Mean field rough differential equations

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Abstract. We provide in this work a robust solution theory for random rough differential equations of mean field type

$$dX_t = V\big(X_t, \mathcal{L}(X_t)\big)dt + \mathrm{F}\big(X_t, \mathcal{L}(X_t)\big)dW_t,$$

where W is a random rough path and $\mathcal{L}(X_t)$ stands for the law of X_t, with mean field interaction in both the drift and diffusivity. Propagation of chaos results for large systems of interacting rough differential equations are obtained as a consequence, with explicit optimal convergence rate. The development of these results requires the introduction of a new rough path-like setting and an associated notion of controlled path. We use crucially Lions' approach to differential calculus on Wasserstein space along the way.

Contents

Introduction

The first works on mean field stochastic dynamics and interacting diffusions / Markov processes have their roots in Kac's simplified approach to kinetic theory [30] and McKean's work [37] on nonlinear parabolic equations. They provide the description of evolutions $(\mu_t)_{t\geqslant 0}$ in the space of probability measures under the form of a pathspace random dynamics

$$\begin{aligned} dX_t(\omega) &= V\big(X_t(\omega), \mu_t\big)dt + \mathrm{F}\big(X_t(\omega), \mu_t\big)dW_t(\omega), \\ \mu_t &:= \mathcal{L}(X_t), \end{aligned} \tag{0.1}$$

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(where $\mathcal{L}(A)$ stands for the law of a random variable A) and relate it to the empirical behaviour of large systems of interacting dynamics. The main emphasis of subsequent works has been on proving propagation of chaos and other limit theorems, and giving stochastic representations of solutions to nonlinear parabolic equations under more and more general settings; see [38, 39, 28, 17, 18, 5, 6] for a tiny sample. Classical stochastic calculus makes sense of equation (0.1), in a probabilistic setting $(\Omega, \mathcal{F}, \mathbb{P})$, only when the process W is a semi-martingale under $\mathbb{P}$, for some filtration, and the integrand is predictable. However, this setting happens to be too restrictive in a number of situations, especially when the diffusivity is random. This prompted several authors to address equation (0.1) by means of rough paths theory. Indeed, one may understand rough paths theory as a natural framework for providing probabilistic models of interacting populations, beyond the realm of Itô calculus. Cass and Lyons [12] did the first study of mean field random rough differential equations and proved the well-posed character of equation (0.1), and propagation of chaos for an associated system of interacting particles, under the assumption that

- there is no mean field interaction in the diffusivity, $\mathrm{F}(x, \mu) = \mathrm{F}(x)$,
- the drift depends linearly on the mean field interaction
$$V(x, \mu) = \int V(x, y)\, \mu(dy),$$
for some function $V(\cdot, \cdot)$ on $\mathbb{R}^d \times \mathbb{R}^d$.

The method of proof of Cass and Lyons depends crucially on both assumptions. Bailleul extended partly these results in [3] by proving well-posedness of the mean field rough differential equation (0.1) in the case where the drift depends nonlinearly on the interaction term and the diffusivity is still independent of the interaction, and by proving an existence result when the diffusivity depends on the interaction. The naive approach to showing well-posedness of equation (0.1) in its general form consists in treating the measure argument as a time argument. However, this is of a rather limited scope since, in this generality, one cannot expect the time dependence in F to be better than $\frac{1}{p}$-Hölder if the rough path W is itself $\frac{1}{p}$-Hölder. Clearly, such a time regularity is not sufficient to make sense of the rough integral $\int \mathrm{F}(\cdots)\, dW$ in the case $p \geqslant 2$. This serious issue explains why, so far in the literature, the coefficient F has been assumed to be a function of the sole variable x.

Including the time component as one of the components of W brings back the study of equation (0.1) to the study of equation

$$\begin{aligned} dX_t(\omega) &= \mathrm{F}\big(X_t(\omega), \mathcal{L}(X_t)\big) dW_t(\omega), \\ \mu_t &:= \mathcal{L}(X_t); \end{aligned} \tag{0.2}$$

this is the precise purpose of the present paper. Treating the drift as part of the diffusivity has the drawback that we shall impose on V some regularity conditions stronger than needed. Our method accommodates the general case but we leave the reader the pleasure of optimizing the details and concentrate on the new features of our approach, working on equation (0.2). The raw driver $\big(W_t(\omega)\big)_{t \geqslant 0}$ will be assumed to take values in some $\mathbb{R}^m$ and to be $\frac{1}{p}$-Hölder continuous, for $p \in [2, 3)$, and the one form F will be an $L(\mathbb{R}^m, \mathbb{R}^d)$-valued function on $\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$, where $\mathcal{P}_2(\mathbb{R}^d)$ is the so-called Wasserstein space of probability measures μ with a finite second-order moment. Inspired by Lions' approach [34, 7, 9] to differential calculus on $\mathcal{P}_2(\mathbb{R}^d)$, one of the key point in our analysis is to lift the function F into a function

$\widehat{\mathrm{F}}$ defined on the space $\mathbb{R}^d \times \mathbb{L}^2\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d\big)$, given by the formula

$$\widehat{\mathrm{F}}\big(x, Z\big) = \mathrm{F}\big(x, \mathcal{L}(Z)\big), \tag{0.3}$$

for $x \in \mathbb{R}^d$ and $Z \in \mathbb{L}^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$. So, we may rewrite equation (0.2) as

$$dX_t(\omega) = \widehat{\mathrm{F}}\big(X_t(\omega), X_t(\cdot)\big)dW_t(\omega). \tag{0.4}$$

We used the notation $X_t(\cdot)$ to distinguish the realization $X_t(\omega)$ of the random variable X_t at point ω from the random variable itself, seen as an element of the space $\mathbb{L}^2\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d\big)$. So, $X_t(\cdot)$ is a random variable, and thus an infinite-dimensional object, whilst $X_t(\omega)$ is a finite-dimensional vector. We feel that this writing is sufficiently explicit to remove the hat over F.

Our main well-posedness result is stated below, in a preliminary form only. The precise statement requires additional ingredients that we introduce later on in the text. In this first formulation

- the quantity $w(\cdot,\cdot) = \big(w(s,t)\big)_{0 \leqslant s < t}$ is a random control function that is used to quantify the regularity of the solution path on subintervals $[s,t]$ of a given finite interval $[0,T]$, using some associated notion of p-variation for the same p as above,
- the quantity $N([0,T], \alpha)$ is some local accumulated variation of the 'rough lift' of W that counts the increments of w of size α over a bounded interval $[0,T]$ for a given positive α;

see Section 1 for the set-up. The regularity assumptions on the diffusivity F are spelled-out in Section 3.

1. Theorem – *Let* F *satisfy the regularity assumptions* **Assumption 1** *and* **Assumption 2**. *Assume there exists a positive time horizon T such that the random variables $w(0,T)$ and $\big(N\big((0,T), \alpha\big)\big)_{\alpha > 0}$ have 'sub' and super exponential tails, respectively,*

- $\mathbb{P}\big(w(0,T) \geqslant t\big) \leqslant c_1 \exp\big(-t^{\varepsilon_1}\big)$,
- $\mathbb{P}\big(N\big([0,T], \alpha\big) \geqslant t\big) \leqslant c_2(\alpha) \exp\big(-t^{1+\varepsilon_2(\alpha)}\big), \quad \alpha > 0,$

for some positive constants c_1 and ε_1 and possibly α-dependent positive constants $c_2(\alpha)$ and $\varepsilon_2(\alpha)$. Then for any d-dimensional square-integrable random variable X_0, the mean field rough differential equation

$$dX_t = \mathrm{F}\big(X_t, \mathcal{L}(X_t)\big)\, dW_t$$

has a unique solution defined on the whole interval $[0,T]$.

Results of that form seem out of reach of the methods used in [12, 3]. Theorem 1 applies in particular to mean field rough differential equations driven by some fractional Brownian motion with Hurst parameter greater than $\frac{1}{3}$, other Gaussian processes or some Markovian rough paths; see Section 1. It is important that the solution depends continuously on the driving 'rough path', in a quantitative sense detailed in Theorem 20. As an example that fits our regularity assumptions, one can solve the above mean field rough differential equation with

$$\mathrm{F}(x, \mu) = \int f(x,y)\mu(dy)$$

for some fuction f of class C_b^2 (meaning that f is bounded and has bounded derivatives of order 1 and 2), or with

$$\mathrm{F}(x, \mu) = g\left(x, \int_{\mathbb{R}^d} y\mu(dy)\right)$$

for some function g of class C_b^2. The Curie-Weiss model, where F is of the form $\mathrm{F}(x,\mu) = \nabla U(x) + \int (x-y)\mu(dy)$, falls outside the scope of what is written here, because of the linear growth rate in x, but is within reach of our method.

One of the difficulties in solving equation (0.2) comes from the fact that it happens not to be sufficient to consider each signal $W_\bullet(\omega)$ as the first level of a rough path; one somehow needs to consider the whole family $\big(W_\bullet(\omega)\big)_{\omega\in\Omega}$ as an infinite-dimensional rough path. This leads us to defining in Section 1 a rough setting where $\big(W_t(\omega), W_t(\cdot)\big)_{0\leqslant t\leqslant T}$ is, for each ω, the first level of a rough path over $\mathbb{R}^m \times \mathbb{L}^q\big(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^m\big)$; seemingly, the natural choice for q, as dictated by the aforementioned lifting procedure of the Wasserstein space, is $q=2$; we shall actually need a larger value. Unlike the seminal works [12, 3] that set the scene in Davie's approach of rough differential equations, such as reshaped by Friz-Victoir and Bailleul respectively, we use here Gubinelli's versatile approach of controlled paths to make sense of equation (0.2). Our mixed finite/infinite dimensional setting introduces an interesting twist in the notion of controlled path presented in Section 2.1. Defining the rough integral of a controlled path with respect to a rough driver is done classically in Section 2.2 using the sewing lemma. We prove stability of a certain class of controlled paths by nonlinear mappings in Section 3.1, which is precisely the place where Lions' differential calculus on $\mathcal{P}_2(\mathbb{R}^d)$ comes in. One then has all the ingredients needed to formulate in Section 3 equation (0.2) as a fixed point problem in some space of controlled paths. Local well-posedness is proved, and sufficient conditions on the law of the driver are given to get well-posedness on any fixed time interval. As expected from any solution theory for rough differential equations, the solution depends continuously on all the parameters in the equation, most notably its law depends continuously on the law of the driving rough path. This point is used in Section 4 to provide a proof of propagation of chaos for an interacting particle system associated with equation (0.2) and quantify the convergence rate; see equation (4.1) for the particle system. Among others, it recovers Sznitman' seminal work [38] on the case where the noise is a Brownian motion. We formulate this result here for the case of Gaussian rough signals and refer the reader to Theorem 22 and Theorem 24 for finer and more general statements.

2. Theorem – *Let W be a continuous centered Gaussian process defined over some time interval $[0,T]$. Assume it has independent components and its covariance function has finite ρ-two dimensional variation, for some $\rho\in[1,3/2)$. Let the diffusivity* F *satisfy* **Assumption 1** *and* **Assumption 2** *and some further mild regularity assumptions satisfied by the above two examples. Then the empirical measure $\frac{1}{n}\sum_{i=1}^n \delta_{X^{i,(n)}(\omega)}$ of the interacting n-particle system associated with the mean field rough differential equation (0.2), converges almost surely to $\mathcal{L}\big(X(\cdot)\big)$. The marginals of the empirical measure converge at the same mean speed in 1-Wasserstein distance as an empirical sample of independent, identically distributed, random variables with the same law as X_0, provided the latter is sufficiently integrable.*

While Lyons formulated his theory in a Banach setting from the begining [35], the theory has mainly been explored for finite dimensional drivers, with the noticeable exception of the works of Ledoux, Lyons and Qian on Banach space valued rough paths [33, 36], Dereich follow-up works [19, 20], Kelly and Melbourne application to homogenization of fast/slow systems of ordinary differential equations [31], and Bailleul and Riedel's work on rough flows [2]. One can see the present work as another illustration of the strength of the theory in its full generality. However, although the underlying rough set-up associated to $(W_t(\omega), W_t(\cdot))_{0\leqslant t\leqslant T}$ is a mixed finite/infinite dimensional object, a solution to the mean field rough differential equation is more than a solution to a rough differential equation driven by an infinite dimensional rough path. Indeed, the mean field structure imposes an additional fixed point condition, which is to identify the finite dimensional component of the solution as the ω-realization of the infinite dimensional component. This is precisely this constraint that makes the equation difficult to solve and that explains the need for a specific analysis.

The present work leaves wide open the question of refining the strong law of large numbers given by the propagation of chaos result stated in Theorem 2 – Theorem 22 in its full form. A central limit theorem for the fluctuations of the empirical measure of the particle system is expected to hold under reasonable conditions on the common law of the rough drivers. Large and moderate deviation results would also be most welcome. In a different direction, it would be interesting to investigate the propagation of chaos phenomenon for systems of interacting rough dynamics subject to a common noise. Very interesting things happen in the Itô setting in relation with mean field games [8, 32]. Also, one would get a more realistic model of natural phenomena by working with systems of particles driven by non-independent noises. Individuals with close initial conditions could have drivers strongly correlated while individuals started far apart could have (almost-)independent drivers. Limit mean field dynamics are likely to be different from the results obtained here – see [14] for a result in this direction in the Itô setting. We invite the reader to make his own mind about these problems.

Notations. We gather here a number of notations that will be used throughout the text.

• We denote by $\mathcal{S}_2$ the simplex $\big\{(s,t)\in[0,\infty)^2 : s\leqslant t\big\}$, and set

$$\mathcal{S}_2^T := \big\{(s,t)\in[0,T]^2 : s\leqslant t\big\}.$$

• We denote by $(\Omega,\mathcal{F},\mathbb{P})$ an atomless Polish probability space, $\mathcal{F}$ standing for the completion of the Borel σ-field under $\mathbb{P}$, and denote by $\langle\cdot\rangle$ the expectation operator and by $\langle\cdot\rangle_r$, for $r\in[1,+\infty]$, the $\mathbb{L}^r$-norm on $(\Omega,\mathcal{F},\mathbb{P})$ and by $\langle\!\langle\cdot\rangle\!\rangle$ and $\langle\!\langle\cdot\rangle\!\rangle_r$ the expectation operator and the $\mathbb{L}^r$-norm on

$$\big(\Omega^2,\mathcal{F}^{\otimes 2},\mathbb{P}^{\otimes 2}\big).$$

Importantly, when r is finite, $\mathbb{L}^r(\Omega,\mathcal{F},\mathbb{P};\mathbb{R})$ is separable as Ω is Polish.

• When dealing with processes $X_\bullet = (X_t)_{t\in I}$, defined on some time interval I, we often write X for $X_\bullet$.

1 – Probabilistic Rough Structure

We define in this section a notion of rough path appropriate for the study of mean field rough differential equations. It happens to be a mixed finite/infinite dimensional object. Throughout the section, we work on a finite time horizon $[0,T]$, for a given $T>0$.

• We define the first level of our rough path structure as an ω-indexed pair of paths

$$\big(W_t(\omega), W_t(\cdot)\big)_{0\leqslant t\leqslant T}, \tag{1.1}$$

where $\big(W_t(\cdot)\big)_{0 \leqslant t \leqslant T}$ is a collection of q-integrable $\mathbb{R}^m$-valued random variables on $(\Omega, \mathcal{F}, \mathbb{P})$, which we regard as a deterministic $\mathbb{L}^q(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^m)$-valued path, for some exponent $q \geqslant 1$, and $\big(W_t(\omega)\big)_{0 \leqslant t \leqslant T}$ stands for the realizations of these random variables along the outcome $\omega \in \Omega$; so the pair (1.1) takes values in $\mathbb{R}^m \times \mathbb{L}^q(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^m)$. As we already explained, a natural choice would be to take $q = 2$, but for technical reasons that will get clear below we shall require $q \geqslant 8$.

The second level of the rough path structure contains a two-parameter path $\big(\mathbb{W}_{s,t}(\omega)\big)_{0 \leqslant s \leqslant t \leqslant T}$ with values in $\mathbb{R}^{m \times m}$, obtained as the ω-realizations of a collection of q-integrable random variables $\big(\mathbb{W}_{s,t}(\cdot)\big)_{0 \leqslant s \leqslant t \leqslant T}$ defined on Ω; importantly, this second level also comprises the sections $\big(\mathbb{W}^{\perp\!\!\!\perp}_{s,t}(\omega, \cdot)\big)_{0 \leqslant s \leqslant t \leqslant T}$ and $\big(\mathbb{W}^{\perp\!\!\!\perp}_{s,t}(\cdot, \omega)\big)_{0 \leqslant s \leqslant t \leqslant T}$ of a collection of $\mathbb{R}^{m \times m}$-valued random variables defined on the product space $\big(\Omega^2, \mathcal{F}^{\otimes 2}, \mathbb{P}^{\otimes 2}\big)$ and considered as a deterministic $\mathbb{L}^q\big(\Omega^2, \mathcal{F}^{\otimes 2}, \mathbb{P}^{\otimes 2}; \mathbb{R}^{m \times m}\big)$-valued path $\big(\mathbb{W}^{\perp\!\!\!\perp}_{s,t}(\cdot, \cdot)\big)_{0 \leqslant s \leqslant t \leqslant T}$. Each $\mathbb{W}^{\perp\!\!\!\perp}_{s,t}(\cdot, \cdot)$, for $(s,t) \in \mathcal{S}_2^T$, belonging to the space $\mathbb{L}^q\big(\Omega^2, \mathcal{F}^{\otimes 2}, \mathbb{P}^{\otimes 2}; \mathbb{R}^{m \times m}\big)$, we have

$$\big\langle \mathbb{W}^{\perp\!\!\!\perp}_{s,t}(\omega, \cdot) \big\rangle_q < \infty, \quad \big\langle \mathbb{W}^{\perp\!\!\!\perp}_{s,t}(\cdot, \omega) \big\rangle_q < \infty, \tag{1.2}$$

for $\mathbb{P}$-almost every $\omega \in \Omega$. Below, we shall assume (1.2) to be true for every $\omega \in \Omega$. This is not such a hindrance since we can modify in a quite systematic way the definition of the rough path structure on the null event where (1.2) fails; this is exemplified in Proposition 4 below. Taken this assumption for granted, we can regard $\Omega \ni \omega \mapsto \mathbb{W}^{\perp\!\!\!\perp}_{s,t}(\omega, \cdot)$ and $\Omega \ni \omega \mapsto \mathbb{W}^{\perp\!\!\!\perp}_{s,t}(\cdot, \omega)$ as random variables with values in $\mathbb{L}^q(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^{m \times m})$: Since $\mathbb{L}^q(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^{m \times m})$ is separable, it suffices to notice from Fubini's theorem that, for any $Z \in \mathbb{L}^q(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^{m \times m})$, $\Omega \ni \omega \mapsto \big\langle \mathbb{W}^{\perp\!\!\!\perp}_{s,t}(\omega, \cdot) - Z \big\rangle_q$ is measurable, and similarly for $\mathbb{W}^{\perp\!\!\!\perp}_{s,t}(\cdot, \omega)$.

Hence, the entire second level has the form of an ω-dependent two-index path with values in $\big(\mathbb{R}^m \times \mathbb{L}^q(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^m)\big)^{\otimes 2}$ and is encoded in matrix form as

$$\begin{pmatrix} \mathbb{W}_{s,t}(\omega) & \mathbb{W}^{\perp\!\!\!\perp}_{s,t}(\omega, \cdot) \\ \mathbb{W}^{\perp\!\!\!\perp}_{s,t}(\cdot, \omega) & \mathbb{W}^{\perp\!\!\!\perp}_{s,t}(\cdot, \cdot) \end{pmatrix}_{0 \leqslant s \leqslant t \leqslant T}. \tag{1.3}$$

Here,

- $\mathbb{W}_{s,t}(\omega)$ is in $(\mathbb{R}^m)^{\otimes 2} \simeq \mathbb{R}^{m \times m}$,
- $\mathbb{W}^{\perp\!\!\!\perp}_{s,t}(\omega, \cdot)$ is in $\mathbb{R}^m \otimes \mathbb{L}^q\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^m\big) \simeq \mathbb{L}^q\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^{m \times m}\big)$,
- $\mathbb{W}^{\perp\!\!\!\perp}_{s,t}(\cdot, \omega)$ is in $\mathbb{L}^q\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^m\big) \otimes \mathbb{R}^m \simeq \mathbb{L}^q\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^{m \times m}\big)$,
- $\mathbb{W}^{\perp\!\!\!\perp}_{s,t}(\cdot, \cdot)$ is in $\mathbb{L}^q\big(\Omega^{\otimes 2}, \mathcal{F}^{\otimes 2}, \mathbb{P}^{\otimes 2}; \mathbb{R}^{m \times m}\big)$, the realizations of which read in the form $\Omega^2 \ni (\omega, \omega') \mapsto \mathbb{W}^{\perp\!\!\!\perp}_{s,t}(\omega, \omega') \in \mathbb{R}^{m \times m}$ and the two sections of which are precisely given by $\mathbb{W}^{\perp\!\!\!\perp}_{s,t}(\omega, \cdot) : \Omega \ni \omega' \mapsto \mathbb{W}^{\perp\!\!\!\perp}_{s,t}(\omega, \omega')$, and $\mathbb{W}^{\perp\!\!\!\perp}_{s,t}(\cdot, \omega) \ni \omega' \mapsto \mathbb{W}^{\perp\!\!\!\perp}_{s,t}(\omega', \omega)$, for $\omega \in \Omega$.

As usual with rough paths, algebraic consistency requires that Chen's relations

$$\begin{aligned} \mathbb{W}_{r,t}(\omega) &= \mathbb{W}_{r,s}(\omega) + \mathbb{W}_{s,t}(\omega) + W_{r,s}(\omega) \otimes W_{s,t}(\omega), \\ \mathbb{W}^{\perp\!\!\!\perp}_{r,t}(\cdot, \omega) &= \mathbb{W}^{\perp\!\!\!\perp}_{r,s}(\cdot, \omega) + \mathbb{W}^{\perp\!\!\!\perp}_{s,t}(\cdot, \omega) + W_{r,s}(\cdot) \otimes W_{s,t}(\omega), \\ \mathbb{W}^{\perp\!\!\!\perp}_{r,t}(\omega, \cdot) &= \mathbb{W}^{\perp\!\!\!\perp}_{r,s}(\omega, \cdot) + \mathbb{W}^{\perp\!\!\!\perp}_{s,t}(\omega, \cdot) + W_{r,s}(\omega) \otimes W_{s,t}(\cdot), \\ \mathbb{W}^{\perp\!\!\!\perp}_{r,t}(\cdot, \cdot) &= \mathbb{W}^{\perp\!\!\!\perp}_{r,s}(\cdot, \cdot) + \mathbb{W}^{\perp\!\!\!\perp}_{s,t}(\cdot, \cdot) + W_{r,s}(\cdot) \otimes W_{s,t}(\cdot), \end{aligned} \tag{1.4}$$

hold for any $0 \leqslant r \leqslant s \leqslant t \leqslant T$. We used here the very convenient notation

$$f_{r,s} := f_s - f_r,$$

for a function f from $[0,\infty)$ into a vector space. In (1.4) and throughout, we denote by $X(\cdot)\otimes Y(\cdot)$, for any two X and Y in $\mathbb{L}^q(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^m)$, the random variable

$$\big(\omega,\omega'\big)\mapsto\big(X_i(\omega)Y_j(\omega')\big)_{1\leqslant i,j\leqslant m}$$

defined on the product space Ω^2. It defines an element of $\mathbb{L}^q\big(\Omega^2,\mathcal{F}^{\otimes 2},\mathbb{P}^{\otimes 2};\mathbb{R}^{m\times m}\big)$.

Remark – *The last three lines in Chen's relations* (1.4) *are somewhat redundant. Assume indeed that we are given a collection of random variables* $\big(\mathbb{W}^{\perp\!\!\!\perp}_{s,t}(\cdot,\cdot)\big)_{0\leqslant s\leqslant t\leqslant T}$ *satisfying the last line of* (1.4)*. Then, for all* $0\leqslant r\leqslant s\leqslant t\leqslant T$ *and for* $\mathbb{P}^{\otimes 2}$*-almost every* $(\omega,\omega')\in\Omega^2$*, it holds*

$$\mathbb{W}^{\perp\!\!\!\perp}_{r,t}(\omega,\omega')=\mathbb{W}^{\perp\!\!\!\perp}_{r,s}(\omega,\omega')+\mathbb{W}^{\perp\!\!\!\perp}_{s,t}(\omega,\omega')+W_{r,s}(\omega)\otimes W_{s,t}(\omega').$$

Clearly, for $\mathbb{P}$*-almost every* $\omega\in\Omega$*, the second and third lines in* (1.4) *hold true as well. This is slightly weaker than the formulation* (1.4) *as, therein, the second and third lines are required to hold for all* $\omega\in\Omega$*. As exemplified in the proof of Proposition 4, one may modify the definition of* $W^{\perp\!\!\!\perp}$ *so that the second and third lines in* (1.4) *hold true for all* ω *and for all* $0\leqslant r\leqslant s\leqslant t\leqslant T$*.*

Definition – *We shall denote by* $\boldsymbol{W}(\omega)$ *the* **rough set-up** *specified by the* ω*-dependent collection of maps given by* (1.1) *and* (1.3)*.*

As for the component $\mathbb{W}^{\perp\!\!\!\perp}$ of $\boldsymbol{W}(\omega)$, the notation $\perp\!\!\!\perp$ is used to indicate, as we shall make it clear below, that $\mathbb{W}^{\perp\!\!\!\perp}_{s,t}(\cdot,\cdot)$ should be thought of as the random variable

$$(\omega,\omega')\mapsto\int_s^t\Big(W_r(\omega)-W_s(\omega)\Big)\otimes dW_r(\omega').$$

Since $\Omega^2\ni(\omega,\omega')\mapsto(W_t(\omega))_{t\geqslant 0}$ and $\Omega^2\ni(\omega,\omega')\mapsto(W_t(\omega'))_{t\geqslant 0}$ are independent under $\mathbb{P}^{\otimes 2}$, we then understand $\mathbb{W}^{\perp\!\!\!\perp}_{s,t}$ as an iterated integral for two independent copies of the noise. While such a construction is elementary for a random C^1 path, the well-defined character of this integral needs to be proved for more general probability measures $\mathbb{P}$.

3. Example – *Let* W *stand for an* $\mathbb{R}^m$*-valued Brownian motion defined on some probability space* $(\Omega,\mathcal{F},\mathbb{P})$*. Denote by* $W_t(\cdot)$ *the equivalence class of* $\Omega\ni\omega\mapsto W_t(\omega)$ *in* $\mathbb{L}^q\big(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^m\big)$*, and extend* W_t *on the product space* $\big(\Omega^2,\mathcal{F}^{\otimes 2},\mathbb{P}^{\otimes 2}\big)$*, setting* $W_t(\omega,\omega'):=W_t(\omega)$*. Define also on the product space the random variable* $W'_t(\omega,\omega'):=W_t(\omega')$*. Then,* W *and* W' *are two independent* m*-dimensional Brownian motions under* $\mathbb{P}^{\otimes 2}$*, and one can construct the time-indexed Stratonovich stochastic integral*

$$\Omega^2\ni(\omega,\omega')\mapsto\left(\left\{\int_s^t(W_r-W_s)\otimes\circ dW'_r\right\}(\omega,\omega')\right)_{0\leqslant s\leqslant t\leqslant T}\in\mathcal{C}\big(\mathcal{S}_2;\mathbb{R}^{m\times m}\big).$$

The stochastic integral is uniquely defined up to an event of zero measure under $\mathbb{P}^{\otimes 2}$*. Up to an exceptional event (of* $(\Omega^2,\mathcal{F}^{\otimes 2},\mathbb{P}^{\otimes 2})$*), we then let*

$$\mathbb{W}^{\perp\!\!\!\perp}_{s,t}(\omega,\omega'):=\left(\int_s^t\big(W_r-W_s\big)\otimes\circ dW'_r\right)(\omega,\omega'),\quad 0\leqslant s\leqslant t\leqslant T.$$

We can specify the definition of $W^{\perp\!\!\!\perp}$ *on the remaining exceptional event and then modify the definition of* W *on a null event of* $(\Omega,\mathcal{F},\mathbb{P})$ *in such a way that Chen's relations* (1.4) *hold everywhere –see the end of the proof of Proposition 4 below for a* *detailed proof of this fact–. The process* $\big(\mathbb{W}_{s,t}(\omega)\big)_{0\leqslant s\leqslant t}$ *is defined in a standard way from a Stratonovich integral defined outside a set of null measure:*

$$\mathbb{W}_{s,t}(\omega) := \left(\int_s^t (W_r - W_s) \otimes \circ dW_r\right)(\omega), \quad 0 \leqslant s \leqslant t \leqslant T.$$

The principle underpinning the above example may be put in a more general framework which will be useful to prove continuity of the Itô-Lyons solution map to the mean field rough differential equation (0.2). We advise the reader to come back to this proposition later on.

4. Proposition – *Let* $(\Xi, \mathcal{G}, \mathbb{Q})$ *be a probability space, and* $W^1 := \big(W^1_t\big)_{0\leqslant t\leqslant T}$ *and* $W^2 := \big(W^2_t\big)_{0\leqslant t\leqslant T}$ *be two independent and identically distributed* $\mathbb{R}^m$*-valued processes defined on* Ξ*. Assume they have continuous trajectories and*

$$\mathbb{E}_{\mathbb{Q}}\left[\sup_{0\leqslant t\leqslant T} |W^1_t|^q\right] < \infty.$$

Let also $\big((W^{i,j}_{s,t})_{0\leqslant s<t\leqslant T}\big)_{i,j=1,2}$ *be four* $\mathbb{R}^m \otimes \mathbb{R}^m \cong \mathbb{R}^{m\times m}$*-valued continuous paths such that*

$$\mathbb{E}_{\mathbb{Q}}\left[\sup_{0\leqslant s<t\leqslant T} |W^{i,j}_{s,t}|^q\right] < \infty,$$

for $i,j = 1,2$*, and* $\big(W^1, W^{1,1}\big)$ *is independent of* W^2*. Last, assume that, for almost every* $\xi \in \Xi$*, the pair*

$$\left(\begin{pmatrix} W^1(\xi) \\ W^2(\xi) \end{pmatrix}, \begin{pmatrix} W^{1,1}(\xi) & W^{1,2}(\xi) \\ W^{2,1}(\xi) & W^{2,2}(\xi) \end{pmatrix}\right)$$

satisfies Chen's relation. Set

$$\Omega := \Xi \times [0,1]$$

with $[0,1]$ *equipped with its Borel* σ*-algebra* $\mathcal{B}\big([0,1]\big)$*, and denote by* Leb *the Lebesgue measure on* $[0,1]$*. Then we can find a triple of random variables* $\big(W, \mathbb{W}, \mathbb{W}^{\perp\!\!\!\perp}\big)$*, the first two components being defined on* $\big(\Omega, \mathcal{F}\otimes\mathcal{B}([0,1]), \mathbb{Q}\otimes\mathrm{Leb}\big)$*, the last component being constructed on the product space, and the whole family satisfying all the above requirements for a rough set-up, such that*

$$\mathbb{P}\Big(\Big\{(\xi,u) : \big(W,\mathbb{W}\big)(\xi,u) = \big(W^1, W^{1,1}\big)(\xi)\Big\}\Big) = 1,$$

and, for $\mathbb{P}$*-almost every* $\omega = (\xi,u)$*, the law of* $W^{\perp\!\!\!\perp}(\cdot,\omega)$ *is the same as the conditional law of* $W^{2,1}$ *given* $\big(W^1(\xi), W^2(\xi), W^{1,1}(\xi)\big)$*.*

Proof – Recall first from Blackwell and Dubins [4] the following form of Skorokhod representation theorem. *There exists a function*

$$\Psi : [0,1] \times \mathcal{P}\Big(\mathcal{C}\big(\mathcal{S}_2^T; \mathbb{R}^m \otimes \mathbb{R}^m\big)\Big) \to \mathcal{C}\big(\mathcal{S}_2^T; \mathbb{R}^m \otimes \mathbb{R}^m\big)$$

such that

- *for every probability* μ *on* $\mathcal{C}(\mathcal{S}_2^T)$*, equipped with its Borel* σ*-field,* $[0,1] \ni u \mapsto \Psi(u,\mu)$ *is a random variable with* μ *as distribution –* $[0,1]$ *being equipped with Lebesgue measure,*
- *the map* Ψ *is measurable.*

Let now $\big(q(w^1,w^2,w^{1,1},\cdot)\big)_{w^1,w^2\in\mathcal{C}([0,T];\mathbb{R}^m);w^{1,1}\in\mathcal{C}(\mathcal{S}_2^T;\mathbb{R}^m\otimes\mathbb{R}^m)}$ be a regular conditional probability of $W^{2,1}$ given $(W^1,W^2,W^{1,1})$. Define on Ω the random variables

$$W(\xi,u) := W^1(\xi), \quad \mathbb{W}(\xi,u) := W^{1,1}(\xi),$$

and, on Ω^2,

$$\begin{aligned}
&W'\big((\xi,u),(\xi',u')\big) := W^1(\xi'),\\
&\mathbb{W}^{\perp\!\!\!\perp}\big((\xi,u),(\xi',u')\big) := \Psi\Big(u', q\big(W^1(\xi'),W^1(\xi),W^{1,1}(\xi'),\cdot\big)\Big).
\end{aligned}$$

Since the law of $\big(W,W',\mathbb{W}\big)$ under $\mathbb{P}^{\otimes 2}$ is the same as the law of $\big(W^1,W^2,W^{1,1}\big)$ under $\mathbb{Q}$, we deduce that the law of $\big(W,W',\mathbb{W},\mathbb{W}^{\top}\big)$ under $\mathbb{P}^{\otimes 2}$, with $\mathbb{W}^{\top}(\omega,\omega') := \mathbb{W}^{\perp\!\!\!\perp}(\omega',\omega)$, is the same as the law of $\big(W^1,W^2,W^{1,1},W^{2,1}\big)$ under $\mathbb{Q}$. In particular, with probability 1 under $\mathbb{P}^{\otimes 2}$, for all $0\leqslant r\leqslant s\leqslant t\leqslant T$,

$$\mathbb{W}^{\top}_{r,t}(\omega,\omega') = \mathbb{W}^{\top}_{r,s}(\omega,\omega') + \mathbb{W}^{\top}_{s,t}(\omega,\omega') + W_{r,s}(\omega')\otimes W_{s,t}(\omega),$$

that is

$$\mathbb{W}^{\perp\!\!\!\perp}_{r,t}(\omega,\omega') = \mathbb{W}^{\perp\!\!\!\perp}_{r,s}(\omega,\omega') + \mathbb{W}^{\perp\!\!\!\perp}_{s,t}(\omega,\omega') + W_{r,s}(\omega)\otimes W_{s,t}(\omega').$$

Call now $A\in\mathcal{F}$ the set of those ω's in Ω for which the above relation fails for ω' in a set of positive probability measure under $\mathbb{P}$. Clearly, $\mathbb{P}(A)=0$. Define in a similar way A' by exchanging the roles of ω and ω'. For $\omega\in A\cup A'$, set $W(\omega)\equiv 0$; and whenever $\omega\in A$ or $\omega'\in A'$, set $\mathbb{W}^{\perp\!\!\!\perp}(\omega,\omega')\equiv 0$. If $\omega\notin A$, we have, by definition of A, the third identity in (1.4) – pay attention that we use the fact that the identity is understood as an equality between classes of random variables that are $\mathbb{P}$-almost surely equal. If $\omega\in A$, it is also true since all the terms are zero. The second identity in (1.4) is checked in the same way. As for the first one, it holds on the complementary $B^{\complement}$ of a null event B. We then replace A by $A\cup B$ and A' by $A'\cup B$ in the previous lines and set $W(\omega)\equiv 0$ and $\mathbb{W}(\omega)\equiv 0$ on $A\cup A'\cup B$. $\rhd$

We use in this work the notion of p-variation to handle the regularity of the various trajectories in hand. The choice of the p-variation, instead of the simplest Hölder (semi-)norm, is dictated by the arguments we use below to prove well-posedness of equation (0.4). As we make it clear in the text, we shall indeed invoke some integrability results due to Cass, Litterer and Lyons [11] which are explicitly based upon the notion of p-variation and are not proved in Hölder (semi-)norm. Several types of p-variations are needed to handle differently the finite and infinite dimensional components of a rough set-up $\boldsymbol{W}$. Throughout, the exponent p is taken in the interval $[2,3)$. For a continuous function $\mathbb{G}$ from the simplex $\mathcal{S}_2^T$ into some $\mathbb{R}^\ell$, we set, for any $p'\geqslant 1$,

$$\|\mathbb{G}\|^{p'}_{[0,T],p'-\mathrm{var}} := \sup_{0=t_0<t_1\cdots<t_n=T}\sum_{i=1}^n |\mathbb{G}_{t_{i-1},t_i}|^{p'},$$

and define for any function g from $[0,T]$ into $\mathbb{R}^\ell$,

$$\|g\|^p_{[0,T],p-\mathrm{var}} := \|\mathbb{G}\|^p_{[0,T],p-\mathrm{var}}$$

as the p-variation semi-norm of its associated two index function $\mathbb{G}_{s,t} := g_t - g_s$. Similarly, for a random variable $\mathbb{G}(\cdot)$ on Ω with values in $\mathcal{C}(\mathcal{S}_2^T;\mathbb{R}^\ell)$, and $p'\geqslant 1$, we define its p'-variation in $\mathbb{L}^q$ as

$$\langle\mathbb{G}(\cdot)\rangle^{p'}_{q;[0,T],p'-\mathrm{var}} := \sup_{0=t_0<t_1\cdots<t_n=T}\sum_{i=1}^n \big\langle\mathbb{G}_{t_{i-1},t_i}(\cdot)\big\rangle^{p'}_q, \tag{1.5}$$

and define for a random variable $G(\cdot)$ on Ω, with values in $\mathcal{C}([0,T];\mathbb{R}^\ell)$

$$\big\langle G(\cdot)\big\rangle^p_{q;[0,T],p-\mathrm{var}} := \big\langle \mathbb{G}(\cdot)\big\rangle^p_{q;[0,T],p-\mathrm{var}},$$

as the p-variation semi-norm in $\mathbb{L}^q$ of its associated two-index function $\mathcal{S}_2^T \ni (s,t) \mapsto \mathbb{G}_{s,t}(\cdot) = G_t(\cdot) - G_s(\cdot)$. Last, for a random variable $\mathbb{G}(\cdot,\cdot)$ from $(\Omega^2, \mathcal{F}^{\otimes 2})$ into $\mathcal{C}(\mathcal{S}_2^T;\mathbb{R}^\ell)$, we set

$$\langle\!\langle \mathbb{G}(\cdot,\cdot)\rangle\!\rangle^{p/2}_{q;[0,T],p/2-\mathrm{var}} := \sup_{0=t_0<t_1\cdots<t_n=T} \sum_{i=1}^n \big\langle\!\big\langle \mathbb{G}_{t_{i-1},t_i}(\cdot,\cdot)\big\rangle\!\big\rangle^{p/2}_q . \tag{1.6}$$

Given these definitions, we require from the rough set-up $\boldsymbol{W}$ that

- For any $\omega \in \Omega$, the path $W(\omega)$ is in the space $\mathcal{C}([0,T];\mathbb{R}^m)$, and the map $W : \Omega \ni \omega \mapsto W(\omega) \in \mathcal{C}([0,T];\mathbb{R}^m)$ is Borel-measurable and q-integrable (meaning that the supremum of W over $[0,T]$ is q-integrable).
- For any $\omega \in \Omega$, the two-index path $\mathbb{W}(\omega)$ is in $\mathcal{C}(\mathcal{S}_2^T;\mathbb{R}^{m\times m})$, and the map $\mathbb{W} : \Omega \ni \omega \mapsto \mathbb{W}(\omega) \in \mathcal{C}(\mathcal{S}_2^T;\mathbb{R}^{m\times m})$ is Borel-measurable and q-integrable (i.e., the supremum of $\mathbb{W}$ over $\mathcal{S}_2^T$ has a finite q-moment).
- For any $(\omega,\omega') \in \Omega^2$, the two-index path $\mathbb{W}^{\perp\!\!\!\perp}(\omega,\omega')$ is an element of the space $\mathcal{C}(\mathcal{S}_2^T;\mathbb{R}^{m\times m})$, and the map $\mathbb{W}^{\perp\!\!\!\perp} : \Omega^2 \ni (\omega,\omega') \mapsto \mathbb{W}^{\perp\!\!\!\perp}(\omega,\omega') \in \mathcal{C}(\mathcal{S}_2^T;\mathbb{R}^{m\times m})$ is Borel-measurable and q-integrable. In particular, for almost every $\omega \in \Omega$, the two-time parameter path $\mathbb{W}^{\perp\!\!\!\perp}(\omega,\cdot)$ is in $\mathcal{C}\big(\mathcal{S}_2^T;\mathbb{L}^q(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^{m\times m})\big)$, and the map $\Omega \ni \omega \mapsto \mathbb{W}^{\perp\!\!\!\perp}(\omega,\cdot)$ is Borel-measurable and q-integrable, and similarly for $\mathbb{W}^{\perp\!\!\!\perp}(\cdot,\omega)$; as before, we assume the latter to be true for every $\omega \in \Omega$. Also, the two-time parameter deterministic path $\mathbb{W}^{\perp\!\!\!\perp}(\cdot,\cdot)$ is a continuous mapping from $\mathcal{S}_2^T$ into $\mathbb{L}^q\big(\Omega^2,\mathcal{F}^{\otimes 2},\mathbb{P}^{\otimes 2};\mathbb{R}^{m\times m}\big)$.

We then set, for all $0 \leqslant s \leqslant t \leqslant T$ and $\omega \in \Omega$,

$$\begin{aligned} v(s,t,\omega) := \big\|W(\omega)\big\|^p_{[s,t],p-\mathrm{var}} &+ \big\langle W(\cdot)\big\rangle^p_{q;[s,t],p-\mathrm{var}} \\ &+ \big\|\mathbb{W}(\omega)\big\|^{p/2}_{[s,t],p/2-\mathrm{var}} + \big\langle \mathbb{W}^{\perp\!\!\!\perp}(\omega,\cdot)\big\rangle^{p/2}_{q;[s,t],p/2-\mathrm{var}} \\ &+ \big\langle \mathbb{W}^{\perp\!\!\!\perp}(\cdot,\omega)\big\rangle^{p/2}_{q;[s,t],p/2-\mathrm{var}} + \big\langle\!\big\langle \mathbb{W}^{\perp\!\!\!\perp}(\cdot,\cdot)\big\rangle\!\big\rangle^{p/2}_{q;[s,t],p/2-\mathrm{var}}, \end{aligned} \tag{1.7}$$

and we assume that, for any positive finite time T and any $\omega \in \Omega$, the quantity $v(0,T,\omega)$ is finite. Importantly, we have the following super-additivity property. For any $0 \leqslant r \leqslant s \leqslant t \leqslant T$, and $\omega \in \Omega$, we have

$$v(r,t,\omega) \geqslant v(r,s,\omega) + v(s,t,\omega).$$

Observe also from [27, Proposition 5.8] that $\omega \mapsto (v(s,t,\omega))_{(s,t)\in\mathcal{S}_2^T}$ is a random variable with values in $\mathcal{C}(\mathcal{S}_2^T;\mathbb{R}_+)$. Throughout the analysis, we assume

$$\big\langle v(0,T,\cdot)\big\rangle_q < \infty,$$

for any rough set-up considered on the interval $[0,T]$. By Lebesgue's dominated convergence theorem, the function

$$\mathcal{S}_2^T \ni (s,t) \mapsto \big\langle v(s,t,\cdot)\big\rangle_q$$

is continuous. *We shall actually assume that it is of bounded variation on* $[0,T]$, *i.e.*,

$$\langle v(\cdot)\rangle_{q;[s,t],1-\mathrm{var}} := \sup_{0\leqslant t_1<\cdots<t_K\leqslant T} \sum_{i=1}^K \langle v(t_{i-1},t_i,\cdot)\rangle_q < \infty.$$

Below, we set

$$w(s,t,\omega) := v(s,t,\omega) + \langle v(\cdot)\rangle_{q;[s,t],1-\mathrm{var}}. \tag{1.8}$$

Note the useful inequality

$$\langle w(s,t,\cdot)\rangle_q \leqslant 2\, w(s,t,\omega), \tag{1.9}$$

and the super-additivity property satisfied by w

$$w(r,t,\omega) \geqslant w(r,s,\omega) + w(s,t,\omega).$$

Below, we often check that $\mathcal{S}_2^T \ni (s,t) \mapsto \langle v(s,t,\cdot)\rangle_q$ is of bounded variation by proving that it is Lipschitz continuous.

5. Example – Gaussian processes – *Start from an $\mathbb{R}^m$-valued collection $W := (W^1, \cdots, W^m)$ of independent and centered continuous Gaussian processes, defined on some finite time interval $[0,T]$, such that the two-dimensional covariance of W is of finite ρ-variation for some $\rho \in [1, 3/2)$ and there exists a constant K such that, for any subinterval $[s,t] \subset [0,T]$ and any $k = 1, \cdots, m$, one has*

$$\sup \sum_{i,j} \Big| \mathbb{E}\Big[\big(W^k_{t_{i+1}} - W^k_{t_i}\big)\big(W^k_{s_{j+1}} - W^k_{s_j}\big)\Big]\Big|^\rho \leqslant K|t-s|, \tag{1.10}$$

where the supremum is taken over all dissections $(t_i)_i$ and $(s_j)_j$ of the interval $[s,t]$. See Definition 5.54 in [27]. This setting includes the case of fractional Brownian motion, with Hurst index greater than $1/4$. Without any loss of generality, we may assume that the process W is constructed on the canonical space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\Omega = \mathcal{W}$, with $\mathcal{W} := \mathcal{C}([0,T]; \mathbb{R}^m)$, $\mathcal{F}$ is the Borel σ-field, and W is the coordinate process. We then denote by $(\Omega, \mathcal{H}, \mathbb{P})$ the abstract Wiener space associated with W, where $\mathcal{H}$ is a Hilbert space, which is automatically embedded in the subspace $\mathcal{C}^{\varrho-\mathrm{var}}\big([0,T]; \mathbb{R}^m\big)$ of $\mathcal{C}\big([0,T]; \mathbb{R}^m\big)$ consisting of continuous paths of finite ϱ-variation. By Theorem 15.34 in [27], we know that, for ω outside an exceptional event, the trajectory $W(\omega)$ may be lifted into a rough path $(W(\omega), \mathbb{W}(\omega))$ with finite p-variation for any $p \in (2\rho, 3)$, namely $W(\omega)$ has a finite p-variation and $\mathbb{W}(\omega)$ has a finite $p/2$-variation. We lift arbitrarily (say onto the zero path) on the null set where the lift is not automatic. The pair $(W, \mathbb{W})$, indexed by ω is part of our rough set-up. In this regard, we recall from Theorem 15.34 and Theorem 7.44 in [27] that the random variables

$$\Omega \ni \omega \mapsto \big\|W(\omega)\big\|_{[0,T],p-\mathrm{var}}, \quad \Omega \ni \omega \mapsto \big\|\mathbb{W}(\omega)\big\|_{[0,T],p/2-\mathrm{var}}, \tag{1.11}$$

have Gaussian tails, and thus have a finite $\mathbb{L}^q$-moment.

One can proceed as follows to construct the other elements

$$\big(\mathbb{W}^{\perp\!\!\!\perp}(\omega,\cdot)\big)_{\omega\in\Omega}, \quad \big(\mathbb{W}^{\perp\!\!\!\perp}(\cdot,\omega)\big)_{\omega\in\Omega}, \quad \mathbb{W}^{\perp\!\!\!\perp}(\cdot,\cdot)$$

of our rough set-up. We extend the space into $(\Omega^2, \mathcal{F}^{\otimes 2}, \mathbb{P}^{\otimes 2})$, with Ω embedded in the first component say, and denote by (W, W') the canonical coordinate process on Ω^2. They are independent and have independent Gaussian components under $\mathbb{P}^2$. The associated abstract Wiener space is nothing but $\big(\Omega^2, \mathcal{H} \oplus \mathcal{H}, \mathbb{P}^{\otimes 2}\big)$. The process (W, W') also satisfies Theorem 15.34 in [27] for the same exponent ρ as before, so, we can enhance (W, W') into a Gaussian rough path, with some arbitrary extension outside the $\mathbb{P}^{\otimes 2}$-exceptional event on which we cannot construct the enhancement. To ease the notations, we merely write $W(\omega)$ for $W(\omega,\omega')$ as it is independent of ω; similarly, we write $W'(\omega')$ for $W'(\omega,\omega')$. Proceeding as before, we call $\big(\mathbb{W}^{\perp\!\!\!\perp}(\omega,\omega')\big)_{\omega,\omega'\in\Omega}$, the upper off-diagonal $m \times m$ block in the decomposition of the second-order tensor of the rough path in the form of a $(2m) \times (2m)$-matrix with *flour blocks of size $m \times m$. Chen's relationship then yields, for $\mathbb{P}^{\otimes 2}$-almost every (ω,ω'),*

$$\mathbb{W}^{\perp\!\!\!\perp}_{r,t}(\omega,\omega') = \mathbb{W}^{\perp\!\!\!\perp}_{r,s}(\omega,\omega') + \mathbb{W}^{\perp\!\!\!\perp}_{s,t}(\omega,\omega') + W_{r,s}(\omega) \otimes W_{s,t}(\omega'),$$

for all $r \leqslant s \leqslant t$. As before, the paths of $\big(\mathbb{W}^{\perp\!\!\!\perp}(\omega,\omega')\big)_{\omega,\omega'\in\Omega}$ are almost surely of finite $p/2$-variation and the $p/2$-variation semi-norm has we know from Theorem 15.33 in [27] that the $1/p$-Hölder semi-norm of $W(\omega)$, which we denote by $\|W(\omega)\|_{[0,T],(1/p)-\text{Hölder}}$, and the $2/p$-Hölder semi-norm of $\mathbb{W}^{\perp\!\!\!\perp}(\omega,\omega')$, which we denote by $\|\mathbb{W}^{\perp\!\!\!\perp}(\omega,\omega')\|_{[0,T],(2/p)-\text{Hölder}}$, have respectively Gaussian and exponential tails, when considered as random variables on the spaces $(\Omega,\mathcal{F},\mathbb{P})$ and $\big(\Omega^2,\mathcal{F}^{\otimes 2},\mathbb{P}^{\otimes 2}\big)$. In particular, for almost every $\omega \in \Omega$, we may consider $\big(\mathbb{W}^{\perp\!\!\!\perp}_{s,t}(\omega,\cdot)\big)_{(s,t)\in\mathcal{S}_2^T}$ as a continuous process with values in $\mathbb{L}^q$. Moreover,

$$\begin{aligned}\big\langle \mathbb{W}^{\perp\!\!\!\perp}(\omega,\cdot)\big\rangle^{p/2}_{q;[0,T],p/2-var} &= \sup_{0=t_0<t_1<\cdots<t_n=T} \sum_{i=1}^n \big\langle \mathbb{W}^{\perp\!\!\!\perp}_{t_{i-1},t_i}(\omega,\cdot)\big\rangle^{p/2}_q \\ &\leqslant T\left\langle \|\mathbb{W}^{\perp\!\!\!\perp}(\omega,\cdot)\|_{[0,T],(2/p)-\text{Hölder}}\right\rangle^{p/2}_q \\ &\leqslant T\left\langle \|\mathbb{W}^{\perp\!\!\!\perp}(\omega,\cdot)\|^{p/2}_{[0,T],(2/p)-\text{Hölder}}\right\rangle_q,\end{aligned}$$

which shows that the left-hand side has finite moments of any order. Arguing in the same way for $\big(\mathbb{W}^{\perp\!\!\!\perp}(\cdot,\omega)\big)_{\omega\in\Omega}$ and for $\mathbb{W}^{\perp\!\!\!\perp}$, we deduce that v in (1.7) is almost surely finite and q-integrable. Obviously, by replacing $[0,T]$ by $[s,t] \subset [0,T]$, we obtain that the q-moment of v is Lipschitz, as required.

All these properties (that hold true on a full event) may be extended to the full set Ω^2 by arguing as in the proof of Proposition 4.

To use that rough set-up in our machinery for solving mean field rough differential equations we need a version of an integrability result of Cass, Litterer and Lyons [11] whose proof is postponed to Appendix A. Given a continuous positive valued function ϖ on $\mathcal{S}_2$, a non-negative parameter s and a positive threshold α, we define inductively a sequence of times setting $\tau_0(s,\alpha) := s$, and

$$\tau^{\varpi}_{n+1}(s,\alpha) := \inf\Big\{u \geqslant \tau^{\varpi}_n(s,\alpha) \ : \ \varpi\big(\tau^{\varpi}_n(s,\alpha),u\big) \geqslant \alpha\Big\}, \tag{1.12}$$

with the understanding that $\inf\varnothing = +\infty$. For $t \geqslant s$, set

$$N_{\varpi}\big([s,t],\alpha\big) := \sup\Big\{n \in \mathbb{N} \ : \ \tau^{\varpi}_n(s,\alpha) \leqslant t\Big\}. \tag{1.13}$$

Below, we call N_ϖ the **local accumulation of** ϖ (of size α if we specify the value of the threshold). When $\varpi(s,t) = w(s,t,\omega)$ with w as in (1.8) and when the framework makes it clear, we just write $N([s,t],\omega,\alpha)$ for $N_\varpi([s,t],\alpha)$. Similarly, we also write $\tau_n(s,\omega,\alpha)$ for $\tau^{\varpi}_n(s,\alpha)$ when $\varpi(s,t) = w(s,t,\omega)$. We will also use the convenient notation

$$\tau^{\varpi}_n(s,t,\alpha) := \tau^{\varpi}_n(s,\alpha) \wedge t.$$

The proof of the following statement is given in Appendix A.1. Recall that a positive random variable A has a Weibull tail with shape parameter $1/\varrho$ if $A^{1/\rho}$ has a Gaussian tail.

6. Theorem – *Let W be a continuous centered Gaussian process, defined over some finite interval $[0,T]$. Assume it has independent components, and denote by $(\mathcal{W},\mathcal{H},\mathbb{P})$ its associated Wiener space. Suppose that the covariance function is of finite two dimensional ϱ-variation for some $\varrho \in [1,3/2)$ and satisfies the Lipschitz estimate (1.10). Then, for $p \in (2\varrho,3)$ and $\alpha > 0$, the process $N(\cdot,\alpha) := (N([0,T],\omega,\alpha))_{\omega\in\Omega}$ associated to the rough-set up built from W has a Weibull tail with shape parameter $1/\varrho$.*

The integrability estimate on N required in Theorem 1 is satisfied in this setting. For the same value of p, the quantity $w(0,T)$ in (1.8) also satisfies the integrability statement of Theorem 1; the latter then applies in the above Gaussian setting. Building on Cass-Ogrodnik's work [13] on Markovian rough paths one can prove a similar result as Theorem 6 for Markovian rough paths.

2 – Controlled Trajectories and Rough Integral

With a rough set-up at hands on a given finite time interval $[0,T]$, one can follow Gubinelli [29] and define an associated notion of controlled path and rough integral. This section is dedicated to that task, for which we follow a now classical approach.

2.1 – Controlled Trajectories

We first define the notion of controlled trajectory for a given outcome $\omega \in \Omega$.

7. Definition – *An ω-dependent continuous $\mathbb{R}^d$-valued path $(X_t(\omega))_{0\leqslant t\leqslant T}$ is called an ω-**controlled path** on $[0,T]$ if its increments can be decomposed as*

$$X_{s,t}(\omega) = \delta_x X_s(\omega) W_{s,t}(\omega) + \mathbb{E}\big[\delta_\mu X_s(\omega,\cdot) W_{s,t}(\cdot)\big] + R^X_{s,t}(\omega), \tag{2.1}$$

where

- $\big(\delta_x X_t(\omega)\big)_{0\leqslant t\leqslant T}$ *belongs to* $\mathcal{C}\big([0,T];\mathbb{R}^{d\times m}\big)$,
- $\big(\delta_\mu X_t(\omega,\cdot)\big)_{0\leqslant t\leqslant T}$ *belongs to* $\mathcal{C}\big([0,T];\mathbb{L}^{4/3}(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^{d\times m})\big)$,
- $\big(R^X_{s,t}(\omega)\big)_{s,t\in\mathcal{S}_2^T}$ *is in* $\mathcal{C}(\mathcal{S}_2^T;\mathbb{R}^d)$,

and

$$|||X(\omega)|||_{\star,[0,T],w,p} := |X_0(\omega)| + \big|\delta_x X_0(\omega)\big| + \big\langle \delta_\mu X_0(\omega,\cdot)\big\rangle_{4/3} + |||X(\omega)|||_{[0,T],p} < \infty,$$

with

$$\begin{aligned}|||X(\omega)|||_{[0,T],w,p} := \|X(\omega)\|_{[0,T],w,p} + \|\delta_x X(\omega)\|_{[0,T],w,p} + \big\langle \delta_\mu X(\omega,\cdot)\big\rangle_{[0,T],w,p,4/3} \\ + \|R^X(\omega)\|_{[0,T],w,p/2},\end{aligned}$$

and

$$\begin{aligned}\|X(\omega)\|_{[0,T],w,p} &:= \sup_{[s,t]\subset[0,T]} \frac{\big|X_{s,t}(\omega)\big|}{w(s,t,\omega)^{1/p}},\\ \|\delta_x X(\omega)\|_{[0,T],w,p} &:= \sup_{[s,t]\subset[0,T]} \frac{\big|\delta_x X_{s,t}(\omega)\big|}{w(s,t,\omega)^{1/p}},\\ \big\langle \delta_\mu X(\omega,\cdot)\big\rangle_{[0,T],w,p,4/3} &:= \sup_{[s,t]\subset[0,T]} \frac{\big\langle \delta_\mu X_{s,t}(\omega,\cdot)\big\rangle_{4/3}}{w(s,t,\omega)^{1/p}},\end{aligned}$$

$$\|R^X(\omega)\|_{[0,T],w,p/2} := \sup_{[s,t]\subset[0,T]} \frac{\big|R^X_{s,t}(\omega)\big|}{w(s,t,\omega)^{2/p}}.$$

We call $\delta_x X(\omega)$ *and* $\delta_\mu X(\omega,\cdot)$ *in the decomposition* (2.1) *the* **derivatives of the controlled path** $X(\omega)$*.*

The value 4/3 is somewhat arbitrary here. The analysis provided below could be managed, if needed, with another exponent strictly greater than 1, but this would require higher values for the exponent q than that one we use in the definition of the rough set-up – recall $q \geqslant 8$. It seems that the value 4/3 is pretty convenient, as 4/3 is the conjugate exponent of 4. It follows from the fact that $\| X(\omega) \|_{\star,[0,T],p}$ is finite that an ω-controlled path is controlled in the usual sense by the first level $\big(W_t(\omega), W_t(\cdot)\big)_{0\leqslant t\leqslant T}$ of our rough set-up, provided the latter is considered as taking values in an infinite dimensional space.

We now define the notion of random controlled trajectory, which consists of a collection of ω-controlled trajectories indexed by the elements of Ω.

8. Definition – *A family of* ω*-controlled paths* $(X(\omega))_{\omega\in\Omega}$ *such that the maps*

$$\begin{aligned}
&\Omega \ni \omega \mapsto \big(X_t(\omega)\big)_{0\leqslant t\leqslant T} \in \mathcal{C}\big([0,T];\mathbb{R}^d\big)\\
&\Omega \ni \omega \mapsto \big(\delta_x X_t(\omega)\big)_{0\leqslant t\leqslant T} \in \mathcal{C}\big([0,T];\mathbb{R}^{d\times m}\big)\\
&\Omega \ni \omega \mapsto \big(\delta_\mu X_t(\omega)\big)_{0\leqslant t\leqslant T} \in \mathcal{C}\big([0,T];\mathbb{L}^{4/3}(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^{d\times m})\big)\\
&\Omega \ni \omega \mapsto \big(R^X_{s,t}(\omega)\big)_{(s,t)\in\mathcal{S}_2^T},
\end{aligned}$$

are measurable and satisfy

$$\big\langle X_0(\cdot)\big\rangle_2 + \big\langle \| X(\cdot) \|_{[0,T],w,p}\big\rangle_8 < \infty \tag{2.2}$$

is called a **random controlled path** *on* $[0,T]$*.*

Note from (1.9) the following elementary fact, whose proof is left to the reader.

9. Lemma – *Let* $\big((X_t(\omega))\big)_{0\leqslant t\leqslant T})_{\omega\in\Omega}$ *be a random controlled path on a time interval* $[0,T]$*. Then, for any* $0\leqslant s<t\leqslant T$*, we have*

$$\begin{aligned}
\big\langle X_{s,t}(\cdot)\big\rangle_2 &\leqslant \Big\langle \| X(\cdot)\|^2_{[0,T],w,p}\, w(s,t,\cdot)^{2/p}\Big\rangle^{1/2}\\
&\leqslant \big\langle \| X(\cdot)\|_{[0,T],w,p}\big\rangle_4 \big\langle w(s,t,\cdot)\big\rangle_4^{1/p} \leqslant 2\big\langle \| X(\cdot)\|_{[0,T],w,p}\big\rangle_4\, w(s,t,\omega)^{1/p}.
\end{aligned}$$

Similarly,

$$\big\langle X_{s,t}(\cdot)\big\rangle_4 \leqslant \big\langle \| X(\cdot)\|_{[0,T],w,p}\big\rangle_8 \big\langle w(s,t,\cdot)\big\rangle_8^{1/p} \leqslant 2\big\langle \| X(\cdot)\|_{[0,T],w,p}\big\rangle_8\, w(s,t,\omega)^{1/p}.$$

A straightforward consequence of Lemma 9 is that a random controlled trajectory induces a continuous path from $[0,T]$ to $L^2(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^d)$.

2.2 – Rough Integral

Set $U := \mathbb{R}^m \times \mathbb{L}^q(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^m)$ and note that $U\otimes U$ can be canonically identified with

$$\begin{aligned}
&\big(\mathbb{R}^m\otimes\mathbb{R}^m\big) \oplus \Big(\mathbb{R}^m \otimes \mathbb{L}^q(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^m)\Big)\\
&\qquad \oplus \Big(\mathbb{L}^q(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^m)\otimes\mathbb{R}^m\Big) \oplus \Big(\mathbb{L}^q(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^m)\otimes\mathbb{L}^q(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^m)\Big).
\end{aligned}$$

We take as a starting point of our analysis the fact that $\boldsymbol{W}(\omega)$ may be considered as a rough path with values in $U \oplus U^{\otimes 2}$, for any given ω. Indeed the first level $\boldsymbol{W}^{(1)}(\omega) := \big(W_t(\omega), W_t(\cdot)\big)_{t \geqslant 0}$ of $\boldsymbol{W}(\omega)$ is a continuous path with values in U and its second level

$$\boldsymbol{W}^{(2)}(\omega) := \begin{pmatrix} \mathbb{W}_{0,t}(\omega) & \mathbb{W}^{\perp\!\!\!\perp}_{0,t}(\omega, \cdot) \\ \mathbb{W}^{\perp\!\!\!\perp}_{0,t}(\cdot, \omega) & \mathbb{W}^{\perp\!\!\!\perp}_{0,t}(\cdot, \cdot) \end{pmatrix}_{t \geqslant 0},$$

is a continuous path with values in $U \otimes U$, with $\mathbb{W}_{0,t}(\omega)$ seen as an element of $\mathbb{R}^m \otimes \mathbb{R}^m$, with $\mathbb{W}^{\perp\!\!\!\perp}_{0,t}(\omega, \cdot)$ seen as an element of $\mathbb{R}^m \otimes \mathbb{L}^q(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^m)$, and $\mathbb{W}^{\perp\!\!\!\perp}_{0,t}(\cdot, \omega)$ seen as an element of $\mathbb{L}^q(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^m) \otimes \mathbb{R}^m$, and $\mathbb{W}^{\perp\!\!\!\perp}_{0,t}(\cdot, \cdot)$ as an element of $\mathbb{L}^q(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^m) \otimes \mathbb{L}^q(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^m)$. Condition (1.4) then reads as Chen's relation for $\boldsymbol{W}(\omega)$.

We can then use Feyel-de la Pradelle' sewing lemma [23], in the form given by Coutin and Lejay in [15, 16], to construct the rough integral of an ω-controlled path and a Banach-valued rough set-up.

10. Theorem – *There exists a universal constant c_0 and, for any $\omega \in \Omega$, there exists a continuous linear map*

$$\big(X_t(\omega)\big)_{0 \leqslant t \leqslant T} \mapsto \left(\int_s^t X_{s,u}(\omega) \otimes d\boldsymbol{W}_u(\omega)\right)_{(s,t) \in \mathcal{S}_2^T}$$

from the space of ω-controlled trajectories equipped with the norm $\|\!| \cdot |\!\|_{\star,[0,T],p}$, onto the space of continuous functions from $\mathcal{S}_2^T$ into $\mathbb{R}^d \otimes \mathbb{R}^m$ with finite norm $\|\cdot\|_{[0,T],w,p/2}$, with w being evaluated along the realization ω, that satisfies for any $0 \leqslant r \leqslant s \leqslant t \leqslant T$ the identity

$$\begin{aligned} &\int_r^t X_{r,u}(\omega) \otimes d\boldsymbol{W}_u(\omega) \\ &\quad = \int_r^s X_{r,u}(\omega) \otimes d\boldsymbol{W}_u(\omega) + \int_s^t X_{s,u}(\omega) \otimes d\boldsymbol{W}_u(\omega) + X_{r,s}(\omega) \otimes W_{s,t}(\omega), \end{aligned}$$

together with the estimate

$$\begin{aligned} &\left| \int_s^t X_{s,u}(\omega) \otimes d\boldsymbol{W}_u(\omega) - \Big\{ \delta_x X_s(\omega) \mathbb{W}_{s,t}(\omega) + \mathbb{E}\big[\delta_\mu X_s(\omega, \cdot) \mathbb{W}^{\perp\!\!\!\perp}_{s,t}(\cdot, \omega)\big] \Big\} \right| \\ &\leqslant c_0 \,\|\!| X(\omega) |\!\|_{[0,T],w,p}\, w(s,t,\omega)^{3/p}. \end{aligned} \tag{2.3}$$

To make notations clear, $\delta_x X_s(\omega)\, \mathbb{W}_{s,t}(\omega)$ is the product of a $d \times m$ matrix and an $m \times m$ matrix, so it gives back a $d \times m$ matrix, with components

$$\big(\delta_x X_s(\omega) \mathbb{W}_{s,t}(\omega)\big)^{i,j} = \sum_{k=1}^m \big(\delta_x X_s^i(\omega)\big)^k \big(\mathbb{W}_{s,t}(\omega)\big)^{k,j},$$

for $i \in \{1, \cdots, d\}$ and $j \in \{1, \cdots, m\}$. We also stress that the notation

$$\mathbb{E}\big[\delta_\mu X_s(\omega, \cdot) \mathbb{W}^{\perp\!\!\!\perp}_{s,t}(\cdot, \omega)\big],$$

which reads as the expectation of a matrix of size $d \times m$, can be also interpreted as a contraction product between an element of $\mathbb{R}^d \otimes \mathbb{L}^{4/3}(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^m)$ and an element of $\mathbb{L}^q(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^m) \otimes \mathbb{R}^m$. While this remark may seem anecdotal it is actually important for the proof below.

Proof – The proof is a consequence of Proposition 2 in Coutin and Lejay's work [15], except for one main fact. In order to use Coutin and Lejay's result, we consider $\boldsymbol{W}(\omega)$ as a rough path with values in $U \oplus U^{\otimes 2}$ and $\big(X(\omega), \delta_x X(\omega), \delta_\mu X(\omega), R^X(\omega)\big)$ as a controlled path; this was explained above. When doing so, the resulting integral is constructed as a process with values in $\mathbb{R}^d \otimes U$, whilst the integral given by the statement of Theorem 10 takes values in $\mathbb{R}^d$. We denote the $\mathbb{R}^d \otimes U$-valued integral by $(I_s^t X_{s,u}(\omega) \otimes d\boldsymbol{W}_u(\omega))_{(s,t)\in\mathcal{S}_2^T}$. We use a simple projection to pass from the infinite dimensional-valued quantity $I_s^t X_{s,u}(\omega) \otimes d\boldsymbol{W}_u(\omega)$ to the finite dimensional-valued quantity $\int_s^t X_{s,u}(\omega) \otimes d\boldsymbol{W}_u(\omega)$. Indeed, we may use the canonical projection from $\mathbb{R}^d \otimes U \cong \big(\mathbb{R}^d \otimes \mathbb{R}^m\big) \oplus \big(\mathbb{R}^d \otimes \mathbb{L}^q(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^m)\big)$ onto $\mathbb{R}^d \otimes \mathbb{R}^m$ to project $I_s^t X_{s,u}(\omega) \otimes d\boldsymbol{W}_u(\omega)$ onto $\int_s^t X_{s,u}(\omega) \otimes d\boldsymbol{W}_u(\omega)$. ▷

As usual, we define an additive process setting

$$\int_s^t X_u(\omega) \otimes d\boldsymbol{W}_u(\omega) := \int_s^t X_{s,u}(\omega) \otimes d\boldsymbol{W}_u(\omega) + X_s(\omega) \otimes W_{s,t}(\omega),$$

for $0 \leqslant t \leqslant T$. We can thus consider the integral process $\big(\int_0^t X_s(\omega) \otimes d\boldsymbol{W}_s(\omega)\big)_{0\leqslant t\leqslant T}$ as an ω-controlled trajectory with values in $\mathbb{R}^{d\times m}$, with x-derivative a linear map from $\mathbb{R}^m$ into $\mathbb{R}^{d\times m}$, and entries

$$\left(\delta_x\left[\int_0^\cdot X_s(\omega) \otimes d\boldsymbol{W}_s(\omega)\right]_t\right)_{(i,j),k} = \big(X_t(\omega)\big)_i \delta_{j,k},$$

for $i \in \{1, \cdots, d\}$ and $j, k \in \{1, \cdots, m\}$, where $\delta_{j,k}$ stands for the usual Kronecker symbol, and with null μ-derivative, namely

$$\delta_\mu\left[\int_0^\cdot X_s(\omega) \otimes d\boldsymbol{W}_s(\omega)\right]_t = 0. \tag{2.4}$$

This property is fundamental for the fixed point formulation of the mean field rough differential equation (0.2). The remainder $R^{\int X \otimes d\boldsymbol{W}}$ can be estimated by combining (2.3) together with the inequality

$$\begin{aligned}
&\Big|\delta_x X_s(\omega)\mathbb{W}_{s,t}(\omega) + \mathbb{E}\big[\delta_\mu X_s(\omega,\cdot)\mathbb{W}^{\perp\!\!\!\perp}_{s,t}(\cdot,\omega)\big]\Big| \\
&\qquad \leqslant \left\{\sup_{r\in[0,T]} |\delta_x X_r(\omega,\cdot)| + \sup_{r\in[0,T]} \langle \delta_\mu X_r(\omega)\rangle_{4/3}\right\} w(s,t,\omega)^{2/p} \\
&\qquad \leqslant \|\!|X(\omega)|\!\|_{\star,[0,T],w,p}\Big(1 + w(0,T,\omega)^{1/p}\Big) w(s,t,\omega)^{2/p},
\end{aligned}$$

so that, with the notation as in Definition 7,

$$\left\|\!\left|\int_0^\cdot X_s(\omega) \otimes d\boldsymbol{W}_s(\omega)\right|\!\right\|_{[0,T],w,p} < \infty. \tag{2.5}$$

When $X(\omega)$ is given as the ω-realization of a random controlled path $(X(\omega'))_{\omega'\in\Omega}$, the integral may be defined for any $\omega' \in \Omega$. For the integral $\int_0^\cdot X_s(\omega) \otimes dW_s$ to define a random controlled path, its $\|\!|\cdot|\!\|_{[0,T],w,p}$-semi-norm needs to have finite 8-th moment. When the trajectory $X(\omega)$ takes in values in $\mathbb{R}^d \otimes \mathbb{R}^m$ rather than $\mathbb{R}^d$, the integral

$$\int_0^t X_s(\omega) \otimes d\boldsymbol{W}_s(\omega) \in \mathbb{R}^d \otimes \mathbb{R}^m \otimes \mathbb{R}^m$$

may be identified with a tuple

$$\left(\left(\int_0^t X_s(\omega) \otimes d\boldsymbol{W}_s(\omega)\right)_{i,j,k}\right)_{(i,j,k)\in\{1,\cdots,d\}\times\{1,\cdots,m\}\times\{1,\cdots,m\}}.$$

We then set for $i \in \{1, \cdots, d\}$

$$\left(\int_0^t X_s(\omega) d\boldsymbol{W}_s(\omega)\right)_i := \sum_{j=1}^m \left(\int_0^t X_s(\omega) \otimes d\boldsymbol{W}_s(\omega)\right)_{i,j,j},$$

and consider $\int_0^t X_s(\omega) d\boldsymbol{W}_s(\omega)$ as an element of $\mathbb{R}^d$.

2.3 – Stability of Controlled Paths under Nonlinear Maps

We show in this section that controlled paths are stable under some nonlinear, sufficiently regular, maps and start by recalling the reader about the regularity notion used when working with functions defined on Wasserstein space. We refer the reader to Lions' lectures [34], to the lecture notes [7] of Cardaliaguet or to Carmona and Delarue's monograph [9, Chapter 5] for basics on the subject.

• Recall $(\Omega, \mathcal{F}, \mathbb{P})$ stands for an atomless probability space, with Ω a Polish space and $\mathcal{F}$ its Borel σ-algebra. Fix a finite dimensional space $E = \mathbb{R}^k$ and denote by $L^2 := \mathbb{L}^2(\Omega, \mathcal{F}, \mathbb{P}; E)$ the space of E-valued random variables on Ω with finite second moment. We equip the space $\mathcal{P}_2(E) := \{\mathcal{L}(Z) \,;\, Z \in L^2\}$ with the 2-Wasserstein distance

$$d_2(\mu_1, \mu_2) := \inf\Big\{\|Z_1 - Z_2\|_2 \,;\, \mathcal{L}(Z_1) = \mu_1,\, \mathcal{L}(Z_2) = \mu_2\Big\}.$$

An $\mathbb{R}^k$-valued function u defined on $\mathcal{P}_2(E)$ is canonically extended into L^2 by setting, for any $Z \in L^2$,

$$U(Z) := u\big(\mathcal{L}(Z)\big).$$

- The function u is then said to be differentiable at $\mu \in \mathcal{P}_2(E)$ if its canonical lift is Fréchet differentiable at some point Z such that $\mathcal{L}(Z) = \mu$; we denote by $\nabla_Z U \in (L^2)^k$ the gradient of U at Z. The function U is then differentiable at any other point $Z' \in L^2$ such that $\mathcal{L}(Z') = \mu$, and the laws of $\nabla_Z U$ and $\nabla_{Z'} U$ are equal, for any such Z'.
- The function u is said to be of class C^1 on some open set O of $\mathcal{P}_2(E)$ if its canonical lift is of class C^1 in some open set of L^2 projecting onto O. It is then of class C^1 in the whole fiber in L^2 above O. If u is of class C^1, then $\nabla_Z U$ is $\sigma(Z)$-measurable and given by an $\mathcal{L}(Z)$-dependent function Du from E to E^k such that
$$\nabla_Z U = (Du)(Z); \tag{2.6}$$
we have in particular $Du \in L^2_\mu(E; E^k) := \mathbb{L}^2(E, \mathcal{B}(E), \mu; E^k)$, where $\mathcal{B}(E)$ is the Borel σ-field on E. In order to emphasize the fact that Du depends upon $\mathcal{L}(Z)$, we shall write $Du(\mathcal{L}(Z))(\cdot)$ instead of $Du(\cdot)$. Sometimes, we shall put an index μ and write $D_\mu u(\mathcal{L}(Z))(\cdot)$ in order to emphasize the fact that the derivative is taken with respect to the measure argument; this will be especially useful for functionals u depending on additional variables. Importantly, this representation is independent of the choice of the probability space $(\Omega, \mathcal{F}, \mathbb{P})$; in fact, it can be easily transported from one probability space to another. (A simple proof of the structural equation (2.6) can be found in [40].)

As an elementary example, think of a real-valued function u of the form $u(\mu) = f\big(\int x^2\mu(dx)\big)$, for which the lift $Z \mapsto U(Z) = f\big(\mathbb{E}[Z^2]\big)$ has differential $(d_Z U)(H) = 2f'\big(\mathbb{E}[Z^2]\big)\,\mathbb{E}[ZH]$ and gradient $2f'\big(\mathbb{E}[Z^2]\big)\,Z$, so $Du(\mu)(z) = 2f'\big(\int x^2\mu(dx)\big)z$ here. We refer to [7] and [9, Chapter 5] for further examples.

• Back to controlled paths. Let F stand here for a map from $\mathbb{R}^d \times \mathbb{L}^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$ into the space $L(\mathbb{R}^m, \mathbb{R}^d) \cong \mathbb{R}^d \otimes \mathbb{R}^m$ of linear mappings from $\mathbb{R}^m$ to $\mathbb{R}^d$. Intuitively, F should be thought of as the lift of the coefficient driving equation (0.2), or, with the same notation as in (0.3), as $\widehat{\mathrm{F}}$ itself, with the slight abuse of notation that it requires to identify F and $\widehat{\mathrm{F}}$. Our goal now is to expand the image of a controlled trajectory by F.

Regularity assumptions 1 – *Assume that* F *is continuously differentiable in the joint variable* (x, Z)*, that* $\partial_x F$ *is also continuously differentiable in* (x, Z) *and that there is some positive finite constant* Λ *such that*

$$\begin{aligned} \sup_{x\in\mathbb{R}^d,\, \mu\in\mathcal{P}_2(\mathbb{R}^d)} \big|\mathrm{F}(x,\mu)\big| \vee \big|\partial_x\mathrm{F}(x,\mu)\big| \vee \big|\partial_x^2\mathrm{F}(x,\mu)\big| \leqslant \Lambda, \\ \sup_{x\in\mathbb{R}^d,\, \mathcal{L}(Z)\in\mathcal{P}_2(\mathbb{R}^d)} \big\|\nabla_Z\mathrm{F}(x,Z)\big\|_2 \vee \big\|\partial_x\nabla_Z\mathrm{F}(x,Z)\big\|_2 \leqslant \Lambda, \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} \nabla_Z\mathrm{F}(x,\cdot) : \mathbb{L}^2(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^d) &\to \mathbb{L}^2(\Omega,\mathcal{F},\mathbb{P}; L(\mathbb{R}^d, \mathbb{R}^d\otimes\mathbb{R}^m)) \\ Z &\mapsto \nabla_Z\mathrm{F}(x,Z) = D_\mu F(x,\mathcal{L}(Z))(Z) \end{aligned}$$

is a Λ*-Lipschitz function of* $Z \in \mathbb{L}^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$*, uniformly in* $x \in \mathbb{R}^d$*.*

Importantly, the $\mathbb{L}^2$-Lipschitz bound required in the second line of (2.7) may be formulated as a Lipschitz bound on $\mathcal{P}_2(\mathbb{R}^d)$ equipped with d_2. Moreover, notice that $\mathbb{L}^2\big(\Omega, \mathcal{F}, \mathbb{P}; L(\mathbb{R}^d, \mathbb{R}^d \otimes \mathbb{R}^m)\big)$ can be identified with $\mathbb{L}^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)^{d\times m}$; also, $\partial_x\mathrm{F}(x, Z)$ and $\nabla_Z\mathrm{F}(x, Z)$ will be considered as random variables with values in $L(\mathbb{R}^d, \mathbb{R}^d \otimes \mathbb{R}^m) \cong \mathbb{R}^d \otimes \mathbb{R}^m \otimes \mathbb{R}^d$. As an example, the functions

$$\mathrm{F}(x,\mu) = \int f(x,y)\mu(dy)$$

for some fuction f of class C_b^2, and

$$\mathrm{F}(x,\mu) = g\left(x, \int y\mu(dy)\right)$$

for some function g of class C_b^2, both satisfy **Regularity assumptions 1**.

We expand below the path $\big(\mathrm{F}(X_t(\omega), Y_t(\cdot))\big)_{0\leqslant t\leqslant T}$, which we write $\mathrm{F}(X(\omega), Y(\cdot))$, where $X(\omega)$ is an ω-controlled path and $Y(\cdot)$ is an $\mathbb{R}^d$-valued random controlled path, both of them being defined on some finite time interval $[0, T]$. Identity (2.4) tells us that a fixed point formulation of the mean field rough differential equation (0.2) will only involve pairs $(X(\omega), Y(\cdot))$ such that

$$\delta_\mu X(\omega) \equiv 0, \qquad \delta_\mu Y(\cdot) \equiv 0, \tag{2.8}$$

which prompts us to restrict ourselves to the case when $X(\omega)$ and Y have null μ-derivatives in the expansion (2.1).

11. Proposition – *Let $X(\omega)$ be an ω-controlled path and $Y(\cdot)$ be an $\mathbb{R}^d$-valued random controlled path. Assume that condition (2.8) hold and we have the ω-*independent *bound*

$$M := \sup_{0\leqslant t\leqslant T}\Big(\big|\delta_x X_t(\omega)\big| \vee \big\langle \delta_x Y_t(\cdot)\big\rangle_\infty\Big) < \infty.$$

Then, $\mathrm{F}\big(X(\omega), Y(\cdot)\big)$ is an ω-controlled path with

$$\delta_x\Big(\mathrm{F}\big(X(\omega), Y(\cdot)\big)\Big)_t = \partial_x \mathrm{F}\big(X_t(\omega), Y_t(\cdot)\big)\,\delta_x X_t(\omega),$$

which is understood as $\big(\partial_{x_\ell}\mathrm{F}^{i,j}\big(X_t(\omega), Y_t(\cdot)\big)\big(\delta_x X_t^\ell(\omega)\big)_k\big)_{i,j,k}$, with $i,k \in \{1,\cdots,d\}$ and $j\in\{1,\cdots,m\}$, and (with a similar interpretation for the product)

$$\begin{aligned}\delta_\mu\Big(\mathrm{F}\big(X(\omega), Y(\cdot)\big)\Big)_t &= \nabla_Z \mathrm{F}\big(X_t(\omega), Y_t(\cdot)\big)\delta_x Y_t(\cdot)\\ &= D_\mu \mathrm{F}\big(X_t(\omega), \mathcal{L}(X_t)\big)\big(X_t(\cdot)\big)\delta_x Y_t(\cdot),\end{aligned}$$

and one can find a constant $C_{\Lambda,M}$, depending only on Λ and M, such that

$$\big\|\!\big|\mathrm{F}\big(X(\omega), Y(\cdot)\big)\big\|\!\big|_{[0,T],w,p} \leqslant C_{\Lambda,M}\Big(1 + \|\!|X(\omega)\|\!|^2_{[0,T],w,p} + \big\langle \|\!|Y(\cdot)\|\!|_{[0,T],w,p}\big\rangle_8^2\Big).$$

Proof – For $0\leqslant s<t$, expand $\mathrm{F}(X(\omega),Y(\cdot))_{s,t}$ into

$$\begin{aligned}\mathrm{F}(X(\omega), Y(\cdot))_{s,t} &= \mathrm{F}\big(X_t(\omega), Y_t(\cdot)\big) - \mathrm{F}\big(X_s(\omega), Y_s(\cdot)\big)\\ &= \Big\{\mathrm{F}\big(X_t(\omega), Y_t(\cdot)\big) - \mathrm{F}\big(X_s(\omega), Y_t(\cdot)\big)\Big\}\\ &\quad +\Big\{\mathrm{F}\big(X_s(\omega), Y_t(\cdot)\big) - \mathrm{F}\big(X_s(\omega), Y_s(\cdot)\big)\Big\}\\ &=: \Big\{\mathbf{(1)} + \mathbf{(2)} + \mathbf{(3)}\Big\} + \Big\{\mathbf{(4)} + \mathbf{(5)}\Big\},\end{aligned}\tag{2.9}$$

where

$$\begin{aligned}\mathbf{(1)} &:= \partial_x \mathrm{F}\big(X_s(\omega), Y_s(\cdot)\big)\Big\{\delta_x X_s(\omega) W_{s,t}(\omega) + R^X_{s,t}(\omega)\Big\},\\ \mathbf{(2)} &:= \int_0^1 \Big[\partial_x \mathrm{F}\Big(X^{(\lambda)}_{s;(s,t)}(\omega), Y_t(\cdot)\Big) - \partial_x \mathrm{F}\Big(X^{(\lambda)}_{s;(s,t)}(\omega), Y_s(\cdot)\Big)\Big] X_{s,t}(\omega)\,d\lambda,\\ \mathbf{(3)} &:= \int_0^1 \Big[\partial_x \mathrm{F}\Big(X^{(\lambda)}_{s;(s,t)}(\omega), Y_s(\cdot)\Big) - \partial_x \mathrm{F}\big(X_s(\omega), Y_s(\cdot)\big)\Big] X_{s,t}(\omega)\,d\lambda,\\ \mathbf{(4)} &:= \Big\langle \nabla_Z \mathrm{F}\big(X_s(\omega), Y_s(\cdot)\big) Y_{s,t}(\cdot)\Big\rangle,\\ \mathbf{(5)} &:= \int_0^1 \Big\langle \Big(\nabla_Z \mathrm{F}\big(X_s(\omega), Y^{(\lambda)}_{s;(s,t)}(\cdot)\big) - \nabla_Z \mathrm{F}\big(X_s(\omega), Y_s(\cdot)\big)\Big) Y_{s,t}(\cdot)\Big\rangle\,d\lambda;\end{aligned}$$

we used here the fact that $X(\omega)$ and $Y(\cdot)$ have null μ-derivative and where we let

$$X^{(\lambda)}_{s;(s,t)}(\omega) = X_s(\omega) + \lambda X_{s,t}(\omega), \quad Y^{(\lambda)}_{s;(s,t)}(\cdot) = Y_s(\cdot) + \lambda Y_{s,t}(\cdot). \tag{2.10}$$

We read on the decomposition (2.9) the formulas for the x and μ-derivatives of $\mathrm{F}(X(\omega), Y(\cdot))$. The remainder $R^{\mathrm{F}(X,Y)}_{s,t}$ in the controlled decomposition of the path $\mathrm{F}(X(\omega), Y(\cdot))$ is

$$\partial_x \mathrm{F}\big(X_s(\omega), Y_s(\cdot)\big) R^X_{s,t}(\omega) + \Big\langle \nabla_Z \mathrm{F}\big(X_s(\omega), Y_s(\cdot)\big) R^Y_{s,t}(\cdot)\Big\rangle + \mathbf{(2)} + \mathbf{(3)} + \mathbf{(5)}. \tag{2.11}$$

We now compute $\big\|\!\big|\mathrm{F}\big(X(\omega), Y(\cdot)\big)\big\|\!\big|_{\star,[0,T],w,p}$.

- We have first from the regularity assumptions on F that the initial conditions for the quantities
$$\mathrm{F}\big(X(\omega),Y(\cdot)\big),\ \delta_x\Big(\mathrm{F}\big(X(\omega),Y(\cdot)\big)\Big),\ \delta_\mu\Big(\mathrm{F}\big(X(\omega),Y(\cdot)\big)\Big),$$
are all bounded above by ΛM.
- **Variation of** $\mathrm{F}(X(\omega),Y(\cdot))$**.** Using the Lipschitz property of F and Lemma 9, we have
$$\begin{aligned}\Big|\big[\mathrm{F}\big(X(\omega),Y(\cdot)\big)\big]_{s,t}\Big| &= \Big|\big[\mathrm{F}\big(X(\omega),Y(\cdot)\big)\big]_t - \big[\mathrm{F}\big(X(\omega),Y(\cdot)\big)\big]_s\Big| \\ &\leqslant \Lambda\left(\big|X_{s,t}(\omega)\big| + \big\langle Y_{s,t}(\cdot)\big\rangle_2\right) \\ &\leqslant 2\Lambda\left(\|\!|X(\omega)|\!\|_{[0,T],w,p} + \big\langle \|\!|Y(\cdot)|\!\|_{[0,T],w,p}\big\rangle_4\right) w(s,t,\omega)^{1/p},\end{aligned}$$
- **Variation of** $\delta_x\big(\mathrm{F}(X(\omega),Y(\cdot))\big)$ **and** $\delta_\mu\big(\mathrm{F}(X(\omega),Y(\cdot))\big)$**.** The Lipschitz properties of $\partial_x\mathrm{F}$ and $\nabla_Z\mathrm{F}(x,\cdot)$ also give
$$\begin{aligned}\Big|\delta_x\big[\mathrm{F}\big(X(\omega),Y(\cdot)\big)\big]_{s,t}\Big| &\leqslant 2\Lambda M\left(\|\!|X(\omega)|\!\|_{[0,T],w,p} + \big\langle \|\!|Y(\cdot)|\!\|_{[0,T],w,p}\big\rangle_4\right) w(s,t,\omega)^{1/p} \\ &\quad + \Lambda\,\|\!|X(\omega)|\!\|_{[0,T],w,p}\, w(s,t,\omega)^{1/p},\end{aligned}$$
and, applying Hölder's inequality with exponents 3/2 and 3,
$$\begin{aligned}&\Big\langle \delta_\mu\big[\mathrm{F}\big(X(\omega),Y(\cdot)\big)\big]_{s,t}\Big\rangle_{4/3} \\ &\leqslant 2\Lambda\big\langle\delta_x Y_t(\cdot)\big\rangle_\infty\left(\|\!|X(\omega)|\!\|_{[0,T],w,p} + \langle \|\!|Y(\cdot)|\!\|_{[0,T],w,p}\rangle_4\right) w(s,t,\omega)^{1/p} \\ &\quad + \Lambda\langle\delta_x Y_{s,t}(\cdot)\rangle_4 \\ &\leqslant 2\Lambda M\Big(\|\!|X(\omega)|\!\|_{[0,T],w,p} + \langle \|\!|Y(\cdot)|\!\|_{[0,T],w,p}\rangle_4\Big) w(s,t,\omega)^{1/p} \\ &\quad + 2\Lambda\big\langle \|\!|Y(\cdot)|\!\|_{[0,T],w,p}\big\rangle_8\, w(s,t,\omega)^{1/p}.\end{aligned}$$
- **Remainder** (2.11)**.** The first two terms in (2.11) are less than
$$\begin{aligned}&\Lambda\,\|\!|X|\!\|_{[0,T],w,p}\, w(s,t,\omega)^{2/p} + \Lambda\big\langle R^Y_{s,t}(\cdot)\big\rangle_2 \\ &\leqslant \Lambda\|\!|X|\!\|_{[0,T],w,p}\, w(s,t,\omega)^{2/p} + \Lambda\big\langle \|\!|Y(\cdot)|\!\|_{[0,T],w,p}\, w(s,t,\cdot)^{2/p}\big\rangle_2 \\ &\leqslant \Lambda\|\!|X|\!\|_{[0,T],w,p}\, w(s,t,\omega)^{2/p} + \Lambda\big\langle \|\!|Y(\cdot)|\!\|_{[0,T],w,p}\big\rangle_4 \big\langle w(s,t,\cdot)\big\rangle_4^{2/p} \\ &\leqslant \Lambda\,\|\!|X|\!\|_{[0,T],w,p}\, w(s,t,\omega)^{2/p} + 2\Lambda\big\langle \|\!|Y(\cdot)|\!\|_{[0,T],w,p}\big\rangle_4\, w(s,t,\omega)^{2/p},\end{aligned}$$
from Lemma 9. We also have
$$\begin{aligned}\big|\mathbf{(2)}\big| &\leqslant \Lambda\,\big|X_{s,t}(\omega)\big|\,\big\langle Y_{s,t}(\cdot)\big\rangle_2 \\ &\leqslant 2\Lambda\,\big\|\!\big|X(\omega)\big|\!\big\|_{[0,T],w,p}\big\langle \|\!|Y(\cdot)|\!\|_{[0,T],w,p}\big\rangle_4\, w(s,t,\omega)^{2/p}.\end{aligned}$$
and
$$\big|\mathbf{(3)}\big| \leqslant \Lambda\,\big|X_{s,t}(\omega)\big|^2 \leqslant \Lambda\,\big\|\!\big|X(\omega)\big|\!\big\|^2_{[0,T],w,p}\, w(s,t,\omega)^{2/p}.$$
Last, since $\nabla_Z\mathrm{F}$ is a Lipchitz function of its second argument,
$$\mathbf{(5)} \leqslant \Lambda\big\langle Y_{s,t}(\cdot)\big\rangle_2^2 \leqslant 4\Lambda\big\langle \|\!|Y(\cdot)|\!\|_{[0,T],w,p}\big\rangle_4^2\, w(s,t,\omega)^{2/p}.$$

$\triangleright$

3 – Solving the Equation

We now have all the tools to formulate the mean field rough differential equation (0.4) (or (0.2)) as a fixed point problem and solve it by Picard iteration. Our definition of the fixed point is given in the form of a two-step procedure: The first step is to write a *frozen* version of the equation, in which the mean field component is seen as a mere exogenous collection of ω-controlled trajectories; the second step is to regard the family of exogenous controlled trajectories as an input and to map it to the collection of controlled trajectories solving the frozen version of the equation. In this way, we define a solution as a collection of ω-controlled trajectories. In order to proceed, recall the generic notation $\big(X(\omega); \delta_x X(\omega); \partial_\mu X(\omega, \cdot)\big)$ for an ω-controlled path and its derivatives; we sometimes abuse notations and talk of $X(\omega)$ as an ω-controlled path.

12. Definition – *Let W together with its enhancement $\boldsymbol{W}$ satisfy the assumption of Section 1 on a finite nontrivial time interval $[0,T]$, and let $Y(\cdot)$ stand for some $\mathbb{R}^d$-valued random controlled path on $[0,T]$, with the property that $\delta_\mu Y(\cdot) \equiv 0$ and that $\sup_{0\leqslant t\leqslant T} \langle \delta_x Y_t(\cdot)\rangle_\infty < \infty$. For a given $\omega \in \Omega$, let $X(\omega)$ be an $\mathbb{R}^d$-valued ω-controlled path on $[0,T]$, with the properties that $\delta_\mu X(\omega) \equiv 0$ and $\sup_{0\leqslant t\leqslant T} |\delta_x X_t(\omega)| < \infty$. We associate to ω and $X(\omega)$ an ω-controlled path by setting*

$$\begin{aligned}&\Gamma\big(\omega, X(\omega), Y(\cdot)\big)\\ &\quad := \left(X_0(\omega) + \int_0^t \mathrm{F}\big(X_s(\omega), Y_s(\cdot)\big) d\boldsymbol{W}_s(\omega)\,;\, \mathrm{F}\big(X_t(\omega), Y_t(\cdot)\big)\,;\, 0\right)_{0\leqslant t\leqslant T}.\end{aligned}$$

A **solution to the mean field rough differential equation**

$$dX_t = \mathrm{F}\big(X_t, \mathcal{L}(X_t)\big)\, d\boldsymbol{W}_t,$$

on the time interval $[0,T]$, with given initial condition $X_0(\cdot) \in L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$ is a random controlled path $X(\cdot)$ starting from $X_0(\cdot)$ and satisfying the same prescription as $Y(\cdot)$, such that for $\mathbb{P}$-almost every ω the path $X(\omega)$ and $\Gamma\big(\omega, X(\omega), X(\cdot)\big)$ coincide.

We should more properly replace $X(\omega)$ in $\Gamma\big(\omega, X(\omega), Y(\cdot)\big)$ by $\big(X(\omega)\,; \delta_x X(\omega)\,; 0\big)$ and $Y(\cdot)$ by $\big(Y(\cdot)\,; \delta_x Y(\cdot)\,; 0\big)$, but we stick to the above lighter notation. Observe also that our formulation bypasses any requirement on the properties of the map Γ itself. To make it clear, we should be indeed tempted to check that, for a random controlled path $X(\cdot)$, the collection $\big(\Gamma(\omega, X(\omega), Y(\cdot))\big)_{\omega\in\Omega}$, for $Y(\cdot)$ as in the statement, is also a random controlled path. Somehow, our definition of a solution avoids this question; however, it should not come as a surprise that, at the end of the day, we need to check this fact carefully; below, we refer to it as the *stability properties of* Γ, see Section 3.1.

What remains of the above definition when $\boldsymbol{W}$ is the Itô or Stratonovich enhancement of a Brownian motion? The key point to connect the above notion of solution to the mean field rough differential equation (0.2) with the standard notion of solution to mean field stochastic differential equation is to observe that the rough integral therein should be, if a solution exists, the limit of the compensated Riemann sums

$$\sum_{j=0}^{K-1}\bigg(\mathrm{F}\big(X_{t_j}(\omega),X_{t_j}(\cdot)\big)W_{t_j,t_{j+1}}(\omega)$$
$$+\,\partial_x\mathrm{F}\big(X_{t_j}(\omega),X_{t_j}(\cdot)\big)\mathrm{F}\big(X_{t_j}(\omega),X_{t_j}(\cdot)\big)\mathbb{W}_{t_j,t_{j+1}}(\omega)$$
$$+\,\Big\langle D_\mu\mathrm{F}\big(X_{t_j}(\omega),X_{t_j}(\cdot)\big)\big(X_{t_j}(\cdot)\big)\mathrm{F}\big(X_{t_j}(\omega),X_{t_j}(\cdot)\big)\mathbb{W}^{\perp\!\!\!\perp}_{t_j,t_{j+1}}(\cdot,\omega)\Big\rangle\bigg),$$

as the step of the dissection $0=t_0<\cdots<t_K=t$ tends to 0. When the solution is constructed by a contraction argument, such as done below, the process $(X_t(\cdot))_{0\leqslant t\leqslant T}$ is adapted with respect to the completion of the filtration $(\mathcal{F}_t)_{0\leqslant t\leqslant T}$ generated by the initial condition $X_0(\cdot)$ and the Brownian motion $W(\cdot)$. Returning if necessary to Example 5, we then check that

$$\mathbb{E}\big[\mathbb{W}^{\perp\!\!\!\perp}_{t_j,t_{j+1}}(\cdot,\omega)\,|\,\mathcal{F}_{t_j}\big]=0,$$

whatever the interpretation of the rough integral, Itô or Stratonovich. Pay attention that the conditional expectation is taken with respect to "$\cdot$", while the element ω is kept frozen. This implies that, for any $j\in\{0,\cdots,K-1\}$, we have

$$\Big\langle D_\mu\mathrm{F}\big(X_{t_j}(\omega),X_{t_j}(\cdot)\big)\big(X_{t_j}(\cdot)\big)\mathrm{F}\big(X_{t_j}(\omega),X_{t_j}(\cdot)\big)\mathbb{W}^{\perp\!\!\!\perp}_{t_j,t_{j+1}}(\cdot,\omega)\Big\rangle=0.$$

This proves that the solution to the rough mean field equation coincides with the solution that is obtained when the equation (0.2) is interpreted in the standard McKean-Vlasov sense.

We formulate here the regularity assumptions on $\mathrm{F}(x,\mu)$ needed to show that Γ satisfies the required stability properties and to run Picard's iteration for proving the well-posed character of the mean field rough differential equation (0.4) (or (0.2)) in small time, or in some given time interval. Recall from (2.6) the definition of $D_\mu\mathrm{F}(x,\cdot)(\cdot)$ as a function from $\mathcal{P}_2(\mathbb{R}^d)\times\mathbb{R}^d$ to $L(\mathbb{R}^d,\mathbb{R}^d\otimes\mathbb{R}^m)\cong\mathbb{R}^d\otimes\mathbb{R}^m\otimes\mathbb{R}^d$ such that $D_\mu\mathrm{F}(x,\mathcal{L}(Z))(Z)=\nabla_Z\mathrm{F}(x,Z)$, where we emphasize the dependence of $D_\mu\mathrm{F}(x,\cdot)$ on $\mu=\mathcal{L}(Z)$ by writing $D_\mu\mathrm{F}(x,\mu)(\cdot)$. In addition to **Regularity assumptions 1**, we make the following assumptions on the interaction-dependent diffusivity F.

Regularity assumptions 2 – • *The function $\partial_x F$ is differentiable in (x,μ) in the same sense as before.*

• *For each $(x,\mu)\in\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)$, there exists a version of $D_\mu\mathrm{F}(x,\mu)(\cdot)\in L^2_\mu(\mathbb{R}^d;\mathbb{R}^d\otimes\mathbb{R}^m)$ such that the map*

$$(x,\mu,z)\mapsto D_\mu\mathrm{F}(x,\mu)(z)$$

from $\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)\times\mathbb{R}^d$ to $\mathbb{R}^d\otimes\mathbb{R}^m\otimes\mathbb{R}^d$ is of class C^1, the derivative in the direction μ being understood as before.

• *The function*

$$\big(x,Z\big)\mapsto\partial_x^2\mathrm{F}\big(x,\mathcal{L}(Z)\big)$$

from $\mathbb{R}^d\times\mathbb{L}^2(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^d)$ to $\mathbb{R}^d\otimes\mathbb{R}^m\otimes\mathbb{R}^d\otimes\mathbb{R}^d\cong L(\mathbb{R}^d\otimes\mathbb{R}^d,\mathbb{R}^d\otimes\mathbb{R}^m)$ is bounded by Λ and Λ-Lipschitz continuous.

• *The following three functions*

$$(x,Z)\mapsto\partial_x D_\mu\mathrm{F}\big(x,\mathcal{L}(Z)\big)(Z(\cdot))$$
$$(x,Z)\mapsto D_\mu\partial_x\mathrm{F}\big(x,\mathcal{L}(Z)\big)(Z(\cdot))$$
$$(x,Z)\mapsto\partial_z D_\mu\mathrm{F}\big(x,\mathcal{L}(Z)\big)(Z(\cdot))$$

from $\mathbb{R}^d \times \mathbb{L}^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$ *to* $\mathbb{L}^2\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d \otimes \mathbb{R}^m \otimes \mathbb{R}^d \otimes \mathbb{R}^d\big)$*, are bounded by* Λ *and* Λ*-Lipschitz continuous. (By Schwarz' theorem, the transpose of* $\partial_x D_\mu \mathrm{F}^{i,j}$ *is in fact equal to* $D_\mu \partial_x \mathrm{F}^{i,j}$*, for any* $i \in \{1, \cdots, d\}$ *and* $j \in \{1, \cdots, m\}$*.)*

- *For each* $\mu \in \mathcal{P}_2(\mathbb{R}^d)$*, we denote by*

$$D_\mu^2 \mathrm{F}(x, \mu)(z, \cdot)$$

the derivative of $D_\mu \mathrm{F}(x, \mu)(z)$ *with respect to* μ *– which is indeed given by a function. For* $z' \in \mathbb{R}^d$*,* $D_\mu^2 \mathrm{F}(x, \mu)(z, z')$ *is an element of* $\mathbb{R}^d \otimes \mathbb{R}^m \otimes \mathbb{R}^d \otimes \mathbb{R}^d$*.*

Denote by $\big(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}}\big)$ *a copy of* $(\Omega, \mathcal{F}, \mathbb{P})$*, and given a random variable* Z *on* $(\Omega, \mathcal{F}, \mathbb{P})$*, write* $\widetilde{Z}$ *for its copy on* $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$*. We assume that the function*

$$(x, Z) \mapsto D_\mu^2 \mathrm{F}\big(x, \mathcal{L}(Z)\big)\big(Z(\cdot), \widetilde{Z}(\cdot)\big),$$

from $\mathbb{R}^d \times \mathbb{L}^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$ *to* $\mathbb{L}^2\big(\Omega \times \widetilde{\Omega}, \mathcal{F} \otimes \widetilde{\mathcal{F}}, \mathbb{P} \otimes \widetilde{\mathbb{P}}; \mathbb{R}^d \otimes \mathbb{R}^m \otimes \mathbb{R}^d \otimes \mathbb{R}^d\big)$*, is bounded by* Λ *and* Λ*-Lipschitz continuous.*

The two functions

$$\mathrm{F}(x, \mu) = \int f(x, y)\mu(dy)$$

for some fuction f of class C_b^2, and

$$\mathrm{F}(x, \mu) = g\left(x, \int y\mu(dy), \int y^2 \mu(dy)\right)$$

for some function g of class C_b^2, both satisfy **Regularity assumptions 2**. We refer to [9, Chapter 5] and [10, Chapter 5] for other examples of functions that satisfy the above assumptions and for sufficient conditions under which these assumptions are satisfied. We feel free to abuse notations and write $Z(\cdot)$ for $\mathcal{L}(Z)$ in the argument of the functions $\partial_x D_\mu \mathrm{F}$, $\partial_z D_\mu \mathrm{F}$ and $D_\mu^2 \mathrm{F}$. We prove in Section 3.1 that the map Γ sends some large ball of its state space into itself for a small enough time horizon T. The contractive character of Γ is proved in Section 3.2, and Section 3.3 is dedicated to proving the well-posed character of equation (0.4) and continuity of the law of its solution with respect to all the parameters in the problems.

3.1 – Stability of Balls by Γ

Recall Λ was introduced in **Regularity assumptions 1** and **2** as a bound on F and some of its derivatives. The following lemma, of a technical nature, brings back the general case to $\Lambda = 1$.

13. Lemma – *There is no loss of generality in assuming* $\Lambda = 1$ *in* **Regularity assumptions 1** *and* **Regularity assumptions 2***.*

Proof – We may indeed change F into $\Lambda^{-1}\mathrm{F}$. Doing so, we need to change in equation (0.4) the driver W into ΛW and $\mathbb{W}$ into $\Lambda^2\mathbb{W}$, and also $\mathbb{W}^{\perp\!\!\!\perp}$ into $\Lambda^2\mathbb{W}^{\perp\!\!\!\perp}$. Importantly, for an ω-controlled path $X(\omega)$ and a random controlled path $Y(\cdot)$ on a segment $[0, T]$, for $T > 0$, this change of variable leaves invariant the definition of the integral

$$\left(\int_0^t \mathrm{F}\big(X_s(\omega), Y_s(\cdot)\big) d\boldsymbol{W}_s(\omega)\right)_{0 \leqslant t \leqslant T}.$$

Indeed, changing the first-level $\boldsymbol{W}^{(1)}$ of the rough set-up into $\boldsymbol{W}^{(1),(\Lambda)} := \Lambda \boldsymbol{W}^{(1)}$ requires to change $\delta_x X(\omega)$ into $\delta_x^{(\Lambda)} X(\omega) := \Lambda^{-1}\delta_x X(\omega)$. Also,

$$\begin{aligned}\delta_x^{(\Lambda)}\big[\Lambda^{-1}\mathrm{F}\big(X(\omega),Y(\cdot)\big)\big]_s &= \Lambda^{-1}\partial_x\mathrm{F}\big(X_s(\omega),Y_s(\cdot)\big)\,\delta_x^{(\Lambda)}X_s(\omega)\\ &= \Lambda^{-2}\partial_x\mathrm{F}\big(X_s(\omega),Y_s(\cdot)\big)\delta_x X_s(\omega)\\ &= \Lambda^{-2}\delta_x\big[\mathrm{F}\big(X(\omega),Y(\cdot)\big)\big]_s,\end{aligned}$$

and, with similar notations,

$$\delta_\mu^{(\Lambda)}\big[\mathrm{F}\big(X(\omega),Y(\cdot)\big)\big]_s = \Lambda^{-2}\delta_\mu\big[\mathrm{F}\big(X(\omega),Y(\cdot)\big)\big]_s.$$

Setting $\boldsymbol{W}^{(2),(\Lambda)} := \Lambda^2\boldsymbol{W}^{(2)}$, for the second level of the rough set up, we then observe that, up to a small remainder,

$$\begin{aligned}&\int_s^t \Lambda^{-1}\mathrm{F}\big(X_u(\omega),Y_u(\cdot)\big)d\boldsymbol{W}_u^{(\Lambda)}(\omega)\\ &\approx \Lambda^{-1}\mathrm{F}\big(X_s(\omega),Y_s(\cdot)\big)W_{s,t}^{(\Lambda)}(\omega) + \delta_x^{(\Lambda)}\big[\Lambda^{-1}\mathrm{F}\big(X(\omega),Y(\cdot)\big)\big]_s\mathbb{W}_{s,t}^{(\Lambda)}(\omega)\\ &\quad + \mathbb{E}\Big[\delta_\mu^{(\Lambda)}\big[\Lambda^{-1}\mathrm{F}\big(X(\omega),Y(\cdot)\big)\big]_s\mathbb{W}_{s,t}^{\perp\!\!\!\perp,(\Lambda)}(\cdot,\omega)\Big]\\ &= \mathrm{F}\big(X_s(\omega),Y_s(\cdot)\big)W_{s,t}(\omega) + \delta_x\big[\mathrm{F}\big(X(\omega),Y(\cdot)\big)\big]_s\mathbb{W}_{s,t}(\omega)\\ &\quad + \mathbb{E}\Big[\delta_\mu\big[\mathrm{F}\big(X(\omega),Y(\cdot)\big)\big]_s\mathbb{W}_{s,t}^{\perp\!\!\!\perp}(\cdot,\omega)\Big].\end{aligned}$$

As the last line is the second order expansion of $\int_s^t \mathrm{F}\big(X_u(\omega),Y_u(\cdot)\big)d\boldsymbol{W}_u(\omega)$, this shows indeed that

$$\int_s^t \Lambda^{-1}\mathrm{F}\big(X_u(\omega),Y_u(\cdot)\big)d\boldsymbol{W}_u^{(\Lambda)}(\omega) = \int_s^t \mathrm{F}\big(X_u(\omega),Y_u(\cdot)\big)d\boldsymbol{W}_u(\omega).$$

$\triangleright$

Recall from identity (1.13) the definition of the local accumulated variation

$$N\big([0,T],\omega;\alpha\big).$$

We use the notations $|\!|\!|\cdot|\!|\!|_{[a,b],w,p}$ and $|\!|\!|\cdot|\!|\!|_{\star,[a,b],w,p}$, for some interval $[a,b]$, to denote a quantity defined in Definition 8 for paths defined on some interval $[a,b]$ rather than on the interval $[0,T]$.

14. Proposition – *Let* F *satisfy* **Regularity assumptions 1** *with* $\Lambda = 1$. *Consider an* ω*-controlled path* $X(\omega)$ *together with a random controlled path* $Y(\cdot)$ *satisfying*

$$\sup_{0\leqslant t\leqslant T}\Big(\big|\delta_x X_t(\omega)\big| \vee \big\langle \delta_x Y_t(\cdot)\big\rangle_\infty\Big) \leqslant 1. \tag{3.1}$$

Assume that there exists a positive constant L *such that we have*

$$\big\langle \|Y(\cdot)\|_{[0,T],w,p}\big\rangle_8^2 \leqslant \sqrt{L}, \qquad \big\langle |\!|\!|Y(\cdot)|\!|\!|_{[0,T],w,p}\big\rangle_8^2 \leqslant L, \tag{3.2}$$

and

$$|\!|\!|X(\omega)|\!|\!|^2_{[t_i,t_{i+1}],w,p} \leqslant \sqrt{L}, \tag{3.3}$$

for all $0 \leqslant i \leqslant N$, *with* $N := N([0,T],\omega,1/(4L))$, *and for the sequence* $\big(t_i := \tau_i(0,T,\omega,1/(4L))\big)_{i=0,\cdots,N+1}$ *given by* (1.12). *Then, these bounds remain true for possibly larger values of* L, *and there exists a universal constant* L_0 *such that the following two properties hold for every* $L \geqslant L_0$.

- *The path* $\Gamma\big(\omega, X(\omega), Y(\cdot)\big)$ *satisfies for each* ω *the size estimate* (3.3)*, and there exist two positive constants* c *and* C_L*, with* c *universal and* C_L *depending only on* L*, such that the following estimates hold for each* ω*:*

$$\big\|\!\big|\Gamma\big(\omega, X(\omega), Y(\cdot)\big)\big\|\!\big|^2_{[0,T],w,p} \leqslant C_L\left\{1 + N\Big([0,T],\omega,1/(4L)\Big)^{2(1-1/p)}\right\},$$

$$\big\|\!\big|\Gamma\big(\omega, X(\omega), Y(\cdot)\big)\big\|\!\big|^2_{\star,[0,T],w,p} \leqslant c\big|X_0(\omega)\big|^2 + C_L\left\{1 + N\Big([0,T],\omega,1/(4L)\Big)^{2(1-1/p)}\right\}; \tag{3.4}$$

- *If* $X(\omega)$ *is the* ω*-realization of a random controlled path* $X(\cdot) = \big(X(\omega')\big)_{\omega'\in\Omega'}$ *such that the estimate* $\big\|\!\big|X(\omega')\big\|\!\big|^2_{[t_i,t_{i+1}],w,p} \leqslant \sqrt{L}$ *holds for all* ω'*, for the* ω'*-dependent partition* $\big(t_i := \tau_i(0,T,\omega',1/(4L))\big)_{i=0,\cdots,N+1}$ *of* $[0,T]$*, with* $N := N([0,T],\omega',1/(4L))$*, and if* T *is small enough to have*

$$\Big\langle N\big([0,T],\cdot,1/(4L)\big) + 1\Big\rangle_8^{2(p-1)/p} \leqslant 2;$$

then

$$\big\langle\|\Gamma(\cdot,X(\cdot),Y)\|_{[0,T],w,p}\big\rangle_8^2 \leqslant \sqrt{L}, \qquad \big\langle\|\!|\Gamma(\cdot,X(\cdot),Y)\|\!|_{[0,T],w,p}\big\rangle_8^2 \leqslant L,$$

and

$$\Big\langle\big\|\!\big|\Gamma(\cdot,X(\cdot),Y)\big\|\!\big|_{\star,[0,T],w,p}\Big\rangle_2^2 \leqslant C_L\Big(1 + \big\langle X_0(\cdot)\big\rangle_2^2\Big).$$

The measurability properties of the function $\omega \mapsto \Gamma\big(\omega, X(\omega), Y(\cdot)\big)$ implicitely required above can all be checked by approximating the integral in the definition of $\Gamma\big(\omega, X(\omega), Y(\cdot)\big)$ by means of (2.3).

Proof – We proceed in three steps.

- For a given $\omega \in \Omega$, consider a subdivision $(t_i)_{0\leqslant i\leqslant N+1}$ of $[0,T]$ such that

$$w(t_i,t_{i+1},\omega) \leqslant 1$$

for all $i \in \{0,\cdots,N\}$, for some integer $N \geqslant 0$. Then, by Proposition 4 in Coutin and Lejay [16] (rearranging the terms therein), we know that

$$\bigg\|\!\bigg|\int_{t_i}^{\cdot} \mathrm{F}\big(X_r(\omega), Y_r(\cdot)\big)d\boldsymbol{W}_r(\omega)\bigg\|\!\bigg|_{[t_i,t_{i+1}],w,p} \leqslant \gamma + \gamma w(t_i,t_{i+1},\omega)^{1/p}\Big\|\!\Big|\mathrm{F}\big(X(\omega),Y(\cdot)\big)\Big\|\!\Big|_{\star,[t_i,t_{i+1}],w,p},$$

for a universal constant $\gamma \geqslant 1$. By Proposition 11 and (3.1), we deduce that

$$\bigg\|\!\bigg|\int_{t_i}^{\cdot} \mathrm{F}\big(X_r(\omega), Y_r(\cdot)\big)d\boldsymbol{W}_r(\omega)\bigg\|\!\bigg|_{[t_i,t_{i+1}],w,p} \leqslant \gamma + C_{1,1}\,\gamma\, w(t_i,t_{i+1},\omega)^{1/p}\Big(1 + \|\!|X\|\!|^2_{[t_i,t_{i+1}],w,p} + \big\langle\|\!|Y(\cdot)\|\!|_{[0,T],w,p}\big\rangle_8^2\Big). \tag{3.5}$$

For a given constant $L \geqslant 1$ that will be fixed later on, assume that we have both $C_{1,1}\,\gamma\, w(t_i,t_{i+1},\omega)^{1/p} \leqslant 1/(4L\gamma) \leqslant 1$ and

$$\big\langle\|Y(\cdot)\|_{[0,T],w,p}\big\rangle_8^2 \leqslant \sqrt{L}, \qquad \big\langle\|\!|Y(\cdot)\|\!|_{[0,T],w,p}\big\rangle_8^2 \leqslant L, \tag{3.6}$$

and

$$\big\|\!\big|X(\omega)\big\|\!\big|^2_{[t_i,t_{i+1}],w,p} \leqslant \sqrt{L}. \tag{3.7}$$

Then

$$\left\| \left\| \int_{t_i}^{\cdot} \mathrm{F}\big(X_r(\omega), Y_r(\cdot)\big) d\boldsymbol{W}_r(\omega) \right\| \right\|_{[t_i,t_{i+1}],w,p} \leqslant 2\gamma. \tag{3.8}$$

Hence,

$$\left\| \left\| \int_{t_i}^{\cdot} \mathrm{F}\big(X_r(\omega), Y_r(\cdot)\big) d\boldsymbol{W}_r(\omega) \right\| \right\|^2_{[t_i,t_{i+1}],w,p} \leqslant 4\gamma^2 < \sqrt{L},$$

if $L > 16\gamma^4$, in which case $\Gamma\big(\omega, X(\omega), Y(\cdot)\big)$ satisfies (3.3).

• We now use a concatenation argument to get an estimate on the whole interval $[0, T]$. For all $s < t$ in $[0, T]$, we have

$$\begin{aligned}
\Big| \big[\Gamma\big(\omega, X(\omega), Y(\cdot)\big)\big]_{s,t} \Big| &\leqslant \sum_{j=0}^{N} \Big| \big[\Gamma\big(\omega, X(\omega), Y(\cdot)\big)\big]_{t'_j,t'_{j+1}} \Big| \\
&\leqslant 2\gamma \sum_{j=0}^{N} w\big(t'_j, t'_{j+1}, \omega\big)^{1/p} \\
&\leqslant 2\gamma \left(\sum_{j=0}^{N} w(t'_j, t'_{j+1}, \omega) \right)^{1/p} \big(N+1\big)^{(p-1)/p} \\
&\leqslant 2\gamma\, w(s,t,\omega)^{1/p} \big(N+1\big)^{(p-1)/p},
\end{aligned}$$

where we let $t'_i = \max(s, \min(t, t_i))$ and where used the super-additivity of w in the last line. In the same way,

$$\begin{aligned}
\Big| \delta_x\big[\Gamma\big(\omega, X(\omega), Y(\cdot)\big)\big]_{s,t} \Big| &\leqslant \sum_{j=0}^{N} \Big| \delta_x\big[\Gamma\big(\omega, X(\omega), Y(\cdot)\big)\big]_{t'_j,t'_{j+1}} \Big| \\
&\leqslant 2\gamma\, w(s,t,\omega)^{1/p} \big(N+1\big)^{(p-1)/p}.
\end{aligned}$$

Setting, with a slight abuse of notation,

$$\mathrm{F}(\omega, \cdot) := \big(\mathrm{F}_r(\omega, \cdot)\big)_{0\leqslant r\leqslant T} := \big(\mathrm{F}(X_r(\omega), Y_r(\cdot))\big)_{0\leqslant r\leqslant T},$$

we have

$$\begin{aligned}
&R^{\Gamma}_{s,t}(\omega) \\
&= \int_s^t \mathrm{F}_r(\omega, \cdot) d\boldsymbol{W}_r(\omega) - \mathrm{F}_s(\omega, \cdot) W_{s,t}(\omega) - \delta_x \mathrm{F}_s(\omega, \cdot) \mathbb{W}_{s,t}(\omega) - \mathbb{E}\big[\delta_\mu \mathrm{F}_s(\omega, \cdot) \mathbb{W}^{\perp\!\!\!\perp}_{s,t}(\cdot, \omega)\big] \\
&= \sum_{j=0}^{N} \int_{t'_j}^{t'_{j+1}} \mathrm{F}_r(\omega, \cdot) d\boldsymbol{W}_r(\omega) \\
&\qquad - \mathrm{F}_s(\omega, \cdot) W_{s,t} - \delta_x \mathrm{F}_s(\omega, \cdot) \mathbb{W}_{s,t}(\omega) - \mathbb{E}\big[\delta_\mu \mathrm{F}_s(\omega, \cdot) \mathbb{W}^{\perp\!\!\!\perp}_{s,t}(\cdot, \omega)\big] \qquad (3.9) \\
&= \sum_{j=0}^{N} \Big\{ R^{\Gamma}_{t'_j,t'_{j+1}}(\omega) + \big(\mathrm{F}_{t'_j}(\omega, \cdot) - \mathrm{F}_s(\omega, \cdot)\big) W_{t'_j,t'_{j+1}}(\omega) \\
&\qquad\qquad + \delta_x \mathrm{F}_{t'_j}(\omega, \cdot) \mathbb{W}_{t'_j,t'_{j+1}}(\omega) + \mathbb{E}\big[\delta_\mu \mathrm{F}_{t'_j}(\omega, \cdot) \mathbb{W}^{\perp\!\!\!\perp}_{t'_j,t'_{j+1}}(\cdot, \omega)\big] \Big\} \\
&\qquad - \delta_x \mathrm{F}_s(\omega, \cdot) \mathbb{W}_{s,t}(\omega) - \mathbb{E}\big[\delta_\mu \mathrm{F}_s(\omega, \cdot) \mathbb{W}^{\perp\!\!\!\perp}_{s,t}(\cdot, \omega)\big],
\end{aligned}$$

where $\delta_x \mathrm{F}_r(\omega, \cdot)$ and $\delta_\mu \mathrm{F}_r(\omega, \cdot)$ stand here for the x and μ-derivatives of the ω controlled path

$$\big(\mathrm{F}_r(\omega, \cdot)\big)_{0\leqslant r\leqslant T}.$$

We recall that the product $\delta_x \mathrm{F}_s(\omega,\cdot)\, \mathbb{W}_{s,t}(\omega)$ is understood as the result of the action of an element of $\mathbb{R}^d \otimes \mathbb{R}^m \otimes \mathbb{R}^m$ onto an element of $\mathbb{R}^m \otimes \mathbb{R}^m$, i.e., as an element of $\mathbb{R}^d$ with i^{th} coordinate

$$\begin{aligned}\big(\delta_x \mathrm{F}_s(\omega,\cdot)\mathbb{W}_{s,t}(\omega)\big)_i &= \delta_x \mathrm{F}_s^{i,\cdot}(\omega,\cdot)\mathbb{W}_{s,t}(\omega)\\ &= \sum_{j,k=1}^m \big(\delta_x \mathrm{F}_s^{i,j}\big)^k(\omega,\cdot)\big(\mathbb{W}_{s,t}(\omega)\big)_{k,j};\end{aligned}$$

a similar notation is used for $\delta_\mu \mathrm{F}$. Above, $\mathrm{F}_s^{i,\cdot}(\omega,\cdot)$ is an m dimensional vector obtained by considering the i^{th} line in the $d \times m$ matrix $(\mathrm{F}_s^{i,j}(\omega,\cdot))_{1\leqslant i\leqslant d, 1\leqslant j\leqslant m}$, and $\delta_x \mathrm{F}_s^{i,\cdot}(\omega,\cdot)$ is an $m \times m$ matrix.

The most difficult term to handle in (3.9) is $\sum_{j=0}^N \big(\mathrm{F}_{t'_j}(\omega,\cdot) - \mathrm{F}_s(\omega,\cdot)\big)\, W_{t'_j,t'_{j+1}}(\omega)$. We first notice that the increments $\mathrm{F}_{t'_j}(\omega,\cdot) - \mathrm{F}_s(\omega,\cdot)$, for $j = 0,\cdots,N$, can be bounded by $\sum_{i=0}^{j-1}\big(|X_{t'_{i+1}}(\omega) - X_{t'_i}(\omega)| + \big\langle Y_{t'_{i+1}}(\cdot) - Y_{t'_i}(\cdot)\big\rangle_2\big)$, since F is 1-Lipschitz continuous. Then, $|X_{t'_{i+1}}(\omega) - X_{t'_i}(\omega)|$ is less than $\|X(\omega)\|_{[t'_i,t'_{i+1}],w,p}\, w(t'_i,t'_{i+1},\omega)^{1/p}$ and, following Lemma 9, $\langle Y_{t'_{i+1}}(\cdot) - Y_{t'_i}(\cdot)\rangle_2 \leqslant 2\big\langle \|Y(\cdot)\|_{[t'_i,t'_{i+1}],w,p}\big\rangle_8\, w(t'_i,t'_{i+1},\omega)^{1/p}$. Invoking the first bound in (3.6) –this is the rationale for it– together with (3.7), we deduce that the sum $\sum_{j=0}^N \big(\mathrm{F}_{t'_j}(\omega,\cdot) - \mathrm{F}_s(\omega,\cdot)\big)\, W_{t'_j,t'_{j+1}}(\omega)$ is bounded by

$$\begin{aligned}&3\gamma\, L^{1/4}\sum_{j=0}^N\left(\sum_{i=0}^N w(t'_i,t'_{i+1},\omega)^{1/p}\right) w(t'_j,t'_{j+1},\omega)^{1/p}\\ &\leqslant 3\,\gamma\, L^{1/4}(N+1)^{2(p-1)/p}\, w(s,t,\omega)^{2/p}.\end{aligned}$$

In order to proceed with the other terms in (3.9), we note that since $|\mathrm{F}|$, $|\partial_x \mathrm{F}|$ and $\big\langle \nabla_Z \mathrm{F}\big\rangle_2$ are less than $\Lambda = 1$, and $|\delta_x X(\omega)| = \big(|\delta_x X_t(\omega)|\big)_{0\leqslant t\leqslant T}$ and $\big\langle \delta_x Y(\cdot)\big\rangle_\infty = \big(\big\langle \delta_x Y_t(\cdot)\big\rangle_\infty\big)_{0\leqslant t\leqslant T}$ are all less than 1, Proposition 11 ensures that

$$\Big|\delta_x\big[\mathrm{F}\big(X(\omega),Y(\cdot)\big)\big]\Big| \vee \Big\langle \delta_\mu[\mathrm{F}(X(\omega),Y(\cdot))]\Big\rangle_2 \vee \Big|\delta_x\big[\Gamma\big(\omega,X(\omega),Y(\cdot)\big)\big]\Big| \leqslant 1.$$

The other terms in the last two lines of (3.9) are easily handled using the above bound. As for the remainder term $R^\Gamma_{t'_j,t'_{j+1}}(\omega)$, it can be estimated by means of (3.8). Finally, one can find a constant C_γ depending only on γ such that

$$\big|R^\Gamma_{s,t}(\omega)\big| \leqslant C_\gamma\big(1+L^{1/4}\big)\,(N+1)^{2(p-1)/p}\, w(s,t,\omega)^{2/p}.$$

Changing the value of C_γ from line to line, we end up with

$$\begin{aligned}\Big\|\Gamma\big(\omega,X(\omega),Y(\cdot)\big)\Big\|^2_{[0,T],w,p} &\leqslant C_\gamma\,(N+1)^{2(p-1)/p},\\ \Big\|\!\Big|\Gamma\big(\omega,X(\omega),Y(\cdot)\big)\Big|\!\Big\|^2_{[0,T],w,p} &\leqslant C_\gamma\,\big(1+\sqrt{L}\big)\,(N+1)^{2(p-1)/p},\end{aligned}$$

which proves the bound (3.4) by choosing the sequence

$$(t_i)_{i=0,\cdots,N+1} = \big(\tau_i(0,T,\omega,1/(4L))\big)_{i=0,\cdots,N+1}$$

defined in (1.12) and $N = N\big([0,T],\omega,1/(4L)\big)$.

• Assume now that $X(\omega)$ is the realization of a random controlled path $X(\cdot) = (X(\omega'))_{\omega'\in\Omega'}$ satisfying the bound (3.3) for any ω', for the ω'-dependent partition

$(t_i)_{i=0,\cdots,N+1}$. Then, integrating with respect to ω the conclusion of the second point we get

$$\Big\langle \big\|\Gamma\big(\cdot, X(\cdot), Y\big)\big\|_{[0,T],w,p} \Big\rangle_8^2 \leqslant C_\gamma \Big\langle N\big([0,T], \cdot, 1/(4L)\big) + 1 \Big\rangle_8^{2(p-1)/p},$$

$$\Big\langle \big\|\!\big|\Gamma\big(\cdot, X(\cdot), Y\big)\big\|\!\big|_{[0,T],w,p} \Big\rangle_8^2 \leqslant C_\gamma \left(1+\sqrt{L}\right) \Big\langle N\big([0,T], \cdot, 1/(4L)\big) + 1 \Big\rangle_8^{2(p-1)/p}.$$

We get the conclusion of the statement if one assumes that

$$\Big\langle N\big([0,T], \cdot, 1/(4L)\big) + 1 \Big\rangle_8^{2(p-1)/p} \leqslant 2,$$

by choosing L such that $2\,C_\gamma \leqslant \sqrt{L}$ and $2\,C_\gamma\,(1+\sqrt{L}) \leqslant L$. $\rhd$

Remark that if $\big\langle N\big([0,1], \cdot, 1/(4L)\big) + 1\big\rangle_8$ is finite, then we can choose $T \leqslant 1$ small enough such that the condition $\big\langle N\big([0,T], \cdot, 1/(4L)\big) + 1\big\rangle_8^{2(p-1)/p} \leqslant 2$ is satisfied. (Since $N\big([0,t], \omega, 1/(4L)\big)$ converges to 0 as $t \searrow 0$, for any $\omega \in \Omega$, the result follows indeed from Lebesgue's dominated convergence theorem.)

3.2 – Contractive Property of Γ

15. Proposition – *Let* F *satisfy* **Regularity assumptions 1** *and* **Regularity assumptions 2** *with* $\Lambda = 1$. *Consider two* ω*-controlled paths* $X(\omega)$ *and* $X'(\omega)$, *defined on a time interval* $[0,T]$, *together with two random controlled paths* $Y(\cdot)$ *and* $Y'(\cdot)$ *satisfying*

$$\big|\delta_x X(\omega)\big| \vee \big|\delta_x X'(\omega)\big| \vee \big\langle \delta_x Y(\cdot)\big\rangle_\infty \vee \big\langle \delta_x Y'(\cdot)\big\rangle_\infty \leqslant 1, \tag{3.10}$$

together with the size estimates

$$\begin{aligned} &\big\langle \|Y(\cdot)\|_{[0,T],w,p}\big\rangle_8^2 \leqslant \sqrt{L_0}, \qquad \big\langle \|\!|Y(\cdot)|\!\|_{[0,T],w,p}\big\rangle_8^2 \leqslant L_0, \\ &\big\langle \|Y'(\cdot)\|_{[0,T],w,p}\big\rangle_8^2 \leqslant \sqrt{L_0}, \qquad \big\langle \|\!|Y'(\cdot)|\!\|_{[0,T],w,p}\big\rangle_8^2 \leqslant L_0, \end{aligned} \tag{3.11}$$

and

$$\big\|\!\big|X(\omega)\big|\!\big\|^2_{[t_i^0,t_{i+1}^0],w,p} \leqslant \sqrt{L_0}, \qquad \big\|\!\big|X'(\omega)\big|\!\big\|^2_{[t_i^0,t_{i+1}^0],w,p} \leqslant \sqrt{L_0}, \tag{3.12}$$

for $i \in \{0, \cdots, N^0\}$, *for* L_0 *given by* Proposition 14, *and* $N^0 = N\big([0,T], \omega, 1/(4L_0)\big)$ *given by* (1.13), *and for the sequence* $\big(t_i^0 = \tau_i(0,T,\omega,1/(4L_0))\big)_{i=0,\cdots,N^0+1}$ *given by* (1.12). *Then, we can find a constant* γ *depending on* L_0 *such that, for any partition* $(t_i)_{i=0,\cdots,N}$ *refining* $(t_i^0)_{i=0,\cdots,N^0}$ *and satisfying* $w(t_i,t_{i+1},\omega) \leqslant 1/(4L)$ *for some* $L \geqslant L_0$, *we have*

$$\begin{aligned} &\left\|\!\left| \int_{t_i}^{\cdot} F\big(X_r(\omega), Y_r(\cdot)\big) d\boldsymbol{W}_r(\omega) - \int_{t_i}^{\cdot} F\big(X'_r(\omega), Y'_r(\cdot)\big) d\boldsymbol{W}_r(\omega) \right|\!\right\|_{[t_i,t_{i+1}],w,p} \\ &\leqslant \gamma\, w(0,t_i,\omega)^{1/p} \left(1 + \frac{1}{4L}\right) \Big(\big\|\!\big|\Delta X(\omega)\big|\!\big\|_{[0,t_i],w,p} + \big\langle \|\!|\Delta Y(\cdot)|\!\|_{[0,T],w,p}\big\rangle_8 \Big) \\ &\qquad + \frac{\gamma}{4L} \Big(\big\|\!\big|\Delta X(\omega)\big|\!\big\|_{[t_i,t_{i+1}],w,p} + \big\langle \|\!|\Delta Y(\cdot)|\!\|_{[0,T],w,p}\big\rangle_8 \Big), \end{aligned}$$

where

$$\Delta X_t(\omega) := X_t(\omega) - X'_t(\omega), \qquad \Delta Y_t(\cdot) := Y_t(\cdot) - Y'_t(\cdot), \quad t \in [0,T].$$

Proof – We get the conclusion after four intermediate steps. Proceeding as in the proof of stability, we consider a subdivision $(t_i)_{i=0,\cdots,N+1}$ of the interval $[0,T]$ such that $w(t_i,t_{i+1},\omega) \leqslant 1/(4L)$, for a frozen value of $\omega \in \Omega$. The value of $L \geqslant L_0$ will be fixed later on. We can assume without any loss of generality that the partition $(t_i)_{i=0,\cdots,N+1}$ refines the partition $\big(t_i^0 = \tau_i(0,T,\omega,1/(4L_0))\big)_{i=0,\cdots,N^0+1}$, where $N^0(\omega) = N\big([0,T],\omega,1/(4L_0)\big)$. Like in the first step of the proof of Proposition 14, we start from the estimate

$$
\begin{aligned}
&\left\|\!\left|\int_{t_i}^{\cdot} \mathrm{F}\big(X_r(\omega),Y_r(\cdot)\big)d\boldsymbol{W}_r(\omega) - \int_{t_i}^{\cdot} \mathrm{F}\big(X'_r(\omega),Y'_r(\cdot)\big)d\boldsymbol{W}_r(\omega)\right|\!\right\|_{[t_i,t_{i+1}],w,p} \\
&\leqslant \gamma\left(\big|X_{t_i}(\omega) - X'_{t_i}(\omega)\big| + \|Y_{t_i}(\cdot) - Y'_{t_i}(\cdot)\|_2\right) \\
&\quad + \gamma\, w(t_i,t_{i+1},\omega)^{1/p}\, \big\|\!\big|\mathrm{F}\big(X(\omega),Y(\cdot)\big) - \mathrm{F}(X'(\omega),Y'(\cdot)\big)\big|\!\big\|_{\star,[t_i,t_{i+1}],w,p},
\end{aligned}
\tag{3.13}
$$

for a universal constant $\gamma \geqslant 1$.

The first point is to bound the quantity $\big\|\!\big|\mathrm{F}\big(X(\omega),Y(\cdot)\big)-\mathrm{F}\big(X'(\omega),Y'(\cdot)\big)\big|\!\big\|_{\star,[t_i,t_{i+1}],w,p}$.

Step 1. We first analyse the term

$$
\begin{aligned}
\Delta\mathrm{F}(\omega,\cdot) &:= \mathrm{F}\big(X(\omega),Y(\cdot)\big) - \mathrm{F}\big(X'(\omega),Y'(\cdot)\big) \\
&:= \Big(\mathrm{F}\big(X_t(\omega),Y_t(\cdot)\big) - \mathrm{F}\big(X'_t(\omega),Y'_t(\cdot)\big)\Big)_{0\leqslant t\leqslant T}.
\end{aligned}
$$

• **Initial condition of $\Delta\mathbf{F}(\omega,\cdot)$ –** As $\big|[\Delta\mathrm{F}(\omega,\cdot)]_{t_i}\big| \leqslant \big(|\Delta X_{t_i}(\omega)| + \langle|\Delta Y_{t_i}(\cdot)|\rangle_2\big)$, we have from Lemma 9 and from the two identities $\Delta X_0(\omega) = 0$ and $\Delta Y_0(\cdot) = 0$

$$
\big|[\Delta\mathrm{F}(\omega,\cdot)]_{t_i}\big| \leqslant 2w(0,t_i,\omega)^{1/p}\left(\|\!|\Delta X(\omega)|\!\|_{[0,t_i],w,p} + \big\langle \|\!|\Delta Y(\cdot)|\!\|_{[0,t_i],w,p}\big\rangle_4\right).
$$

• **Variation of $\Delta\mathbf{F}(\omega,\cdot)$.** Using the notations (2.10) together with similar ones for the processes tagged with a *prime*, we have

$$
\begin{aligned}
&\big[\Delta\mathrm{F}(\omega,\cdot)\big]_{s,t} \\
&= \int_0^1 \Big\{\partial_x\mathrm{F}\Big(X^{(\lambda)}_{s;(s,t)}(\omega),Y^{(\lambda)}_{s;(s,t)}(\cdot)\Big)X_{s,t}(\omega) - \partial_x\mathrm{F}\Big(X^{(\lambda)\prime}_{s;(s,t)}(\omega),Y^{(\lambda)\prime}_{s;(s,t)}(\cdot)\Big)X'_{s,t}(\omega)\Big\}d\lambda \\
&+ \int_0^1 \mathbb{E}\Big\{\nabla_Z\mathrm{F}\Big(X^{(\lambda)}_{s;(s,t)}(\omega),Y^{(\lambda)}_{s;(s,t)}(\cdot)\Big)\,Y_{s,t}(\cdot) - \nabla_Z\mathrm{F}\Big(X^{(\lambda)\prime}_{s;(s,t)}(\omega),Y^{(\lambda)\prime}_{s;(s,t)}(\cdot)\Big)\,Y'_{s,t}(\cdot)\Big\}d\lambda.
\end{aligned}
$$

We now use the following three facts. First, we recall once again that $X_0(\omega) = X'_0(\omega)$ and $Y_0(\cdot) = Y'_0(\cdot)$; second, we know from **Regularity assumptions 1** that, for any $x \in \mathbb{R}^d$ and $Z \in \mathbb{L}^2(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^d)$, the quantities $|\partial_x\mathrm{F}(x,Z)|$ and $\langle\nabla_Z\mathrm{F}(x,Z)\rangle_2$ are bounded by 1; last, the two mappings $(x,Z)\mapsto\partial_x\mathrm{F}(x,Z)$ and $(x,Z)\mapsto\nabla_Z\mathrm{F}(x,Z)$ are 1-Lipschitz continuous. Hence, we get, for a new value of the universal constant γ, and for s,t in the interval $[t_i,t_{i+1}]$, the estimate

$$
\begin{aligned}
\big|[\Delta\mathrm{F}(\omega,\cdot)]_{s,t}\big| &\leqslant \big|\Delta X_{s,t}(\omega)\big| + \big\langle\Delta Y_{s,t}(\cdot)\big\rangle_2 \\
&\quad + \Big(|X_{s,t}(\omega)| + \big\langle Y_{s,t}(\cdot)\big\rangle_2\Big) \\
&\qquad \times \Big\{|\Delta X_s(\omega)| + \langle\Delta Y_s(\cdot)\rangle_2 + |\Delta X_{s,t}(\omega)| + \big\langle\Delta Y_{s,t}(\cdot)\big\rangle_2\Big\} \\
&\leqslant \boldsymbol{(a)} + \boldsymbol{(b)},
\end{aligned}
$$

where

$$
\boldsymbol{(a)} := \gamma\, w(s,t,\omega)^{1/p}\left(\|\!|\Delta X(\omega)|\!\|_{[t_i,t_{i+1}],w,p} + \big\langle\|\!|\Delta Y(\cdot)|\!\|_{[t_i,t_{i+1}],w,p}\big\rangle_4\right),
$$

and $\boldsymbol{(b)} = \boldsymbol{(b_1)} \times \boldsymbol{(b_2)}$ with

$$\begin{aligned}
\boldsymbol{(b_1)} &:= \gamma\, w(s,t,\omega)^{1/p} \Big(|||X(\omega)|||_{[t_i,t_{i+1}],w,p} + \big\langle |||Y(\cdot)|||_{[t_i,t_{i+1}],w,p} \big\rangle_4 \Big) \\
\boldsymbol{(b_2)} &:= w(0,t_i,\omega)^{1/p} \Big(|||\Delta X(\omega)|||_{[0,t_i],w,p} + \big\langle |||\Delta Y(\cdot)|||_{[0,t_i],w,p} \big\rangle_4 \Big) \\
&\quad + w(t_i,t_{i+1},\omega)^{1/p} \Big(|||\Delta X(\omega)|||_{[t_i,t_{i+1}],w,p} + \big\langle |||\Delta Y(\cdot)|||_{[t_i,t_{i+1}],w,p} \big\rangle_4 \Big).
\end{aligned}$$

It follows that we have

$$\begin{aligned}
\big\|\Delta \mathrm{F}(\omega,\cdot)\big\|_{[t_i,t_{i+1}],w,p} &\leqslant \gamma\Big(|||\Delta X(\omega)|||_{[t_i,t_{i+1}],w,p} + \big\langle |||\Delta Y(\cdot)|||_{[t_i,t_{i+1}],w,p} \big\rangle_4 \Big) \\
&\quad + \gamma\Big(|||X(\omega)|||_{[t_i,t_{i+1}],w,p} + \big\langle |||Y(\cdot)|||_{[t_i,t_{i+1}],w,p} \big\rangle_4 \Big) \times \boldsymbol{(b_2)}.
\end{aligned}$$

Allowing the constant γ to depend on L_0 and using (3.11) and (3.12), we get

$$\begin{aligned}
\big\|\Delta \mathrm{F}(\omega,\cdot)\big\|_{[t_i,t_{i+1}],w,p} &\leqslant \gamma\Big(|||\Delta X(\omega)|||_{[t_i,t_{i+1}],w,p} + \big\langle |||\Delta Y(\cdot)|||_{[t_i,t_{i+1}],w,p} \big\rangle_4 \Big) \\
&\quad + \gamma\, w(0,t_i,\omega)^{1/p} \Big(|||\Delta X(\omega)|||_{[0,t_i],w,p} + \big\langle |||\Delta Y(\cdot)|||_{[0,t_i],w,p} \big\rangle_4 \Big).
\end{aligned}$$

Step 2 – *We now handle the Gubinelli derivative* $\delta_x[\Delta \mathrm{F}(\omega,\cdot)]$. We start from the algebraic identity

$$\begin{aligned}
\delta_x[\Delta \mathrm{F}(\omega,\cdot)]_t &= \big[\partial_x \mathrm{F}\big(X_t(\omega),Y_t(\cdot)\big) - \partial_x \mathrm{F}\big(X'_t(\omega),Y'_t(\cdot)\big)\big]\, \delta_x X_t(\omega) \\
&\quad + \partial_x \mathrm{F}\big(X'_t(\omega),Y'_t(\cdot)\big)\, \Delta\delta_x X_t(\omega).
\end{aligned}$$

• **Initial condition of $\delta_x\big[\Delta\mathbf{F}(\omega,\cdot)\big]$.** Combining **Regularity assumptions 1** and (3.10), we obtain the estimate

$$\begin{aligned}
\Big|\delta_x\big[\Delta \mathrm{F}(\omega,\cdot)\big]_{t_i}\Big| &\leqslant \big|\delta_x \Delta X_{t_i}(\omega)\big| + \big|\Delta X_{t_i}(\omega)\big| + \big\langle \Delta Y_{t_i}(\cdot)\big\rangle_2 \\
&\leqslant \gamma\, w(0,t_i,\omega)^{1/p} \Big(|||\Delta X(\omega)|||_{[0,t_i],w,p} + \big\langle |||\Delta Y(\cdot)|||_{[0,t_i],w,p} \big\rangle_4 \Big).
\end{aligned}$$

• **Variation of $\partial_x\big[\Delta\mathbf{F}(\omega,\cdot)\big]$.** Similarly,

$$\begin{aligned}
\Big|\delta_x\big[\Delta \mathrm{F}(\omega,\cdot)\big]_{s,t}\Big| &\leqslant \big|[\delta_x X(\omega)]_{s,t}\big| \Big(|\Delta X_s(\omega)| + \big\langle \Delta Y_s(\cdot)\big\rangle_2 \Big) \\
&\quad + \Big|\big[\partial_x \mathrm{F}\big(X(\omega),Y(\cdot)\big) - \partial_x \mathrm{F}\big(X'(\omega),Y'(\cdot)\big)\big]_{s,t}\Big| \\
&\quad + \Big|\big[\Delta\delta_x X(\omega)\big]_{s,t}\Big| + \big|\Delta\delta_x X_s(\omega)\big| \Big|\big[\partial_x \mathrm{F}\big(X'(\omega),Y'(\cdot)\big)\big]_{s,t}\Big|.
\end{aligned} \tag{3.14}$$

The second term in the right-hand side is handled as $[\Delta\mathrm{F}(\omega,\cdot)]_{s,t}$ in the first step, with s and t in $[t_i,t_{i+1}]$. Observing by linearity that $\Delta\delta_x X(\omega) = \delta_x \Delta X(\omega)$, the third term is seen to be less than $w(s,t,\omega)^{1/p}\, |||\Delta X(\omega)|||_{[t_i,t_{i+1}],w,p}$. The term $\big|\Delta\delta_x X_s(\omega)\big|\big|[\partial_x \mathrm{F}(X'(\omega),Y'(\cdot))]_{s,t}\big|$ may be bounded above by

$$\begin{aligned}
&\gamma\, w(s,t,\omega)^{1/p} \Big(w(0,t_i,\omega)^{1/p} |||\Delta X(\omega)|||_{[0,t_i],w,p} + w(t_i,t_{i+1},\omega)^{1/p} |||\Delta X(\omega)|||_{[t_i,t_{i+1}],w,p} \Big) \\
&\quad \times \Big(|||X'(\omega)|||_{[t_i,t_{i+1}],w,p} + \big\langle |||Y'(\cdot)|||_{[t_i,t_{i+1}],w,p} \big\rangle_4 \Big) \\
&\leqslant \gamma\, w(s,t,\omega)^{1/p} \Big(w(0,t_i,\omega)^{1/p} |||\Delta X(\omega)|||_{[0,t_i],w,p} + |||\Delta X(\omega)|||_{[t_i,t_{i+1}],w,p} \Big),
\end{aligned}$$

where we used again (3.11) and (3.12). Now, the first term in (3.14) is less than

$$\begin{aligned}&\gamma\, w(s,t,\omega)^{1/p}\, \vert\!\vert\!\vert X \vert\!\vert\!\vert_{[t_i,t_{i+1}],w,p}\\ &\quad \times \Big\{ w(0,t_i,\omega)^{1/p} \Big(\vert\!\vert\!\vert \Delta X(\omega) \vert\!\vert\!\vert_{[0,t_i],w,p} + \big\langle \vert\!\vert\!\vert \Delta Y(\cdot) \vert\!\vert\!\vert_{[0,t_i],w,p} \big\rangle_4 \Big)\\ &\qquad + w(t_i,t_{i+1},\omega)^{1/p} \Big(\vert\!\vert\!\vert \Delta X(\omega) \vert\!\vert\!\vert_{[t_i,t_{i+1}],w,p} + \big\langle \vert\!\vert\!\vert \Delta Y(\cdot) \vert\!\vert\!\vert_{[t_i,t_{i+1}],w,p} \big\rangle_4 \Big) \Big\}.\end{aligned}$$

Hence, by (3.12),

$$\begin{aligned}&\big| [\delta_x X(\omega)]_{s,t} \big| \Big(|\Delta X_s(\omega)| + \langle |\Delta Y_s(\cdot)| \rangle_2 \Big)\\ &\leqslant \gamma\, w(s,t,\omega)^{1/p} \Big\{ w(0,t_i,\omega)^{1/p} \Big(\vert\!\vert\!\vert \Delta X(\omega) \vert\!\vert\!\vert_{[0,t_i],w,p} + \big\langle \vert\!\vert\!\vert \Delta Y(\cdot) \vert\!\vert\!\vert_{[0,t_i],w,p} \big\rangle_4 \Big)\\ &\quad + \Big(\vert\!\vert\!\vert \Delta X(\omega) \vert\!\vert\!\vert_{[t_i,t_{i+1}],w,p} + \big\langle \vert\!\vert\!\vert \Delta Y(\cdot) \vert\!\vert\!\vert_{[t_i,t_{i+1}],w,p} \big\rangle_4 \Big) \Big\}.\end{aligned}$$

So, the final bound for $\big\| \delta_x \big[\Delta \mathrm{F}(\omega,\cdot) \big] \big\|_{[t_i,t_{i+1}],w,p}$ is

$$\begin{aligned}&\gamma \Big(\vert\!\vert\!\vert \Delta X(\omega) \vert\!\vert\!\vert_{[t_i,t_{i+1}],w,p} + \big\langle \vert\!\vert\!\vert \Delta Y(\cdot) \vert\!\vert\!\vert_{[t_i,t_{i+1}],w,p} \big\rangle_4 \Big)\\ &\qquad + \gamma\, w(0,t_i,\omega)^{1/p} \Big(\vert\!\vert\!\vert \Delta X(\omega) \vert\!\vert\!\vert_{[0,t_i],w,p} + \big\langle \vert\!\vert\!\vert \Delta Y(\cdot) \vert\!\vert\!\vert_{[0,t_i],w,p} \big\rangle_4 \Big),\end{aligned}$$

which yields the same bound as in the first step.

Step 3 – We now handle the other Gubinelli derivative $\delta_\mu\big[\Delta \mathrm{F}(\omega,\cdot)\big]$, for which we have

$$\begin{aligned}\delta_\mu\big[\Delta \mathrm{F}(\omega,\cdot)\big]_t &= \Big[\nabla_Z \mathrm{F}\big(X_t(\omega), Y_t(\cdot)\big) - \nabla_Z \mathrm{F}\big(X'_t(\omega), Y'_t(\cdot)\big) \Big] \delta_x Y_t(\cdot)\\ &\quad + \nabla_Z \mathrm{F}\big(X'_t(\omega), Y'_t(\cdot)\big)\, \Delta \delta_x Y_t(\cdot).\end{aligned}$$

• **Initial condition of $\delta_\mu\big[\Delta\mathbf{F}(\omega,\cdot)\big]$.** Proceeding as before,

$$\begin{aligned}\Big\langle \delta_\mu[\Delta \mathrm{F}(\omega,\cdot)]_{t_i} \Big\rangle_{4/3} &\leqslant \big| \Delta X_{t_i}(\omega) \big| + \big\langle \Delta Y_{t_i}(\cdot) \big\rangle_4 + \big\langle \delta_x \Delta Y_{t_i}(\cdot) \big\rangle_4\\ &\leqslant \gamma\, w(0,t_i,\omega)^{1/p} \Big(\vert\!\vert\!\vert \Delta X(\omega) \vert\!\vert\!\vert_{[0,t_i],w,p} + \big\langle \vert\!\vert\!\vert \Delta Y(\cdot) \vert\!\vert\!\vert_{[0,t_i],w,p} \big\rangle_8 \Big),\end{aligned}$$

where we used the Hölder inequality

$$\begin{aligned}&\mathbb{E}\Big[\big| \Delta \delta_x Y_t(\cdot) \big|^{4/3} \big| \nabla_Z \mathrm{F}\big(X'_t(\omega), Y'_t(\cdot)\big) \big|^{4/3} \Big]^{3/4}\\ &\qquad \leqslant \mathbb{E}\Big[\big| \Delta \delta_x Y_t(\cdot) \big|^4 \Big]^{1/4} \mathbb{E}\Big[\big| \nabla_Z \mathrm{F}\big(X'_t(\omega), Y'_t(\cdot)\big) \big|^2 \Big]^{1/2},\end{aligned}$$

with exponents 3 and 3/2.

• **Variation of $\partial_\mu[\Delta\mathbf{F}(\omega,\cdot)]$.** Using again Hölder inequality with exponents 3 and 3/2, we get

$$\begin{aligned}&\Big\langle \big[\delta_\mu[\Delta \mathrm{F}(\omega,\cdot)] \big]_{s,t} \Big\rangle_{4/3}\\ &\leqslant \big\langle [\delta_x Y(\cdot)]_{s,t} \big\rangle_4 \Big(|\Delta X_s(\omega)| + \big\langle \Delta Y_s(\cdot) \big\rangle_2 \Big)\\ &\quad + \Big\langle \Big[\nabla_Z \mathrm{F}\big(X(\omega), Y(\cdot)\big) - \nabla_Z \mathrm{F}\big(X'(\omega), Y'(\cdot)\big) \Big]_{s,t} \Big\rangle_{4/3}\\ &\quad + \big\langle [\Delta \delta_x Y(\cdot)]_{s,t} \big\rangle_4 + \big\langle \Delta \delta_x Y_s(\cdot) \big\rangle_4 \Big\langle \Big[\nabla_Z \mathrm{F}\big(X'(\omega), Y'(\cdot)\big) \Big]_{s,t} \Big\rangle_2.\end{aligned} \tag{3.15}$$

Thanks to Lemma 9, the third term is less than $2w(s,t,\omega)^{1/p}\big\langle \|\!|\Delta Y(\cdot)\|\!|_{[t_i,t_{i+1}],w,p}\big\rangle_8$. As for the fourth term, we have

$$
\begin{aligned}
&\big\langle \Delta\delta_x Y_s(\cdot)\big\rangle_4 \Big\langle \big[\nabla_Z \mathrm{F}\big(X(\omega),Y(\cdot)\big)\big]_{s,t}\Big\rangle_2 \\
&\leqslant \gamma w(s,t,\omega)^{1/p}\Big(\|\!|\Delta X(\omega)\|\!|_{[t_i,t_{i+1}],w,p} + \big\langle \|\!|\Delta Y(\cdot)\|\!|_{[t_i,t_{i+1}],w,p}\big\rangle_4\Big) \\
&\quad \times \Big\{ w(0,t_i,\omega)^{1/p}\big\langle \|\!|\Delta Y(\cdot)\|\!|_{[0,t_i],w,p}\big\rangle_8 + \big\langle \|\!|\Delta Y(\cdot)\|\!|_{[t_i,t_{i+1}],w,p}\big\rangle_8\Big\} \\
&\leqslant \gamma\, w(s,t,\omega)^{1/p}\Big(w(0,t_i,\omega)^{1/p}\big\langle \|\!|\Delta Y(\cdot)\|\!|_{[0,t_i],w,p}\big\rangle_8 + \big\langle \|\!|\Delta Y(\cdot)\|\!|_{[t_i,t_{i+1}],w,p}\big\rangle_8\Big),
\end{aligned}
$$

where we used (3.11).

Observing as before that $\Delta\delta_x Y(\cdot) = \delta_x \Delta Y(\cdot)$, the third term in (3.15) is seen to be less than $2w(s,t,\omega)^{1/p}\big\langle \|\!|\Delta Y(\cdot)\|\!|_{[t_i,t_{i+1}],w,p}\big\rangle_8$.

We now handle the first term in (3.15). Proceeding as in the second step, we have

$$
\begin{aligned}
&\big\langle [\delta_x Y(\cdot)]_{s,t}\big\rangle_4 \Big(|\Delta X_s(\omega)| + \big\langle \Delta Y_s(\cdot)\big\rangle_2\Big) \\
&\leqslant \gamma\, w(s,t,\omega)^{1/p}\Big\{ w(0,t_i,\omega)^{1/p}\Big(\|\!|\Delta X(\omega)\|\!|_{[0,t_i],w,p} + \big\langle \|\!|\Delta Y(\cdot)\|\!|_{[0,t_i],w,p}\big\rangle_4\Big) \\
&\qquad\qquad + \Big(\|\!|\Delta X(\omega)\|\!|_{[t_i,t_{i+1}],w,p} + \big\langle \|\!|\Delta Y(\cdot)\|\!|_{[t_i,t_{i+1}],w,p}\big\rangle_8\Big)\Big\}.
\end{aligned}
$$

As for the second term in (3.15), we write $\big[\nabla_Z \mathrm{F}\big(X(\omega),Y(\cdot)\big) - \nabla_Z \mathrm{F}\big(X'(\omega),Y'(\cdot)\big)\big]_{s,t}$ in the form $\big[D_\mu \mathrm{F}\big(X(\omega),Y(\cdot)\big)\big(Y(\cdot)\big) - D_\mu \mathrm{F}\big(X'(\omega),Y'(\cdot)\big)\big(Y'(\cdot)\big)\big]_{s,t}$ and then expand it as

$$
\begin{aligned}
&\int_0^1 \Big\{ \partial_x D_\mu \mathrm{F}\Big(X^{(\lambda)}_{s;(s,t)}(\omega), Y^{(\lambda)}_{s;(s,t)}(\cdot)\Big)\Big(Y^{(\lambda)}_{s;(s,t)}(\cdot)\Big) X_{s,t}(\omega) \\
&\qquad - \partial_x D_\mu \mathrm{F}\Big(X^{(\lambda)\prime}_{s;(s,t)}(\omega), Y^{(\lambda)\prime}_{s;(s,t)}(\cdot)\Big)\Big(Y^{(\lambda)\prime}_{s;(s,t)}(\cdot)\Big) X'_{s,t}(\omega)\Big\} d\lambda \\
&+ \int_0^1 \Big\{ \partial_z D_\mu \mathrm{F}\Big(X^{(\lambda)}_{s;(s,t)}(\omega), Y^{(\lambda)}_{s;(s,t)}(\cdot)\Big)\Big(Y^{(\lambda)}_{s;(s,t)}(\cdot)\Big) Y_{s,t}(\cdot) \\
&\qquad - \partial_z D_\mu \mathrm{F}\Big(X^{(\lambda)\prime}_{s;(s,t)}(\omega), Y^{(\lambda)\prime}_{s;(s,t)}(\cdot)\Big)\Big(Y^{(\lambda)\prime}_{s;(s,t)}(\cdot)\Big) Y'_{s,t}(\cdot)\Big\} d\lambda \\
&+ \int_0^1 \tilde{\mathbb{E}}\Big\{ D^2_\mu \mathrm{F}\Big(X^{(\lambda)}_{s;(s,t)}(\omega), Y^{(\lambda)}_{s;(s,t)}(\cdot)\Big)\Big(Y^{(\lambda)}_{s;(s,t)}(\cdot), \tilde{Y}^{(\lambda)}_{s;(s,t)}\Big) \tilde{Y}_{s,t}(\cdot) \\
&\qquad - \tilde{\mathbb{E}}\Big\{ D^2_\mu \mathrm{F}\Big(X^{(\lambda)\prime}_{s;(s,t)}(\omega), Y^{(\lambda)\prime}_{s;(s,t)}(\cdot)\Big)\Big(Y^{(\lambda)\prime}_{s;(s,t)}(\cdot), \tilde{Y}^{(\lambda)\prime}_{s;(s,t)}\Big) \tilde{Y}'_{s,t}(\cdot)\Big\} d\lambda,
\end{aligned}
$$

where the symbol $\sim$ is used to denote independent copies of the various random variables and where, as before, we used the notation (2.10), with an obvious analogue for the processes tagged with a *prime* or a *tilde*. By using Hölder inequality with exponents 3 and $3/2$, we get

$$
\begin{aligned}
&\Big\langle \big[\nabla_Z \mathrm{F}\big(X(\omega),Y(\cdot)\big) - \nabla_Z \mathrm{F}\big(X'(\omega),Y'(\cdot)\big)\big]_{s,t}\Big\rangle_{4/3} \\
&\leqslant \gamma\Big\{ \big|\Delta X_{s,t}(\omega)\big| + \big\langle \Delta Y_{s,t}(\cdot)\big\rangle_4 \\
&\quad + |X_{s,t}(\omega)|\Big(\big|\Delta X_s(\omega)\big| + \big\langle \Delta Y_s(\cdot)\big\rangle_2 + \big|\Delta X_{s,t}(\omega)\big| + \big\langle \Delta Y_{s,t}(\cdot)\big\rangle_2\Big) \\
&\quad + \big\langle Y_{s,t}(\cdot)\big\rangle_4 \Big(\big|\Delta X_s(\omega)\big| + \big\langle \Delta Y_s(\cdot)\big\rangle_2 + \big|\Delta X_{s,t}(\omega)\big| + \big\langle \Delta Y_{s,t}(\cdot)\big\rangle_2\Big)\Big\},
\end{aligned}
$$

where, to get the first line, we used the fact that $\partial_x D_\mu \mathrm{F}$ and $\partial_z D_\mu \mathrm{F}$ and the function

$$\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \ni (x,\mu) \mapsto \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \big|D^2\mathrm{F}(x,\mu)(z,z')\big|^2 \mu(dz)\mu(dz'),$$

are bounded by $\Lambda = 1$. We end up with the same bound as in the first and second steps, namely

$$\begin{aligned}
&\big\langle \delta_\mu[\Delta\mathrm{F}(\omega,\cdot)]\big\rangle_{[t_i,t_{i+1}],w,p,4/3} \\
&\leqslant \gamma\Big(\vert\kern-0.25ex\vert\kern-0.25ex\vert \Delta X(\omega)\vert\kern-0.25ex\vert\kern-0.25ex\vert_{[t_i,t_{i+1}],w,p} + \big\langle \vert\kern-0.25ex\vert\kern-0.25ex\vert \Delta Y(\cdot)\vert\kern-0.25ex\vert\kern-0.25ex\vert_{[t_i,t_{i+1}],w,p}\big\rangle_8\Big) \\
&\quad + \gamma\, w(0,t_i,\omega)^{1/p}\Big(\vert\kern-0.25ex\vert\kern-0.25ex\vert \Delta X(\omega)\vert\kern-0.25ex\vert\kern-0.25ex\vert_{[0,t_i],w,p} + \big\langle \vert\kern-0.25ex\vert\kern-0.25ex\vert \Delta Y(\cdot)\vert\kern-0.25ex\vert\kern-0.25ex\vert_{[0,t_i],w,p}\big\rangle_8\Big).
\end{aligned}$$

Step 4 – We use (2.11) to write the remainder term $R^{\Delta\mathrm{F}}$ in the form

$$\begin{aligned}
R^{\Delta\mathrm{F}}_{s,t} &= \Big(\partial_x\mathrm{F}\big(X_s(\omega),Y_s(\cdot)\big) - \partial_x\mathrm{F}\big(X'_s(\omega),Y'_s(\cdot)\big)\Big) R^X_{s,t}(\omega) \\
&\quad + \partial_x\mathrm{F}\big(X'_s(\omega),Y'_s(\cdot)\big)\Big(R^X_{s,t}(\omega) - R^{X'}_{s,t}(\omega)\Big) \\
&\quad + \mathbb{E}\Big[\Big(\nabla_Z\mathrm{F}\big(X_s(\omega),Y_s(\cdot)\big) - \nabla_Z\mathrm{F}\big(X'_s(\omega),Y'_s(\cdot)\big)\Big) R^Y_{s,t}(\cdot)\Big] \\
&\quad + \mathbb{E}\Big[\nabla_Z\mathrm{F}\big(X'_s(\omega),Y'_s(\cdot)\big)\Big(R^Y_{s,t}(\cdot) - R^{Y'}_{s,t}(\cdot)\Big)\Big] \\
&\quad + \mathbf{(2)} - \mathbf{(2')} + \mathbf{(3)} - \mathbf{(3')} + \mathbf{(5)} - \mathbf{(5')},
\end{aligned}$$

with

$$\begin{aligned}
\mathbf{(2)} &:= \int_0^1 \Big\{\partial_x\mathrm{F}\Big(X^{(\lambda)}_{s;(s,t)}(\omega),Y_t(\cdot)\Big) - \partial_x\mathrm{F}\Big(X^{(\lambda)}_{s;(s,t)}(\omega),Y_s(\cdot)\Big)\Big\} X_{s,t}(\omega)\,d\lambda, \\
\mathbf{(2')} &:= \int_0^1 \Big\{\partial_x\mathrm{F}\Big(X^{(\lambda)\prime}_{s;(s,t)}(\omega),Y'_t(\cdot)\Big) - \partial_x\mathrm{F}\Big(X^{(\lambda)\prime}_{s;(s,t)}(\omega),Y'_s(\cdot)\Big)\Big\} X'_{s,t}(\omega)\,d\lambda, \\
\mathbf{(3)} &:= \int_0^1 \Big\{\partial_x\mathrm{F}\Big(X^{(\lambda)}_{s;(s,t)}(\omega),Y_s(\cdot)\Big) - \partial_x\mathrm{F}\big(X_s(\omega),Y_s(\cdot)\big)\Big\} X_{s,t}(\omega)\,d\lambda, \\
\mathbf{(3')} &:= \int_0^1 \Big\{\partial_x\mathrm{F}\Big(X^{(\lambda)\prime}_{s;(s,t)}(\omega),Y'_s(\cdot)\Big) - \partial_x\mathrm{F}\big(X'_s(\omega),Y'_s(\cdot)\big)\Big\} X'_{s,t}(\omega)\,d\lambda, \\
\mathbf{(5)} &:= \int_0^1 \Big\langle \Big\{\nabla_Z\mathrm{F}\big(X_s(\omega),Y^{(\lambda)}_{s;(s,t)}(\cdot)\big) - \nabla_Z\mathrm{F}\big(X_s(\omega),Y_s(\cdot)\big)\Big\} Y_{s,t}(\cdot)\Big\rangle\,d\lambda \\
\mathbf{(5')} &:= \int_0^1 \Big\langle \Big\{\nabla_Z\mathrm{F}\big(X'_s(\omega),Y^{(\lambda)\prime}_{s;(s,t)}(\cdot)\big) - \nabla_Z\mathrm{F}\big(X'_s(\omega),Y'_s(\cdot)\big)\Big\} Y'_{s,t}(\cdot)\Big\rangle\,d\lambda.
\end{aligned}$$

We start with the analysis of the first fourth lines in $R^{\Delta\mathrm{F}}$. Proceeding as before, the first line is less than

$$\begin{aligned}
&\Big|\Big[\partial_x\mathrm{F}\big(X_s(\omega),Y_s(\cdot)\big) - \partial_x\mathrm{F}\big(X'_s(\omega),Y'_s(\cdot)\big)\Big] R^X_{s,t}(\omega)\Big| \\
&\leqslant \gamma\, w(s,t,\omega)^{2/p}\Big\{ w(0,t_i)^{1/p}\Big(\vert\kern-0.25ex\vert\kern-0.25ex\vert \Delta X(\omega)\vert\kern-0.25ex\vert\kern-0.25ex\vert_{[0,t_i],w,p} + \big\langle \vert\kern-0.25ex\vert\kern-0.25ex\vert \Delta Y(\cdot)\vert\kern-0.25ex\vert\kern-0.25ex\vert_{[0,t_i],w,p}\big\rangle_8\Big) \\
&\qquad\qquad + \Big(\vert\kern-0.25ex\vert\kern-0.25ex\vert \Delta X(\omega)\vert\kern-0.25ex\vert\kern-0.25ex\vert_{[t_i,t_{i+1}],w,p} + \big\langle \vert\kern-0.25ex\vert\kern-0.25ex\vert \Delta Y(\cdot)\vert\kern-0.25ex\vert\kern-0.25ex\vert_{[t_i,t_{i+1}],w,p}\big\rangle_8\Big)\Big\}.
\end{aligned}$$

We also have

$$\Big|\partial_x\mathrm{F}\big(X'_s(\omega),Y'_s(\cdot)\big)\Big(R^X_{s,t}(\omega) - R^{X'}_{s,t}(\omega)\Big)\Big| \leqslant w(s,t,\omega)^{2/p}\, \vert\kern-0.25ex\vert\kern-0.25ex\vert \Delta X(\omega)\vert\kern-0.25ex\vert\kern-0.25ex\vert_{[t_i,t_{i+1}],w,p}.$$

Similarly,

$$\begin{aligned}
&\Big|\mathbb{E}\Big[\Big(\nabla_Z \mathrm{F}\big(X_s(\omega), Y_s(\cdot)\big) - \nabla_Z \mathrm{F}\big(X'_s(\omega), Y'_s(\cdot)\big)\Big) R^Y_{s,t}(\cdot)\Big]\Big| \\
&\leqslant \gamma\, w(s,t,\omega)^{2/p} \Big\{ w(0,t_i)^{1/p} \Big(|\!|\!| \Delta X(\omega) |\!|\!|_{[0,t_i],w,p} + \big\langle |\!|\!| \Delta Y(\cdot) |\!|\!|_{[0,t_i],w,p} \big\rangle_8 \Big) \\
&\qquad\qquad + \Big(|\!|\!| \Delta X(\omega) |\!|\!|_{[t_i,t_{i+1}],w,p} + \big\langle |\!|\!| \Delta Y(\cdot) |\!|\!|_{[t_i,t_{i+1}],w,p} \big\rangle_8 \Big) \Big\},
\end{aligned}$$

and

$$\Big|\mathbb{E}\Big[\nabla_Z \mathrm{F}\big(X_s(\omega), Y_s(\cdot)\big) \Big(R^Y_{s,t}(\cdot) - R^{Y'}_{s,t}(\cdot)\Big)\Big]\Big| \leqslant 2 w(s,t,\omega)^{2/p} \big\langle |\!|\!| \Delta Y(\cdot) |\!|\!|_{[t_i,t_{i+1}],w,p} \big\rangle_8.$$

Now, $\big|\mathbf{(2)} - \mathbf{(2')}\big|$ is bounded above by

$$\begin{aligned}
&\gamma\, w(s,t,\omega)^{2/p}\, \big\|\Delta X(\omega)\big\|_{[t_i,t_{i+1}],w,p} \\
&\quad + \gamma\, w(s,t,\omega)^{1/p} \int_0^1\!\!\int_0^1 \Big| \Big\langle \nabla_Z \partial_x \mathrm{F}\Big(X^{(\lambda)}_{s;(s,t)}(\omega), Y^{(\lambda)}_{s;(s,t)}(\cdot)\Big) Y_{s,t}(\cdot) \Big\rangle \\
&\qquad\qquad - \Big\langle \nabla_Z \partial_x \mathrm{F}\Big(X^{(\lambda)\prime}_{s;(s,t)}(\omega), Y^{(\lambda)\prime}_{s;(s,t)}(\cdot)\Big) Y'_{s,t}(\cdot) \Big\rangle \Big| d\lambda d\lambda',
\end{aligned}$$

so $\big|\mathbf{(2)} - \mathbf{(2')}\big|$ is bounded above by

$$\begin{aligned}
&\gamma\, w(s,t,\omega)^{2/p}\, \big\|\Delta X(\omega)\big\|_{[t_i,t_{i+1}],w,p} \\
&\quad + \gamma\, w(s,t,\omega)^{2/p} \Big\{ w(0,t_i,\omega)^{1/p} \Big(|\!|\!| \Delta X(\omega) |\!|\!|_{[0,t_i],w,p} + \big\langle |\!|\!| \Delta Y(\cdot) |\!|\!|_{[0,t_i],w,p} \big\rangle_8 \Big) \\
&\qquad\qquad + \big\|\Delta X(\omega)\big\|_{[t_i,t_{i+1}],w,p} + \big\langle |\!|\!| \Delta Y(\cdot) |\!|\!|_{[t_i,t_{i+1}],w,p} \big\rangle_8 \Big\}.
\end{aligned}$$

The difference $\mathbf{(3)} - \mathbf{(3')}$ can be handled in the same way. We end up with the term $\mathbf{(5)} - \mathbf{(5')}$. As $Y_{s,t}$ and $Y'_{s,t}$ may be estimated in $\mathbb{L}^4$, it suffices to control both

$$\mathbf{(5a)} := \nabla_Z \mathrm{F}\big(X_s(\omega), Y^{(\lambda)}_{s;(s,t)}(\cdot)\big) - \nabla_Z \mathrm{F}\big(X_s(\omega), Y_s(\cdot)\big)$$

and

$$\begin{aligned}
\mathbf{(5a)} - \mathbf{(5a')} := &\Big(\nabla_Z \mathrm{F}\big(X_s(\omega), Y^{(\lambda)}_{s;(s,t)}(\cdot)\big) - \nabla_Z \mathrm{F}\big(X_s(\omega), Y_s(\cdot)\big)\Big) \\
&- \Big(\nabla_Z \mathrm{F}\big(X'_s(\omega), Y^{(\lambda)\prime}_{s;(s,t)}(\cdot)\big) - \nabla_Z \mathrm{F}\big(X'_s(\omega), Y'_s(\cdot)\big)\Big),
\end{aligned}$$

in $\mathbb{L}^{4/3}$. We have first

$$\big\langle \mathbf{(5a)} \big\rangle_{\mathbb{L}^{4/3}} \leqslant \big\langle \mathbf{(5a)} \big\rangle_{\mathbb{L}^2} \leqslant \gamma\, w(s,t,\omega)^{1/p},$$

In order to estimate **(5a)-(5a')**, we rewrite **(5a)** in the form

$$\begin{aligned}
\mathbf{(5a)} &= D_\mu \mathrm{F}\Big(X_s(\omega), Y^{(\lambda)}_{s;(s,t)}(\cdot)\Big)\Big(Y^{(\lambda)}_{s;(s,t)}(\cdot)\Big) - D_\mu \mathrm{F}\Big(X_s(\omega), Y_s(\cdot)\Big)\big(Y_s(\cdot)\big) \\
&= \lambda \int_0^1 \partial_z D_\mu \mathrm{F}\Big(X_s(\omega), Y^{(\lambda\lambda')}_{s;(s,t)}(\cdot)\Big)\Big(Y^{(\lambda\lambda')}_{s;(s,t)}(\cdot)\Big) Y_{s,t}(\cdot) d\lambda' \\
&\quad + \lambda \int_0^1 \widetilde{\mathbb{E}}\Big[D^2_\mu \mathrm{F}\Big(X_s(\omega), Y^{(\lambda\lambda')}_{s;(s,t)}(\cdot)\Big)\Big(Y^{(\lambda\lambda')}_{s;(s,t)}(\cdot), \widetilde{Y}^{(\lambda\lambda')}_{s;(s,t)}(\cdot)\Big) \widetilde{Y}_{s,t}(\cdot)\Big] d\lambda',
\end{aligned}$$

with the symbol $\sim$ used to denote independent copies of various random variables. Then, using Hölder inequality with exponents 3 and 3/2 as in the first lines of the third step, we obtain that $\big\langle \textbf{(5a)-(5a')} \big\rangle_{\mathbb{L}^{4/3}}$ is bounded above by

$$
\begin{aligned}
&\gamma\, w(s,t,\omega)^{1/p} \big\langle \| \Delta Y(\cdot) \|_{[t_i,t_{i+1}],w,p} \big\rangle_8 \\
&\quad + \gamma\, w(s,t,\omega)^{1/p} \Big\{ w(0,t_i,\omega)^{1/p} \Big(\| \Delta X(\omega) \|_{[0,t_i],w,p} + \big\langle \| \Delta Y(\cdot) \|_{[0,t_i],w,p} \big\rangle_8 \Big) \\
&\qquad\qquad + \big\| \Delta X(\omega) \big\|_{[t_i,t_{i+1}],w,p} + \big\langle \| \Delta Y(\cdot) \|_{[t_i,t_{i+1}],w,p} \big\rangle_8 \Big\}.
\end{aligned}
$$

and end up with the bound

$$
\begin{aligned}
\Big\| R^{\Delta \mathrm{F}}(\omega) \Big\|_{[t_i,t_{i+1}],w,p/2} &\leqslant \gamma \Big\{ w(0,t_i,\omega)^{1/p} \Big(\| \Delta X(\omega) \|_{[0,t_i],w,p} + \big\langle \| \Delta Y(\cdot) \|_{[0,t_i],w,p} \big\rangle_8 \Big) \\
&\qquad + \big\| \Delta X(\omega) \big\|_{[t_i,t_{i+1}],w,p} + \big\langle \| \Delta Y(\cdot) \|_{[t_i,t_{i+1}],w,p} \big\rangle_8 \Big\}.
\end{aligned}
$$

Conclusion. Plugging the conclusion of the previous steps into equation (3.13), we get

$$
\begin{aligned}
&\Big\| \int_{t_i}^{\cdot} \mathrm{F}\big(X_r(\omega), Y_r(\cdot)\big) d\boldsymbol{W}_r(\omega) - \int_{t_i}^{\cdot} \mathrm{F}\big(X'_r(\omega), Y'_r(\cdot)\big) d\boldsymbol{W}_r(\omega) \Big\|_{[t_i,t_{i+1}],w,p} \\
&\leqslant \gamma \Big(\big| X_{t_i}(\omega) - X'_{t_i}(\omega) \big| + \big\| Y_{t_i}(\cdot) - Y'_{t_i}(\cdot) \big\|_2 \Big) \\
&\quad + \gamma\, w(t_i,t_{i+1},\omega)^{1/p} \big\| \mathrm{F}(X(\omega),Y(\cdot)) - \mathrm{F}(X'(\omega),Y'(\cdot)) \big\|_{\star,[t_i,t_{i+1}],w,p} \\
&\leqslant \gamma\, w(0,t_i,\omega)^{1/p} \Big(\| \Delta X(\omega) \|_{[0,t_i],w,p} + \big\langle \| \Delta Y(\cdot) \|_{[0,t_i],w,p} \big\rangle_8 \Big) \\
&\quad + \gamma\, w(t_i,t_{i+1},\omega)^{1/p} \Big\{ \Big(\| \Delta X(\omega) \|_{[t_i,t_{i+1}],w,p} + \big\langle \| \Delta Y(\cdot) \|_{[t_i,t_{i+1}],w,p} \big\rangle_8 \Big) \\
&\qquad + w(0,t_i,\omega)^{1/p} \Big(\| \Delta X(\omega) \|_{[0,t_i],w,p} + \big\langle \| \Delta Y(\cdot) \|_{[0,t_i],w,p} \big\rangle_8 \Big) \Big\}.
\end{aligned}
\tag{3.16}
$$

Choosing the subdivision such that $w(t_i,t_{i+1},\omega)^{1/p} \leqslant 1/(4L)$, we finally get

$$
\begin{aligned}
&\Big\| \int_{t_i}^{\cdot} \mathrm{F}\big(X_r(\omega), Y_r(\cdot)\big) d\boldsymbol{W}_r(\omega) - \int_{t_i}^{\cdot} \mathrm{F}\big(X'_r(\omega), Y'_r(\cdot)\big) d\boldsymbol{W}_r(\omega) \Big\|_{[t_i,t_{i+1}],w,p} \\
&\leqslant \gamma\, w(0,t_i,\omega)^{1/p} \left(1 + \frac{1}{4L} \right) \Big(\big\| \Delta X(\omega) \big\|_{[0,t_i],w,p} + \big\langle \| \Delta Y(\cdot) \|_{[0,T],w,p} \big\rangle_8 \Big) \\
&\quad + \frac{\gamma}{4L} \Big\{ \| \Delta X(\omega) \|_{[t_i,t_{i+1}],w,p} + \big\langle \| \Delta Y(\cdot) \|_{[0,T],w,p} \big\rangle_8 \Big\},
\end{aligned}
$$

which completes the proof. ▷

3.3 – Well-posedness

We first prove a well-posedness result in small time from which our global in time result, Theorem 1, follows. Recall from (1.7) and (1.8) the definition of $w(0,T)$, and from Definition 12 the fact that the map Γ depends on $X_0(\omega)$; recall also from Lemma 13 that there is no loss of generality in assuming $\Lambda = 1$ in (2.7) – this explains the bound for $\partial_x X(\omega)$ in the statement below.

16. Theorem – *Let* F *satisfy* **Regularity assumptions 1** *and* **Regularity assumptions 2** *with* $\Lambda = 1$. *Assume there exists a positive time horizon* T *such that the random variables* $w(0,T,\cdot)$ *and* $\big(N\big([0,T],\cdot,\alpha\big)\big)_{\alpha>0}$ *have 'sub' and super exponential tails* *respectively*

$$\begin{aligned} \mathbb{P}\big(w(0,T,\cdot) \geqslant t\big) &\leqslant c_1 \exp\big(-t^{\varepsilon_1}\big), \\ \mathbb{P}\big(N([0,T],\cdot,\alpha) \geqslant t\big) &\leqslant c_2(\alpha) \exp\big(-t^{1+\varepsilon_2(\alpha)}\big), \end{aligned} \tag{3.17}$$

for some positive constants c_1 *and* ε_1*, and possibly* α*-dependent positive constants* $c_2(\alpha)$ *and* $\varepsilon_2(\alpha)$*. Then, there exist a positive random variable* A *satisfying*

$$\Big\langle A(\cdot)^{N([0,T],\cdot,1/(4L))} \Big\rangle_1 < \infty,$$

together with three positive reals L_0*,* L *and* η *with the following property. For any* $0 \leqslant S \leqslant T$ *such that*

$$\Big\langle N\big([0,S],\cdot,1/(4L_0)\big) + 1 \Big\rangle_8^{2(p-1)/p} \leqslant 2, \tag{3.18}$$

and

$$\Big\langle A(\cdot)^{N([0,S],\cdot,1/(4L))} \Big\rangle_1 \leqslant \eta, \tag{3.19}$$

and for any d*-dimensional random square-integrable variable* X_0*, there exists a random controlled path* $X(\cdot) = (X(\omega))_{\omega\in\Omega}$ *defined on the time interval* $[0,S]$ *satisfying the estimates*

$$\big\langle \delta_x X(\cdot) \big\rangle_\infty \leqslant 1,$$

and

$$\Big\langle \| X(\cdot) \|_{[0,S],w,p} \Big\rangle_8^2 < \infty,$$

such that, for every $\omega \in \Omega$*, the paths* $X(\omega)$ *and* $\Gamma(\omega, X(\omega), X(\cdot))$ *coincide on* $[0,S]$*. Any other random controlled path* $X'(\cdot)$ *with* $X_0' = X_0$ *almost surely, and such that the paths* $X'(\omega)$ *and* $\Gamma\big(\omega, X'(\omega), X'(\cdot)\big)$ *coincide almost surely, satisfies*

$$\mathbb{P}\Big(\| X(\cdot) - X'(\cdot) \|_{\star,[0,S],w,p} = 0 \Big) = 1.$$

Proof – We construct a fixed point of the map Γ, in the sense of Definition 12, as the limit of the following Picard sequence

$$\big(X^{n+1}(\omega); \delta_x X^{n+1}(\omega); 0\big) := \Gamma\Big(\omega, \big(X^n(\omega); \delta_x X^n(\omega); 0\big), \big(X^n(\omega'); \delta_x X^n(\omega'); 0\big)_{\omega'\in\Omega}\Big),$$

started from

$$\Big(X^0(\omega); \partial_x X^0(\omega); 0\Big) \equiv \big(X_0(\omega); 0; 0\big),$$

for each $\omega \in \Omega$. Importantly, we deduce from the tail estimates (3.17) that Proposition 14 applies iteratively: Following the discussion that comes right after the statement of Proposition 14, each $X^n(\cdot) = (X^n(\omega))_{\omega\in\Omega}$, $n \geqslant 1$, is a random controlled trajectory.

Step 1. Instead of working with S such that $\big\langle N([0,S]\cdot, 1/(4L_0)) + 1 \big\rangle_8^{2(p-1)/p} \leqslant 2$, we can assume, using (3.17), that $\big\langle N([0,T],\cdot,1/(4L_0)) + 1 \big\rangle_8^{2(p-1)/p} \leqslant 2$, with L_0 as in the statement of Proposition 14. We deduce that, at any rank $n \geqslant 1$, both

X^n and X^{n-1} satisfy the esimates (3.11) and (3.12). Hence, by Proposition 15, the quantity $\big|\!\big|\!\big|(X^{n+1}-X^n)(\omega)\big|\!\big|\!\big|_{[t_i,t_{i+1}],w,p}$, is bounded above by

$$\begin{aligned}
&\gamma\, w(0,t_i,\omega)^{1/p}\Big(1+\frac{1}{4L}\Big)\Big\{\big|\!\big|\!\big|(X^n-X^{n-1})(\omega)\big|\!\big|\!\big|_{[0,t_i],w,p}\\
&\qquad\qquad\qquad\qquad + \Big\langle |\!|\!|(X^n-X^{n-1})(\cdot)|\!|\!|_{[0,T],w,p}\Big\rangle_8\Big\}\\
&+\frac{\gamma}{4L}\Big\{\big|\!\big|\!\big|(X^n-X^{n-1})(\omega)\big|\!\big|\!\big|_{[t_i,t_{i+1}],w,p}+\Big\langle |\!|\!|(X^n-X^{n-1})(\cdot)|\!|\!|_{[0,T],w,p}\Big\rangle_8\Big\},
\end{aligned}$$

for any $n\geqslant 1$ and for a sequence $(t_i)_{i=0,\cdots,N}$ as in the statement of Proposition 15. We start with the case $i=0$. The above bound yields, for all $n\geqslant 1$,

$$\begin{aligned}
&\big|\!\big|\!\big|(X^{n+1}-X^n)(\omega)\big|\!\big|\!\big|_{[0,t_1],w,p}\\
&\quad\leqslant \frac{3\gamma}{4L}\Big\{\big|\!\big|\!\big|(X^n-X^{n-1})(\omega)\big|\!\big|\!\big|_{[0,t_1],w,p}+\Big\langle |\!|\!|(X^n-X^{n-1})(\cdot)|\!|\!|_{[0,T],w,p}\Big\rangle_8\Big\},
\end{aligned}$$

so we have, for any $n\geqslant 1$,

$$\begin{aligned}
&\big|\!\big|\!\big|(X^{n+1}-X^n)(\omega)\big|\!\big|\!\big|_{[0,t_1],w,p}\\
&\leqslant \Big(\frac{3\gamma}{4L}\Big)^n\big|\!\big|\!\big|X^1(\omega)\big|\!\big|\!\big|_{[0,t_1],w,p}+\sum_{k=1}^n\Big(\frac{3\gamma}{4L}\Big)^{n+1-k}\Big\langle |\!|\!|(X^k-X^{k-1})(\cdot)|\!|\!|_{[0,T],w,p}\Big\rangle_8.
\end{aligned}\tag{3.20}$$

We proceed with a similar computation when $i\geqslant 1$. We have, for $n\geqslant 1$,

$$\begin{aligned}
&\big|\!\big|\!\big|(X^{n+1}-X^n)(\omega)\big|\!\big|\!\big|_{[t_i,t_{i+1}],w,p}\\
&\leqslant \Big(\frac{\gamma}{4L}\Big)^n\big|\!\big|\!\big|X^1(\omega)\big|\!\big|\!\big|_{[t_i,t_{i+1}],w,p}\\
&\quad+\sum_{k=1}^n\Big(\frac{\gamma}{4L}\Big)^{n+1-k}\Big[\gamma w(0,t_i,\omega)^{1/p}\Big(1+\frac{1}{4L}\Big)\big|\!\big|\!\big|(X^k-X^{k-1})(\omega)\big|\!\big|\!\big|_{[0,t_i],w,p}\Big]\\
&\quad+\sum_{k=1}^n\Big(\frac{\gamma}{4L}\Big)^{n+1-k}\Big[\gamma\Big\{\frac{1}{4L}+w(0,t_i,\omega)^{1/p}\Big(1+\frac{1}{4L}\Big)\Big\}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\times\Big\langle |\!|\!|(X^k-X^{k-1})(\cdot)|\!|\!|_{[0,T],w,p}\Big\rangle_8\Big].
\end{aligned}$$

Following the second bullet point in the proof of Proposition 14, we can prove that, for a new value of γ,

$$\begin{aligned}
&\big|\!\big|\!\big|(X^{n+1}-X^n)(\omega)\big|\!\big|\!\big|_{[0,t_{i+1}],w,p}\\
&\qquad\leqslant \gamma\big|\!\big|\!\big|(X^{n+1}-X^n)(\omega)\big|\!\big|\!\big|_{[0,t_i],w,p}+\gamma\big|\!\big|\!\big|(X^{n+1}-X^n)(\omega)\big|\!\big|\!\big|_{[t_i,t_{i+1}],w,p},
\end{aligned}$$

so

$$\begin{aligned}
&\big|\!\big|\!\big|(X^{n+1}-X^n)(\omega)\big|\!\big|\!\big|_{[0,t_{i+1}],w,p}\\
&\leqslant \gamma\,\big|\!\big|\!\big|(X^{n+1}-X^n)(\omega)\big|\!\big|\!\big|_{[0,t_i],w,p}+\gamma\Big(\frac{\gamma}{4L}\Big)^n\big|\!\big|\!\big|X^1(\omega)\big|\!\big|\!\big|_{[t_i,t_{i+1}],w,p}\\
&\quad+\gamma\sum_{k=1}^n\Big(\frac{\gamma}{4L}\Big)^{n+1-k}\Big[\gamma\, w(0,t_i,\omega)^{1/p}\Big(1+\frac{1}{4L}\Big)\big|\!\big|\!\big|(X^k-X^{k-1})(\omega)\big|\!\big|\!\big|_{[0,t_i],w,p}\Big]\\
&\quad+\gamma\sum_{k=1}^n\Big(\frac{\gamma}{4L}\Big)^{n+1-k}\Big[\gamma\Big\{\frac{1}{4L}+w(0,t_i,\omega)^{1/p}\Big(1+\frac{1}{4L}\Big)\Big\}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\times\Big\langle |\!|\!|(X^k-X^{k-1})(\cdot)|\!|\!|_{[0,T],w,p}\Big\rangle_8\Big],
\end{aligned}$$

which we can rewrite as

$$
\begin{aligned}
&\left\|\left|(X^{n+1}-X^n)(\omega)\right|\right\|_{[0,t_{i+1}],w,p} \\
&\leqslant \gamma^2\zeta(\omega)\left\{\sum_{k=1}^{n+1}\Big(\frac{\gamma}{4L}\Big)^{n+1-k}\left\|\left|(X^k-X^{k-1})(\omega)\right|\right\|_{[0,t_i],w,p}+\Big(\frac{\gamma}{4L}\Big)^n\left\|\left|X^1(\omega)\right|\right\|_{[t_i,t_{i+1}],w,p}\right. \\
&\qquad\left.+\sum_{k=1}^{n}\Big(\frac{\gamma}{4L}\Big)^{n+1-k}\Big\langle\left\|\left|(X^k-X^{k-1})(\cdot)\right|\right\|_{[0,T],w,p}\Big\rangle_8\right\},
\end{aligned}
$$

provided we choose $\gamma \geqslant 1$, and with

$$
\zeta(\omega) := 1 + w(0,T,\omega)^{1/p}\Big(1+\frac{1}{4L}\Big).
$$

Step 2. Combine the above estimate together with (3.20) to get

$$
\begin{aligned}
&\left\|\left|(X^{n+1}-X^n)(\omega)\right|\right\|_{[0,t_2],w,p} \\
&\leqslant \gamma^2\zeta(\omega)\sum_{k=1}^{n+1}\Big(\frac{\gamma}{4L}\Big)^{n+1-k}\Big(\frac{3\gamma}{4L}\Big)^{k-1}\left\|\left|X^1(\omega)\right|\right\|_{[0,t_1],w,p} \\
&\quad+\gamma^2\zeta(\omega)\Big(\frac{\gamma}{4L}\Big)^n\left\|\left|X^1(\omega)\right|\right\|_{[t_i,t_{i+1}],w,p} \\
&\quad+\gamma^2\zeta(\omega)\sum_{k=1}^{n}\Big(\frac{\gamma}{4L}\Big)^{n+1-k}\sum_{i=1}^{k}\Big(\frac{3\gamma}{4L}\Big)^{k+1-i}\Big\langle\left\|\left|(X^i-X^{i-1})(\cdot)\right|\right\|_{[0,T],w,p}\Big\rangle_8 \\
&\quad+\gamma^2\zeta(\omega)\sum_{k=1}^{n}\Big(\frac{\gamma}{4L}\Big)^{n+1-k}\Big\langle\left\|\left|(X^k-X^{k-1})(\cdot)\right|\right\|_{[0,T],w,p}\Big\rangle_8.
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
&\left\|\left|(X^{n+1}-X^n)(\omega)\right|\right\|_{[0,t_2],w,p} \\
&\leqslant \gamma^2\zeta(\omega)\Big(\frac{3\gamma}{4L}\Big)^n\Big(1+\sum_{k=1}^{n+1}\big(\frac{1}{3}\big)^{n+1-k}\Big)\left\|\left|X^1(\omega)\right|\right\|_{[0,t_2],w,p} \\
&\quad+\gamma^2\zeta(\omega)\Big(\frac{\gamma}{4L}\Big)^{n+1}\sum_{i=1}^{n}\Big(\frac{3\gamma}{4L}\Big)^{1-i}\Big\langle\left\|\left|(X^i-X^{i-1})(\cdot)\right|\right\|_{[0,T],w,p}\Big\rangle_8\sum_{k=i}^{n}3^k \\
&\quad+\gamma^2\zeta(\omega)\sum_{k=1}^{n}\Big(\frac{\gamma}{4L}\Big)^{n+1-k}\Big\langle\left\|\left|(X^k-X^{k-1})(\cdot)\right|\right\|_{[0,T],w,p}\Big\rangle_8.
\end{aligned}
$$

Therefore, using the bound $\sum_{k=i}^n 3^k \leqslant 3^{n+1}/2$, we deduce

$$
\begin{aligned}
&\left\|\left|(X^{n+1}-X^n)(\omega)\right|\right\|_{[0,t_2],w,p} \\
&\leqslant 3\gamma^2\zeta(\omega)\Big(\frac{3\gamma}{4L}\Big)^n\left\|\left|X^1(\omega)\right|\right\|_{[0,t_2],w,p} \\
&\quad+3\gamma^2\zeta(\omega)\sum_{i=1}^{n}\Big(\frac{3\gamma}{4L}\Big)^{n+1-i}\Big\langle\left\|\left|(X^i-X^{i-1})(\cdot)\right|\right\|_{[0,T],w,p}\Big\rangle_8.
\end{aligned}
$$

We here assumed that L was chosen big enough to have $3\gamma < 4L$. The above inequality may be summed up into

$$\begin{aligned}\big\|\!\big|(X^{n+1} - X^n)(\omega)\big\|\!\big|_{[0,t_2],w,p} &\leqslant c_2(\omega)\Big(\frac{3\gamma}{4L}\Big)^n \big\|X^1(\omega)\big\|_{[0,t_2],w,p} \\ &\quad + c_2(\omega)\sum_{i=1}^{n}\Big(\frac{3\gamma}{4L}\Big)^{n+1-i}\Big\langle \|\!|(X^i - X^{i-1})(\cdot)\|\!|_{[0,T],w,p}\Big\rangle_8,\end{aligned}$$

where $c_2(\omega){:} = 3\gamma^2\zeta(\omega)$. Set now

$$c_i(\omega) := \big(3\gamma^2\zeta(\omega)\big)^{i-1}.$$

Comparing the previous estimate of $\big\|\!\big|(X^{n+1} - X^n)(\omega)\big\|\!\big|_{[0,t_2],w,p}$ with (3.20) and iterating over the time index t_i from the conclusion of the first step, we obtain

$$\begin{aligned}\big\|\!\big|(X^{n+1} - X^n)(\omega)\big\|\!\big|_{[0,t_i],w,p} &\leqslant c_i(\omega)\left(\frac{3\gamma}{4L}\right)^n \big\|X^1(\omega)\big\|_{[0,t_i],w,p} \\ &\quad + c_i(\omega)\sum_{k=1}^{n}\Big(\frac{3\gamma}{4L}\Big)^{n+1-k}\Big\langle \|\!|(X^k - X^{k-1})(\cdot)\|\!|_{[0,T],w,p}\Big\rangle_8,\end{aligned}$$

as long as $t_i \leqslant T$.

Step 3. Noting that we can take the number of points N in the statement of Theorem 15 less than $N_0\big([0,T],\omega,1/(4L_0)\big) + N_0\big([0,T],\omega,1/(4L)\big) \leqslant 2N\big([0,T],\omega,1/(4L_0)\big)$, where we recall the definition (1.13) of $N\big([0,T],\omega,1/(4L)\big)$, we deduce that

$$\begin{aligned}&\big\|\!\big|(X^{n+1} - X^n)(\omega)\big\|\!\big|_{[0,T],w,p} \\ &\leqslant \Big(3\gamma^2\zeta(\omega)\Big)^{2N(\omega,1/(4L))}\Big(\frac{3\gamma}{4L}\Big)^n \big\|X^1(\omega)\big\|_{[0,T],w,p} \\ &\quad + \Big(3\gamma^2\zeta(\omega)\Big)^{2N(\omega,1/(4L))}\sum_{k=1}^{n}\Big(\frac{3\gamma}{4L}\Big)^{n+1-k}\Big\langle \|\!|(X^k - X^{k-1})(\cdot)\|\!|_{[0,T],w,p}\Big\rangle_8,\end{aligned} \tag{3.21}$$

where we let $N\big(\omega,1/(4L)\big) := N\big([0,T],\omega,1/(4L)\big)$. It follows from the assumed tail behaviour of the random variables $N\big(\cdot,1/(4L)\big)$ and $w(0,T,\cdot)$ that we have, for $a > 1$ and any integer k the upper bound

$$\begin{aligned}\mathbb{P}\Big(\big\{\omega\in\Omega : \zeta^{2N(\omega,1/(4L))}(\omega) \geqslant a\big\}\Big) &\leqslant \mathbb{P}\big(N(\cdot,1/(4L)) \geqslant k\big) + \mathbb{P}\big(\zeta^2 \geqslant a^{1/k}\big) \\ &\leqslant c\exp\big(-k^{1+\varepsilon_2}\big) + c\exp\left(-\frac{a^{\varepsilon_1/(2k)}}{c}\right),\end{aligned} \tag{3.22}$$

for a constant $c \geqslant 1$ depending on L and with $\varepsilon_2 = \varepsilon_2(1/(4L))$. Choosing $k = (\ln a)^{1/(1+\varepsilon_2/2)}$ then gives

$$\forall \ell\in\mathbb{N}\backslash\{0\},\quad \mathbb{P}\Big(\Big\{\omega\in\Omega : \zeta^{2N(\omega,1/(4L))}(\omega) \geqslant a\Big\}\Big) \leqslant C_\ell a^{-\ell},$$

for a constant C_ℓ depending on ℓ, from which we deduce that

$$\Big\langle \big(3\gamma^2\zeta\big)^{2N(\cdot,1/(4L))}\Big\rangle_{16} < \infty.$$

Set now $A := (3\gamma^2\zeta)^{2N(\cdot,1/(4L)}$. Importantly, A depends on the time horizon T through γ,ζ and $N(\cdot,1/4L)$ (and this on L as well). In order to emphasize the dependance upon the time argument, we expand the notation and write

$$A_T := (3\gamma_T^2\zeta_T)^{2N([0,T],\cdot,1/(4L))}.$$

Clearly,

$$A_S \leqslant (3\gamma_T^2\zeta_T)^{2N([0,S],\cdot,1/(4L))},$$

since γ_T and ζ_T are greater than 1. Since the quantity $N\big([0,S],\cdot,1/(4L)\big)$ tends to 0 as S tends to 0, we have

$$\lim_{S\searrow 0}\Big\langle \big(3\gamma_T^2\zeta_T\big)^{2N([0,S],\cdot,1/(4L))}\Big\rangle_{16} = 1,$$

so

$$\lim_{S\searrow 0}\big\langle A_S\big\rangle_{16} = 1.$$

Hence, taking the $\mathbb{L}^8$ norm in (3.21) with T replaced by S, there is a quantity $\delta(S)$ with zero limit as S goes to 0 such that

$$\begin{aligned}
&\Big\langle \|\!|(X^{n+1}-X^n)(\cdot)\|\!|_{[0,S],w,p}\Big\rangle_8\\
&\leqslant \big(1+\delta(S)\big)\Big(\frac{3\gamma}{4L}\Big)^n\Big\langle \|\!|X^1(\cdot)\|\!|_{[0,S],w,p}\Big\rangle_{16}\\
&\quad+\big(1+\delta(S)\big)\sum_{i=1}^{n}\Big(\frac{3\gamma}{4L}\Big)^{n+1-i}\Big\langle \|\!|(X^i-X^{i-1})(\cdot)\|\!|_{[0,S],w,p}\Big\rangle_8\\
&= \big(1+\delta(S)\big)\Big(\frac{3\gamma}{4L}\Big)^n\Big\langle \|\!|X^1(\cdot)\|\!|_{[0,S],w,p}\Big\rangle_{16}\\
&\quad+\big(1+\delta(S)\big)\sum_{i=0}^{n-1}\Big(\frac{3\gamma}{4L}\Big)^{n-i}\Big\langle \|\!|(X^{i+1}-X^i)(\cdot)\|\!|_{[0,S],w,p}\Big\rangle_8,
\end{aligned}$$

so we have

$$\begin{aligned}
&\sum_{k=0}^{n}\Big(\frac{3\gamma}{4L}\Big)^{(n-k)/2}\Big\langle \|\!|(X^{k+1}-X^k)(\cdot)\|\!|_{[0,S],w,p}\Big\rangle_8\\
&\leqslant \big(1+\delta(S)\big)\sum_{k=0}^{n}\Big(\frac{3\gamma}{4L}\Big)^{(n-k)/2}\Big(\frac{3\gamma}{4L}\Big)^{k}\Big\langle \|\!|X^1(\cdot)\|\!|_{[0,S],w,p}\Big\rangle_{16}\\
&\quad+\big(1+\delta(S)\big)\sum_{i=0}^{n-1}\Big(\frac{3\gamma}{4L}\Big)^{(n-i)/2}\Big\langle \|\!|(X^{i+1}-X^i)(\cdot)\|\!|_{[0,S],w,p}\Big\rangle_8\sum_{k=i+1}^{n}\Big(\frac{3\gamma}{4L}\Big)^{(k-i)/2}\\
&\leqslant \big(1+\delta(S)\big)\Big(\frac{3\gamma}{4L}\Big)^{n/2}\sum_{k=0}^{n}\Big(\frac{3\gamma}{4L}\Big)^{k/2}\Big\langle \|\!|X^1(\cdot)\|\!|_{[0,S],w,p}\Big\rangle_{16}\\
&\quad+\frac{1+\delta(S)}{1-\sqrt{3\gamma/(4L)}}\Big(\frac{3\gamma}{4L}\Big)^{1/2}\sum_{i=0}^{n}\Big(\frac{3\gamma}{4L}\Big)^{(n-i)/2}\Big\langle \|\!|(X^{i+1}-X^i)(\cdot)\|\!|_{[0,S],w,p}\Big\rangle_8.
\end{aligned}$$

Without any loss of generality, we can assume that $3\gamma/(4L) \leqslant 1/16$, so

$$\frac{1}{1-\sqrt{3\gamma/(4L)}}\Big(\frac{3\gamma}{4L}\Big)^{1/2} < 1,$$

and we can choose S small enough such that

$$a := \frac{1+\delta(S)}{1-\sqrt{3\gamma/(4L)}}\Big(\frac{3\gamma}{4L}\Big)^{1/2} < 1.$$

Then, we can find a positive constant C such that

$$\sum_{k=0}^{n}\Big(\frac{3\gamma}{4L}\Big)^{(n-k)/2}\Big\langle \big|\!\big|\!\big|(X^{k+1}-X^k)(\cdot)\big|\!\big|\!\big|_{[0,S],w,p}\Big\rangle_8$$
$$\leqslant C\Big(\frac{3\gamma}{4L}\Big)^{n/2}\Big\langle \big|\!\big|\!\big|X^1(\cdot)\big|\!\big|\!\big|_{[0,S],w,p}\Big\rangle_{16} + a\sum_{i=0}^{n}\Big(\frac{3\gamma}{4L}\Big)^{(n-i)/2}\Big\langle \big|\!\big|\!\big|(X^{i+1}-X^i)(\cdot)\big|\!\big|\!\big|_{[0,S],w,p}\Big\rangle_8.$$

Changing the value of C if necessary, we obtain

$$\sum_{k=0}^{n}\Big(\frac{3\gamma}{4L}\Big)^{(n-k)/2}\Big\langle \big|\!\big|\!\big|(X^{k+1}-X^k)(\cdot)\big|\!\big|\!\big|_{[0,S],w,p}\Big\rangle_8 \leqslant C\Big(\frac{3\gamma}{4L}\Big)^{n/2}\Big\langle \big|\!\big|\!\big|X^1(\cdot)\big|\!\big|\!\big|_{[0,S],w,p}\Big\rangle_{16},$$

which leads to

$$\Big\langle \big|\!\big|\!\big|(X^{n+1}-X^n)(\cdot)\big|\!\big|\!\big|_{[0,S],w,p}\Big\rangle_8 \leqslant C\,a^n\Big\langle \big|\!\big|\!\big|X^1(\cdot)\big|\!\big|\!\big|_{[0,S],w,p}\Big\rangle_{16}.$$

It then follows from (3.21) that we eventually have

$$\begin{aligned}&\big|\!\big|\!\big|(X^{n+1}-X^n)(\omega)\big|\!\big|\!\big|_{[0,S],w,p}\\ &\leqslant \Big(3\gamma^2\zeta(\omega)\Big)^{2N([0,T],\omega,1/4L)}\Big(\frac{3\gamma}{4L}\Big)^n\big|\!\big|\!\big|X^1(\omega)\big|\!\big|\!\big|_{[0,T],w,p}\\ &\quad + C\big(3\gamma^2\zeta(\omega)\big)^{2N([0,T],\omega,1/4L)}a^n\big\langle \big|\!\big|\!\big|X^1(\cdot)\big|\!\big|\!\big|_{[0,S],w,p}\big\rangle_{16}\sum_{i=1}^{n}\Big(\frac{3\gamma}{4La}\Big)^{n-i}.\end{aligned}$$

As we can assume that $3\gamma < 4La$, we can change the value of C and get

$$\begin{aligned}&\big|\!\big|\!\big|(X^{n+1}-X^n)(\omega)\big|\!\big|\!\big|_{[0,S],w,p}\\ &\leqslant \big(3\gamma^2\zeta(\omega)\big)^{2N([0,T],\omega,1/4L)}\Big(\frac{3\gamma}{4L}\Big)^n\,\big|\!\big|\!\big|X^1(\omega)\big|\!\big|\!\big|_{[0,T],w,p}\\ &\quad + C\big(3\gamma^2\zeta(\omega)\big)^{2N([0,T]\omega,1/4L)}\,a^n\,\big\langle \big|\!\big|\!\big|X^1(\cdot)\big|\!\big|\!\big|_{[0,S],w,p}\big\rangle_{16}.\end{aligned} \tag{3.23}$$

In order to conclude, we notice the following two facts. First, the above estimate remains true if we replace $\big|\!\big|\!\big|(X^{n+1}-X^n)(\omega)\big|\!\big|\!\big|_{[0,S],w,p}$ by $\big|\!\big|\!\big|(X^{n+1}-X^n)(\omega)\big|\!\big|\!\big|_{\star,[0,S],w,p}$ in the left-hand side. Second, Proposition 14 guarantees that $\big\langle \big|\!\big|\!\big|X^1(\cdot)\big|\!\big|\!\big|_{[0,S],w,p}\big\rangle_{16} < \infty$. Using a Cauchy like argument, we deduce that, for any $\omega \in \Omega$, the sequence $\big(X^n(\omega), \partial_x X^n, R^{X^n}\big)_{n\geqslant 0}$ is convergent for the norm $|\!|\!|\cdot|\!|\!|_{\star,[0,S],w,p}$. Using Proposition 15, the limit is a fixed point of Γ as required.

Uniqueness – Let $\big(X'(\cdot);\delta_x X'(\cdot);0\big)$ stand for another fixed point of Γ, with

$$\delta_x X'(\omega) = \mathrm{F}\big(X'(\omega), X'(\cdot)\big), \quad \omega \in \Omega,$$

and $\big\langle \delta_x X'(\cdot)\big\rangle_\infty \leqslant 1$ together with $\big\langle |\!|\!|X'(\cdot)|\!|\!|_{[0,T],w,p}\big\rangle_8 < \infty$. A careful inspection of the proof of Proposition 15 shows that the conclusion still holds true if we increase the value of the constant L_0. Hence, we can assume that

$$\big\langle \|Y'(\cdot)\|_{[0,T],w,p}\big\rangle_8^2 \leqslant \sqrt{L_0}, \qquad \big\langle |\!|\!|Y'(\cdot)|\!|\!|_{[0,T],w,p}\big\rangle_8^2 \leqslant L_0.$$

By (3.5), we can also assume that

$$\big|\!\big|\!\big|X(\omega)\big|\!\big|\!\big|^2_{[t_i^0,t_{i+1}^0],w,p} \leqslant \sqrt{L_0},$$

Therefore, we can duplicate the analysis of the convergence sequence, replacing $X^{n+1} - X^n$ by $X - X'$. Similar to (3.21), $\big|\!\big|\!\big|(X-X')(\omega)\big|\!\big|\!\big|_{[0,T],w,p}$ is bounded above by

$$\Big(3\gamma^2\zeta(\omega)\Big)^{2N(\omega,1/(4L))}\Big(\frac{3\gamma}{4L}\Big)^n \big\|\!\big|(X-X')(\omega)\big\|\!\big|_{[0,T],w,p} + \Big(3\gamma^2\zeta(\omega)\Big)^{2N(\omega,1/(4L))}\sum_{i=1}^n\Big(\frac{3\gamma}{4L}\Big)^{n+1-i}\big\langle \|\!|(X-X')(\cdot)\|\!|_{[0,T],w,p}\big\rangle_8.$$

Letting n tend to ∞, this yields

$$\big\|\!\big|(X-X')(\omega)\big\|\!\big|_{[0,T],w,p} \leqslant \big(3\gamma^2\zeta(\omega)\big)^{2N(\omega,1/(4L))}\frac{3\gamma/(4L)}{1-3\gamma/(4L)}\Big\langle \|\!|(X-X')(\cdot)\|\!|_{[0,T],w,p}\Big\rangle_8.$$

Taking the $\mathbb{L}^8$ norm, we deduce that uniqueness holds in small time. ▷

Applying iteratively Theorem 16 along a sequence of times $(S_0=0,\cdots,S_\ell=T)$ satisfying

$$\Big\langle N([S_{j-1},S_j],\cdot,1/(4L_0))+1\Big\rangle_8^{2(p-1)/p}\leqslant 2, \quad \text{and} \quad \Big\langle A(\cdot)^{N([S_{j-1},S_j],\cdot,1/(4L))}\Big\rangle_1\leqslant \eta,$$

the mean field rough differential equation is seen to have a unique solution defined on the whole interval $[0,T]$. This is Theorem 1.

3.4 – Uniqueness in Law on Strong Rough Set-Ups

Since the solution given by Theorem 16 is constructed by Picard iteration on each interval $[S_{j-1},S_j]$, for $j=1,\cdots,\ell$, we should expect its law to be somehow independent of the probability space used to build the rough set-up $\boldsymbol{W}$. However, although it seems to be a relevant concept in our context, uniqueness in law requires some care as the rough set-up explicitly depends upon the underlying probability space $(\Omega,\mathcal{F},\mathbb{P})$; recall indeed that the random variables $\Omega\ni\omega\mapsto\mathbb{W}^{\perp\!\!\!\perp}(\omega,\cdot)$ and $\Omega\ni\omega\mapsto\mathbb{W}^{\perp\!\!\!\perp}(\cdot,\omega)$ are not only defined on $(\Omega,\mathcal{F},\mathbb{P})$ but also take values in $\mathbb{L}^q(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^m)$. The fact that the arrival spaces of both random variables explicitly depend upon the probability space is a serious drawback to get a form of weak uniqueness. It is thus relevant to identify the canonical information in the rough set-up that is needed to determine the law of the solution. To do so, we keep track of the information required at each step of the Picard iteration used in the proof of Theorem 16. To this end, recall from the estimate (2.3) on rough integrals the expansion

$$\begin{aligned} X^{n+1}_{t_i}(\omega) &= X_0(\omega)+\sum_{j=1}^i \mathrm{F}\big(X^n_{t_{j-1}}(\omega),X^n_{t_{j-1}}(\cdot)\big)W_{t_{j-1},t_j}(\omega)\\ &\quad+\sum_{j=1}^i \partial_x\mathrm{F}\big(X^n_{t_{j-1}}(\omega),X^n_{t_{j-1}}(\cdot)\big)\Big(\mathrm{F}\big(X^n_{t_{j-1}}(\omega),X^n_{t_{j-1}}(\cdot)\big)\mathbb{W}_{t_{j-1},t_j}(\omega)\Big)\\ &\quad+\sum_{j=1}^i\Big\langle D_\mu\mathrm{F}\big(X^n_{t_{j-1}}(\omega),X^n_{t_{j-1}}(\cdot)\big)\big(X^n_{t_{j-1}}(\cdot)\big)\Big(\mathrm{F}\big(X^n_{t_{j-1}}(\cdot),X^n_{t_{j-1}}(\cdot)\big)\mathbb{W}^{\perp\!\!\!\perp}_{t_{j-1},t_j}(\cdot,\omega)\Big)\Big\rangle\\ &\quad+\sum_{j=1}^i S^{n+1}_{t_{j-1},t_j}(\omega); \end{aligned}$$

it holds true for any subdivision $0 = t_0 < \cdots < t_K = T$, the last term converging to 0 as the step size of the subdivision tends to 0. Hence, if we assume that the $\mathcal{C}([0,T];\mathbb{R}^d)$-valued random variable $X^n(\cdot)$ is measurable with respect to the σ-field generated by some variable Θ^n with values in an auxiliary Polish space $\mathcal{S}_n$, we have that $X^{n+1}(\omega)$ is the image, by a measurable function, of

$$\Big(X_0(\omega), W(\omega), \mathbb{W}(\omega), \Theta^n(\omega), \mathcal{L}\big(\Theta^n(\cdot), \mathbb{W}^{\perp\!\!\!\perp}(\cdot,\omega)\big)\Big).$$

The random variable right above takes values in

$$\mathbb{R}^d \times \mathcal{C}\big([0,T];\mathbb{R}^m\big) \times \mathcal{C}\big(\mathcal{S}_2^T;\mathbb{R}^m \otimes \mathbb{R}^m\big) \times \mathcal{S}_n \times \mathcal{P}\Big(\mathcal{S}_n \times \mathcal{C}\big(\mathcal{S}_2^T;\mathbb{R}^m \otimes \mathbb{R}^m\big)\Big),$$

the last factor being equipped with the standard topology of weak convergence. Noticing that $\mathcal{S}_0$ can be chosen as $\{0\}$ and $\Theta^0(\cdot)$ as $\Theta^0(\cdot) \equiv 0$, this defines a countable sequence of Polish space-valued random variables; basically, the law of the whole sequence suffices to determine the law of the solution to (0.2).

Although this approach could be made entirely rigorous to address uniqueness in law in the upmost general framework, all the examples we have enter in fact a simpler setting. Somehow, the problem we face with weak uniqueness is the same as the one we encountered in the example of a rough set-up given by Proposition 4. The difficulty is indeed to reconstruct the iterated integral $\mathbb{W}^{\perp\!\!\!\perp}(\omega',\omega)$ from the observation of $W(\omega)$, $W(\omega')$ and $\mathbb{W}(\omega)$; in the proof of Proposition 4, this is made at the price of an extra source of randomness. When addressing weak uniqueness, this extra source of randomness has to be identified in a canonical way; this is exactly what the above iterative procedure, based on the sequence $(\Theta^n)_{n\geqslant 0}$, does. Interestingly (and fortunately), all this cumbersome construction becomes trivial when $\mathbb{W}^{\perp\!\!\!\perp}(\omega',\omega)$ can be (almost surely) written as the image of $\big(W(\omega), W(\omega')\big)$ by a measurable function. In that case, there is no need of an extra source of randomness. Equivalently, all the $\big(\Theta^n, \mathcal{S}_n\big)_{n\geqslant 1}$ can be chosen as $\big(\Theta^n \equiv (X_0, W^n, \mathbb{W}^n), \mathcal{S}_n = \mathbb{R}^d \times \mathcal{C}([0,T];\mathbb{R}^m) \times \mathcal{C}(\mathcal{S}_2^T;\mathbb{R}^m \otimes \mathbb{R}^m)\big)_{n\geqslant 1}$. Indeed, $\mathcal{L}\big(\mathbb{W}^{\perp\!\!\!\perp}(\cdot,\omega)\big)$ writes, for almost every $\omega \in \Omega$, as the image of $W(\omega)$ by a measurable function. Importantly, both Examples 3 and 5 fall within this case. More generally, in the framework of Proposition 4, we can write $W^{2,1}$ as the almost sure image of $\big(W^1, W^2\big)$ by a measurable function from $\mathcal{C}\big([0,T];\mathbb{R}^m\big)^2$ into $\mathcal{C}\big(\mathcal{S}_2^T;\mathbb{R}^m \otimes \mathbb{R}^m\big)$, when, for almost every $\xi \in \Xi$, the quantity $W^{2,1}(\xi)$ can be approximated by the iterated integral of mollified versions of $W^1(\xi)$ and $W^2(\xi)$, provided the mollification procedure defines a measurable map from $\mathcal{C}([0,T];\mathbb{R}^m)$ into itself. This is for instance the case with linear interpolation or convolution by a smooth kernel.

17. Proposition – *Within the framework of* Proposition *4, define, for* $1 \leqslant i \leqslant 2$, *and for all* $n \geqslant 0$, *the linear interpolation* $W^{i,n}$ *of* W^i *at dyadic points* $\big(t_n^k = kT/2^n\big)_{k=0,\cdots,2^n-1}$ *of* $[0,T]$, *namely, set*

$$W_t^{i,n}(\xi) = \sum_{k=0}^{2^n-1} \left(W_{t_n^k}^i(\xi) + W_{t_n^k,t_n^{k+1}}^i(\xi) \frac{2^n(t-t_n^k)}{T} \right) \mathbf{1}_{[t_n^k,t_n^{k+1})}(t).$$

If for $\mathbb{Q}$*-almost every* $\xi \in \Xi$, *for all* $(s,t) \in \mathcal{S}_2^T$,

$$W_{s,t}^{2,1}(\xi) = \lim_{n\to\infty} \int_{s,t} \Big(W_r^{2,n}(\xi) - W_s^{2,n}(\xi) \Big) \otimes dW_r^{1,n}(\xi),$$

then there exists a measurable function $\mathcal{I}$ from $\mathcal{C}([0,T];\mathbb{R}^m)^2$ into $\mathcal{C}\big(\mathcal{S}_2^T;\mathbb{R}^m\otimes\mathbb{R}^m\big)$ such that

$$\mathbb{Q}\Big(\Big\{\xi\in\Xi : W^{2,1}(\xi)=\mathcal{I}\big(W^2(\xi),W^1(\xi)\big)\Big\}\Big)=1.$$

The scope of Proposition 17 is limited to so-called *geometric rough paths*, but the underlying principle is actually more general. This prompts us to introduce the following definition.

18. Definition – *A* **rough set-up**, *as defined in Section 1, is called* **strong** *if there exists a measurable mapping $\mathcal{I}$ from $\mathcal{C}\big([0,T];\mathbb{R}^m\big)^2$ into $\mathcal{C}\big(\mathcal{S}_2^T;\mathbb{R}^m\otimes\mathbb{R}^m\big)$ such that*

$$\mathbb{P}^{\otimes 2}\Big(\big\{(\omega,\omega')\in\Omega^2 : \mathbb{W}^{\perp\!\!\!\perp}(\omega,\omega')=\mathcal{I}\big(W(\omega),W(\omega')\big)\big\}\Big)=1. \tag{3.24}$$

So, Proposition 17 provides a typical instance of strong set-up, which covers in particular Examples 3 and 5. However, it is worth mentioning that strong set-ups may not fall within the scope of Proposition 17, since the latter is limited to geometric rough paths. This is for instance the case if in Proposition 4 we take $W^1(\cdot)$ and $W^2(\cdot)$ to be two independent Brownian motions and $\mathbb{W}^{2,1}(\cdot,\cdot)$ to be the Itô integral between $W^2(\cdot)$ and $W^1(\cdot)$ rather than their Stratonovich integral. Also, we refer the reader to Deuschel and al. [21] for a related use of the notion of strong set-up, although the terminology *strong* does not appear therein.

Proposition 4 sheds a light on the rationale for the word *strong* in Definition 18. Here *strong* has the same meaning as in the theory of strong solutions to stochastic differential equations: The second level $W^{2,1}$ of the rough-path is a measurable function of (W^2,W^1). In contrast, the general set-up considered in the statement of Proposition 4 may not be strong as $W^{2,1}$ may carry, in addition to (W^1,W^2), an additional external independent randomization. If this additional randomization is not trivial, the set-up should be called *weak*. An instance is given by the collection of real-valued rough paths:

$$\begin{aligned} &W^1(\xi)=W^2(\xi)\equiv 0, \quad W^{1,1}(\xi)\equiv 0,\\ &W^{2,1}_{s,t}(\xi)=a(\xi)(t-s), \quad (s,t)\in\mathcal{S}_2^T, \end{aligned}$$

for ξ in a probability space $(\Xi,\mathcal{G},\mathbb{Q})$, where a is a real-valued random variable on $(\Xi,\mathcal{G},\mathbb{Q})$. If the support of a does not reduce to one point, then the set-up induced by $\big(W^1(\cdot),W^2(\cdot),W^{1,1}(\cdot),W^{1,2}(\cdot)\big)$ is strictly weak. We now have all the ingredients to formulate a weak uniqueness property.

19. Theorem – *Let $X_0(\cdot):=\big(X_0(\omega)\big)_{\omega\in\Omega}$ and $X_0'(\cdot):=\big(X_0'(\omega)\big)_{\omega\in\Omega'}$ and*

$$\begin{aligned} &\boldsymbol{W}(\cdot):=\big(W(\omega),\mathbb{W}(\omega),\mathbb{W}^{\perp\!\!\!\perp}(\omega,\omega')\big)_{\omega\in\Omega,\omega'\in\Omega},\\ &\boldsymbol{W}'(\cdot):=\big(W'(\omega),\mathbb{W}'(\omega),\mathbb{W}^{\perp\!\!\!\perp,\prime}(\omega,\omega')\big)_{\omega\in\Omega',\omega'\in\Omega'}, \end{aligned}$$

be two square integrable initial conditions and two strong rough set-ups with the same parameters m, p and q, defined on two probability spaces $(\Omega,\mathcal{F},\mathbb{P})$ and $(\Omega',\mathcal{F}',\mathbb{P}')$, and such that the random variables

$$\begin{aligned} &\Omega^2\ni(\omega,\omega')\mapsto\big(X_0(\omega),W(\omega),\mathbb{W}(\omega),W^{\perp\!\!\!\perp}(\omega,\omega')\big),\\ &(\Omega')^2\ni(\omega,\omega')\mapsto\big(X_0'(\omega),W'(\omega),\mathbb{W}'(\omega),W^{\perp\!\!\!\perp,\prime}(\omega,\omega')\big), \end{aligned}$$

have the same law on $\mathbb{R}^d\times\mathcal{C}([0,T];\mathbb{R}^m)\times\mathcal{C}(\mathcal{S}_2^T;\mathbb{R}^m\otimes\mathbb{R}^m)\times\mathcal{C}(\mathcal{S}_2^T;\mathbb{R}^m\otimes\mathbb{R}^m)$. Then, the corresponding two solutions $\big(X(\omega)\big)_{\omega\in\Omega}$ and $\big(X'(\omega)\big)_{\omega\in\Omega'}$ to (0.2) have the same law on $\mathcal{C}([0,T];\mathbb{R}^m)$.

As the two set-ups have the same law, we can use the same mapping $\mathcal{I}$ in the representations (3.24) of $\mathbb{W}^{\perp\!\!\!\perp}$ and of $\mathbb{W}^{\perp\!\!\!\perp,\prime}$.

3.5 – Continuity of the Itô-Lyons Map

As expected from a robust solution theory of differential equations, we have continuity of the solution with respect to the parameters in the equation, most notably the rough set-up itself. The next statement quantifies that fact.

20. Theorem – *Let* F *satisfy the same assumptions as in* Theorem 16*. Given a time interval* $[0,T]$ *and a sequence of probability spaces* $(\Omega_n, \mathcal{F}_n, \mathbb{P}_n)$*, indexed by* $n \in \mathbb{N}$*, let, for any* n*,* $X_0^n(\cdot) := (X_0^n(\omega_n))_{\omega_n \in \Omega_n}$ *be an* $\mathbb{R}^d$*-valued square-integrable initial condition and*

$$\boldsymbol{W}^n(\cdot) := \Big(W^n(\omega_n), \mathbb{W}^n(\omega_n), \mathbb{W}^{n,\perp\!\!\!\perp}(\omega_n, \omega_n')\Big)_{\omega_n, \omega_n' \in \Omega_n}$$

be an m*-dimensional rough set-up with corresponding control* w^n *and local accumulated variation* N^n*, for fixed values of* $p \in [2,3)$ *and* $q > 8$*. Assume that*

- *for positive constants* ε_1, c_1 *and* $(\varepsilon_2(\alpha), c_2(\alpha))_{\alpha>0}$*, the tail assumption* (3.17) *hold for* w^n *and* N^n*, for all* $n \geqslant 0$*;*
- *associating* v^n *with each* $\boldsymbol{W}^n(\cdot)$ *as in* (1.7)*, the functions*

$$\big(\mathcal{S}_2^T \ni (s,t) \mapsto \langle v^n(s,t,\cdot)\rangle_{2q}\big)_{n\geqslant 0}$$

  are uniformly Lipschitz continuous;

Assume also that there exist, on another probability space $(\Omega, \mathcal{F}, \mathbb{P})$*, a square integrable initial condition* $X_0(\cdot)$ *with values in* $\mathbb{R}^d$ *and a strong rough set-up*

$$\boldsymbol{W}(\cdot) := \Big(W(\omega), \mathbb{W}(\omega), \mathbb{W}^{\perp\!\!\!\perp}(\omega, \omega')\Big)_{\omega, \omega' \in \Omega}$$

with values in $\mathbb{R}^m$*, such that*

- *The collection* $\big(\mathbb{P}_n \circ (|X_0^n(\cdot)|^2)^{-1}\big)_{n\geqslant 0}$ *is uniformly integrable.*
- *The law under the probability measure* $\mathbb{P}_n^{\otimes 2}$ *of the random variable* $\Omega_n^2 \ni (\omega_n, \omega_n') \mapsto \big(X_0^n(\omega_n), W^n(\omega_n), \mathbb{W}_n(\omega_n), \mathbb{W}_n^{\perp\!\!\!\perp}(\omega_n, \omega_n')\big)$*, seen as a random variable with values in* $\mathcal{C}([0,T];\mathbb{R}^m) \times \big\{\mathcal{C}(\mathcal{S}_2^T; \mathbb{R}^m \otimes \mathbb{R}^m)\big\}^2$*, converges in the weak sense to the law of* $\Omega^2 \ni (\omega, \omega') \mapsto \big(X_0(\omega), W(\omega), \mathbb{W}(\omega_n), \mathbb{W}^{\perp\!\!\!\perp}(\omega, \omega')\big)$*.*

Then, $\boldsymbol{W}(\cdot)$ *satisfies the requirements of* Theorem 16 *for some* $p' \in (p,3)$ *and* $q' \in [8,q)$*. Moreover, if* $X^n(\cdot)$*, resp.* $X(\cdot)$*, is the solution of the mean field rough differential equation driven by* $\boldsymbol{W}^n(\cdot)$*, resp.* $\boldsymbol{W}(\cdot)$*, then* $X^n(\cdot)$ *converges in law to* $X(\cdot)$ *on* $\mathcal{C}([0,T];\mathbb{R}^d)$*.*

The rationale for the framework and the assumptions used in the statement of Theorem 20 is two-fold. First, it allows for a proof based on compactness arguments; in particular, the proof completely bypasses any lengthy stability estimate of the paths with respect to the rough structure, which, in our extended framework, would be especially cumbersome. Also, this compactness argument is pretty interesting in itself and complements quite well Subsection 3.4 on weak uniqueness; noticeably, it allows the set-ups to be supported by different probability spaces. Second, our formulation of the continuity of the Itô-Lyons map turns out to be well-fitted to the applications we have in mind, see the next section.

The assumption that the limiting rough set-up is strong is tailored-made to the compactness arguments we use below; indeed, our strategy is to prove that the sequence of laws induced by the solutions to the equations (0.2), when driven by the $(\boldsymbol{W}^n(\cdot))_{n\geqslant 0}$'s, are tight. Even if this procedure is quite simple, it also requires to pass to the weak limit along the laws of the rough set-ups $(\boldsymbol{W}^n(\cdot))_{n\geqslant 0}$ and identify the limiting law. As explained in Subsection 3.4, this is much easier to come when the set-ups are strong; hence the assumption.

Proof – Throughout the proof, we call $p \in [2,3)$ and $q > 8$ the fixed indices used to define the set-ups and, in particular, to control the variations in the definition (3.17) of each w^n, $n \geqslant 0$. This is important because, at some points of the proof, we will use other values $p' > p$ and $q' < q$.

Step 1. This step is dedicated to the proof of several key properties on the tightness of the sequence $(\boldsymbol{W}^n(\cdot))_{n\geqslant 0}$.

1a. For any $n \geqslant 0$, we introduce the modulus of continuity of $(W^n(\cdot), \mathbb{W}^n(\cdot), \mathbb{W}^{n,\perp\!\!\!\perp}(\cdot))$, namely we let, for any $\delta > 0$,

$$\begin{aligned}\varsigma^n\big(\delta, \omega_n, \omega_n'\big) := &\sup_{|s-t|\leqslant \delta} |W_t^n(\omega_n) - W_s^n(\omega_n)| \\ &+ \sup_{|s-t|+|s'-t'|\leqslant \delta} \big|\mathbb{W}^n_{s',t'}(\omega_n) - \mathbb{W}^n_{s,t}(\omega_n)\big| \\ &+ \sup_{|s-t|+|s'-t'|\leqslant \delta} \big|\mathbb{W}^{n,\perp\!\!\!\perp}_{s',t'}(\omega_n,\omega_n') - \mathbb{W}^{n,\perp\!\!\!\perp}_{s',t'}(\omega_n,\omega_n')\big|,\end{aligned}$$

where $(\omega_n, \omega_n') \in \Omega_n^2$.

Since the laws of the processes $(W^n(\cdot), \mathbb{W}^n(\cdot), \mathbb{W}^{n,\perp\!\!\!\perp}(\cdot,\cdot))_{n\geqslant 0}$ are tight in the space $\mathcal{C}([0,T];\mathbb{R}^m) \times \big\{\mathcal{C}(\mathcal{S}_2^T;\mathbb{R}^m \otimes \mathbb{R}^m)\big\}^2$, we deduce that

$$\forall \varepsilon > 0, \quad \lim_{\delta \searrow 0} \sup_{n\geqslant 0} \mathbb{P}_n^{\otimes 2}\Big(\big\{(\omega_n,\omega_n') \in \Omega_n^2 : \varsigma_n\big(\delta,\omega_n,\omega_n'\big) \geqslant \varepsilon\big\}\Big) = 0.$$

1b. We now prove that, for any $q' \in [8,q)$, the laws of the processes $\big(\Omega_n \ni \omega_n \mapsto \langle \mathbb{W}^{n,\perp\!\!\!\perp}(\omega_n,\cdot)\rangle_{q'}\big)_{n\geqslant 0}$ are tight, and similarly for the laws of the processes $\big(\Omega_n \ni \omega_n \mapsto \langle \mathbb{W}^{n,\perp\!\!\!\perp}(\cdot,\omega_n)\rangle_{q'}\big)_{n\geqslant 0}$.

Obviously, we have, for any $\omega_n \in \Omega_n$,

$$\sup_{(s,t)\in\mathcal{S}_2^T} \mapsto \big\langle \mathbb{W}^{n,\perp\!\!\!\perp}_{s,t}(\omega_n,\cdot)\big\rangle_q \leqslant w^n(0,T,\omega_n).$$

By the first bullet point in the assumption, the tails of the right-hand side are uniformly dominated. So,

$$\lim_{A\to\infty} \sup_{n\geqslant 0} \mathbb{P}\Big(\Big\{\omega_n \in \Omega_n : \sup_{(s,t)\in\mathcal{S}_2^T} \big\langle \mathbb{W}^{n,\perp\!\!\!\perp}_{s,t}(\omega_n,\cdot)\big\rangle_q \geqslant A\Big\}\Big) = 0,$$

which is one first step in the proof of tightness.

For any $a > 0$, we now consider the following event:

$$E_n(\delta,a) := \Big\{\omega_n \in \Omega_n : \mathbb{P}_n\Big(\big\{\omega_n' \in \Omega_n : \varsigma_n(\delta,\omega_n,\omega_n') \geqslant \varepsilon\big\}\Big) \geqslant a\Big\}.$$

By Markov's inequality and then Fubini's theorem,

$$\mathbb{P}_n\big(E_n(\delta,a)\big) \leqslant a^{-1}\mathbb{P}_n^{\otimes 2}\Big(\big\{(\omega_n,\omega_n') \in \Omega_n^2 : \varsigma_n(\delta,\omega_n,\omega_n') \geqslant \varepsilon\big\}\Big).$$

Clearly, for any $\varepsilon > 0$, we can find a collection of positive reals $(a_\varepsilon(\delta))_{\delta>0}$ such that

$$\lim_{\delta \searrow 0} a_\varepsilon(\delta) = 0, \quad \text{and} \quad \lim_{\delta \searrow 0} \mathbb{P}_n\Big(E_n\big(\delta, a_\varepsilon(\delta)\big)\Big) = 0.$$

Take now $\omega_n \in E_n(\delta, a_\varepsilon(\delta))^{\complement}$ such that

$$\sup_{(s,t)\in \mathcal{S}_2^T} \big\langle \mathbb{W}_{s,t}^{n,\perp\!\!\!\perp}(\omega_n, \cdot)\big\rangle_q \leqslant A,$$

for a given $A > 0$. Then, for any $q' \in [8, q)$ and $(s,t), (s',t') \in \mathcal{S}_2^T$,

$$\begin{aligned}\Big|\big\langle \mathbb{W}_{s',t'}^{n,\perp\!\!\!\perp}(\omega_n, \cdot)\big\rangle_{q'} - \big\langle \mathbb{W}_{s,t}^{n,\perp\!\!\!\perp}(\omega_n, \cdot)\big\rangle_{q'}\Big| &\leqslant \Big\langle \mathbb{W}_{s',t'}^{n,\perp\!\!\!\perp}(\omega_n, \cdot) - \mathbb{W}_{s,t}^{n,\perp\!\!\!\perp}(\omega_n, \cdot)\Big\rangle_{q'} \\ &\leqslant \varepsilon + A a_\varepsilon(\delta)^{1-q'/q}.\end{aligned}$$

For A fixed and δ small enough, the right-hand side is less than 2ε. We easily deduce that

$$\forall \varepsilon > 0, \quad \lim_{\delta \searrow 0} \sup_{n \geqslant 0} \mathbb{P}_n\bigg(\Big\{\omega_n \in \Omega_n : \Big|\big\langle \mathbb{W}_{s',t'}^{n,\perp\!\!\!\perp}(\omega_n, \cdot)\big\rangle_{q'} - \big\langle \mathbb{W}_{s,t}^{n,\perp\!\!\!\perp}(\omega_n, \cdot)\big\rangle_{q'}\Big| \geqslant \varepsilon\Big\}\bigg) = 0.$$

Of course, we can proceed in a similar way for $\big(\Omega_n \ni \omega_n \mapsto \langle \mathbb{W}^{n,\perp\!\!\!\perp}(\cdot, \omega_n)\rangle_{q'}\big)_{n \geqslant 0}$. In fact, the same argument shows that the deterministic functions $\big(\langle W^n(\cdot)\rangle_{q'}\big)_{n\geqslant 0}$ and $\big(\langle\!\langle W^{n,\perp}(\cdot,\cdot)\rangle\!\rangle_{q'}\big)_{n\geqslant 0}$ are relatively compact in $\mathcal{C}([0,T];\mathbb{R})$ and $\mathcal{C}(\mathcal{S}_2^T;\mathbb{R})$.

1c. For each of the following family of processes, we know that the corresponding family of laws is tight in $\mathcal{C}(\mathcal{S}_2^T;\mathbb{R})$ and that the associated family of p-variations over $[0,T]$ has tight laws in $\mathbb{R}$ (because of the first item in the assumption):

- $\Big(\Omega_n \ni \omega_n \mapsto \big(|(W_t^n - W_s^n)(\omega_n)|\big)_{(s,t)\in\mathcal{S}_2^T}\Big)_{n\geqslant 0}$;
- $\Big(\Omega_n \ni \omega_n \mapsto \big(|\mathbb{W}_{s,t}^n(\omega_n)|\big)_{(s,t)\in\mathcal{S}_T^2}\Big)_{n\geqslant 0}$;
- $\Big(\Omega_n \ni \omega_n \mapsto \big(\big\langle \mathbb{W}_{s,t}^{n,\perp\!\!\!\perp}(\omega_n,\cdot)\big\rangle_{q'}\big)_{(s,t)\in\mathcal{S}_T^2}\Big)_{n\geqslant 0}$;
- $\Big(\Omega_n \ni \omega_n \mapsto \big(\big\langle \mathbb{W}_{s,t}^{n,\perp\!\!\!\perp}(\cdot,\omega_n)\big\rangle_{q'}\big)_{(s,t)\in\mathcal{S}_T^2}\Big)_{n\geqslant 0}$.

As a consequence, we can apply Lemma 21 below, with any $p' \in (p,3)$ instead of p itself, and with $Z_{s,t}^n(\omega)$ equal to one the above process.

We proceed similarly with the deterministic sequences

- $\Big(\big(z_{s,t}^n = \big\langle (W_t^n - W_s^n)(\cdot)\big\rangle_{q'}\big)_{(s,t)\in\mathcal{S}_2^T}\Big)_{n\geqslant 0}$;
- $\Big(\big(z_{s,t}^n = \langle\!\langle \mathbb{W}_{s,t}^{n,\perp\!\!\!\perp}(\cdot,\cdot)\rangle\!\rangle_{q'}\big)_{(s,t)\in\mathcal{S}_2^T}\Big)_{n\geqslant 0}$.

We deduce that, for any $p' \in (p,3)$, the sequence of probability measures $\Big(\mathbb{P}\circ(\mathcal{S}_2^T \ni (s,t) \mapsto v^{n,\prime}(s,t,\cdot))^{-1}\Big)_{n\geqslant 0}$ is tight in $\mathcal{C}(\mathcal{S}_T^2;\mathbb{R})$ and that

$$\forall \varepsilon > 0, \quad \lim_{\delta \to 0} \sup_{n\geqslant 0} \mathbb{P}_n\left(\sup_{(s,t)\in\mathcal{S}_2^T: t-s\leqslant\delta} v^{n,\prime}(s,t,\cdot) > \varepsilon\right) = 0,$$

where $v^{n,\prime}$ is associated with $\boldsymbol{W}^n(\cdot)$ through (1.7) and where we put a prime in the notation to emphasize the fact that we use the pair of parameters (p',q') instead of (p,q).

1d. Obviously, $v^{n,\prime}(s,t,\cdot) \leqslant (v^n(s,t,\cdot))^{p'/p}$. Since $p'/p \leqslant 2$ and

- the tails of $w^n \geqslant v^n$ decay faster than any polynomial function, uniformly in $n \geqslant 0$;
- the function $\mathcal{S}_2^T \ni (s,t) \mapsto \langle v^n(s,t,\cdot)\rangle_{2q}$ is Lipschitz continuous, uniformly in $n \geqslant 0$;

we deduce that $(s,t) \mapsto \langle v^{n,\prime}(s,t,\cdot)\rangle_q$ is Lipschitz continuous, uniformly in $n \geqslant 0$. Hence,

$$\forall \varepsilon > 0, \quad \lim_{\delta \to 0} \sup_{n \geqslant 0} \mathbb{P}_n \left(\sup_{(s,t) \in \mathcal{S}_2^T : t-s \leqslant \delta} w^{n,\prime}(s,t,\cdot) > \varepsilon \right) = 0,$$

where, as above, we put a prime in the notation $w^{n,\prime}$ to emphasize the fact that the rough set-up is driven by the parameters (p',q'). Importantly, we deduce from the bound $(v^{n,\prime}(0,T,\cdot))^{1/p'} \leqslant (v^n(0,T,\cdot))^{1/p}$ that, similar to w^n and N^n (the latter is associated with w^n through (1.13)), the function $w^{n,\prime}$ and the corresponding local accumulated variation $N^{n,\prime}$ (given by (1.13) with $\varpi = w^{n,\prime}$) satisfy the tail assumption (3.17), uniformly in $n \geqslant 0$. The bound on the tails of $N^{n,\prime}$ is easily obtained by comparison with the tails of N^n.

Step 2.

2a. The next step is to observe, as a corollary of the proof of Theorem 16, that there exist a constant C and a real $S > 0$ such that, for all $n \geqslant 0$,

$$\left\langle \| X^n(\cdot) \|_{[0,S],w^{n,\prime},p'} \right\rangle_8 \leqslant C.$$

The fact that C and S can be chosen independently of n is a consequence of the fact that the tails of N^n and w^n are controlled uniformly in $n \geqslant 0$. Here S is chosen small enough so that the two constraints (3.18) and (3.19) appearing in the statement are satisfied, uniformly in $n \geqslant 0$.

2b. Arguing as in the derivation of Theorem 1 from the statement of Theorem 16, we can iterate the argument and construct a sequence of deterministic times $0 = S_0 < S = S_1 < \ldots < S_K = T$, for some deterministic $K \geqslant 1$, such that, for all $n \geqslant 0$ and all $j \in \{0, \cdots, K-1\}$,

$$\left\langle \| X^n(\cdot) \|_{[S_j,S_{j+1}],w^{n,\prime},p'} \right\rangle_8 \leqslant C.$$

Up to a modification of the constant C, we deduce that, for all $n \geqslant 1$,

$$\left\langle \| X^n(\cdot) \|_{[0,T],w^{n,\prime},p'} \right\rangle_8 \leqslant C.$$

Recalling that $\big(\mathbb{P}_n \circ (|X_0^n(\cdot)|^2)^{-1}\big)_{n \geqslant 0}$ is uniformly integrable, it is easily checked that $\big(\mathbb{P}_n \circ (\sup_{0 \leqslant t \leqslant T} |X_t^n(\cdot)|^2)^{-1}\big)_{n \geqslant 0}$ is also uniformly integrable.

2c. As another result of the previous step, for any $\varepsilon > 0$, we can find $a > 0$ such that

$$\sup_{n \geqslant 0} \mathbb{P}_n \Big(\| X^n(\cdot) \|_{[0,T],w^{n,\prime},p'} > a \Big) \leqslant \varepsilon,$$

from which, together with **1d**, we deduce that

$$\forall a > 0, \quad \exists \varepsilon > 0 \ : \ \sup_{n \geqslant 0} \mathbb{P}_n \Big(\forall (s,t) \in \mathcal{S}_2^T, |X_{s,t}^n|^{p'} > a w^{n,\prime}(s,t) \Big) \leqslant \varepsilon.$$

Combining with the conclusion of the first step, this yields

$$\forall \varepsilon > 0, \quad \lim_{\delta \to 0} \sup_{n \geqslant 0} \mathbb{P}_n \left(\sup_{(s,t) \in \mathcal{S}_2^T : t-s \leqslant \delta} |X_{s,t}^n| > \varepsilon \right) = 0.$$

From the conclusion of **2b**, the sequence $\big(\mathbb{P}_n \circ (X^n(\cdot))^{-1}\big)_{n\geqslant 0}$ is tight in $\mathcal{C}\big([0,T];\mathbb{R}^d\big)$.

Step 3.

3a. As a consequence of the assumptions of Theorem 20 and of Step 2, we have the following tightness properties:

- The families of distributions $\big(\mathbb{P}_n \circ (W^n(\cdot))^{-1}\big)_{n\geqslant 0}$ and $\big(\mathbb{P}_n \circ (X^n(\cdot))^{-1}\big)_{n\geqslant 0}$ are tight in $\mathcal{C}\big([0,T];\mathbb{R}^m\big)$ and in $\mathcal{C}\big([0,T];\mathbb{R}^d\big)$;
- the family of distributions $\big(\mathbb{P}_n \circ (\mathbb{W}^n)^{-1}(\cdot)\big)_{n\geqslant 0}$ is tight in $\mathcal{C}\big(\mathcal{S}_2^T;\mathbb{R}^m \otimes \mathbb{R}^m\big)$;
- the family

$$\Big(\mathbb{P}_n^{\otimes 2} \circ \Big(\Omega_n^2 \ni (\omega_n,\omega_n') \mapsto \mathbb{W}^{n,\perp\!\!\!\perp}(\omega_n,\omega_n') \in \mathcal{C}(\mathcal{S}_2^T;\mathbb{R}^m \otimes \mathbb{R}^m)\Big)^{-1}\Big)_{n\geqslant 0}$$

  is tight in $\mathcal{C}\big(\mathcal{S}_2^T;\mathbb{R}^m \otimes \mathbb{R}^m\big)$;
- the family

$$\Big(\mathbb{P}_n \circ \Big(v^{n,\prime}(\omega_n) : \Omega_n \ni \omega_n \mapsto \big(\mathcal{S}_2^T \ni (s,t) \mapsto v^{n,\prime}(s,t,\omega_n)\big) \in \mathcal{C}(\mathcal{S}_2^T;\mathbb{R})\Big)^{-1}\Big)_{n\geqslant 0}$$

  is tight in $\mathcal{C}\big(\mathcal{S}_2^T;\mathbb{R}\big)$;

3b. By Skorokhod's representation theorem, we can find an auxiliary Polish probability space $\big(\widehat{\Omega},\widehat{\mathcal{F}},\widehat{\mathbb{P}}\big)$, such that, up to a subsequence, the following convergence holds for $\widehat{\mathbb{P}}$-almost every $\widehat{\omega} \in \widehat{\Omega}$. We have

$$\begin{aligned}&\lim_{n\to\infty} \Big(\widehat{W}^{n,1}(\widehat{\omega}),\widehat{W}^{n,2}(\widehat{\omega}),\widehat{\mathbb{W}}^{n,1,1}(\widehat{\omega}),\widehat{\mathbb{W}}^{n,2,1}(\widehat{\omega}),\widehat{v}^{n,1,\prime}(\widehat{\omega}),\widehat{v}^{n,2,\prime}(\widehat{\omega}),\widehat{X}^{n,1}(\widehat{\omega}),\widehat{X}^{n,2}(\widehat{\omega})\Big)\\ &= \Big(\widehat{W}^{1}(\widehat{\omega}),\widehat{W}^{2}(\widehat{\omega}),\widehat{\mathbb{W}}^{1,1}(\widehat{\omega}),\widehat{\mathbb{W}}^{2,1}(\widehat{\omega}),\widehat{v}^{1,\prime}(\widehat{\omega}),\widehat{v}^{2,\prime}(\widehat{\omega}),\widehat{X}^{1}(\widehat{\omega}),\widehat{X}^{2}(\widehat{\omega})\Big), \end{aligned} \tag{3.25}$$

where $\big(\widehat{W}^{n,1},\widehat{W}^{n,2},\widehat{\mathbb{W}}^{n,1,1},\widehat{\mathbb{W}}^{n,1,2},\widehat{v}^{n,1,\prime}(\widehat{\omega}),\widehat{v}^{n,2,\prime}(\widehat{\omega}),\widehat{X}^{n,1}(\widehat{\omega}),\widehat{X}^{n,2}(\widehat{\omega})\big)$ has the same law as the random variable

$$\begin{aligned}&\Omega_n^2 \ni (\omega_n,\omega_n')\\ &\quad\mapsto \Big(W^n(\omega_n),W^n(\omega_n'),\mathbb{W}^n(\omega_n),\mathbb{W}^{n,\perp\!\!\!\perp}(\omega_n,\omega_n'),v^{n,\prime}(\omega_n),v^{n,\prime}(\omega_n'),X^n(\omega_n),X^n(\omega_n')\Big),\end{aligned}$$

which takes values in

$$\big\{\mathcal{C}([0,T];\mathbb{R}^m)\big\}^2 \times \big\{\mathcal{C}(\mathcal{S}_2^T;\mathbb{R}^m\otimes\mathbb{R}^m)\big\}^2 \times \big\{\mathcal{C}([0,T];\mathbb{R}^d)\big\}^2 \times \big\{\mathcal{C}(\mathcal{S}_2^T;\mathbb{R})\big\}^2,$$

and where $\big(\widehat{W}^1(\cdot),\widehat{W}^2(\cdot),\widehat{\mathbb{W}}^{1,1}(\cdot),\widehat{\mathbb{W}}^{2,1}(\cdot),X_0^1(\cdot)\big)$ has the same law as the random variable

$$\Omega^2 \ni (\omega,\omega') \mapsto \Big(W(\omega),W(\omega'),\mathbb{W}(\omega),\mathbb{W}^{\perp\!\!\!\perp}(\omega',\omega),X_0(\omega)\Big). \tag{3.26}$$

3c. At this point of the proof, the difficulty is that $\big(\widehat{W}^1(\cdot),\widehat{W}^2(\cdot),\widehat{\mathbb{W}}^{1,1}(\cdot),\widehat{\mathbb{W}}^{2,1}(\cdot)\big)$ does not form a rough set-up. Still, we have the following two properties. First, using the fact that the limiting set-up is strong, we have

$$\widehat{\mathbb{P}}\Big(\Big\{\widehat{\omega} \in \widehat{\Omega} : \mathbb{W}^{2,1}(\widehat{\omega}) = \mathcal{I}\big(W^2(\widehat{\omega}),W^1(\widehat{\omega})\big)\Big\}\Big) = 1,$$

for a measurable mapping $\mathcal{I} : \mathcal{C}([0,T];\mathbb{R}^m)^2 \to \mathcal{C}(\mathcal{S}_2^T;\mathbb{R}^m\otimes\mathbb{R}^m)$, which follows from the identification with the law of (3.26). Also, passing to the limit in Chen's relations satisfied by each $\boldsymbol{W}^n$, we have, for $\widehat{\mathbb{P}}$ almost every $\hat{\omega} \in \widehat{\Omega}$, and all $0 \leqslant r \leqslant s \leqslant t \leqslant T$,

$$\widehat{W}^{1,1}_{r,t}(\hat{\omega}) = \widehat{W}^{1,1}_{r,s}(\hat{\omega}) + \widehat{W}^{1,1}_{s,t}(\hat{\omega}) + \widehat{W}^{1}_{r,s}(\hat{\omega}) \otimes \widehat{W}^{1}_{s,t}(\hat{\omega}),$$
$$\widehat{W}^{2,1}_{r,t}(\hat{\omega}) = \widehat{W}^{2,1}_{r,s}(\hat{\omega}) + \widehat{W}^{2,1}_{s,t}(\hat{\omega}) + \widehat{W}^{2}_{r,s}(\hat{\omega}) \otimes \widehat{W}^{1}_{s,t}(\hat{\omega}).$$

By preservation of independence under weak limit, $(\widehat{W}^2, \widehat{X}^2)$ is independent of $\big(\widehat{W}^1, \widehat{W}^{1,1}, \widehat{X}^1, \hat{v}^{1,\prime}\big)$. Following the proof of Proposition 4, in a simpler setting here since the limiting rough set-up is strong, we can find:

- four random variables $\widehat{W}(\cdot)$, $\widehat{\mathbb{W}}(\cdot)$, $\hat{v}'(\cdot)$ and $\widehat{X}(\cdot)$ from $\big(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathbb{P}}\big)$ into the spaces $\mathcal{C}([0,T];\mathbb{R}^m)$, $\mathcal{C}\big(\mathcal{S}_2^T;\mathbb{R}^m \otimes \mathbb{R}^m\big)$, $\mathcal{C}\big(\mathcal{S}_2^T;\mathbb{R}\big)$ and $\mathcal{C}([0,T];\mathbb{R}^d)$ such that
$$\widehat{\mathbb{P}}\Big(\Big\{\hat{\omega} \in \widehat{\Omega} : \big(\widehat{W}, \widehat{\mathbb{W}}, \hat{v}', \widehat{X}\big)(\hat{\omega}) = \big(W^1, W^{1,1}, \hat{v}^{1,\prime}, \widehat{X}^1\big)(\hat{\omega})\Big\}\Big) = 1;$$
- a random variable $\widehat{\mathbb{W}}^{\perp\!\!\!\perp}(\cdot,\cdot)$ from $\big(\widehat{\Omega}^2, \widehat{\mathcal{F}}^{\otimes 2}, \widehat{\mathbb{P}}^{\otimes 2}\big)$ into $\mathcal{C}\big(\mathcal{S}_2^T;\mathbb{R}^m \otimes \mathbb{R}^m\big)$ such that
$$\widehat{\mathbb{P}}^{\otimes 2}\Big(\Big\{(\hat{\omega}, \hat{\omega}') \in \widehat{\Omega}^2 : \widehat{\mathbb{W}}^{\perp\!\!\!\perp}(\hat{\omega}, \hat{\omega}') = \mathcal{I}\big(\widehat{W}(\hat{\omega}), \widehat{W}(\hat{\omega}')\big)\Big\}\Big) = 1; \tag{3.27}$$

the rough set-up $\widehat{\boldsymbol{W}}(\cdot) := \big(\widehat{W}(\cdot), \widehat{\mathbb{W}}(\cdot), \widehat{\mathbb{W}}^{\perp\!\!\!\perp}(\cdot,\cdot)\big)$ satisfying (1.4) with probability 1 and $\widehat{\Omega}^2 \ni (\hat{\omega}, \hat{\omega}') \mapsto \big(\widehat{W}(\hat{\omega}), \widehat{W}(\hat{\omega}'), \widehat{\mathbb{W}}(\hat{\omega}), \widehat{\mathbb{W}}^{\perp\!\!\!\perp}(\hat{\omega}', \hat{\omega}), \hat{v}'(\hat{\omega}), \hat{v}'(\hat{\omega}'), \widehat{X}(\hat{\omega}), \widehat{X}(\hat{\omega}')\big)$ having the same law as $\big(\widehat{W}^1(\cdot), \widehat{W}^2(\cdot), \widehat{W}^{1,1}(\cdot), \widehat{W}^{2,1}(\cdot), \hat{v}^{1,\prime}(\cdot), \hat{v}^{2,\prime}(\cdot), \widehat{X}^1(\cdot), \widehat{X}^2(\cdot)\big)$ on the product space

$$\big\{\mathcal{C}\big([0,T];\mathbb{R}^m\big)\big\}^2 \times \big\{\mathcal{C}\big(\mathcal{S}_2^T;\mathbb{R}^m \otimes \mathbb{R}^m\big)\big\}^2 \times \big\{\mathcal{C}\big(\mathcal{S}_2^T;\mathbb{R}\big)\big\}^2 \times \big\{\mathcal{C}\big([0,T];\mathbb{R}^d\big)\big\}^2.$$

3d. We now check that $\widehat{\boldsymbol{W}}(\cdot)$ satisfies the required regularity properties.

We start with the variations of $\widehat{W}(\hat{\omega})$, $\langle \widehat{W}(\cdot)\rangle_{q'}$, $\widehat{\mathbb{W}}(\hat{\omega})$, $\langle \widehat{\mathbb{W}}^{\perp\!\!\!\perp}(\hat{\omega},\cdot)\rangle_{q'}$, $\langle \widehat{\mathbb{W}}^{\perp\!\!\!\perp}(\cdot,\hat{\omega})\rangle_{q'}$ and $\langle\!\langle \widehat{\mathbb{W}}^{\perp\!\!\!\perp}(\cdot,\cdot)\rangle\!\rangle_{q'}$. To do so, we recall that, for almost every $\hat{\omega} \in \widehat{\Omega}$, $\hat{v}'(\hat{\omega})$ is the limit of $\hat{v}^{n,\prime}(\hat{\omega})$. By passage to the limit, $\hat{v}'$ inherits the super-additive property of the $(v^{n,\prime})_{n\geqslant 0}$'s and its tails satisfy (uniformly in $n \geqslant 0$) a bound similar to that satisfied by the $(v^n)_{n\geqslant 0}$'s in the first item of the assumption, see **1d**. Also, $\mathcal{S}_2^T \ni (s,t) \mapsto \langle v'(s,t,\cdot)\rangle_{q'}$ is Lipschitz.

Using once more the passage to the limit, we get that, for almost every $\hat{\omega} \in \widehat{\Omega}$, for any $(s,t) \in \mathcal{S}_2^T$, $|\widehat{W}_{s,t}(\hat{\omega})|^{p'} \leqslant v'(s,t,\omega)$, from which we deduce that the p'-variation of $\widehat{W}(\hat{\omega})$ is dominated (in an obvious sense) by $\hat{v}'$. A similar augment applies for $\langle \widehat{W}(\hat{\omega})\rangle_{q'}$, $\widehat{\mathbb{W}}(\hat{\omega})$ and $\langle\!\langle \widehat{\mathbb{W}}^{\perp\!\!\!\perp}(\cdot,\cdot)\rangle\!\rangle_{q'}$.

It thus remains to handle $\big\langle \widehat{\mathbb{W}}^{\perp\!\!\!\perp}(\hat{\omega},\cdot)\big\rangle_{q'}$ and $\big\langle \widehat{\mathbb{W}}^{\perp\!\!\!\perp}(\cdot,\hat{\omega})\big\rangle_{q'}$. Observe first from Fatou's lemma that

$$\Big\langle\!\!\Big\langle \sup_{(s,t)\in\mathcal{S}_2^T} \big|\widehat{\mathbb{W}}^{\perp\!\!\!\perp}_{s,t}(\cdot,\cdot)\big| \Big\rangle\!\!\Big\rangle_{q'} < \infty. \tag{3.28}$$

Hence, arguing as in the presentation of a rough set-up, see Section 1, we can consider

$$\widehat{\Omega} \ni \hat{\omega} \mapsto \widehat{W}^{\perp\!\!\!\perp}(\hat{\omega},\cdot)\mathbf{1}_{\{\langle \sup_{t\in[0,T]} |\widehat{\mathbb{W}}^{\perp\!\!\!\perp}(\hat{\omega},\cdot)|\rangle_{q'} < \infty\}},$$
$$\text{and} \quad \widehat{\Omega} \ni \hat{\omega} \mapsto \widehat{W}^{\perp\!\!\!\perp}(\cdot,\hat{\omega})\mathbf{1}_{\{\langle \sup_{t\in[0,T]} |\widehat{\mathbb{W}}^{\perp\!\!\!\perp}(\cdot,\hat{\omega})|\rangle_{q'} < \infty\}},$$

as random variables with values in the spaces

$$\mathcal{C}\big(\mathcal{S}_2^T;\mathbb{R}^m \otimes \mathbb{L}^q(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathbb{P}};\mathbb{R}^m)\big), \quad \text{and} \quad \mathcal{C}\big(\mathcal{S}_2^T;\mathbb{L}^q(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathbb{P}};\mathbb{R}^m) \otimes \mathbb{R}^m\big).$$

Continuity of the preceding two paths follows from the fact that $\widehat{W}^{\perp\!\!\!\perp}$ has continuous paths and from the bound (3.28), which makes licit the application of Lebesgue's dominated convergence theorem to prove continuity. In order to control the variations, we proceed as follows. For any non-negative valued bounded continuous function g on $\mathcal{C}([0,T];\mathbb{R}^m) \times \mathcal{C}(\mathcal{S}_2^T;\mathbb{R})$ and for every $(s,t) \in \mathcal{S}_2^T$, we have

$$\begin{aligned}
&\int_{\widehat{\Omega}} \Big[g\big(\widehat{W}(\widehat{\omega}), \widehat{v}'(\widehat{\omega})\big) \big\langle \widehat{W}^{\perp\!\!\!\perp}_{s,t}(\widehat{\omega}, \cdot)\big\rangle_{q'}^{q'} \Big] d\widehat{\mathbb{P}}(\widehat{\omega}) \\
&= \int_{\widehat{\Omega}^2} \Big[g\big(\widehat{W}(\widehat{\omega}'), \widehat{v}'(\widehat{\omega}')\big) \big(\widehat{W}^{\perp\!\!\!\perp}_{s,t}(\widehat{\omega}', \widehat{\omega})\big)^{q'} \Big] d\widehat{\mathbb{P}}^{\otimes 2}(\widehat{\omega}, \widehat{\omega}') \\
&= \lim_{n\to\infty} \int_{\Omega_n^2} \Big[g\big(W^n(\omega_n'), v^{n,\prime}(\omega_n')\big) \big(W^{n,\perp\!\!\!\perp}_{s,t}(\omega_n, \omega_n')\big)^{q'} \Big] d\mathbb{P}_n^{\otimes 2}(\omega_n', \omega_n),
\end{aligned}$$

where we used Fubini's theorem to pass from the first to the second line together with (3.25) to pass from the second to the third line. Now, we use the very definition of $v^{n,\prime}$ and the second item in the assumption to deduce that

$$\begin{aligned}
&\int_{\widehat{\Omega}} \Big[g\big(\widehat{W}(\widehat{\omega}), \widehat{v}'(\widehat{\omega})\big) \langle \widehat{W}^{\perp\!\!\!\perp}_{s,t}(\widehat{\omega}, \cdot)\rangle_{q'}^{q'} \Big] d\widehat{\mathbb{P}}(\widehat{\omega}) \\
&\leqslant \lim_{n\to\infty} \int_{\Omega_n} \Big[g\big(W^n(\omega_n), v^{n,\prime}(\omega_n)\big) \big(v^{n,\prime}(s,t,\omega_n)\big)^{q'/p'} \Big] d\mathbb{P}_n(\omega_n) \\
&= \int_{\widehat{\Omega}} \Big[g\big(\widehat{W}(\widehat{\omega}), \widehat{v}'(\widehat{\omega})\big) \big(v^{n,\prime}(s,t,\widehat{\omega})\big)^{q'} \Big] d\widehat{\mathbb{P}}(\widehat{\omega}).
\end{aligned}$$

Recalling from (3.27) that $\widehat{\Omega} \ni \widehat{\omega} \mapsto \langle \widehat{W}^{\perp\!\!\!\perp}_{s,t}(\widehat{\omega}, \cdot)\rangle_{q'/p'}$ is $\sigma\{\widehat{W}(\cdot)\}$-measurable, we get, for any $(s,t) \in \mathcal{S}_2^T$ and for almost every $\widehat{\omega} \in \widehat{\Omega}$,

$$\langle \widehat{W}^{\perp\!\!\!\perp}_{s,t}(\widehat{\omega}, \cdot)\rangle_{q'}^{p'} \leqslant v^{n,\prime}(s,t,\widehat{\omega}).$$

By continuity, this holds for almost every $\widehat{\omega} \in \widehat{\Omega}$, for all $(s,t) \in \mathcal{S}_2^T$. The same holds for $\langle \widehat{W}^{\perp\!\!\!\perp}_{s,t}(\cdot, \widehat{\omega})\rangle_{q'}$.

Associating with the rough set-up $\widehat{\boldsymbol{W}}$ a (random) control function $\overline{v}'$ through the definition (1.7) with (p,q) replaced by (p',q'), we deduce that, for $\widehat{\mathbb{P}}$-almost every $\widehat{\omega} \in \widehat{\Omega}$, for all $(s,t) \in \mathcal{S}_2^T$, $\overline{v}'(s,t,\widehat{\omega})$ is less than $\widehat{v}'(s,t,\widehat{\omega})$.

Modifying the definition of the set-up on the possibly non-empty null event where one of the aforementioned properties fails (see the proof of Proposition 4 for details), we can assume without any loss of generality that, for any $\widehat{\omega} \in \widehat{\Omega}$, the variation of $\widehat{\boldsymbol{W}}(\widehat{\omega})$ is dominated by $\widehat{v}'(\widehat{\omega})$ and that the latter is finite for all $\widehat{\omega} \in \widehat{\Omega}$. Also, we can assume that Chen's relationship, see (1.4), is satisfied for every $\widehat{\omega} \in \widehat{\Omega}$.

3e. We let $\widehat{w}'(s,t,\widehat{\omega}) := \widehat{v}'(s,t,\widehat{\omega}) + C(t-s)$, where C is the Lipschitz constant in the second item of the assumption. Clearly, $\widehat{w}'$ satisfies the first tail estimate in (3.17). Moreover, if we associate with $\widehat{w}'$ the (random) local accumulation $\widehat{N}'(\cdot,\alpha) := N_{\widehat{w}'}([0,T],\alpha)$ as in (1.13), then, by lower semicontinuity of the local accumulation (see [21, Lemma 4.2]), $\widehat{N}'(\cdot,\alpha)$ satisfies the second tail estimate in (3.17). Obviousy, the same holds for the counter $\bar{N}'(\cdot,\alpha)$ associated with $\overline{v}'(\cdot)$. This completes the proof of the fact that $\widehat{\boldsymbol{W}}(\cdot)$ satisfies all the requirements of Theorem 16.

Step 4.

4a. For each $n \geqslant 0$, we define $\delta_x \widehat{X}^n(\cdot)$ and $R^{\widehat{X}^n}(\cdot)$ as

$$\delta_x \widehat{X}^n_t(\widehat{\omega}) = \mathrm{F}\big(\widehat{X}^n_t(\widehat{\omega}), \mathcal{L}(X^n_t)\big), \quad t \in [0,T], \quad \widehat{\omega} \in \widehat{\Omega},$$
$$\widehat{R}^{\widehat{X}^n}_{s,t}(\widehat{\omega}) = \widehat{X}^n_t(\widehat{\omega}) - \widehat{X}^n_s(\widehat{\omega}) - \delta_x \widehat{X}^n_s(\widehat{\omega})\widehat{W}^n_{s,t}(\widehat{\omega}), \quad (s,t) \in \mathcal{S}_2^T, \quad \widehat{\omega} \in \widehat{\Omega},$$

from which we easily deduce that $\big(\delta_x \widehat{X}^n(\cdot), \widehat{R}^{\widehat{X}^n}(\cdot)\big)_{n \geqslant 0}$ converges with probability to 1 to $\big(\delta_x \widehat{X}(\cdot), \widehat{R}^{\widehat{X}}(\cdot)\big)$ defined as

$$\delta_x \widehat{X}_t(\widehat{\omega}) := \mathrm{F}\big(\widehat{X}_t(\widehat{\omega}), \mathcal{L}(\widehat{X}_t)\big), \quad t \in [0,T], \quad \widehat{\omega} \in \widehat{\Omega},$$
$$\widehat{R}^{\widehat{X}}_{s,t}(\widehat{\omega}) = \widehat{X}_t(\widehat{\omega}) - \widehat{X}_s(\widehat{\omega}) - \delta_x \widehat{X}_s(\widehat{\omega})\widehat{W}_{s,t}(\widehat{\omega}), \quad (s,t) \in \mathcal{S}_2^T, \quad \widehat{\omega} \in \widehat{\Omega}.$$

In order to pass to the limit in the measure argument of F, we use the fact that, for any $t \in [0,T]$, $(\mathcal{L}(X^n_t))_{n \geqslant 0}$ converges in the weak sense to $\mathcal{L}(\widehat{X}_t)$. By the uniform integrability property **2b**, the convergence also holds in 2-Wasserstein distance d_2. By continuity of F with respect to d_2, we easily conclude.

4b. By the second step, the sequence $\big(\mathbb{P}_n \circ (\|\!| X^n(\cdot) \|\!|_{[0,T],w^{n,\prime},p'})^{-1}\big)_{n \geqslant 0}$ is tight in $\mathbb{R}$, where, without any loss of generality, we take $w^{n,\prime}(s,t,\omega_n) = v^{n,\prime}(s,t,\omega_n) + C(t-s)$, for the same C as in **3e**.

So, using the fact that $\mathbb{P}_n \circ \big(X^n(\cdot), \delta_x X^n(\cdot), R^{X^n}(\cdot), v^{n,\prime}(\cdot)\big)^{-1}$ has, for each $n \geqslant 0$, the same law as $\widehat{\mathbb{P}} \circ \big(\widehat{X}^n(\cdot), \delta_x \widehat{X}^n(\cdot), \widehat{R}^{\widehat{X}^n}(\cdot), \widehat{v}^{n,\prime}(\cdot)\big)^{-1}$, we can assume that the sequence $\big(\|\!| \widehat{X}^n(\cdot) \|\!|_{[0,T],\widehat{w}^{n,\prime},p'}\big)_{n \geqslant 0}$ is almost surely convergent, where $\widehat{w}^{n,\prime}(s,t,\widehat{\omega}) = \widehat{v}^{n,\prime}(s,t,\widehat{\omega}) + C(t-s)$.

Moreover, by identity in law of $\big(W^n(\cdot), X^n(\cdot)\big)$ under $\mathbb{P}_n$ and of $\big(\widehat{W}^n(\cdot), \widehat{X}^n(\cdot)\big)$ under $\widehat{\mathbb{P}}$, we have, for $\widehat{\mathbb{P}}$-almost every $\widehat{\omega} \in \widehat{\Omega}$, for any $(s,t) \in \mathcal{S}_2^T$,

$$|\widehat{X}^n_{s,t}(\widehat{\omega})| \leqslant \|\!| \widehat{X}^n(\widehat{\omega}) \|\!|_{[0,T],\widehat{w}^{n,\prime},p'} \big(\widehat{w}^{n,\prime}(s,t,\widehat{\omega})\big)^{1/p'}.$$

By **3c**, we get, for $\widehat{\mathbb{P}}$-almost every $\widehat{\omega} \in \widehat{\Omega}$, for all $(s,t) \in \mathcal{S}_2^T$,

$$|\widehat{X}_{s,t}(\widehat{\omega})| \leqslant \Big(\lim_{n \to \infty} \|\!| \widehat{X}^n(\widehat{\omega}) \|\!|_{[0,T],\widehat{w}^{n,\prime},p'}\Big) \big(\widehat{w}'(s,t,\widehat{\omega})\big)^{1/p'},$$

Proceeding similarly for $\delta_x \widehat{X}^n(\cdot)$ and $R^{\widehat{X}^n}(\cdot)$, we deduce that, for $\widehat{\mathbb{P}}$-almost every $\widehat{\omega} \in \widehat{\Omega}$,

$$\|\!| \widehat{X}(\widehat{\omega}) \|\!|_{[0,T],\widehat{w}',p'} \leqslant \lim_{n \to \infty} \|\!| X^n(\widehat{\omega}) \|\!|_{[0,T],\widehat{w}^{n,\prime},p'},$$

which shows in particular by Fatou's lemma, see step **2b**, that

$$\Big\langle \|\!| \widehat{X}(\cdot) \|\!|_{[0,T],\widehat{w}_{p'},p'} \Big\rangle_8 < \infty.$$

Although $\widehat{v}'(\widehat{\omega})$ (and thus $\widehat{w}'(\widehat{\omega})$) is not associated with $\widehat{\boldsymbol{W}}(\widehat{\omega})$ through (1.7), we shall say that, for almost every $\widehat{\omega} \in \widehat{\Omega}$, $\widehat{X}(\widehat{\omega})$ is an $\widehat{\omega}$-controlled trajectory for the rough set-up $\boldsymbol{W}(\cdot)$.

Step 5.

5a. So far, we have constructed $\big(\widehat{X}(\widehat{\omega}); \mathrm{F}\big(\widehat{X}(\widehat{\omega}), \widehat{X}(\cdot)\big); 0\big)$ as an $\widehat{\omega}$-controlled trajectory for the limit rough set-up $\boldsymbol{W}(\cdot)$, but for $\widehat{\omega}$ in a full event $\widehat{\Omega}' \subset \widehat{\Omega}$. For free, we can modify the definition of $\widehat{X}(\widehat{\omega})$ for $\widehat{\omega} \in \widehat{\Omega} \backslash \widehat{\Omega}'$ and define $\delta_x \widehat{X}(\widehat{\omega})$ accordingly so that $\big(\widehat{X}(\widehat{\omega}); \delta_x \widehat{X}(\widehat{\omega}); 0\big)$ is an $\widehat{\omega}$-controlled trajectory for any $\widehat{\omega}$. Then, the collection $\big(\widehat{X}(\widehat{\omega})\big)_{\widehat{\omega} \in \widehat{\Omega}}$ forms a random controlled trajectory.

5b. In order to conclude, it remains to identify $\big(\widehat{X}(\widehat{\omega}); \mathrm{F}\big(\widehat{X}(\widehat{\omega}), \widehat{X}(\cdot)\big); 0\big)$, for $\widehat{\mathbb{P}}$ almost every $\widehat{\omega} \in \widehat{\Omega}$, with

$$\Gamma_{\widehat{\boldsymbol{W}}}\Big(\widehat{X}(\widehat{\omega}); \mathrm{F}\big(\widehat{X}(\widehat{\omega}), \widehat{X}(\cdot)\big); 0\Big),$$

where the index $\widehat{\boldsymbol{W}}$ in $\Gamma_{\widehat{\boldsymbol{W}}}$ is to emphasize the rough set-up upon which the map Γ in Definition 12 is constructed. To do so, we recall from (2.3) the expansion

$$\begin{aligned}
&X^n_{t_i}(\omega_n) \\
&= X^n_0(\omega_n) + \sum_{j=1}^{i} \mathrm{F}\big(X^n_{t_{j-1}}(\omega_n), \mathcal{L}(X^n_{t_{j-1}})\big) W^n_{t_{j-1},t_j}(\omega_n) \\
&+ \sum_{j=1}^{i} \partial_x \mathrm{F}\big(X^n_{t_{j-1}}(\omega_n), \mathcal{L}(X^n_{t_{j-1}})\big) \Big(\mathrm{F}\big(X^n_{t_{j-1}}(\omega_n), \mathcal{L}(X^n_{t_{j-1}})\big) \mathbb{W}^n_{t_{j-1},t_j}(\omega_n)\Big) \\
&+ \sum_{j=1}^{i} \Big\langle D_\mu \mathrm{F}\big(X^n_{t_{j-1}}(\omega_n), \mathcal{L}(X^n_{t_{j-1}})\big)\big(X^n_{t_{j-1}}(\cdot)\big) \Big(\mathrm{F}\big(X^n_{t_{j-1}}(\cdot), \mathcal{L}(X^n_{t_{j-1}})\big) \mathbb{W}^{n,\perp\!\!\!\perp}_{t_{j-1},t_j}(\cdot, \omega_n)\Big)\Big\rangle \\
&+ \sum_{j=1}^{i} S^n_{t_{j-1},t_j}(\omega_n),
\end{aligned} \tag{3.29}$$

that holds true for any $\omega_n \in \Omega_n$, any $n \geqslant 0$ and any subdivision $0 = t_0 < t_1 < \cdots < t_K = T$, with $K \geqslant 1$, and with (see Theorem 10, Proposition 11 and **2b**)

$$\big|S^n_{t_{j-1},t_j}(\omega_n)\big| \leqslant C\Big(1 + \|\!|X^n(\omega_n)\|\!|^2_{[0,T],w^{n,\prime},p}\Big)\, w^{n,\prime}(t_{j-1}, t_j, \omega_n)^{3/p'}.$$

In order to pass to the limit in (3.29), we consider a non-negative valued bounded continuous function g on $\mathcal{C}([0,T];\mathbb{R}^m) \times \mathcal{C}(\mathcal{S}^T_2;\mathbb{R}^m \otimes \mathbb{R}^m) \times \mathcal{C}(\mathcal{S}^T_2;\mathbb{R}) \times \mathcal{C}([0,T];\mathbb{R}^d)$. We then multiply both sides of (3.29) by $g\big(W^n(\omega_n), \mathbb{W}^n(\omega_n), v^{n,\prime}(\omega_n), X^n(\omega_n)\big)$ and integrate ω_n with respect to $\mathbb{P}_n$. It is absolutely obvious that

$$\lim_{n\to\infty} \mathbb{E}_n\Big[g\big(W^n(\cdot), \mathbb{W}^n(\cdot), v^{n,\prime}(\cdot), X^n(\cdot)\big) X^n_{t_i}(\cdot)\Big] = \widehat{\mathbb{E}}\Big[g\big(\widehat{W}(\cdot), \widehat{\mathbb{W}}(\cdot), \widehat{v}^{n,\prime}(\cdot), \widehat{X}(\cdot)\big) \widehat{X}_{t_i}(\cdot)\Big],$$

and similarly with t_i replaced by 0. In the same way,

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{E}_n\Big[g\big(W^n(\cdot), \mathbb{W}^n(\cdot), v^{n,\prime}(\cdot), X^n(\cdot)\big) \mathrm{F}\big(X^n_{t_{j-1}}(\cdot), \mathcal{L}(X^n_{t_{j-1}})\big) W^n_{t_{j-1},t_j}(\cdot)\Big] \\
&= \widehat{\mathbb{E}}\Big[g\big(\widehat{W}(\cdot), \widehat{\mathbb{W}}(\cdot), \widehat{v}'(\cdot), \widehat{X}(\cdot)\big) \mathrm{F}\big(\widehat{X}_{t_{j-1}}(\cdot), \mathcal{L}(\widehat{X}_{t_{j-1}})\big) \widehat{W}_{t_{j-1},t_j}(\cdot)\Big],
\end{aligned}$$

and similarly for the terms on the second line. As for the fifth term in the right-hand side, we have

$$\begin{aligned}
&\limsup_{n\to\infty} \mathbb{E}_n\Big[g\big(W^n(\cdot), \mathbb{W}^n(\cdot), v^{n,\prime}(\cdot), X^n(\cdot)\big) S^n_{t_{j-1},t_j}(\cdot)\Big] \\
&\leqslant C \limsup_{n\to\infty} \mathbb{E}_n\Big[g\big(W^n(\cdot), \mathbb{W}^n(\cdot), v^{n,\prime}(\cdot), X^n(\cdot)\big) \Big(1 + \|\!|X^n(\cdot)\|\!|^2_{[0,T],w^{n,\prime},p}\Big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \times w^{n,\prime}(t_{j-1}, t_j, \cdot)^{3/p'}\Big].
\end{aligned}$$

Transferring the right-hand side into an expectation on $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathbb{P}})$ and using obvious uniform integrability properties, see **2b**, we deduce from **4b** that

$$\begin{aligned}
&\limsup_{n\to\infty} \mathbb{E}_n\Big[g\big(W^n(\cdot), \mathbb{W}^n(\cdot), v^{n,\prime}(\cdot), X^n(\cdot)\big) |S^n_{t_{j-1},t_j}(\cdot)|\Big] \\
&\leqslant C\widehat{\mathbb{E}}\Big[g\big(\widehat{W}(\cdot), \widehat{\mathbb{W}}(\cdot), \widehat{v}'(\cdot), \widehat{X}(\cdot)\big) \Big(1 + \lim_{n\to\infty} \|\!|X^n(\cdot)\|\!|^2_{[0,T],w^{n,\prime},p}\Big)\, w'(t_{j-1}, t_j, \cdot)^{3/p'}\Big].
\end{aligned}$$

Of course, the most difficult term to treat in (3.29) is the fourth one in the right-hand side. This can be done by using Fubini's theorem:

$$\begin{aligned}
&\int_{\Omega_n} d\mathbb{P}_n(\omega_n)\Big[g\big(W^n(\omega_n),\mathbb{W}^n(\omega_n),v^{n,\prime}(\omega_n),X^n(\omega_n)\big)\\
&\cdot\Big\langle D_\mu \mathrm{F}\big(X^n_{t_{j-1}}(\omega_n),\mathcal{L}(X^n_{t_{j-1}})\big)\big(X^n_{t_{j-1}}(\cdot)\big)\Big(\mathrm{F}\big(X^n_{t_{j-1}}(\cdot),\mathcal{L}(X^n_{t_{j-1}})\big)\mathbb{W}^{n,\perp\!\!\!\perp}_{t_{j-1},t_j}(\cdot,\omega_n)\Big)\Big\rangle\Big]\\
&=\int_{\Omega_n^2} d\mathbb{P}_n^{\otimes 2}(\omega_n,\omega_n')\Big[g\big(W^n(\omega_n),\mathbb{W}^n(\omega_n),v^{n,\prime}(\omega_n),X^n(\omega_n)\big)\\
&\cdot D_\mu \mathrm{F}\big(X^n_{t_{j-1}}(\omega_n),\mathcal{L}(X^n_{t_{j-1}})\big)\big(X^n_{t_{j-1}}(\omega_n')\big)\Big(\mathrm{F}\big(X^n_{t_{j-1}}(\omega_n'),\mathcal{L}(X^n_{t_{j-1}})\big)\mathbb{W}^{n,\perp\!\!\!\perp}_{t_{j-1},t_j}(\omega_n',\omega_n)\Big)\Big]\\
&=\widehat{\mathbb{E}}\Big[g\big(\widehat{W}^{n,1}(\cdot),\widehat{\mathbb{W}}^{n,1}(\cdot),\widehat{v}^{1,n,\prime}(\cdot),\widehat{X}^{n,1}(\cdot)\big)\\
&\cdot D_\mu \mathrm{F}\big(\widehat{X}^{n,1}_{t_{j-1}}(\cdot),\mathcal{L}(X^n_{t_{j-1}})\big)\big(\widehat{X}^{n,2}_{t_{j-1}}(\cdot)\big)\Big(\mathrm{F}\big(\widehat{X}^{n,2}_{t_{j-1}}(\cdot),\mathcal{L}(X^n_{t_{j-1}})\big)\widehat{\mathbb{W}}^{n,2,1}_{t_{j-1},t_j}(\cdot)\Big)\Big].
\end{aligned}$$

We now use (3.25) in order to pass to the limit. The only slight difficult is that we must ensure that the regularity conditions satisfied by $D_\mu\mathrm{F}$ are compatible with the almost sure convergence property (3.25). Recall indeed that the continuity property **Regularity assumptions 1** is formulated in $\mathbb{L}_2$; at first sight, it seems needed to assume that the pair $(\widehat{X}^{n,2}_{t_{j-1}}(\cdot),\widehat{X}^{2}_{t_{j-1}}(\cdot))$ is independent of $(\widehat{X}^{n,1}_{t_{j-1}}(\cdot),\widehat{X}^{1}_{t_{j-1}}(\cdot))$ in order to take full advantage of it. In fact, we can overcome this difficulty by invoking [9, Proposition 5.36], which basically asserts that the mapping $v\mapsto D_\mu\mathrm{F}(x,\mu)(v)$ is Lipschitz continuous, uniformly in x and μ, see Section 5.3.4 for more details. The latter guarantees that, for almost every $\widehat{\omega}\in\widehat{\Omega}$,

$$\lim_{n\to\infty} D_\mu\mathrm{F}\big(\widehat{X}^{n,1}_{t_{j-1}}(\widehat{\omega}),\mathcal{L}(X^n_{t_{j-1}})\big)\big(\widehat{X}^{n,2}_{t_{j-1}}(\widehat{\omega})\big)=D_\mu\mathrm{F}\big(\widehat{X}^{1}_{t_{j-1}}(\widehat{\omega}),\mathcal{L}(\widehat{X}_{t_{j-1}})\big)\big(\widehat{X}^{2}_{t_{j-1}}(\widehat{\omega})\big).$$

So, the limit of the summand on the fourth line of (3.29) is

$$\begin{aligned}
&\widehat{\mathbb{E}}\Big[g\big(\widehat{W}^{1}(\cdot),\widehat{\mathbb{W}}^{1}(\cdot),\widehat{v}^{1,\prime}(\cdot),\widehat{X}^{1}(\cdot)\big)\\
&\quad\cdot D_\mu \mathrm{F}\big(\widehat{X}^{1}_{t_{j-1}}(\cdot),\mathcal{L}(\widehat{X}^1_{t_{j-1}})\big)\big(\widehat{X}^{2}_{t_{j-1}}(\cdot)\big)\Big(\mathrm{F}\big(\widehat{X}^{2}_{t_{j-1}}(\cdot),\mathcal{L}(\widehat{X}^1_{t_{j-1}})\big)\widehat{\mathbb{W}}^{2,1}_{t_{j-1},t_j}(\cdot)\Big)\Big],
\end{aligned}$$

and our reconstruction of the limiting set-up permits to rewrite it in the form

$$\begin{aligned}
&\int_{\widehat{\Omega}} d\widehat{\mathbb{P}}(\widehat{\omega})\Big[g\big(\widehat{W}(\widehat{\omega}),\widehat{\mathbb{W}}(\widehat{\omega}),\widehat{v}'(\widehat{\omega}),\widehat{X}(\widehat{\omega})\big)\\
&\cdot\Big\langle D_\mu \mathrm{F}\big(\widehat{X}_{t_{j-1}}(\widehat{\omega}),\mathcal{L}(\widehat{X}_{t_{j-1}})\big)\big(\widehat{X}_{t_{j-1}}(\cdot)\big)\Big(\mathrm{F}\big(\widehat{X}_{t_{j-1}}(\cdot),\mathcal{L}(\widehat{X}_{t_{j-1}})\big)\widehat{\mathbb{W}}^{\perp\!\!\!\perp}_{t_{j-1},t_j}(\cdot,\omega)\Big)\Big\rangle\Big].
\end{aligned}$$

Importantly, since the limiting set-up is strong, the term in bracket in the last line is $\sigma\{\widehat{W},\widehat{X}\}$-measurable.

5c. Let now

$$\begin{aligned}
\mathcal{J}(\widehat{\omega}) &:= \widehat{X}_{t_i}(\widehat{\omega})-\widehat{X}_0(\widehat{\omega})\\
&-\sum_{j=1}^{i}\mathrm{F}\big(\widehat{X}_{t_{j-1}}(\widehat{\omega}),\mathcal{L}(\widehat{X}_{t_{j-1}})\big)\widehat{W}_{t_{j-1},t_j}(\widehat{\omega})\\
&-\sum_{j=1}^{i}\partial_x\mathrm{F}\big(\widehat{X}_{t_{j-1}}(\widehat{\omega}),\mathcal{L}(\widehat{X}_{t_{j-1}})\big)\Big(\mathrm{F}\big(\widehat{X}_{t_{j-1}}(\widehat{\omega}),\mathcal{L}(\widehat{X}_{t_{j-1}})\big)\widehat{\mathbb{W}}_{t_{j-1},t_j}(\widehat{\omega})\Big)\\
&-\sum_{j=1}^{i}\Big\langle D_\mu \mathrm{F}\big(\widehat{X}_{t_{j-1}}(\widehat{\omega}),\mathcal{L}(\widehat{X}_{t_{j-1}})\big)\big(\widehat{X}_{t_{j-1}}(\cdot)\big)\Big(\mathrm{F}\big(\widehat{X}_{t_{j-1}}(\cdot),\mathcal{L}(\widehat{X}_{t_{j-1}})\big)\widehat{\mathbb{W}}^{\perp\!\!\!\perp}_{t_{j-1},t_j}(\cdot,\widehat{\omega})\Big)\Big\rangle.
\end{aligned}$$

By the conclusion of **5b**, it is $\sigma\{\widehat{W}, \widehat{\mathbb{W}}, \widehat{X}\}$-measurable and it satisfies, for any g as in the previous step,

$$\widehat{\mathbb{E}}\big[g\big(\widehat{W}(\cdot), \widehat{\mathbb{W}}(\cdot), \widehat{v}'(\cdot), \widehat{X}(\cdot)\big)\mathcal{J}(\cdot)\big]$$

$$\leqslant \widehat{\mathbb{E}}\Big[g\big(\widehat{W}(\cdot), \widehat{\mathbb{W}}(\cdot), \widehat{v}'(\cdot), \widehat{X}(\cdot)\big)\Big(1 + \lim_{n\to\infty} \|\!|X^n(\cdot)|\!\|^2_{[0,T],w^{n,\prime},p}\Big) \sum_{j=1}^{i} \widehat{w}'(t_{j-1}, t_j, \cdot)^{3/p'}\Big].$$

Therefore, for $\widehat{\mathbb{P}}$-almost every $\widehat{\omega}$,

$$\mathcal{J}(\widehat{\omega}) \leqslant C\Big(\sum_{j=1}^{i} \widehat{w}'(t_{j-1}, t_j)^{3/p'}\Big)\widehat{\mathbb{E}}\Big[\lim_{n\to\infty} \|\!|X^n(\cdot)|\!\|^2_{[0,T],w^{n,\prime},p} \,|\, \sigma\{\widehat{W}, \widehat{\mathbb{W}}, \widehat{v}', \widehat{X}\}\Big].$$

By the super-additivity property of $\widehat{w}'$, this suffices to identify $\widehat{X}_t(\widehat{\omega})$ with $\widehat{X}_0(\widehat{\omega}) + \int_0^t \mathrm{F}(\widehat{X}_s(\omega), \widehat{X}_s(\cdot))d\widehat{\boldsymbol{W}}_s(\omega)$. Note that this is true although the functionals $\widehat{v}'(\widehat{\omega})$ and $\widehat{w}'(\widehat{\omega})$ that control the variations of $\widehat{X}$ are not associated with $\widehat{\boldsymbol{W}}(\widehat{\omega})$ through (1.7); the sole fact that $\widehat{v}'(\widehat{\omega})$ dominates $\overline{v}'(\widehat{\omega})$ (which is associated with $\widehat{\boldsymbol{W}}(\widehat{\omega})$ through (1.7)) suffices.

Again, the sole domination of $\overline{v}'(\widehat{\omega})$ by $\widehat{v}'(\widehat{\omega})$, the latter satisfying the required tail properties in Theorem 16, suffices to duplicate the uniqueness argument. In words, $\widehat{X}(\cdot)$ is the solution to the mean field rough equation driven by $\widehat{\boldsymbol{W}}$ and, by uniqueness in law, $\widehat{X}(\cdot)$ has the same law as $X(\cdot)$. ▷

We used the following lemma in the proof of Theorem 20.

21. Lemma – *For a separable Banach space $(E, |\cdot|)$, call $\mathcal{C}_0^{p-\mathrm{var}}(\mathcal{S}_2^T; E)$ the space of continuous paths G from $\mathcal{S}_2^T$ into E that are null on the diagonal, i.e. $G_{t,t} = 0$ for all $t \in [0,T]$, and have a finite p-variation, i.e.*

$$\|G\|^p_{[0,T],p-\mathrm{var}} = \sup_{0\leqslant t_1<\cdots<t_N=T} \sum_{i=0}^{N-1} |G_{t_i,t_{i+1}}|^p < \infty.$$

For each $n \geqslant 0$, let $\big(Z^n = (Z^n_{s,t})_{s,t\in\mathcal{S}_2^T}\big)_{n\geqslant 0}$ be a process defined on $(\Omega_n, \mathcal{F}_n, \mathbb{P}_n)$ with trajectories in $\mathcal{C}^{p-\mathrm{var}}\big(\mathcal{S}_2^T; E\big)$. Assume that

- *the family of distributions $\big(\mathbb{P}_n \circ (Z^n)^{-1}\big)_{n\geqslant 0}$ is tight in $\mathcal{C}(\mathcal{S}_2^T; E)$;*
- *the family of distributions $\big(\mathbb{P} \circ (\|Z^n\|_{[0,T],p-\mathrm{var}})^{-1}\big)_{n\geqslant 0}$ is tight in $\mathbb{R}$.*

Then, for $p' > p$

- *the family of distributions $\big(\mathbb{P} \circ (\mathcal{S}_2^T \ni (s,t) \mapsto \|Z^n\|_{[s,t],p'-\mathrm{var}} \in \mathbb{R})^{-1}\big)_{n\geqslant 0}$ is tight in $\mathcal{C}(\mathcal{S}_2^T; \mathbb{R})$. In particular, for any $\varepsilon > 0$, there exists $\delta > 0$, such that*

$$\mathbb{P}\left(\sup_{(s,t)\in\mathcal{S}_2^T: t-s\leqslant\delta} \|Z^n\|_{[s,t],p'-\mathrm{var}} > \varepsilon\right) < \varepsilon;$$

Proof – Take a compact subset $\mathcal{K}$ of $\mathcal{C}(\mathcal{S}_2^T; E)$ and a sequence $(z^n)_{n\geqslant 0}$ with values in $\mathcal{K}$ such that

$$\sup_{n\geqslant 1} \|z^n\|_{[0,T],p-\mathrm{var}} < \infty.$$

Up to a subsequence, the sequence $(z^n)_{n\geqslant 0}$ converges in $\mathcal{C}(\mathcal{S}_2^T; E)$. Obviously, the limit z is in $\mathcal{C}_0^{p-\mathrm{var}}(\mathcal{S}_2^T; E)$. Now, by the same argument as in the proof of Proposition

5.5 in [27], we have

$$\sum_{i=0}^{N-1} |(z^n - z)_{t_i,t_{i+1}}|^{p'} \leqslant \sup_{(s,t)\in\mathcal{S}_2^T} |(z^n - z)_{s,t}|^{p'-p} \sum_{i=0}^{N-1} |(z^n - z)_{t_i,t_{i+1}}|^{p},$$

for any subdivision $0 = t_0 < \cdots < t_N = T$. Taking the supremum over such subdivisions, we deduce that $(z^n)_{n\geqslant 0}$ converges to z in $\mathcal{C}_0^{p'-\mathrm{var}}(\mathcal{S}_2^T; E)$, which proves that

$$\begin{aligned}\limsup_{n\to\infty} \sup_{(s,t)\in\mathcal{S}_2^T} \Big| \|z^n\|_{[s,t],p'-\mathrm{var}} - \|z\|_{[s,t],p'-\mathrm{var}} \Big| &\leqslant \limsup_{n\to\infty} \sup_{(s,t)\in\mathcal{S}_2^T} \|z^n - z\|_{[s,t],p'-\mathrm{var}} \\ &\leqslant \lim_{n\to\infty} \|z^n - z\|_{[0,T],p'-\mathrm{var}} = 0.\end{aligned}$$

Hence the family $(\mathcal{S}_2^T \ni (s,t) \mapsto \|z\|_{[s,t],p'-\mathrm{var}})_{z\in\mathcal{K}}$ is relatively compact for the uniform topology. In particular, it is equicontinuous. Using the fact that $\|z\|_{[t,t],p'-\mathrm{var}} = 0$ for each $t \in [0,T]$, we deduce that

$$\lim_{\delta\searrow 0} \sup_{|t-s|\leqslant\delta} \|z\|_{[s,t],p'-\mathrm{var}} = \lim_{\delta\searrow 0} \sup_{|t-s|\leqslant\delta} \Big| \|z\|_{[s,t],p'-\mathrm{var}} - \|z\|_{[t,t],p'-\mathrm{var}} \Big| = 0.$$

This proof is easily completed. ▷

4 – Particle System and Propagation of Chaos

We now have all the ingredients to write down our limiting mean field rough differential equation as the limit of a system of particles driven by rough signals.

4.1 – Empirical Rough Set-Up

Loosely speaking, the finite particle system associated with (0.1) has the form

$$X_t^i(\omega) = X_0^i(\omega) + \int_0^t \mathrm{F}\big(X_s^i(\omega), \mu_s^n(\omega)\big) dW_s^i(\omega), \quad t \geqslant 0, \tag{4.1}$$

for $1 \leqslant i \leqslant n$, where $(X_0^i(\cdot))_{1\leqslant i\leqslant n}$ is a collection of $\mathbb{R}^d$-valued independent and identically distributed variables with the same distribution as X_0 in the statement of Theorem 16 and $\big(W_0^i(\cdot)\big)_{1\leqslant i\leqslant n}$ is a collection of $\mathbb{R}^m$-valued independent and identically distributed processes with the same distribution on the space of continuous functions as $W(\cdot)$ in Theorem 16. All of them are constructed on a single probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Obviously, equation (4.1) must be understood as a rough differential equation driven by an $(n \times m)$-dimensional signal $\big(W^1(\omega), \cdots, W^n(\omega)\big)$, and with $\big(X^1(\omega), \cdots, X^n(\omega)\big)$ as $(n \times d)$-dimensional output. This requires that we lift $\big(W^1(\omega), \cdots, W^n(\omega)\big)$ into an enhanced rough set-up $\boldsymbol{W}^{(n)}(\omega)$. In order to do so, it suffices to define the various iterated integrals. Without any loss of generality, we can assume that, instead of $\big(W^1(\cdot), \cdots, W^n(\cdot)\big)$, we have in fact n independent copies $\big(W^i(\cdot), \mathbb{W}^i(\cdot)\big)_{1\leqslant i\leqslant n}$ of the pair $\big(W(\cdot), \mathbb{W}(\cdot)\big)$, where $\mathbb{W}(\omega)$ is the iterated integral of $W(\omega)$, see Section 1 for details; and, in fact we assume that $\big(X_0^i(\cdot), W^i(\cdot), \mathbb{W}^i(\cdot)\big)_{1\leqslant i\leqslant n}$ are n independent copies of $\big(X_0(\cdot), W(\cdot), \mathbb{W}(\cdot)\big)$. For sure, $\mathbb{W}^i(\omega)$ is understood as the iterated integral of $W^i(\omega)$. However, this does not suffice as we also need to define the iterated integrals of $W^j(\omega)$ with respect to $W^i(\omega)$, for $j \neq i$. We do so under the additional assumption that $\boldsymbol{W}$ is a strong set-up, namely under the assumption that there is a measurable map giving $\mathbb{W}^{i,j}(\omega)$ from $W^i(\omega)$ and $W^j(\omega)$,

$$\mathbb{W}^{i,j}(\omega) = \mathcal{I}\big(W^i(\omega), W^j(\omega)\big), \quad i \neq j,$$

see Definition 18. If we require $\mathbb{P}^{\otimes 2}\big(\{(\omega,\omega') : \|\mathbb{W}^{\perp\!\!\!\perp}(\omega,\omega')\|_{[0,T],p/2-\mathrm{var}} < \infty\}\big)$ in Definition 18, then it is pretty clear that, for almost every $\omega \in \Omega$,

$$\begin{aligned} \boldsymbol{W}^{(n)}(\omega) &= \Big(\big(W^i(\omega)\big)_{1\leqslant i\leqslant n}, \big(\mathbb{W}^{i,j}(\omega)\big)_{1\leqslant i,j\leqslant n}\Big) \\ &=: \Big(W^{(n)}(\omega), \mathbb{W}^{(n)}(\omega)\Big), \end{aligned}$$

is a rough path, with the convention that $\mathbb{W}^{i,i}(\omega) = \mathbb{W}^i(\omega)$, for $i \in \{1,\cdots,n\}$. As explained in Proposition 4, we may change the definition of the whole collection $\big((W^i(\omega))_{1\leqslant i\leqslant n}, (\mathbb{W}^{i,j}(\omega))_{1\leqslant i,j\leqslant n}\big)$ on a $\mathbb{P}$-null set so that $\boldsymbol{W}^{(n)}$ is in fact a rough path for any $\omega \in \Omega$.

• The striking fact of the analysis was already noticed by Cass and Lyons in their seminal work [12]. The quantity $\mathbb{W}^{(n)}(\omega)$ may be seen as a rough set-up defined on a finite probability space for any fixed $\omega \in \Omega$; we call it the **empirical rough set-up**. To make it clear, observe that, throughout Section 1, the rough structure is supported by the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ itself. Here, ω is fixed, and we see the probability space as

$$\left(\{1,\cdots,n\}, \mathcal{P}\big(\{1,\cdots,n\}\big), \frac{1}{n}\sum_{i=1}^n \delta_i\right),$$

where $\mathcal{P}(\{1,\cdots,n\})$ denotes the collection of subsets of $\{1,\cdots,n\}$. The reader may object that such a probability space is not atomless whilst we explicitly assumed $(\Omega,\mathcal{F},\mathbb{P})$ to be atomless in the introduction; actually, the reader must realize that, in the paper, the atomless property is just used to guarantee that, for any probability measure μ on a given Polish space S, the probability space $(\Omega,\mathcal{F},\mathbb{P})$ carries an S-valued random variable with μ as distribution. So, it is not a hindrance that $\{1,\cdots,n\}$ is finite. Hence, in comparison, with (1.3), the role played by $\omega \in \Omega$ is played by $i \in \{1,\cdots,n\}$ and the matrix (1.3) must read

$$\begin{pmatrix} \mathbb{W}^{i,i}_{s,t}(\omega) & \mathbb{W}^{i,\bullet}_{s,t}(\omega) \\ \mathbb{W}^{\bullet,i}_{s,t}(\omega) & \mathbb{W}^{\bullet,\bullet}_{s,t}(\omega) \end{pmatrix}_{0\leqslant s\leqslant t\leqslant T}, \tag{4.2}$$

where $\mathbb{W}^{i,\bullet}_{s,t}(\omega)$ is understood as $\{1,\cdots,n\} \ni j \mapsto \mathbb{W}^{i,j}_{s,t}(\omega)$, $\mathbb{W}^{\bullet,i}_{s,t}(\omega)$ as $\{1,\cdots,n\} \ni j \mapsto \mathbb{W}^{j,i}_{s,t}(\omega)$ and $\mathbb{W}^{\bullet,\bullet}_{s,t}(\omega)$ as $\{1,\cdots,n\} \ni (i,j) \mapsto \mathbb{W}^{i,j}_{s,t}(\omega)$.

In the same spirit, the variation function v in (1.7) is

$$\begin{aligned} v^{i,n}(s,t,\omega) := {} & \big\|W^i(\omega)\big\|^p_{[s,t],p-\mathrm{var}} + {}^{(n)}\big[W^\bullet(\omega)\big]^p_{q;[s,t],p-\mathrm{var}} \\ & + \big\|\mathbb{W}^i(\omega)\big\|^{p/2}_{[s,t],p/2-\mathrm{var}} + {}^{(n)}\big[\mathbb{W}^{i,\bullet}(\omega)\big]^{p/2}_{q;[s,t],p/2-\mathrm{var}} \\ & + {}^{(n)}\big[\mathbb{W}^{\bullet,i}(\omega)\big]^{p/2}_{q;[s,t],p/2-\mathrm{var}} + {}^{(n)}\big[\!\big[\mathbb{W}^{\bullet,\bullet}(\omega)\big]\!\big]^{p/2}_{q;[s,t],p/2-\mathrm{var}}, \end{aligned} \tag{4.3}$$

where we used the notations

$${}^{(n)}\big[X^\bullet\big]_q = \left(\frac{1}{n}\sum_{j=1}^n |X^j|^q\right)^{1/q}, \quad {}^{(n)}\big[\!\big[X^{\bullet,\bullet}\big]\!\big]_q = \left(\frac{1}{n^2}\sum_{j,k=1}^n |X^{j,k}|^q\right)^{1/q},$$

the corresponding p-variation being defined as in (1.5) and (1.6). In order to check that $\boldsymbol{W}^{(n)}(\omega)$ defines a rough set-up, it remains to check that it satisfies (1.8).

To do so, we now let

$$\begin{aligned}
\left\|W^i(\omega)\right\|_{[s,t],(1/p)-\text{Hölder}} &= \sup_{[s',t']\subset[s,t]} \frac{|W^i_{t'}(\omega) - W^i_{s'}(\omega)|}{|t'-s'|^{1/p}} \\
\left\|\mathbb{W}^i(\omega)\right\|_{[s,t],(2/p)-\text{Hölder}} &= \sup_{[s',t']\subset[s,t]} \frac{|\mathbb{W}^i_{s',t'}(\omega)|}{|t'-s'|^{2/p}}, \\
\left\|\mathbb{W}^{i,j}(\omega)\right\|_{[s,t],(2/p)-\text{Hölder}} &= \sup_{[s',t']\subset[s,t]} \frac{|\mathbb{W}^{i,j}_{s',t'}(\omega)|}{|t'-s'|^{2/p}},
\end{aligned}$$

stand for the standard Hölder semi-norms of the rough path, see e.g. Theorem 11.9 in [25]. Then, we can find a universal positive constant c such that

$$\begin{aligned}
v_p^{i,n}(s,t,\omega) \leqslant c\Big\{&\left\|W^i(\omega)\right\|^p_{[s,t],(1/p)-\text{Hölder}} + \left\|\mathbb{W}^i(\omega)\right\|^{p/2}_{[s,t],(2/p)-\text{Hölder}} \\
&+ {}^{(n)}\!\big(\left\|W^\bullet(\omega)\right\|^p_{[s,t],(1/p)-\text{Hölder}}\big)_q \\
&+ {}^{(n)}\!\big(\left\|\mathbb{W}^{i,\bullet}(\omega)\right\|^{p/2}_{[s,t],(2/p)-\text{Hölder}}\big)_q \\
&+ {}^{(n)}\!\big(\left\|\mathbb{W}^{\bullet,i}(\omega)\right\|^{p/2}_{[s,t],(2/p)-\text{Hölder}}\big)_q \\
&+ {}^{(n)}\!\big(\!\big(\left\|\mathbb{W}^{\bullet,\bullet}(\omega)\right\|^{p/2}_{[s,t],(2/p)-\text{Hölder}}\big)\!\big)_q\Big\}(t-s).
\end{aligned} \tag{4.4}$$

Taking the empirical mean over $i \in \{1,\cdots,n\}$ and invoking the law of large numbers, we deduce that, for almost every $\omega \in \Omega$,

$$\begin{aligned}
&\limsup_{n\geqslant 1} \sup_{0\leqslant s<t\leqslant T} \frac{{}^{(n)}\!\big(v_p^{\bullet,n}(s,t,\omega)\big)_q}{t-s} \\
&\quad\leqslant c\,\mathbb{E}\Big[\left\|W(\cdot)\right\|^{pq}_{[0,T],(1/p)-\text{Hölder}} + \left\|\mathbb{W}(\cdot)\right\|^{pq/2}_{[0,T],(2/p)-\text{Hölder}} \\
&\qquad\qquad + \left\|\mathbb{W}^{\perp\!\!\!\perp}(\cdot,\cdot)\right\|^{pq/2}_{[0,T],(2/p)-\text{Hölder}}\Big]^{1/q},
\end{aligned} \tag{4.5}$$

for a new value of the constant c. Observe that, in order to derive (4.5), the law of large numbers can be directly applied to each of the first five terms in the right-hand side of (4.4), since each of them can be put in the form $\mathcal{J}\big(W^i(\omega)\big)$, for a suitable form of the functional $\mathcal{J}$. Differently, the last term in (4.4) requires a modicum of care as it reads

$$\frac{1}{n^2}\sum_{j,k=1}^n \mathcal{J}\big(W^j(\omega), W^k(\omega)\big). \tag{4.6}$$

Still, we let the reader check that, provided that the summand in the above right-hand side is integrable, the limit is $\mathbb{E}[\mathcal{J}(W^j(\cdot), W^k(\cdot))]$. Hence (4.5). Now, if the right-hand side of (4.5) is finite, then

$$\sup_{n\geqslant 1}\sup_{0\leqslant s<t\leqslant T} \frac{{}^{(n)}\!\big(v_p^{\bullet,n}(s,t,\omega)\big)_q}{t-s} < \infty,$$

which guarantees that the 1-variation in the mean in (1.8) is uniformly controlled in $n \geqslant 1$, the mean therein being understood as the mean on the probability space $\big(\{1,\cdots,n\}, \mathcal{P}(\{1,\cdots,n\}), \frac{1}{n}\sum_{i=1}^n \delta_i\big)$. Here are two examples under which the above assumption holds true.

Example 1 – Assume that the index q used in (1.7) satisfies the inequality

$$q > \frac{1}{1 - p/3},$$

and that, for some constant $K_T \geqslant 0$, $\langle v(s,t,\cdot)\rangle_q \leqslant K_T(t-s)$ for $(s,t) \in \mathcal{S}_2^T$. Then, we get the bounds

$$\begin{aligned}
&\mathbb{E}\big[|(W_t - W_s)(\cdot)|^{pq}\big] \leqslant K_T\,|t-s|^q,\\
&\mathbb{E}\big[|\mathbb{W}_{s,t}(\cdot)|^{pq/2}\big] \leqslant K_T\,|t-s|^q,\\
&\mathbb{E}^{\otimes 2}\Big[|\mathbb{W}^{\perp\!\!\!\perp}_{s,t}(\cdot,\cdot)|^{pq/2}\Big] \leqslant K_T\,|t-s|^q.
\end{aligned}$$

(We write here and below $\mathbb{E}^{\otimes 2}$ for the expectation operator with respect to $\mathbb{P}^{\otimes 2}$.) By Kolmogorov's criterion for rough paths, Theorem 3.1 in [25], we deduce that W has paths that are $(1-1/q)/p > 1/3$-Hölder continuous. Similarly, $\mathbb{W}$ and $\mathbb{W}^{\perp\!\!\!\perp}$ have paths that are $2(1-1/q)/p > 2/3$-Hölder continuous and

$$\mathbb{E}^{\otimes 2}\Big[\|W(\cdot)\|^{pq}_{[0,T],(1/p')-\text{Hölder}} + \|\mathbb{W}(\cdot)\|^{pq/2}_{[0,T],(2/p')-\text{Hölder}} + \|\mathbb{W}^{\perp\!\!\!\perp}(\cdot,\cdot)\|^{pq/2}_{[0,T],(2/p')-\text{Hölder}}\Big] < \infty.$$

So, the empirical rough set-up satisfies the required conditions provided we replace p by p' and $\langle v(s,t,\cdot)\rangle_{qp'/p} \leqslant K_T(t-s)$, for all $(s,t) \in \mathcal{S}_2^T$.

Example 2 – Another instance is given by Example 5. With the same notation as therein, $\|W(\cdot)\|_{[0,T],(1/p)-\text{Hölder}}$ has Gaussian tails and $\|\mathbb{W}(\cdot)\|_{[0,T],(2/p)-\text{Hölder}}$ and $\|\mathbb{W}^{\perp\!\!\!\perp}(\cdot,\cdot)\|_{[0,T],(2/p)-\text{Hölder}}$ have exponential tails; see Theorem 11.9 in [25]. This suffices to conclude.

• Now that we have defined the empirical rough set-up, we must make clear the meaning given to the rough differential equation (0.2) in Definition 12 when the rough set-up therein is precisely the empirical rough set-up. We call the corresponding rough differential equation the *empirical rough differential equation*.

For a given $\omega \in \Omega$, the probability space that carries the empirical rough-set up is $\big(\{1,\cdots,n\},\mathcal{P}(\{1,\cdots,n\},\frac{1}{n}\sum_{i=1}^n \delta_i\big)$. Despite the fact it is not atomless, whilst $(\Omega,\mathcal{F},\mathbb{P})$ is, Theorem 16 applies and guarantees existence and uniqueness of a solution to the empirical rough differential equation must. In this regard, observe that the square integrability requirement on the initial condition takes the simple form

$$\frac{1}{n}\sum_{i=1}^n |X_0^i(\omega)|^2 < \infty,$$

which is obviously satisfied (at least for ω in a full event). The solution reads in the form of a n-tuple $X^{(n)}(\omega) = (X^i(\omega))_{1\leqslant i\leqslant n}$ in $\mathcal{C}([0,T];\mathbb{R}^d)^n$. Each $X^i(\omega)$ is controlled, in standard Gubinelli's sense, by the enhanced rough path $\big(W^i(\omega),\mathbb{W}^i(\omega)\big)$. The coefficient driving the equation for $X^i(\omega)$ reads

$$\mathrm{F}\left(X_t^i(\omega), X_t^{\theta_n(\cdot)}(\omega)\right), \quad t \in [0,T],$$

where $\theta_n(\cdot)$ is a uniformly distributed random variable on the probability space $\big(\{1,\cdots,n\},\mathcal{P}(\{1,\cdots,n\},\frac{1}{n}\sum_{i=1}^n \delta_i\big)$. Here the dot in the notation $X_t^{\theta_n(\cdot)}(\omega)$ refers to the current element in $\{1,\cdots,n\}$. As a result, the law of $X_t^{\theta_n(\cdot)}(\omega)$ must be understood as the empirical distribution $\mu_t^n(\omega)$.

The key fact in our analysis lies in the interpretation of the two Gubinelli derivatives $\delta_x[\mathrm{F}(X^i(\omega), X^{\theta_n(\cdot)}(\omega))]$ and $\delta_\mu[\mathrm{F}(X^i(\omega), X^{\theta_n(\cdot)}(\omega))]$ in Proposition 11. First, it is elementary to check that

$$\begin{aligned}\delta_x\Big(\mathrm{F}\big(X^i(\omega), X^{\theta_n(\cdot)}(\omega)\big)\Big)_t &= \partial_x\mathrm{F}\big(X^i_t(\omega), X^{\theta_n(\cdot)}_t(\omega)\big)\delta_x X^i_t(\omega) \\ &= \partial_x\mathrm{F}\big(X^i_t(\omega), \mu^n_t(\omega)\big)\delta_x X^i_t(\omega),\end{aligned} \tag{4.7}$$

where $\delta_x X^i(\omega)$ is the standard derivative of $X^i(\omega)$ with respect to $\big(W^i(\omega), \mathbb{W}^i(\omega)\big)$. More interestingly, we have

$$\delta_\mu\Big(\mathrm{F}\big(X^i(\omega), X^{\theta_n(\cdot)}(\omega)\big)\Big)_t = D_\mu\mathrm{F}\big(X^i_t(\omega), \mu^n_t(\omega)\big)\big(X^{\theta_n(\cdot)}_t(\omega)\big)\,\delta_x X^{\theta_n(\cdot)}_t(\omega), \tag{4.8}$$

and the right-hand side may identified with an n-tuple

$$\Big(D_\mu\mathrm{F}\big(X^i_t(\omega), \mu^n_t(\omega)\big)(X^j_t(\omega))\delta_x X^j_t(\omega)\Big)_{1\leqslant j\leqslant n}.$$

So, we get

$$\delta_\mu\Big(\mathrm{F}\big(X^i(\omega), X^{\theta_n(\cdot)}(\omega)\big)\Big)_t = \frac{1}{n}\sum_{j=1}^n D_\mu\mathrm{F}\big(X^j_t(\omega), \mu^n_t\big)\big(X^j_t(\omega)\big)\,\delta_x X^j_t(\omega).$$

This shows that the integral

$$\int_0^t \mathrm{F}\Big(X^i_s(\omega), X^{\theta_n(\cdot)}_s(\omega)\Big)d\boldsymbol{W}^{(n)}_s(\omega)$$

is the limit of the compensated Riemann sums

$$\begin{aligned}\sum_{k=0}^{K-1}\Bigg(&\mathrm{F}\big(X^i_{t_k}(\omega), X^{\theta_n(\cdot)}_{t_k}(\omega)\big)W^i_{t_k,t_{k+1}}(\omega) \\ &+ \partial_x\mathrm{F}\big(X^i_{t_k}(\omega), X^{\theta_n(\cdot)}_{t_k}(\omega)\big)\mathrm{F}\big(X^i_{t_k}(\omega), X^{\theta_n(\cdot)}_{t_k}(\omega)\big)\mathbb{W}^i_{t_k,t_{k+1}}(\omega) \\ &+ \frac{1}{n}\sum_{j=1}^n D_\mu\mathrm{F}\big(X^j_{t_k}(\omega), \mu^n_t(\omega)\big)(X^j_{t_k}(\omega))\mathrm{F}\big(X^j_{t_k}(\omega), X^{\theta_n(\cdot)}_{t_k}(\omega)\big)\mathbb{W}^{i,j}_{t_k,t_{k+1}}(\omega)\Bigg),\end{aligned} \tag{4.9}$$

as the mesh of the dissection $0 = t_0 < \cdots < t_K = t$ tends to 0. This allows to compare the latter quantity with (4.1) if we intepret the integral with respect to $W^i(\omega)$ as a rough integral with respect to $\boldsymbol{W}^{(n)}(\omega)$, and consider the leading coefficient $\mathrm{F}(X^i_t(\omega), \mu^n_t(\omega))$ as a *standard* Euclidean function of the tuple $X^{(n)}_t(\omega) = \big(X^1_t(\omega), \cdots, X^n_t(\omega)\big)$ and if we understand the integral therein as the integral with respect to the rough driver $\boldsymbol{W}^{(n)}(\omega)$. Indeed, under the standing **Regularity assumptions 1** and **2**, the function

$$f^i : (\mathbb{R}^d)^n \ni \big(x^1, \cdots, x^n\big) \mapsto \mathrm{F}\left(x^i, \frac{1}{n}\sum_{k=1}^n x^k\right)$$

is $\mathcal{C}^2$ with Lipschitz derivatives and

$$\partial_{x^j} f^i\big(x^1, \cdots, x^n\big) = \frac{1}{n}D_\mu\mathrm{F}\left(x^i, \frac{1}{n}\sum_{k=1}^n x^k\right)(x^j),$$

for $j \neq i$, and

$$\partial_{x^i} f^i\big(x^1, \cdots, x^n\big) = \partial_x\mathrm{F}\left(x^i, \frac{1}{n}\sum_{\ell=1}^n x^\ell\right) + \frac{1}{n}D_\mu\mathrm{F}\left(x^i, \frac{1}{n}\sum_{k=1}^n x^k\right)(x^i);$$

see Chapter 5 in [9]. Therefore, (4.1) is uniquely solvable in the classical sense and the above formulas for the derivatives show that the rough integral therein may be approximated by the same Riemann sum as in (4.9). This proves that the solution to (4.1), when the latter is understood as a rough differential equation driven by the enhanced setting above $(W^1(\omega), \cdots, W^n(\omega))$, coincides with the solution of the empirical version of (0.2), when the latter is understood as a mean field rough differential equation driven by the empirical rough set up.

4.2 – Propagation of Chaos

We now have all the ingredients to prove that the solution to (4.1) converges, in some sense, to the solution of the rough mean field equation (0.2) when the rough set-up is interpreted as originally explained in Section 1. This should read as *propagation of chaos*. The statement takes the following form.

22. Theorem – *On top of the assumptions of* Theorem 16, *assume that the rough set-up* $\boldsymbol{W}$ *is strong. Assume also that*

- *there exists a real* $\varepsilon_1 > 0$ *such that*

$$\mathbb{E}\Big[\exp\Big(\|W(\cdot)\|^{\varepsilon_1}_{[0,T],(1/p)-\text{Hölder}}\Big)\Big] + \mathbb{E}\Big[\exp\Big(\|\mathbb{W}(\cdot)\|^{\varepsilon_1/2}_{[0,T],(2/p)-\text{Hölder}}\Big)\Big] + \mathbb{E}^{\otimes 2}\Big[\exp\Big(\|\mathbb{W}^{\perp\!\!\!\perp}(\cdot,\cdot)\|^{\varepsilon_1/2}_{[0,T],(2/p)-\text{Hölder}}\Big)\Big] < \infty.$$

- *for almost every* $\omega \in \Omega$*, for any* $\alpha > 0$*, there exists a constant* $\varepsilon_2 > 0$ *such that, for all* $n \geqslant 1$*,*

$$\sup_{n\geqslant 1} \frac{1}{n}\sum_{i=1}^{n} \exp\Big(N^{i,n}(0,T,\omega,\alpha)^{1+\varepsilon_2}\Big) < \infty,$$

where $N^{i,n}(0,T,\omega,\alpha)$ *is defined as the local accumulation*

$$N^{i,n}([0,T],\omega,\alpha) := N_{\varpi}([0,T],\alpha),$$

when $\varpi(s,t) = v_p^{i,n}(s,t,\omega)$*, see* (1.13)*.*

Then, for almost every $\omega \in \Omega$*,*

$$\frac{1}{n}\sum_{i=1}^{n} \delta_{X^{i,(n)}(\omega)} \to \mathcal{L}\big(X(\cdot)\big),$$

where $X^{(n)}(\omega)$ *is the solution to* (4.1) *and* $X(\cdot)$ *is the solution to* (0.2)*, the convergence being the convergence in law on* $\mathcal{C}\big([0,T];\mathbb{R}^d\big)$*. Moreover, for any fixed* $k \geqslant 1$*, the law of* $\big(X^{1,(n)}(\cdot), \cdots, X^{k,(n)}(\cdot)\big) := X^{(n)}(\cdot)$ *converges to* $\mathcal{L}\big(X(\cdot)\big)^{\otimes k}$*.*

Strangely enough, and somewhat disappointingly, we did not manage to provide a generic simple condition on the limiting set-up $\boldsymbol{W}$ that forces the empirical set-ups to satisfy the estimate of the second item in the assumptions right above. Still, as pointed out in Theorem 23 below, we can check by hand that this condition is indeed satisfied in the Gaussian case, see Example 5 and the subsequent Theorem 6, which serve us as a benchmark throughout the article. The main difficulty in proving Theorem 22 is in controlling the accumulated local variation of the empirical rough set-up.

Proof – The key tool for passing to the limit is the continuity Theorem 20. To make the notations clear, we write $X_0^{i,(n)}$ for X^i, $W^{i,(n)}$ for W^i, $\mathbb{W}^{i,(n)}$ for $\mathbb{W}^i$ and $\mathbb{W}^{i,j,(n)}$ for $\mathbb{W}^{i,j}$.

Step 1. As a starting point, observe that, from the law of large numbers, for any real-valued bounded and measurable function f on

$$\mathbb{R}^d \times \mathcal{C}_{p-\mathrm{var}}\big([0,T];\mathbb{R}^m\big) \times \Big\{\mathcal{C}^2_{p/2-\mathrm{var}}\big(\mathcal{S}_2^T;\mathbb{R}^m \otimes \mathbb{R}^m\big)\Big\}^2,$$

for almost every $\omega \in \Omega$, we have (see (4.6))

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{n^2} \sum_{i,j=1}^n f\Big(X_0^{i,(n)}(\omega), W^{i,(n)}(\omega), \mathbb{W}^{i,(n)}(\omega), \mathbb{W}^{i,j,(n)}(\omega)\Big) \\
&= \mathbb{E}\Big[f\big(X_0(\cdot), W(\cdot), \mathbb{W}(\cdot), \mathbb{W}^{\perp\!\!\!\perp}(\cdot,\cdot)\big)\Big].
\end{aligned}$$

In fact, for $p' > p$, the spaces $\mathcal{C}_{p-\mathrm{var}}([0,T];\mathbb{R}^m)$ and $\mathcal{C}_{p/2-\mathrm{var}}(\mathcal{S}_2^T;\mathbb{R}^m \otimes \mathbb{R}^m)$ embed in Polish subspaces $\mathcal{C}^0_{p'-\mathrm{var}}([0,T];\mathbb{R}^m)$ and $\mathcal{C}^0_{p'/2-\mathrm{var}}(\mathcal{S}_2^T;\mathbb{R}^m \otimes \mathbb{R}^m)$ of $\mathcal{C}_{p'-\mathrm{var}}([0,T];\mathbb{R}^m)$ and $\mathcal{C}_{p'/2-\mathrm{var}}(\mathcal{S}_2^T;\mathbb{R}^m \otimes \mathbb{R}^m)$, respectively; see for instance [27, Proposition 5.38]. The above is true for any real-valued bounded and continuous function f on

$$\mathbb{R}^d \times \mathcal{C}^0_{p-\mathrm{var}}\big([0,T];\mathbb{R}^m\big) \times \big\{\mathcal{C}^0_{p/2-\mathrm{var}}(\mathcal{S}_2^T;\mathbb{R}^m \otimes \mathbb{R}^m)\big\}^2.$$

By choosing f in a countable convergence determining class, we deduce that, there exists a full subset $E \subset \Omega$, whose precise definition may change from line to line as long as E remains of probability 1, and such that, for any $\omega \in E$, the sequence of probability measures

$$\pi_n(\omega) = \left(\frac{1}{n^2} \sum_{i,j=1}^n \delta_{(X_0^{i,(n)}(\omega), W^{i,(n)}(\omega), \mathbb{W}^{i,(n)}(\omega), \mathbb{W}^{i,j,(n)}(\omega))}\right)_{n\geqslant 1}$$

converges in the weak sense to $(X_0(\cdot), W(\cdot), \mathbb{W}(\cdot), \mathbb{W}^{\perp\!\!\!\perp}(\cdot,\cdot))$ on $\mathbb{R}^d \times \mathcal{C}\big([0,T];\mathbb{R}^m\big) \times \big\{\mathcal{C}(\mathcal{S}_2^T;\mathbb{R}^m \otimes \mathbb{R}^m)\big\}^2$.

Step 2. Our strategy now relies on Theorem 20. The third item in the statement is a consequence of the law of large numbers. As for the fourth item, it follows directly from the previous strep.

We now have a look at $v_p^{i,n}(s,t,\omega)$ in (4.3). We already know that

$$\begin{aligned}
&\limsup_{n\geqslant 1} \sup_{0\leqslant s<t\leqslant T} \frac{{}^{(n)}\big(v_p^{\bullet,n}(s,t,\omega)\big)_{2pq}}{t-s} \\
&\leqslant c\,\mathbb{E}\Big[\big\|W(\cdot)\big\|^{pq}_{[0,T],(1/p)-\text{Hölder}} + \big\|\mathbb{W}(\cdot)\big\|^{pq}_{[0,T],(2/p)-\text{Hölder}} \\
&\qquad\qquad + \big\|\mathbb{W}^{\perp\!\!\!\perp}(\cdot,\cdot)\big\|^{pq}_{[0,T],(2/p)-\text{Hölder}}\Big]^{1/q},
\end{aligned}$$

which proves the second item in the statement of Theorem 20. We end up with the proof of the first item. By (4.4), there exists a constant c' such that, for any $\varepsilon > 0$, the quantity

$$\sup_{n\geqslant 1} {}^{(n)}\Big(\exp\big([v_p^{\bullet,n}(0,T,\omega)]^{\varepsilon}\big)\Big)_1 \tag{4.10}$$

is finite if

$$\sup_{n\geqslant 1}\frac{1}{n}\sum_{i=1}^{n}\exp\Big(\|W^i(\omega)\|^{c'p\varepsilon}_{[0,T],(1/p)-\text{Hölder}}\Big)<\infty,$$
$$\sup_{n\geqslant 1}\frac{1}{n}\sum_{i=1}^{n}\exp\Big(\|\mathbb{W}^i(\omega)\|^{c'p\varepsilon/2}_{[0,T],(2/p)-\text{Hölder}}\Big)<\infty,$$
$$\sup_{n\geqslant 1}\frac{1}{n}\sum_{i=1}^{n}\exp\Big({}^{(n)}\Big(\|\mathbb{W}^{\bullet,i}(\omega)\|^{p/2}_{[0,T],(2/p)-\text{Hölder}}\Big)_q^{c'\varepsilon}\Big)<\infty,$$
$$\sup_{n\geqslant 1}\frac{1}{n}\sum_{i=1}^{n}\exp\Big({}^{(n)}\Big(\|\mathbb{W}^{\bullet,i}(\omega)\|^{p/2}_{[0,T],(2/p)-\text{Hölder}}\Big)_q^{c'\varepsilon}\Big)<\infty.$$

By the law of large numbers, the first two lines hold true on a full event if $c'p\varepsilon<\varepsilon_1$. As for the third and fourth lines, we use the following trick. Notice that the function

$$(0,+\infty)\ni x\mapsto \exp\big(x^{c'\varepsilon/q}\big), \tag{4.11}$$

is convex on $[A_\varepsilon,\infty)$, for some $A_\varepsilon>0$. Therefore, Jensen's inequality says that, in order to check the third line, it suffices to prove that

$$\sup_{n\geqslant 1}\frac{1}{n^2}\sum_{i,j=1}^{n}\exp\Big[\Big(A_\varepsilon^{c'\varepsilon/q}\vee\|\mathbb{W}^{i,j}(\omega)\|^{c'p\varepsilon/2}_{[0,T],(2/p)-\text{Hölder}}\Big)\Big]<\infty, \tag{4.12}$$

and similarly for the last line. Obviously, under the standing assumption, the latter holds true with probability 1 provided that $c'p\varepsilon<\varepsilon_1$. This proves (4.10). In the statement of Theorem 20, this proves the condition related to the tails of w^n by a standard application of Markov inequality.

The bound on the local accumulation in the first item of Theorem 4.10 is a consequence of the second item in the standing assumption. Indeed, we let the reader check that it suffices to work with the local accumulation associated with v^n instead of the local accumulation associated with w^n, see if needed the inequality (A.1) in Appendix.

Step 3. Theorem 20 says that, for a fixed $\omega\in E$, the solutions associated with the rough set-ups $\big(\boldsymbol{W}^{(n)}(\omega)\big)_{n\geqslant 1}$ converge in law to the solution associated with the limiting rough set-up, i.e., the empirical law of the solutions associated with the $\big(\boldsymbol{W}^{(n)}\big)_{n\geqslant 1}$ converges to the law of the solution of the mean field equation, which is exactly to say that, for any $\omega\in E$,

$$\frac{1}{n}\sum_{i=1}^{n}\delta_{X^{i,(n)}(\omega)}\to\mathcal{L}\big(X(\cdot)\big),$$

where $X(\cdot)$ is the solution to (0.2). Here, the convergence is the convergence in law on $\mathcal{C}([0,T];\mathbb{R}^d)$. By Proposition 2.2 in [38], we deduce that, for any fixed $k\geqslant 1$, the law of $\big(X^{1,(n)},\cdots,X^{k,(n)}\big)$ converges to $\mathcal{L}\big(X(\cdot)\big)^{\otimes k}$. ▷

As an example of application, we have the following statement, proved in Appendix A.2.

23. Theorem – *Let W be a continuous centered Gaussian process, defined over some finite interval $[0,T]$. Assume it has independent components. Suppose that the covariance function is of finite ϱ-two dimensional variation for some $\varrho\in[1,3/2)$. Then, for $p\in(2\varrho,3)$, the conditions of* Theorem 22 *are satisfied.*

4.3 – Rate of Convergence

The goal of this subsection is to elucidate the rate of convergence in the convergence result stated in Theorem 22. Note the use of the Wasserstein W_1-distance in the regularity assumption required from F in the statement.

24. Theorem – *On top of the assumption of* Theorem 22, *assume that*

- *The first and second order derivatives of* F, $(x,\mu) \mapsto \partial_x \mathrm{F}(x,\mu)$, $(x,\mu,z) \mapsto D_\mu \mathrm{F}(x,\mu)(z)$, $(x,\mu,z) \mapsto \partial_x D_\mu \mathrm{F}(x,\mu,z)$ *and* $(x,\mu,z,z') \mapsto D^2_\mu \mathrm{F}(x,\mu,z,z')$, *are bounded on the whole space and are Lipschitz continuous with respect to all the variables, the Lipschitz property in the direction* μ *being understood with respect to* the W_1-Wasserstein distance*;*
- *for any* $\alpha > 0$, *there exists a constant* $\varepsilon_2 > 0$ *such that, for any* $n \geqslant 1$, *for any* $p' \in (1/3, 1/p)$, *and any random variables* $\tau, \tau' : \Omega \to [0,T]$, *with* $\mathbb{P}(\tau < \tau') = 1$, *we have*
$$\sup_{n\geqslant 1}\sup_{1\leqslant i\leqslant n} \mathbb{E}\left[\exp\left[\left(\frac{\widehat{N}^{i,n}\big([\tau,\tau'],\omega,\alpha\big)}{\tau'-\tau}\right)^{1+\varepsilon_2}\right]\right] < \infty,$$
where $\widehat{N}^{i,n}\big([\tau,\tau'],\omega,\alpha\big)$ *is defined as the local accumulation*
$$\widehat{N}^{i,n}\big([\tau,\tau'],\omega,\alpha\big) := N_\varpi\big([\tau,\tau'],\alpha\big)$$
when $\varpi = \widehat{w}^{i,n}_{p'}$ *with*

$$\begin{aligned}
\widehat{w}^{i,n}_{p'}(s,t,\omega) &:= w^{i,n}_{p'}(s,t,\omega) + \widehat{v}^{i,n}_{p'}(s,t,\omega) + {}^{(n)}\big(\widehat{v}^{\bullet,n}_{p'}(\omega)\big)_{q;[s,t],1-\mathrm{var}} + (t-s),\\
w^{i,n}_{p'}(s,t,\omega) &:= v^{i,n}_{p'}(s,t,\omega) + {}^{(n)}\big(v^{\bullet,n}_{p'}(\omega)\big)_{q;[s,t],1-\mathrm{var}},\\
\widehat{v}^{i,n}_{p'}(s,t,\omega) &= \big\langle \mathbb{W}^{i,\perp\!\!\!\perp}(\omega,\cdot)\big\rangle^{p'/2}_{q;[s,t],p'/2-\mathrm{var}} + \big\langle \mathbb{W}^{i,\perp\!\!\!\perp}(\cdot,\omega)\big\rangle^{p'/2}_{q;[s,t],p'/2-\mathrm{var}}.
\end{aligned} \tag{4.13}$$

Then, for any $r \geqslant 1$, *there exists an exponent* $q(r) \geqslant 8$ *such that, if* $q \geqslant q(r)$, *with* q *as in* Section 1, *and* $X_0(\cdot)$ *is in* $\mathbb{L}^{q(r)}$, *then*
$$\sup_{1\leqslant i\leqslant n} \mathbb{E}\left[\sup_{0\leqslant t\leqslant T} \big|\overline{X}^i_t - X^{i,(n)}_t\big|^r\right]^{1/r} \leqslant C\eta_n,$$
for a constant C *independent of* n, *and* $\eta_n = n^{-1/2}$ *if* $d=1$, $\eta_n = n^{-1/2}\ln(1+n)$ *if* $d=2$ *and* $\eta_n = n^{-1/d}$ *if* $d \geqslant 3$.

Let us make a few remarks on this statement before embarking on its proof.

- We refer to [9, Chapter 5] for examples of a function F satisfying the first item in the assumptions of the statement.
- The rate which is obtained corresponds to the usual rate for the convergence in the 1-Wasserstein distance of an empirical sample of independent, identically distributed, random variables toward the limiting common distribution.
- As before, Theorem 24 applies when W is a continuous centered Gaussian process defined over some finite interval $[0,T]$ with independent components and with a covariance function that is of finite ϱ-two dimensional variation for some $\varrho \in [1, 3/2)$, see Theorem 22. The proof is pretty similar to that of Theorem 22 given in Appendix. In order to check the second item in the statement, the trick is to notice that all the bounds we have for the local accumulation on $[0,T]$ depend linearly on T. Put differently, we can provide bounds for

quantities of the form $N_{\varpi}([0,T],\cdot,\alpha)/T$, where ϖ denotes the corresponding function in hand. In order to do so, we can treat separately the local accumulation associated to each of the terms entering the definition of $\widehat{w}_{p'}^{i,n}$, see (A.1). As for $v_{p'}^{i,n}$, the computations fit exactly those performed in the proof of Theorem 22. As for $\hat{v}_{p'}^{i,n}$, the proof derives from Theorem 6. In order to handle the local accumulations associated to $\mathcal{S}_2^T \ni (s,t) \mapsto {}^{(n)}[\![v_{p'}^{\bullet;n}(\omega)]\!]_{q;[s,t],1-\mathrm{var}}$ and $\mathcal{S}_2^T \ni (s,t) \mapsto {}^{(n)}[\![\widehat{v}_{p'}^{\bullet;n}(\omega)]\!]_{q;[s,t],1-\mathrm{var}}$, it is necessary to slightly adapt the proof of Theorem 22; the arguments are left to the reader. For sure, the local accumulation associated to the additional $t-s$ in $\widehat{w}_{p'}^{i,n}(s,t,\omega)$ is easily taken.

- By inspecting the proof of the theorem, we could make explicit the value of $q(r)$, but we feel that it would not be so useful.

Proof – The proof consists in a variation of Sznitman's original coupling argument, see [38]. To do so, we recall that, on the probability space $(\Omega,\mathcal{F},\mathbb{P})$, the pairs $\big(W^1(\cdot),\mathbb{W}^1(\cdot)\big),\cdots,\big(W^n(\cdot),\mathbb{W}^n(\cdot)\big)$ are n independent copies of $\big(W(\cdot),\mathbb{W}(\cdot)\big)$. For each $i \in \{1,\cdots,n\}$, the pair $\big(W^i(\cdot),\mathbb{W}^i(\cdot)\big)$ is completed into a rough set-up $\overline{\boldsymbol{W}}^i(\cdot) := \big(W^i(\cdot),\mathbb{W}^i(\cdot),\mathbb{W}^{i,\perp\!\!\!\perp}(\cdot,\cdot)\big)$, with

$$\mathbb{W}^{i,\perp\!\!\!\perp}(\omega,\omega') = \mathcal{I}\big(W^i(\omega),W^i(\omega')\big), \quad (\omega,\omega') \in \Omega^2.$$

Here we put a bar on the symbol $\overline{\boldsymbol{W}}^i$ in order to distinguish it from the finite-dimensional rough set-up $\boldsymbol{W}^{(n)}(\omega)$ that lies above $\big(W^1(\omega),\cdots,W^n(\omega)\big)$. The second-order level of $\boldsymbol{W}^{(n)}$ is made of $(\mathbb{W}^i)_{1\leqslant i\leqslant n}$ and of $\big(\mathbb{W}^{i,j} = \mathcal{I}(W^i,W^j)\big)_{1\leqslant i\neq j\leqslant n}$, see (4.2). To make the notations more homogeneous, we write $\boldsymbol{W}^{i,i}(\omega)$ for $\boldsymbol{W}^i(\omega)$.

We also consider n independent copies $\big(X_0^1(\cdot),\cdots,X_0^n(\cdot)\big)$ of the initial condition $X_0(\cdot)$, the two n-tuples $\big(\overline{\boldsymbol{W}}^1(\cdot),\cdots,\overline{\boldsymbol{W}}^n(\cdot)\big)$ and $\big(X_0^1(\cdot),\cdots,X_0^n(\cdot)\big)$ being assumed to be independent. With each $(X_0^i(\cdot),\overline{\boldsymbol{W}}^i(\cdot))$, we associate the corresponding solution $\overline{X}^i(\cdot)$ to the mean field equation (0.2). Of course, the n-tuples $\Omega \ni \omega \mapsto \big(X_0^i(\omega),W^i(\omega),\mathbb{W}^i(\omega),\mathbb{W}^{i,\perp\!\!\!\perp}(\cdot,\omega),\overline{X}^i(\omega)\big)_{1\leqslant i\leqslant n}$ are independent, where $\Omega \ni \omega \mapsto \big(\mathbb{W}_t^{i,\perp\!\!\!\perp}(\cdot,\omega)\big)_{0\leqslant t\leqslant T}$ is considered as a process with values in $\mathbb{L}^q(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^d)$. We then let

$$\overline{\mu}_t^n(\omega) = \frac{1}{n}\sum_{i=1}^n \delta_{\overline{X}_t^i(\omega)}, \quad t \in [0,T], \quad \omega \in \Omega.$$

Observe that, for each $i \in \{1,\cdots,n\}$ and any $\omega \in \Omega$, we can define the integral process

$$\bigg(\int_0^t \mathrm{F}\big(\overline{X}_s^i(\omega),\overline{\mu}_s^n(\omega)\big)\,d\boldsymbol{W}_s^{i,(n)}(\omega)\bigg)_{0\leqslant t\leqslant T},$$

where the label i in the notation $\boldsymbol{W}^{i,(n)}(\omega)$ is here to indicate that the integral only involves $\big(W^i(\omega),(\mathbb{W}^{j,i}(\omega))_{1\leqslant j\leqslant n}\big)$. So, the symbol $\boldsymbol{W}^{i,(n)}(\omega)$ must be understood as $\big(W^i(\omega),(\mathbb{W}^{j,i}(\omega))_{1\leqslant j\leqslant n}\big)$. The fact that the integral may be defined with respect to $\big(W^i(\omega),(\mathbb{W}^{j,i}(\omega))_{1\leqslant j\leqslant n}\big)$ follows from the fact that $\overline{X}^j(\omega)$, for each $j \in \{1,\cdots,n\}$ and each $\omega \in \Omega$, is controlled by the variations of the sole $W^j(\omega)$.

Step 1. The first step is to compare

$$\int_0^t \mathrm{F}\big(\overline{X}_s^i(\omega), \mathcal{L}(X_s)\big) d\overline{\boldsymbol{W}}_s^i(\omega) \quad \text{and} \quad \int_0^t \mathrm{F}\big(\overline{X}_s^i(\omega), \overline{\mu}_s^n(\omega)\big) d\boldsymbol{W}_s^{i,(n)}(\omega),$$

for $t \in [0,T]$. What makes the proof non-trivial is the fact that the rough set-ups used in the first and the second integrals are not the same. So, in order to compare the two of them, we need to come back to the original constructions of the two integrals. To simplify notations, and for $0 \leqslant t \leqslant T$, set

$$\overline{F}_t^i(\omega) := \mathrm{F}\big(\overline{X}_t^i(\omega), \mathcal{L}(X_t)\big)$$

and

$$F_t^{i,n}(\omega) := \mathrm{F}\big(\overline{X}_t^i(\omega), \overline{\mu}_t^n(\omega)\big).$$

For sure, $\big(\overline{F}_t^i(\omega)\big)_{0 \leqslant t \leqslant T}$ is ω-controlled by $\overline{\boldsymbol{W}}^i(\omega)$ and the collection indexed by $\omega \in \Omega$ is a random path controlled by $\overline{\boldsymbol{W}}^i$, see Definition 8 for a reminder. The corresponding Gubinelli derivatives are denoted by $\big(\delta_x \overline{F}_t^i(\omega), \delta_\mu \overline{F}_t^i(\omega, \cdot)\big)_{0 \leqslant t \leqslant T}$, see Proposition 11. Similarly, $(F_t^{i,n}(\omega))_{0 \leqslant t \leqslant T}$ is controlled by $\boldsymbol{W}^{i,(n)}(\omega)$ and Gubinelli derivatives are denoted by $\big(\delta_x F_t^{i,n}(\omega), \big(\delta_\mu F_t^{i,j,n}(\omega)\big)_{1 \leqslant j \leqslant n}\big)_{0 \leqslant t \leqslant T}$, see Subsection 4.1. To make it clear, set

$$\begin{aligned} \delta_x \overline{F}_t^i(\omega) &:= \partial_x \mathrm{F}\Big(\overline{X}_t^i(\omega), \mathcal{L}(X_t)\Big)\ \mathrm{F}\big(\overline{X}_t^i(\omega), \mathcal{L}(X_t)\big), \\ \delta_\mu \overline{F}_t^i(\omega, \cdot) &:= D_\mu \mathrm{F}\Big(\overline{X}_t^i(\omega), \mathcal{L}(X_t)\Big)\ \big(X_t(\cdot)\big)\ \mathrm{F}\big(X_t(\cdot), \mathcal{L}(X_t)\big), \end{aligned} \tag{4.14}$$

where $X(\cdot)$ is the solution to the mean field equation (0.2) when driven by $\boldsymbol{W}(\cdot) = \big(W(\cdot), \mathbb{W}(\cdot), \mathbb{W}^{\perp\!\!\!\perp}(\cdot,\cdot)\big)$. We also let

$$\begin{aligned} \delta_x F_t^{i,n}(\omega) &:= \partial_x \mathrm{F}\big(\overline{X}_t^i(\omega), \overline{\mu}_t^n(\omega)\big) \mathrm{F}\big(\overline{X}_t^i(\omega), \overline{\mu}_t^n(\omega)\big), \\ \delta_\mu F_t^{i,j,n}(\omega) &:= D_\mu \mathrm{F}\big(\overline{X}_t^i(\omega), \overline{\mu}_t^n(\omega)\big)\big(\overline{X}_t^j(\omega)\big) \mathrm{F}\big(\overline{X}_t^j(\omega), \overline{\mu}_t^n(\omega)\big). \end{aligned} \tag{4.15}$$

Given these definitions, and for a subdivision $\Delta = \{s = t_0 < t_1 < \cdots < t_K = t\}$, set

$$\begin{aligned} \overline{\mathcal{I}}_{s,t}^{i,\Delta}(\omega) := \sum_{k=0}^{K-1} \Big\{ &\overline{F}_{t_k}^i(\omega) W_{t_k,t_{k+1}}^i(\omega) + \delta_x \overline{F}_{t_k}^i(\omega) \mathbb{W}_{t_k,t_{k+1}}^i(\omega) \\ &+ \mathbb{E}\big[\delta_\mu \overline{F}_{t_k}^i(\omega, \cdot) \mathbb{W}_{t_k,t_{k+1}}^{i,\perp\!\!\!\perp}(\cdot, \omega)\big] \Big\}, \\ \mathcal{I}_{s,t}^{i,n,\Delta}(\omega) := \sum_{k=0}^{K-1} \Big\{ &F_{t_k}^{i,n}(\omega) W_{t_k,t_{k+1}}^i(\omega) + \delta_x F_{t_k}^{i,n}(\omega) \mathbb{W}_{t_k,t_{k+1}}^i(\omega) \\ &+ \frac{1}{n} \sum_{j=1}^n \delta_\mu F_{t_k}^{i,j,n}(\omega) \mathbb{W}_{t_k,t_{k+1}}^{j,i}(\omega) \Big\}. \end{aligned}$$

We denote the summand in the first sum by $\overline{\mathcal{I}}_{\{t_k,t_{k+1}\}}^{i,\partial}(\omega)$ and the summand in the second sum by $\mathcal{I}_{\{t_k,t_{k+1}\}}^{i,n,\partial}(\omega)$. By Lemma 25 proved in Appendix A.3, we can find, for any $\varrho \geqslant 8$, a constant C and an exponent $\varrho' \geqslant q$ independent of n and K such that, when $X_0(\cdot) \in \mathbb{L}^{\varrho'}$, it holds for any $k \in \{1, \cdots, K-1\}$,

$$\Big\langle \Big\{ \mathcal{I}_{s,t}^{i,n,\Delta}(\cdot) - \mathcal{I}_{s,t}^{i,n,\Delta'}(\cdot) \Big\} - \Big\{ \overline{\mathcal{I}}_{s,t}^{i,\Delta}(\cdot) - \overline{\mathcal{I}}_{s,t}^{i,\Delta'}(\cdot) \Big\} \Big\rangle_\varrho \leqslant C \eta_n \big\langle\!\big\langle w^+(t_{k-1}, t_{k+1}, \cdot, \cdot) \big\rangle\!\big\rangle_{\varrho'}^{3/p},$$

where

$$\Delta' := \Delta \backslash \{t_k\}$$

and

$$w^+(s,t,\omega,\omega') := w(s,t,\omega) + \|\mathbb{W}^{\perp\!\!\!\perp}(\omega,\omega')\|_{[s,t],p/2-\mathrm{var}}^{p/2}.$$

Following (4.4), we know that the right hand side is less than

$$C\eta_n\Big[\big\langle \|W(\cdot)\|_{[0,T],(1/p)-\text{Hölder}}\big\rangle_{p\varrho'} + \big\langle \|\mathbb{W}(\cdot)\|_{[0,T],(2/p)-\text{Hölder}}\big\rangle_{p\varrho'}^{1/2} \\ + \big\langle\!\big\langle \|\mathbb{W}^{\perp\!\!\!\perp}(\cdot,\cdot)\|_{[0,T],(2/p)-\text{Hölder}}\big\rangle\!\big\rangle_{p\varrho'}^{1/2}\Big]^3 (t_{k+1}-t_k)^{3/p},$$

but by assumption all the expectations are finite. Now we can choose t_k such that $|t_{k+1}-t_k| \leqslant 2|t-s|/K$. We get

$$\Big\langle \Big\{\mathcal{I}_{s,t}^{i,n,\Delta}(\cdot) - \mathcal{I}_{s,t}^{i,n,\Delta'}(\cdot)\Big\} - \Big\{\overline{\mathcal{I}}_{s,t}^{i,\Delta}(\cdot) - \overline{\mathcal{I}}_{s,t}^{i,\Delta'}(\cdot)\Big\}\Big\rangle_\varrho \leqslant C\eta_n\Big(\frac{2(t-s)}{K}\Big)^{3/p},$$

the constant C being allowed to increase from line to line as long as it remains independent of n and K. Letting $t^{(1)} = t_k$ and applying iteratively the above bound to a sequence of meshes of the form $\Delta\backslash\{t^{(1)}\}$, $\Delta\backslash\{t^{(1)},t^{(2)}\}$, $\dots$, and then letting K tend to ∞, we deduce that

$$\Big\langle \int_s^t F_r^{i,n}(\cdot)d\boldsymbol{W}_r^{i,(n)}(\cdot) - \int_s^t \overline{F}_r^i(\cdot)d\overline{\boldsymbol{W}}_r^i(\cdot) - \Big\{\mathcal{I}_{\{s,t\}}^{i,n,\partial} - \overline{\mathcal{I}}_{\{s,t\}}^{i,\partial}\Big\}\Big\rangle_\varrho \\ \leqslant C\eta_n(t-s)^{3/p}. \tag{4.16}$$

By a straightforward adaptation of the first two steps in the proof of Lemma 25, we have in a similar way

$$\Big\langle \mathcal{I}_{\{s,t\}}^{i,n,\partial} - \overline{\mathcal{I}}_{\{s,t\}}^{i,\partial}\Big\rangle_\varrho \leqslant C\eta_n(t-s)^{1/p},$$

from which we deduce that

$$\Big\langle \int_s^t F_r^{i,n}(\cdot)d\boldsymbol{W}_r^{i,(n)}(\cdot) - \int_s^t \overline{F}_r^i(\cdot)d\overline{\boldsymbol{W}}_r^i(\cdot)\Big\rangle_\varrho \leqslant C\eta_n(t-s)^{1/p}.$$

Similarly, following again the proof of the first step in the proof of Lemma 25, we get

$$\big\langle \big[F^{i,n}(\cdot) - \overline{F}^i(\cdot)\big]_{s,t}\big\rangle_\varrho \leqslant C\eta_n(t-s)^{1/p},$$

and, noting that

$$R_{s,t}^{\int F^{i,n}d\boldsymbol{W}^{i,(n)}}(\omega) \\ = \int_s^t F_r^{i,n}(\omega)d\boldsymbol{W}_r^{i,(n)}(\omega) - \mathcal{I}_{s,t}^{i,n,\partial}(\omega) + \delta_x F_s^{i,n}(\omega)\mathbb{W}_{s,t}^i(\omega) + \frac{1}{n}\sum_{j=1}^n \delta_\mu F_s^{i,j,n}(\omega)\mathbb{W}_{s,t}^{j,i}(\omega),$$

$$R_{s,t}^{\int \overline{F}^i d\overline{\boldsymbol{W}}^i}(\omega) \\ = \int_s^t \overline{F}_r^i(\omega)d\overline{\boldsymbol{W}}_r^i(\omega) - \overline{\mathcal{I}}_{s,t}^{i,\partial}(\omega) + \delta_x\overline{F}_s^i(\omega)\mathbb{W}_{s,t}^i(\omega) + \mathbb{E}\big[\delta_\mu\overline{F}_s^i(\omega,\cdot)\mathbb{W}_{s,t}^{i,\perp\!\!\!\perp}(\cdot,\omega)\big],$$

we deduce in a similar manner, using in addition (4.16), that

$$\Big\langle R_{s,t}^{\int F^{i,n}d\boldsymbol{W}^{i,(n)}}(\cdot) - R_{s,t}^{\int \overline{F}^i d\overline{\boldsymbol{W}}^i}(\cdot)\Big\rangle_\varrho \leqslant C\eta_n(t-s)^{2/p}.$$

So, fixing $i \in \{1,\cdots,n\}$, choosing ϱ large enough and applying a suitable version of Kolmogorov's theorem (see for instance Theorem 3.1 in [25]), we can find $p' \in (p,3)$

such that

$$
\begin{aligned}
&\left|\int_s^t F_r^{i,n}(\omega) d\boldsymbol{W}_r^{i,(n)} - \int_s^t \overline{F}_r^i(\omega) d\overline{\boldsymbol{W}}_r^i(\omega)\right| \leqslant \theta^{i,n}(\omega)(t-s)^{1/p'}, \\
&\left|\left[F^{i,n}(\omega) - \overline{F}^i(\omega)\right]_{s,t}\right| \leqslant \theta^{i,n}(\omega)(t-s)^{1/p'}, \\
&\left|R_{s,t}^{\int F^{i,n} d\boldsymbol{W}^{i,(n)}}(\omega) - R_{s,t}^{\int \overline{F}^i d\overline{\boldsymbol{W}}^i}(\omega)\right| \leqslant \theta^{i,n}(\omega)(t-s)^{2/p'},
\end{aligned}
\tag{4.17}
$$

with $\langle \theta^{i,n}(\cdot)\rangle_\varrho \leqslant C\eta_n$, for a new value of the constant C.

Observe now that the empirical control associated with our empirical rough set-up and with the exponent p' reads

$$
w_{p'}^{i,n}(s,t,\omega) := v_{p'}^{i,n}(s,t,\omega) + {}^{(n)}\!\left[\!\left[v_{p'}^{\bullet,n}(\omega)\right]\!\right]_{q;[s,t],1-\mathrm{var}},
$$

where we used the same notation as in (4.3). In fact, there is no loss of generality in changing the definition of $w_{p'}^{i,n}$ into

$$
w_{p'}^{i,n}(s,t,\omega) := v_{p'}^{i,n}(s,t,\omega) + {}^{(n)}\!\left[\!\left[v_{p'}^{\bullet,n}(\omega)\right]\!\right]_{q;[s,t],1-\mathrm{var}} + (t-s), \tag{4.18}
$$

which permits to replace $(t-s)^{1/p'}$ by $w_{p'}^{i,n}(s,t,\omega)^{1/p'}$ in the inequalities (4.17). Hence,

$$
\left\| \! \left| \int_0^\cdot F_r^{i,n}(\omega) d\boldsymbol{W}_r^{i,(n)} - \int_0^\cdot \overline{F}_r^i(\omega) d\overline{\boldsymbol{W}}_r^i(\omega) \right| \! \right\|_{[0,T],w_{p'}^{i,n},p'} \leqslant \theta^{i,n}(\omega).
$$

Step 2. We now make use of Proposition 15 to compare

$$
\int_0^t \mathrm{F}\big(X_s^{i,(n)}(\omega), \mu_s^n(\omega)\big) d\overline{\boldsymbol{W}}_s^i(\omega) \quad \text{and} \quad \int_0^t \mathrm{F}\big(\overline{X}_s^i(\omega), \overline{\mu}_s^n(\omega)\big) d\boldsymbol{W}_s^{i,(n)}(\omega),
$$

where

$$
\mu_s^n(\omega) := \frac{1}{n}\sum_{j=1}^n \delta_{X_s^{j,(n)}(\omega)}.
$$

To simplify the notations, we just write X^i for $X^{i,(n)}$ and $\boldsymbol{W}^i$ for $\boldsymbol{W}^{i,(n)}$. We then apply Proposition 15 with

$$
\big(X(\omega), Y(\cdot)\big) = \big(X^i(\omega), X^\bullet(\omega)\big), \quad \big(X'(\omega), Y'(\cdot)\big) = \big(\overline{X}^i(\omega), \overline{X}^\bullet(\omega)\big), \tag{4.19}
$$

the underlying set-up being understood as the empirical rough set-up for a given realization ω. The difficulty here is that the variations of these two solutions are controlled by two different functionals w, see (2.1). This is the rationale for introducing $\widehat{w}_{p'}^{i,n}$ in (4.13). Obviously, $\widehat{w}^{i,n}(\cdot,\cdot,\omega)$ (we remove the index p' for simplicity) is not the *natural* control functional associated with $\boldsymbol{W}^i(\omega)$, but it is greater than $w_{p'}^{i,n}(s,t,\omega)$ and it satisfies

$$
{}^{(n)}\!\big[\widehat{w}^{\bullet,n}(s,t,\omega)\big]_q \leqslant 2\widehat{w}^{i,n}(s,t,\omega),
$$

which suffices to duplicate the proof of Proposition 15 with $w_{p'}^{i,n}(s,t,\omega)$ replaced by $\widehat{w}^{i,n}(s,t,\omega)$. The resulting semi-norm that must be used to control the difference $\big(X(\omega)-X'(\omega), Y(\cdot)-Y'(\cdot)\big) = \big(X^i(\omega)-\overline{X}^i(\omega), X^\bullet(\omega)-\overline{X}^\bullet(\omega)\big)$ on a given interval $[s,t]$ is $\|\!|\cdot|\!\|_{[s,t],\widehat{w}^{i,n},p'}$. Of course the fact that we no longer use the *natural* control functional prompts us to use the local accumulation $\widehat{N}^{i,n}\big([0,T],\omega,\alpha\big)$ defined in the statement.

By construction of the processes $\big(X^i(\omega)\big)_{i=1,\cdots,n}$ as the solution of the empirical rough equation, the pair $\big(X(\omega),Y(\cdot)\big) = \big(X^i(\omega),X^\bullet(\omega)\big)$ in (4.19) automatically satisfies the first bound in (3.12) with $w = \widehat{w}^{i,n}$; implicitly, this means that we perform the same construction as in the proof of Theorem 16 using therein the empirical rough-set up and the control functionals $\big(\widehat{w}^{i,n}\big)_{i=1,\cdots,n}$. In particular, the sequence of points $\big(t^0_\ell = \tau_\ell(0,T,\omega,1/(4L_0))\big)_{\ell=0,\cdots,N^0+1}$ in the statement of Proposition 15 is understood as with respect to $\widehat{w}^{i,n}$. Also, by the last part in the statement of Proposition 14, we know that $Y(\cdot) = X^\bullet(\omega)$ satisfies condition (3.11) with respect to ${}^{(n)}\big[\cdot\big]_8$ if we assume that T satisfies

$$
{}^{(n)}\Big[\,\widehat{N}^{\bullet,n}\big([0,T],\omega,1/(4L_0)\big)\,\Big]_8 \leqslant c, \tag{4.20}
$$

for a deterministic constant c, independent of n, L_0 and T.

In fact, following (4.4) and using the additional $t-s$ in the definition (4.13), $\widehat{w}^{i,n}$ dominates (up to a multiplicative constant) the control $\overline{w}^i$ associated to $\overline{\boldsymbol{W}}^i$ through (1.8). Moreover, we have

$$
\big\langle \widehat{w}^{i,n}(s,t,\cdot)\big\rangle_q \leqslant C(t-s),
$$

for a constant C independent of i, n, s and t. Although $C \geqslant 2$, this permits to use $\widehat{w}^{i,n}(s,t,\cdot)$ as control functional when working with the rough set-up $\overline{\boldsymbol{W}}^i$. This is an important point as it says that the pair $\big(X'(\omega),Y'(\cdot)\big) = \big(\overline{X}^i(\omega),\overline{X}^\bullet(\omega)\big)$ in (4.19) satisfies the second bound in (3.12) with $w = \widehat{w}^{i,n}$. Also, invoking the first line in (3.4) for each $i \in \{1,\cdots,n\}$, we deduce that $Y'(\cdot) = \overline{X}^\bullet(\omega)$ satisfies condition (3.11) with respect to ${}^{(n)}\big[\cdot\big]_8$ provided that that (4.20) holds true. Possibly, this requires to work with a larger value of the threshold L_0 in the statement of Proposition 15, but this is not a hindrance.

Then, by Proposition 15, we obtain, for a given $L \geqslant L_0$,

$$
\begin{aligned}
&\Big\|\Big| \int_{t_k}^{\cdot} \mathrm{F}\big(X^i_r(\omega),\mu^n(\omega)\big)d\boldsymbol{W}^i_r(\omega) - \int_{t_k}^{\cdot} \mathrm{F}\big(\overline{X}^i_r(\omega),\overline{\mu}^n(\omega)\big)d\boldsymbol{W}^i_r(\omega)\Big|\Big\|_{[t_k,t_{k+1}],\widehat{w}^{i,n},p'} \\
&\leqslant \gamma\,\widehat{w}^{i,n}(0,t_k,\omega)^{1/p'}\bigg(\big\|\big|(X^i-\overline{X}^i)(\omega)\big|\big\|_{[0,t_k],\widehat{w}^{i,n},p'} \\
&\qquad\qquad + {}^{(n)}\Big[\Big(\big\|\big|(X^\bullet-\overline{X}^\bullet)(\omega)\big|\big\|_{[0,T],\widehat{w}^{\bullet,n},p'}\Big)\Big]_8\bigg) \\
&\quad + \frac{\gamma}{4L}\bigg(\big\|\big|(X^i-\overline{X}^i)(\omega)\big|\big\|_{[t_k,t_{k+1}],\widehat{w}^{i,n},p'} + {}^{(n)}\Big[\Big(\big\|\big|(X^\bullet-\overline{X}^\bullet)(\omega)\big|\big\|_{[t_k,t_{k+1}],\widehat{w}^{\bullet,n},p'}\Big)\Big]_8\bigg),
\end{aligned}
$$

where $\widehat{w}^{i,n}(t_k,t_{k+1},\omega)^{1/p'} = 1/(4L)$ as long as $k < N^{i,n}([0,T],\omega,1/(4L))$. The point now is to insert the conclusion of the first step. We get

$$
\begin{aligned}
&\big\|\big|(X^i-\overline{X}^i)(\omega)\big|\big\|_{[t_k,t_{k+1}],\widehat{w}^{i,n},p'} \\
&\leqslant \gamma\,\widehat{w}^{i,n}(0,t_k,\omega)^{1/p'}\bigg(\big\|\big|(X^i-\overline{X}^i)(\omega)\big|\big\|_{[0,t_k],\widehat{w}^{i,n},p'} \\
&\qquad\qquad + {}^{(n)}\Big[\Big(\big\|\big|(X^\bullet-\overline{X}^\bullet)(\omega)\big|\big\|_{[0,T],\widehat{w}^{\bullet,n},p'}\Big)\Big]_8\bigg) + \theta^{i,n}(\omega) \\
&\quad + \frac{\gamma}{4L}\bigg(\big\|\big|(X^i-\overline{X}^i)(\omega)\big|\big\|_{[t_k,t_{k+1}],\widehat{w}^{i,n},p'} + {}^{(n)}\Big[\Big(\big\|\big|(X^\bullet-\overline{X}^\bullet)(\omega)\big|\big\|_{[t_k,t_{k+1}],\widehat{w}^{\bullet,n},p'}\Big)\Big]_8\bigg).
\end{aligned}
$$

If $\gamma/(4L) \leqslant 1/2$, we get

$$\begin{aligned} &\big\|\big(X^i - \overline{X}^i\big)(\omega)\big\|_{[t_k,t_{k+1}],\widehat{w}^{i,n},p'} \\ &\leqslant 2\gamma\left(\frac{1}{L} + \widehat{w}^{i,n}(0,t_k,\omega)^{1/p'}\right)\left(\big\|\big(X^i - \overline{X}^i\big)(\omega)\big\|_{[0,t_k],\widehat{w}^{i,n},p'}\right. \\ &\qquad\left. + {}^{(n)}\Big(\!\!\Big(\big\|\big(X^\bullet - \overline{X}^\bullet\big)(\omega)\big\|_{[0,T],\widehat{w}^{\bullet,n},p'}\Big)\!\!\Big)_8\right) + 2\theta^{i,n}(\omega), \end{aligned} \tag{4.21}$$

and then, allowing the value of the constant c to increase from line to line, as long as it remains independent of n, L_0 and T, we get

$$\begin{aligned} &\big\|\big(X^i - \overline{X}^i\big)(\omega)\big\|_{[0,t_{k+1}],\widehat{w}^{i,n},p'} \\ &\leqslant c\left(1 + \zeta_T^{i,n}(\omega)\right)\big\|\big(X^i - \overline{X}^i\big)(\omega)\big\|_{[0,t_k],\widehat{w}^{i,n},p'} \\ &\qquad + c\,\zeta_T^{i,n}(\omega)^{(n)}\Big(\!\!\Big(\big\|\big(X^\bullet - \overline{X}^\bullet\big)(\omega)\big\|_{[0,T],\widehat{w}^{\bullet,n},p'}\Big)\!\!\Big)_8 + c\theta^{i,n}(\omega), \end{aligned}$$

with

$$\zeta_T^{i,n}(\omega) := \frac{1}{L} + w_{p'}^{i,n}(0,T,\omega)^{1/p'}.$$

So, by induction,

$$\begin{aligned} &\big\|\big(X^i - \overline{X}^i\big)(\omega)\big\|_{[0,t_{k+1}],\widehat{w}^{i,n},p'} \\ &\leqslant c\Big(\sum_{\ell=0}^{k}\big[c\big(1 + \zeta_T^{i,n}(\omega)\big)\big]^\ell\Big)\left(\zeta_T^{i,n}(\omega)\,{}^{(n)}\Big(\!\!\Big(\big\|\big(X^\bullet - \overline{X}^\bullet\big)(\omega)\big\|_{[0,T],\widehat{w}^{\bullet,n},p'}\Big)\!\!\Big)_8 + \theta^{i,n}(\omega)\right). \end{aligned}$$

In the end,

$$\begin{aligned} &\big\|\big(X^i - \overline{X}^i\big)(\omega)\big\|_{[0,T],\widehat{w}^{i,n},p'} \\ &\leqslant c\big[c\big(1 + \zeta_T^{i,n}(\omega)\big)\big]^{\widehat{N}^{i,n}([0,T],\omega,1/(4L))+1} \\ &\qquad \times\left(\zeta_T^{i,n}(\omega)\,{}^{(n)}\Big(\!\!\Big(\big\|\big(X^\bullet - \overline{X}^\bullet\big)(\omega)\big\|_{[0,T],\widehat{w}^{\bullet,n},p'}\Big)\!\!\Big)_8 + \theta^{i,n}(\omega)\right). \end{aligned} \tag{4.22}$$

Hence, using the shorten notation $\widehat{N}_T^{i,n}(\omega)$ for $\widehat{N}^{i,n}([0,T],\omega,1/(4L))$, we obtain

$$\begin{aligned} &{}^{(n)}\Big(\!\!\Big(\big\|\big(X^\bullet - \overline{X}^\bullet\big)(\omega)\big\|_{[0,T],\widehat{w}^{\bullet,n},p'}\Big)\!\!\Big)_8 \\ &\leqslant {}^{(n)}\Big(\!\!\Big(\big[c^2\left(1 + \zeta_T^{\bullet,n}(\omega)\right)\big]^{\widehat{N}_T^{\bullet,n}(\omega)+1}\zeta_T^{\bullet,n}(\omega)\Big)\!\!\Big)_8 \\ &\qquad \times {}^{(n)}\Big(\!\!\Big(\big\|\big(X^\bullet - \overline{X}^\bullet\big)(\omega)\big\|_{[0,T],\widehat{w}^{\bullet,n},p'}\Big)\!\!\Big)_8 \\ &\quad + {}^{(n)}\Big(\!\!\Big(\big[c^2\left(1 + \zeta_T^{\bullet,n}(\omega)\right)\big]^{\widehat{N}_T^{\bullet,n}(\omega)+1}\theta^{\bullet,n}(\omega)\Big)\!\!\Big)_8. \end{aligned} \tag{4.23}$$

Step 3. The key quantity of interest in (4.23) is the multiplicative factor in the second line, which we denote by

$$\Psi_T^n(\omega) := {}^{(n)}\Big(\!\!\Big(\big[c^2\left(1 + \zeta_T^{\bullet,n}(\omega)\right)\big]^{\widehat{N}^{\bullet,n}(\omega)+1}\zeta_T^{\bullet,n}(\omega)\Big)\!\!\Big)_8.$$

In particular, letting

$$\Theta_T^n(\omega) := {}^{(n)}\Big(\!\!\Big(\big[c^2\left(1 + \zeta_T^{\bullet,n}(\omega)\right)\big]^{\widehat{N}^{\bullet,n}(\omega)+1}\theta^{\bullet,n}(\omega)\Big)\!\!\Big)_8,$$

we deduce from (4.23) that

$$
\begin{aligned}
&{}^{(n)}\Big[\!\Big[\, \big|\!\big|\!\big| \big(X^{\bullet} - \overline{X}^{\bullet}\big)(\omega) \big|\!\big|\!\big|_{[0,T],\widehat{w}^{\bullet,n},p'} \Big]\!\Big]_{8} \\
&\qquad \leqslant \Psi_T^n(\omega)\, {}^{(n)}\Big[\!\Big[\, \big|\!\big|\!\big| \big(X^{\bullet} - \overline{X}^{\bullet}\big)(\omega) \big|\!\big|\!\big|_{[0,T],\widehat{w}^{\bullet,n},p'} \Big]\!\Big]_{8} + \Theta_T^n(\omega).
\end{aligned}
\tag{4.24}
$$

Here comes the key point. The variable ω being frozen, we can choose T small enough, depending on ω, and L large enough, deterministically, such that $\Psi_T^n(\omega) \leqslant 1/2$ and (4.20) holds true. The proof is made clear below. Take it for granted for a while and deduce that

$$
{}^{(n)}\Big[\!\Big[\, \big|\!\big|\!\big| \big(X^{\bullet} - \overline{X}^{\bullet}\big)(\omega) \big|\!\big|\!\big|_{[0,T],\widehat{w}^{\bullet,n},p'} \Big]\!\Big]_{8} \leqslant c'\Theta^n(\omega),
$$

for a new constant c'. The above inequality sounds really close to the desired result, except for the fact that it is on a small interval $[0,T]$ only. The purpose is thus to iterate it in order to cover any given time interval.

Step 4. In order to iterate in a proper way, we change our notation: While we keep T for the deterministic time horizon given in the statement, we use the latter τ instead of T in the previous analysis. Put differently, τ will stand for the (random) time horizon such that Ψ_τ is small enough. More precisely, we consider a random dissection $0 = \tau_0 < \tau_1 < \cdots < \tau_\Lambda = T$ of the interval $[0,T]$ by Λ subintervals.

We need to go back to the proof of Proposition 15. Assume indeed that we have an estimate for

$$
\mathcal{E}_{\tau_\ell}^{i,n}(\omega) := \Big(1 + \widehat{w}^{i,n}(0,T,\omega)^{1/p'}\Big)\, \big|\!\big|\!\big| \big(X^{i} - \overline{X}^{i}\big)(\omega) \big|\!\big|\!\big|_{[0,\tau_\ell],\widehat{w}^{i,n},p'},
$$

for some $\ell \leqslant \Lambda$. Then, in order to duplicate the second step, we must consider a new dissection $\tau_\ell = t_0 < t_1 < \cdots < t_K = \tau_{\ell+1}$ of the interval $[\tau_\ell, \tau_{\ell+1}]$ with the property that $K = \widehat{N}^{i,n}\big([\tau_\ell,\tau_{\ell+1}],\omega,1/(4L)\big) + 1$ and that $\widehat{w}^{i,n}(t_k,t_{k+1},\omega) = 1/(4L)$ if $t_k < K$. The key point is to apply a relevant version of (3.16), but with τ_ℓ instead of 0 as initial time. This requires a modicum of care as $X^i(\omega)$ and $\overline{X}^i(\omega)$ do not coincide at time τ_ℓ. We let the reader adapt the proof accordingly and check that the following holds true

$$
\begin{aligned}
&\Big|\!\Big|\!\Big| \int_{t_k}^{\cdot} \mathrm{F}\big(X_r^i(\omega),\mu_r^n(\omega)\big)\, d\boldsymbol{W}_r^i(\omega) - \int_{t_k}^{\cdot} \mathrm{F}\big(\overline{X}_r^i(\omega),\overline{\mu}_r^n(\omega)\big)\, d\boldsymbol{W}_r^i(\omega) \Big|\!\Big|\!\Big|_{[t_k,t_{k+1}],\widehat{w}^{i,n},p'} \\
&\leqslant \gamma\, \widehat{w}^{i,n}(\tau_\ell,\tau_{\ell+1},\omega)^{1/p'} \Big\{ \big|\!\big|\!\big| \big(X^{i} - \overline{X}^{i}\big)(\omega) \big|\!\big|\!\big|_{[\tau_\ell,t_k],\widehat{w}^{i,n},p'} \\
&\qquad\qquad + {}^{(n)}\Big[\!\Big[\, \big|\!\big|\!\big| \big(X^{\bullet} - \overline{X}^{\bullet}\big)(\omega) \big|\!\big|\!\big|_{[\tau_\ell,\tau_{\ell+1}],\widehat{w}^{\bullet,n},p'} \Big]\!\Big]_{8} \Big\} \\
&\quad + \frac{\gamma}{4L} \wedge \widehat{w}^{i,n}(\tau_\ell,\tau_{\ell+1},\omega)^{1/p'} \Big\{ \big|\!\big|\!\big| \big(X^{i} - \overline{X}^{i}\big)(\omega) \big|\!\big|\!\big|_{[t_k,t_{k+1}],\widehat{w}^{i,n},p'} \\
&\qquad\qquad + {}^{(n)}\Big[\!\Big[\, \big|\!\big|\!\big| \big(X^{\bullet} - \overline{X}^{\bullet}\big)(\omega) \big|\!\big|\!\big|_{[t_k,t_{k+1}],\widehat{w}^{\bullet,n},p'} \Big]\!\Big]_{8} \Big\} \\
&\quad + \gamma \Big[\mathcal{E}_{\tau_\ell}^{i,n}(\omega) + {}^{(n)}\big[\!\big[\mathcal{E}_{\tau_\ell}^{\bullet,n}(\omega) \big]\!\big]_{8} \Big],
\end{aligned}
$$

provided the analogue of (4.20) holds true, namely

$$
{}^{(n)}\Big[\!\Big[\, \widehat{N}^{\bullet,n}\big([\tau_\ell,\tau_{\ell+1}],\omega,1/(4L_0)\big) \Big]\!\Big]_{8} \leqslant c.
$$

Then, proceeding as in the second step,

$$\begin{aligned}
&\big\| (X^i - \overline{X}^i)(\omega) \big\|_{[t_k,t_{k+1}],\widehat{w}^{i,n},p'} \\
&\leqslant c\, \widehat{w}^{i,n}(\tau_\ell,\tau_{\ell+1},\omega)^{1/p'} \Big\{ \big\| (X^i - \overline{X}^i)(\omega) \big\|_{[\tau_\ell,t_k],\widehat{w}^{i,n},p'} \\
&\qquad\qquad + {}^{(n)}\Big[\!\!\Big[\big\| (X^\bullet - \overline{X}^\bullet)(\omega) \big\|_{[\tau_\ell,\tau_{\ell+1}],\widehat{w}^{\bullet,n},p'} \Big]\!\!\Big]_8 \Big\} \\
&\quad + \gamma \Big\{ \mathcal{E}^{i,n}_{\tau_\ell}(\omega) + {}^{(n)}[\![\mathcal{E}^{\bullet,n}_{\tau_\ell}(\omega)]\!]_8 + \theta^{i,n}(\omega) \Big\}.
\end{aligned}$$

In the end, we are in the same situation as in (4.21), but with new $\zeta^{i,n}_T$ and $\widehat{N}^{i,n}_T$. Here, we let

$$\begin{aligned}
\zeta^{i,n}_\ell(\omega) &:= \widehat{w}^{i,n}(\tau_\ell,\tau_{\ell+1},\omega)^{1/p'}, \\
\widehat{N}^{i,n}_\ell(\omega) &:= \widehat{N}^{i,n}\left([\tau_\ell,\tau_{\ell+1}], \omega, \frac{1}{4L} \right).
\end{aligned}$$

Following (4.22), we obtain

$$\begin{aligned}
&\big\| (X^i - \overline{X}^i)(\omega) \big\|_{[\tau_\ell,\tau_{\ell+1}],\widehat{w}^{i,n},p'} \\
&\leqslant c \big[c\big(1+\zeta^{i,n}_\ell(\omega)\big) \big]^{\widehat{N}^{i,n}_\ell(\omega)+1} \\
&\quad \times \Big\{ \zeta^{i,n}_\ell(\omega)\, {}^{(n)}\Big[\!\!\Big[\big\| (X^\bullet - \overline{X}^\bullet)(\omega) \big\|_{[\tau_\ell,\tau_{\ell+1}],\widehat{w}^{\bullet,n},p'} \Big]\!\!\Big]_8 \\
&\qquad\qquad + \theta^{i,n}(\omega) + \mathcal{E}^{i,n}_{\tau_\ell} + {}^{(n)}[\![\mathcal{E}^{\bullet,n}_{\tau_\ell}(\omega)]\!]_8 \Big\}.
\end{aligned} \tag{4.25}$$

Hence,

$$\begin{aligned}
&{}^{(n)}\Big[\!\!\Big[\big\| (X^\bullet - \overline{X}^\bullet)(\omega) \big\|_{[\tau_\ell,\tau_{\ell+1}],\widehat{w}^{\bullet,n},p'} \Big]\!\!\Big]_8 \\
&\qquad \leqslant \Psi^n_\ell(\omega) \times {}^{(n)}\Big[\!\!\Big[\big\| (X^\bullet - \overline{X}^\bullet)(\omega) \big\|_{[\tau_\ell,\tau_{\ell+1}],\widehat{w}^{\bullet,n},p'} \Big]\!\!\Big]_8 + \Theta^n_\ell(\omega),
\end{aligned}$$

with

$$\begin{aligned}
\Psi^n_\ell(\omega) &:= {}^{(n)}\Big[\!\!\Big[\big[c^2 \left(1+\zeta^{\bullet,n}_\ell(\omega)\right) \big]^{\widehat{N}^{\bullet,n}_\ell(\omega)+1} \zeta^{\bullet,n}_\ell(\omega) \Big]\!\!\Big]_8, \\
\Theta^n_\ell(\omega) &:= {}^{(n)}\Big[\!\!\Big[\big[c^2 \left(1+\zeta^{\bullet,n}_\ell(\omega)\right) \big]^{\widehat{N}^{\bullet,n}_\ell(\omega)+1} \Big(\theta^{\bullet,n}(\omega) + \mathcal{E}^{\bullet,n}_{\tau_\ell}(\omega) + {}^{(n)}[\![\mathcal{E}^{\bullet,n}_{\tau_\ell}(\omega)]\!]_8 \Big) \Big]\!\!\Big]_8.
\end{aligned}$$

If we can choose $\tau_{\ell+1} - \tau_\ell$ such that $\Psi^n_\ell(\omega) \leqslant 1/2$, then we get

$${}^{(n)}\Big[\!\!\Big[\big\| (X^\bullet - \overline{X}^\bullet)(\omega) \big\|_{[\tau_\ell,\tau_{\ell+1}],\widehat{w}^{\bullet,n},p'} \Big]\!\!\Big]_8 \leqslant 2\,\Theta^n_\ell(\omega).$$

Eventually, returning to (4.25) and modifying the value of the constant c, we deduce

$$\begin{aligned}
&\big\| (X^i - \overline{X}^i)(\omega) \big\|_{[\tau_\ell,\tau_{\ell+1}],\widehat{w}^{i,n},p'} \\
&\quad \leqslant c \big[c\big(1+\zeta^{i,n}_\ell(\omega)\big) \big]^{\widehat{N}^{i,n}_\ell(\omega)+1} \Big(\zeta^{i,n}_\ell(\omega)\, \Theta^n_\ell(\omega) + \theta^{i,n}(\omega) + \mathcal{E}^{i,n}_{\tau_\ell} + {}^{(n)}[\![\mathcal{E}^{\bullet,n}_{\tau_\ell}(\omega)]\!]_8 \Big),
\end{aligned}$$

and then

$$\mathcal{E}^{i,n}_{\tau_{\ell+1}}(\omega) \leqslant \kappa^{i,n}_\ell(\omega) \Big(\zeta^{i,n}_\ell(\omega)\, \Theta^n_\ell(\omega) + \theta^{i,n}(\omega) + \mathcal{E}^{i,n}_{\tau_\ell}(\omega) + {}^{(n)}[\![\mathcal{E}^{\bullet,n}_{\tau_\ell}(\omega)]\!]_8 \Big),$$

with (using the fact that $c > 1$)

$$\kappa_\ell^{i,n}(\omega) := c^2\left(1 + \widehat{w}^{i,n}(0,T,\omega)^{1/p'}\right)\left[c^2\big(1+\zeta_\ell^{i,n}(\omega)\big)\right]^{\widehat{N}_\ell^{i,n}(\omega)+1}.$$

This yields to the following global bound:

$$\mathcal{E}^{i,n}_{\tau_{\ell+1}}(\omega) \leqslant \sum_{k=0}^{\ell} \mathcal{K}^{i,n}_{k,\ell}(\omega)\,\kappa_k^{i,n}(\omega)\left[\zeta_k^{i,n}(\omega)\Theta_k^n(\omega) + \theta^{i,n}(\omega) + {}^{(n)}\!\big[\!\big[\mathcal{E}^{\bullet,n}_{\tau_k}(\omega)\big]\!\big]_8\right], \qquad (4.26)$$

with

$$\mathcal{K}^{i,n}_{k,\ell}(\omega) := \prod_{j=k}^{\ell-1}\kappa_j^{i,n}(\omega), \quad \mathcal{K}^{i,n}_{\ell,\ell}(\omega) = 1.$$

Observe that

$$\begin{aligned}\mathcal{K}^{i,n}_{k,\ell}(\omega) &\leqslant c^{2(\ell-k)}\prod_{j=k}^{\ell-1}\big(1+\widehat{w}^{i,n}(0,T,\omega)^{1/p'}\big)\big[c^2\big(1+\zeta_j^{i,n}(\omega)\big)\big]^{\widehat{N}_j^{i,n}(\omega)}\\ &\leqslant c^{2(\ell-k)+2\widehat{N}^{i,n}_{k,\ell}(\omega)}\big(1+\widehat{w}^{i,n}(0,T,\omega)^{1/p'}\big)^{\ell-k+\widehat{N}^{i,n}_{k,\ell}(\omega)},\end{aligned}$$

with the shortened notation $\widehat{N}^{i,n}_{k,\ell}(\omega) := \widehat{N}^{i,n}\big([\tau_k,\tau_\ell],\omega,1/(4L)\big)$, and that

$$\mathcal{K}^{i,n}_{k,\ell}(\omega)\kappa_k^{i,n}(\omega) \leqslant c^{2(\ell+1-k)+4\widehat{N}^{i,n}_{k,\ell}(\omega)}\left(1+\widehat{w}^{i,n}(0,T,\omega)^{1/p'}\right)^{\ell+1-k+2\widehat{N}^{i,n}_{k,\ell}(\omega)}.$$

From (4.26), we deduce that for any $r > 8$, we can find a constant $q(r)$ such that

$$\begin{aligned}{}^{(n)}\!\big[\!\big[\mathcal{E}^{\bullet,n}_{\tau_{\ell+1}}(\omega)\big]\!\big]_r \leqslant \sum_{k=0}^{\ell}\Big\{&{}^{(n)}\!\big[\!\big[\mathcal{K}^{\bullet,n}_{k,\ell}\,\kappa_k^{\bullet,n}\big]\!\big]_{q(r)} \times \left(1 + {}^{(n)}\!\big[\!\big[\widehat{w}^{\bullet,n}(0,T,\omega)^{1/p'}\big]\!\big]_{q(r)}\right)\\ &\times\left(1 + {}^{(n)}\!\left[\!\left[\left[c^2\left(1+\zeta_k^{\bullet,n}(\omega)\right)\right]^{\widehat{N}_k^{\bullet,n}(\omega)+1}\right]\!\right]_{q(r)}\right)\\ &\times\left({}^{(n)}\!\big[\!\big[\theta^{\bullet,n}(\omega)\big]\!\big]_{q(r)} + {}^{(n)}\!\big[\!\big[\mathcal{E}^{\bullet,n}_{\tau_k}(\omega)\big]\!\big]_r\right)\Big\}.\end{aligned}$$

which we rewrite in the form

$$a_{\ell+1} \leqslant \sum_{k=0}^{\ell} g_{k,\ell}\big(b + a_k\big),$$

with

$$a_\ell := {}^{(n)}\!\big[\!\big[\mathcal{E}^{\bullet,n}_{\tau_\ell}(\omega)\big]\!\big]_r, \quad g_{k,\ell} := 4\times\left({}^{(n)}\!\big[\!\big[\mathcal{K}^{\bullet,n}_{k,\ell}\,\kappa_k^{\bullet,n}\big]\!\big]_{q(r)}\right)^3, \quad b := {}^{(n)}\!\big[\!\big[\theta^{\bullet,n}(\omega)\big]\!\big]_{q(r)}.$$

Hence,

$$a_\ell \leqslant b\sum_{j=1}^{\ell}\ \sum_{0\leqslant k_1\leqslant\cdots\leqslant k_j\leqslant k_{j+1}=\ell}\ \prod_{i=1}^{j} g_{k_i,k_{i+1}}.$$

Now, we can find $q'(r) \geqslant 1$ such that, for $1 \leqslant i \leqslant j \leqslant \ell \leqslant \Lambda$, where Λ is the number of subintervals in the dissection $0 = \tau_0 < \tau_1 < \cdots < \tau_\Lambda = T$ of $[0,T]$,

$$\begin{aligned}&{}^{(n)}\!\big[\!\big[\mathcal{K}^{\bullet,n}_{k_i,k_{i+1}}\,\kappa_{k_i}^{\bullet,n}\big]\!\big]_{q(r)}\\ &\leqslant {}^{(n)}\!\left[\!\left[\left(c^2\big(1+\widehat{w}^{\bullet,n}(0,T,\omega)^{1/p'}\big)\right)^{\Lambda+1}\right]\!\right]_{q'(r)}^{(k_{i+1}-k_i+1)/(\Lambda+1)}\\ &\quad\times{}^{(n)}\!\left[\!\left[\left(c^2\big(1+\widehat{w}^{\bullet,n}(0,T,\omega)^{1/p'}\big)\right)^{2T\widehat{N}^{\bullet,n}_{k_i,k_{i+1}}/(\tau_{k_{i+1}}-\tau_{k_i})}\right]\!\right]_{q'(r)}^{(\tau_{k_{i+1}}-\tau_{k_i})/T}.\end{aligned}$$

Hence, by Young's inequality

$$\prod_{i=1}^{j} g_{k_i,k_{i+1}} \leqslant {}^{(n)}\Big[\!\!\Big[\Big((2c)^2\big(1+\widehat{w}^{\bullet,n}(0,T,\omega)^{1/p'}\big)\Big)^{\Lambda+1} \Big]\!\!\Big]^{6}_{q'(r)} \\ \times \sum_{i=1}^{j} {}^{(n)}\Big[\!\!\Big[\Big(c\big(1+\widehat{w}^{\bullet,n}(0,T,\omega)^{1/p'}\big)\Big)^{2T\widehat{N}^{\bullet,n}_{k_i,k_{i+1}}/(\tau_{k_{i+1}}-\tau_{k_i})} \Big]\!\!\Big]^{3}_{q'(r)}.$$

Finally,

$$a_\Lambda \leqslant {}^{(n)}\big[\!\big[\theta^{\bullet,n}(\omega)\big]\!\big]_{q(r)} \times \big(2\Lambda\big)^{\Lambda+1} \times {}^{(n)}\Big[\!\!\Big[\Big((2c)^2\big(1+\widehat{w}^{\bullet,n}(0,T,\omega)^{1/p'}\big)\Big)^{\Lambda+1} \Big]\!\!\Big]^{6}_{q'(r)} \\ \times \sum_{i=1}^{\Lambda} {}^{(n)}\Big[\!\!\Big[\Big(c\big(1+\widehat{w}^{\bullet,n}(0,T,\omega)^{1/p'}\big)\Big)^{2T\widehat{N}^{\bullet,n}_{k_i,k_{i+1}}/(\tau_{k_{i+1}}-\tau_{k_i})} \Big]\!\!\Big]^{3}_{q'(r)}.$$

Step 5. Repeating (4.11) and (4.12), we can find a real $\varepsilon_1 > 0$, independent of n, such that $\sup_{i=1,\cdots,n} \mathbb{E}\big[\exp\big(\widehat{w}^{i,n}(0,T,\cdot)^{\varepsilon_1}\big)\big] \leqslant C$, for a constant C independent of n. Hence, following (3.22) in the proof of Theorem 16, we deduce that, for any $\varrho > 0$,

$$\mathbb{E}\bigg[\Big[\!\!\Big[\Big(c\big(1+\widehat{w}^{\bullet,n}(0,T,\omega)^{1/p'}\big)\Big)^{2T\widehat{N}^{\bullet,n}_{k_p,k_{p+1}}/(\tau_{k_{p+1}}-\tau_{k_p})} \Big]\!\!\Big]^{\varrho}_{q'(r)} \bigg] \leqslant C_{\varrho,r},$$

for some constant $C_{\varrho,r}$ only depending on r and ϱ, the value of which is allowed to increase from line to line.

• We prove below that the number Λ of subintervals in the dissection $0 = \tau_0 < \tau_1 < \cdots < \tau_\Lambda = T$ of $[0,T]$ has Weibull tails with shape parameter $\lambda > 1/2$, with λ independent of n and the Weibull tails uniformly controlled in $n \geqslant 1$. Hence, following (3.22) in the proof of Theorem 16, we deduce that, for any $\varrho > 0$,

$$\mathbb{E}\left[\left(\sum_{i=1}^{\Lambda} \Big[\!\!\Big[\Big(c\big(1+\widehat{w}^{\bullet,n}(0,T,\omega)^{1/p'}\big)\Big)^{2T\widehat{N}^{\bullet,n}_{k_i,k_{i+1}}/(\tau_{k_{i+1}}-\tau_{k_i})} \Big]\!\!\Big]^{3}_{q'(r)} \right)^{\varrho} \right] \\ \leqslant C^{1/2}_{2\varrho,r} \sum_{J\geqslant 0} \mathbb{P}\big(\Lambda \geqslant J\big)^{1/2} J^{\varrho-1} \leqslant C_{\varrho,r}.$$

Similarly,

$$\mathbb{E}\bigg[{}^{(n)}\Big[\!\!\Big[\Big((2c)^2\big(1+\widehat{w}^{\bullet,n}(0,T,\omega)^{1/p'}\big)\Big)^{\Lambda+1} \Big]\!\!\Big]^{\varrho}_{q'(r)} \bigg] \leqslant C_{\varrho,r}.$$

Returning to the conclusion of the fourth step and observing that Λ^Λ has finite moments of any order since Λ has Weibull tails with shape parameter $\lambda > 1/2$, we deduce that, for a possibly new value of $q(r)$,

$$\big\langle a_\Lambda(\cdot)\big\rangle_r \leqslant C_r \big\langle \theta^{1,n}(\cdot)\big\rangle_{q(r)},$$

where C_r only depends on r. It remains to observe that $a_\Lambda(\omega)$ is equal to

$$a_\Lambda(\omega) = {}^{(n)}\big[\!\big[\mathcal{E}^{\bullet,n}_T(\omega)\big]\!\big]_r,$$

which is less than η_n by the conclusion of the first step. Inserting into (4.26), we easily complete the proof.

- We now justify the fact that Λ has Weibull tails. We use the following bound

$$\begin{aligned}
\Psi_\ell^n(\omega) &\leqslant {}^{(n)}\Big(\!\!\Big(\big[c^2\left(1+\zeta_\ell^{\bullet,n}(\omega)\right)\big]^{\widehat{N}_\ell^{\bullet,n}(\omega)+1} \Big)\!\!\Big)_{16} \, {}^{(n)}\Big(\!\!\Big(\zeta_\ell^{\bullet,n}(\omega) \Big)\!\!\Big)_{16} \\
&\leqslant c^2 \, {}^{(n)}\Big(\!\!\Big(\big[c^2\left(1+\widehat{w}^{\bullet,n}(0,T,\omega)^{1/p'}\right)\big]^{\widehat{N}_\ell^{\bullet,n}(\omega)} \Big)\!\!\Big)_{32} \\
&\quad \times \Big({}^{(n)}\Big(\!\!\Big(\widehat{w}^{\bullet,n}(\tau_\ell,\tau_{\ell+1},\omega)^{1/p'} \Big)\!\!\Big)_{32} + {}^{(n)}\Big(\!\!\Big(\widehat{w}^{\bullet,n}(\tau_\ell,\tau_{\ell+1},\omega)^{1/p'} \Big)\!\!\Big)_{32}^2 \Big).
\end{aligned}$$

For sure, this shows that we can choose $\delta_\ell(\omega) := \tau_{\ell+1}(\omega) - \tau_\ell(\omega)$ small enough such that

$$\Psi_\ell^n(\omega) \leqslant 2.$$

Moreover, by Hölder inequality, we obtain, for any $a > 0$

$$\begin{aligned}
\Psi_\ell^n &\leqslant c^2 \, {}^{(n)}\Big(\!\!\Big(\big[c^2\left(1+\widehat{w}^{\bullet,n}(0,T,\omega)^{1/p'}\right)\big]^{\widehat{N}_\ell^{\bullet,n}(\omega)/\delta_\ell} \Big)\!\!\Big)_{32}^{\delta_\ell} \\
&\quad \times \Big({}^{(n)}\Big(\!\!\Big(\widehat{w}^{\bullet,n}(\tau_\ell,\tau_{\ell+1},\omega)^{1/p'} \Big)\!\!\Big)_{32} + {}^{(n)}\Big(\!\!\Big(\widehat{w}^{\bullet,n}(\tau_\ell,\tau_{\ell+1},\omega)^{1/p'} \Big)\!\!\Big)_{32}^2 \Big) \\
&\leqslant a\, c^4 \, {}^{(n)}\Big(\!\!\Big(\big[c^2\left(1+\widehat{w}^{\bullet,n}(0,T,\omega)^{1/p'}\right)\big]^{\widehat{N}_\ell^{\bullet,n}(\omega)/\delta_\ell} \Big)\!\!\Big)_{32}^{2\delta_\ell} \\
&\quad + \frac{1}{a} \Big({}^{(n)}\Big(\!\!\Big(\widehat{w}^{\bullet,n}(\tau_\ell,\tau_{\ell+1},\omega)^{1/p'} \Big)\!\!\Big)_{32} + {}^{(n)}\Big(\!\!\Big(\widehat{w}^{\bullet,n}(\tau_\ell,\tau_{\ell+1},\omega)^{1/p'} \Big)\!\!\Big)_{32}^2 \Big)^2.
\end{aligned}$$

Call for a while Λ the ω-dependent minimal number of steps such that $\Psi_\Lambda(\omega) \geqslant 2$. By (A.1) in appendix, we can prove that Λ has Weibull tails if the local accumulation associated with each of the two terms above have also Weibull tails, with the same shape parameter. As for the term on the last line, this is precisely the assumption we have (whatever the value of a), see the beginning of this step and assume that $q \geqslant 32$ in (4.3). It thus remains to handle the local accumulation of the term in the penultimate line. So, we can regard δ_ℓ as if Ψ_ℓ was exactly equal to the term in the penultimate line. We then observe that, for $ac^4 < 1$ and $A > 0$,

$$\begin{aligned}
\mathbb{P}\left(\delta_\ell \leqslant \frac{1}{A}\right) &\leqslant \mathbb{P}\left(\left(ac^4\right)^{1/A} \, {}^{(n)}\Big(\!\!\Big(\big[c^2\left(1+\widehat{w}^{\bullet,n}(0,T,\omega)^{1/p'}\right)\big]^{\widehat{N}_\ell^{\bullet,n}(\omega)/\delta_\ell} \Big)\!\!\Big)_{32}^{1/A} \geqslant 1 \right) \\
&= \mathbb{P}\left({}^{(n)}\Big(\!\!\Big(\big[c^2\left(1+\widehat{w}^{\bullet,n}(0,T,\omega)^{1/p'}\right)\big]^{\widehat{N}_\ell^{\bullet,n}(\omega)/\delta_\ell} \Big)\!\!\Big)_{32} \geqslant (ac^4)^{-A} \right).
\end{aligned}$$

We now introduce the function

$$f(x) = \exp\big(\ln(x)^{1+\varepsilon}\big), \quad x > 1;$$

it is non-decreasing on $[1,\infty)$ and convex on $[e,\infty)$. By Markov inequality, for $c > 1$,

$$\begin{aligned}
&\mathbb{P}\left(\delta_\ell \leqslant \frac{1}{A}\right) \\
&\leqslant \exp\left(-\big(\ln[(ac^4)^{-32A}]\big)^{1+\varepsilon_2'}\right) \mathbb{E}\left[f\left(\frac{1}{n}\sum_{i=1}^n e\Big[c^2\Big(1+\widehat{w}^{i,n}(0,T,\cdot)^{1/p'}\Big)\Big]^{32\widehat{N}_\ell^{i,n}/\delta_\ell} \right) \right] \\
&\leqslant \exp\left(-\big(\ln[(ac^4)^{-32A}]\big)^{1+\varepsilon_2'}\right) \frac{1}{n}\sum_{i=1}^n \mathbb{E}\left[f\left(e\Big[c^2\Big(1+\widehat{w}^{i,n}(0,T,\cdot)^{1/p'}\Big)\Big]^{32\widehat{N}_\ell^{i,n}/\delta_\ell} \right) \right],
\end{aligned}$$

where $1+\varepsilon_2' < (1+\varepsilon_2)/(1+\varepsilon_2/2)$, where ε_2 is such that that $\widehat{N}_\ell^{i,n}(\cdot)/\delta_\ell(\cdot)$ has Weibull tails with shape parameter $1/2(1+\varepsilon_2)$, uniformly in $n, \ell \geqslant 1$, and $\varepsilon = \varepsilon_2'$ in the definition of f. Therefore, following (3.22) again,

$$\mathbb{P}\left(\delta_\ell \leqslant \frac{1}{A}\right) \leqslant C \exp\Big(-\big(-32\ln(ac^4)\big)^{1+\varepsilon_2'} A^{1+\varepsilon_2'}\Big).$$

Now,

$$\begin{aligned}\mathbb{P}\big(\Lambda > \ell\big) &\leqslant \mathbb{P}\big(\delta_1 + \cdots + \delta_\ell \leqslant T\big) \\ &\leqslant \sum_{i=1}^{\ell} \mathbb{P}\left(\delta_i \leqslant \frac{T}{\ell}\right) \\ &\leqslant C\ell \exp\Big(- \big(-32\ln(ac^4)\big)^{1+\varepsilon_2'} (\ell/T)^{1+\varepsilon_2'}\Big),\end{aligned}$$

which shows that Λ has a Weibull tail.

In fact, Λ needs also to take into account the condition

$$^{(n)}\Big[\!\!\Big[\, \widehat{N}^{\bullet,n}\big([\tau_\ell, \tau_{\ell+1}], \omega, 1/(4L_0)\big) \,\Big]\!\!\Big]_8 \leqslant c.$$

Using again the lower bound (A.1), we can assume that Λ only counts the number of ℓ for which the above inequality is actually an equality. Then, we can repeat the same proof as above by using the fact that

$$c = {}^{(n)}\Big[\!\!\Big[\, \frac{\widehat{N}^{\bullet,n}\big([\tau_\ell, \tau_{\ell+1}], \omega, 1/(4L_0)\big)}{\delta_\ell} \,\Big]\!\!\Big]_8 \delta_\ell$$

and by recalling that

$$^{(n)}\Big[\!\!\Big[\, \frac{\widehat{N}^{\bullet,n}\big([\tau_\ell, \tau_{\ell+1}], \omega, 1/(4L_0)\big)}{\delta_\ell} \,\Big]\!\!\Big]_8$$

has Weibull tails with shape parameter strictly greater than 1/2, which follows from the convexity of the function $[0, +\infty) \ni x \mapsto \exp(x^{1+\varepsilon})$, for $\varepsilon > 0$. This permits to provide an upper bound for $\mathbb{P}(\delta_\ell \leqslant 1/A)$. ▷

A – Integrability and Auxiliary Estimates

We prove in this appendix a number of auxiliary results that we left aside to keep focused on the main problems at hand. Thus we prove in Appendix A.1 the version of Cass, Litterer and Lyons' integrability estimate on the accumulated local variation of a rough path under the form needed here, Theorem 6. In Appendix A.2, we provide a proof of Theorem 23 showing propagation of chaos for an interacting particle system driven by Gaussian rough paths. Appendix A.3 is dedicated to proving a crucial moment estimate for some quantity of interest in the proof of the convergence rate in the propagation of chaos result, Theorem 24. This is where the convergence rate η_n appears.

A.1 – Proof of Theorem 6

We provide here the proof of Theorem 6; this statement allows to use our well-posedness result for the mean field rough differential equation (0.4) when W is some Gaussian or Markovian rough path. We follow the proof of Theorem 11.7 in [25]. Throughout the proof, we use the same notations as in the statement of Theorem 6. The following statement provides the required analogue of Proposition 6.2 in [11].

Recall that $v(s,t,\omega)$ in (1.7) consists in six different terms. It is an easy exercice to check that it suffices to control the local accumulation associated with each of this six terms. To make it clear, we have the following property. If, for a given threshold $\alpha > 0$ and for any two continuous functions $v_1 : \mathcal{S}_2^T \to \mathbb{R}_+$ and $v_2 : \mathcal{S}_2^T \to \mathbb{R}_+$, set

$$N_i(\alpha) := N_{v_i}\big([0,T],\alpha\big),$$

for $1 \leqslant i \leqslant 2$; see (1.13) for the original definition. Then

$$\max\Big(N_1\Big(\frac{\alpha}{2}\Big), N_2\Big(\frac{\alpha}{2}\Big)\Big) \geqslant \frac{N(\alpha)}{2}. \tag{A.1}$$

For sure, the result is true with the first and third terms in (1.7) as this fits the original property established in [11]. Also, it is obviously true for the second and sixth terms since they are completely deterministic. Hence, the only difficulty is to control the local accumulation associated with the fourth and fifth terms.

The strategy is as follows. As we work with Gaussian rough paths, the set-up, as defined in Section 1, is strong. So, we can transfer it to any arbitrarily fixed probability space (provided that the letter is rich enough). Hence, we can choose Ω as the canonical path space $\mathcal{W}$, see the notation used in the statement of Theorem 6.

We denote by $\boldsymbol{W}(\omega,\omega')$ the enhanced Gaussian rough path associated to

$$\big(W(\omega), W'(\omega')\big)$$

along the lines of Example 5, for $\mathbb{P}^{\otimes 2}$-almost every $(\omega,\omega') \in \Omega^2$. To make it clear, the second level of $\boldsymbol{W}(\omega,\omega')$ reads

$$\boldsymbol{W}^{[2]}(\omega,\omega') := \begin{pmatrix} \mathbb{W}(\omega) & \mathcal{I}\big(W(\omega), W'(\omega')\big) \\ \mathcal{I}\big(W'(\omega'), W(\omega)\big) & \mathbb{W}(\omega') \end{pmatrix},$$

where $\mathcal{I}$ is as in Definition 18, and where we used the same symbol $\boldsymbol{W}$ as in Section 1 for the enhanced path although the meaning here is not exactly the same. Here, $\boldsymbol{W}(\omega,\omega')$ is a function of both ω and ω' and takes values in $\mathbb{R}^{2m}\oplus(\mathbb{R}^{2m})^{\otimes 2}$. Following Section 3 in [11], see also (11.5) in [25], we define, for $h \oplus k \in \mathcal{H} \oplus \mathcal{H}$ the translated rough path $(T_{h\oplus k}\boldsymbol{W})(\omega,\omega')$. We then recall that, with probability 1 under $\mathbb{P}^{\otimes 2}$,

$$T_{h\oplus k}\boldsymbol{W}(\omega,\omega') = \boldsymbol{W}(\omega+h,\omega'+k).$$

Following the argument given in Proposition 6.2 in [11], see also Theorem 11.4 in [25], we have, for any $h \in \mathcal{H}$ and any $(s,t) \in \mathcal{S}_2^T$,

$$[\![\boldsymbol{W}(\omega,\omega')]\!]^p_{[s,t],p-\mathrm{var}} \leqslant c\Big([\![T_{h\oplus 0}\boldsymbol{W}(\omega,\omega')\ ]\!]^p_{[s,t],p-\mathrm{var}} + \|h\|^p_{[s,t],\varrho-\mathrm{var}}\Big),$$

where we recall that $1/p + 1/\varrho > 1$ and c only depends on p and ϱ, and where

$$[\![\boldsymbol{W}(\omega,\omega')]\!]_{[s,t],p-\mathrm{var}} = \|(W,W')(\omega,\omega')\|_{[s,t],p-\mathrm{var}} + \sqrt{\|\boldsymbol{W}^{[2]}(\omega,\omega')\|_{[s,t],(p/2)-\mathrm{var}}},$$

and similarly for $[\![T_{h\oplus 0}\boldsymbol{W}(\omega,\omega')]\!]_{[s,t],p-\mathrm{var}}$. Taking the power q, allowing the constant c to depend on q and integrating with respect to ω', we get

$$\Big\langle \|\mathbb{W}^{\perp\!\!\!\perp}(\omega,\cdot)\|^{p/2}_{[s,t],(p/2)-\mathrm{var}}\Big\rangle_q \leqslant c\Big(\Big\langle \|T_{h\oplus 0}\boldsymbol{W}(\omega,\cdot)\|^p_{[s,t],p-\mathrm{var}}\Big\rangle_q + \|h\|^p_{[s,t],\varrho-\mathrm{var}}\Big),$$

and then

$$\Big\langle \|\mathbb{W}^{\perp\!\!\!\perp}(\omega,\cdot)\|_{[s,t],(p/2)-\mathrm{var}}\Big\rangle_q^{p/2} \leqslant c\Big(\Big\langle \|T_{h\oplus 0}\boldsymbol{W}(\omega,\cdot)\|^p_{[s,t],p-\mathrm{var}}\Big\rangle_q + \|h\|^p_{[s,t],\varrho-\mathrm{var}}\Big).$$

We now recall the notation

$$\begin{aligned}[\![\boldsymbol{W}(\omega,\omega')]\!]_{[s,t],(1/p)-\text{Hölder}} &= \|(W,W')(\omega,\omega')\|_{[s,t],(1/p)-\text{Hölder}} \\ &\quad + \sqrt{\|\boldsymbol{W}^{(2)}(\omega,\omega')\|_{[s,t],(2/p)-\text{Hölder}}},\end{aligned}$$

for the standard Hölder semi-norm of the rough path, see Theorem 11.9 in [25]. Then,

$$\Big\langle \|\mathbb{W}^{\perp\!\!\!\perp}(\omega,\cdot)\|_{[s,t],(p/2)-\mathrm{var}} \Big\rangle_q^{p/2}$$
$$\leqslant c\Big(\Big\langle \|T_{h\oplus 0}\boldsymbol{W}(\omega,\cdot)\|^p_{[0,T],(1/p)-\text{Hölder}} \Big\rangle_q (t-s) + \|h\|^p_{[s,t],\varrho-\mathrm{var}} \Big).$$

Therefore,

$$\Big\langle \mathbb{W}^{\perp\!\!\!\perp}(\omega,\cdot) \Big\rangle_{q;[s,t],(p/2)-\mathrm{var}}^{p/2}$$
$$\leqslant c\Big(\Big\langle \|T_{h\oplus 0}\boldsymbol{W}(\omega,\cdot)\|^p_{[0,T],(1/p)-\text{Hölder}} \Big\rangle_q (t-s) + \|h\|^p_{[s,t],\varrho-\mathrm{var}} \Big).$$

Observe that if the left-hand side is equal to or less than α, we can easily replace $\|h\|^p_{[s,t],\varrho-\mathrm{var}}$ by $\|h\|^\varrho_{[s,t],\varrho-\mathrm{var}}$ up to a modification of the constant c. Since $\varrho \leqslant p$, this is obviously the case when $\|h\|_{[s,t],\varrho-\mathrm{var}} \leqslant 1$. When $\|h\|_{[s,t],\varrho-\mathrm{var}} \geqslant 1$, we can easily modify the constant c in order to preserve the inequality. Define now

$$N([0,T],\omega,\alpha) := N_\varpi([0,T],\alpha),$$

when

$$\varpi(s,t) = \Big\langle \mathbb{W}^{\perp\!\!\!\perp}(\omega,\cdot) \Big\rangle_{q;[s,t],(p/2)-\mathrm{var}}^{p/2}.$$

Then,

$$N([0,T],\omega,\alpha)\alpha \leqslant c\Big(\Big\langle \|T_{h\oplus 0}\boldsymbol{W}(\omega,\cdot)\|^p_{[0,T],(1/p)-\text{Hölder}} \Big\rangle_q T + \|h\|^\varrho_{[0,T],\varrho-\mathrm{var}} \Big).$$

By Proposition 11.2 in [25], we get

$$N([0,T],\omega,\alpha)\alpha \leqslant c\Big(\Big\langle \|T_{h\oplus 0}\boldsymbol{W}(\omega,\cdot)\|^p_{[0,T],(1/p)-\text{Hölder}} \Big\rangle_q T + \|h\|^\varrho_{\mathcal{H}} T \Big),$$

where $\|\cdot\|_{\mathcal{H}}$ is the standard norm on the reproducing Hilbert space $\mathcal{H}$, see again for instance Appendix D in [27]. We then conclude by recalling that the quantity $\langle\!\langle \|\boldsymbol{W}(\cdot,\cdot)\|^p_{[0,T],(1/p)-\text{Hölder}} \rangle\!\rangle_q$ is finite, by observing that the set

$$E := \Big\{ (\omega,\omega') \in \Omega^2 : T_{h\oplus 0}\boldsymbol{W}(\omega,\omega') = \boldsymbol{W}(\omega+h,\omega'), \quad h \in \mathcal{H} \Big\},$$

is of full $\mathbb{P}^{\otimes 2}$-probability measure, see Theorem 11.9 in [25], and then by invoking Theorem 11.7 in [25].

As for the conclusion of the statement (the fact that the tails of $w(0,T)$ satisfy the required decay), it suffices to duplicate the convexity argument used in (4.11) and (4.12).

A.2 – Proof of Theorem 23

Theorem 23 asserts that the assumptions of Theorem 22 ensuring propagation of chaos for the interacting particle system associated with the mean field rough differential equation (0.2) are satisfied in the Gaussian framework specified in its statement. We only prove here that we can control the empirical local accumulation as the other requirements in the statement of Theorem 22 are easily checked. Following the proof of Theorem 6 in Subsection A.1, we may focus on the local accumulation of each of the various terms in (4.3).

Step 1. The first step is to consider the local accumulation $\tilde{N}^i\big([0,T],\omega,\alpha\big)$ associated with $\|W^i(\omega)\|^p_{[s,t],p-\mathrm{var}} + \|\mathbb{W}^i(\omega)\|^{p/2}_{[s,t],p/2-\mathrm{var}}$, namely

$$\tilde{N}^i([0,T],\omega,\alpha) := N_{\varpi}\big([0,T],\alpha\big),$$

when

$$\varpi(s,t) = \|W^i(\omega)\|^p_{[s,t],p-\mathrm{var}} + \|\mathbb{W}^i(\omega)\|^{p/2}_{[s,t],p/2-\mathrm{var}}.$$

We recall from Theorem 6 that each $\tilde{N}^i\big([0,T],\omega,\alpha\big)$ has Weibull tails with $2/\rho$ as shape parameter, uniformly in i, in the sense that there exists $a>0$ such that

$$\sup_{1\leqslant i\leqslant n} \mathbb{E}\Big[\exp\Big(a\big[\tilde{N}^i([0,T],\cdot,\alpha)\big]^{2/\varrho}\Big)\Big] < \infty. \tag{A.2}$$

Then, by the L^4-version of the law of large numbers, which here applies because the variables $\big(\tilde{N}^i([0,T],\cdot,\alpha)\big)_{i=1,\cdots,n}$ are independent, we get

$$\mathbb{P}\bigg(\omega\in\Omega : \frac{1}{n}\sum_{i=1}^n \exp\Big(\frac{a}{4}\big[\tilde{N}^i([0,T],\omega,\alpha)\big]^{2/\varrho}\Big) \geqslant 1 + \mathbb{E}\Big[\exp\Big(\frac{a}{4}\big[\tilde{N}^1([0,T],\cdot,\alpha)\big]^{2/\varrho}\Big)\Big]\bigg)$$
$$\leqslant \frac{C}{n^2},$$

for a constant C independent of n. By Borel-Cantelli Lemma, we then obtain that, with probability 1, there exists a rank n_0 such that, for any $n\geqslant n_0$,

$$\frac{1}{n}\sum_{i=1}^n \exp\Big(\frac{a}{4}\big[\tilde{N}^i([0,T],\omega,\alpha)\big]^{2/\varrho}\Big) \leqslant 1 + \mathbb{E}\Big[\exp\Big(\frac{a}{4}\big[\tilde{N}^1([0,T],\omega,\alpha)\big]^{2/\varrho}\Big)\Big],$$

which suffices to complete the proof for the first and third terms in (4.3).

Step 2. We now focus on the local accumulation of the fourth and fifth terms in (4.3).

We use the same notation as in Subsection 4.1 and proceed as in the proof of Theorem 6. Consider the Gaussian process $(W^1,\cdots,W^n)$, with abstract Wiener space $\big(\mathcal{W}^n,\mathcal{H}^{\oplus n},\mathbb{P}^{\otimes n}\big)$. As before, we call, for $\omega=(\omega_i)_{i=1}^n\in\Omega^n$ and for $\boldsymbol{h}=\oplus_{i=1}^n h_i\in\mathcal{H}^{\oplus n}$ set

$$T_{\boldsymbol{h}}\boldsymbol{W}^{(n)}(\omega) = T_{\oplus_{i=1}^n h_i}\boldsymbol{W}^{(n)}(\omega)$$

for the translated rough path along h. Then,

$$\begin{aligned}
&\|\mathbb{W}^{i,j}(\omega)\|^{p/2}_{[s,t],(p/2)-\mathrm{var}}\\
&\quad\leqslant c\Big(\|(T_{\boldsymbol{h}}\mathbb{W})^{i,j}(\omega)\|^{p/2}_{[s,t],(p/2)-\mathrm{var}} + \|(T_{\boldsymbol{h}}W)^i(\omega)\|^p_{[s,t],p-\mathrm{var}} + \|(T_{\boldsymbol{h}}W)^j(\omega)\|^p_{[s,t],p-\mathrm{var}}\\
&\qquad + \|h^i\|^p_{[s,t],\varrho-\mathrm{var}} + \|h^j\|^p_{[s,t],\varrho-\mathrm{var}}\Big).
\end{aligned}$$

Importantly, the constant c is independent of n. Below, it is allowed to increase from line to line as long as it remains independent of n. So,

$$\begin{aligned}
&{}^{(n)}\Big(\|\mathbb{W}^{i,\bullet}(\omega)\|^{p/2}_{[s,t],(p/2)-\mathrm{var}}\Big)_q\\
&\quad\leqslant c\bigg\{{}^{(n)}\Big(\|(T_{\boldsymbol{h}}\mathbb{W})^{i,\bullet}(\omega)\|^{p/2}_{[s,t],(p/2)-\mathrm{var}}\Big)_q + {}^{(n)}\Big(\|(T_{\boldsymbol{h}}W)^{\bullet}(\omega)\|^p_{[s,t],p-\mathrm{var}}\Big)_q\\
&\qquad + \|(T_{\boldsymbol{h}}W)^i(\omega)\|^p_{[s,t],p-\mathrm{var}} + \|h^i\|^p_{[s,t],\varrho-\mathrm{var}} + {}^{(n)}\Big(\|h^{\bullet}\|^p_{[s,t],\varrho-\mathrm{var}}\Big)_q\bigg\}.
\end{aligned}$$

And then, proceeding as in the proof of Theorem 6 and applying Proposition 11.2 in [25] together with the fact that $p > 2\varrho$, we obtain

$$
\begin{aligned}
&{}^{(n)}\Big(\mathbb{W}^{i,\bullet}_{s,t}(\omega)\Big)^{p/2}_q \\
&\quad \leqslant c\left\{ {}^{(n)}\Big(\big\|(T_{\boldsymbol{h}}\boldsymbol{W})^{i,\bullet}(\omega)\big\|^p_{[0,T],(1/p)-\text{Hölder}}\Big)_q + \|h^i\|^{\varrho}_{\mathcal{H}} + {}^{(n)}\Big(\|h^\bullet\|^{\varrho}_{\mathcal{H}}\Big)_q \right\}(t-s),
\end{aligned}
\tag{A.3}
$$

at least when the left-hand side is less than or equal to α. Similarly,

$$
\begin{aligned}
&{}^{(n)}\Big(\mathbb{W}^{i,\bullet}(\omega)\Big)^{p/2}_{q;[s,t],(p/2)-\text{var}} \\
&\qquad \leqslant c\left\{ {}^{(n)}\Big(\big\|(T_{\boldsymbol{h}}\boldsymbol{W})^{i,\bullet}(\omega)\big\|^p_{[0,T],(1/p)-\text{Hölder}}\Big)_q + \|\boldsymbol{h}\|^{\varrho}_{\mathcal{H}^{\oplus n}} \right\}(t-s),
\end{aligned}
$$

when the left-hand side is less than or equal to α. Define now

$$
N^{i,n,\perp\!\!\!\perp}\big([0,T],\omega,\alpha\big) := N_{\varpi}\big([0,T],\alpha\big),
$$

when

$$
\varpi(s,t) = {}^{(n)}\big(\mathbb{W}^{i,\bullet}(\omega)\big)^{p/2}_{q;[s,t],(p/2)-\text{var}}.
$$

Then,

$$
N^{i,n,\perp\!\!\!\perp}\big([0,T],\omega,\alpha\big)\alpha \leqslant c\left\{ {}^{(n)}\Big(\big\|(T_{\boldsymbol{h}}\boldsymbol{W})^{i,\bullet}(\omega)\big\|^p_{[0,T],(1/p)-\text{Hölder}}\Big)_q + \|\boldsymbol{h}\|^{\varrho}_{\mathcal{H}^{\oplus n}} \right\} T.
$$

We then apply Theorem 11.7 in [25] but on the space $(\mathcal{W}^{\otimes n}, \mathcal{H}^{\oplus n}, \mathbb{P}^{\otimes n})$. Importantly, we observe that

$$
\mathbb{E}\Big[{}^{(n)}\Big(\big\|(T_{\boldsymbol{h}}\boldsymbol{W})^{i,\bullet}(\omega)\big\|^p_{[0,T],(1/p)-\text{Hölder}}\Big)_q \Big]
$$

is bounded by a constant c, independent of i and n, which proves that the local accumulation $N^{i,n,\perp\!\!\!\perp}([0,T],\cdot,\alpha)$ has a Weibull distribution with shape parameter $1/\varrho$.

Step 3. The fact that $N^{i,n,\perp\!\!\!\perp}\big([0,T],\cdot,\alpha\big)$ has Weibull tails does not suffice for our purpose. Indeed, differently from the variables $\tilde{N}^{i,n}\big([0,T],\cdot,\alpha\big)$, the variables $N^{i,n,\perp\!\!\!\perp}\big([0,T],\cdot,\alpha\big)$ are independent, which prevents us from a straightforward application of the law of large numbers as done in the first step. In order to overcome this difficulty, we must revisit the above argument and prove that the variables $\big(N^{i,n,\perp\!\!\!\perp}([0,T],\cdot,\alpha)\big)_{i=1,\cdots,n}$ are in fact nearly independent, in a sense that is made clear below. In order to do so, go back to (A.3) and observe that, since $\varrho < 2$,

$$
{}^{(n)}\Big(\|h^\bullet\|^{\varrho}_{\mathcal{H}}\Big)_q = \bigg(\frac{1}{n}\sum_{j=1}^n \|h^j\|^{\varrho q}_{\mathcal{H}}\bigg)^{1/q} \leqslant \bigg(\frac{1}{n}\sum_{j=1}^n \|h^j\|^{2}_{\mathcal{H}}\bigg)^{\varrho/(2q)} \leqslant n^{-\varrho/(2q)}\,\|\boldsymbol{h}\|^{\varrho}_{\mathcal{H}^{\oplus n}}.
$$

The trick now is to use the additional factor $n^{-\varrho/(2q)}$, but to benefit in a full way of this additional decay, we assume that $h^i = 0$, in which case (A.3) becomes

$$
\begin{aligned}
&{}^{(n)}\Big(\mathbb{W}^{i,\bullet}_{s,t}(\omega)\Big)^{p/2}_q \\
&\quad \leqslant c\Big\{ {}^{(n)}\Big(\big\|(T_{\boldsymbol{h}}\mathbb{W})^{i,\bullet}(\omega)\big\|^{p/2}_{[s,t],(p/2)-\text{var}}\Big)_q + {}^{(n)}\Big(\big\|(T_{\boldsymbol{h}}W)^{\bullet}(\omega)\big\|^{p}_{[s,t],p-\text{var}}\Big)_q \\
&\qquad + \big\|(T_{\boldsymbol{h}}W)^{i}(\omega)\big\|^{p}_{[s,t],p-\text{var}} + n^{-\varrho/(2q)}\,\|\boldsymbol{h}\|^{\varrho}_{\mathcal{H}^{\oplus n}}(t-s)\Big\},
\end{aligned}
$$

at least in the case when the left-hand side is less than or equal to α. In fact, the above inequality must be considered as an inequality on the smaller space $\big(\mathcal{W}^{n-1}, \mathcal{H}^{\oplus(n-1)}, \mathbb{P}^{\otimes(n-1)}\big)$ containing the $(n-1)$-tuple $\big(\omega_1, \cdots, \omega_{i-1}, \omega_{i+1}, \cdots, \omega_n\big)$, which we denote by ω^{-i}, while the value of ω_i is frozen. In particular, since $h^i = 0$, the vector $\boldsymbol{h}$ can be identified with $\boldsymbol{h}^{-i} = (h_1, \cdots, h_{i-1}, h_{i+1}, \cdots, h_n)$ and $\|\boldsymbol{h}\|_{\mathcal{H}^{\oplus n}}$ is then equal to $\|\boldsymbol{h}_{i-1}\|_{\mathcal{H}^{\oplus n}}$. As before, take now, τ_ℓ and $\tau_{\ell+1}$ such that

$$
{}^{(n)}\Big(\mathbb{W}^{i,\bullet}_{\tau_\ell,\tau_{\ell+1}}(\omega) \Big)^{p/2}_q = \alpha.
$$

Then, at least one of the two inequalities below holds true

$$
\begin{aligned}
& c\bigg\{ {}^{(n)}\Big(\big\|(T_{\boldsymbol{h}}\mathbb{W})^{i,\bullet}(\omega)\big\|^{p/2}_{[\tau_\ell,\tau_{\ell+1}],(p/2)-\mathrm{var}} \Big)_q + {}^{(n)}\Big(\big\|(T_{\boldsymbol{h}}W)^{\bullet}(\omega)\big\|^{p}_{[\tau_\ell,\tau_{\ell+1}],p-\mathrm{var}} \Big)_q \\
& \qquad + \big\|(T_{\boldsymbol{h}}W)^{i}(\omega)\big\|^{p}_{[\tau_\ell,\tau_{\ell+1}],p-\mathrm{var}} \bigg\} \geqslant \frac{\alpha}{2}, \\
& n^{-\varrho/(2q)} \, \|\boldsymbol{h}^{-i}\|^{\varrho}_{\mathcal{H}^{\oplus(n-1)}} \, (\tau_{\ell+1} - \tau_\ell) \geqslant \frac{\alpha}{2}.
\end{aligned}
$$

Therefore, denoting the left-hand side in the first line by $g_{h,[\tau_\ell,\tau_{\ell+1}]}(\omega)$, we get

$$
\begin{aligned}
& {}^{(n)}\Big(\mathbb{W}^{i,\bullet}_{\tau_\ell,\tau_{\ell+1}}(\omega) \Big)^{p/2}_q \\
& \qquad \leqslant \alpha \, \mathbf{1}_{\{g_{h,[\tau_\ell,\tau_{\ell+1}]}(\omega) \geqslant \alpha/(2c)\}} + n^{-\varrho/(2q)} \, \|\boldsymbol{h}^{-i}\|^{\varrho}_{\mathcal{H}^{\oplus(n-1)}} \, (t_{\ell+1} - t_\ell).
\end{aligned}
$$

So, we get, with probability 1

$$
N^{i,n,\perp\!\!\!\perp}\big([0,T], \omega, \alpha\big) \leqslant N^{i,n}\Big([0,T], \omega + \boldsymbol{h}^{-i}, \frac{\alpha}{2c}\Big) + n^{-\varrho/(2q)} \, \|\boldsymbol{h}^{-i}\|^{\varrho}_{\mathcal{H}^{\oplus(n-1)}} \, T, \tag{A.4}
$$

where $N^{i,n}$ is the full-fledged local accumulation defined in the statement of Theorem 22.

The important point here is that $N^{i,n}\big([0,T], \cdot, \alpha/(2c)\big)$ has Weibull tails with shape parameter $2/\varrho$, uniformly in $n \geqslant 1$, as a consequence of the first step, the second step and fourth step below – the fourth step is actually a duplication of the second step. Hence, there exist a positive constant a and a non-negative constant C such that

$$
\mathbb{E}\bigg[\exp\bigg(a\Big[N^{i,n}\Big([0,T], \cdot, \frac{\alpha}{2c}\Big)\Big]^{2/\varrho} \bigg) \bigg] \leqslant C.
$$

Set

$$
f(\omega_i) = \inf\bigg\{ r > 0 : \mathbb{P}^{\otimes(n-1)}\Big(\omega^{-i} : N^{i,n}\Big([0,T], (\omega^{-i}, \omega_i), \frac{\alpha}{2c}\Big) > r \Big) \leqslant \frac{1}{2} \bigg\}.
$$

In the right-hand side, we wrote ω under the form (ω^{-i}, ω_i) to specify the fact the random variable is seen on the smaller space $\mathcal{W}^{n-1}$. For any $A > 0$,

$$
\begin{aligned}
& \Big\{ \omega_i : f(\omega_i) \geqslant A \Big\} \\
& \subset \bigg\{ \omega_i : \mathbb{P}^{\otimes(n-1)}\Big(\omega^{-i} : N^{i,n}\Big([0,T], (\omega^{-i}, \omega_i), \frac{\alpha}{2c}\Big) \geqslant A \Big) \geqslant \frac{1}{2} \bigg\} \\
& \subset \bigg\{ \omega_i : \mathbb{E}^{\otimes(n-1)}\Big[\exp\Big(a\Big(N^{i,n}\big([0,T], (\cdot, \omega_i)\big)\Big)^{2/\varrho} \Big) \Big] \geqslant \frac{1}{2} \exp\big(aA^{2/\varrho}\big) \bigg\}.
\end{aligned}
$$

So,

$$
\mathbb{P}\big(f \geqslant A\big) \leqslant 2 \exp\big(-aA^{2/\varrho}\big),
$$

from which we deduce that f has Weibull tails with shape parameter $\varrho' > 1$, uniformly in n.

Returning to (A.4) and subtracting $f(\omega_i)$ to both sides, we get

$$\begin{aligned}&\Big(N^{i,n,\perp\!\!\!\perp}\big([0,T],\omega,\alpha\big) - f(\omega_i)\Big)_+\\&\quad\leqslant \Big(N^{i,n}\Big([0,T],\omega + \boldsymbol{h}^{-i}, \frac{\alpha}{2c}\Big) - f(\omega_i)\Big)_+ + n^{-\varrho/(2q)}\,\|\boldsymbol{h}^{-i}\|^{\varrho}_{\mathcal{H}^{\oplus(n-1)}}\,T.\end{aligned}$$

Now, we can apply Theorem 11.7 in [25] with $a = 0$ and $\widehat{a} \geqslant 0$ on the smaller space $\mathcal{W}^{n-1}$ containing ω^{-i}. We deduce that there exist $a > 0$ and $C \geqslant 0$, independent of n, such that

$$\mathbb{E}^{\otimes(n-1)}\bigg[\exp\bigg(an^{1/q}\Big(\Big[N^{i,n,\perp\!\!\!\perp}\big([0,T],(\cdot,\omega_i),\alpha\big) - f(\omega_i)\Big]_+\Big)^{2/\varrho}\bigg)\bigg] \leqslant C.$$

Taking expectation and rewriting $g(\omega_i)$ in the form $f(W_i(\omega))$, we get

$$\mathbb{E}^{\otimes n}\bigg[\exp\bigg(an^{1/q}\Big(\Big[N^{i,n,\perp\!\!\!\perp}\big([0,T],\cdot,\alpha\big) - f(W_i(\cdot))\Big]_+\Big)^{2/\varrho}\bigg)\bigg] \leqslant C.$$

Therefore, by Jensen's inequality,

$$\mathbb{E}\bigg[\bigg(\frac{1}{n}\sum_{i=1}^n \exp\bigg(a\Big(\Big[N^{i,n,\perp\!\!\!\perp}\big([0,T],\cdot,\alpha\big) - f(W_i(\cdot))\Big]_+\Big)^{2/\varrho}\bigg)\bigg)^{n^{1/q}}\bigg] \leqslant C.$$

Therefore, for any $A \geqslant 1$,

$$\mathbb{P}\bigg(\frac{1}{n}\sum_{i=1}^n \exp\bigg(a\Big(\Big[N^{i,n,\perp\!\!\!\perp}\big([0,T],\omega,\alpha\big) - f(W_i(\cdot)))\Big]_+\Big)^{2/\varrho}\bigg) \geqslant A\bigg) \leqslant CA^{-n^{1/q}},$$

and by Borel-Cantelli lemma, we deduce that, with probability 1, there exists a rank n_0 such that for $n \geqslant n_0$,

$$\frac{1}{n}\sum_{i=1}^n \exp\bigg(a\Big(\Big[N^{i,n,\perp\!\!\!\perp}\big([0,T],\omega,\alpha\big) - f(W_i(\cdot)))\Big]_+\Big)^{2/\varrho}\bigg) \leqslant A.$$

It suffices to duplicate the first step to conclude, this time with the random variables $\big(f(W_i)\big)_{i=1,\cdots,n}$ which have Weibull tails with shape parameter $\varrho' > 1$, uniformly in n. Assuming without of loss of generality that $\varrho' \geqslant \varrho$, we complete the proof by Cauchy-Schwarz inequality together with the fact that

$$\begin{aligned}&\Big(N^{i,n,\perp\!\!\!\perp}\big([0,T],\omega,\alpha\big)\Big)^{2/\varrho'}\\&\quad\leqslant C\Big(\Big[N^{i,n,\perp\!\!\!\perp}\big([0,T],\omega,\alpha\big) - f(W_i(\cdot)))\Big]_+\Big)^{2/\varrho'} + C\Big(f(W_i(\cdot))\Big)^{2/\varrho'}.\end{aligned}$$

Step 4. We now turn to the local accumulation of the second and sixth terms in (4.3). Proceeding as the second step, we get

$$\begin{aligned}&{}^{(n)}\Big(\!\Big(\,\mathbb{W}^{\bullet,\bullet}(\omega)\Big)\!\Big)^{p/2}_{q;[s,t],(p/2)-\mathrm{var}}\\&\quad\leqslant c\,\bigg\{{}^{(n)}\Big(\!\Big(\,\|(T_{\boldsymbol{h}}\boldsymbol{W})^{\bullet,\bullet}(\omega)\|^p_{[0,T],(1/p)-\text{Hölder}}\Big)\!\Big)_q + n^{-\varrho/(2q)}\,\|\boldsymbol{h}\|^{\varrho}_{\mathcal{H}^{\oplus n}}\bigg\}\,(t-s),\end{aligned}$$

at least when the left-hand side is less than or equal to α. Importantly, the coefficient in front of $\|\boldsymbol{h}\|^{\varrho}_{\mathcal{H}^{\oplus n}}$ holds for all $\boldsymbol{h}$. So, the context is simpler than in the two previous steps. We then conclude as in the second step as for the tails and, using the additional $n^{-\varrho/(2q)}$, we can implement the same Borel-Cantelli argument as in the third step.

A.3 – An Auxiliary Estimate

We prove in this appendix an auxiliary estimate that was used in Step 1 of the proof of Theorem 24; this is where the convergence rate η_n for the propagation of chaos appears. Recall the definition of the terms $\mathcal{I}^{i,n,\partial}_{\{r,s\}}(\cdot)$ and $\overline{\mathcal{I}}^{i,\partial}_{\{r,s\}}(\cdot)$, given after equation (4.15) in Step 1 of the proof of Theorem 24.

25. Lemma – *Fix $\varrho \geqslant 8$. Then, there exist an exponent ϱ' and a constant c such that, if $X_0(\cdot)$ is ϱ'-integrable, then, for any $n \geqslant 1$, $i \in \{1, \cdots, n\}$ and $0 \leqslant r \leqslant s \leqslant t \leqslant T$,*

$$\Big\langle \Big\{ \mathcal{I}^{i,n,\partial}_{\{r,s\}}(\cdot) + \mathcal{I}^{i,n,\partial}_{\{s,t\}}(\cdot) - \mathcal{I}^{i,n,\partial}_{\{r,t\}}(\cdot) \Big\} - \Big\{ \overline{\mathcal{I}}^{i,\partial}_{\{r,s\}}(\cdot) + \overline{\mathcal{I}}^{i,\partial}_{\{s,t\}}(\cdot) - \overline{\mathcal{I}}^{i,\partial}_{\{r,t\}}(\cdot) \Big\} \Big\rangle_{\varrho}$$
$$\leqslant C\,\eta_n \left\langle\!\left\langle w^+(r,t,\cdot,\cdot) \right\rangle\!\right\rangle_{\varrho'}^{3/p},$$

where $(\eta_n)_{n\geqslant 1}$ is as in the statement of Theorem 24 and

$$w^+(r,t,\omega,\omega') := w(r,t,\omega) + \|\mathbb{W}^{\perp\!\!\!\perp}(\omega,\omega')\|^{p/2}_{[r,t],p/2-\mathrm{var}}.$$

Proof – Throughout the proof, we use the following notations. For each $i \in \{1, \cdots, n\}$, we call $\overline{w}^i$ the control associated with $\boldsymbol{W}^i(\cdot)$ through (1.8). For $j \in \{1, \cdots, n\}$, we also let

$$\overline{w}^{i,j}(s,t,\omega) := \big\|W^{i,j}(\omega)\big\|^p_{[s,t],p-\mathrm{var}}.$$

We also make an intense use of Lemma 26 below, giving the convergence rate of the empirical measure of a sample of independent, identically distributed random variables towards their common law. By (3.4), we know that, under the standing assumption, $\sup_{0\leqslant t\leqslant T}\big|X_t(\cdot)\big|$ and $\big|\!\big|\!\big|X(\cdot)\big|\!\big|\!\big|_{[0,T],w,p}$ are in $\mathbb{L}^\rho$ as soon as $X_0(\cdot)$ is in $\mathbb{L}^\rho$. We then compute

$$\begin{aligned}
&\Big\{ \mathcal{I}^{i,n,\partial}_{\{r,s\}}(\omega) + \mathcal{I}^{i,n,\partial}_{\{s,t\}}(\omega) - \mathcal{I}^{i,n,\partial}_{\{r,t\}}(\omega) \Big\} - \Big\{ \overline{\mathcal{I}}^{i,\partial}_{\{r,s\}}(\omega) + \overline{\mathcal{I}}^{i,\partial}_{\{s,t\}}(\omega) - \overline{\mathcal{I}}^{i,\partial}_{\{r,t\}}(\omega) \Big\} \\
&= \Big(R^{F^{i,n}}_{r,s}(\omega) - R^{\overline{F}^i}_{r,s}(\omega) \Big) W^i_{s,t}(\omega) + \Big(\delta_x F^{i,n}_{r,s}(\omega) - \delta_x \overline{F}^i_{r,s}(\omega) \Big) \mathbb{W}^i_{s,t}(\omega) \\
&\quad + \left(\frac{1}{n}\sum_{j=1}^n \delta_\mu F^{i,j,n}_{r,s}(\omega) \mathbb{W}^{j,i}_{s,t}(\omega) - \mathbb{E}\Big[\delta_\mu \overline{F}^i_{r,s}(\omega,\cdot) \mathbb{W}^{i,\perp\!\!\!\perp}_{s,t}(\cdot,\omega) \Big] \right),
\end{aligned}$$

where

$$\begin{aligned}
R^{F^{i,n}}_{r,s}(\omega) &:= F^{i,n}_s(\omega) - F^{i,n}_r(\omega) - \delta_x F^{i,n}_r(\omega) W^i_{r,s}(\omega) - \frac{1}{n}\sum_{j=1}^n \delta_\mu F^{i,j,n}_r(\omega) W^j_{r,s}(\omega), \\
R^{\overline{F}^i}_{r,s}(\omega) &:= \overline{F}^i_s(\omega) - \overline{F}^i_r(\omega) - \delta_x \overline{F}^i_r(\omega) W^i_{r,s}(\omega) - \mathbb{E}\Big[\delta_\mu \overline{F}^i_r(\omega,\cdot) W_{r,s}(\cdot) \Big].
\end{aligned} \tag{A.5}$$

Following (4.14) and (4.15), we define differentiable functions G_x and G_μ of their arguments setting

$$\begin{aligned}
\delta_x F^{i,n}_t(\omega) &=: G_x\big(\overline{X}^i_t(\omega), \overline{\mu}^n_t(\omega)\big), \\
\delta_x \overline{F}^i_t(\omega,\omega') &=: G_x\big(\overline{X}^i_t(\omega), \mathcal{L}(X_t)\big),
\end{aligned}$$

and

$$\delta_\mu F_t^{i,j,n}(\omega) =: G_\mu\big(\overline{X}_t^i(\omega), \overline{\mu}_t^n(\omega)\big)\big(\overline{X}_t^j(\omega)\big),$$
$$\delta_\mu \overline{F}_t^i(\omega,\omega') =: G_\mu\big(\overline{X}_t^i(\omega), \mathcal{L}(X_t)\big)\big(X_t(\omega')\big).$$

Finally, we can write the whole difference in the form

$$\begin{aligned}
&\Big\{\mathcal{I}^\partial_{\{r,s\}}(\omega) + \mathcal{I}^\partial_{\{s,t\}}(\omega) - \mathcal{I}^\partial_{\{r,t\}}(\omega)\Big\} - \Big\{\overline{\mathcal{I}}^\partial_{\{r,s\}}(\omega) + \overline{\mathcal{I}}^\partial_{\{s,t\}}(\omega) - \overline{\mathcal{I}}^\partial_{\{r,t\}}(\omega)\Big\}\\
&= \big(R^{F^{i,n}}_{r,s}(\omega) - R^{\overline{F}^i}_{r,s}(\omega)\big)W^i_{s,t}(\omega)\\
&\quad + \Big[G_x\big(\overline{X}^i(\omega), \overline{\mu}^n(\omega)\big) - G_x\big(\overline{X}^i(\omega), \mathcal{L}(X)\big)\Big]_{r,s}\mathbb{W}^i_{s,t}(\omega) \qquad\qquad (A.6)\\
&\quad + \frac{1}{n}\sum_{j=1}^n \Big[G_\mu\big(\overline{X}^i(\omega), \overline{\mu}^n(\omega)\big)\big(\overline{X}^j(\omega)\big) - G_\mu\big(\overline{X}^i(\omega), \mathcal{L}(X)\big)\big(X^j(\omega)\big)\Big]_{r,s}\mathbb{W}^{j,i}_{s,t}(\omega)\\
&\quad + \frac{1}{n}\sum_{j=1}^n \Big[G_\mu\big(\overline{X}^i(\omega), \mathcal{L}(X)\big)\big(\overline{X}^j(\omega)\big)\Big]_{r,s}\mathbb{W}^{j,i}_{s,t}(\omega) - \mathbb{E}\Big[\delta_\mu F^i_{r,s}(\omega,\cdot)\mathbb{W}^{i,\perp\!\!\!\perp}_{s,t}(\cdot,\omega)\Big].
\end{aligned}$$

Step 1. Observe that

$$\begin{aligned}
&\Big[G_x\big(\overline{X}^i(\omega), \overline{\mu}^n(\omega)\big)\Big]_{r,s}\\
&= \int_0^1 \partial_x G_x\Big(\overline{X}^{i,(\lambda)}_{r;(r,s)}(\omega), \overline{\mu}^{n,\lambda}_{r;(r,s)}(\omega)\Big)\overline{X}^i_{r,s}(\omega)d\lambda\\
&\quad + \frac{1}{n}\sum_{j=1}^n \int_0^1 D_\mu G_x\Big(\overline{X}^{i,(\lambda)}_{r;(r,s)}(\omega), \overline{\mu}^{n,\lambda}_{r;(r,s)}(\omega)\Big)\big(\overline{X}^{j,(\lambda)}_{r;(r,s)}(\omega)\big)\overline{X}^j_{r,s}(\omega)d\lambda\\
&= \int_0^1 \partial_x G_x\Big(\overline{X}^{i,(\lambda)}_{r;(r,s)}(\omega), \overline{\mu}^{n,\lambda}_{r;(r,s)}(\omega)\Big)\overline{X}^i_{r,s}(\omega)d\lambda\\
&\quad + \int_{\mathbb{R}^{2d}}\Big[\int_0^1 D_\mu G_x\Big(\overline{X}^{i,(\lambda)}_{r;(r,s)}(\omega), \overline{\mu}^{n,\lambda}_{r;(r,s)}(\omega)\Big)(y)zd\lambda\Big]d\overline{\nu}^{n,\lambda}_{r;(r,s)}(\omega;y,z)
\end{aligned}$$

where

$$\overline{\mu}^{n,(\lambda)}_{r;(r,s)}(\omega) := \frac{1}{n}\sum_{j=1}^n \delta_{\overline{X}^{j,(\lambda)}_{r;(r,s)}(\omega)}, \quad \overline{\nu}^{n,(\lambda)}_{s;(s,t)}(\omega) := \frac{1}{n}\sum_{j=1}^n \delta_{\big(\overline{X}^{j,(\lambda)}_{r;(r,s)}(\omega), \overline{X}^j_{r,s}(\omega)\big)},$$

with

$$\overline{X}^{j,(\lambda)}_{r;(r,s)}(\omega) := \overline{X}^j_r(\omega) + \lambda\overline{X}^j_{r,s}(\omega).$$

Proceeding similarly with $\big[G_x\big(\overline{X}^i(\omega), \mathcal{L}(X)\big)\big]_{r,s}$, we get

$$\begin{aligned}
&\Big[G_x\big(\overline{X}^i(\omega), \overline{\mu}^n(\omega)\big) - G_x\big(\overline{X}^i(\omega), \mathcal{L}(X)\big)\Big]_{r,s}\\
&= \int_0^1 \Big[\partial_x G_x\Big(\overline{X}^{i,(\lambda)}_{r;(r,s)}(\omega), \overline{\mu}^{n,(\lambda)}_{r;(r,s)}(\omega)\Big) - \partial_x G_x\Big(\overline{X}^{i,(\lambda)}_{r;(r,s)}(\omega), \mathcal{L}\big(X^{(\lambda)}_{r;(r,s)}\big)\Big)\Big]\overline{X}^i_{r,s}(\omega)\,d\lambda\\
&\quad + \int_{\mathbb{R}^{2d}}\Big[\int_0^1 D_\mu G_x\Big(\overline{X}^{i,(\lambda)}_{r;(r,s)}(\omega), \overline{\mu}^{n,(\lambda)}_{r;(r,s)}(\omega)\Big)(y)zd\lambda\Big]d\overline{\nu}^{n,(\lambda)}_{r;(r,s)}(\omega;y,z)\\
&\quad - \int_{\mathbb{R}^{2d}}\Big[\int_0^1 D_\mu G_x\Big(\overline{X}^{i,(\lambda)}_{r;(r,s)}(\omega), \mathcal{L}\big(X^{(\lambda)}_{r;(r,s)}\big)\Big)(y)zd\lambda\Big]d\mathcal{L}\big(X^{(\lambda)}_{r;(r,s)}, X_{r,s}\big)(y,z),
\end{aligned}$$

where, as before, $X^{(\lambda)}_{r;(r,s)}(\omega) = X_r(\omega) + \lambda X_{r,s}(\omega)$. Splitting the last two terms in the above expansion into

$$
\begin{aligned}
&\int_{\mathbb{R}^{2d}} \Big[\int_0^1 D_\mu G_x\Big(\overline{X}^{i,(\lambda)}_{r;(r,s)}(\omega), \overline{\mu}^{n,(\lambda)}_{r;(r,s)}(\omega)\Big)(y) z d\lambda \Big] d\overline{\nu}^{n,(\lambda)}_{r;(r,s)}(\omega; y, z) \\
&\quad - \int_{\mathbb{R}^{2d}} \Big[\int_0^1 D_\mu G_x\Big(\overline{X}^{i,(\lambda)}_{r;(r,s)}(\omega), \mathcal{L}\big(X^{(\lambda)}_{r;(r,s)}\big)\Big)(y) z d\lambda \Big] d\mathcal{L}\big(X^{(\lambda)}_{r;(r,s)}, X_{r,s}\big)(y, z) \\
&= \int_{\mathbb{R}^{2d}} \Big[\int_0^1 D_\mu G_x\Big(\overline{X}^{i,(\lambda)}_{r;(r,s)}(\omega), \overline{\mu}^{n,(\lambda)}_{r;(r,s)}(\omega)\Big)(y) z d\lambda \Big] d\overline{\nu}^{n,(\lambda)}_{r;(r,s)}(\omega; y, z) \\
&\quad - \int_{\mathbb{R}^{2d}} \Big[\int_0^1 D_\mu G_x\Big(\overline{X}^{i,(\lambda)}_{r;(r,s)}(\omega), \mathcal{L}\big(X^{(\lambda)}_{r;(r,s)}\big)\Big)(y) z d\lambda \Big] d\overline{\nu}^{n,(\lambda)}_{r;(r,s)}(\omega; y, z) \\
&\quad + \int_{\mathbb{R}^{2d}} \Big[\int_0^1 D_\mu G_x\Big(\overline{X}^{i,(\lambda)}_{r;(r,s)}(\omega), \mathcal{L}\big(X^{(\lambda)}_{r;(r,s)}\big)\Big)(y) z d\lambda \Big] d\overline{\nu}^{n,(\lambda)}_{r;(r,s)}(\omega; y, z) \\
&\quad - \int_{\mathbb{R}^{2d}} \Big[\int_0^1 D_\mu G_x\Big(\overline{X}^{i,(\lambda)}_{r;(r,s)}(\omega), \mathcal{L}\big(X^{(\lambda)}_{r;(r,s)}\big)\Big)(y) z d\lambda \Big] d\mathcal{L}\big(X^{(\lambda)}_{r;(r,s)}, X_{r,s}\big)(y, z),
\end{aligned}
$$

we get

$$
\begin{aligned}
&\Big| \Big[G_x\big(\overline{X}^i(\omega), \overline{\mu}^n(\omega)\big) - G_x\big(\overline{X}^i(\omega), \mathcal{L}(X)\big) \Big]_{r,s} \Big| \\
&\leqslant c \int_0^1 W_1\Big(\overline{\mu}^{n,(\lambda)}_{r;(r,s)}(\omega), \mathcal{L}\big(X^{(\lambda)}_{r;(r,s)}\big)\Big) d\lambda \\
&\quad \times \Bigg(|\!|\!| \overline{X}^i(\omega) |\!|\!|_{[0,T],\overline{w}^i,p}\, \overline{w}^i(r,s,\omega)^{1/p} + \frac{1}{n} \sum_{k=1}^n |\!|\!| \overline{X}^k(\omega) |\!|\!|_{[0,T],\overline{w}^k,p}\, \overline{w}^k(r,s,\omega)^{1/p} \Bigg) \\
&\quad + c \Big| \mathcal{S}^{i,n}_{r,s}\big(\omega, |\overline{X}^{\bullet}_{r,s}(\omega)|\big) \Big|,
\end{aligned}
$$

where $\mathcal{S}^{i,n}_{r,s}\big(\omega, |\overline{X}^{\bullet}_{r,s}(\omega)|\big)$ is the n-empirical mean of n random variables that are dominated by $\big(|\overline{X}^j_{r,s}(\omega)|\big)_{j=1,\cdots,n}$ and $n-1$ of which are conditionally centered and conditionally independent given the realization of the path $(\overline{X}^i, W^i, \mathbb{W}^i)$. Recalling (1.9) and allowing the value of the constant c to increase from line to line, we obtain

$$
\begin{aligned}
&\Big| \Big[G_x\Big(\overline{X}^i(\omega), \overline{\mu}^n(\omega)\Big) - G_x\Big(\overline{X}^i(\omega), \mathcal{L}(X)\Big) \Big]_{r,s} \mathbb{W}^i_{s,t}(\omega) \Big| \\
&\leqslant c \int_0^1 W_1\Big(\overline{\mu}^{n,(\lambda)}_{r;(r,s)}(\omega), \mathcal{L}\big(X^{(\lambda)}_{r;(r,s)}\big)\Big) d\lambda \\
&\times \Bigg[|\!|\!| \overline{X}^i(\omega) |\!|\!|_{[0,T],\overline{w}^i,p} + \Bigg(\frac{1}{n} \sum_{k=1}^n |\!|\!| \overline{X}^k(\omega) |\!|\!|^2_{[0,T],\overline{w}^k,p} \Bigg)^{1/2} \Bigg] \\
&\times \Bigg[\overline{w}^i(r,t,\omega)^{3/p} + \Bigg(\frac{1}{n} \sum_{k=1}^n \overline{w}^k(r,t,\omega)^{2/p} \Bigg)^{3/2} \Bigg] + c \Big| \mathcal{S}^{i,n}_{r,s}\big(\omega, |\overline{X}^{\bullet}_{r,s}(\omega)|\big) \Big| \overline{w}^i(r,t,\omega)^{2/p}.
\end{aligned}
$$

In order to conclude for the second term in the right-hand side of (A.6), it suffices to recall from Rosenthal's inequality (applied under the conditional probability given the realization of the path $(\overline{X}^i, W^i, \mathbb{W}^i))$ that

$$\Big\langle \mathcal{S}^{i,n}_{r,s}\big(\cdot, |\overline{X}^{\bullet}_{r,s}(\cdot)|\big) \Big\rangle_{3\varrho/2} \leqslant c\, n^{-1/2} \Big\langle \| X(\cdot) \|_{[0,T],w,p} w(r,s,\cdot)^{1/p} \Big\rangle_{3\varrho/2} \leqslant c\, n^{-1/2} \big\langle \| X(\cdot) \|_{[0,T],w,p} \big\rangle_{3\varrho} \big\langle w(r,t,\cdot) \big\rangle^{1/p}_{3\varrho}.$$

If ρ is large enough, we deduce from Lemma 26 that

$$\begin{aligned}
&\Big\langle \Big[G_x\big(\overline{X}^i(\cdot), \overline{\mu}^n(\cdot)\big) - G_x\big(\overline{X}^i(\cdot), \mathcal{L}(X)\big) \Big]_{r,s} \mathbb{W}^i_{s,t}(\cdot) \Big\rangle_{\varrho} \\
&\leqslant c \left(\int_0^1 \Big\langle W_1\Big(\overline{\mu}^{n,(\lambda)}_{r;(r,s)}(\cdot), \mathcal{L}\big(X^{(\lambda)}_{r;(r,s)}\big) \Big) \Big\rangle_{3\varrho} d\lambda \right) \big\langle \| X(\cdot) \|_{[0,T],w,p} \big\rangle_{6\varrho} \big\langle w(r,t,\cdot) \big\rangle^{3/p}_{6\varrho} \\
&\quad + c\, n^{-1/2} \big\langle \| X(\cdot) \|_{[0,T],w,p} \big\rangle_{3\varrho} \big\langle w(r,t,\cdot) \big\rangle^{3/p}_{3\varrho} \\
&\leqslant c\, \eta_n \Big(1 + \big\langle \sup_{0\leqslant u\leqslant T} |X_u(\cdot)| \big\rangle_{3\varrho} \Big) \big\langle \| X(\cdot) \|_{[0,T],w,p} \big\rangle^2_{6\varrho} \big\langle\!\big\langle w^+(r,t,\cdot,\cdot) \big\rangle\!\big\rangle^{3/p}_{6\varrho}.
\end{aligned}$$

Step 2. By the same argument, we have

$$\begin{aligned}
&\Big| \Big[G_\mu\Big(\overline{X}^i(\omega), \overline{\mu}^n(\omega)\Big)\big(\overline{X}^j(\omega)\big) - G_\mu\Big(\overline{X}^i(\omega), \mathcal{L}(X)\Big)\big(\overline{X}^j(\omega)\big) \Big]_{r,s} \mathbb{W}^{j,i}_{s,t}(\omega) \Big| \\
&\leqslant c \left(\int_0^1 W_1\Big(\overline{\mu}^{n,(\lambda)}_{r;(r,s)}(\omega), \mathcal{L}\big(X^{(\lambda)}_{r;(r,s)}\big) \Big) d\lambda \right) \overline{w}^{j,i}(s,t,\omega)^{2/p} \\
&\quad \times \left[\| \overline{X}^i(\omega) \|_{[0,T],\overline{w}^i,p} + \| \overline{X}^j(\omega) \|_{[0,T],\overline{w}^j,p} + \left(\frac{1}{n} \sum_{k=1}^n \| \overline{X}^k(\omega) \|^2_{[0,T],\overline{w}^k,p} \right)^{1/2} \right] \\
&\quad \times \left[\overline{w}^i(r,s,\omega)^{1/p} + \overline{w}^j(r,s,\omega)^{1/p} + \left(\frac{1}{n} \sum_{k=1}^n \overline{w}^k(r,s,\omega)^{2/p} \right)^{1/2} \right] \\
&\quad + c \Big| \mathcal{S}^{i,j,n}_{r,s}\big(\omega, |\overline{X}^{\bullet}_{r,s}(\omega)|\big) \Big| \, \overline{w}^{j,i}(s,t,\omega)^{2/p},
\end{aligned}$$

where

$$\Big\langle \mathcal{S}^{i,j,n}_{r,s}\big(\cdot, |\overline{X}^{\bullet}_{r,s}(\cdot)|\big) \Big\rangle_{3\varrho/2} \leqslant c\, n^{-1/2} \big\langle \| X \|_{[0,T],w,p} w(r,s,\cdot)^{1/p} \big\rangle_{3\varrho/2} \leqslant c\, n^{-1/2} \big\langle \| X \|_{[0,T],w,p} \rangle_{3\varrho} \big\langle w(r,t,\cdot) \big\rangle^{1/p}_{3\varrho}.$$

Observing that $\langle \overline{w}^{j,i}(s,t,\cdot)^{2/p} \rangle_{3\varrho} \leqslant \langle\!\langle w^+(r,t,\cdot,\cdot) \rangle\!\rangle^{2/p}_{3\varrho}$ – this is the rationale for introducing w^+, and taking expectation, we get

$$\begin{aligned}
&\Big\langle \Big[G_\mu\big(\overline{X}^i(\cdot), \overline{\mu}^n(\cdot)\big)\big(\overline{X}^j(\cdot)\big) - G_\mu\big(\overline{X}^i(\cdot), \mathcal{L}(X)\big)\big(\overline{X}^j(\cdot)\big) \Big]_{r,s} \mathbb{W}^{j,i}_{s,t}(\omega) \Big\rangle_{\varrho} \\
&\leqslant c \left(\int_0^1 \Big\langle W_1\Big(\overline{\mu}^{n,(\lambda)}_{r;(r,s)}(\cdot), \mathcal{L}\big(X^{(\lambda)}_{r;(r,s)}\big) \Big) \Big\rangle_{3\varrho} d\lambda \right) \big\langle \| X(\cdot) \|_{[0,T],w,p} \big\rangle_{6\varrho} \big\langle\!\big\langle w^+(r,t,\cdot,\cdot) \big\rangle\!\big\rangle^{3/p}_{6\varrho} \\
&\quad + c\, n^{-1/2} \big\langle \| X(\cdot) \|_{[0,T],w,p} \big\rangle_{3\varrho} \big\langle\!\big\langle w^+(r,t,\cdot,\cdot) \big\rangle\!\big\rangle^{3/p}_{3\varrho}.
\end{aligned}$$

Taking the mean over j, we obtain for upper bound for the third term in the right-hand side of (A.6) the quantity

$$\begin{aligned}
&\Bigl\langle \frac{1}{n}\sum_{j=1}^{n}\Bigl[G_\mu\bigl(\overline{X}^i(\cdot),\overline{\mu}^n(\cdot)\bigr)\bigl(\overline{X}^j(\cdot)\bigr)-G_\mu\bigl(\overline{X}^i(\cdot),\mathcal{L}(X)\bigr)\bigl(\overline{X}^j(\cdot)\bigr)\Bigr]_{r,s}\mathbb{W}^{j,i}_{s,t}(\omega)\Bigr\rangle_\varrho \\
&\leqslant c\left(\int_0^1\Bigl\langle W_1\Bigl(\overline{\mu}^{n,(\lambda)}_{r;(r,s)}(\cdot),\mathcal{L}\bigl(X^{(\lambda)}_{r;(r,s)}\bigr)\Bigr)\Bigr\rangle_{3\varrho}d\lambda\right)\bigl\langle \|\!|X(\cdot)|\!\|_{[0,T],w,p}\bigr\rangle_{6\varrho}\bigl\langle\!\bigl\langle w^+(r,t,\cdot,\cdot)\bigr\rangle\!\bigr\rangle^{3/p}_{6\varrho} \\
&\quad + c\,n^{-1/2}\bigl\langle \|\!|X(\cdot)|\!\|_{[0,T],w,p}\bigr\rangle_{3\varrho}\bigl\langle\!\bigl\langle w^+(r,t,\cdot,\cdot)\bigr\rangle\!\bigr\rangle^{3/p}_{3\varrho}.
\end{aligned}$$

By Lemma 26, we get the same bound as in the first step.

Step 3. We now turn to the last term in the right-hand side of (A.6). It reads as the empirical mean of n random variables, $n-1$ of which are conditionally centered and conditionally independent given the realization of the paths $(\overline{X}^i, W^i, \mathbb{W}^i)$, namely

$$\frac{1}{n}\sum_{j=1}^{n}\bigl[G_\mu\bigl(\overline{X}^i(\omega),\mathcal{L}(X)\bigr)\bigl(\overline{X}^j(\omega)\bigr)\bigr]_{r,s}\mathbb{W}^{j,i}_{s,t}(\omega)-\mathbb{E}\bigl[\delta_\mu\overline{F}^i_{r,s}(\omega,\cdot)\mathbb{W}^{i,\perp\!\!\!\perp}_{s,t}(\cdot,\omega)\bigr].$$

Invoking Rosenthal's inequality once again (in a conditional form), it suffices to compute the L^ϱ norm of

$$\bigl[G_\mu\bigl(\overline{X}^i(\omega),\mathcal{L}(X)\bigr)\bigl(\overline{X}^j(\omega)\bigr)\bigr]_{r,s}\mathbb{W}^{j,i}_{s,t}(\omega).$$

Proceeding as before, it is less than $c\bigl\langle \|\!|\overline{X}(\cdot)|\!\|_{[0,T],w,p}\bigr\rangle_{3\varrho}\bigl\langle\!\bigl\langle w^+(r,t,\cdot,\cdot)\bigr\rangle\!\bigr\rangle^{3/p}_{3\varrho}$. So,

$$\begin{aligned}
&\Bigl\langle \frac{1}{n}\sum_{j=1}^{n}\Bigl[G_\mu\bigl(\overline{X}^i(\omega),\mathcal{L}(X)\bigr)\bigl(\overline{X}^j(\omega)\bigr)\Bigr]_{r,s}\mathbb{W}^{j,i}_{s,t}(\omega)-\mathbb{E}\Bigl[\delta_\mu\overline{F}^i_{r,s}(\omega,\cdot)\mathbb{W}^{i,\perp\!\!\!\perp}_{s,t}(\cdot,\omega)\Bigr]\Bigr\rangle_\varrho \\
&\leqslant c\,n^{-1/2}\bigl\langle \|\!|X(\cdot)|\!\|_{[0,T],w,p}\bigr\rangle_{3\varrho}\bigl\langle\!\bigl\langle w^+(r,t,\cdot,\cdot)\bigr\rangle\!\bigr\rangle^{3/p}_{3\varrho}.
\end{aligned}$$

We conclude as before, by invoking Lemma 26.

Step 4. We now handle the remainders in (A.6). By expanding (A.5) and by using similar notations for the remainders in the expansion of each $\bigl(\overline{X}^j\bigr)_{j=1,\cdots,n}$, we have

$$\begin{aligned}
&R^{F^{i,n}}_{r,s}(\omega) \\
&= \partial_x\mathrm{F}\Bigl(\overline{X}^i_r(\omega),\overline{\mu}^n_r(\omega)\Bigr)R^{\overline{X}^i}_{r,s}(\omega)+\frac{1}{n}\sum_{j=1}^{n}D_\mu\mathrm{F}\Bigl(\overline{X}^i_r(\omega),\overline{\mu}^n_r(\omega)\Bigr)\bigl(\overline{X}^j_r(\omega)\bigr)R^{\overline{X}^j}_{r,s}(\omega) \\
&\quad+\int_0^1\Bigl[\partial_x\mathrm{F}\Bigl(\overline{X}^{i,(\lambda)}_{r;(r,s)}(\omega),\overline{\mu}^{n,(\lambda)}_{r;(r,s)}(\omega)\Bigr)-\partial_x\mathrm{F}\Bigl(\overline{X}^i_r(\omega),\overline{\mu}^n_r(\omega)\Bigr)\Bigr]\overline{X}^i_{r,s}(\omega)\,d\lambda \qquad (A.7)\\
&\quad+\frac{1}{n}\sum_{j=1}^{n}\int_0^1\Bigl[D_\mu\mathrm{F}\Bigl(\overline{X}^{i,(\lambda)}_{r;(r,s)}(\omega),\overline{\mu}^{n,(\lambda)}_{r;(r,s)}\Bigr)\Bigl(\overline{X}^{j,(\lambda)}_{r;(r,s)}(\omega)\Bigr) \\
&\qquad\qquad -D_\mu\mathrm{F}\Bigl(\overline{X}^i_r(\omega),\overline{\mu}^n_r\Bigr)\bigl(\overline{X}^j_r(\omega)\bigr)\Bigr]\,\overline{X}^j_{r,s}(\omega)\,d\lambda.
\end{aligned}$$

Expanding $R^{\overline{F}^i}_{r,s}(\omega)$ in a similar way, we have to investigate four terms in order to estimate the difference $R^{F^{i,n}}_{r,s}(\omega)-R^{\overline{F}^i}_{r,s}(\omega)$. The first term corresponds to the first term in the right-hand side of (A.7)

$$\begin{aligned}&\Big|\Big[\partial_x \mathrm{F}\Big(\overline{X}^i_r(\omega),\overline{\mu}^n_r(\omega)\Big)-\partial_x \mathrm{F}\Big(\overline{X}^i_r(\omega),\mathcal{L}(X_r)\Big)\Big]\,R^{\overline{X}^i}_{r,s}(\omega)\Big|\\&\leqslant c\,W_1\Big(\overline{\mu}^n_r(\omega),\mathcal{L}(X_r)\Big)\,\big\|\!\big|\overline{X}^i(\cdot)\big\|\!\big|_{[0,T],\overline{w}^i,p}\,\overline{w}^i(r,s,\omega)^{2/p}.\end{aligned}$$

Then, we must recall that, in the first line of the right-hand side in (A.6), the difference $R^{F^{i,n}}_{r,s}(\omega)-R^{\overline{F}^i}_{r,s}(\omega)$ is multiplied by $W^i_{s,t}(\omega)$, which is less than $\overline{w}^i(s,t,\omega)^{1/p}$. In other words, we must multiply both sides in the above inequality by $\overline{w}^i(r,t,\omega)^{1/p}$. By Cauchy Schwarz inequality, the $\mathbb{L}^\varrho$ norm of the resulting bound is less than $c\big\langle W_1(\overline{\mu}^n_r(\cdot),\mathcal{L}(X_r)\big\rangle_{3\varrho}\big\langle\|\!|X(\cdot)\|\!|_{[0,T],w,p}\big\rangle_{3\varrho}\big\langle w(r,t,\cdot)\big\rangle^{3/p}_{6\varrho}$.

The second term that we have to handle corresponds to the second term in the right-hand side of (A.7). With an obvious definition for $R^X(\cdot)$, it reads

$$\bigg|\frac{1}{n}\sum_{j=1}^n D_\mu \mathrm{F}\Big(\overline{X}^i_r(\omega),\overline{\mu}^n_r(\omega)\Big)\big(\overline{X}^j_r(\omega)\big)R^{\overline{X}^j}_{r,s}(\omega)-\Big\langle D_\mu \mathrm{F}\Big(\overline{X}^i_r(\omega),\mathcal{L}(X_r)\Big)\big(X_r(\cdot)\big)R^{X}_{r,s}(\cdot)\Big\rangle\bigg|.$$

Proceeding exactly as in the first step, we get

$$\begin{aligned}&\bigg|\frac{1}{n}\sum_{j=1}^n D_\mu \mathrm{F}\Big(\overline{X}^i_r(\omega),\overline{\mu}^n_r(\omega)\Big)\big(\overline{X}^j_r(\omega)\big)R^{\overline{X}^j}_{r,s}(\omega)-\Big\langle D_\mu \mathrm{F}\Big(\overline{X}^i_r(\omega),\mathcal{L}(X_r)\Big)\big(X_r(\cdot)\big)R^{\overline{X}}_{r,s}(\cdot)\Big\rangle\bigg|\\&\leqslant c\,W_1\Big(\overline{\mu}^n_r(\omega),\mathcal{L}(X_r)\Big)\,\bigg(\frac{1}{n}\sum_{j=1}^n\big|R^{\overline{X}^j}_{r,s}(\omega)\big|\bigg)+c\Big|\mathcal{S}^{i,n}_{r,s}\Big(\omega,\big|R^{\overline{X}^\bullet}_{r,s}(\omega)\big|\Big)\Big|,\end{aligned}$$

where $\mathcal{S}^{i,n}_{r,s}\big(\omega,|R^{\overline{X}^\bullet}_{r,s}(\omega)|\big)$ is the n-empirical mean of n random variables that are dominated by $\big(|R^{\overline{X}^j}_{r,s}(\omega)|\big)_{j=1,\cdots,n}$ and $n-1$ of which are conditionally centered and conditionally independent given the realization of the path $(\overline{X}^i,W^i,\mathbb{W}^i)$. Hence, the $\mathbb{L}^\varrho$ norm of the right-hand side, after multiplication as before by $\overline{w}^i(s,t,\omega)^{1/p}$, is less than

$$c\left(\Big\langle W_1\Big(\overline{\mu}^n_r(\cdot),\mathcal{L}(X_r)\Big)\Big\rangle_{3\varrho}+n^{-1/2}\right)\big\langle\|\!|X(\cdot)\|\!|_{[0,T],w,p}\big\rangle_{6\varrho}\big\langle w(r,t,\cdot)\big\rangle^{3/p}_{6\varrho}.$$

As for the third term in the right-hand side of (A.7), it fits exactly, up to the additional factor $\overline{X}^i_{r,s}(\omega)$, the analysis provided in the first step. So we get as an upper bound for its $\mathbb{L}^\varrho$ norm, after multiplication by $\overline{w}^i(s,t,\omega)^{1/p}$, the quantity

$$\begin{aligned}&c\left(\int_0^1\!\!\int_0^1\Big\langle W_1\Big(\overline{\mu}^{n,(\lambda\lambda')}_{r;(r,s)}(\cdot),\mathcal{L}\big(X^{(\lambda\lambda')}_{r;(r,s)}\big)\Big)\Big\rangle_{3\varrho}d\lambda d\lambda'\right)\big\langle\|\!|X(\cdot)\|\!|_{[0,T],w,p}\big\rangle^2_{6\varrho}\big\langle w(r,t,\cdot)\big\rangle^{3/p}_{6\varrho}\\&\quad+c\,n^{-1/2}\big\langle\|\!|X(\cdot)\|\!|_{[0,T],w,p}\big\rangle^2_{6\varrho}\big\langle w(r,t,\cdot)\big\rangle^{3/p}_{6\varrho}.\end{aligned}$$

Following Step 2, we get exactly a similar bound for the fourth term in the right-hand side of (A.7). Applying once again Lemma 26 completes the proof. ▷

26. Lemma – *There exists a real $q_d\geqslant 1$ such that, for any $q\geqslant q_d$ and any probability measure μ on $\mathbb{R}^d$ satisfying $M_q(\mu):=\big(\int_{\mathbb{R}^d}|x|^q\mu(dx)\big)^{1/q}<\infty$, it holds*

$$\mathbb{E}\Big[W_1\big(\mu^n(\cdot),\mu\big)^{q/2}\Big]^{2/q}\leqslant c_{q,d}\,M_q(\mu)\,\eta_n,$$

for a constant $c_{q,d}$ depending on q and d, where $(\eta_n)_{n\geqslant 1}$ is as in the statement of Theorem 24 *and $\mu^n(\cdot)$ is the empirical distribution of n independent identically distributed random variables $\big(X^1(\cdot),\cdots,X^n(\cdot)\big)$ of law μ, namely*

$$\mu^n(\omega)=\frac{1}{n}\sum_{i=1}^n \delta_{X^i(\omega)}.$$

Proof – Without any loss of generality, we can assume that $M_q(\mu)=1$, see the argument in [9, Chapter 5]. Then, by [24, Theorem 2], we obtain, for $d\geqslant 3$,

$$\mathbb{P}\Big(W_1\big(\mu^n(\cdot),\mu\big)\geqslant A\eta_n\Big)\leqslant C\exp\Big(-cn\eta_n^dA^d\Big)+Cn\big(nA\eta_n\big)^{-q/2},$$

in which case the result easily follows. When $d=1$, we have

$$\mathbb{P}\Big(W_1\big(\mu^n(\cdot),\mu\big)\geqslant A\eta_n\Big)\leqslant C\exp\Big(-cn\eta_n^2A^2\Big)+Cn\big(nA\eta_n\big)^{-q/2},$$

and the result follows as well by our choice of η_n. Finally, when $d=2$,

$$\mathbb{P}\Big(W_1\big(\mu^n(\cdot),\mu\big)\geqslant A\eta_n\Big)\leqslant C\exp\left(-\frac{cn\eta_n^2A^2}{(\ln(2+A^{-1}\eta_n^{-1}))^2}\right)+Cn(nA\eta_n)^{-q/2}.$$

Assuming without any loss of generality that $A\geqslant 1$, we have $\ln(2+A^{-1}\eta_n^{-1})\leqslant \ln(2+\eta_n^{-1})=\ln(1+2\eta_n)-\ln(\eta_n)$, which is less than $-2\ln(\eta_n)$ for n large enough. Given our choice of η_n, we have $-\ln(\eta_n)=\ln(n)/2-\ln(\ln(1+n))$, which is less than $\ln(n)/2$. Hence, modifying the value of the constant c, we get, for $A\geqslant 1$ and for n large enough, independently of the value of A, we get the bound

$$\mathbb{P}\Big(W_1\big(\mu^n(\cdot),\mu\big)\geqslant A\eta_n\Big)\leqslant C\exp\left(-\frac{cA^2\ln(1+n)^2}{(\ln(n))^2}\right)+Cn(nA\eta_n)^{-q/2},$$

which suffices to complete the proof. ▷

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