_m}))\}$. Thus, by proposition 4, $\{w : w \in \text{Mod}(\mu) \text{ and } \forall i \in \{1 \dots m\} \prod_i w \in \text{Mod}(\Delta_{\prod_i \mu}(\prec_{T_i}))\} = \text{Mod}(\mu) \cap \{w : \prod_1 w \in \text{Mod}(\Delta_{\prod_1 \mu}(\prec_{T_1}^k))\} \cap \dots \cap \{w : \prod_m w \in \text{Mod}(\Delta_{\prod_m \mu}(\prec_{T_m}^k))\}$. This allows us to conclude $\{w : w \in \text{Mod}(\mu) \text{ and } \forall i \in \{1 \dots m\} \prod_i w \in \text{Mod}(\Delta_{\prod_i \mu}(\prec_{T_i}))\} = \{w : w \in \text{Mod}(\mu) \text{ and } \forall i \in \{1 \dots m\} \prod_i w \in \text{Mod}(\Delta_{\prod_i \mu}(\prec_{T_i}^k))\}$*

- Suppose $\{w : w \in \text{Mod}(\mu) \text{ and } \forall i \in \{1 \dots m\} \prod_i w \in \text{Mod}(\Delta_{\prod_i \mu}(\prec_{T_i}))\} = \emptyset$. Then, by the previous remark, we also have $\{w : w \in \text{Mod}(\mu) \text{ and } \forall i \in \{1 \dots m\} \prod_i w \in \text{Mod}(\Delta_{\prod_i \mu}(\prec_{T_i}^k))\} = \emptyset$. Applying definition 18 twice, we conclude $\text{Mod}(\Theta_\mu(\prec_{T_1} \dots \prec_{T_m})) = \text{Mod}(\mu)$ and $\text{Mod}(\Theta_\mu(\prec_{T_1}^k \dots \prec_{T_m}^k)) = \text{Mod}(\mu)$. Finally, $\text{Mod}(\Theta_\mu(\prec_{T_1} \dots \prec_{T_m})) = \text{Mod}(\Theta_\mu(\prec_{T_1}^k \dots \prec_{T_m}^k))$.
- Assume now that $\{w : w \in \text{Mod}(\mu) \text{ and } \forall i \in \{1 \dots m\} \prod_i w \in \text{Mod}(\Delta_{\prod_i \mu}(\prec_{T_i}))\} \neq \emptyset$. We prove the result by induction on the number of topics.

- Assume that there is only one topic. We have $Mod(\Theta_\mu(\prec)) = Mod(\Delta_\mu(\prec))$ and $Mod(\Theta_\mu(\prec^k)) = Mod(\Delta_\mu(\prec^k))$. Moreover proposition 4 implies that $Mod(\Delta_\mu(\prec)) = Mod(\Delta_\mu(\prec^k))$. Thus finally, $Mod(\Theta_\mu(\prec)) = Mod(\Theta_\mu(\prec^k))$
- Assume that the property is true for $t - 1$ topics. Consider t topics.
By definition 18, we can write $Mod(\Theta_\mu(\prec_{T_1} \dots \prec_t)) = Mod(\Theta_\mu(\prec_{T_1} \dots \prec_{T_{t-1}})) \cap Mod(\Theta_\mu(\prec_{T_t}))$. We can then apply the induction hypothesis on both subsets and write: $Mod(\Theta_\mu(\prec_{T_1} \dots \prec_{T_t})) = Mod(\Theta_\mu(\prec_{T_1}^k \dots \prec_{T_{t-1}}^k)) \cap Mod(\Theta_\mu(\prec_{T_t}^k))$. Finally $Mod(\Theta_\mu(\prec_{T_1} \dots \prec_{T_t})) = Mod(\Theta_\mu(\prec_{T_1}^k \dots \prec_{T_t}^k))$

Proposition 17. *When there is only one topic and only one order on a multi-set of formulas, then the Topic-dependant Influence-Based Opinion Diffusion Sequence, as defined in definition 19, is identical to the Influence-Based Opinion Diffusion Sequence as defined in definition 4.*

Proof 17. We prove $\varphi_1 \models \mu \implies Mod(\Theta_{\varphi_1 \vee \dots \vee \varphi_n}(\varphi_1 \prec \dots \prec \varphi_n)) = Mod(\Delta_\mu(\varphi_1 \prec \dots \prec \varphi_n))$

Definition 18 implies $Mod(\Theta_{\varphi_1 \vee \dots \vee \varphi_n}(\varphi_1 \prec \dots \prec \varphi_n)) = Mod(\Delta_{\varphi_1 \vee \dots \vee \varphi_n}(\varphi_1 \prec \dots \prec \varphi_n))$. Proposition 2, allows us to conclude $Mod(\Delta_\mu(\varphi_1 \prec \dots \prec \varphi_n)) = Mod(\Delta_{\varphi_1}(\varphi_1 \prec \dots \prec \varphi_n))$ (since $\varphi_1 \models \mu$). Proposition 2 also allows us to conclude $Mod(\Delta_{\varphi_1 \vee \dots \vee \varphi_n}(\varphi_1 \prec \dots \prec \varphi_n)) = Mod(\Delta_{\varphi_1}(\varphi_1 \prec \dots \prec \varphi_n))$ (since $\varphi_1 \models \varphi_1 \vee \dots \vee \varphi_n$). From these three equalities, we conclude $Mod(\Theta_{\varphi_1 \vee \dots \vee \varphi_n}(\varphi_1 \prec \dots \prec \varphi_n)) = Mod(\Delta_\mu(\varphi_1 \prec \dots \prec \varphi_n))$.

Proposition 18. *In a TIODS, the opinion sequence of any agent is cyclic i.e., the opinion sequence of any agent i is characterized by an initial segment $B_i^0 \dots B_i^{b-1}$ and a cycle $B_i^b \dots B_i^e$ which will be repeated up as infinitum.*

Proof 18. *Here again, the propositional language is finite, the number of agents is finite, the integrity constraint and the graph of influences topic-by-topic do not change over the diffusion process.*

Proposition 19. *Let $S = (A, \mu, B, Inf)$ a TIODS. Then $\forall i \in A \forall s \in \mathbb{N} B_i^s \models \mu$.*

Proof 19. *We prove this by induction on s .*

- For $s = 0$ we have $\forall i \in A B_i^0 \models \mu$ by definition of TIODS.
- Suppose that the property is true till step $s - 1$.

Consider $B_i^s = \Theta_{B_{i_1}^{s-1} \vee \dots \vee B_{i_n}^{s-1}}(\prec_{T_1} \dots \prec_{T_m})$. Because Θ satisfies proposition 15 (second item), we have $B_i^s \models B_{i_1}^{s-1} \vee \dots \vee B_{i_n}^{s-1}$. Thus, by induction hypothesis, we have $B_i^s \models \mu$.

Proposition 20. *Let $S = (A, \mu, B, Inf)$ a TIODS. Then $\forall i \in A \forall s \in \mathbb{N} B_i^s$ is consistent.*

Proof 20. We prove this by induction on s .

- For $s = 0$ we have $\forall i \in A$ B_i^0 is consistent by definition of TIODS.
- Suppose that the property is true till step $s - 1$.

Consider $B_i^s = \Theta_{B_{i_1}^{s-1} \vee \dots \vee B_{i_{n_i}}^{s-1}}(\prec_{T_1} \dots \prec_{T_m})$. The induction hypothesis allows us to conclude that $B_{i_1}^{s-1} \vee \dots \vee B_{i_{n_i}}^{s-1}$ is consistent. Because Θ satisfies proposition 15 (second item), we conclude B_i^s is consistent.

Proposition 21. Let $S = (A, \mu, B, Inf)$ a TIODS and φ a consistent formula of L . If $\forall i \in A, \varphi \models B_i$ then $\forall i \in A \forall s \in \mathbb{N}, \varphi \models B_i^s$.

Proof 21. We prove this by induction on s . Suppose that φ is a consistent formula such that $\forall i \in A, \varphi \models B_i$

- For $s = 0$, obviously $\varphi \models B_i^0$
- Assume now that the property holds till $s - 1$. Let $i \in A$ and $i_1 \dots i_{n_i}$ its influencers. We have $\forall j = 1 \dots n_i, \varphi \models B_{i_j}^{s-1}$. This implies that $\bigwedge_{j=1 \dots n_i} B_{i_j}^{s-1} \wedge (\bigvee_{j=1 \dots n_i} B_{i_j}^{s-1})$ is consistent. Because Topic-dependent Importance Merging Operators satisfy proposition 15 (third item), we can conclude $\models B_i^s \leftrightarrow \bigwedge_{j=1 \dots n_i} B_{i_j}^{s-1} \wedge (\bigvee_{j=1 \dots n_i} B_{i_j}^{s-1})$ i.e., $\models B_i^s \leftrightarrow \bigwedge_{j=1 \dots n_i} B_{i_j}^{s-1}$ and finally, $\varphi \models B_i^s$.

Proposition 22. Let $S = (A, \mu, B, Inf)$ a TIODS and φ a consistent formula of L . If $\forall i \in A, B_i \models \varphi$ then $\forall i \in A, \forall s \in \mathbb{N}, B_i^s \models \varphi$.

Proof 22. We prove this by induction.

- For $s = 0$ the property holds.
- Assume that the property holds till $s - 1$. Let $i \in A$ and $i_1 \dots i_{n_i}$ its influencers. Because Θ satisfy proposition 15 (first item), we have: $B_i^s \models B_{i_1} \vee \dots \vee B_{i_{n_i}}$. By induction hypothesis, $\forall j, B_{i_j}^{s-1} \models \varphi$. Then we conclude $B_i^s \models \varphi$.

Proposition 23. TIODS which are DAG with self-loops converge.

Proof 23. The proof is similar to proof 10. This result is a consequence of proposition 16.