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► **To cite this version:**

| Ismael Lemhadri. Market impact in a latent order book. 2018. hal-01711192v1

HAL Id: hal-01711192

<https://hal.science/hal-01711192v1>

Preprint submitted on 16 Feb 2018 (v1), last revised 21 Feb 2018 (v2)

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Market impact in a latent order book

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Abstract

In this article we revisit the classical problem of market impact, reviewing the recent literature both empirical and theoretical. We propose a new agent-based model where market participants follow a mean-reversion process. Our model, inspired from [DBMB], draws from the so-called mean field approach in Statistical Mechanics and Physics ([Lasry and Lions]). It assumes that a large number of “agents” interact in the order book and by taking the “continuum limit” (when the number of agents goes to infinity), we obtain a set of nonlinear partial differential equations. And we explicitly solve them using Fourier analysis. One could talk as well of a “micro-macro” approach of equilibrium, where the market price is the consequence of each (“microscopic”) agent behaving with respect to his preferences and to global (“macroscopic”) information. This model accomodates very well the concept of market impact, leading to an integral equation satisfied by the impacted price. We prove the existence of a solution to this integral equation. And we establish that the impact of a metaorder decreases when either the volatility of the underlying asset or the agents’ reassessment intensity increases. In addition, we give various limiting cases, examples and possible extensions.

Keywords: agent-based models, latent order book, price formation, market impact, reaction-diffusion, optimal execution strategies, market microstructure, mean-field games.

Résumé

Nous nous intéressons ici au problème classique de l’impact de marché. Nous commençons par synthétiser la littérature récente à la fois théorique et empirique. Nous proposons ensuite un nouveau modèle multi-agents pour le carnet d’ordre où les agents obéissent à un processus de retour à la moyenne. Notre modèle, inspiré de [DBMB] et de la théorie des “jeux à champ moyen” popularisée par [Lasry and Lions], conduit à une équation aux dérivées partielles que nous résolvons explicitement par analyse de Fourier. Nous introduisons ensuite un métaordre et en déduisons une équation intégrale définissant l’impact de marché. Puis nous prouvons l’existence d’une solution à cette équation et nous explicitons son comportement dans différents régimes, retrouvant ainsi un impact concave. En particulier, nous montrons que l’impact décroît avec la volatilité de l’actif et la force de rappel. Enfin, nous explorons plusieurs cas limites au travers d’exemples numériques et d’approximations asymptotiques.

Mots-clés: modèles multi-agents, carnet d’ordres latent, formation des prix, impact de marché, réaction-diffusion, stratégies optimales d’exécution, microstructure de marché, jeux à champ moyen.

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Disclaimer

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Organization of the paper

The first section reviews some basic facts about modern financial markets. It can be skipped by the experienced reader. First, it introduces the reader to the notion of order book and gives a short historical summary of its evolution. It also presents the problems of liquidity risk and market impact.

The next section reviews our main reference, [DBMB], summarizing its main ingredients and findings. The remaining sections present our contributions, both theoretical and empirical.

In section 3 we generalize the original model by allowing agents to cancel some of their orders and submit new orders.

Section 4 is devoted to the mathematical analysis of the impacted price equation, and the existence of a solution is established using Banach's fixed-point theorem.

In section 5, we discuss various approximations stated without proof in [DBMB]. We explicit the assumptions under which they are valid and summarize the resulting formulations of the impacted price and the cost of execution. In particular, we show how concave impact arises in different trading regimes.

In section 6, we propose our main improvement of the model, allowing the agents to exhibit a mean-reversion behavior towards the "efficient" price. We show how a metaorder impacts the price in this new framework. And we analyse mathematically the resulting equations.

We conclude by indicating various directions and potential extensions that can be (or need to be) investigated.

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1 Introduction

1.1 Basic facts

The limit order book forms the basis behind most modern financial networks. **Passive orders** correspond to outstanding orders placed in the book and awaiting executing. These orders can only be placed on a predetermined price grid whose size is called the **tick size**. Therefore a limit order corresponds to a *disclosed buy/sell intention* for a fixed volume and at a fixed price. One of the most common ways of ordering orders that fall at the same price level is the first-in-first-out” time priority queue. The **bid** corresponds to the highest available price for a buy limit order, and the **ask** to the lowest available price for a sell limit order.

A trader who wishes to buy or sell a certain volume can also place an **aggressive order** that consumes the best available limit orders. For instance, an aggressive buy order first consumes the order at the best ask price, then the ask price plus one tick, and so on until executing the order’s volume.

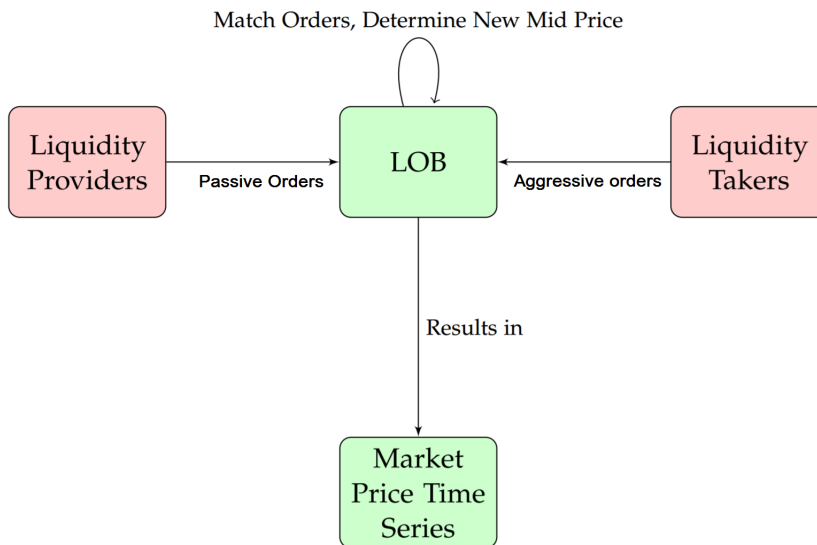


Figure 1.1: Illustration of the activity and agent types within a limit order book

Passive limit orders may be viewed as the fundamental building block of the order book. Their interaction with aggressive orders determines the bid and the ask and gives rise to the very notion of price. The main advantage of aggressive orders is that they are executed immediately. Their cost, however, depends on the structure of the orderbook. On the other hand, the cost of a passive order is determined in advance, but its execution time is uncertain. Choosing between passive and aggressive orders is generally a tradeoff between the price and the execution time, but a passive order also exposes its sender to *adverse selection* (as defined in section 1.5.3).

Typically, if the current best bid/ask prices in the book are b_t, a_t , and a new limit buy order at price x is submitted, then:

- if $x \geq a_t$, the transaction is executed immediately (entirely or partly depending on the available volume and the specific type of order sent) by matching its volume with the available ask liquidity.
- if $b_t < x < a_t$, the transaction reduces the bid-ask spread from $a_t - b_t$ to $a_t - x$ but is not executed.

- if $x \leq b_t$, the transaction is queued below existing orders in the bid book.

The evolution of the price when such a transaction is executed depends on the order matching rules of the LOB. If the total size available for sale at x was higher than the volume of the new transaction, then the new market price is usually set to x . Price changes also occur whenever the total volume available at a price x is cancelled.

To help traders assess the current state of the market, the exchanges typically provide electronic feeds that display the current state of the order book, i.e the different prices and the corresponding available volumes.

1.2 Structure of the order book

A usual order book allows the following types of actions:

- New limit sell order
- New limit buy order
- New market buy order
- New market sell order
- Cancellation of an existing order in the book

The figure shows four stages of an order book, connected by arrows (⇒). Each table has columns for Price and Size, and rows for Ask and Bid orders.

	Price	Size
Ask	9.12	1,525
	9.11	3,624
	9.10	4,123
	9.09	1,235
	9.08	3,287
Bid	9.07	4,895
	9.06	3,645
	9.05	2,004
	9.04	3,230
	9.03	7,246

⇒

	Price	Size
Ask	9.12	1,525
	9.11	3,624
	9.10	4,123
	9.09	1,235
	9.08	3,287
Bid	9.07	3,695
	9.06	3,645
	9.05	2,004
	9.04	3,230
	9.03	7,246

⇒

	Price	Size
Ask	9.12	1,525
	9.11	3,624
	9.10	4,123
	9.09	1,235
	9.08	3,687
Bid	9.07	3,695
	9.06	3,645
	9.05	2,004
	9.04	3,230
	9.03	7,246

⇒

	Price	Size
Ask	9.12	1,525
	9.11	3,624
	9.10	4,100
	9.09	1,235
	9.08	3,687
Bid	9.07	3,695
	9.06	3,645
	9.05	2,004
	9.04	3,230
	9.03	7,246

Figure 1.3: A market sell order with size of 1200, a limit ask order with size of 400 at 9.08, and a cancellation of 23 shares of limit ask order at 9.10, in sequence.

In practice, a wide variety of orders are possible. In 2007, a new European directive called MiFID (for Markets in Financial Instruments Directive) was enacted with the goal of increasing competition between exchanges, which at the time corresponded to the large European stock markets¹. This has led to the emergence of new exchanges that set different trading rules in order to attract more clients and bring more liquidity. Worldwide, exchanges decided to drop transaction fees for large volume traders, and created new types of limit orders: e.g iceberg orders, orders placed in a dark pool (where the entire liquidity is hidden until it is executed, i.e. market participants can send orders but cannot read the state of the book), orders indexed on the NBBO², etc.

¹A revision of this directive called Mifid II is set to take effect in 2018.

²National Best Bid and Offer

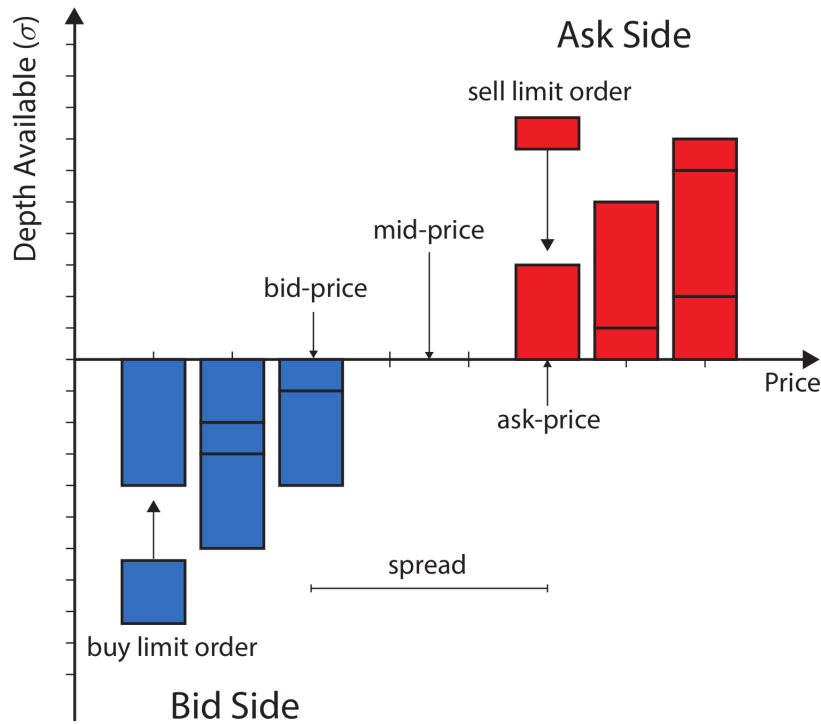


Figure 1.2: Illustration of the concept of order book. The vertical axis represents the available volume for any given price, and each order is represented by a rectangle with height proportional to the volume. In a price-time priority market, a new incoming order enters the queue below the existing orders at the same price, and priority is given to the orders with the highest bid/the lowest ask. The difference between the highest bid-price and the lowest ask-price is the spread. Source: [Gould et al.]

1.3 How to define the “price”?

It is clear that the dynamics of the orderbook at the micro-level are not accurately represented by the Black-Scholes price. Because the very notion of market price is ill-defined in the framework of orderbooks, several definitions can be adopted:

1. The price at which transactions happen. This definition, perhaps the most intuitive, comes with two drawbacks. The first is that it is only defined at the (discrete) times of transactions. The second is that transactions take place from both sides of the spread, leading to a relatively high mean-reversion at small time scales.
2. An average of transaction prices. Also called *Volume-Weighted Average Price* or *Time-Weighted Average Prices*, depending on whether the mean is taken volume-wise or time-wise. This definition of prices is often used as a benchmark for assessing a broker’s execution strategy.
3. The *mid-price*, defined as the middle of the spread. This “virtual” price allows to somewhat denoise the microstructure noise due to the bid-ask spread.
4. The *micro-price*, also computed as an average of the bid and the ask, but weighted by the inverse of the volumes. This definition is used as an indicator in the context of price prediction, as the imbalance of the orderbook should give more information than just the transaction price: if there are many more buyers in the bid side than sellers in the ask size, one can expect the price to rise in the near future.

1.4 Overview of the market ecosystem and participants

Historically, the task of supplying liquidity by permanently maintaining limit orders in the book was assumed by designated market makers who, in exchange for this service, kept a spread, namely, they offered to buy at a price lower than their sell price, leading to profit on each transaction. All other actors were liquidity takers, being forced to interact with a market maker. In reality, the idea of market makers profiting on each transaction due to the bid-ask spread is hindered by the challenge of *adverse selection*: if an informed trader has an accurate prediction about the future evolution of the price (while the market maker is less informed and has not updated his quotes accordingly), he can profit by entering a transaction with them.

To reduce his information mismatch, the market maker seeks to process any new piece of public information as soon as possible, and re-adjust the quotes they are offering accordingly. By reducing the latency between different trading platforms up to the speed of light, this type of arbitrage has allowed cross-market synchronization to happen at the scale of microseconds.

The market-making business was profoundly affected by the *decimalization* rule that came into effect in 2001 in the United States, which required stock exchanges to quote prices with a minimum tick size of 0.01\$ instead of the prevailing $\frac{1}{16}$ th of a dollar. As the bid-ask spread got narrower, this led to a drastic reduction in profit for traditional market-makers. Since then, the market-making business has greatly expanded. In particular, high-frequency traders have come in and, through the execution of massive volumes, they were capable of operating at such small incremental margins. The defining difference between a human trader and an HFT is that the latter 1) has very short portfolio holding periods, 2) operates at a very short time scale, where human traders cannot compete. The speed requirements of HFT problems require significant hardware and software investments. However, the recent technological developments have lowered the barriers to entry into HFT. Therefore, the market-making business has become widely diversified and fragmented, and the ever-increasing competition between liquidity providers led to even more abrupt reductions in the spread size: from ≈ 70 basepoints³ between 1900 and 1980, it has shrunk to a few basepoints today. We refer the reader to [Loveless1, Loveless2] for a concise summary of the historical development and technological challenges of high-frequency trading.

Due to the nature of their activity, market-makers are not typically affected by long-term trends in prices, as they operate on a much more “local” level. On the other hand, the second class of market participants, that of large investors, e.g. institutional investors, is typically in the opposing situation. This class roughly encompasses brokerage firms, asset management funds, traditional investment banks and hedge funds, all acting on behalf of their clients. For our purposes, we need not make a specific distinction between these actors. To understand the difficulty that such an investor faces when executing a large order, we need to introduce the notion of liquidity risk.

1.5 Liquidity risk and market impact

The concept of “liquidity risk” reflects the extra cost incurred during a buy (resp. sell) order that is due to the scarcity of supply (resp. demand). In the most extreme cases, it can even mean that it is impossible to trade an asset due to the absence of counterparty. One striking example is that during the subprimes crisis, products such as CDOs became practically unsaleable. Another example is that it can be impossible to convert certain currencies that are very weak in US dollars or euros.

It is the case, however, that the majority of the assets on a given market are liquid enough to allow to buy and sell small volumes at a price very close to the listed price. Therefore liquidity risk is often insignificant for a small trader. When a trader wishes to execute large volumes, however, the impact of their orders is noticeable and must be taken account as an additional cost.

³1 bp = 1 basepoint = 0.01% of the trading currency

In summary, when a trader seeks to execute a large transaction, the scarcity of instantaneous liquidity means that the order must be executed incrementally. Hence the importance of devising execution strategies that minimize the cost to the investor: given a time horizon and a volume, the investor/ the broker distributes the execution over time so as to obtain the best average execution price. To do so, he can decide to add limit orders to the book, or to send market orders. It is important to note that, generally speaking, the market reacts to *all types of orders* and not just to market orders.

The concept of price impact is fundamental when it comes to designing execution strategies for large orders. Since the available liquidity at a given time is not sufficient to absorb the entire order, the trader must split his order into several chunks to be executed incrementally. Every time a small order - also called **child order** - is executed, the price is mechanically pushed in its direction, making the average execution price higher than the decision price and leading to the notion of **execution shortfall**. If the market can “guess” that the trader intends to buy (resp. sell) large quantities, he can be outrun by informed traders who push the price up (resp. down) in order to benefit from his orders. Without such price pressure, all trading strategies would be infinitely scalable, as the cost of trading would remain unchanged despite the size of the trade increasing. This does not seem plausible and contradicts the fact that at any given time, there are only limited amounts of liquidity available in the real order book.

Understanding the determinants of impact is crucial for several reasons:

- From a theoretical point of view, modeling the impact means understanding how prices change and how much they reflect some “fundamental price”. It requires to develop a “micro-model” for the statistics of prices.
- From a practical point of view, price impact can represent a large fraction of transaction execution costs. Assessing the impact of a trading strategy is of utmost importance for quantitative asset managers, since too much trading (in volume and/or frequency) can significantly deteriorate the performance of a strategy, or even turn a winning strategy into a money-losing one.
- From the regulatory point of view, acknowledging the existence of impact means that fair value accounting with mark-to-market prices is over-optimistic. A second important point is that trading costs may prevent investors from executing certain trades, so that a better understanding of market impact should lead to new market microstructure regulations that lower certain trading costs to improve the allocation of investors [Foucault and Menkveld]. Finally, it is important because it explains the connections between market design and systemic risk.

1.5.1 The empirical square root law

Let $I(Q)$ be the average price variation after executing a volume Q : $\mathcal{I}(Q) = \langle p_T - p_0 | Q \rangle$, where p_0 (resp. p_T) is the price of the first (resp. last) child order. Then the typical market impact “law” reads:

$$\mathcal{I}(Q) = Y\sigma\sqrt{\frac{Q}{V}} \quad (1.1)$$

where Q is the total executed volume, V is the (average) daily traded volume, σ the daily volatility, and Y a homogeneization constant of order 1.

This near universal empirical law has been empirically verified for very diverse markets and instruments: stocks [Lehalle et al.], futures contracts, options [Tóth et al.], and does not seem to depend on geographical zone or time period (as can be checked by comparing [Barra, Almgren], that use pre-2004 data, to [Mastromatteo et al., Deremble], that use post-2007 data).

The square-root law implies that the impact of trading only depends on the traded volume, and not on the duration of execution and the execution path. Nevertheless, real data shows that impact

does depend on the execution path. This “law” should therefore be seen as a (good) first-order approximation and a way to benchmark market impact models. It indicates that markets are inherently fragile: the impact of vanishingly small volumes is disproportionately high. While this result may seem counterintuitive and is in sharp contrast with the classical economic literature (which asserts linear impact, [Kyle]), it is perfectly in accordance with the fact that the instantaneous supply of liquidity is limited in real markets.

1.5.2 Why concave impact?

This empirical law asserts that impact is non-additive but strictly concave: after having traded a volume $\frac{Q}{2}$, the next $\frac{Q}{2}$ will have less impact on the price.

One justification for this is to consider impact as a *price surprise* due to new information. If some traders have superior information about the “fair price” of the asset in the future, then the observation of an excess buy demand allows the market to guess this information. The fact that this surprise decreases marginally with the metaorder’s volume leads to a concave impact.

The model we develop provides an alternative vision where impact is a purely mechanical phenomenon, the universal mechanism by which prices respond to traded volumes. The concavity means that there must be additional volume available deeper in the book, which is however not observable and only appears when we push the price. This justifies the elaboration of a model where orders to buy/sell are not always visible but only reveal themselves as the transaction price moves closer to their limit price, which leads to the framework of **latent order books**.

1.5.3 Modelling strategies

Modeling or not market impact depends on the underlying problem settings and definitely depends on the time horizon of the trade. Within the Black-Scholes framework where there are no transaction costs and no bid-ask spread, liquidity risk is ignored, i.e we assume infinite liquidity at the market price. This framework is reasonable and has proven successful, since the pioneering work of Black-Scholes [BS] and Merton [M] to tackle problems in derivatives pricing and hedging. Despite the limitation they suffer, they may be enough for an investor who wishes to assess its impact over a long period (of a few weeks at least), and who would estimate it roughly as a quantity that only depends on the traded volume.

However, when the estimation of impact becomes critical at lower time horizons, e.g when the impact becomes the main factor that prevents a trading strategy from being scalable, the need arises for a more realistic model of stock prices, in which can be incorporated the stylized facts and universal statistical features observed on all financial assets.

Classical models: from Kyle to the propagators

The first and simplest market impact model is probably due to [Kyle]. It assumes that impact is *permanent* and linear both in time and in the traded volume. A single trade of volume v and sign ϵ leads to a price move $\Delta p = \lambda \epsilon v$ for some constant λ . This leads to a total price change between times 0 and T : $p_T = p_0 + \sum_{0 \leq t \leq T} \lambda \epsilon_t v_t$. If the price $(p_t)_t$ is to follow a random walk, then the signs of the trades $(\epsilon_t)_t$ should be uncorrelated. However, real data shows that orders signs are correlated on longer timescales. Furthermore, while the assumptions of linearity can be justified within the Kyle model, real data shows that the impact of trading follows a near square-root law.

This is closely related to a general family of models that fall under the category of **propagator models** [Almgren, Bouchaud0, JoR32, Ow]. Given a positive, non-increasing function $\mathcal{G} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, the idea is to write the impact of a series of trades with given volume and sign $\left((q_s, \epsilon_s) \right)_{s \geq 0}$ on the price at time t as $\sum_{s < t} \epsilon_s q_s \mathcal{G}(t - s)$. This can be stated equivalently in the continuous case as

$\mathcal{I}_t = \int_0^t ds \mathcal{G}(t-s)m_s$ where m_s represents the instantaneous trading rate. The requirement that \mathcal{G} be non-increasing represents the decay of impact with time. The fact that $\lim_{+\infty} \mathcal{G} = 0$ means that the impact vanishes at longer timescales [see 5.1 for a discussion on transient and permanent impact]. [?] provides a good summary of propagator models. The propagator family has the advantage of being easy to design and to lead to tractable analytical results. However, it does not provide any explanation for the origin of impact, which fundamentally lies in the price formation process. This is what the last class of models aims at improving.

This last category is more structural and can be described as **statistical models of supply and demand** [DBMB, Mastromatteo et al., Deremble, DB]. Such models start by describing the spatio-temporal evolution of the order book which leads *en passant* to an accurate description of market impact. This is the category to which our main paper belongs through the concept of the *latent order book*.

A new idea: the latent order book

When working at the mesoscopic time-scale, the study of the visible order book is no longer sufficient, as it does not reflect the true supply and demand in the market. In reality, the visible order book mostly displays the activity of high-frequency participants, whereas the intentions of low-frequency actors remain hidden up till times very close to their execution. The fundamental reason for this is the asymmetry between liquidity providers and liquidity takers. In [Glosten et Milgrom], Glosten and Milgrom first introduced the notion of **adverse selection**, the idea being that submitting limit orders exposes the trader to the possibility that another trader, who has superior private information about the future evolution of the price, engages in a transaction with her. Yet an order can only take place if a liquidity provider has already submitted a limit order book. Therefore, liquidity providers must factor in these adverse selection costs when submitting their limit orders, hence determining the bid-ask spread.

As a consequence of the adverse selection risks, most of the participants' intentions are not clearly displayed in the order book, leading to low visible liquidity and market order splitting. It is in this regard that was introduced the concept of **the latent order book** as a means to describe the true supply and demand of financial markets by collecting the expected/intended trading volumes and prices for all market participants.

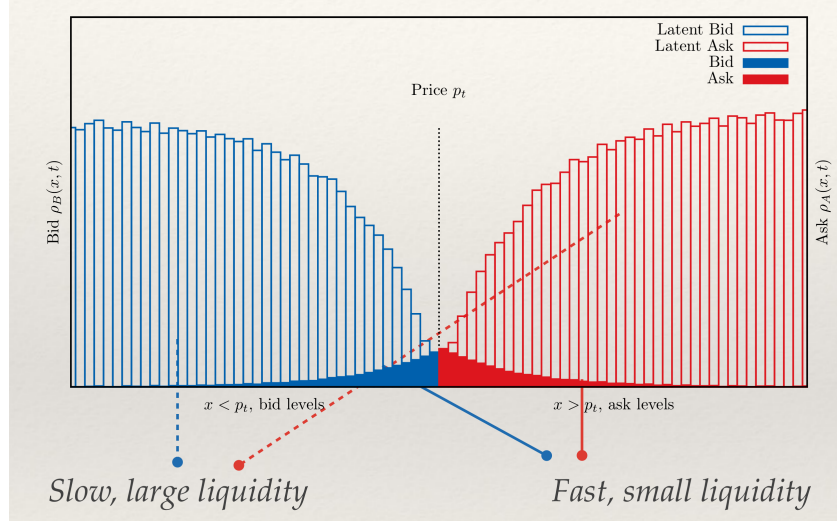


Figure 1.4: Latent *vs* real order book. Source: I. Mastromatteo

Notations.

LOB stands for *limit order book* and LLOB for *latent limit order book*.

δ denotes the Dirac distribution.

We use the French notation for open and closed intervals, e.g $[a, b[$ is closed to the left and open to the right.

The convolution of two square-integrable complex-valued functions is the function $f_1 \star f_2 : x \mapsto \int_{-\infty}^{+\infty} f_1(y)f_2(x - y)dy$.

Whenever working with finite difference schemes, ΔT and Δx generically stand for time and space discretization steps, respectively. Since our schemes are unconditionnally stable, we do not enforce any particular conditions on the discretization.

2 The latent order book model

2.1 Introduction and notations

The main motivation behind the concept of *latent order book* is that the *real*, displayed order books contain imperfect information, making a realistic and comprehensive analysis very difficult. The latent limit order book, first introduced in the paper [Deremble], attempts to circumvent these game-theoretic issues. Our model, [DBMB], takes place in this LLOB framework. It builds upon coupled continuous reaction-diffusion equations for the dynamics of the bid and the ask sides of the latent order book and is amenable to exact analytical treatment of the price trajectory for any execution profile. In the case of slow execution, we recover a linear propagator that relates past orders to the current price through a decay kernel.

At any given price x , at time t , we let ρ_B (resp. ρ_A) be the density of buy (resp. sell) orders. The current market price is defined as the point such that $\rho_B(p_t, t) = \rho_A(p_t, t)$.

Assuming that the order book is symmetric from the bid and ask regions, let us take the example of the bid density, whose evolution is modeled with four components:

- The addition of new buy orders with a term $\lambda \mathbf{1}_{\{x < p_t\}}$;
- the cancellation of already existing orders with a term $-\nu \rho_B$;
- the matching of already existing orders: whenever the market price changes, this leads certain existing orders to be activated as they cross the bid-ask boundary. This reaction term is given by $\kappa \rho_A \rho_B$, and
- the mechanical evolution due to random fluctuations in prices, which leads to a drift-diffusion at the diffusion rate D .

2.2 The drift-diffusion term

We postulate that a large number of infinitesimal agents with *heterogeneous beliefs*, interact in the market. The idea is that each agent has its own assessment of the “perfect price”, and this assessment evolves with time. This evolution is modeled as a drift-diffusion phenomenon: the drift is due to external information (e.g public news), whereas the diffusion refers to the endogeneous movements of the agents. Once more, we do not consider a single seller or buyer but continua of them through their densities and the price is then determined by a dynamic equilibrium.

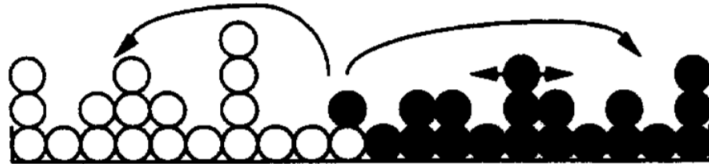


Figure 2.1: A white “particle” diffusing from the bid side, and a black particle diffusing from the ask side, annihilate when they bump into each other near the market price. The annihilated particles correspond to a transaction happening in the (real) order book. Source: [Bak]

Let $p_{i,t}$ denote the *reservation price* (i.e the “perfect price” at which he is willing to trade) of the i -th agent. Then the reservation price evolves between times t and $t + dt$ following the dynamics

$$p_{i,t+dt} = p_{i,t} + \beta_i \xi_t + \eta_{i,t}.$$

Here the drift ξ_t represents the new (“fundamental”) information, common to all market participants, and β_i the sensitivity of the i -th agent to the news. The second term $\eta_{i,t}$ (e.g a centered Gaussian variable) refers to the diffusion. It can be interpreted as a noise because the agents only have an approximate estimate $\hat{\xi}_{i,t}$ of the “fundamental” value ξ_t .

In the “continuum limit” where the number of agents goes to infinity, we obtain the dynamics of the density of the buy (and ask):

$$\rho(x, t + dt) = \int_{\beta} \mathbb{P}(d\beta) \int_{\eta} \mathbb{P}(d\eta) \int_{-\infty}^{+\infty} dy \rho(y, t) \delta(x - y - \beta\xi_t - \eta) = \int_{\beta} \int_{\eta} \rho(x - \beta\xi_t - \eta, t) \Pi(\beta) R(\eta) d\beta d\eta,$$

where Π and R are the probability density functions of β and η respectively.

Consequently, a second-order expansion leads to

$$\rho(x, t + dt) = \rho(x, t) - \frac{\partial \rho}{\partial x}(x, t) (\xi_t \mathbb{E}[\beta] + \mathbb{E}[\eta]) + \frac{1}{2} \frac{\partial^2 \rho}{\partial x^2}(x, t) (\xi_t^2 \mathbb{E}[\beta^2] + \mathbb{E}[\eta^2]).$$

It is reasonable to assume that the agents’ assessments sum up to an unbiased assessment, i.e $\mathbb{E}[\beta] = 1, \mathbb{E}[\eta] = 0$.

Letting $\xi_t = V_t dt$ and $\text{Var}(\eta) = 2Ddt = \sigma^2 dt$, we may neglect the term ξ_t^2 . We obtain the following partial differential equation:

$$\frac{\partial \rho}{\partial t}(x, t) = -V_t \frac{\partial \rho}{\partial x}(x, t) + D \frac{\partial^2 \rho}{\partial x^2}(x, t).$$

V_t is the instantaneous movement in the fundamental price, so that the fundamental price is $\hat{p}_t = \int_0^t V_s ds$. \hat{p}_t can be taken as, e.g, an additive Brownian motion.⁴

2.3 The order book in the absence of meta-orders

The previous ingredients sum up to

$$\begin{cases} \frac{\partial \rho_B}{\partial t}(x, t) = -V_t \frac{\partial \rho_A}{\partial x}(x, t) + D \frac{\partial^2 \rho_A}{\partial x^2}(x, t) + \lambda 1_{\{x \leq p_t\}} - \nu \rho_B(x, t) - \kappa \rho_A(x, t) \rho_B(x, t) \\ \frac{\partial \rho_A}{\partial t}(x, t) = -V_t \frac{\partial \rho_A}{\partial x}(x, t) + D \frac{\partial^2 \rho_A}{\partial x^2}(x, t) + \lambda 1_{\{x \geq p_t\}} - \nu \rho_A(x, t) - \kappa \rho_A(x, t) \rho_B(x, t) \end{cases}$$

If we now introduce $\varphi = \rho_B - \rho_A$, we obtain the simpler PDE

$$\frac{\partial \varphi}{\partial t}(x, t) = -V_t \frac{\partial \varphi}{\partial x}(x, t) + D \frac{\partial^2 \varphi}{\partial x^2}(x, t) + \lambda \text{sign}(p_t - x) - \nu \varphi(x, t),$$

where the market price p_t is defined by the equality $\varphi(p_t, t) = 0$.

At this stage, we can perform a change of reference frame by letting $y = x - \hat{p}_t$ and $g(y, t) = \varphi(y + \hat{p}_t, t)$. If we still denote g by φ (for notational simplicity) this leads to:

$$\frac{\partial \varphi}{\partial t}(y, t) = D \frac{\partial^2 \varphi}{\partial y^2}(y, t) + \lambda \text{sign}(p_t - \hat{p}_t - y) - \nu \varphi(y, t). \quad (2.1)$$

This equation, which is at the core of the present model, describes the structural evolution of the LLOB around the fundamental price \hat{p}_t . However, \hat{p}_t no longer appears in the above equation, meaning that the evolution of the order book can be treated independently from the dynamics of the price itself, as long as it is studied in the reference frame of the price. Even if the price is random, e.g. a Brownian motion, the shape of the order book is still deterministic. This is due to the fact that we allow microscopic orders. Still, a relationship exists between the two through the factor D , which is related to the variance of the price process.

⁴Note that in this case, the change of reference performed in section 2.3 leads to an Itô term which adds up to D .

In the absence of market impact, it should hold that $p_t = \hat{p}_t$. This allows to study the stationary density φ_{st} obtained by letting $t \rightarrow +\infty$, which satisfies

$$|D \frac{\partial^2 \varphi_{st}}{\partial y^2}(y, t) = \lambda \text{sign}(y) + \nu \varphi_{st}(y, t). \quad (2.2)$$

Since the LLOB is symmetric in the stationary case, φ_{st} is an odd function, and $\varphi_{st}(0) = 0$. Therefore, $\varphi_{st}(y \geq 0) = -\frac{\lambda}{\nu}(1 - e^{-\gamma y})$ and $\varphi_{st}(y \leq 0) = \frac{\lambda}{\nu}(1 - e^{\gamma y})$, where $\gamma = \sqrt{\frac{\nu}{D}}$.

2.4 Derivation of the impact equation

The cancellation rate ν defines the “memory” of the order book as $\tau_\nu = \nu^{-1}$: for times much larger than τ_ν , all orders in the LLOB have been cancelled and replaced by other ones so that its “memory” has been wiped. This time scale is of crucial importance because concave impact can hold only in the case when the duration of the metaorder satisfies $T \ll \tau_\nu$, so that the LLOB retains the “information” that a metaorder is being executed. We therefore restrict the dynamics to the limit $\nu \rightarrow 0$ such that times are small compared to τ_ν . Since the deposition rate must not exceed the cancellation rate too much in a balanced order book, this means we should also take the limit $\lambda \rightarrow 0$. Note that in the regime where $\nu \rightarrow 0$, φ_{st} becomes a simple linear function of y . This observation leads us to enforce a specific boundary condition on φ . If we simultaneously shrink the cancellation and deposition rates $\lambda, \nu \rightarrow 0$ in 2.1, the evolution becomes

$$\frac{\partial \varphi}{\partial t}(y, t) = D \frac{\partial^2 \varphi}{\partial y^2}(y, t), \quad (2.3)$$

which is just the **heat equation**.

This linear approximation can be justified by recalling the empirical square root law. If we assume that the current order book $\varphi = \rho_B - \rho_A$ is linear, say $\varphi(x) = -L(x - p_0)$, then buying a volume Q moves the price from 0 to the price p_+ such that $\int_{p_0}^{p_+} \varphi_{st}(y) dy = Q$, i.e $p_+ = \sqrt{2\frac{Q}{L}}$, hence a square-root impact. Note that the equilibrium price p_0 is well-defined here only if execution takes place instantaneously, or at least that the characteristic duration at which the equilibrium moves is much larger than the duration of the metaorder. In practice, however, meta-orders are split into several child orders and there is no equilibrium price at which the market pauses. The diffusive nature of prices means that between the beginning and the end of execution, the price shift $p_T - p_0$ is due to many other variations than the impact of our trade. Consequently, the above argument is only valid **locally** to estimate the impact of one child order, which concludes the justification of the linear approximation.

We can explicitly add the contribution of the metaorder as follows. We consider some fixed horizon $T > 0$. When the agent executes a metaorder (given by its execution rate, a continuous function $m : [0, T] \rightarrow \mathbb{R}$) then the evolution becomes

$$\frac{\partial \varphi}{\partial t}(y, t) = D \frac{\partial^2 \varphi}{\partial y^2}(y, t) + m_t \delta(y - y_t), \quad (2.4)$$

with the time boundary condition

$$\varphi|_{t=0}(y) = -Ly \text{ on } \mathbb{R}.$$

Note that $y - y_t = x - p_t$, therefore the Dirac term simply means that the child order at time t is executed at the market price $x = p_t$. The boundary condition is equivalent to assuming that far from the current price, the orderbook replenishes at a constant rate, so that $\frac{\partial \varphi}{\partial y}(y, t) \xrightarrow{y \rightarrow \pm\infty} -L$, where $L = |\frac{\partial \varphi_{st}}{\partial y}(y = 0)| = \frac{\lambda \gamma}{\nu} = \frac{\lambda}{\sqrt{\nu D}}$. We obtain the solution of the equation 2.4 as

$$\varphi(y, t) = -Ly + \int_0^t \frac{dsm_s}{\sqrt{4\pi D(t-s)}} e^{-\frac{(y-y_s)^2}{4D(t-s)}},$$

leading to the following integral equation for the impacted price:

$$y_t = \frac{1}{L} \int_0^t \frac{dsm_s}{\sqrt{4\pi D(t-s)}} e^{-\frac{(y_t-y_s)^2}{4D(t-s)}}. \quad (2.5)$$

3 The deposition-cancellation model

In order to obtain the main differential equation (2.4), the authors neglected the effect of deposition and cancellation of new orders. Our contribution here is to re-integrate them and deduce the dynamics of the LLOB and the impacted price.

The full PDE reads:

$$\frac{\partial \varphi}{\partial t}(y, t) = -\nu \varphi + D \frac{\partial^2 \varphi}{\partial y^2}(y, t) + \lambda \text{sign}(y) + m_t \delta(y - y_t). \quad (3.1)$$

Using the initial condition $\varphi(y, 0) = -Ly$, this equation can be solved explicitly by working in Fourier space. The explicit steps of the derivation are outlined in section 6.2.2. Letting $g(y, s) = e^{-\nu s} \left((\lambda \text{sign}(y) + m_s \delta(y - y_s)) \right)$ and $E_1 : (y, t) \mapsto e^{-\nu t} \frac{1}{\sqrt{4\pi D t}} e^{-\frac{y^2}{4Dt}}$, the solution takes the form

$$\varphi(y, t) = \left(\varphi(\cdot, 0) \star E_1(\cdot, t) \right)(y) + \int_0^t (g(\cdot, s) \star E_1(\cdot, t - s))(y) ds.$$

Hence

$$\varphi(y, t) = -Ly e^{-\nu t} + \int_0^t \frac{ds e^{-\nu(t-s)}}{\sqrt{4\pi D(t-s)}} \left(m_s e^{-\frac{(y-y_s)^2}{4D(t-s)}} + \lambda \left(\int_0^{+\infty} dx e^{-\frac{(y-x)^2}{4D(t-s)}} - \int_{-\infty}^0 dx e^{-\frac{(y-x)^2}{4D(t-s)}} \right) \right),$$

i.e

$$\varphi(y, t) = -Ly e^{-\nu t} + \int_0^t ds e^{-\nu(t-s)} \left(\frac{m_s e^{-\frac{(y-y_s)^2}{4D(t-s)}}}{\sqrt{4\pi D(t-s)}} + \lambda (1 - 2\phi_{y, 2D(t-s)}(0)) \right),$$

where $\phi_{x,V}$ is the cumulative density function of the Gaussian distribution $\mathcal{N}(x, V)$.

Consequently, the impacted price satisfies the integral equation

$$y_t = \frac{1}{L} \int_0^t ds \left(\frac{m_s e^{\nu s} e^{-\frac{(y_t - y_s)^2}{4D(t-s)}}}{\sqrt{4\pi D(t-s)}} + \lambda (1 - 2\phi_{y_t, 2D(t-s)}(0)) \right).$$

- The influence of the cancellation rate ν is clear: it simply yields the usual impact equation with the trading rate $(m_s e^{\nu s})_s$ instead of $(m_s)_s$.
- The influence of the deposition rate λ is more subtle. Let us make a first approximation and consider the first term of the right-hand side constant, to focus on the equation $y = C + \lambda \int_0^t ds (1 - 2\phi_{y, 2D(t-s)})$. We can check numerically that its solution decreases to 0 as λ increases, which is compatible with the intuition that higher deposition rates decrease the impact because they contribute to the replenishment of the order book (thus reducing its imbalance and increasing the instantaneous liquidity).

4 Existence and uniqueness of the impacted price

Despite the absence of an explicit solution to the integral equation 2.5, we have the

Theorem 4.1. *Let $m : [0, T] \rightarrow \mathbb{R}$ be a continuous execution strategy. Then the impacted price equation 2.5 admits a solution y defined over $[0, +\infty[$.*

Proof. We work in $E := \left(\mathcal{C}(I, \mathbb{R}), \|\cdot\|_\infty \right)$, the space of real-valued continuous functions on $I := [0, T]$.

Let $T > 0$ and $m = (m_t)_{0 \leq t \leq T}$ be a fixed function of E . For any $\epsilon \geq 0$, consider the function $F_\epsilon : E \times E \rightarrow E$ defined by

$$F_\epsilon(x, y)(t) = \int_0^{t-\epsilon} \frac{ds m_s}{\sqrt{t-s}} e^{-\frac{(x_t - y_s)^2}{t-s}}$$

for all $y \in E$. We have dropped the constants from equation 2.5 for readability but they do not affect the proof.

The first step is to show that the partial function $F_0(x, \cdot)$ has a fixed point $y(x)$ for any function $x \in E$. To do so we use Banach's fixed point theorem to prove that for all $\epsilon > 0$, $F_\epsilon(x, \cdot)$ has a unique fixed point $y_\epsilon(x)$. Then we prove that $y_\epsilon(x)$ converges to a fixed point of $F_0(x, \cdot)$ as $\epsilon \rightarrow 0$.

Given y and \tilde{y} in E and $s, t \in I$ with $s < t$, we have:

$$\left| e^{-\frac{(x_t - \tilde{y}_s)^2}{t-s}} - e^{-\frac{(x_t - y_s)^2}{t-s}} \right| = \frac{2}{t-s} \left| \int_{y_s}^{\tilde{y}_s} du (u - x_t) e^{-\frac{(u - x_t)^2}{t-s}} \right| \leq \frac{1}{e^{1/4}} \frac{1}{t-s} |y_s - \tilde{y}_s|.$$

The equality is just the fundamental theorem of calculus applied to the function $y \mapsto e^{-\frac{(x_t - y)^2}{2(t-s)}}$ between y_s and \tilde{y}_s , and for the inequality we have used the fact that $x e^{-x^2} \leq \frac{1}{2e^{1/4}}$ which is valid for all $x \geq 0$. This leads to

$$\begin{aligned} |F_\epsilon(x, y)(t) - F_\epsilon(x, \tilde{y})(t)| &\leq \frac{1}{e^{1/4}} \int_0^{t-\epsilon} ds \frac{m_s}{t-s} \int_{y_s}^{\tilde{y}_s} du \frac{(u - x_t)}{\sqrt{t-s}} e^{-\frac{(u - x_t)^2}{t-s}} \\ &\leq \frac{1}{e^{1/4}} \frac{1}{\epsilon} \|m\|_\infty \int_0^{t-\epsilon} ds |y_s - \tilde{y}_s|. \end{aligned}$$

Therefore, we have the inequality $|F_\epsilon^n(x, y)(t) - F_\epsilon^n(x, \tilde{y})(t)| \leq \frac{C}{\epsilon} \int_0^{t-\epsilon} ds |F_\epsilon^{n-1}(y)(t) - F_\epsilon^{n-1}(\tilde{y})(t)|$, where $C = \frac{1}{e^{1/4}} \|m\|_\infty$, which gives

$$\|F_\epsilon^n(x, y)(t) - F_\epsilon^n(x, \tilde{y})(t)\|_\infty \leq \frac{1}{n!} \left(\frac{C(t-\epsilon)}{\epsilon} \right)^n \|y - \tilde{y}\|_\infty.$$

In the right-hand side we find the general term of an exponential series, therefore it is < 1 for sufficiently large n (since it tends to 0 as $n \rightarrow \infty$). For such a choice of n , $F_\epsilon^n(x, \cdot)$ is therefore a contraction in the (complete) space E . This guarantees that $F_\epsilon(x, \cdot)$ has a unique fixed point $y_\epsilon(x) \in E$.

Observe that the family $(y_\epsilon(x))_{\epsilon \in]0, T[}$ is uniformly bounded for the $\|\cdot\|_\infty$ norm. Therefore, for any $t \in I$, we can find a sequence $(\epsilon_n)_n$ which tends to 0 and such that $y_{\epsilon_n}(x)(t)$ converges as $n \rightarrow \infty$. Denoting its limit $y_0(x)(t)$, we obtain a measurable, bounded function $y_0(x)$.

Since for all $t \in I$,

$$\begin{aligned} |y_\epsilon(x)(t) - F_0(x, y_\epsilon(x))(t)| &= |F_\epsilon(x, y_\epsilon(x))(t) - F_0(x, y_\epsilon(x))(t)| \\ &= \left| \int_{t-\epsilon}^t \frac{ds m_s}{\sqrt{t-s}} \left(e^{-\frac{(x - y_\epsilon(x)(s))^2}{t-s}} \right) \right| \\ &\leq 2 \|m_\infty\| \sqrt{\epsilon} \end{aligned}$$

we obtain by taking the limit $\epsilon \rightarrow 0$ that $y_0(x)$ is a fixed point of $F_0(x, \cdot)$, which concludes the first step. Note that in fact y_0 must be continuous since it is the integral of a measurable function (and in fact is even \mathcal{C}^∞).

The second idea is to fix a time step δ and start from any function $x_0 \in \mathcal{C}([0, \delta], \mathbb{R})$, then iterate this fixed point procedure in order to build a solution of equation 2.5. Starting from x_0 , build the corresponding fixed point $x_1 = y(x_0)$, and repeat this procedure $n = \lfloor \frac{T}{\delta} \rfloor$ times. Then concatenate these functions together, i.e consider the function y_δ defined on $[0, T]$ by $y_\delta(t) = x_k(t - k\delta)$ where $k = \lfloor \frac{t}{\delta} \rfloor$. By construction, the function y_δ satisfies

$$y_\delta(t) = \int_0^t \frac{ds m_s}{\sqrt{t-s}} e^{-\frac{(y_\delta(t-\delta) - y_s)^2}{t-s}}$$

We obtain the desired solution by letting $\delta \rightarrow 0$. □

Possible extensions: does the impact equation admit a **unique** solution? Intuitively we expect that the answer be positive. If so, it would be interesting to see if this solution depends continuously on the execution profile, and more generally to study its regularity depending on the regularity of the execution strategy.

5 Asymptotic expansions and approximations

The goal of this section is to derive simple expressions for the market impact equation and to justify the approximations performed in [DBMB]. While the general impact equation is likely intractable, we are able to simplify it considerably by making a few assumptions on the trading rates. These assumptions are general enough as they only distinguish between small and large trading rates, the reference for comparison being given by $J := LD$. The parameter J represents the average number of transactions (or rate of trading) in the stationary orderbook. Our findings allow to recover once more the square-root law, and yield a $\frac{3}{2}$ power-law of the volume for the total trading cost.

5.1 Transient or permanent impact?

Suppose that certain agents are able to successfully forecast short term price movements and use this information for their trading. For instance, if the agent correctly predicted (or was otherwise informed) that the price is about to rise, he is more likely to buy as an anticipation of this movement. This should result in measurable correlation between trades and price changes, even if the trades by themselves have absolutely *no effect* on the prices. Therefore, the information processed by investors, driving their decisions, leads to *permanent* price changes so that the market adjusts the asset to its new “fundamental” value. This vision of market impact is purely based on *information* does not assume any mechanical impact.

So far, the model was able to reflect uninformed trading only: impact is seen as a *temporary*, statistical effect due to order flow fluctuations and the liquidity imbalance following a metaorder. This temporary effect reflects an important aspect of the market structure, namely the difference between short-term and long-term supply. If a trader speeds up his buy trades, he depletes the short-term supply and increases the immediate cost for additional trades. As more time elapses, supply gradually recovers and the price witnesses a *mean-reversion* to its initial value.

Ideally, a model for market impact would take into account both the mechanical and informational aspects, or equivalently, it would display both a transient and a permanent component. For an adept of the mechanical vision, permanent impact is seen as the accumulation over time of the mechanical effects. For an adept of the informational vision, mechanical impact is a noise that reflects the activity of uninformed traders. This distinction gives a double interpretation of impact: on the one hand, market impact is a **friction**, and on the other it is the process by which prices adjust to **new information**.

Whilst the latent orderbook framework so far has only explained mechanical impact, it would be very interesting to incorporate permanent impact. To do so we would need to elaborate the **random drift** and its **correlations with volume**, so that $(m_s)_s$ would be a distribution endogenized together with the one of V_s . We leave this question as an extension for future work.

5.2 Closed-form expressions

5.2.1 Constant trading rate

The equation 2.5 can not be solved analytically in general. However, it can be solved explicitly in the case of a constant trading rate $m \equiv m_0$ by looking for a solution of the form $y_t = A\sqrt{Dt}$. By making the change of variable $s = tu$, such a function is a solution if and only if A satisfies:

$$A = \frac{m_0}{J} \int_0^1 \frac{du}{\sqrt{4\pi(1-u)}} e^{-\frac{A^2(1-\sqrt{u})}{4(1+\sqrt{u})}}, \quad (5.1)$$

where $J = LD$. Small constant trading rate For a small trading rate such that $m_0 \ll J$, we can approximate A by first assuming $A \ll 1$, then neglecting the exponential term, which leads to

$A \approx \frac{1}{\sqrt{\pi}} \frac{m_0}{J}$ and $y_t = \frac{1}{\sqrt{\pi}} m_0 \frac{\sqrt{DT}}{J}$. This can be rewritten as $y_t = \sqrt{\frac{m_0 D}{\pi J}} \sqrt{Q_t}$ where Q_t is the cumulative traded volume up till time t .

We have recovered the empirical, universal square-root law and this serves as a first confirmation of our latent order book model. In fact, this same law can be recovered in several different regimes as is proved below, the difference lying in the constant pre-factor. Large constant trading rate For a large trading rate such that $m_0 \gg J$, we can approximate A by restricting the integral to u close to 1, i.e writing $A \approx \frac{m_0}{J} \int_{1-\epsilon}^1 \frac{du}{\sqrt{4\pi(1-u)}} e^{-\frac{A^2(1-\sqrt{u})}{4(1+\sqrt{u})}}$ for some ϵ . If we let $u = 1 - s$, we approximate $\int_{1-\epsilon}^1 \frac{du}{\sqrt{4\pi(1-u)}} e^{-\frac{A^2(1-\sqrt{u})}{4(1+\sqrt{u})}} \approx \int_0^\epsilon \frac{ds}{\sqrt{s}} e^{-A^2 \frac{s/2}{2}}$. A good choice of ϵ is such that $e^{-A^2 \frac{s}{4}} \ll 1$ vanishes beyond when $x > \epsilon$. Now, changing the variables again to $v = \frac{A^2}{4}s$ gives $A^2 \approx \frac{1}{\sqrt{\pi}} \frac{2m_0}{J} \int_0^{A^2 \epsilon/4} dv \frac{e^{-v}}{\sqrt{v}}$. Finally, since $\int_0^{A^2 \frac{s}{4}} dv \frac{e^{-v}}{\sqrt{v}} \approx \Gamma(1/2) = \sqrt{\pi}$, this leads to $A^2 = \frac{2m_0}{J}$, hence $A = \sqrt{\frac{2m_0}{J}}$ (since A must be positive) and $y_t = \sqrt{\frac{2Dm_0 t}{J}}$, which are exactly the values stated in [1]. In particular, $y_T = \sqrt{\frac{2}{L} Q}$.

5.2.2 Small trading rate

A small trading rate allows to drop the exponential term in 2.5, leading to the impact

$$y_t = \frac{1}{L} \int_0^t \frac{dsm_s}{\sqrt{4\pi D(t-s)}}$$

We observe that the impact is linear and accumulates with a square-root decay kernel. This impact equation actually falls within the broad family of **propagator models**. This class of equations has been investigated extensively in the literature, see in particular [?] and [?].

5.2.3 Large trading rate

When the trading rate satisfies $m \gg J$, the integral equation 2.5 can be simplified by keeping only the values of s that are close to t . Using a first order expansion $y_s = y_t + (s - t)y'_t + o(s - t)$, we write $y_t \approx \frac{1}{L} \int_{t-\alpha}^t \frac{dsm_s}{\sqrt{4\pi D(t-s)}} e^{-\frac{y_t'^2(t-s)}{4D}}$, for some (small) value of α .

With the change of variable $v = y_t'^2 \frac{t-s}{4D}$, this gives $y_t = \frac{1}{Ly_t' \sqrt{\pi}} \int_0^\beta m_{t - \frac{4Dv}{y_t'^2}} \frac{dv e^{-v}}{\sqrt{v}}$ where $\beta = y_t'^2 \frac{\alpha}{4D}$. At this stage, we can assume that α is small enough for the expansion $m_{t - \frac{4Dv}{y_t'^2}} \approx m_t - \frac{4Dv}{y_t'^2} m_t'$ to be valid for all $v \in [0, \beta]$. Approximating the two integrals by $\Gamma(1/2) = \sqrt{\pi}$ and $\Gamma(3/2) = \frac{1}{2}\sqrt{\pi}$ respectively, we arrive at the first-order approximation:

$$y_t y_t' \approx \frac{1}{L} \left(m_t - \frac{2Dm_t'}{y_t'^2} \right).$$

Keeping the term of order 0 only, this leads to $\frac{1}{2}y_t^2 = \frac{1}{L} \int_0^t ds m_s e^{vs}$, i.e

$$y_t = \sqrt{\frac{2}{L} Q_t}, \tag{5.2}$$

where $Q_t = \int_0^t ds m_s y_s$ is the cumulative traded volume until time t . In particular, $y_T = \sqrt{\frac{2}{L} Q}$ which confirms the value derived above in the particular case of a large constant trading rate.

Note that for higher-order approximations to remain valid, one should expand both m and y around t , leading to the equation (54) in [DBMB].

5.3 The cost of impact

The cost of an execution strategy m is defined by $\mathcal{C} = \int_0^T dt m_t y_t$. We illustrate the derivation of the cost in the case of constant trading rates.

The cost becomes $\mathcal{C} = A \int_0^T dt m_0 \sqrt{Dt} = \frac{2}{3} A m_0 \sqrt{D} T^{3/2}$.

- When $m_0 \ll J$, this gives $\mathcal{C} = \frac{2}{3\sqrt{\pi}} \frac{\sqrt{D}}{J} m_0^2 T^{3/2} = \frac{2}{3\sqrt{\pi}} \frac{\sqrt{D}}{J} m_0^{1/2} Q^{3/2}$.
- For $m_0 \gg J$, the cost becomes $\mathcal{C} = 2\sqrt{2} \frac{\sqrt{D}}{J} Q^{3/2}$, which only depends on the total volume Q .

Since D was defined as the variance of the noise η , this is in line with the wide empirical consensus in the literature that the impact-induced costs are on the order of $\sigma Q^{3/2} V^{1/2}$, where σ is the daily volatility and V the daily traded volume.

5.4 Extension to the deposition/cancellation model

We have seen that nonzero deposition rates simply led to consider the rescaled volume $\widehat{Q} = \int_0^T ds m_s e^{\nu s}$ instead of $Q = \int_0^T ds m_s e^{\nu s}$. Therefore, the approximations and analytical estimations we have derived here remain valid so long as the rescaled volume satisfies the required assumptions. In particular, we expect the square root law to remain approximately valid in several regimes.

5.5 Why a stronger diffusion leads to smaller impact

In this section we explicit the influence of the diffusion σ on the impacted price. Intuitively, we can foretell that a higher diffusion decreases the impact due to its ‘‘smoothing effect’’ that replenishes the order book liquidity, thus increasing the resistance towards a price impact.

To simplify the treatment further we take the case of a constant trading rate. We have seen in Section 5.2.1 that y_t takes the form $y_t = A\sqrt{Dt}$, where $D \propto \sigma^2$ and A is a constant that depends on D and on the metaorder m .

Recall that A satisfies an integral equation given in 5.1. Differentiating this equation with respect to A yields $\partial_D A = -m_0 \int_0^1 \frac{du}{\sqrt{1-u}} e^{-A^2 \frac{1-\sqrt{u}}{1+\sqrt{u}}} \left(\frac{1}{D^2} + 2 \frac{A}{D} \partial_D A \frac{1-\sqrt{u}}{1+\sqrt{u}} \right)$. Grouping the terms together we obtain that $\partial_D A (1 + 2 \frac{A}{D} \int_0^1 \frac{du}{\sqrt{1-u}} \frac{1-\sqrt{u}}{1+\sqrt{u}}) = -\frac{A}{D}$. Since $A > 0$, this implies that $\partial_D A < 0$, and in fact we even have $\partial_D A (1 + *) = -\frac{A}{D} \implies \partial_D (DA) = - * \partial_D A \leq 0$, hence the impact $y_t \propto A\sqrt{D}$ decreases with σ , which concludes the proof.

5.5.1 Price manipulation in changing markets

Section VIII of [DBMB] shows that the model is consistent with the principle of no-dynamic-arbitrage, that is, a ‘‘pump and dump’’ strategy that sells the asset with the aim of lowering its price before buying it again (or conversely by buying before selling) cannot lead to positive profits. The authors show that for any *round trip strategy*, defined by an execution path $(m_t)_t$ such that $\int_0^T dt m_t = 0$, the expected cost of trading satisfies $\mathcal{C} := \int_0^T dt m_t y_t \geq 0$. This is the definition of price manipulation according to [Huberman and Stanzl]. The idea is that if such a strategy existed, then by repeating it infinitely many times, we would obtain using the law of large numbers an arbitrage almost surely in the usual meaning.

However, price manipulation is not proven to be impossible in real markets. Consider for instance the example of FX trading. Forex trading hours move around the world so that at any time of the day, it is possible to trade FX instruments, either in London, New York, Tokyo. It is observed that during the Tokyo shift, the liquidity of the EUR/USD is relatively weak, meaning that a price impact

could be higher at these times. A manipulator could thus perform a round trip trade that buys large quantities during the Asian shifts and dumps them during the London/NY hours.

This observation can be accounted for in the extended model where nonzero deposit/cancellation rates are allowed. It is enough for this example to allow piecewise-constant cancellation rates, and zero deposit rates. The lower-liquidity setting comes with a higher cancellation rate $\nu_{Asia} > \nu_{NY}$. For the sake of computational simplicity, let us assume a constant small trading rate $\pm m_0 \ll J$, and take the execution strategy

$$m(t) = \begin{cases} m_0 & \text{if } 0 \leq t < t_{NY} \\ -m_0 & \text{if } t_{NY} \leq t \leq T \end{cases}$$

where t_{NY} is the opening time of the New York FX platform. This leads to the impact $y_t = \frac{m_0}{L} \left(\int_0^{t \wedge t_{NY}} ds e^{\nu_{Asia} s} - \int_{t \wedge t_{NY}}^{t \wedge T} ds e^{\nu_{NY} s} \right)$, and it is immediate to check that the corresponding cost is negative.

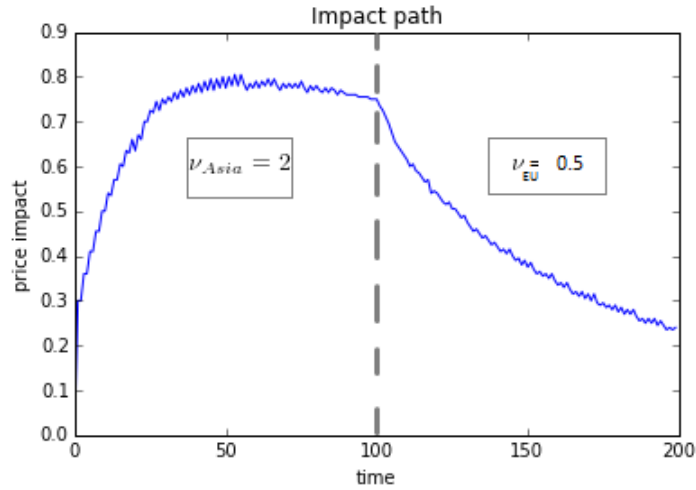


Figure 5.1: Illustration of a price manipulation strategy when the cancellation parameter ν is allowed to vary. Here, a trader can profit from the changing dynamics to keep prices up despite his selling. The simulation was performed using the scheme 6.6 from section 6.3.

6 The Mean Reverted LLOB

6.1 Introduction

We now allow the agents to adjust their current price towards the *latent/hidden* price of the asset (or rather their estimation of it). The idea is to maintain the same diffusive behavior and to complement it with a mean-reversion process. A motivation for this work is that the original dynamics are purely diffusive, and in this sense, resemble the “zero-intelligence” model of [Farmer], raising doubts about its ability to account for the rationality of agents.

To obtain the new price dynamics we start by writing the microscopic evolution in a non-rigorous way and transform it into a partial differential equation.

We write that each agent reassesses its price from $p_{i,t}$ to $p_{i,t+dt} = p_{i,t} + \eta_{i,t} - \kappa(p_{i,t} - B_t)dt$, where the process $(B_t)_t$ represents a **reference price**, either exogeneous (e.g a Brownian motion) or endogeneous (e.g, take the current market price and plug it in, leading to a feedback loop). The noise variables $\eta_{i,t} \stackrel{\text{i.i.d}}{\sim} \mathcal{N}(0, \sigma^2 dt)$ are the noise in the agents’ estimations of the fundamental price. Finally $\kappa > 0$ quantifies the return force towards B_t .

To obtain the corresponding partial differential equation, note that $p_{i,t} = \frac{1}{1-\kappa dt}(p_{i,t+dt} - \eta_{i,t} + \kappa B_t dt) \approx p_{i,t+dt} - \eta_{i,t} + \kappa(p_{i,t+dt} - B_t)dt$. Now perform a second-order expansion:

$$\begin{aligned} \varphi(x, t + dt) &= \int \mathbb{P}(\eta) \int dy \varphi(y, t) \delta(x - \eta + \kappa(x - B_t)dt - y) \\ &\approx \varphi(x) + \left(0 + \kappa(x - B_t)\right) \partial_x \varphi(x, t) dt + \frac{1}{2} \sigma^2 \partial_{xx} \varphi(x, t), \end{aligned}$$

so that the the density of agents evolves according to the PDE:

$$\partial_t \varphi(x, t) = \kappa(x - B_t) \partial_x \varphi(x, t) + \frac{1}{2} \sigma^2 \partial_{xx} \varphi(x, t), \forall y \in \mathbb{R}, \forall t \geq 0. \quad (6.1)$$

This means that the agents reassess their price all the more as they are far from the reference price B_t , and the intensity of the reassessments is controlled by the parameter κ .

6.2 Analytical resolution

Due to the presence of the reference price B_t , the dynamics of the orderbook 6.1 are no longer linear. However, we can change the reference frame to get rid of the reference price, thereby considerably simplifying the PDE. To do so we perform the change of variable $y = x - f(t)$, where $f(t) := \kappa \int_0^t ds B_s e^{-\kappa(t-s)}$. This means that we define a new function $\phi(y, t) = \varphi(y + f(t), t)$. Since f satisfies the differential equation $f' + \kappa f = \kappa B_t$, ϕ satisfies the **linear** PDE

$$\partial_t \phi(y, t) = \kappa y \partial_y \phi(y, t) + \frac{1}{2} \sigma^2 \partial_{yy} \phi(y, t). \quad (6.2)$$

6.2.1 First approach: separation of variables

Here we are interested in special solutions of 6.2 of the form $\phi(y, t) = f(y)g(t)$, leading to the system

$$\begin{cases} g'(t) &= cg(t) \\ \kappa y f'(y) + \frac{\sigma^2}{2} f''(y) &= cf(y) \end{cases}$$

for some $c \in \mathbb{R}$. We only keep $C = 0$ as an acceptable parameter since at large times, the orderbook is expected to not collapse to 0 or explode to $+\infty$. De facto this ensures that $g \equiv \text{cte}$, so we are looking for a stationary solution, i.e $\phi_{st}(y) = cf(y)$ and $\kappa y f' + \frac{\sigma^2}{2} f'' = 0$. A straightforward calculation yields $\phi_{st}(y) = c_0 + c_1 \int_{-\infty}^y dx e^{-\frac{\kappa x^2}{\sigma^2}}$.

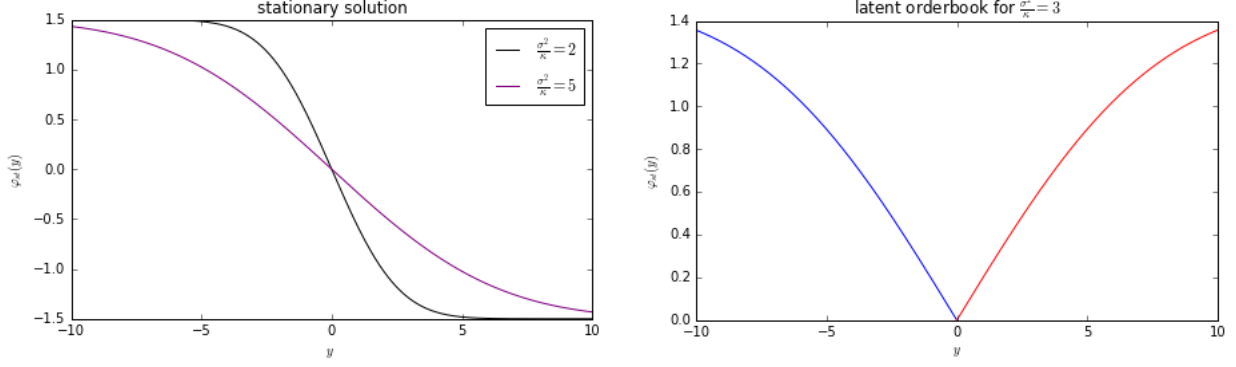


Figure 6.1: *Left*: an example of stationary solution ϕ_{st} plotted for $c_0 = 1.5$ and $c_1 = -3\sqrt{\frac{\kappa}{\pi\sigma^2}}$ and $\frac{\sigma^2}{\kappa} \in \{2, 5\}$. *Right*: the corresponding orderbook, where the blue (resp. the red) curve represents the bid side (resp. the ask side). In the new reference frame the equilibrium price is 0.

Going back to the original reference frame simply yields

$$\varphi_{st}(x, t) = c_0 + c_1 \int_{-\infty}^{x-f(t)} dx e^{-\frac{\kappa(x-f(t))^2}{\sigma^2}}.$$

6.2.2 Full resolution

Our first approach has consisted in performing a simple change of variable that linearized the equation and allowed for variable separation. We can actually simplify the equation 6.1 further with the change of variable $y = e^{\kappa t}(x - f(t))$. That is, by letting $\psi(y, t) = \varphi(e^{-\kappa t}y + f(t), t)$, the equation becomes

$$\partial_t \psi(y, t) = \frac{\sigma^2}{2} e^{2\kappa t} \partial_{yy} \psi(y, t), \quad (6.3)$$

The first differential equation 6.1 was difficult to tackle directly because of the non-linear term, the reference price B_t around which the mean-reversion occurs. Our change of variable can be decomposed into two steps:

- the first change, $y_0 = x - f(t)$, translates the “coordinates” to follow the reference price.
- the second change, $y_1 = e^{\kappa t}y_0$, is a change of time scale. Under this change the price evolves as a simple diffusion process (at a time-dependent diffusion rate).

For more readability we have decided to keep our results in the new reference frame, and the conversion from ψ to φ is left to the reader.

All that is left now is to solve 6.3. To this end we recall some elementary facts about the Fourier transform. Given an integrable function $f \in L^1(\mathbb{R}, \mathbb{C})$, its Fourier transform is defined by $\mathcal{F}(f) : k \mapsto \int_{-\infty}^k dy f(y) e^{-iky}$. If $\hat{f} := \mathcal{F}(f) \in L^1(\mathbb{R}, \mathbb{C})$, in particular if f is continuous, the inverse Fourier transform $\mathcal{F}^{-1}(\hat{f}) : y \mapsto \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk f(k) e^{iky}$ is well defined and satisfies $\mathcal{F}^{-1}(\hat{f}) = f$ almost everywhere.

The Fourier transform is a very convenient tool to solve (partial) differential equations and we illustrate its application below on the resolution of equation 6.3.

We Fourierize the space variable, leading to the PDE

$$\partial_t \hat{\psi}(k, t) = -\frac{\sigma^2}{2} e^{2\kappa t} k^2 \hat{\psi}(k, t),$$

where we used the fact that $\mathcal{F}[\partial_{yy}\psi] = -k^2\psi$.

The solution to this last equation is straightforward by variation of constants: $\hat{\psi}(k, t) = g(k)e^{-\frac{C(0,t)\sigma^2 k^2}{2}}$, where $g = \mathcal{F}[\psi_0]$ (where the initial condition $\psi_0 = \varphi_0 := \varphi(\cdot, t=0)$ is a continuous function over \mathbb{R}) and

$$C(s, t) := \int_s^t du e^{2\kappa u} = \frac{1}{2\kappa}(e^{2\kappa t} - e^{2\kappa s})$$

for $0 \leq s \leq t$.

By going back to the space domain, and since the transform of the product of two functions is the convolution of their transforms, the solution is given by

$$\psi(y, t) = \int_{-\infty}^{+\infty} \frac{du}{\sqrt{2\pi C(0, t)\sigma^2}} \varphi_0(u) e^{-\frac{(u-xy)}{2C(0,t)\sigma^2}},$$

Note that this solution is well-defined iff the integrand is in L^1 for all $t \geq 0$. This condition is not very restrictive and allows for a large choice of initial conditions (and in particular linear functions).

The price evolves with a diffusion rate (i.e the variance of the heat kernel) that increases exponentially with time, which is just due to the exponential change of variable.

We conclude by noting that we indeed recover the usual impact profile when we let $\kappa \rightarrow 0$.

6.2.3 The impacted price

What market price does our model predict during the execution of a metaorder?

The previous equation can be very easily tailored to answer this question. The PDE in the presence of a metaorder $m \in \mathcal{C}([0, T])$, executed at the current market price, denoted x_t , becomes

$$\partial_t \varphi(x, t) = \kappa(x - B_t) \partial_x \varphi(x, t) + \frac{1}{2} \sigma^2 \partial_{xx} \varphi(x, t) + m_t \delta(x - x_t),$$

where δ is the Dirac distribution.

This means that in the new frame of reference we have:

$$\partial_t \psi(y, t) = \frac{\sigma^2}{2} e^{2\kappa t} \partial_{yy} \psi(y, t) + m_t \delta(e^{-\kappa t} y - x_t).$$

Since the Dirac distribution $\xi \mapsto \delta(\xi)$ is homogeneous of degree -1 with respect to ξ , we may rewrite this equation as

$$\partial_t \psi(y, t) = \frac{\sigma^2}{2} e^{2\kappa t} \partial_{yy} \psi(y, t) + m_t e^{\kappa t} \delta(y - y_t).$$

where $y_t := e^{\kappa t}(x_t - f(t))$ is the market price in the new reference frame, i.e the zero of $\psi(\cdot, t)$ (whose existence and unicity depend on the initial condition and that we admit at this stage). Observe that the metaorder has become $m_t e^{\kappa t}$ instead of m_t . This is a natural consequence of our change of variable which made the new space variable depend on time.

Going again to Fourier domain gives $\partial_t \hat{\psi}(k, t) = -\frac{\sigma^2}{2} e^{2\kappa t} k^2 \hat{\psi}(k, t) + m_t e^{\kappa t} e^{-iky_t}$, which is solved by a variation of constants and gives $\hat{\psi}(k, t) = g(k) e^{-\frac{\sigma^2 C(0,t)k^2}{2}} + \int_0^t ds m_s e^{\kappa s} e^{-iky_s - \frac{\sigma^2 C(s,t)k^2}{2}}$, where g is again the Fourier transform of the initial condition.

Going back to the space domain therefore leads

$$\psi(y, t) = \frac{1}{\sqrt{2\pi C(0, t)\sigma^2}} (\psi_0 * e^{-\frac{x^2}{2C(t)\sigma^2}})(y) + \int_0^t ds m_s e^{\kappa s} \mathcal{F}^{-1}[e^{-iky_s - \frac{\sigma^2 C(s,t)k^2}{2}}](y).$$

For a linear initial condition $\psi_0(y) = \varphi_0(y) = -Ly$, this gives

$$\psi(y, t) = -Ly + \int_0^t \frac{dsm_s e^{\kappa s}}{\sqrt{2\pi C(s, t)\sigma^2}} e^{-\frac{(y-y_s)^2}{2\sigma^2 C(s, t)}},$$

hence the impacted price satisfies the integral equation

$$y_t = \frac{1}{L} \int_0^t \frac{dsm_s e^{\kappa s}}{\sqrt{2\pi C(s, t)\sigma^2}} e^{-\frac{(y_t-y_s)^2}{2\sigma^2 C(s, t)}}. \quad (6.4)$$

Existence of a solution to 6.4

In Section 4 we proved the existence of a solution to the impact equation 2.5. The proof can be easily adapted to show that 6.4 enjoys an existence property as well. Since for every $0 \leq s < t$, $\frac{e^{\kappa s}}{\sqrt{e^{2\kappa t} - e^{2\kappa s}}} = \frac{1}{\sqrt{e^{2\kappa(t-s)} - 1}} \leq \frac{1}{\sqrt{2\kappa(t-s)}}$ (as a consequence of the inequality $e^x \geq 1 + x$ applied to $x = t - s \geq 0$), the proof remains valid by replacing $t - s$ everywhere with $C(s, t)$, and we therefore have the

Theorem 6.1. *Let $m \in \mathcal{C}([0, T], \mathbb{R})$ be any execution strategy. Then the integral equation 6.4 admits a solution y defined over $[0, +\infty[$.*

6.2.4 The limit of small trading rates

In the limit where $\|m\|_\infty \ll L\sigma$, we make the approximation of small impacts: $(y_t - y_s)^2 \ll C(s, t)$ that allows to drop the exponential term in 6.4. We therefore obtain the **linear propagator**:

$$y_t = \frac{1}{L} \int_0^t \frac{dsm_s e^{\kappa s}}{\sqrt{2\pi C(s, t)\sigma^2}}$$

To have an insight about the behavior of this impacted price, we take a constant trading rate $m \equiv m_0$. Then a straightforward calculation with the change of variable $v = e^{-\kappa(t-s)}$ yields the following **concave impact**:

$$y_t = \frac{1}{L\sigma\sqrt{\kappa\pi}} \int_{e^{-\kappa t}}^1 \frac{dv}{\sqrt{1-v^2}} = \frac{1}{L\sigma\sqrt{\kappa\pi}} \left(\frac{\pi}{2} - \arcsin(e^{-\kappa t}) \right) \quad (6.5)$$

Since $\arcsin x \underset{x \rightarrow 1}{\sim} \frac{\pi}{2} - \sqrt{2(1-x)}$, we obtain in the short time scales $y_t \underset{t \rightarrow 0}{\sim} \frac{1}{L\sigma} \sqrt{2t}$, i.e **we recover the original square-root law**. In the larger time scales, however, the impact converges to a finite nonzero value: $y_t \underset{t \rightarrow +\infty}{\rightarrow} \frac{\sqrt{\pi}}{L\sigma\sqrt{2\kappa}}$, in contrast with the divergence of the impact in the original model. This is explained by the fact that here agents keep reassessing their price towards the fundamental value (i.e 0 in this new reference frame), providing resistance to price increases by the means of added liquidity.

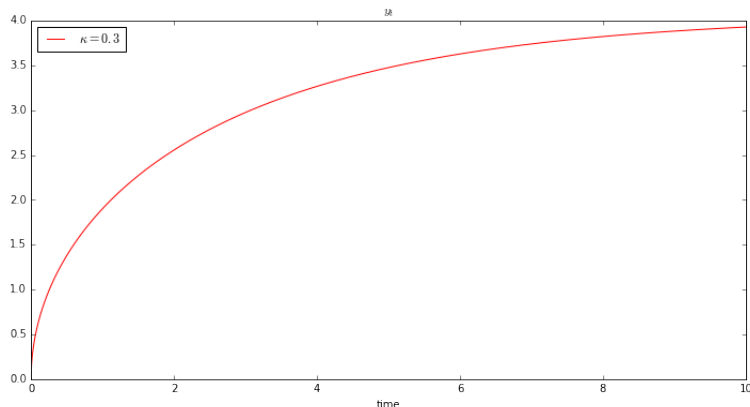


Figure 6.2: For very small trading rates, we have recovered a special linear propagator that leads to concave impact. This propagator replicates the square-root behavior at the shorter times, and converges to a finite value as $t \rightarrow \infty$.

Let us emphasize that this is the impacted price in the new reference frame, whereas in the original reference frame it is given by $x_t = e^{-\kappa t}(y_t + f(t))$. Surprisingly, the latter decreases invariably to 0 due to the exponential shrinkage. This somewhat unexpected behaviour (discussed in greater detail in the conclusion) might indicate that our mean-reversion approach was too strong, and we leave further investigation of this phenomenon as a priority for future work.

6.2.5 The influence of diffusion and mean-reversion

The objective of this section here is to explicit the influence 6.4 of the microscopic parameters governing the PDE 6.1 : the volatility σ and the intensity of reversion κ , both on the “mispricing” of the market and on the impacted price.

On mispricing

We can speak of “convergence” of the market price to the efficient price in the sense that the stronger the value of κ , the closer $f(t)$ is to the “true” underlying movement B_t . To see this, observe that an integration by parts (where the Ito term is 0 since the integrand is a deterministic function of time) leads to

$$f(t) = \kappa \int_0^t ds e^{-\kappa(t-s)} B_s = \int_0^t d(e^{-\kappa(t-s)}) B_s = B_t - \int_0^t e^{-\kappa(t-s)} dB_s.$$

Therefore the “mispricing”

$$B_t - f(t) = \int_0^t e^{-\kappa(t-s)} dB_s$$

is a centered Gaussian variable with variance $\nu(\kappa, t) = \int_0^t e^{-2\kappa(t-s)} ds$, which clearly decreases with κ (for all t).

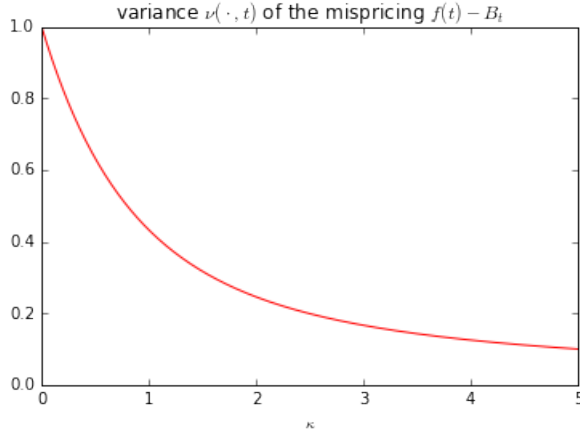


Figure 6.3: The stronger the mean-reversion, the smaller the mispricing (displayed here for $t = 1$).

On the impacted price

Observe that the original framework corresponds to $\kappa \rightarrow 0$ so that $C(s, t) \underset{\kappa \rightarrow 0}{\sim} t - s$, allowing to recover exactly the original impact equation 2.5. And we have seen in section 5.5 that in the original model **the impacted price decreases with diffusion**.

To get an intuition about our new model, we first look at the stationary solution ϕ_{st} plotted in 6.1 (in the reference frame of B_t). We see that an increase in the volatility σ leads to decreasing liquidity around the market price, making the market less robust to small perturbations. The reversion parameter κ has the inverse effect as it drives the liquidity towards the price. We therefore would like to show that **the impact decreases with κ** as well.

Intuitively, a higher κ strengthens the reversion and thus diminishes the impact. Similarly, a strong volatility should lead to diffusion of impact. This is explained because the diffusive price jumps, which act as a *smoothing mechanism*, occur more frequently.

These results can be proved using the integral equation 6.4 to deduce the sign of $\partial_\kappa y_t$, using arguments similar to those of section 5.5. Note that by taking the special case of small constant trading rates, where the impact is explicitly given by equation 6.5, it is immediate to see that the impact indeed decreases with κ .

6.3 Numerical experiments

6.3.1 The order book

Our objective here is to confirm numerically the analytical formulations derived above. To this end we build a Crank-Nicolson finite difference scheme. We use the first and second order differentiation

matrices $B = \begin{pmatrix} 0 & 0 & \dots & \dots & 0 \\ 0 & -1 & 1 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & -1 & 1 \\ 0 & \dots & \dots & 0 & 0 \end{pmatrix}$ and $A = \begin{pmatrix} 0 & 0 & \dots & \dots & 0 \\ 1 & -2 & 1 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 1 & -2 & 1 \\ 0 & \dots & \dots & 0 & 0 \end{pmatrix}$, respectively. The first and last

row of A and B enforce reflective Dirichlet boundary conditions, namely $\partial_x \varphi(-M, t) = \partial_x \varphi(M, t) = 0, \forall t$ for B and $\varphi(-M, t) = -LM, \varphi(M, t) = LM, \forall t \geq 0$ for A .

Our scheme reads:

$$\left(I - \frac{\Delta T \cdot D}{2\Delta x^2} A\right) U^{k+1} = \left(I + \frac{\Delta T \cdot D}{2\Delta x^2} A + \frac{\Delta T \cdot \kappa}{\Delta x} (X - B_t \tilde{\mathbf{1}})^T B\right) U^k, \quad (6.6)$$

where $X = \begin{pmatrix} -M \\ -M + \Delta x \\ \dots \\ M - \Delta x \\ M \end{pmatrix}$.

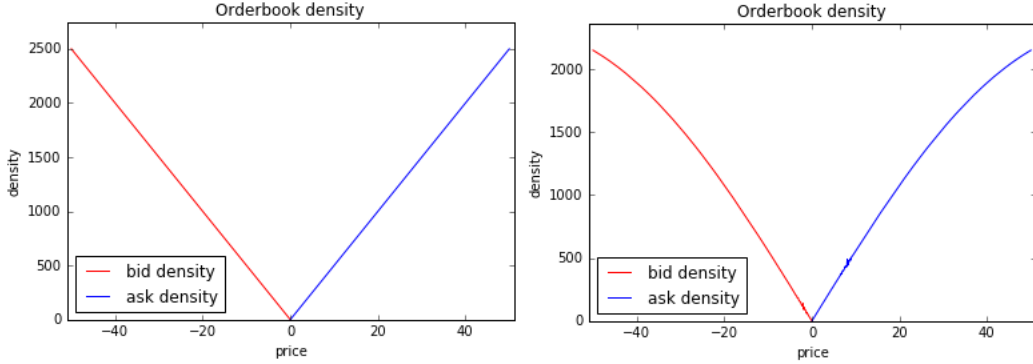


Figure 6.4: Starting from a linear initial condition $\varphi(y, 0) = -50y$ (plotted on the left), 1500 iterations of the Crank-Nicolson scheme 6.6 are performed and the final result is plotted on the right. This simulation confirms the stationary shape predicted using the separation of variables. Here $\kappa = 0.05$ and $B_t \equiv 0$.

Using this simulation, we know the shape of the order book at any point in time. Hence we can immediately recover the market price, which is just the zero of the density function φ . In the absence of a metaorder, we observe that the market price simply follows the dynamics of B_t .

6.3.2 The mean reversion

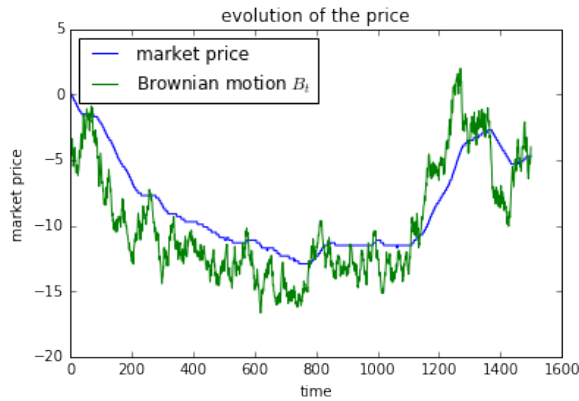


Figure 6.5: When mean-reversion is sufficiently strong (with $\kappa = 0.5$ here), The market price closely follows the evolution of the reference price B_t (taken here as an affine transformation of a Brownian motion).

The change of variable we derived in Section 6.2.2 led us to a diffusion equation, which foretells that the real market price should “converge” not to B but to f . Note that if we write $f(t) \approx \hat{f}(t) := \frac{\int_0^t dB_s e^{\kappa s}}{\int_0^t ds e^{\kappa s}}$, f may be seen as a weighted average of all the past values of B , with $\tau := \frac{1}{\kappa}$ defining the “memory” of the averaging.

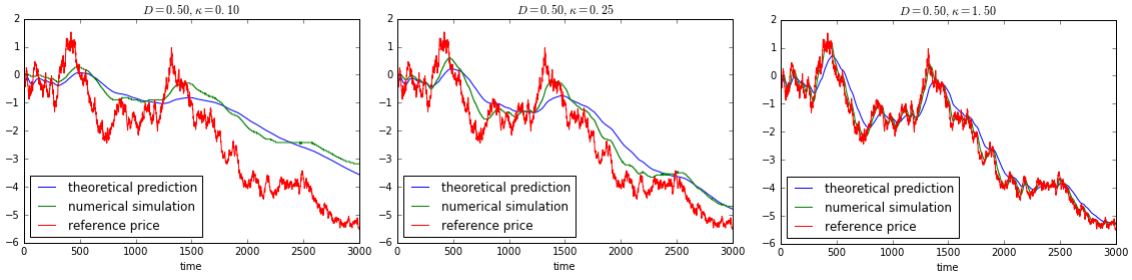


Figure 6.6: The reference price B_t , the observed market price with the Crank Nicolson scheme, and the theoretical prediction $f(t)$. We observe a very good fit for higher values of κ , in line with the analysis of mispricing conducted in 6.3. The time unit is one simulation step.

6.3.3 Impacted price

Incorporating a buy (resp. sell) metaorder is simply done by consuming the corresponding volume from the best available ask (resp. bid). This allows to observe the evolution of the price in the presence of a metaorder.

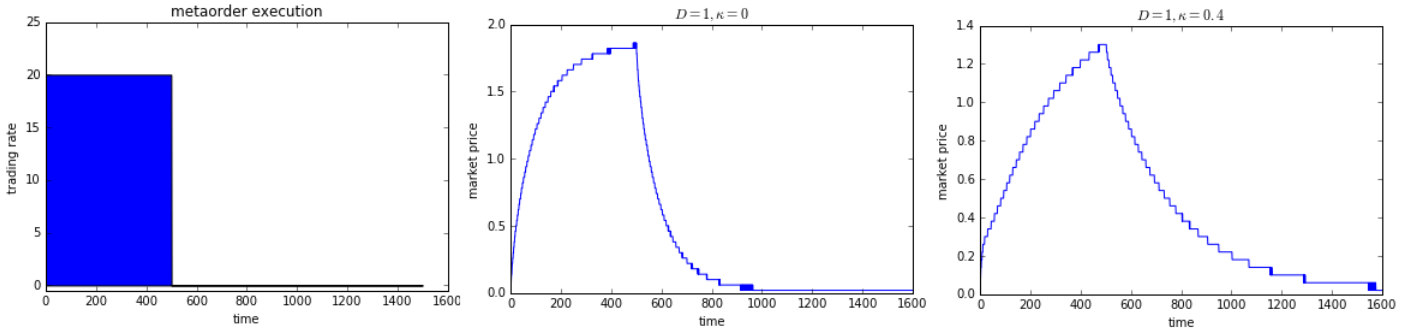


Figure 6.7: Evolution of the price during a buy metaorder. *Left*: the metaorder’s execution profile. *Middle*: the resulting impacted price in the absence of mean-reversion. *Right*: the resulting impacted price when B_t is 0 (amounting to the first change of reference frame). Observe that the impact decreases when $\kappa > 0$ due to the attraction towards the origin.

The graph below confirms the prediction of section 6.2.5 : higher diffusion leads to smaller impact.

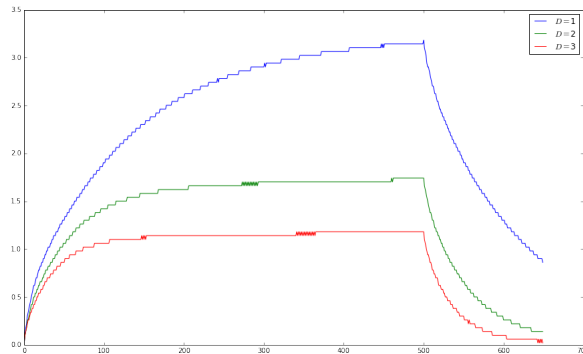


Figure 6.8: Evolution of the impacted price $(y_t)_t$ for a constant buy metaorder executed during $[0, 500]$, proving that the volatility has a negative effect. Here $\kappa = 1$.

Another interesting simulation consists in taking an exogeneous B_t that opposes the direction of the

metaorder.

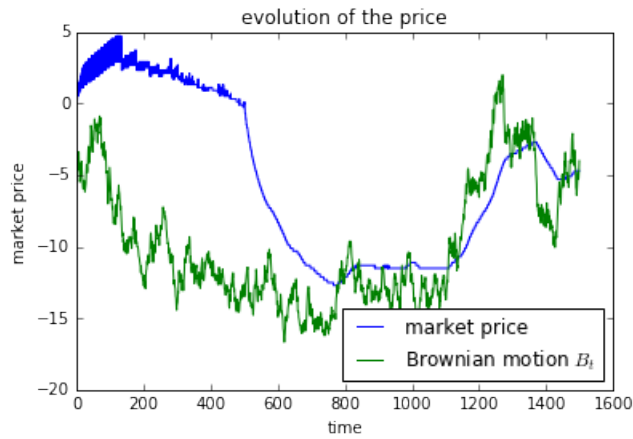


Figure 6.9: Evolution of the price for the same metaorder as Figure 6.7 but with B_t as in Figure 6.5. Observe the tradeoff between the positive push of the metaorder and the negative push of B_t . During the first 500 seconds, the effect of the metaorder dominates, but the price reverts to negative territory as soon as its execution has finished.

7 Conclusion and discussion

We have carried out an extensive analysis of the latent order book model, based on the seminal work of [Deremble] and [DBMB]. We have first investigated the mathematical properties of the price dynamics and have proved the existence of a solution to the impacted price equation. We have confirmed our theoretical predictions by extensive numerical experiments that rely on a finite-difference scheme to discretize the partial differential equation. Our results have led us to question the assumptions of absence of arbitrage and we have shown how a real-world manipulation strategy can take place within our framework.

Our essential building block, the latent order book, uses only minimal ingredients, thereby confirming the universality of the concave impact law. Our model has constructed impact as the consequence of two opposing phenomena: liquidity consumption, which increases the cost of trading and the spread, and diffusion which pushes the prices back towards a mean-reversion equilibrium.

It is worth mentioning that we have recently confirmed our findings on Bitcoin data [Lemhadri], demonstrating that the salient characteristics of concave impact remain valid even on such an amateur marketplace (at least at this time). Even when individual metaorders cannot be systematically detected (due to the anonymity enforced on exchanges), even in the absence of a notion of fundamental value (which makes little sense as of today on the Bitcoin market), the persistence of impact suggests that a robust self-organizing mechanism is at work, which is well incorporated in the latent order book model. Contrary to *ad-hoc* stochastic models of prices, this modelling strategy offers a much deeper understanding of price formation and can certainly be extended to other economic situations.

We have complemented the initial latent order book model by suggesting that the diffusive motion can be tailored from a basic random walk to an Ornstein-Uhlenbeck process. This allowed us to incorporate a phenomenon of mean-reversion towards the “fundamental price” as estimated by the market participants. We then quantified the mispricing between the real market price and the fundamental price, and we observed the complex interplay between liquidity taking with metaorders and liquidity provision by mean-reverting agents. And we analyzed mathematically the resulting equations. In particular, we have established that the impact of a metaorder decreases when either the volatility of the underlying asset or the agents’ reassessment intensity (quantified by κ) increase.

We see numerous ways in which this work can be complemented. First and foremost, the resolution of the mean-reverted model indicates that impact decreases to 0 for any constant trading strategy. This is because by bounding the exponential term in equation 6.4 by 1, we obtain the arcsine shape that corresponds to the limit of small rates, which collapses to zero when going back to the original reference frame. This contrasts notably with the original LLOB model and may be interpreted as the informational content of the metaorder being discovered by market participants. When this information discovery is over, the increase in price ceases, eventually leading to null permanent impact. However, this result is surprising since our modelling did not incorporate any informational component. It would therefore be advisable to compare these results to variations of our model and see if the impact decrease still persists. For instance, instead of taking a latent Brownian (known to all market participants) as the origin of mean-reversion, take the market price itself. This type of feedback loop would make the model more realistic, however it is likely to lose analytical tractability.

In addition, the vast majority of price impact models assume that there is only one large investor (the rest of the market evolving through a martingale). This assumption is reasonable when a market brings together a large number of agents with heterogeneous beliefs, but it can (and should) be questioned if other large investors happen to trade at the same time. It would be very interesting to extend the framework to allow for different metaorders to interact simultaneously in the orderbook. Such problems have important connections with game theory and are certainly of great interest to practitioners. One possible way to do this would be to fix the random drift and its correlation with metaorder volumes, so that an execution strategy be seen as a distribution, endogenized together with the one of the drift.

The tremendous complexity of real-world markets makes it all too clear that our model is only a rough approximation of the dynamics of the orderbook, and suggests that many other practical problems (e.g the market-maker's problem, the presence of transaction costs and other market design peculiarities) are yet to be incorporated. Building a detailed full-scale model of order flow would certainly be welcome for future work, however our main objective was to show how a simplified model allows to capture the most salient characteristics of price formation: concave price impact and locally small liquidity.

As a concluding remark, we note that our work has important practical consequences for market regulators as it supports more than ever the idea of impact-adjusted valuation of assets.

Acknowledgements

The author would like to express his gratitude to Pierre Laffitte, for giving him the opportunity to work on this subject and for fruitful mentoring. He also would like to thank Jiayu Cai for many interesting discussions.

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