Convergence of a Degenerate Microscopic Dynamics of the Porous Medium Equation

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CONVERGENCE OF A DEGENERATE MICROSCOPIC DYNAMICS TO THE POROUS MEDIUM EQUATION

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Abstract. We derive the porous medium equation from an interacting particle system which belongs to the family of exclusion processes, with nearest neighbor exchanges. The particles follow a degenerate dynamics, in the sense that the jump rates can vanish for certain configurations, and there exist blocked configurations that cannot evolve. In [7] it was proved that the macroscopic density profile in the hydrodynamic limit is governed by the porous medium equation (PME), for initial densities uniformly bounded away from 0 and 1. In this paper we consider the more general case where the density can take those extreme values. In this context, the PME solutions display a richer behavior, like moving interfaces, finite speed of propagation and breaking of regularity. As a consequence, the standard techniques that are commonly used to prove this hydrodynamic limits cannot be straightforwardly applied to our case. We present here a way to generalize the *relative entropy method*, by involving approximations of solutions to the hydrodynamic equation, instead of exact solutions.

1. Introduction

The derivation of *macroscopic* partial differential equations from *microscopic* interacting particle systems has aroused an intense research activity in the past few decades. In particular, the family of conservative interacting particle systems with exclusion-type constraints is rich enough to provide significant results. One aims at showing that the macroscopic density profile for these models under time rescaling evolves according to some deterministic partial differential equation. The space-time scaling limit procedure which is at play here is called *hydrodynamic limit*. The simplest example in that family is the *symmetric simple exclusion process* (SSEP), for which the macroscopic hydrodynamic equation is the linear heat equation [9, Chapter 2.2].

In [7], Gonçalves *et al.* designed an exclusion process with *local kinetic constraints*, in order to obtain the *porous medium equation* (PME) as the macroscopic limit equation. The class of kinetically constrained lattice gases has been introduced in the physical literature in the 1980's (we refer to [1, 11] for a review) and is used to model liquid/glass transitions. The PME is a partial differential equation which reads in dimension one as

$$\partial_t \rho = \partial_{uu}(\rho^m), \tag{1.1}$$

where m is an positive integer which satisfies $m \geqslant 2$. The PME belongs to the class of diffusion equations, with diffusion coefficient $D(\rho) = m\rho^{m-1}$. Since $D(\rho)$ vanishes as $\rho \to 0$, the PME is not parabolic, and its solutions can be compactly supported at each fixed time, the boundary of the positivity set $\{\rho > 0\}$ moving at finite speed. Another important feature is that if the initial condition ρ^{ini} of (1.1) is allowed to vanish, then the solution $\rho(t,u)$ can have gradient discontinuities across the interfaces which separate

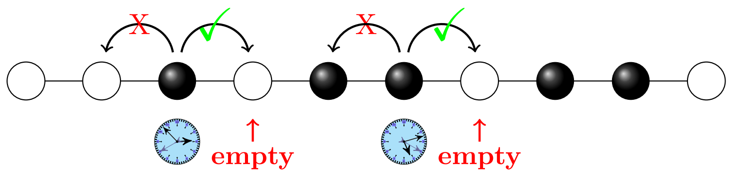

FIGURE 1. Allowed jumps are denoted by ✓. Forbidden jumps are denoted by X.

the positivity set $\{\rho > 0\}$ from its complement. We refer to the monograph [16] for an extended presentation of the mathematical properties of the PME.

We consider in this paper the following particle system (given by [7]). The setting is one-dimensional and periodic: particles are distributed on the points of the finite torus of size N denoted by $\mathbb{T}_N = \mathbb{Z}/N\mathbb{Z}$. We impose the exclusion restriction: no two particles can occupy the same site. A particle at x jumps to an empty neighboring site, say $x+1$, at rate 2 if there are particles at $x-1$ and $x+2$, at rate 1 if there is only one particle in $\{x-1, x+2\}$, and rate 0 else. The jump rate from $x+1$ to x is given by the same rule.

As explained in [7], this constrained exclusion process permits to derive the PME (1.1) with $m = 2$, when the process is accelerated in the diffusive time scale tN^2. However, in that paper the authors need to assume that the initial profile $\rho^{\rm ini}$ is uniformly bounded away from 0 and 1, namely that it satisfies an ellipticity condition of the form $0 < c_- \leqslant \rho^{\rm ini} \leqslant c_+ < 1$. With this assumption, the PME is uniformly parabolic and in particular does not display its more interesting features: finite speed of propagation and gradient discontinuities. The authors in [7] manage to circle around the problem by perturbing the microscopic dynamics with a slowed SSEP. This way, they gain ergodicity of the Markov process and can derive the PME using the well-known *entropy method* introduced in [8].

In this paper, we do not assume the ellipticity condition on $\rho^{\rm ini}$ and we keep the original model described above. We believe this is the first derivation of a moving boundary problem from a conservative and degenerate microscopic dynamics (see [6, 15] for derivations in non-conservative or non-degenerate settings). Our choice of initial condition makes the entropy method and the *relative entropy method* fail (these techniques are explained in detail in [9]). Indeed, the lack of ergodicity breaks any hope to use the entropy method and the special features of the PME are a serious obstacle to using the relative entropy method. Let us explain why and describe how we manage anyway here.

The relative entropy method was introduced for the first time by Yau [17], and its main idea is the following: since the particle system has a family of product invariant measures indexed by the density (here, the Bernoulli product measure ν_ρ^N), one can use the non-homogeneous product measures $\nu_{\rho(t,u)}^N$ with slowly varying parameter associated with the solution $\rho(t,u)$ to (1.1), and compare it to the state at macroscopic time t of the diffusively accelerated Markov process. The latter is denoted below by μ_t^N, it is a probability law on $\{0,1\}^{\mathbb{T}_N}$. If one expects the PME to be the correct hydrodynamic equation, these two measures should be close, and this can be seen from the investigation of the time evolution of the relative entropy $H(\mu_t^N|\nu_{\rho(t,u)}^N)$.

In our case, two obstacles appear straight away. The first one is that $\rho(t,u)$ can take values 0 and 1, and therefore the above entropy will generally be infinite. The second one is that the solution $\rho(t,u)$ has poor analytic properties as soon as ρ^{ini} vanishes, which will complicate the control of the time evolution of the entropy. To remove these obstacles, we modify the original investigation by considering an approximation of $\rho(t,u)$, denoting ahead by $\rho_N(t,u)$, which satisfies two important properties:

(i) it is bounded away from 0 and 1 and regular;
(ii) the sequence (ρ_N) uniformly converges to ρ on compactly supported time intervals.

As we will see in the text, these two properties are not enough to apply straightforwardly Yau's method: we also need sharp controls on several derivatives of ρ. Moreover, the usual *one-block estimate* (which is at the core of the relative entropy method) requires understanding the interface between the positivity set of ρ and its complement. These are the main ingredients of our proof.

Finally, note that our result could be easily generalized to the case $m \geqslant 3$, as in [7]. More generally, the idea of plugging an approximation of the solution into the relative entropy method should apply to other degenerate particle systems and allow to derive other degenerate parabolic equations. The additional work with respect to what we present here would be to derive the corresponding analytic estimates on the solution to the PME (see mainly Proposition 3.3 and the estimates in Section 4). The complexity of this program in higher dimensions is the reason we kept $d = 1$.

Here follows an outline of the paper. In Section 2, we introduce and define the model with its notations, and we state our hydrodynamic limit result. In Section 3, we start with recalling some specificities of the solutions to the porous medium equation, then we give an crucial property of the boundary of the positivity set. We also define an approximation of the solution ρ_N and study its convergence. Finally we expose the strategy of the proof of the hydrodynamic limit through the control of $H(\mu_t^N|\nu_{\rho_N(t,u)}^N)$, which generalizes the usual relative entropy method. The estimates that we need about the derivatives of ρ_N are proved in Section 4. The proof of the hydrodynamic limit, and in particular the one-block estimate, is completed in Section 5.

2. Hydrodynamics limits

2.1. **Context.** Let us introduce with more details the microscopic dynamics which was first given in [7], and which we described in the introduction. For any $x \in \mathbb{T}_N$, we set $\eta(x) = 1$ if x is occupied, and $\eta(x) = 0$ if x is empty, which makes our state space $\{0,1\}^{\mathbb{T}_N}$. The dynamics can be entirely encoded by the infinitesimal generator $\mathcal{L}_N$ which acts on functions $f : \{0,1\}^{\mathbb{T}_N} \to \mathbb{R}$ as

$$\mathcal{L}_N f(\eta) := \sum_{\substack{x,y\in\mathbb{T}_N \\ |x-y|=1}} r_{x,y}(\eta)\eta(x)(1-\eta(y))\big(f(\eta^{x,y}) - f(\eta)\big), \tag{2.1}$$

where

$$r_{x,x+1}(\eta) = r_{x+1,x}(\eta) = \eta(x-1) + \eta(x+2),$$

and

$$\eta^{x,y}(z) = \begin{cases} \eta(y) & \text{if } z = x, \\ \eta(x) & \text{if } z = y, \\ \eta(z) & \text{otherwise.} \end{cases}$$

The initial configuration is random, distributed according to some initial probability measure μ_0^N on $\{0,1\}^{\mathbb{T}_N}$. We denote by $(\eta_t^N)_{t\geqslant 0}$ the Markov process generated by $N^2\mathcal{L}_N$ (note that it is equivalent to accelerate time by a factor N^2) and starting from the initial state μ_0^N. For any fixed $t \geqslant 0$, the probability law of $\{\eta_t^N(x)\ ;\ x \in \mathbb{T}_N\}$ on the state space $\{0,1\}^{\mathbb{T}_N}$ is denoted by μ_t^N.

In the following we also denote by $\mathbb{P}_{\mu_0^N}$ the probability measure on the space of trajectories $\mathcal{D}(\mathbb{R}_+, \{0,1\}^{\mathbb{T}_N})$ induced by the initial state μ_0^N and the accelerated Markov process $(\eta_t^N)_{t\geqslant 0}$. Its corresponding expectation is denoted by $\mathbb{E}_{\mu_0^N}$.

2.2. **Product Bernoulli measures.** For any $\alpha \in [0,1]$, let ν_α^N be the Bernoulli product measure on $\{0,1\}^{\mathbb{T}_N}$ with marginal at site $x \in \mathbb{T}_N$ given by

$$\nu_\alpha^N\{\eta\ :\ \eta(x) = 1\} = \alpha.$$

In other words, we put a particle at each site x with probability α, independently of the other sites. Similarly, we define ν_α as the Bernoulli product measure on $\{0,1\}^{\mathbb{Z}}$. We denote by E_α the expectation with respect to ν_α, and note that $E_\alpha[\eta(0)] = \alpha$. One can easily check that the product measures $\{\nu_\alpha^N\ ;\ \alpha \in [0,1]\}$ are *reversible* for the Markov process (η_t^N).

As the size N of the system goes to ∞, the discrete torus $\mathbb{T}_N$ tends to the full lattice $\mathbb{Z}$. Therefore, we will need to consider functions on the space $\{0,1\}^{\mathbb{Z}}$. Let $\varphi : \{0,1\}^{\mathbb{Z}} \to \mathbb{R}$ be a *local* function, in the sense that $\varphi(\eta)$ depends on η only through a finite number of coordinates, and therefore φ is necessarily bounded. We then denote by $\overline{\varphi}(\alpha)$ its average with respect to the measure ν_α:

$$\overline{\varphi}(\alpha) := E_\alpha[\varphi(\eta)].$$

Note that $\alpha \mapsto \overline{\varphi}(\alpha)$ is continuous for every local function φ.

The one-dimensional continuous torus is denoted by $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Let us now define the *non-homogeneous product measure* $\nu_{\rho(\cdot)}^N$ on $\{0,1\}^{\mathbb{T}_N}$ associated with a density profile $\rho : \mathbb{T} \to [0,1]$, whose marginal at site $x \in \mathbb{T}_N$ is given by

$$\nu_{\rho(\cdot)}^N\{\eta\ :\ \eta(x) = 1\} = 1 - \nu_{\rho(\cdot)}^N\{\eta\ :\ \eta(x) = 0\} = \rho\big(\tfrac{x}{N}\big). \tag{2.2}$$

We denote by $\mathbb{E}_{\rho(\cdot)}^N$ the expectation with respect to $\nu_{\rho(\cdot)}^N$. If $\rho(\cdot)$ is continuous on $\mathbb{T}$ and if $\varphi : \{0,1\}^{\mathbb{Z}} \to \mathbb{R}$ is local, then the following Riemann convergence holds:

$$\frac{1}{N}\sum_{x\in\mathbb{T}_N} \mathbb{E}_{\rho(\cdot)}^N\big[\tau_x\varphi(\eta)\big] \xrightarrow[N\to\infty]{} \int_{\mathbb{T}} E_{\rho(u)}\big[\varphi(\eta)\big]\,\mathrm{d}u = \int_{\mathbb{T}} \overline{\varphi}\big(\rho(u)\big)\,\mathrm{d}u. \tag{2.3}$$

Moreover, if a sequence of continuous profiles $\rho_N(\cdot)$ converges uniformly to $\rho(\cdot)$ on $\mathbb{T}$, then

$$\frac{1}{N}\sum_{x\in\mathbb{T}_N} \mathbb{E}_{\rho_N(\cdot)}^N\big[\tau_x\varphi(\eta)\big] \xrightarrow[N\to\infty]{} \int_{\mathbb{T}} E_{\rho(u)}\big[\varphi(\eta)\big]\,\mathrm{d}u. \tag{2.4}$$

The last convergence property will be used several times in the paper.

2.3. **Statement of the main result.** Let $\rho^{\rm ini} \in L^\infty(\mathbb{T};[0,1])$ be an initial density profile. Our goal is to consider the hydrodynamic limit of the microscopic dynamics described in Section 2. As already pointed out by Gonçalves *et al.* [7], the underlying macroscopic equation is expected to be the *porous medium equation* (PME)

$$\begin{cases} \partial_t \rho = \partial_{uu}(\rho^2) & \text{in } (0,\infty)\times\mathbb{T}, \\ \rho_{|t=0} = \rho^{\rm ini} & \text{in } \mathbb{T}. \end{cases} \tag{2.5}$$

This equation is of degenerate parabolic type. It is well known that the notion of strong solution —i.e., $\rho \in C^{1,2}(\mathbb{R}_+ \times \mathbb{T})$— is not suitable to get the well-posedness of the problem (2.5) unless $\rho^{\rm ini}$ remains bounded away from 0. Indeed, the space derivative of ρ may be discontinuous at the boundary of the set $\{\rho > 0\}$ (see for instance [16]). This motivates the introduction of the following notion of weak solutions.

Definition 2.1. *A function $\rho \in L^\infty(\mathbb{R}_+\times\mathbb{T};[0,1])$ is said to be a weak solution to (2.5) corresponding to the initial profile $\rho^{\rm ini}$ if $\partial_u(\rho^2) \in L^2(\mathbb{R}_+\times\mathbb{T})$ and*

$$\iint\limits_{\mathbb{R}_+\times\mathbb{T}} \rho\, \partial_t \xi \, {\rm d}u{\rm d}t + \int_{\mathbb{T}} \rho^{\rm ini}\, \xi(0,\cdot){\rm d}u - \iint\limits_{\mathbb{R}_+\times\mathbb{T}} \partial_u(\rho^2)\partial_u\xi\, {\rm d}u{\rm d}t = 0, \qquad \textit{for all } \xi \in C^1_c(\mathbb{R}_+\times\mathbb{T}). \tag{2.6}$$

What we call a weak solution corresponds to what is called an energy solution in Vazquez' monograph (see [16, Section 5.3.2]). The classical existence theory based on compactness arguments (see for instance [16, Therorem 5.5]) can be extended to our periodic setting without any difficulty. The uniqueness of the weak solution and the fact that they remain bounded between 0 and 1 are consequences of the following L^1-contraction/comparison principle (see [16, Proposition 6.1]): let $\rho^{\rm ini}$ and $\check\rho^{\rm ini}$ be two initial profiles in $L^\infty(\mathbb{T};[0,1])$, and let ρ and $\check\rho$ be corresponding weak solutions, then

$$\int_{\mathbb{T}} (\rho(t,u) - \check\rho(t,u))^+ {\rm d}u \leqslant \int_{\mathbb{T}} (\rho^{\rm ini}(u) - \check\rho^{\rm ini}(u))^+ {\rm d}u, \qquad \text{for any } t \geqslant 0, \tag{2.7}$$

where $a^+ = \max(a,0)$ denotes the positive part of a. In the above relation, we have used the fact that any weak solution to (2.5) belongs to $\mathcal{C}(\mathbb{R}_+; L^1(\mathbb{T}))$ (see for instance [3]).

In what follows, we assume that:

- the initial profile $\rho^{\rm ini}$ is Lipchitz continuous, namely there exists $C_{\rm Lip} > 0$ such that
$$\left\| \partial_u \rho^{\rm ini} \right\|_\infty \leqslant C_{\rm Lip}, \tag{2.8}$$
where $\|\cdot\|_\infty$ denotes the usual L^∞-norm;
- the set
$$\mathcal{P}_0 := \left\{ u \in \mathbb{T}\ ;\ \rho^{\rm ini}(u) > 0 \right\} \tag{2.9}$$
has a finite number of connected components.

Note that this assumption is much less restrictive than the one given in [7], where $\rho^{\rm ini}$ was supposed to be uniformly bounded away from 0 and 1. In particular, we authorize vanishing initial profiles. Our main result reads as follows:

Theorem 2.1. *We assume that the initial microscopic system of particles* $\{\eta_0(x)\ ;\ x \in \mathbb{T}_N\}$ *is distributed according to* $\mu_0^N = \nu^N_{\rho^{\rm ini}(\cdot)}$. *Then, the following* local equilibrium convergence *holds at any macroscopic time* $t > 0$*: for any continuous function* $G : \mathbb{T} \to \mathbb{R}$*, any local function* $\varphi : \{0,1\}^{\mathbb{Z}} \to \mathbb{R}$

$$\lim_{N\to\infty} \mathbb{E}_{\mu_0^N}\left[\left|\frac{1}{N}\sum_{x\in\mathbb{T}_N} G\Big(\frac{x}{N}\Big)\tau_x\varphi(\eta_t^N) - \int_{\mathbb{T}} G(u)\overline{\varphi}\big(\rho(t,u)\big)\mathrm{d}u\right|\right] = 0, \tag{2.10}$$

where ρ *is the unique weak solution of* (2.5) *in the sense of Definition 2.1.*

3. Porous medium equation: analytic results

Let us give here some properties of the solution to the (PME) to be used in the sequel. Sometimes we prove the results only partially, and we invite the reader to check the details of the proofs in the monograph [16] written by J.L.Vazquez. Precise references will be given for each result.

If the porous medium equation starts with an initial profile which vanishes, then the solution at any later time can have discontinuous gradients across the *interfaces* at which the function becomes positive. This is a problem when one tries to prove hydrodynamic limits. The best way to tackle discontinuity problems is to slightly perturb the initial condition, by making it positive, and bounded away from 1.

In Section 3.1, we state some properties of the PME starting from an initial profile which can lead to singularities at positive times. In Section 3.2 we modify the initial condition so as to regularize the solution of the PME and gain better control estimates. In Section 3.3 we expose the strategy to prove Theorem 2.1.

In the following we denote by $\|\cdot\|_p$ the usual L^p-norm, whenever the integration spaces are clear to the reader. Otherwise, the $L^p(\Omega)$-norm will be denoted by $\|\cdot\|_{L^p(\Omega)}$.

3.1. The porous medium equation (PME).

We start with recalling some properties of the unique weak solution $\rho(t,u)$ to (2.5). Our first statement is related to the continuity of the weak solutions to the porous medium equation. Such a regularity result can be deduced from [16, Section 7.7]. It is also a straightforward consequence of the forthcoming Proposition 3.6.

Proposition 3.1 (Regularity of the solution)**.** *The unique weak solution to* (2.5) *is continuous on* $\mathbb{R}_+ \times \mathbb{T}$.

Let us denote by $\mathring{A}$ the interior of the subset $A \subset \mathbb{T}$ and by $\overline{A}$ its closure. For all $t \geqslant 0$, we denote by

$$\mathcal{P}_t := \big\{u \in \mathbb{T}\ :\ \rho(t,u) > 0\big\}$$

the positivity set of $\rho(t,\cdot)$, which is an open subset of $\mathbb{T}$ since $\rho(t,\cdot)$ is continuous. Finally we denote by

$$\Gamma_t := \partial\mathcal{P}_t = \overline{\mathcal{P}_t}\backslash\mathcal{P}_t \tag{3.1}$$

the *interface* between the positivity set $\mathcal{P}_t$ of $\rho(t,\cdot)$ and the complementary

$$\mathcal{Z}_t := \overbrace{\{u \in \mathbb{T}\ :\ \rho(t,u) = 0\}}^{\circ} = \mathbb{T}\backslash\overline{\mathcal{P}_t} \tag{3.2}$$

of its support. Note that Γ_t is closed, and is a nowhere dense set, but it can *a priori* have positive Lebesgue measure. Actually, we will prove in Lemma 3.4 below that from our assumption (2.9) on $\mathcal{P}_0$, this does not happen and $\mathrm{Leb}(\Gamma_t)=0$ for any $t>0$.

Remark 3.2. *Let us underline that the derivatives of ρ can have jump discontinuities on the so-called* free boundary

$$\bigcup_{t\in[0,T]}\{t\}\times\Gamma_t,$$

but ρ is smooth outside of this set. We refer the reader to [16, Chapter 14] *for the general theory and also* [16, Chapter 4] *for several examples.*

In what follows, the notation Leb stands for the usual Lebesgue measure restricted on $\mathbb{T}$, and $|B|$ denotes the cardinality of the discrete subset $B\subset\mathbb{T}_N$.

Proposition 3.3 (Positivity intervals)**.** *For any $\delta>0$ and $t\in[0,T]$ we denote by*

$$\Gamma_t(\delta)=\overline{\Big\{u\in\mathbb{T}\;;\;0<\rho(t,u)<\delta\Big\}}. \tag{3.3}$$

We have

$$\int_0^T \mathrm{Leb}\left(\Gamma_t(\delta)\right)\mathrm{d}t\xrightarrow[\delta\to0]{}0. \tag{3.4}$$

Proof. The proof follows from the following technical lemma, which we will prove ahead in Appendix B.

Lemma 3.4 (Connected components of the positivity set)**.** *For any $t>0$, $\mathcal{P}_t$ has a finite number of connected components.*

With this lemma, since $\mathcal{P}_t$ has a finite number of connected components for any $t>0$, Γ_t is a finite union of points, and therefore $\mathrm{Leb}(\Gamma_t)=0$. Since

$$\Gamma_t=\bigcap_{\delta>0}\overline{\Big\{u\in\mathbb{T}\;;\;0<\rho(t,u)<\delta\Big\}}=\bigcap_{\delta>0}\Gamma_t(\delta),$$

it follows from the monotonicity of the Lebesgue measure that

$$0=\mathrm{Leb}(\Gamma_t)=\lim_{\delta\to0}\mathrm{Leb}(\Gamma_t(\delta)),\qquad\text{for any } t\in[0,T].$$

Moreover, since $\Gamma_t(\delta)\subset\mathbb{T}$, we get that $\mathrm{Leb}(\Gamma_t(\delta))\leqslant1$ for all $t\in[0,T]$. Hence (3.4) follows from Lebesgue's dominated convergence Theorem. □

3.2. **The regularized initial condition.** In order to prove Theorem 2.1, we need to introduce a *regularized approximate solution* to the PME. This is the goal of this section.

Let $(\varepsilon_N)_{N\in\mathbb{N}}$ be a vanishing sequence such that $\varepsilon_N\in(0,\frac12)$. The speed at which $\varepsilon_N\to0$ will be made more precise later on.

Let $h\in\mathcal{C}^\infty(\mathbb{R};\mathbb{R}_+)$ be such that

$$(i)\quad \mathrm{Supp}(h)\subset(-1,1),\qquad \int_{\mathbb{R}}h(y)\mathrm{d}y=1, \tag{3.5}$$

$$(ii)\quad h(y)=h(-y), \tag{3.6}$$

$$(iii)\quad \partial_y h(y)\leqslant0,\qquad\text{if } y\geqslant0. \tag{3.7}$$

Denote $C_h:=\|h\|_\infty$. It follows from (3.6) and (3.7) that $\|\partial_y h\|_1=2C_h$. Let us define

(1) the *regularizing approximation of the unit*:

$$h_N(y) = \varepsilon_N^{-1}\, h(\varepsilon_N^{-1} y)$$

which satisfies $\mathrm{Supp}(h_N) \subset (-\varepsilon_N, \varepsilon_N)$,

$$\|h_N\|_1 = 1, \qquad \|h_N\|_\infty = \frac{C_h}{\varepsilon_N}, \qquad \|\partial_y h_N\|_1 = \frac{2C_h}{\varepsilon_N}, \tag{3.8}$$

(2) the *truncated and regularized initial data* $\rho_N^{\rm ini} : \mathbb{T} \to [\varepsilon_N, 1-\varepsilon_N]$ defined by

$$\rho_N^{\rm ini} = \widetilde{\rho}_N^{\rm ini} \star h_N, \tag{3.9}$$

where $\star$ is the usual convolution product on $\mathbb{T}$ and $\widetilde{\rho}_N^{\rm ini}$ is defined as

$$\widetilde{\rho}_N^{\rm ini} = \max\big\{\varepsilon_N\ ;\ \min\big(1-\varepsilon_N; \rho^{\rm ini}\big)\big\}.$$

The initial data $\rho_N^{\rm ini}$ satisfies the following three properties:

(a) *Regularity*: $\rho_N^{\rm ini}$ is smooth on $\mathbb{T}$,

(b) *Boundedness*:

$$\varepsilon_N \leqslant \rho_N^{\rm ini} \leqslant 1-\varepsilon_N, \qquad \text{and} \qquad \|\partial_u \rho_N^{\rm ini}\|_\infty \leqslant C_{\rm Lip}, \tag{3.10}$$

(c) *Uniform convergence towards* $\rho^{\rm ini}$:

$$\|\rho_N^{\rm ini} - \rho^{\rm ini}\|_\infty \leqslant (C_{\rm Lip}+1)\varepsilon_N \xrightarrow[N\to\infty]{} 0. \tag{3.11}$$

Let us now define the *regularized solution* ρ_N on $\mathbb{R}_+ \times \mathbb{T}$ as the solution to

$$\begin{aligned} &\partial_t \rho_N(t,u) = \partial_{uu}\big(\rho_N^2\big)(t,u), \qquad (t,u) \in \mathbb{R}_+ \times \mathbb{T}, \\ &\rho_N(0,u) = \rho_N^{\rm ini}(u). \end{aligned} \tag{3.12}$$

This solution will play a central role in the proof of Theorem 2.1. Let start here with two major properties of ρ_N.

Proposition 3.5. *Problem* (3.12) *admits a unique strong solution* $\rho_N \in \mathcal{C}^\infty([0,T]\times\mathbb{T})$ *which satisfies*

$$\varepsilon_N \leqslant \rho_N \leqslant 1-\varepsilon_N.$$

Proof. The uniqueness of the weak (then strong) solution follows from the monotonicity of the porous medium equation, which yields L^1-contraction and a comparison principle (see for instance [4]). It follows from this comparison principle that $\varepsilon_N \leqslant \rho_N \leqslant 1-\varepsilon_N$ a.e. in $[0,T]\times\mathbb{T}$. Therefore, the solution remains bounded away from the degeneracy $\rho = 0$ of the PME (2.5). The problem (3.12) is then uniformly parabolic. It follows from the classical regularity theory for parabolic equations (see for instance [10]) that ρ_N is smooth. See also [16, Theorem 3.1, Proposition 12.13]. □

Proposition 3.6 (Uniform convergence)**.** *The sequence* $(\rho_N)_{N\in\mathbb{N}}$ *converges uniformly in* $[0,T]\times\mathbb{T}$ *towards the unique weak solution to* (2.5).

Proof. It follows from the comparison principle (2.7) that

$$\int_{\mathbb{T}} |\rho(t,u) - \rho_N(t,u)| \mathrm{d}u \leqslant \int_{\mathbb{T}} |\rho^{\rm ini}(u) - \rho_N^{\rm ini}(u)| \mathrm{d}u, \qquad \text{for any } \ t \in [0,T].$$

Hence, we deduce from estimate (3.11) that $(\rho_N)_{N\in\mathbb{N}}$ converges in $\mathcal{C}([0,T]; L^1(\mathbb{T}))$ towards ρ. Therefore, it suffices to show that $(\rho_N)_{N\in\mathbb{N}}$ is relatively compact in $\mathcal{C}([0,T]\times\mathbb{T})$ to conclude the proof of Proposition 3.6 thanks to the uniqueness of the limit value. Our proof mainly follows the program of [16, Section 7.7]. We first need to introduce the *fractional Sobolev spaces* $H^s(\mathbb{T})$. We refer to [5] for an overview on fractional Sobolev spaces. Since we are in the simple situation where the domain is the one-dimensional torus, such spaces are very easy to define and to manipulate with Fourier series (see also our Appendix A).

Definition 3.1. *Let $s \in [0,1]$. A function $\rho : \mathbb{T} \to \mathbb{R}$ belongs to $H^s(\mathbb{T})$ iff*

$$\|\rho\|_{H^s(\mathbb{T})} := \left(\sum_{k\in\mathbb{Z}} \left(1 + 4\pi^2 |k|^2\right)^s \, |\widehat{\rho}_k|^2 \right)^{\frac{1}{2}} < \infty,$$

where the Fourier coefficient $\widehat{\rho}_k$ reads as

$$\widehat{\rho}_k := \int_{\mathbb{T}} \rho(u) e^{-i2\pi k u} \mathrm{d}u. \tag{3.13}$$

From Parseval's relation, we have

$$\|\rho\|^2_{H^1(\mathbb{T})} = \|\rho\|^2_{L^2(\mathbb{T})} + \|\partial_u \rho\|^2_{L^2(\mathbb{T})}.$$

Multiply the PME (3.12) by $\partial_t \left(\rho_N^2\right)$ and then integrate over $(0, t^\star) \times \mathbb{T}$ for some arbitrary $t^\star \in [0, T]$ to get

$$A_N(t^\star) + B_N(t^\star) = 0,$$

where

$$A_N(t^\star) = \iint_{(0,t^\star)\times\mathbb{T}} \partial_t \rho_N \; \partial_t \left(\rho_N^2\right) \mathrm{d}t\mathrm{d}u, \qquad B_N(t^\star) = \iint_{(0,t^\star)\times\mathbb{T}} \partial_u \left(\rho_N^2\right) \; \partial_{ut} \left(\rho_N^2\right) \mathrm{d}t\mathrm{d}u.$$

The bound $|\rho_N| \leqslant 1$ yields

$$A_N(t^\star) \geqslant \frac{1}{2} \iint_{(0,t^\star)\times\mathbb{T}} |\partial_t v_N|^2 \, \mathrm{d}t\mathrm{d}u,$$

where we have set $v_N := (\rho_N)^2$. On the other hand,

$$B_N(t^\star) = \frac{1}{2} \int_{\mathbb{T}} \left|\partial_u v_N(t^\star, u)\right|^2 \mathrm{d}u - \frac{1}{2} \int_{\mathbb{T}} \left|\partial_u \left(\left(\rho_N^{\mathrm{ini}}\right)^2\right)\right|^2 \mathrm{d}u.$$

(3.10) provides that

$$\frac{1}{2} \int_{\mathbb{T}} \left|\partial_u \left(\rho_N^{\mathrm{ini}}\right)^2\right|^2 \mathrm{d}u \leqslant 2C^2_{\mathrm{Lip}}.$$

Hence, we obtain that

$$\iint_{(0,t^\star)\times\mathbb{T}} |\partial_t v_N|^2 \, \mathrm{d}t\mathrm{d}u + \int_{\mathbb{T}} |\partial_u v_N(t^\star, u)|^2 \, \mathrm{d}u \leqslant 4C^2_{\mathrm{Lip}}, \qquad \text{for any } \; t^\star \in [0, T].$$

To sum up, we have the following (uniform w.r.t. N) estimates on the sequence $(v_N)_N$:

$$\|v_N\|_\infty \leqslant 1, \tag{3.14}$$

$$\|\partial_t v_N\|_{L^2((0,T)\times\mathbb{T})} \leqslant 2C_{\text{Lip}}, \tag{3.15}$$

$$\sup_{t\in[0,T]} \|\partial_u v_N(t,\cdot)\|_{L^2(\mathbb{T})} \leqslant 2C_{\text{Lip}}. \tag{3.16}$$

It follows from (3.16) and the Cauchy-Schwarz inequality that

$$|v_N(t,u) - v_N(t,\widehat{u})| \leqslant 2C_{\text{Lip}}|u-\widehat{u}|^{\frac12}, \qquad \text{for any } \ u,\widehat{u}\in\mathbb{T},\ t\in[0,T]. \tag{3.17}$$

Similarly, we deduce from (3.14) and (3.15) that $(v_N)_N$ is uniformly bounded in the space $\mathcal{C}^{0,\frac12}\big([0,T];L^2(\mathbb{T})\big)$, i.e.,

$$\|v_N(t) - v_N(\widehat{t})\|_{L^2(\mathbb{T})} \leqslant 2C_{\text{Lip}}|t-\widehat{t}|^{\frac12}, \qquad \text{for any } \ t,\widehat{t}\in[0,T]. \tag{3.18}$$

Using Hölder's inequality and Definition 3.1, we get that

$$\|v_N(t) - v_N(\widehat{t})\|_{H^s(\mathbb{T})} \leqslant \|v_N(t) - v_N(\widehat{t})\|_{H^1(\mathbb{T})}^s \|v_N(t) - v_N(\widehat{t})\|_{L^2(\mathbb{T})}^{1-s}, \qquad \text{for any } \ t,\widehat{t}\in[0,T].$$

Combining it with (3.16) and (3.18), this provides

$$\|v_N(t) - v_N(\widehat{t})\|_{H^s(\mathbb{T})} \leqslant C|t-\widehat{t}|^{\frac{1-s}{2}}, \qquad \text{for any } \ t,\widehat{t}\in[0,T],$$

where $C = 2C_{\text{Lip}}(T+4)^{\frac{s}{2}}$.

Choosing $s\in(\frac12,1)$ and using the continuous embedding of $H^s(\mathbb{T})$ in $\mathcal{C}(\mathbb{T})$ (which is proved, for the sake of completeness, in Appendix A), we get that

$$|v_N(t,u) - v_N(\widehat{t},u)| \leqslant C|t-\widehat{t}|^{\frac{1-s}{2}}, \qquad \text{for any } \ u\in\mathbb{T},\ t,\widehat{t}\in[0,T]. \tag{3.19}$$

The combination of (3.17) with (3.19) provides: for any $u,\widehat{u}\in\mathbb{T}$, and $t,\widehat{t}\in[0,T]$,

$$\begin{aligned} |v_N(t,u) - v_N(\widehat{t},\widehat{u})| &\leqslant |v_N(t,u) - v_N(\widehat{t},u)| + |v_N(\widehat{t},u) - v_N(\widehat{t},\widehat{u})| \\ &\leqslant \max\big\{C, 2(C_{\text{Lip}})^{\frac12}\big\}\ \Big(|t-\widehat{t}|^{\frac{1-s}{2}} + |u-\widehat{u}|^{\frac12}\Big). \end{aligned}$$

Therefore, one can apply Arzela-Ascoli's Theorem and claim that $(v_N)_N$ is relatively compact in $\mathcal{C}([0,T]\times\mathbb{T})$, and thus so is $(\rho_N)_N = \big(\sqrt{v_N}\big)_N$. This ends the proof. ☐

3.3. **Strategy of the proof.** In the following, for any probability measures μ,ν on $\{0,1\}^{\mathbb{T}_N}$ we denote by $H(\mu|\nu)$ the relative entropy of μ with respect to ν, defined as usual by

$$H(\mu|\nu) = \sup_f \left\{ \int f d\mu - \log\int e^f d\nu \right\},$$

where the supremum is carried over all real valued functions. The following *entropy inequality* is going to be useful: for any $\gamma>0$, we have

$$\int f d\mu \leqslant \frac{1}{\gamma}\Big(\log\int e^{\gamma f} d\nu + H(\mu|\nu)\Big). \tag{3.20}$$

Recall that we denote by $\mathbb{E}^N_{\rho_N(t,\cdot)}$ the expectation with respect to the non-homogeneous Bernoulli product measure $\nu^N_{\rho_N(t,\cdot)}$. Fix $\alpha \in (0,1)$ and an invariant measure ν_α. We introduce the density

$$\psi^N_t(\eta) := \frac{\mathrm{d}\nu^N_{\rho_N(t,\cdot)}}{\mathrm{d}\nu_\alpha}(\eta) = \frac{1}{\mathsf{Z}^N_t} \exp\bigg(\sum_{x\in\mathbb{T}_N} \eta(x)\, \lambda_N\Big(t, \frac{x}{N}\Big)\bigg),$$

where

$$\lambda_N(t,u) = \log\bigg(\frac{\rho_N(t,u)(1-\alpha)}{\alpha(1-\rho_N(t,u))}\bigg), \tag{3.21}$$

and Z^N_t is the normalization constant. Note that λ_N is well defined thanks to Proposition 3.5. Recall moreover that μ^N_t is the distribution of the accelerated process at time tN^2 and denote its density with respect to ν_α as

$$f^N_t := \frac{\mathrm{d}\mu^N_t}{\mathrm{d}\nu_\alpha}.$$

Finally, we are interested in the relative entropy

$$\mathcal{H}_N(t) := H\big(\mu^N_t | \nu^N_{\rho_N(t,\cdot)}\big) = \int f^N_t(\eta) \log\bigg(\frac{f^N_t(\eta)}{\psi^N_t(\eta)}\bigg) \mathrm{d}\nu_\alpha(\eta). \tag{3.22}$$

The proof of Theorem 2.1 is based on the investigation of the time evolution of that entropy $\mathcal{H}_N(t)$. This strategy is inspired by the *relative entropy method* which is exposed in details for instance in [9, Chapter 6]. However, in our case the standard method cannot work: the usual scheme works with the relative entropy of μ^N_t with respect to the product measure $\nu^N_{\rho(t,\cdot)}$, associated with the *true* weak solution of the PME (2.5). As we have seen in Section 3.1, this solution has poor regularity properties, and more importantly, it can vanish on non-trivial intervals. This would make the relative entropy take infinite values for presumably long times.

This is why we work with a different relative entropy: here, $\mathcal{H}_N(t)$ defined in (3.22) involves the non-homogeneous product measure $\nu^N_{\rho_N(t,\cdot)}$, which is associated with the *regularized* solution ρ_N, defined in (3.12). Since ρ_N is smooth and bounded away from 0 and 1, the relative entropy is always finite. Since (ρ_N) uniformly converges to ρ on $[0,T]\times\mathbb{T}$, one might believe that the arguments of [9] can be easily adapted. However, one needs much more than uniform convergence. In particular, sharp controls on the derivatives of ρ_N are also needed, as explained in the rest of the paper.

Let us conclude this section with two important results concerning $\mathcal{H}_N(t)$. At the end of this paragraph we will show how do they imply Theorem 2.1. First of all, at $t=0$, the initial relative entropy is of order $N\varepsilon_N|\log\varepsilon_N|$ as $N\to\infty$, namely:

Lemma 3.7.

$$\mathcal{H}_N(0) = H\big(\mu^N_0 | \nu^N_{\rho^{\rm ini}_N(\cdot)}\big) = H\big(\nu^N_{\rho^{\rm ini}} | \nu^N_{\rho^{\rm ini}_N(\cdot)}\big) = O(N\varepsilon_N|\log\varepsilon_N|) = o(N), \qquad \textit{as } N\to\infty.$$

This lemma is proved in Section 5.1. Next, we are able to control the entropy production on a finite time interval, thanks to all the sharp estimates that we will obtain in Section 4. This is where we need to make an assumption on the convergence speed of (ε_N). From now on we suppose that

$$\lim_{N\to\infty} N\varepsilon_N^6 = +\infty. \tag{3.23}$$

Proposition 3.8 (Entropy production)**.** *Assuming* (3.23)*, there exists a constant* $\kappa > 0$ *such that*

$$\mathcal{H}_N(T) \leqslant \kappa \int_0^T \mathcal{H}_N(s)\mathrm{d}s + o_T(N),$$

where $o_T(N)$ *stands for a sequence of real numbers* $C_{T,N}$ *such that* $C_{T,N}/N \to 0$ *as* $N \to \infty$*.*

We prove this result in Section 5.2.

From Gronwall's inequality and Lemma 3.7, we conclude:

Corollary 3.9. *For any* $t > 0$*,*

$$H\big(\mu_t^N \mid \nu^N_{\rho_N(t,\cdot)}\big) = \mathcal{H}_N(t) = o_t(N), \qquad \textit{as } N \to \infty.$$

Then, one has to prove that Corollary 3.9 is sufficient to show the local equilibrium result (2.10) stated in Theorem 2.1. To do so, one needs to know that the approximate solution $\rho_N(t,\cdot)$ converges uniformly to $\rho(t,\cdot)$ in $\mathbb{T}$ (which does hold from Proposition 3.6), and that the solution $\rho(t,\cdot)$ is continuous. We have all in hands to conclude the proof of Theorem 2.1:

Proof of Theorem 2.1. One has to compute the limit of the left hand side of (2.10). For the sake of clarity, we assume that the local function φ only depends on the configuration value at 0, namely: $\varphi(\eta) = \varphi(\eta(0))$. Recall that we want to prove that the expectation

$$\mathbb{E}_{\mu_0^N}\left[\left|\frac{1}{N}\sum_{x\in\mathbb{T}_N} G\Big(\frac{x}{N}\Big)\tau_x\varphi(\eta_t^N) - \int_{\mathbb{T}} G(u)\overline{\varphi}\big(\rho(t,u)\big)\mathrm{d}u\right|\right] \tag{3.24}$$

vanishes as $N \to \infty$. Note that G and $\rho(t,\cdot)$ are continuous and bounded. Then, for any fixed $t > 0$, we easily replace

$$\int_{\mathbb{T}} G(u)\overline{\varphi}\big(\rho(t,u)\big)\mathrm{d}u \quad \text{with} \quad \tfrac{1}{N}\sum_{x\in\mathbb{T}_N} G\big(\tfrac{x}{N}\big)\overline{\varphi}\big(\rho(t,\tfrac{x}{N})\big),$$

paying a small price of order $o_t(1)$. Next, we perform an integration by parts, and we bound as follows:

$$\begin{aligned}
&\int\Big|\tfrac{1}{N}\sum_{x\in\mathbb{T}_N} G\big(\tfrac{x}{N}\big)\varphi(\eta(x)) - \tfrac{1}{N}\sum_{x\in\mathbb{T}_N} G\big(\tfrac{x}{N}\big)\overline{\varphi}\big(\rho(t,\tfrac{x}{N})\big)\Big|\mathrm{d}\mu_t^N(\eta)\\
&\leqslant \int \tfrac{1}{N}\sum_{x\in\mathbb{T}_N}\Big|\tfrac{1}{2\ell+1}\sum_{|y-x|\leqslant\ell}\big(G\big(\tfrac{y}{N}\big) - G\big(\tfrac{x}{N}\big)\big)\varphi(\eta(y))\Big|\mathrm{d}\mu_t^N(\eta) &(3.25)\\
&\quad + \int \tfrac{1}{N}\sum_{x\in\mathbb{T}_N}\Big|\tfrac{1}{2\ell+1}G\big(\tfrac{x}{N}\big)\sum_{|y-x|\leqslant\ell}\big(\varphi(\eta(y)) - \overline{\varphi}\big(\rho(t,\tfrac{x}{N})\big)\big)\Big|\mathrm{d}\mu_t^N(\eta). &(3.26)
\end{aligned}$$

Since G is smooth, the first limit (3.25) vanishes as $N \to \infty$ and then $\ell \to \infty$. Since G is bounded, (3.24) vanishes if we are able to prove that

$$\limsup_{\ell\to\infty}\limsup_{N\to\infty}\int\Big(\frac{1}{N}\sum_{x\in\mathbb{T}_N}\Big|\frac{1}{2\ell+1}\sum_{|y-x|\leqslant\ell}\varphi(\eta(y)) - \overline{\varphi}\big(\rho(t,\tfrac{x}{N})\big)\Big|\Big)\mathrm{d}\mu_t^N(\eta) = 0.$$

By the entropy inequality (3.20), for every $\gamma > 0$, we bound the expectation under the previous limit by

$$\frac{\mathcal{H}_N(t)}{\gamma N} + \frac{1}{\gamma N}\log \mathbb{E}^N_{\rho_N(t,\cdot)}\Big[\exp\Big(\gamma \sum_{x\in\mathbb{T}_N}\Big|\frac{1}{2\ell+1}\sum_{|y-x|\leqslant \ell}\varphi(\eta(y)) - \overline{\varphi}\big(\rho\big(t,\tfrac{x}{N}\big)\big)\Big|\Big)\Big].$$

From Corollary 3.9, the first term above vanishes as $N\to\infty$. As for the second term, we use the fact that $\nu^N_{\rho_N(t,\cdot)}$ is a product measure, and from Hölder's inequality we bound it from above by

$$\frac{1}{\gamma N}\sum_{x\in\mathbb{T}_N}\frac{1}{2\ell+1}\log \mathbb{E}^N_{\rho_N(t,\cdot)}\Big[\exp\Big(\gamma\Big|\sum_{|y-x|\leqslant\ell}\varphi(\eta(y)) - \overline{\varphi}\big(\rho\big(t,\tfrac{x}{N}\big)\big)\Big|\Big)\Big]. \tag{3.27}$$

Since the profile $\rho(t,\cdot)$ is continuous on $\mathbb{T}$, and the function $\rho_N(t,\cdot)$ converges uniformly to $\rho(t,\cdot)$ (from Proposition 3.6) we deduce that (3.27) converges as $N\to\infty$ to

$$\frac{1}{\gamma}\int_{\mathbb{T}}\frac{1}{2\ell+1}\log E_{\rho(t,u)}\Big[\exp\Big(\gamma\Big|\sum_{|y|\leqslant\ell}\varphi(\eta(y)) - \overline{\varphi}(\rho(t,u))\Big|\Big)\Big]\,\mathrm{d}u,$$

see also (2.4). To conclude the proof, we proceed as in [9, Chapter 6.1]: use the inequalities $e^x \leqslant 1 + x + \frac12 x^2 e^{|x|}$ and $\log(1+x)\leqslant x$. Finally, choose $\gamma = \varepsilon/(2\ell+1)$. From the law of large numbers, last expression vanishes as $\ell\to\infty$ and then $\varepsilon\to 0$. □

4. Norm bounds: statement and proof

In this section we state and prove the bounds on the derivatives of the regularized solution, that are needed for Proposition 3.8. The latter will be proved further in Section 5.

Proposition 4.1. *For any $N\in\mathbb{N}$,*

$$\sup_{(t,u)\in[0,T]\times\mathbb{T}}\big|\partial_u\rho_N(t,u)\big| \leqslant C_{\mathrm{Lip}},$$

where C_{Lip} has been defined in (2.8).

Proof. Let us define

$$w_N := \partial_u\rho_N + C_{\mathrm{Lip}}, \qquad w_N^{\mathrm{ini}} := \partial_u\rho_N^{\mathrm{ini}} + C_{\mathrm{Lip}} \qquad \text{for any } \ N\in\mathbb{N}, \tag{4.1}$$

then $w_N^{\mathrm{ini}} \geqslant 0$ thanks to (3.10). Moreover, w_N is smooth and satisfies $w_N(0,\cdot) = w_N^{\mathrm{ini}}$ and

$$\partial_t w_N - \partial_u\left(2\rho_N\partial_u w_N + 2(w_N)^2 - 4C_{\mathrm{Lip}}w_N\right) = 0, \qquad \text{for any } N\geqslant 0. \tag{4.2}$$

Multiplying (4.2) by $-w_N^- = \min(0,w_N)$ and integrating over $\mathbb{T}$ yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}}\frac{(w_N^-)^2}{2}\mathrm{d}u + \int_{\mathbb{T}}2\rho_N\left(\partial_u w_N^-\right)^2\mathrm{d}u - \int_{\mathbb{T}}\partial_u\Psi(w_N)\mathrm{d}u = 0,$$

where we have set

$$\Psi(y) = \begin{cases}\frac23 y^3 - 2C_{\mathrm{Lip}}y^2 & \text{if } y\leqslant 0,\\ 0 & \text{otherwise.}\end{cases}$$

Indeed, it follows from the chain-rule property (see for instance [13, Lemma 1.1]) that

$$-w_N^-\,\partial_t w_N = \tfrac12\partial_t\left(w_N^-\right)^2, \qquad -\partial_u w_N^-\,\partial_u w_N = \left(\partial_u w_N^-\right)^2,$$

and

$$\left(2(w_N)^2 - 4C_{\mathrm{Lip}} w_N\right) \partial_u w_N^- = -\partial_u \Psi(w_N).$$

Since w_N is periodic, one gets

$$\int_{\mathbb{T}} \partial_u \Psi(w_N) \mathrm{d}u = 0.$$

Therefore, the function

$$t \mapsto \int_{\mathbb{T}} \frac{\left(w_N^-\right)^2 (t,u)}{2} \mathrm{d}u$$

is non-increasing, takes nonnegative values and vanishes for $t = 0$. Therefore, it is constant equal to 0, hence $w_N(\cdot, t) \geqslant 0$ for all $t \geqslant 0$. Owing to the definition (4.1) of w_N, we get that $\partial_u \rho_N \geqslant -C_{\mathrm{Lip}}$. Proving that $\partial_u \rho_N \leqslant C_{\mathrm{Lip}}$ is similar. □

Proposition 4.2. *For any $N \in \mathbb{N}$,*

$$\iint_{[0,T]\times\mathbb{T}} \left|\partial_{uu}\rho_N(t,u)\right|^2 \mathrm{d}t\mathrm{d}u \leqslant \frac{(C_{\mathrm{Lip}})^2}{4\varepsilon_N}.$$

Proof. If $f_N = \partial_u \rho_N$ and $f_N^{\mathrm{ini}} = \partial_u \rho_N^{\mathrm{ini}}$, then f_N satisfies the equation

$$\partial_t f_N - \partial_u \left(2\rho_N \partial_u f_N + 2 f_N^2\right) = 0, \qquad \text{for any} \quad N \in \mathbb{N}. \tag{4.3}$$

Multiplying (4.3) by f_N and integrating over $\mathbb{T}$ yields

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{T}} \frac{|f_N|^2}{2} \mathrm{d}u + 2 \int_{\mathbb{T}} \rho_N \left|\partial_u f_N\right|^2 \mathrm{d}u + \frac{2}{3} \int_{\mathbb{T}} \partial_u (f_N)^3 \, \mathrm{d}u = 0. \tag{4.4}$$

Since f_N is periodic, the third integral in the above equality is equal to 0. Therefore, integrating (4.4) w.r.t. time over $[0,T]$ yields

$$\iint_{[0,T]\times\mathbb{T}} \rho_N \left|\partial_u f_N\right|^2 \mathrm{d}t\mathrm{d}u \leqslant \frac{1}{4} \int_{\mathbb{T}} |f_N^{\mathrm{ini}}|^2 \mathrm{d}u, \qquad \text{for any} \quad N \in \mathbb{N}.$$

Using that $\rho_N \geqslant \varepsilon_N$ (see Proposition 3.5), one gets that

$$\iint_{[0,T]\times\mathbb{T}} \left|\partial_{uu}\rho_N\right|^2 \mathrm{d}t\mathrm{d}u = \iint_{[0,T]\times\mathbb{T}} \left|\partial_u f_N\right|^2 \mathrm{d}t\mathrm{d}u \leqslant \frac{1}{4\varepsilon_N} \int_{\mathbb{T}} |\partial_u \rho_N^{\mathrm{ini}}|^2 \, \mathrm{d}u \leqslant \frac{(C_{\mathrm{Lip}})^2}{4\varepsilon_N},$$

for any $N \in \mathbb{N}$. □

Proposition 4.3. *There exists $C_0(T) > 0$ such that, for any $N \in \mathbb{N}$,*

$$\sup_{t\in[0,T]} \int_{\mathbb{T}} \left|\partial_{uu}\rho_N(t,u)\right|^2 \mathrm{d}u \leqslant \frac{C_0(T)}{\varepsilon_N^2} \tag{4.5}$$

$$\iint_{[0,T]\times\mathbb{T}} \left|\partial_{uuu}\rho_N(t,u)\right|^2 \mathrm{d}t\mathrm{d}u \leqslant \frac{C_0(T)}{\varepsilon_N^3}. \tag{4.6}$$

Proof. For all $N \in \mathbb{N}$, we set $g_N = \partial_{uu}\rho_N$, then it satisfies

$$\partial_t g_N - \partial_u \left(2\rho_N \partial_u g_N + 6 f_N g_N\right) = 0, \tag{4.7}$$

where $f_N = \partial_u \rho_N$, and $(g_N)_{|t=0} = \partial_{uu}\rho_N^{\rm ini}$. Multiplying (4.7) by g_N and integrating over $(0,t^\star)\times\mathbb{T}$ for some arbitrary $t^\star \in [0,T]$ provides

$$\frac{1}{2}\int_{\mathbb{T}} |g_N|^2(t^\star,u)\mathrm{d}u - \frac{1}{2}\int_{\mathbb{T}} \left|\partial_{uu}\rho_N^{\rm ini}\right|^2 \mathrm{d}u + \iint\limits_{(0,t^\star)\times\mathbb{T}} 2\rho_N|\partial_u g_N|^2 \mathrm{d}t\mathrm{d}u + \iint\limits_{(0,t^\star)\times\mathbb{T}} 6 f_N g_N \partial_u g_N \mathrm{d}t\mathrm{d}u = 0.$$

Using the elementary inequality

$$6 f_N g_N \partial_u g_N \geqslant -\frac{6|f_N|^2|g_N|^2}{\rho_N} - \frac{3}{2}\rho_N |\partial_u g_N|^2,$$

one gets that

$$\int_{\mathbb{T}} |g_N|^2(t^\star,u)\mathrm{d}u + \iint\limits_{(0,t^\star)\times\mathbb{T}} \rho_N|\partial_u g_N|^2 \mathrm{d}t\mathrm{d}u \leqslant A_N + B_N(t^\star), \tag{4.8}$$

where we have set

$$A_N = \int_{\mathbb{T}} \left|\partial_{uu}\rho_N^{\rm ini}\right|^2 \mathrm{d}u, \qquad B_N(t^\star) = 12 \iint\limits_{(0,t^\star)\times\mathbb{T}} \frac{|f_N|^2|g_N|^2}{\rho_N}\mathrm{d}t\mathrm{d}u.$$

The lower bound $\rho_N \geqslant \varepsilon_N$ (see Proposition 3.5) together with the L^∞-estimate on f_N (given in Proposition 4.1) ensure that

$$B_N(t^\star) \leqslant \frac{12 C_{\rm Lip}^2}{\varepsilon_N} \iint\limits_{(0,t^\star)\times\mathbb{T}} |g_N|^2 \mathrm{d}t\mathrm{d}u.$$

Then it follows from Proposition 4.2 that

$$B_N(t^\star) \leqslant \frac{3C_{\rm Lip}^4}{\varepsilon_N^2}, \qquad \text{for any} \quad t^\star \in [0,T]. \tag{4.9}$$

Let us now focus on A_N. One has clearly $A_N \leqslant \left\|\partial_{uu}\rho_N^{\rm ini}\right\|_\infty^2$. The definition (3.9) of $\rho_N^{\rm ini}$ implies that

$$\left|\partial_{uu}\rho_N^{\rm ini}(u)\right| = \left|\int_{\mathbb{T}} \partial_u\tilde\rho_N^{\rm ini}(y)\partial_u h_N(u-y)\mathrm{d}y\right| \leqslant \left\|\partial_u\tilde\rho_N^{\rm ini}\right\|_\infty \left\|\partial_u h_N\right\|_1, \qquad \text{for any} \quad u \in \mathbb{T}.$$

Therefore, we deduce from (3.8) and (3.10) that

$$A_N \leqslant 4\Big(\frac{C_{\rm Lip} C_h}{\varepsilon_N}\Big)^2. \tag{4.10}$$

From (4.8) we get

$$\sup_{t\in[0,T]} \int_{\mathbb{T}} \left|\partial_{uu}\rho_N(t,u)\right|^2 \mathrm{d}u = \sup_{t\in[0,T]} \int_{\mathbb{T}} |g_N|^2(t,u)\mathrm{d}u \leqslant A_N + \sup_{t\in[0,T]} B_N(t).$$

The combination of (4.9) and (4.10) leads to (4.5) with $C_0 = C_{\rm Lip}^2(3C_{\rm Lip}^2 + 4C_h^2)$. In the same way, from (4.8), and using the fact that $\rho_N \geqslant \varepsilon_N$ (recall Proposition 3.5), we have:

$$\iint\limits_{[0,T]\times\mathbb{T}} \big|\partial_{uuu}\rho_N(t,u)\big|^2 \,\mathrm{d}t\mathrm{d}u = \iint\limits_{[0,T]\times\mathbb{T}} \big|\partial_u g_N(t,u)\big|^2 \,\mathrm{d}t\mathrm{d}u \leqslant \frac{1}{\varepsilon_N}\big(A_N + B_N(T)\big),$$

and similarly we obtain (4.6). □

5. Relative entropy estimates

In this section we prove Lemma 3.7 and Proposition 3.8.

5.1. Proof of Lemma 3.7.

We say that a configuration $\eta \in \{0,1\}^{\mathbb{T}_N}$ is *ρ-compatible* with a profile $\rho : \mathbb{T} \to [0,1]$ if

$$\eta(x) = \rho\big(\tfrac{x}{N}\big) \quad \text{whenever } \rho\big(\tfrac{x}{N}\big) = 0 \text{ or } 1.$$

Recall Definition 2.2. Since $\rho_N^{\rm ini} \in [\varepsilon_N, 1-\varepsilon_N]$, we can easily compute

$$\mathcal{H}_N(0) = \sum_{\eta\ \rho^{\rm ini}\text{–comp.}} \nu^N_{\rho^{\rm ini}}(\eta)\Bigg\{ \sum_{x:\ \rho^{\rm ini}(\frac{x}{N})=0} \log\frac{1}{1-\rho_N^{\rm ini}(\frac{x}{N})} + \sum_{x:\ \rho^{\rm ini}(\frac{x}{N})=1} \log\frac{1}{\rho_N^{\rm ini}(\frac{x}{N})}$$
$$+ \sum_{x:\ \rho^{\rm ini}(\frac{x}{N})\in(0,1)} \left(\eta(x)\log\frac{\rho^{\rm ini}(\frac{x}{N})}{\rho_N^{\rm ini}(\frac{x}{N})} + (1-\eta(x))\log\frac{1-\rho^{\rm ini}(\frac{x}{N})}{1-\rho_N^{\rm ini}(\frac{x}{N})}\right)\Bigg\},$$

where the first sum is over configurations $\eta \in \{0,1\}^{\mathbb{T}_N}$ compatible with the density profile $\rho^{\rm ini}$. Then,

$$\mathcal{H}_N(0) = \sum_{x:\ \rho^{\rm ini}(\frac{x}{N})=0} \log\frac{1}{1-\rho_N^{\rm ini}(\frac{x}{N})} + \sum_{x:\ \rho^{\rm ini}(\frac{x}{N})=1} \log\frac{1}{\rho_N^{\rm ini}(\frac{x}{N})} \tag{5.1}$$
$$+ \sum_{x:\ \rho^{\rm ini}(\frac{x}{N})\in(0,1)} \left(\rho^{\rm ini}\big(\tfrac{x}{N}\big)\log\frac{\rho^{\rm ini}(\frac{x}{N})}{\rho_N^{\rm ini}(\frac{x}{N})} + \big(1-\rho^{\rm ini}\big(\tfrac{x}{N}\big)\big)\log\frac{1-\rho^{\rm ini}(\frac{x}{N})}{1-\rho_N^{\rm ini}(\frac{x}{N})}\right) \tag{5.2}$$

The lemma then follows from (3.11): indeed, there exists $C>0$ such that for all $x \in \mathbb{T}_N$,

$$\rho^{\rm ini}\big(\tfrac{x}{N}\big) = 0 \implies \left|\log\frac{1}{1-\rho_N^{\rm ini}(\frac{x}{N})}\right| \leqslant C\varepsilon_N, \tag{5.3}$$
$$\rho^{\rm ini}\big(\tfrac{x}{N}\big) = 1 \implies \left|\log\frac{1}{\rho_N^{\rm ini}(\frac{x}{N})}\right| \leqslant C\varepsilon_N. \tag{5.4}$$

Therefore, we can bound (5.1) by $CN\varepsilon_N$. In order to bound the first term in (5.2), note that (using again (3.11)) there exists $C>0$ such that

- if $\rho^{\rm ini}(\frac{x}{N}) \leqslant 2\varepsilon_N(C_{\rm Lip}+1)$, then

$$\rho^{\rm ini}\big(\tfrac{x}{N}\big)\log\frac{\rho^{\rm ini}(\frac{x}{N})}{\rho_N^{\rm ini}(\frac{x}{N})} \leqslant C\varepsilon_N|\log\varepsilon_N|,$$

- if $\rho^{\rm ini}(\frac{x}{N}) > 2\varepsilon_N(C_{\rm Lip}+1)$, then
$$\left|\frac{\rho^{\rm ini}(\frac{x}{N}) - \rho^{\rm ini}_N(\frac{x}{N})}{\rho^{\rm ini}(\frac{x}{N})}\right| < \frac{1}{2}$$
and
$$\left|\rho^{\rm ini}(\tfrac{x}{N})\log\frac{\rho^{\rm ini}(\frac{x}{N})}{\rho^{\rm ini}_N(\frac{x}{N})}\right| = \left|\rho^{\rm ini}(\tfrac{x}{N})\log\left(1-\frac{\rho^{\rm ini}(\frac{x}{N}) - \rho^{\rm ini}_N(\frac{x}{N})}{\rho^{\rm ini}(\frac{x}{N})}\right)\right|$$
$$\leqslant C\left|\rho^{\rm ini}(\tfrac{x}{N}) - \rho^{\rm ini}_N(\tfrac{x}{N})\right| \leqslant C(C_{\rm Lip}+1)\varepsilon_N.$$

The second term in (5.2) is bounded similarly. Lemma 3.7 follows.

We now turn to the proof of Proposition 3.8, which is the central result of this work.

5.2. **Entropy production.** First of all, the following well-known entropy estimate is due to Yau [17]:
$$\partial_t \mathcal{H}_N(t) \leqslant \int\left\{\frac{N^2\mathcal{L}_N\psi_t^N}{\psi_t^N} - \partial_t \log(\psi_t^N)\right\}{\rm d}\mu_t^N.$$
Let us denote
$$h(\eta) := \eta(0)\eta(1) + \eta(0)\eta(-1) - \eta(-1)\eta(1)$$
$$g(\eta) := \tfrac{1}{2}(\eta(-1)+\eta(2))(\eta(0)-\eta(1))^2.$$
Note that $\overline{g}(\rho) = 2\rho^2(1-\rho)$ and $\overline{h}(\rho) = \rho^2$, and also $|h(\eta)| \leqslant 2$ and $|g(\eta)| \leqslant 1$ for any η. We first prove the following technical result:

Lemma 5.1. *Assume* (3.23)*, namely* $\varepsilon_N^6 N \to \infty$*. Then*

$$\int\left\{\frac{N^2\mathcal{L}_N\psi_t^N}{\psi_t^N} - \partial_t\log(\psi_t^N)\right\}{\rm d}\mu_t^N$$
$$= \int\sum_{x\in\mathbb{T}_N}\partial_{uu}\lambda_N(t,\tfrac{x}{N})\Big\{\tau_x h(\eta) - \overline{h}\big(\rho_N(t,\tfrac{x}{N})\big) - \overline{h}'\big(\rho_N(t,\tfrac{x}{N})\big)\Big(\eta(x) - \rho_N(t,\tfrac{x}{N})\Big)\Big\}{\rm d}\mu_t^N \tag{5.5}$$
$$+ \int\sum_{x\in\mathbb{T}_N}(\partial_u\lambda_N)^2(t,\tfrac{x}{N})\Big\{\tau_x g(\eta) - \overline{g}\big(\rho_N(t,\tfrac{x}{N})\big) - \overline{g}'\big(\rho_N(t,\tfrac{x}{N})\big)\Big(\eta(x) - \rho_N(t,\tfrac{x}{N})\Big)\Big\}{\rm d}\mu_t^N \tag{5.6}$$
$$+\,\delta(t,N),$$
where
$$\frac{1}{N}\left|\int_0^T \delta(t,N){\rm d}t\right| \xrightarrow[N\to\infty]{} 0.$$

Proof. Fix $t\in[0,T]$. For the sake of brevity we denote $\lambda_x^N := \lambda_N(t,\frac{x}{N})$.

STEP 1 – PART COMING FROM THE GENERATOR: First we have

$$\frac{N^2\mathcal{L}_N\psi_t^N}{\psi_t^N} = N^2\sum_{x\in\mathbb{T}_N}\big(\eta(x-1)+\eta(x+2)\big)\eta(x)\big(1-\eta(x+1)\big)\Big(e^{\lambda_{x+1}^N-\lambda_x^N}-1\Big) \tag{5.7}$$

$$+N^2\sum_{x\in\mathbb{T}_N}\big(\eta(x-1)+\eta(x+2)\big)\eta(x+1)\big(1-\eta(x)\big)\Big(e^{\lambda_x^N-\lambda_{x+1}^N}-1\Big). \tag{5.8}$$

In (5.7) and (5.8) we write the exponential as the infinite sum: $e^z-1=\sum_{k\geqslant 1}\frac{z^k}{k!}$. The first order term ($k=1$) gives:

$$N^2\sum_{x\in\mathbb{T}_N}\big(\eta(x-1)+\eta(x+2)\big)\big(\eta(x)-\eta(x+1)\big)\big(\lambda_{x+1}^N-\lambda_x^N\big)$$

$$= N^2\sum_{x\in\mathbb{T}_N}\tau_x h(\eta)\big(\lambda_{x+1}^N+\lambda_{x-1}^N-2\lambda_x^N\big). \tag{5.9}$$

In order to replace the discrete Laplacian by its continuous version, let us estimate the following error

$$r_N(t) := \bigg|\int\sum_{x\in\mathbb{T}_N}\tau_x h(\eta)\,\Big(N^2\big(\lambda_{x+1}^N+\lambda_{x-1}^N-2\lambda_x^N\big)-\partial_{uu}\lambda_N\big(t,\tfrac{x}{N}\big)\Big)\,\mathrm{d}\mu_t^N\bigg|.$$

$$\leqslant 2\sum_{x\in\mathbb{T}_N}\bigg|\Big(N^2\big(\lambda_{x+1}^N+\lambda_{x-1}^N-2\lambda_x^N\big)-\partial_{uu}\lambda_N\big(t,\tfrac{x}{N}\big)\Big)\bigg|,$$

where the last inequality comes from the fact $|h(\eta)|\leqslant 2$. We use the Taylor formula for the smooth function $u\mapsto\lambda_N(t,u)$ in order to obtain

$$N^2(\lambda_{x+1}^N+\lambda_{x-1}^N-2\lambda_x^N)-\partial_{uu}\lambda_N\big(t,\tfrac{x}{N}\big)$$

$$=\frac{N^2}{2}\int_{\frac{x}{N}}^{\frac{x+1}{N}}\partial_{uuu}\lambda_N(t,u)\big(\tfrac{x+1}{N}-u\big)^2\,\mathrm{d}u-\frac{N^2}{2}\int_{\frac{x-1}{N}}^{\frac{x}{N}}\partial_{uuu}\lambda_N(t,u)\big(\tfrac{x-1}{N}-u\big)^2\,\mathrm{d}u. \tag{5.10}$$

We start with the first integral in (5.10). The second one is very similar and the same argument will work. We use several times the Cauchy-Schwarz inequality in order to write

$$N^2\sum_{x\in\mathbb{T}_N}\bigg|\int_{\frac{x}{N}}^{\frac{x+1}{N}}\partial_{uuu}\lambda_N(t,u)\big(\tfrac{x+1}{N}-u\big)^2\,\mathrm{d}u\bigg|$$

$$\leqslant N^2\sum_{x\in\mathbb{T}_N}\bigg\{\bigg(\int_{\frac{x}{N}}^{\frac{x+1}{N}}\big|\partial_{uuu}\lambda_N\big|^2(t,u)\,\mathrm{d}u\bigg)^{\frac12}\bigg(\int_{\frac{x}{N}}^{\frac{x+1}{N}}\big(\tfrac{x+1}{N}-u\big)^4\,\mathrm{d}u\bigg)^{\frac12}\bigg\}$$

$$\leqslant\frac{N^2}{\sqrt5 N^{\frac52}}\sum_{x\in\mathbb{T}_N}\bigg(\int_{\frac{x}{N}}^{\frac{x+1}{N}}\big|\partial_{uuu}\lambda_N\big|^2(t,u)\,\mathrm{d}u\bigg)^{\frac12}$$

$$\leqslant\frac{N^2}{\sqrt5 N^{\frac52}}\sqrt N\bigg\{\sum_{x\in\mathbb{T}_N}\int_{\frac{x}{N}}^{\frac{x+1}{N}}\big|\partial_{uuu}\lambda_N\big|^2(t,u)\,\mathrm{d}u\bigg\}^{\frac12}$$

$$=\frac{1}{\sqrt5}\big\|\partial_{uuu}\lambda_N(t,\cdot)\big\|_2. \tag{5.11}$$

Since λ_N is a function of ρ_N, one can easily obtain some norm bounds on λ_N and its derivatives, using the ones that we got in Section 4. This is done in Appendix C. Precisely in Proposition C.2, we prove that

$$\iint_{[0,T]\times\mathbb{T}} \big|\partial_{uuu}\lambda_N\big|^2(t,u)\,\mathrm{d}t\mathrm{d}u \leqslant \frac{C}{\varepsilon_N^6},$$

for some $C>0$. We let the reader repeat the argument for the second integral in (5.10), and deduce the following:

$$\int_0^T r_N(t)\,\mathrm{d}t \leqslant \frac{C'\sqrt{T}}{\varepsilon_N^3}, \tag{5.12}$$

for some $C'>0$. From the assumption $\varepsilon_N^3 N \to \infty$, we then have $\frac{1}{N}\int_0^T r_N(t)\mathrm{d}t \to 0$. Therefore, the first order term ($k=1$) gives the first contribution in (5.5), namely

$$\int \sum_{x\in\mathbb{T}_N} \partial_{uu}\lambda_N\big(t,\tfrac{x}{N}\big)\tau_x h(\eta)\,\mathrm{d}\mu_t^N$$

plus an error $r_N(t)$ that we include in $\delta(t,N)$.

In the same way, the second order term ($k=2$) gives

$$N^2\sum_{x\in\mathbb{T}_N}\frac{1}{2}\big(\eta(x-1)+\eta(x+2)\big)\big(\eta(x)-2\eta(x)\eta(x+1)+\eta(x+1)\big)\big(\lambda_{x+1}^N-\lambda_x^N\big)^2$$
$$= N^2\sum_{x\in\mathbb{T}_N}\tau_x g(\eta)\big(\lambda_{x+1}^N-\lambda_x^N\big)^2. \tag{5.13}$$

We want here to estimate the error

$$s_N(t) := \bigg|\int\sum_{x\in\mathbb{T}_N}\tau_x g(\eta)\Big(N^2(\lambda_{x+1}^N-\lambda_x^N)^2-(\partial_u\lambda_N)^2\big(t,\tfrac{x}{N}\big)\Big)\mathrm{d}\mu_t^N\bigg|.$$

As before, the Taylor formula and the Cauchy-Schwarz inequality allows us to bound

$$\begin{aligned}
s_N(t) &\leqslant 2N\sum_{x\in\mathbb{T}_N}\bigg|\Big\{\partial_u\lambda_N\big(t,\tfrac{x}{N}\big)\int_{\frac{x}{N}}^{\frac{x+1}{N}}\partial_{uu}\lambda_N(t,u)\big(\tfrac{x+1}{N}-u\big)\,\mathrm{d}u\Big\}\bigg| \\
&\qquad + N^2\sum_{x\in\mathbb{T}_N}\Big\{\int_{\frac{x}{N}}^{\frac{x+1}{N}}\partial_{uu}\lambda_N(t,u)\big(\tfrac{x+1}{N}-u\big)\,\mathrm{d}u\Big\}^2 \\
&\leqslant \frac{2N}{\sqrt{3}N^{3/2}}\big\|\partial_u\lambda_N(t,\cdot)\big\|_\infty\sum_{x\in\mathbb{T}_N}\Big\{\int_{\frac{x}{N}}^{\frac{x+1}{N}}\big|\partial_{uu}\lambda_N(t,u)\big|^2\,\mathrm{d}u\Big\}^{\frac12} \\
&\qquad + \frac{N^2}{3N^3}\sum_{x\in\mathbb{T}_N}\int_{\frac{x}{N}}^{\frac{x+1}{N}}\big|\partial_{uu}\lambda_N(t,u)\big|^2\,\mathrm{d}u \\
&\leqslant \frac{2}{\sqrt{3}}\big\|\partial_u\lambda_N(t,\cdot)\big\|_\infty\,\big\|\partial_{uu}\lambda_N(t,\cdot)\big\|_2+\frac{1}{3N}\,\big\|\partial_{uu}\lambda_N(t,\cdot)\big\|_2^2 \\
&\leqslant C\bigg(\frac{1}{\varepsilon_N^3}+\frac{1}{N\varepsilon_N^4}\bigg)
\end{aligned}$$

for some $C > 0$, where the last inequality follows from Proposition C.2. Therefore, we also get that $\frac{1}{N}\int_0^T s_N(t)\,\mathrm{d}t \to 0$, and the second order term gives the first contribution in (5.6), namely

$$\int \sum_{x\in\mathbb{T}_N} (\partial_u\lambda_N)^2\big(t,\tfrac{x}{N}\big)\tau_x g(\eta)\,\mathrm{d}\mu_t^N,$$

plus that error $s_N(t)$ that we include in $\delta(t,N)$.

Finally, we show that none of the higher order terms ($k \geqslant 3$) contributes and they are all included in $\delta(t,N)$. Precisely, we estimate

$$N^2\int_0^T \frac{1}{N}\sum_{x\in\mathbb{T}_N}\sum_{k\geqslant 3}\frac{|\lambda_{x+1}^N-\lambda_x^N|^k}{k!}\,\mathrm{d}t \tag{5.14}$$

and show that this quantity vanishes as $N\to\infty$. Using Proposition C.2, we bound (5.14) from above by

$$\begin{aligned} N^2\int_0^T\sum_{k\geqslant 3}\frac{\|\partial_u\lambda_N(t,\cdot)\|_\infty^k}{k!\,N^k}\,\mathrm{d}t &\leqslant TN^2\sum_{k\geqslant 3}\frac{C^k}{k!\,(N\varepsilon_N)^k}\\ &= TN^2\Big(e^{C/(N\varepsilon_N)}-\frac{C^2}{2(N\varepsilon_N)^2}-\frac{C}{N\varepsilon_N}-1\Big). \end{aligned}$$

with $C = C_{\mathrm{Lip}}/2$. For any $x\in[0,1]$ we have $e^x-\frac{x^2}{2}-x-1\leqslant x^3$, therefore the last expression above is bounded by

$$TC^3\,\frac{N^2}{(N\varepsilon_N)^3} = \frac{TC^3}{N\varepsilon_N^3}\xrightarrow[N\to\infty]{}0,$$

from assumption (3.23).

Step 2 – Part coming from $\log(\psi_t^N)$: The term with $\log(\psi_t^N)$ can be explicitly computed as

$$\begin{aligned}\partial_t\log(\psi_t^N) &= \sum_{x\in\mathbb{T}_N}\partial_t\lambda_N\big(t,\tfrac{x}{N}\big)\Big[\eta(x)-\int\eta(x)\psi_t^N(\eta)d\nu_\alpha(\eta)\Big]\\ &= \sum_{x\in\mathbb{T}_N}\partial_t\lambda_N\big(t,\tfrac{x}{N}\big)\Big[\eta(x)-\rho_N\big(t,\tfrac{x}{N}\big)\Big].\end{aligned}$$

A straightforward computation (see Proposition C.1) gives

$$\partial_t\lambda_N = \partial_{uu}\lambda_N\,\overline{h}'(\rho_N)+(\partial_u\lambda_N)^2\,\overline{g}'(\rho_N).$$

Therefore, this term appears exactly on that form in (5.5) and (5.6).

Step 3 – Additional term: Note that in (5.5) and (5.6) there is an extra term, that does not appear from the previous computations. Therefore, we have to substract it, and use the triangular inequality to estimate it. We show that that term is actually of order $o(N)$ when integrated in time between 0 and T, and therefore goes in $\delta(t,N)$. Indeed, the extra term reads

$$\sum_{x\in\mathbb{T}_N} F_N\big(t,\tfrac{x}{N}\big)$$

where
$$F_N(t,u) := \partial_{uu}\lambda_N(t,u)\,\overline{h}(\rho_N(t,u)) + (\partial_u\lambda_N)^2(t,u)\,\overline{g}(\rho_N(t,u)).$$
We want to show that
$$\frac{1}{N}\bigg|\int_0^T \sum_{x\in\mathbb{T}_N} F_N\big(t,\tfrac{x}{N}\big)\,\mathrm{d}t\bigg| \xrightarrow[N\to\infty]{} 0. \tag{5.15}$$
First, note that, for any $t>0$,
$$\int_{\mathbb{T}} F_N(t,u)\,\mathrm{d}u = \int_{\mathbb{T}} \partial_u\Big(\frac{\rho_N\,\partial_u\rho_N}{1-\rho_N}\Big)(t,u)\,\mathrm{d}u = 0,$$
Therefore, to prove (5.15) it is enough to prove that the following quantity vanishes:
$$\int_0^T \Big|\frac{1}{N}\sum_{x\in\mathbb{T}_N} F_N\big(t,\tfrac{x}{N}\big) - \int_{\mathbb{T}} F_N(t,u)\,\mathrm{d}u\Big|\mathrm{d}t \leqslant \int_0^T \sum_{x\in\mathbb{T}_N}\int_{\frac{x}{N}}^{\frac{x+1}{N}} \Big|F_N\big(t,\tfrac{x}{N}\big) - F_N(t,u)\Big|\,\mathrm{d}u\mathrm{d}t.$$
From the Cauchy-Schwarz inequality, we have for any $k\in\mathbb{T}_N$ and $u\in[\frac{x}{N},\frac{x+1}{N}]$,
$$\int_0^T \Big|F_N\big(t,\tfrac{x}{N}\big) - F_N(t,u)\Big|\,\mathrm{d}t \leqslant \int_0^T\int_{\frac{x}{N}}^{u} \big|\partial_u F_N(t,u)\big|\,\mathrm{d}u\mathrm{d}t \leqslant \sqrt{T}\big|u-\tfrac{x}{N}\big|^{\frac12}\big\|\partial_u F_N\big\|_{L^2([0,T]\times\mathbb{T})}.$$
One can check that
$$\partial_u F_N = \frac{\rho_N\,\partial_{uuu}\rho_N}{1-\rho_N} + \frac{3\partial_{uu}\rho_N\,\partial_u\rho_N}{(1-\rho_N)^2} + \frac{2(\partial_u\rho_N)^3}{(1-\rho_N)^3}.$$
Therefore, from all the results of Section 4, we get
$$\big\|\partial_u F_N\big\|_{L^2([0,T]\times\mathbb{T})} \leqslant \frac{C(T)}{\varepsilon_N^3}.$$
Finally we have
$$\int_0^T \Big|\frac{1}{N}\sum_{x\in\mathbb{T}_N} F_N\big(t,\tfrac{x}{N}\big) - \int_{\mathbb{T}} F_N(t,u)\,\mathrm{d}u\Big|\mathrm{d}t \leqslant \frac{C(T)}{N^{\frac12}\,\varepsilon_N^3},$$
which vanishes as $N\to\infty$ from assumption (3.23). □

5.3. **Average over large boxes.** To end the proof of Proposition 3.8, we want to take advantage of the Taylor expansion that seems to arise in (5.5) and (5.6). Note that the factor in front of $(\eta(x)-\rho_N(t,\frac{x}{N}))$ in that expression can be simplified as:
$$\partial_t\lambda_N(t,u) = \partial_{uu}\lambda_N(t,u)\overline{h}'(\rho_N(t,u)) + (\partial_u\lambda_N)^2(t,u)\overline{g}'(\rho_N(t,u)).$$
First of all, we are going to replace $\eta(x)$ by its empirical average over large boxes. More precisely, let us estimate the error (integrated in time) made by this replacement, which writes as follows
$$\varepsilon_{N,\ell}(T) := \bigg|\int_0^T\int \sum_{x\in\mathbb{T}_N} \partial_t\lambda_N\big(t,\tfrac{x}{N}\big)\Big(\eta(x)-\eta^{(\ell)}(x)\Big)\,\mathrm{d}\mu_t^N\,\mathrm{d}t\bigg|,$$
where for any $\ell\in\mathbb{N}$, we denote by $\eta^{(\ell)}(x)$ the space average of the configuration η on the box of size $2\ell+1$ centered around x:
$$\eta^{(\ell)}(x) = \frac{1}{2\ell+1}\sum_{|y-x|\leqslant\ell}\eta(y).$$

Performing an integration by parts, using the Taylor formula and the Cauchy-Schwarz inequality, one can easily show that for any $\ell \in \mathbb{N}$, there exists a constant $C(\ell) > 0$ such that

$$\varepsilon_{N,\ell}(T) \leqslant C(\ell)\bigg(\iint_{[0,T]\times\mathbb{T}} \big|\partial_u\partial_t\lambda_N(t,u)\big|^2 \,\mathrm{d}t\mathrm{d}u\bigg)^{\frac12} \leqslant \frac{C(\ell)}{\varepsilon_N^3},$$

the last inequality following from Proposition C.2. Therefore, under assumption (3.23),

$$\lim_{\ell\to\infty}\lim_{N\to\infty}\frac{\varepsilon_{N,\ell}(T)}{N} = 0.$$

The next step consists in replacing in (5.5) the local function $\tau_x h(\eta)$ by the spatial average

$$\frac{1}{2\ell+1}\sum_{|y-x|\leqslant\ell}\tau_y h(\eta)$$

for ℓ sufficiently large and then by its mean value $\overline{h}(\eta^{(\ell)}(x))$. In the same way, in (5.6) we will replace $\tau_x g(\eta)$ by $\overline{g}(\eta^{(\ell)}(x))$. This step is more involved, and is done thanks to the *one-block estimate* proved in the following section. Once again, because of the degeneracy of the limit profile $\rho(t,\cdot)$ (which can vanish), new arguments are needed w.r.t. [7].

5.4. **The one-block estimate.**

Lemma 5.2 (One-block estimate)**.** *Let $\varepsilon > 0$. For every local function $\psi : \{0,1\}^{\mathbb{Z}} \to \mathbb{R}$ there exists $\gamma_0 > 0$ and $L_0 < \infty$ such that: for all $\ell \geqslant L_0$ there exists $N_0 = N_0(\ell)$ such that for any $N \geqslant N_0$ we have*

$$\int_0^T \int \frac1N \sum_{x\in\mathbb{T}_N} \tau_x V_{\ell,\psi}(\eta) f_t^N(\eta)\nu_\alpha(\mathrm{d}\eta)\mathrm{d}t \leqslant \frac{1}{\gamma_0 N}\int_0^T \mathcal{H}_N(t)\mathrm{d}t + \varepsilon, \tag{5.16}$$

where

$$V_{\ell,\psi}(\eta) := \bigg|\frac{1}{2\ell+1}\sum_{|y|\leqslant\ell}\tau_y\psi(\eta) - \overline{\psi}\big(\eta^{(\ell)}(0)\big)\bigg|.$$

We will apply Lemma 5.2 with $\psi(\eta) = h(\eta)$ and $g(\eta)$.

Proof of Lemma 5.2. For $x \in \mathbb{T}_N$, $\ell \in \mathbb{N}$, let

$$\mathcal{Q}_{x,\ell} = \Big\{\eta\ :\ \sum_{y=x-\ell}^{x+\ell-1}\eta(x)\eta(x+1) \geqslant 1\Big\}$$

the set of configurations in which there are two neighbouring particles within distance ℓ of x (in particular the box of radius ℓ around x contains a *mobile cluster*). We split the left hand side in (5.16) into

$$\int_0^T \int \frac1N \sum_{x\in\mathbb{T}_N} \tau_x V_{\ell,\psi}(\eta)\mathbf{1}_{\mathcal{Q}_{x,\ell}}(\eta) f_t^N(\eta)\nu_\alpha(\mathrm{d}\eta)\mathrm{d}t \tag{5.17}$$

$$+\int_0^T \int \frac1N \sum_{x\in\mathbb{T}_N} \tau_x V_{\ell,\psi}(\eta)\mathbf{1}_{\mathcal{Q}_{x,\ell}^c}(\eta) f_t^N(\eta)\nu_\alpha(\mathrm{d}\eta)\mathrm{d}t. \tag{5.18}$$

As indicated in [7], the restriction to the irreducible set $\mathcal{Q}_{x,\ell}$ in (5.17) allows us to repeat standard arguments, and to conclude that

$$\limsup_{\ell\to\infty}\limsup_{N\to\infty}\int_0^T\int\frac{1}{N}\sum_{x\in\mathbb{T}_N}\tau_xV_{\ell,\psi}(\eta)\mathbf{1}_{\mathcal{Q}_{x,\ell}}(\eta)f_t^N(\eta)\nu_\alpha(\mathrm{d}\eta)\mathrm{d}t=0. \tag{5.19}$$

Let us now deal with the other term (5.18). By the entropy inequality (3.20), the term inside the time integral $\int_0^T$ can be bounded above by

$$\frac{H\big(\mu_t^N|\nu^N_{\rho_N(t,\cdot)}\big)}{\gamma N}+\frac{1}{\gamma N}\log\int\exp\Big(\gamma\sum_{x\in\mathbb{T}_N}\tau_xV_{\ell,\psi}(\eta)\mathbf{1}_{\mathcal{Q}^c_{x,\ell}}(\eta)\Big)\nu^N_{\rho_N(t,\cdot)}(\mathrm{d}\eta) \tag{5.20}$$

for any $\gamma>0$. Recall that $\varepsilon>0$ is fixed. We need to show that we can choose $\gamma>0$ such that

$$\limsup_{\ell\to\infty}\limsup_{N\to\infty}\int_0^T\frac{1}{\gamma N}\log\int\exp\Big(\gamma\sum_{x\in\mathbb{T}_N}\tau_xV_{\ell,\psi}(\eta)\mathbf{1}_{\mathcal{Q}^c_{x,\ell}}(\eta)\Big)\nu^N_{\rho_N(t,\cdot)}(\mathrm{d}\eta)\mathrm{d}t\leqslant\varepsilon. \tag{5.21}$$

Now, contrary to [7], we made no assumption to ensure that $\nu^N_{\rho(t,\cdot)}(\mathcal{Q}^c_{x,\ell})$ decays exponentially in ℓ for all x. In fact, this is plain wrong when $\rho(t,\cdot)$ vanishes on an interval.

Let ℓ_0 be such that the support of ψ is contained in $\{-\ell_0,\dots,\ell_0\}$ and $C:=2\|\psi\|_\infty$ (which clearly does not depend on ℓ). From the uniform convergence stated and proved in Proposition 3.6, we know that there exists a vanishing sequence of positive numbers $(\delta_N^{(1)})$ such that: for any $u\in\mathbb{T}$, any $t\in[0,T]$, and $N\in\mathbb{N}$,

$$\big|\rho_N(t,u)-\rho(t,u)\big|\leqslant\delta_N^{(1)}. \tag{5.22}$$

Since $\rho_N\geqslant\varepsilon_N$ (see Proposition 3.5) while ρ can be equal to 0, it is natural to impose that $\delta_N^{(1)}\geqslant\varepsilon_N$. Without loss of generality, we can assume that the sequence $\big(\delta_N^{(1)}\big)$ is decreasing. Moreover, the sequence (ρ_N) is equicontinuous on $[0,T]\times\mathbb{T}$, and therefore, there exists a nondecreasing continuous modulus of continuity $w:[0,1]\to\mathbb{R}_+$ with $w(0)=0$ such that for any $u,v\in\mathbb{T}$, $t\in[0,T]$, $\varepsilon>0$ and $N\in\mathbb{N}$,

$$|u-v|\leqslant\varepsilon\quad\Rightarrow\quad\big|\rho_N(t,u)-\rho_N(t,v)\big|\leqslant w(\varepsilon). \tag{5.23}$$

Let us denote

$$\delta_N:=\delta_N^{(1)}+w\big(\tfrac{\ell+\ell_0+1}{N}\big)\xrightarrow[N\to\infty]{}0, \tag{5.24}$$

then it follows from the monotonicity of $\big(\delta_N^{(1)}\big)_N$ and of w that $(\delta_N)_N$ is decreasing.

We are going to split $\mathbb{T}_N$ into three sets of points: the *good* ones, the *almost zeroes* and the *bad* ones. Namely, for any $\delta>0$, and any vanishing sequence (α_N) such that $\alpha_N\leqslant\delta$, let

$$G_t^{N,\ell}(\delta)\;:=\;\Big\{x\in\mathbb{T}_N\;:\;\rho_N(t,\cdot)\geqslant\delta\quad\text{on }\big[\tfrac{x-\ell-\ell_0}{N},\tfrac{x+\ell+\ell_0}{N}\big]\Big\}, \tag{5.25}$$

$$Z_t^{N,\ell}(\alpha_N)\;:=\;\Big\{x\in\mathbb{T}_N\;:\;\rho_N(t,\cdot)\leqslant\alpha_N\quad\text{on }\big[\tfrac{x-\ell-\ell_0}{N},\tfrac{x+\ell+\ell_0}{N}\big]\Big\}, \tag{5.26}$$

$$B_t^{N,\ell}(\delta,\alpha_N)\;:=\;\mathbb{T}_N\backslash\big(G_t^{N,\ell}(\delta)\cup Z_t^{N,\ell}(\alpha_N)\big). \tag{5.27}$$

The parameters $\delta > 0$ and $\alpha_N \to 0$ will be chosen ahead. We want to study the limit as $N \to \infty$ of the cardinality of these sets of points (renormalized by N). For that purpose, let us introduce the following sets: for any $\delta > 0$, $t \in [0,T]$, let

$$\begin{aligned} \mathcal{G}_t(\delta) &:= \Big\{ u \in \mathbb{T} \; : \; \rho(t,u) \geqslant \delta \Big\}, \\ \mathcal{Z}_t(\delta) &:= \overbrace{\Big\{ u \in \mathbb{T} \; : \; \rho(t,u) = 0 \Big\}}^{\circ} \cup \Gamma_t(\delta), \\ \mathcal{B}_t(\delta) &:= \Big\{ u \in \mathbb{T} \; : \; 0 < \rho(t,u) < \delta \Big\} \subset \Gamma_t(\delta), \end{aligned}$$

where $\Gamma_t(\delta)$ has been defined in (3.3). Note first that

$$\mathbb{T} \backslash \big(\mathcal{G}_t(\delta) \cup \mathcal{Z}_t \big) = \mathcal{B}_t(\delta) \cup \Gamma_t,$$

where $\mathcal{Z}_t$ and Γ_t have been defined respectively in (3.2) and (3.1). Therefore, since $\mathrm{Leb}(\Gamma_t) = 0$ (recall the proof of Proposition 3.3) the two remaining sets above have the same Lebesgue measure:

$$\mathrm{Leb}\big(\mathcal{B}_t(\delta)\big) = \mathrm{Leb}\Big(\mathbb{T} \backslash \big(\mathcal{G}_t(\delta) \cup \mathcal{Z}_t\big)\Big). \tag{5.28}$$

We will make use of the following lemma:

Lemma 5.3. *Recall that δ_N has been defined in (5.24). For any $\ell, \ell_0 \in \mathbb{N}$ fixed, and $\delta > 0$, the following convergences hold:*

$$\lim_{N\to\infty} \frac{1}{N} \Big| G_t^{N,\ell}(\delta - \delta_N) \Big| = \mathrm{Leb}\big(\mathcal{G}_t(\delta)\big) \tag{5.29}$$

$$\lim_{N\to\infty} \frac{1}{N} \Big| Z_t^{N,\ell}(\delta_N) \Big| = \mathrm{Leb}\big(\mathcal{Z}_t\big) \tag{5.30}$$

and therefore from (5.28):

$$\lim_{N\to\infty} \frac{1}{N} \Big| B_t^{N,\ell}(\delta - \delta_N, \delta_N) \Big| = \mathrm{Leb}\big(\mathcal{B}_t(\delta)\big).$$

We will prove Lemma 5.3 further. Let us first end the proof of Lemma 5.2, more precisely of (5.16). Fix $\delta > 0$ as a parameter that will vanish at the end of this paragraph, *after* letting $N \to \infty$ and $\ell \to \infty$. Take the expression under the limit in the left hand side of (5.21), and take N sufficiently large such that $\delta - \delta_N > \delta_N$. We divide the sum that appears there into three sums:

- one over $B_t^{N,\ell}(\delta - \delta_N, \delta_N)$,
- one over $Z_t^{N,\ell}(\delta_N)$,
- and the last one over $G_t^{N,\ell}(\delta - \delta_N)$,

since by definition their union gives $\mathbb{T}_N$. We bound each sum as follows: first, since $V_{\ell,\psi}(\eta)$ is bounded by C, we have

$$\sum_{x \in B_t^{N,\ell}(\delta - \delta_N, \delta_N)} \tau_x V_{\ell,\psi}(\eta) \mathbf{1}_{\mathcal{Q}_{x,\ell}^c}(\eta) \leqslant C \Big| B_t^{N,\ell}(\delta - \delta_N, \delta_N) \Big|.$$

Then note that the two sums over $Z_t^{N,\ell}(\delta_N)$ and $G_t^{N,\ell}(\delta - \delta_N)$ are functions with disjoint supports; since the measure is product, the average factorizes. To bound the term with the sum over $Z_t^{N,\ell}(\delta_N)$, note that, if $\eta(x+y)=0$ for all $|y|\leqslant \ell+\ell_0$, then $\tau_x V_{\ell,\psi}(\eta)=0$. Moreover, if a non-decreasing function[1] has support in $\big\{x\in\mathbb{T}_N:\ \rho_N(t,\frac{x}{N})\leqslant\delta_N\big\}$, we can replace $\nu^N_{\rho_N(t,\cdot)}$ by the homogeneous measure ν_{δ_N} when overestimating its expectation (as we do in the second inequality below). Consequently, we can bound

$$\begin{aligned}
&\int \exp\Bigg(\gamma\sum_{x\in Z_t^{N,\ell}(\delta_N)}\tau_xV_{\ell,\psi}(\eta)\mathbf{1}_{\mathcal{Q}^c_{x,\ell}}(\eta)\Bigg)\nu^N_{\rho_N(t,\cdot)}(\mathrm{d}\eta)\\
&\qquad\leqslant \int\exp\Bigg(\gamma C\sum_{x\in Z_t^{N,\ell}(\delta_N)}\mathbf{1}_{\{\exists\ |y|\leqslant\ell+\ell_0:\ \eta(x+y)=1\}}\Bigg)\nu^N_{\rho_N(t,\cdot)}(\mathrm{d}\eta)\\
&\qquad\leqslant \int\exp\Bigg(\gamma C\sum_{x\in Z_t^{N,\ell}(\delta_N)}\mathbf{1}_{\{\exists\ |y|\leqslant\ell+\ell_0:\ \eta(x+y)=1\}}\Bigg)\nu_{\delta_N}(\mathrm{d}\eta)\\
&\qquad\leqslant \int\exp\Bigg(\gamma C\sum_{y\in\mathbb{T}_N}\eta(y)(2\ell+2\ell_0+1)\Bigg)\nu_{\delta_N}(\mathrm{d}\eta)\\
&\qquad\leqslant \Big(\delta_N\big(e^{\gamma C(2\ell+2\ell_0+1)}-1\big)+1\Big)^N.
\end{aligned}$$

Finally, for any $x\in G_t^{N,\ell}(\delta-\delta_N)$, and any $t\in[0,T]$, we know that

$$\nu^N_{\rho_N(t,\cdot)}\big(\mathcal{Q}^c_{x,\ell}\big)\leqslant\big(1-(\delta-\delta_N)^2\big)^\ell.$$

Therefore, we bound the term under the limit in (5.21) as follows:

$$\begin{aligned}
&\int_0^T\frac{1}{\gamma N}\log\int\exp\Bigg(\gamma\sum_{x\in\mathbb{T}_N}\tau_xV_{\ell,\psi}(\eta)\mathbf{1}_{\mathcal{Q}^c_{x,\ell}}(\eta)\Bigg)\nu^N_{\rho_N(t,\cdot)}(\mathrm{d}\eta)\mathrm{d}t\\
&\quad\leqslant\int_0^T\frac{C\big|B_t^{N,\ell}(\delta-\delta_N,\delta_N)\big|}{N}\mathrm{d}t\\
&\qquad+\frac{T}{\gamma}\log\big(1+\big(e^{\gamma C(2\ell+2\ell_0+1)}-1\big)\delta_N\big)\\
&\qquad+\int_0^T\frac{1}{\gamma N(2\ell+1)}\sum_{x\in G_t^{N,\ell}(\delta-\delta_N)}\log\Big(\nu^N_{\rho_N(t,\cdot)}\big(\mathcal{Q}^c_{x,\ell}\big)\big(\exp(\gamma(2\ell+1)C)-1\big)+1\Big)\mathrm{d}t\\
&\quad\leqslant\int_0^T\frac{C\big|B_t^{N,\ell}(\delta-\delta_N,\delta_N)\big|}{N}\mathrm{d}t && (5.31)\\
&\qquad+\frac{T}{\gamma}\log\big(1+\big(e^{\gamma C(2\ell+2\ell_0+1)}-1\big)\delta_N\big) && (5.32)\\
&\qquad+\frac{T}{\gamma(2\ell+1)}\Big(\exp(\gamma(2\ell+1)C)-1\Big)\big(1-(\delta-\delta_N)^2\big)^\ell. && (5.33)
\end{aligned}$$

[1] A function $f:\{0,1\}^{\mathbb{T}_N}\to\mathbb{R}$ is said to be non-decreasing if $f(\eta)\leqslant f(\eta')$ as soon as $\eta\preccurlyeq\eta'$, where $\preccurlyeq$ denotes the coordinate-wise order in $\{0,1\}^{\mathbb{T}_N}$

We first take $N \to \infty$, then $\ell \to \infty$ and then $\delta \to 0$. We treat each term separately: from Lemma 5.3, Fatou's lemma, and the fact that $\mathcal{B}_t(\delta) \subset \Gamma_t(\delta)$ we obtain:

$$\begin{aligned}\lim_{\delta\to 0}\limsup_{\ell\to\infty}\limsup_{N\to\infty}\int_0^T \frac{1}{N}\Big|B_t^{N,\ell}(\delta-\delta_N,\delta_N)\Big|\,\mathrm{d}t &\leqslant \lim_{\delta\to 0}\int_0^T \mathrm{Leb}\big(\mathcal{B}_t(\delta)\big)\,\mathrm{d}t\\ &\leqslant \lim_{\delta\to 0}\int_0^T \mathrm{Leb}\big(\Gamma_t(\delta)\big)\,\mathrm{d}t = 0,\end{aligned}$$

where the last equality follows from Proposition 3.3. The second term (5.32) easily vanishes since $\delta_N \to 0$. Finally, for the last term (5.33), we choose $\gamma > 0$ such that $2\gamma C + \log(1-\delta^2) < 0$ and the result follows. □

We now go back to the proof of Lemma 5.3 .

Proof of Lemma 5.3. The proof is based on the following fact: for any $\delta > 0$ and N sufficiently large,

$$\mathcal{G}_t(\delta) \subset \frac{1}{N}G_t^{N,\ell}(\delta-\delta_N) \subset \mathcal{G}_t(\delta-2\delta_N), \tag{5.34}$$

$$\mathcal{Z}_t \subset \frac{1}{N}Z_t^{N,\ell}(\delta_N) \subset \mathcal{Z}_t(2\delta_N). \tag{5.35}$$

Let us prove the first inclusion in (5.34), namely: if $u \in \mathcal{G}_t(\delta)$ then, for any N sufficiently large, $\lfloor uN\rfloor \in G_t^{N,\ell}(\delta-\delta_N)$.

Let $u \in \mathcal{G}_t(\delta)$ and $y \in \big[\frac{\lfloor uN\rfloor-\ell-\ell_0}{N}, \frac{\lfloor uN\rfloor+\ell+\ell_0}{N}\big]$, which implies $|y-u| \leqslant \frac{\ell+\ell_0+1}{N}$.

Using (5.22) and (5.23) we get

$$\rho_N(t,y) \geqslant \rho_N(t,u) - w\big(\tfrac{\ell+\ell_0+1}{N}\big) \geqslant \rho(t,u) - \delta_N^{(1)} - w\big(\tfrac{\ell+\ell_0+1}{N}\big) \geqslant \delta-\delta_N,$$

which proves the claim. The same argument works to prove the symmetric inclusion, namely: if $x \in G_t^{N,\ell}(\delta-\delta_N)$ then $\frac{x}{N} \in \mathcal{G}_t(\delta-2\delta_N)$. As a result, (5.34) follows. The proof of the second series of inclusions (5.35) is very similar and we let the reader to check it.

In order to prove (5.29), it is enough to use (5.34) and to show that the Lebesgue measures converge, namely:

$$\mathrm{Leb}\big(\mathcal{G}_t(\delta-2\delta_N)\big) \xrightarrow[N\to\infty]{} \mathrm{Leb}\big(\mathcal{G}_t(\delta)\big).$$

This is indeed the case since (δ_N) is a decreasing sequence and therefore the family of sets $\big(\mathcal{G}_t(\delta-2\delta_N)\big)$ is decreasing for inclusion.

Therefore, thanks to the continuity of the Lebesgue measure, there holds

$$\mathrm{Leb}(\mathcal{G}_t(\delta)) = \mathrm{Leb}\Big(\bigcap_{N\geqslant 1}\mathcal{G}_t(\delta-2\delta_N)\Big) = \lim_{N\to\infty}\mathrm{Leb}\big(\mathcal{G}_t(\delta-2\delta_N)\big).$$

A very similar argument can be worked out to prove (5.30). For that case, first note that

$$\mathcal{Z}_t \cup \Gamma_t = \bigcap_{N\geqslant 1}\mathcal{Z}_t(2\delta_N),$$

and then one is able to conclude the proof, since the boundary set Γ_t satisfies $\mathrm{Leb}(\Gamma_t) = 0$ (from Proposition 3.3). □

5.5. **Conclusion.** Putting together the computation of the entropy production in Lemma 5.1, and then the replacements done in Section 5.3 and Lemma 5.2, up to know we have proved the following:

Corollary 5.4. *There exists $\gamma_0 > 0$ and $\ell_0 \in \mathbb{N}$, such that, for any $\ell \geqslant \ell_0$, there exists $N_0 = N_0(\ell_0)$ such that, for any $N \geqslant N_0$,*

$$\mathcal{H}_N(T) - \mathcal{H}_N(0) \leqslant \frac{1}{\gamma_0} \int_0^T \mathcal{H}_N(t)\,\mathrm{d}t + \varepsilon_T(N,\ell) \tag{5.36}$$

$$+ \int_0^T \int \sum_{x\in\mathbb{T}_N} \partial_{uu}\lambda_N\big(t,\tfrac{x}{N}\big)\,\overline{H}\Big(\eta^{(\ell)}(x), \rho\big(t,\tfrac{x}{N}\big)\Big)\mathrm{d}\mu_t^N\,\mathrm{d}t \tag{5.37}$$

$$+ \int_0^T \int \sum_{x\in\mathbb{T}_N} (\partial_u\lambda_N)^2\big(t,\tfrac{x}{N}\big)\,\overline{G}\Big(\eta^{(\ell)}(x), \rho\big(t,\tfrac{x}{N}\big)\Big)\mathrm{d}\mu_t^N\,\mathrm{d}t, \tag{5.38}$$

where

$$\overline{H}(a,b) := \overline{h}(a) - \overline{h}(b) - \overline{h}'(b)(a-b)$$
$$\overline{G}(a,b) := \overline{g}(a) - \overline{g}(b) - \overline{g}'(b)(a-b)$$

and

$$\limsup_{\ell\to\infty}\limsup_{N\to\infty} \frac{\varepsilon_T(N,\ell)}{N} = 0.$$

In this last paragraph we show that (5.37) and (5.38) are bounded from above by a constant times (5.36). We treat only (5.37), the same argument works for (5.38). Note that applying the entropy inequality, we can bound (5.37) above by

$$\frac{1}{\gamma}\int_0^T \mathcal{H}_N(t)\,\mathrm{d}t + \frac{1}{\gamma}\int_0^T \log \mathbb{E}^N_{\rho_N(t,\cdot)}\Big[\exp\Big\{\gamma \sum_{x\in\mathbb{T}_N} \partial_{uu}\lambda_N\big(t,\tfrac{x}{N}\big)\overline{H}\big(\eta^{(\ell)}(x), \rho_N\big(t,\tfrac{x}{N}\big)\big)\Big\}\Big]\,\mathrm{d}t,$$

for any $\gamma > 0$. The first term will be added to (5.36). A large deviation argument will allow us to chose $\gamma > 0$ such that the second term vanishes:

Lemma 5.5 (Large deviation estimate)**.** *There exists $\gamma > 0$ such that, for all $t \in [0,T]$,*

$$\limsup_{\ell\to\infty}\limsup_{N\to\infty} \frac{1}{N}\log \mathbb{E}^N_{\rho_N(t,\cdot)}\Big[\exp\Big\{\gamma \sum_{x\in\mathbb{T}_N} \partial_{uu}\lambda_N\big(t,\tfrac{x}{N}\big)\overline{H}\big(\eta^{(\ell)}(x), \rho_N\big(t,\tfrac{x}{N}\big)\big)\Big\}\Big] \leqslant 0.$$

Proof. We follow the lines of [9, Chapter 6], where the rough argument is well exposed and now standard. The main difference here consists in the presence of the approximate solution ρ_N instead of ρ. A Riemann-type convergence like in (2.4) will be enough to conclude. □

This concludes the proof of Proposition 3.8.

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Appendix A. Continuous embedding of $H^s(\mathbb{T})$ into $\mathcal{C}(\mathbb{T})$

Proposition A.1. *For any $s \in (\frac{1}{2}, 1)$, the Sobolev space $H^s(\mathbb{T})$ is continuously embedded into $\mathcal{C}(\mathbb{T})$.*

Proof. In the periodic setting, the proof is very simple, so that we give it here for the sake of completeness. Let us assume $s > \frac{1}{2}$ and let $\rho \in H^s(\mathbb{T})$. Define

$$\rho_K(u) := \sum_{|k| \leqslant K} \widehat{\rho}_k e^{i2\pi k u}, \qquad \text{for any } u \in \mathbb{T},$$

where $\widehat{\rho}_k$ has been defined in (3.13). For any $K \in \mathbb{N}$, ρ_K is continuous on $\mathbb{T}$. Moreover,

$$\begin{aligned} |\rho(u) - \rho_K(u)| &\leqslant \sum_{|k| \geqslant K+1} |\widehat{\rho}_k| \\ &\leqslant \Big(\sum_{|k| \geqslant K+1} (1 + 4\pi^2 |k|^2)^s |\widehat{\rho}_k|^2 \Big)^{\frac{1}{2}} \Big(\sum_{|k| \geqslant K+1} (1 + 4\pi^2 |k|^2)^{-s} \Big)^{\frac{1}{2}} \\ &\leqslant \|\rho\|_{H^s(\mathbb{T})} \, R(K), \end{aligned}$$

where $R(K) \to 0$ as $K \to \infty$, as the rest of a convergent sum, since $s > \frac{1}{2}$. We deduce that

$$\|\rho - \rho_K\|_\infty \xrightarrow[K \to \infty]{} 0. \tag{A.1}$$

Thus ρ is continuous as the limit of the sequence $(\rho_K)_K$ of continuous functions.

Moreover, from (A.1), we know that, for any $u \in \mathbb{T}$,

$$\rho(u) = \sum_{k \in \mathbb{Z}} \widehat{\rho}_k e^{i2\pi k u},$$

and similarly as before, from the Cauchy-Schwarz inequality we easily get

$$\|\rho\|_\infty \leqslant C \|\rho\|_{H^s(\mathbb{T})}, \qquad \text{with } C = \Big(\sum_{k \in \mathbb{Z}} (1 + 4\pi^2 k^2)^{-s} \Big)^{\frac{1}{2}} < \infty.$$

□

Appendix B. Connected components of the positivity set

In this section we prove Lemma 3.4, that we recall here for the reader's convenience:

Lemma B.1 (Connected components of the positivity set)**.** *Denote by $\mathcal{I}_t$ the set of the connected components of $\mathcal{P}_t$ for $t \geqslant 0$. Then one can build an injective mapping from $\mathcal{I}_t$ to $\mathcal{I}_s$ for all $t \geqslant s \geqslant 0$. In particular, the function $t \mapsto \#\mathcal{I}_t$ is non-increasing.*

Proof. Let $t > 0$ and let $(a, b) \in \mathcal{I}_t$, i.e., $\rho(t, a) = \rho(t, b) = 0$ and $\rho(t, u) > 0$ for $u \in (a, b)$. Following [16, Proposition 14.1], the mapping $t \mapsto \mathcal{P}_t$ is monotone:

$$\mathcal{P}_s \subset \mathcal{P}_t, \qquad \text{for all } (s, t) \in [0, T]^2 \ \text{ such that } s \leqslant t.$$

This ensures that

$$\rho(s, a) = \rho(s, b) = 0, \qquad \forall s \in [0, t]. \tag{B.1}$$

Since $0 \leqslant \rho \leqslant 1$ and $|\partial_u \rho| \leqslant C_{\text{Lip}}$, the weak formulation (2.6) still holds for test functions ξ of the form $\xi(\tau, u) = \theta(\tau)\zeta(u)$ with $\theta \in L^1 \cap BV(\mathbb{R}_+)$ and compactly supported, and

$\zeta \in H^1(\mathbb{T})$ thanks to the density of $C^1([0,T])$ in $BV(0,T)$ and of $C^1(\mathbb{T})$ in $H^1(\mathbb{T})$ for the respective weak-$\star$ and weak topologies. Here, $BV(\mathbb{R}_+)$ denotes the set of real valued functions of bounded variations on $\mathbb{R}_+$, i.e., functions $t \mapsto \theta(t)$ such that $\partial_t\theta$ is a finite Radon measure on $\mathbb{R}_+$. Fix $s \in [0,t)$ and $\varepsilon \in (0,(b-a)/2)$, then choose $\xi = \theta\zeta_\varepsilon$ with

$$\theta(\tau) = \mathbf{1}_{(s,t)}(\tau) \qquad \text{and} \qquad \zeta_\varepsilon(u) = \max\left(0, \min\left(1, \frac{u-a}{\varepsilon}, \frac{b-u}{\varepsilon}\right)\right)$$

in the weak formulation (2.6). This provides

$$\int_{\mathbb{T}} \rho(s,u)\zeta_\varepsilon(u)\mathrm{d}u = \int_{\mathbb{T}} \rho(t,u)\zeta_\varepsilon(u)\mathrm{d}u + \int_0^\tau \frac{1}{\varepsilon}\int_a^{a+\varepsilon} \partial_x\rho^2 \mathrm{d}u\mathrm{d}\tau - \int_0^\tau \frac{1}{\varepsilon}\int_{b-\varepsilon}^b \partial_x\rho^2\mathrm{d}u\mathrm{d}\tau.$$

Using (B.1), one gets that

$$\int_{\mathbb{T}} \rho(s,u)\zeta_\varepsilon(u)\mathrm{d}u = \int_{\mathbb{T}} \rho(t,u)\zeta_\varepsilon(u)\mathrm{d}u + \frac{1}{\varepsilon}\int_0^\tau \left(\rho^2(\tau,a+\varepsilon) + \rho^2(\tau,b-\varepsilon)\right)\mathrm{d}\tau. \tag{B.2}$$

It follows from (B.1) and on the Lipschitz continuity of ρ that

$$0 \leqslant \rho^2(\tau,a+\varepsilon) \leqslant (C_{\mathrm{Lip}}\varepsilon)^2, \qquad 0 \leqslant \rho^2(\tau,b-\varepsilon) \leqslant (C_{\mathrm{Lip}}\varepsilon)^2.$$

These estimates together with the convergence of ζ_ε in $L^1(\mathbb{T})$ towards $\mathbf{1}_{(a,b)}$ allow to pass to the limit $\varepsilon \to 0$ in (B.2), leading to

$$\int_a^b \rho(s,u)\mathrm{d}u = \int_a^b \rho(t,u)\mathrm{d}u > 0, \qquad \forall s \in [0,t].$$

Since $\rho(s,\cdot)$ is continuous and because of (B.1), this implies that there exists (at least) one interval $(\alpha,\beta) \subset (a,b)$ such that $\rho(s,\alpha) = \rho(s,\beta) = 0$ and $\rho(s,u) > 0$ on (α,β). Such an interval (α,β) belongs to $\mathcal{I}_s$, and the mapping from $\mathcal{I}_t$ to $\mathcal{I}_s$ sending (a,b) to (α,β) is injective. □

Appendix C. Technical results

C.1. **Derivatives.** Let $\rho : \mathbb{R}_+ \times \mathbb{T} \to (0,1)$ and define

$$\lambda = \log\left(\frac{\rho}{1-\rho}\right).$$

Proposition C.1. *We have*

$$\partial_u\lambda = \frac{\partial_u\rho}{\rho(1-\rho)} \tag{C.1}$$

$$\partial_{uu}\lambda = \frac{\partial_{uu}\rho}{\rho(1-\rho)} - \frac{(\partial_u\rho)^2(1-2\rho)}{\rho^2(1-\rho)^2} \tag{C.2}$$

$$\partial_{uuu}\lambda = \frac{\partial_{uuu}\rho}{\rho(1-\rho)} - 3\frac{\partial_{uu}\rho\partial_u\rho(1-2\rho)}{\rho^2(1-\rho)^2} + 2\frac{(\partial_u\rho)^3(1-3\rho+3\rho^2)}{\rho^3(1-\rho)^3} \tag{C.3}$$

Therefore, if ρ is solution to the porous medium equation $\partial_t\rho = \partial_{uu}(\rho^2)$ then

$$\partial_t\lambda = 2\rho\,\partial_{uu}\lambda + 2\rho(2-3\rho)\,(\partial_u\lambda)^2. \tag{C.4}$$

C.2. **Technical estimates on λ_N.** Here we get some technical bounds on the norms of λ_N and its derivatives, where λ_N has been defined in function of ρ_N in (3.21).

Proposition C.2. *For any $N \in \mathbb{N}$,*

$$\sup_{(t,u)\in[0,T]\times\mathbb{T}} \big|\partial_u \lambda_N(t,u)\big| \leqslant \frac{C_{\rm Lip}}{\varepsilon_N}. \tag{C.5}$$

Moreover, there exists $C>0$ such that for any $t \in [0,T]$,

$$\int_{\mathbb{T}} \big|\partial_{uu}\lambda_N\big|^2(t,u)\, \mathrm{d}u \leqslant \frac{C}{(\varepsilon_N)^4} \tag{C.6}$$

and finally

$$\iint_{[0,T]\times\mathbb{T}} \big|\partial_{uuu}\lambda_N\big|^2(t,u)\, \mathrm{d}t\mathrm{d}u \leqslant \frac{C}{(\varepsilon_N)^6} \tag{C.7}$$

$$\iint_{[0,T]\times\mathbb{T}} \big|\partial_u\partial_t\lambda_N\big|^2(t,u)\, \mathrm{d}t\mathrm{d}u \leqslant \frac{C}{(\varepsilon_N)^6}. \tag{C.8}$$

Proof. The first bound (C.5) is straightforward from Proposition 4.1. Now we use Proposition C.1. First,

$$\big\|\partial_{uu}\lambda_N(t,\cdot)\big\|_2^2 \leqslant C\bigg(\frac{\big\|\partial_{uu}\rho_N\big\|_2^2}{\varepsilon_N^2} + \frac{\big\|\partial_u\rho_N\big\|_\infty^4}{\varepsilon_N^4}\bigg),$$

therefore (C.6) follows from Proposition 4.1 and Proposition 4.3. Note that

$$\big|\partial_{uuu}\lambda_N\big|^2 \leqslant C\bigg(\frac{\big|\partial_{uuu}\,\rho_N\big|^2}{\varepsilon_N^2} + \frac{\big|\partial_{uu}\rho_N\big|^2\,\big|\partial_u\rho_N\big|^2}{\varepsilon_N^4} + \frac{\big|\partial_u\rho_N\big|^6}{\varepsilon_N^6}\bigg).$$

From Proposition 4.1, Proposition 4.2 and Proposition 4.3, we easily obtain (C.7). Finally, to get (C.8), we use (C.4) and we obtain

$$\big|\partial_u\partial_t\lambda_N\big|^2 \leqslant C\Big(\big|\partial_u\rho_N\big|^2\big|\partial_{uu}\lambda_N\big|^2 + \big|\partial_{uuu}\lambda_N\big|^2 + \big|\partial_u\lambda_N\big|^4\big|\partial_u\rho_N\big|^2 + \big|\partial_u\lambda_N\big|^2\big|\partial_{uu}\lambda_N\big|^2\Big),$$

and therefore, from Proposition 4.1 and the first three bounds (C.5), (C.6) and (C.7), we get (C.8). ☐

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