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Simone Baldi[1], Giorgio Valmorbida[2], Antonis Papachristodoulou[2] and Elias B. Kosmatopoulos[3]

***Abstract*— This work proposes an online policy iteration procedure for the synthesis of sub-optimal control laws for uncertain Linear Time Invariant (LTI) Asymptotically Null-Controllable with Bounded Inputs (ANCBI) systems. The proposed policy iteration method relies on: a policy evaluation step with a piecewise quadratic Lyapunov function in both the state and the deadzone functions of the input signals; a policy improvement step which guarantees at the same time close to optimality (exploitation) and persistence of excitation (exploration). The proposed approach guarantees convergence of the trajectory to a neighborhood around the origin. Besides, the trajectories can be made arbitrarily close to the optimal one provided that the rate at which the the value function and the control policy are updated is fast enough. The solution to the inequalities required to hold at each policy evaluation step can be efficiently implemented with semidefinite programming (SDP) solvers. A numerical example illustrates the results.**

I. INTRODUCTION

This work proposes an online policy iteration procedure for the synthesis of sub-optimal and practically stabilizing control policies for uncertain Linear Time Invariant (LTI) Asymptotically Null-Controllable with Bounded Inputs (ANCBI) systems. This class includes systems with eigenvalues on the imaginary axis (possibly repeated) but no pole with positive real part. The proposed policy iteration relies is appropriately modified so as to take into account the input saturation function: in particular, the policy evaluation step exploits a class of piecewise quadratic Lyapunov functions which is non-differentiable, but continuous, and depending both on the state and the deadzone function. The policy improvement step is based on a piecewise control policy: the solution of the policy improvement step requires the evaluation of the estimate of the derivative of the Lyapunov function under different candidate control laws, and the resulting mechanism guarantees at the same time close to optimality (exploitation) and persistence of excitation (exploration). The proposed approach guarantees convergence of the trajectory to a neighborhood of the origin. The solution to the inequalities which are required to hold at each step of the policy evaluation is obtained with the solution to semidefinite programmes (SDP).

Synthesis of globally stabilizing control laws for linear saturating systems is a nontrivial problem: even for Linear Time Invariant (LTI) Asymptotically Null-controllable with Bounded Inputs (ANCBI) systems it has been demonstrated with simple examples, that such a class can not, in general, be stabilized by static linear feedback [1]. Different methods to compute globally asymptotically stabilizing nonlinear control laws for ANCBI systems have been proposed [2], [3]. While global stability may not be achieved with linear control laws, strategies for semi-global (exponential) stabilization were presented in [4] (see also the semi-global results for exponentially unstable plants in [5]). However, semi-global results rely on low-gain strategies that may lead to poor performance (in terms of closed-loop convergence rate). In order to obtain faster transients, scheduled [6] and nonlinear control laws [7] have also been proposed in the context of semi-global stabilization. However optimality with respect to other criteria than the convergence rate, has not been explored. In the aforementioned approaches the plant is assumed to be known and the control synthesis is performed offline. Online extension via predictive techniques is considered in [8].

Online techniques for adaptive control of uncertain input-saturated systems have mainly focused on the problem of guaranteeing global stability [9], [10] without optimality considerations: these schemes guarantee global stability via a continuous-time direct adaptive controller. More recently, approaches to optimal control of input-saturated systems have been developed, with the aim of approximating the optimal solution to the Hamilton-Jacobi-Bellman equation. Since some knowledge of the dynamics is required to implement these techniques, online estimation must be employed. Interesting approaches, yet not taking input-saturation into account, are [11], [12], where actor-critic structures are combined with a third network meant to approximate the unknown system dynamics. Actor-critic structures are updated in such a way to approximate the optimal control solution and the optimal value function respectively. For constrained-input systems some offline [13] and online [14], [15], [16] actor-critic methods based on neural networks have been proposed, where however, the input saturation is assumed to be a sigmoidal continuous and differentiable function.

The paper is organized as follows: Section II presents the problem formulation; Section III recalls an offline policy iteration approach for input saturated systems; Section IV presents the estimation scheme for the uncertain system

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dynamics; Section V contains the online policy iteration approach for input-saturated systems, and Section VI the numerical implementation of the policy evaluation step. The numerical example in Section VII demonstrates the effectiveness of the proposed approach.

II. Problem Formulation

We study the class of uncertain LTI Asymptotically Null-Controllable with Bounded Inputs (ANCBI) systems in the presence of input saturation, which consists of the set of dynamic linear systems without exponentially unstable modes. Consider the input-saturated system

$$\dot{x} = A(\Theta^*)x + B(\Theta^*)sat\left(u(x)\right), \quad x(0) = x_0, \tag{1}$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $A(\Theta^*) \in \mathbb{R}^{n\times n}$ and $B(\Theta^*) \in \mathbb{R}^{n\times m}$, $\max(\Re(\lambda(A))) \leq 0$. Both A and B are assumed to be matrices with unknown entries represented by Θ^*. The function $sat : \mathbb{R}^m \to \mathscr{U} \subset \mathbb{R}^m$ is a vector saturation function, with entries satisfying

$$\left(sat\left(u(x)\right)\right)_j = \begin{cases} \bar{u}_j, & \text{if} \quad u_j > \bar{u}_j \\ u_j, & \text{if} \quad \underline{u}_j \leq u_j \leq \bar{u}_j \\ \underline{u}_j, & \text{if} \quad \underline{u}_j > u_j \end{cases}$$

with $\bar{u}_j$ and $\underline{u}_j$ the upper and lower bound of the j-th input, respectively. The set of admissible inputs is defined as

$$\mathscr{U} := \left\{u \in \mathbb{R}^m | \underline{u}_j \leq u_j \leq \bar{u}_j, j = 1,\ldots,m\right\}.$$

In the following, for convenience of notation, we introduce the dead-zone function $dz(u(x)) := u(x) - sat\left(u(x)\right)$, and rewrite (1) as

$$\dot{x} = A(\Theta^*)x + B(\Theta^*)u(x) - B(\Theta^*)dz\left(u(x)\right), \quad x(0) = x_0. \tag{2}$$

We also introduce a cost function for the system (2) in the form

$$J = \int_0^\infty L(x,u)dt = \int_0^\infty x'Qx + \left(sat\left(u\right)\right)'Rsat\left(u\right)dt, \tag{3}$$

where the prime denote transpose. To address the parametric uncertainty in the system, we will develop an adaptive control policy, combined with a parametric adaptation law taking the following form

$$\dot{\hat{\Theta}}(t) = p(\hat{\Theta}(t), \Xi(t)), \quad \hat{\Theta}(0) = \Theta_0 \tag{4a}$$

$$\dot{\Xi}(t) = g(\Xi(t), x(t), u(t)), \quad \xi(0) = \xi_0 \tag{4b}$$

$$V(t) = s(\hat{\Theta}(t), u(t)), \tag{4c}$$

$$u^+(t) = h(\hat{\Theta}(t), V(t)), \tag{4d}$$

where $\hat{\Theta}$ are the estimates of Θ^*, Ξ are auxiliary variables used for estimation, V indicates the value function, and $u^+(t)$ indicates the feedback law to be used in the time interval $[t+k\delta t, t+(k+1)\delta t]$, where δt is the sampling time. The mappings, p, g, s, h will be designed to guarantee the convergence of the state in a neighborhood of the origin and to optimize the cost (3).

Let us introduce the definitions below:

Definition 1: [Practical stability [17]] Given a nonlinear system $\dot{x} = f(x)$, with $f(0) = 0$, the origin of the system is *practically stable* if, for given $(c, \bar{c})$ with $0 < c < \bar{c}$, every solution $x(t, x_0)$ of the system satisfies

$$\|x_0\| < c \Rightarrow \|x(t, x_0)\| < \bar{c}, \quad t \geq t_0$$

for some $t_0 \in \mathbb{R}_+$.

Definition 2: [Asymptotic minimization] Given a function $J(\vartheta)$ and

$$\vartheta^* = \arg\min_{\vartheta} J(\vartheta)$$

the sequence $\{\vartheta_k\}$ asymptotically minimizes J if $\lim_{k\to\infty} \vartheta_k = \vartheta^*$.

The objective of the control problem can be stated as:

Problem 1: Design the functions $p(\cdot,\cdot)$, $g(\cdot,\cdot,\cdot)$, $s(\cdot,\cdot)$ and $h(\cdot,\cdot)$ such that the closed-loop (2)-(4) guarantees the practical stability of the origin of (2) and the asymptotic minimization of the cost (3).

In the following, multidimensional vectors are intended as column vectors, while the gradient of a scalar quantity with respect to a vector is intended as a row vector. We introduce the sector condition pertaining to the deadzone presented in [18]. The deadzone function $dz(u(x))$ satisfies the following sector inequality We recall the sector conditions presented in [18]:

$$dz'(u(x))\Pi_1\left(u(x) - dz\left(u(x)\right)\right) \geq 0, \quad \forall\, x \in \mathbb{R}^n. \tag{5}$$

implying that the deadzone function is contained in the sector $[0, I]$. Furthermore, define $\phi(x) := \frac{d\, dz(u(x))}{dt}$ satisfying

$$\phi(x) = \begin{cases} 0 & \text{if } dz(u(x)) = 0 \\ \dot{u}(x) & \text{if } dz\left(u(x)\right) \neq 0, \end{cases} \tag{6}$$

which can be expressed by the two equalities

$$\phi'(x)\Pi_2\left(\dot{u}(x) - \phi(x)\right) = 0 \tag{7}$$

$$dz'\left(u(x)\right)\Pi_3\left(\dot{u}(x) - \phi(x)\right) = 0. \tag{8}$$

where $\Pi_1, \Pi_2, \Pi_3 \in \mathbb{R}^{m\times m}_{diag}$ are diagonal matrices, and Π_1 is positive definite. Due to the monotonicity of the saturation and the deadzone functions, we also have that the following inequality holds for two arbitrary control policies $u(x)$ and $v(x)$

$$\left(dz\left(u(x)\right) - dz\left(v(x)\right)\right)'\left(sat\left(u(x)\right) - sat\left(v(x)\right)\right) > 0. \tag{9}$$

We adopt the well-known result from optimal control theory [19, Chap.3], that states that the optimal control policy $u^o(x)$ that minimizes (3) satisfies

$$u^o = \arg\min_{u(\cdot)\in\mathscr{U}} \left\{\frac{d\,V^o}{dx}\left(Ax + Bu\right) + L(x,u)\right\}, \tag{10}$$

where $V^o(x)$ is the value function (or cost-to-go function) that solves the Hamilton-Jacobi-Bellman (HJB) equation

$$\min_{u(\cdot)\in\mathscr{U}} \left\{\frac{d\,V^o}{dx}\left(Ax + Bu\right) + L(x,u)\right\} = 0. \tag{11}$$

In order to have a well-posed problem we make the following assumption

Assumption 1: There exists a globally stabilizing control policy $\bar{u}$ for the system (1).

According to standard converse-Lyapunov results [20] Assumption 1 implies the existence of a continuous, positive definite, radially unbounded control Lyapunov function (CLF) $\bar{V}:\mathbb{R}^n\to\mathbb{R}_+$ which satisfies

$$\min_{u(\cdot)\in\mathscr{U}}\left\{\frac{d\,\bar{V}}{dx}(Ax+Bu)\right\}<0,\quad \forall x\neq 0.$$

The following lemma relates the CLF to the uncontrollable region of system (1):

Lemma 1: Assumption 1 implies that there exist positive constants ε_i, $i=1,2,3$ *such that the following condition holds, for all* $x\in\mathbb{R}^n$,

$$\left|\frac{d\,\bar{V}}{dx}(x)B\right|<\varepsilon_1 \text{ and } |x|>\varepsilon_3 \Rightarrow \frac{d\,\bar{V}}{dx}(x)Ax<-\varepsilon_2. \quad (12)$$

Let us define the *uncontrollable region* of (1) to be the subset $\mathscr{R}$ defined according to

$$\mathscr{R}=\left\{x\in\mathbb{R}^n : |x|>\varepsilon_3 \text{ and } \left|\frac{\partial\bar{V}}{\partial x}(x)B\right|<\varepsilon_1\right\}.$$

Note that condition (12) implies that for $x\in\mathscr{R}$, the choice $u=0$ guarantees that $\dot{\bar{V}}<0$.

III. Offline Policy Iterations under Saturation Constraints

The iterative strategy in Algorithm 1 was presented in [21] and provides an offline solution to Problem 1.

Algorithm 1 Modified policy iteration

1: *Initialize*:
2: $c\leftarrow 0$.
3: $\bar{u}^c_{pw}\leftarrow u^0$.
4: $u^c_{pw}\leftarrow u^0$.
5: *Policy evaluation*:
6: Given u^c_{pw}, **solve** for $V^c(x)=W^c(x,dz(u^c(x)))$
7:
$$\frac{d\,V^c(x)}{dx}\left(Ax+Bsat\left(u^c_{pw}\right)\right)+L(x,u^c_{pw})=0 \quad (13)$$
8: *Feasibility*:
9: With $W^c(x,dz(u^c(x)))$ of *Policy evaluation*, **check**
10:
$$\frac{d\,V^c(x)}{dx}\left(Ax+Bsat\left(u^c\right)\right)+L(x,u^c_{pw})<0 \quad (14)$$
11: **if** (13) is *feasible*, $\bar{u}^c(x)\leftarrow u^c(x)$
12: **else** $\bar{u}^c(x)\leftarrow u^{(c-1)}(x)$
13: *Policy improvement*:
14: **Update** the piecewise control policy
15:
$$u^{c+1}_{pw}=\begin{cases}-\frac{1}{2}R^{-1}B'\left.\frac{\partial W^{c\,\prime}}{\partial x}\right|_{q^c=0} & \text{in } \Xi^c_1\cup\Xi^c_2\cup\Xi^c_3\\ \bar{u}^c & \text{in } \Xi^c_4\end{cases} \quad (15)$$
16:
17: **if** $\Delta W^c(x(0)):=W^c(x(0))-W^{(c-1)}(x(0))<\delta$, **STOP**
18: **else** $c\leftarrow c+1$, **goto** *Policy improvement*.

In the algorithm, the policy evaluation and the policy iteration steps are performed based on the piecewise value function and piecewise control policy defined as follows. Given the value function $V^c(x)=W^c(x,dz(u^c))$ solution of (13), define the following *approximated* policy improvement

$$u^{c+1}_{ap}(x) \;=\; -\frac{1}{2}R^{-1}B'\left.\frac{\partial W^{c\,\prime}}{\partial x}\right|_{dz(u^c)=0}, \quad (16)$$

where the term *approximated* policy improvement indicated that (16) coincides with the solution of

$$\arg\min_{u(\cdot)\in\mathscr{U}}\left\{\frac{d\,V^c}{dx}(Ax+Bu)+L(x,u)\right\}, \quad (17)$$

only in a subset of the state space, i.e. in the set

$$\Omega^c(x)=\{x : dz(u^c(x))=0\}. \quad (18)$$

See [21] for details.

Furthermore, the policy $u^{c+1}_{ap}(x)$ in (16) defines the following set,

$$\Omega^{c+1}(x)=\left\{x : dz(u^{c+1}_{ap}(x))=0\right\}. \quad (19)$$

which in general differs from the previously defined (18).

In order to discuss the properties of the policy (16) let us define the following state-space partition arising from the sets Ω^c and Ω^{c+1} defined in (18)-(19)

$$\begin{array}{lcll}\Xi^c_1 & := & \Omega^c\cap\Omega^{c+1} & \text{(Region 1)}\\ \Xi^c_2 & := & \Omega^c\backslash\Omega^{c+1} & \text{(Region 2)}\\ \Xi^c_3 & := & \Omega^{c+1}\backslash\Omega^c & \text{(Region 3)}\\ \Xi^c_4 & := & \mathbb{R}^n\backslash(\Omega^c\cup\Omega^{c+1}) & \text{(Region 4)}\end{array}$$

and satisfying $\cup_i\Xi^c_i=\mathbb{R}^n$ and $\Xi^c_i\cap\Xi^c_j=\emptyset$, $i\neq j$.

Remark 1: The set Ξ^c_1 *corresponds to the set in which both control policies respect the saturation bounds, and* Ξ^c_4, *to the set in which both control policies exceed the saturation bounds. Notice that* $\Xi^c_2=\emptyset$, *whenever* $\Omega^c\subset\Omega^{c+1}$ *and* $\Xi^c_3=\emptyset$, *whenever* $\Omega^{c+1}\subset\Omega^c$.

To study the stability properties of the policy u^{c+1}_{ap}, given a globally stabilizing policy u^c and a value function W^c that certify global stability, we define the piecewise policy

$$sat\left(u^{c+1}_{pw}\right)=\begin{cases}sat\left(u^{c+1}_{ap}\right) & \text{in } \Xi^c_1\cup\Xi^c_2\cup\Xi^c_3\\ sat\left(u^c\right) & \text{in } \Xi^c_4\end{cases} \quad (20)$$

with the value function

$$W^c_{pw}:=\begin{cases}W^c_{un} & \text{in } \Xi^c_1\cup\Xi^c_2\cup\Xi^c_3\\ W^c & \text{in } \Xi^c_4\end{cases} \quad (21)$$

where $W^c_{un}\in\mathscr{C}^1$ is the unsaturated value function defined as

$$W^c_{un}(x):=W^c(x,0). \quad (22)$$

We obtain the following result

Proposition 1: The piecewise value function (21) certifies the global stability of the piecewise control policy (20).

Proof: See [21]. ■

IV. Estimation of the system dynamics

The results of Section III require the knowledge of matrices A and B. Its extension to the uncertain system (1) requires an online parameter estimator. This task will be performed with standard techniques for parameter estimation. To this purpose we write (1) as

$$\dot{x} = A_m x + (A - A_m)x + Bsat(u), \tag{23}$$

with A_m a Hurwitz matrix. We use the series-parallel parametric model [22] to obtain

$$\dot{\hat{x}} = A_m \hat{x} + (\hat{A} - A_m)x + \hat{B}sat(u), \tag{24}$$

where $\hat{x}$ is the state of the parametric model and $\hat{A}$, $\hat{B}$ are the matrices to be estimated. In order to develop a linear-in-the-parameters model for (23) we filter every component of $\dot{x}$, x and $sat(u)$ with a stable filter $\lambda/(s+\lambda)$, $\lambda > 0$

$$z_f = \frac{s\lambda}{s+\lambda}x, \tag{25a}$$

$$x_f = \frac{\lambda}{s+\lambda}x, \tag{25b}$$

$$v_f = \frac{\lambda}{s+\lambda}sat(u). \tag{25c}$$

We thus obtain

$$z_f = A_m x_f + (A - A_m)x_f + Bv_f, \tag{26}$$

and similarly for (24)

$$\hat{z}_f = A_m \hat{x}_f + (\hat{A} - A_m)x_f + \hat{B}v_f, \tag{27}$$

where z_f, x_f, and v_f are all measurable signals to be used for the estimator. After collecting all the entries of A and B in $\Theta^* = [A\ B]'$ and defining $\hat{\Theta} = \left[\hat{A}\ \hat{B}\right]'$[1], we adopt a parameter estimator based on integral cost and gradient update [22], so as to obtain

$$\dot{\hat{\Theta}} = \mathsf{P}\left(-\gamma \bar{R}\hat{\Theta} - \gamma \bar{Q}\right), \quad \hat{\Theta}(0) = \Theta_0 \tag{28a}$$

$$\dot{\bar{R}} = -\beta \bar{R} + \left[x_f'\ \ v_f'\right]'\left[x_f'\ \ v_f'\right], \quad \bar{R}(0) = 0 \tag{28b}$$

$$\dot{\bar{Q}} = -\beta \bar{Q} - \left[x_f'\ \ v_f'\right]' z_f', \quad \bar{Q}(0) = 0 \tag{28c}$$

where β and γ are positive constants and P denotes a projection operator which has to be designed to keep the estimates inside a convex set.

The estimation law (28) satisfies the following properties:

Theorem 1: [22]

i) $\varepsilon := z_f - \hat{z}_f \in \mathscr{L}_2 \cap \mathscr{L}_\infty$

ii) $\lim_{t\to\infty} \left|\dot{\hat{\Theta}}\right| = 0$

iii) if $\left[x_f'\ \ v_f'\right]'$ is persistently exciting, then $\hat{\Theta} \to \Theta^*$ exponentially and the rate of convergence increases with γ.

[1] A parameter estimator can be developed also in the case where only a subset of entries of A and B needs to be estimated, by bringing to the left-hand side of (26) and (27) all the quantities that are known and do not need to be estimated.

V. Online Policy Iterations under saturation constraints

Algorithm 1 is now revised for online implementation. Differently from Section IV, the iterations are not implemented offline at each step c, $c \in \mathbb{Z}_+$, but online at each time instant t_k, $t_k = 0, \Delta t, 2\Delta t, \ldots$, where Δt is the update sample time. The proposed algorithm is shown in Algorithm 2.

In Algorithm 2, t_k^+ indicates the instant of time at which the previous policy is updated, $\left\{\pm u_{(j)}^k, j = 1, \ldots, n\right\}$ indicates a set of candidate policies, $\hat{V}_{(\pm j)}^k(t_k)$ in (35) indicates the estimates of the derivative of the value function calculated at time t_k with the corresponding policy $\pm u_{(j)}^k$. Furthermore, the candidate control policies $\pm u_k^{(j)}$ are calculated as follows

$$h^k(\zeta, x) = -\frac{1}{2}R^{-1}\hat{B}'\left.\frac{\partial V^{k\prime}}{\partial x}\right|_{dz(u^{(k-1)})=0} \tag{37}$$

$$\pm u_{(j)}^k = h^k(\zeta \pm \Delta\zeta_{(j)}, x), \tag{38}$$

where ζ are the coefficients of the expression in (37), and $\Delta\zeta^{(j)}$ are zero-mean random vectors in $[-2\alpha_k, -\alpha_k]^n \cup [\alpha_k, 2\alpha_k]^n$ satisfying

$$\left|\left[\Delta\zeta^{(1)}, \ldots, \Delta\zeta^{(n)}\right]\right|^{-1} < \frac{\Xi}{\alpha_k}, \tag{39}$$

where α_k is a positive sequence and Ξ is a finite positive number independent of α_k. The following result is given.

Theorem 2: Let Δt be sufficiently small. Then, for arbitrary small $\bar{\alpha} > 0$, there exist finite positive constants β_1, β_2, γ_l and a finite positive integer $\bar{h} = \mathscr{O}\left(\frac{1}{\gamma}\right)$ such that the following condition holds: if α^k satisfies

$$\begin{array}{ll} 0 < \alpha^k \le \beta_2 & \text{if } \left|\frac{dV^k}{dx}\hat{B}\right| < \hat{\varepsilon}_1 \text{ or } k \le \bar{h} \\ \alpha_k \ge \beta_1 & \text{otherwise} \end{array}$$

where $\hat{\varepsilon}_1$ is a positive design constant satisfying

$$\frac{1}{4}\varepsilon_1 < \hat{\varepsilon}_1 \le \frac{1}{2}\varepsilon_1$$

and the adaptive gain γ of the estimator satisfies

$$\gamma \ge \gamma_l$$

then, the proposed adaptive control scheme guarantees that the closed-loop solutions are bounded and, moreover,

$$\limsup_{t\to\infty} |x(t)| \le \varepsilon_3, \text{ w.p.1}$$

and

$$-\bar{\alpha} < \dot{\mathscr{L}}(t_k^+) < 0, \text{ if } x_k \notin \mathscr{R} \text{ or } (x_k, \hat{\theta}_k) \notin \mathscr{S}_k, \text{ w.p.1}$$

where

$$\dot{\mathscr{L}}(t_k^+) = \min_{u(\cdot)\in\mathscr{U}}\left\{\frac{dV^o}{dx}(Ax + Bu) + L(x,u)\right\}$$

and $\mathscr{S}_k$ is a subset of $\mathbb{R}^n \times \mathbb{R}^{n\times(n+m)}$ that satisfies $\mathscr{S}_k = \emptyset, \forall k > \bar{h}$.

Proof: See Appendix. ■

Algorithm 2 Online policy iteration

Initialize:

$k \leftarrow 0$.

$\bar{u}^k_{pw} \leftarrow u^0$.

$u^k_{pw} \leftarrow u^0$.

Policy evaluation:

Given u^k_{pw}, $\hat{A}^{(k-1)} = \hat{A}(t_{k-1})$, $\hat{B}^{(k-1)} = \hat{B}(t_{k-1})$,

solve for $V^k(x) = W^k(x, dz\left(u^k(x)\right))$

$$\frac{d\,V^k(x)}{dx}\left(\hat{A}^{(k-1)}x + \hat{B}^{(k-1)} sat\left(u^{(k-1)}_{pw}\right)\right) + L(x, u^{(k-1)}_{pw}) = 0 \tag{29}$$

$$\frac{d\,V^k(x)}{dx}\left(\bar{A}x + \bar{B} sat\left(u^{(k-1)}_{pw}\right)\right) + L(x, u^{(k-1)}_{pw}) < 0 \tag{30}$$

$$\bar{A} = \hat{A}^{(k-1)} + \Delta A, \bar{B} = \hat{B}^{(k-1)} + \Delta B \text{ with } \Delta A, \Delta B \in \mathscr{N}^k \tag{31}$$

$$\mathscr{N}^k = \left\{\Delta A, \Delta B \mid \Delta A' \Delta A \prec \eta^k_1 I,\ \Delta B' \Delta B \prec \eta^k_2 I\right\}, \tag{32}$$

Feasibility:

With $W^k(x, dz\left(u^k(x)\right))$ of *Policy evaluation*, **check**

$$\frac{d\,V^k(x)}{dx}\left(\hat{A}^{(k-1)}x + \hat{A}^{(k-1)} sat\left(u^k\right)\right) + L(x, u^k_{pw}) < 0 \tag{33}$$

if (13) is *feasible*, $\bar{u}^k(x) \leftarrow u^k(x)$

else $\bar{u}^k(x) \leftarrow u^{(k-1)}(x)$

Estimation:

Update the estimates $\hat{A}(t_k)$, $\hat{B}(t_k)$ according to (28), with $\mathsf{P} = \mathsf{P}_k$ the projection operator that keeps the estimate inside the set $\mathscr{N}^k$.

Policy improvement:

Update the piecewise control policy

$$u(t^+_k) = \arg \min_{\pm u^k_{(j)}, j=1,\ldots,n} \hat{V}^k_{(\pm j)}(t_k), \tag{34}$$

$$\begin{aligned}\hat{V}^k_{(\pm j)}(t_k) \quad = \quad & \frac{\partial V_k}{\partial x}'\left[\hat{A}(t_k)x(t_k) + \hat{B}(t_k)(\pm u^k_{(j)})\right] \\ & + Q(x(t_k)) + u^k_{(j)} R u^k_{(j)},\end{aligned} \tag{35}$$

$$u^{k+1}_{pw} = \begin{cases} u(t^+_k) & \text{in } \Xi^k_1 \cup \Xi^k_2 \cup \Xi^k_3 \\ \bar{u}^k & \text{in } \Xi^k_4 \end{cases} \tag{36}$$

if $\Delta W^k(x(0)) := W^k(x(0)) - W^{(k-1)}(x(0)) < \delta$, **STOP** updating W and u

else goto *Policy evaluation*.

$k \leftarrow k+1$

Remark 2: Each policy evaluation step (29)-(30) returns a set of plants that are stabilized by the control law u^k_{pw}. Such a set is given by $\mathscr{N}^k$ in (32)*. This set is used in the estimation law (28) to project the estimate. This approach resembles the so-called 'certainty equivalence principle' of adaptive control [22], where the control policy is stabilizing for the estimated plant and it is updated by solving the underlying control problem for the estimated plant.*

Remark 3: The rationale behind (34) *is that among a set of possible candidate control laws, the one that minimizes* (35)*, i.e. that more closely satisfies the HJB equation is chosen. This choice guarantees the so-called 'exploitation task' of the control policy. Furthermore, the candidate control policies are generated randomly so as to satisfy condition* (39)*: this guarantees the so-called 'exploration task' of the control policy, i.e. persistence of excitation and convergence of the estimates to their real value. It can be shown that the Bernoulli distribution satisfy condition* (39) *[23]: other distributions (segmented uniform, U-shaped) are also possible [24, Sect. 7.3].*

VI. Numerical implementation of policy evaluation

In the following we provide a sufficient condition for calculating a (piecewise) certificate $W(x, dz(u^k))$ for global stability of the system (1) with the piecewise defined policy u^k_{pw}. To this purpose let us define $q^k = dz(u^k)$. Given a policy u^k_{pw} defined as

$$u^k_{pw} = \begin{cases} u^k(x) & x \in \cup_{i\in\{1,2,3\}} \Xi^{(k-1)}_i \\ u^{(k-1)}(x) & x \in \Xi^{(k-1)}_4, \end{cases} \tag{40}$$

find $W(x, q^k)$, $\Pi^{pw}_i(\xi)$ satisfying

$$\begin{aligned} p_1(\xi) = & -\frac{\partial W}{\partial x}\left(\hat{A}^{k-1}x + \hat{B}^{k-1}\left(u^k_{pw} - q^k_{pw}\right)\right) - \frac{\partial W}{\partial q^k}\phi^{k/pw} \\ & - x'Q(x)x - (u^k_{pw} - q^k_{pw})'R(x)(u^k_{pw} - q^k_{pw}) \\ & + q^{k\prime}_{pw}\Pi^{pw}_1(\xi)(u^k_{pw} - q^k_{pw}) + \phi^{k/pw\prime}\Pi^{pw}_2(\xi)(\dot{u}^{k/pw} - \phi^{k/pw}) \\ & + q^{k\prime}\Pi^{pw}_3(\xi)(\dot{u}^{k/pw} - \phi^{k/pw}) \succeq 0 \quad \forall x \in \mathbb{R}^n \end{aligned} \tag{41}$$

and

$$\Pi^{pw}_1(\xi) \succeq 0$$

with

$$\xi = \begin{bmatrix} x' & u^{k\prime}_{pw} & q^{k\prime}_{pw} & \phi^{k/pw\prime} \end{bmatrix}',$$

where $\phi^{k/pw}$ denotes the time-derivative of q^k along the trajectories corresponding to policy u^k_{pw} and Π^{pw}_1. The expression for $\dot{u}^{k/pw}$ is

$$\dot{u}^{k/pw} = \frac{d\,u^k(x)}{dx}\left(Ax + B\left(u^k_{pw} - q^k_{pw}\right)\right).$$

Finally, (30)-(32) can be rewritten using the following property

$$
\begin{aligned}
&-\frac{\partial W}{\partial x}\left((\hat{A}^{k-1}+\Delta A)x+(\hat{B}^{k-1}+\Delta B)\left(u_{pw}^k-q_{pw}^k\right)\right)\\
&\quad=-\frac{\partial W}{\partial x}\left(\hat{A}^{k-1}x+\hat{B}^{k-1}\left(u_{pw}^k-q_{pw}^k\right)\right)\\
&\quad-\left(\frac{1}{\varepsilon_1}\frac{\partial W}{\partial x}-\varepsilon_1\Delta Ax\right)'\left(\frac{1}{\varepsilon_1}\frac{\partial W}{\partial x}-\varepsilon_1\Delta Ax\right)-\\
&\left(\frac{1}{\varepsilon_2}\frac{\partial W}{\partial x}-\varepsilon_2\Delta B\left(u_{pw}^k-q_{pw}^k\right)\right)'\left(\frac{1}{\varepsilon_2}\frac{\partial W}{\partial x}-\varepsilon_2\Delta B\left(u_{pw}^k-q_{pw}^k\right)\right)\\
&\quad+\left(\frac{1}{\varepsilon_1^2}+\frac{1}{\varepsilon_2^2}\right)\frac{\partial W}{\partial x}'\frac{\partial W}{\partial x}+\varepsilon_1^2x\Delta A'\Delta Ax\\
&\quad+\varepsilon_2^2\left(u_{pw}^k-q_{pw}^k\right)'\Delta B'\Delta B\left(u_{pw}^k-q_{pw}^k\right)\\
&\leq-\frac{\partial W}{\partial x}\left(\hat{A}^{k-1}x+\hat{B}^{k-1}\left(u_{pw}^k-q_{pw}^k\right)\right)+\left(\frac{1}{\varepsilon_1^2}+\frac{1}{\varepsilon_2^2}\right)\frac{\partial W}{\partial x}'\frac{\partial W}{\partial x}\\
&\quad+\varepsilon_1^2\eta_1^2x'x+\varepsilon_2^2\eta_2^2\left(u_{pw}^k-q_{pw}^k\right)'\left(u_{pw}^k-q_{pw}^k\right) \qquad (42)
\end{aligned}
$$

By using (42) and the Schur complement, (30)-(32) results in an expression $p_2(\xi)$ linear with respect to the optimization variables. The relations (5)-(7) can be used to obtain a sufficient condition for the negativity of $\frac{dW}{dt}+L(x,u)$. Notice that, due to (40), the variable u_{pw}^k in (41) satisfies

$$(u_{pw}^k-u^k)=0 \quad x\in\cup_{i\in\{1,2,3\}}\Xi_i^{(k-1)} \qquad (43)$$

$$(u_{pw}^k-u^{(k-1)})=0 \quad x\in\Xi_4^{(k-1)}. \qquad (44)$$

Therefore, considering the representation of sets

$$
\begin{aligned}
&\cup_{i\in\{1,2,3\}}\Xi_i^{(k-1)}=\left\{x\in\mathbb{R}^n|p_{0j}(x,\underline{u},\overline{u})\geq 0,j=1,\ldots,n_0\right\}\\
&\Xi_4^{(k-1)}=\left\{x\in\mathbb{R}^n|p_{4j}(x,\underline{u},\overline{u})\geq 0,j=1,\ldots,n_4\right\}
\end{aligned} \qquad (45)
$$

we obtain the following sufficient condition for (41) and (42) to hold with control policy (40)

$$
\begin{aligned}
&p_1(\xi)+p_2(\xi)+\Pi_{41}(\xi)(u_{pw}^k-u^k)+\\
&\qquad\sum_{j=1}^{n_0}m_0(\xi)p_{0j}(x,\underline{u},\overline{u})\succeq 0
\end{aligned} \qquad (46)
$$

$$
\begin{aligned}
&p_1(\xi)+p_2(\xi)+\Pi_{42}(\xi)(u_{pw}^k-u^{(k-1)})+\\
&\qquad\sum_{j=1}^{n_4}m_4(\xi)p_{4j}(x,\underline{u},\overline{u})\succeq 0
\end{aligned} \qquad (47)
$$

$$\Pi_1\succeq 0 \qquad (48)$$

Since $W(x,q^k)$ is an upper bound of the value function (given that the current policy may not be the optimal one), we solve

$$
\begin{aligned}
&\text{minimize } W(x(0),dz(u^k(x(0))))\\
&\text{subject to } (46)-(48)
\end{aligned}
$$

that minimizes the cost-to-go function with initial state $x(0)$.

VII. Numerical example

In the following, we present a numerical example to illustrate the results obtained via the proposed policy iterations. The procedure has been implemented in SOSTOOLS [25] and the formulated SDPs were solved with SeDuMi [26]. The dimension of the example helps to illustrate the results by plotting the computed value functions and the time-evolution of the control policies. It is also worth mentioning that as the number of variables and the degrees of the involved polynomials increase, the dimensions of the related SDPs can be large.

Consider the following 1-input 1-state system

$$\dot{x}(t)=-ax+bsat(u(t)),\quad x(0)=-1 \qquad (49)$$

with a and b two positive and unknown constants. The saturation bounds are $-0.5\leq u\leq 0.5$ and the initial globally stabilizing (but not optimal) state-feedback $u(x)=-0.3x$. For this system we consider the cost as in (3) with $Q=1$ and $R=1$. For $a=1$, $b=1$, $\hat{a}(0)=2$, $\hat{b}(0)=1.5$, $\beta=3$, $\lambda=3$, $\Gamma=10$, $\alpha_k=10$, $\Delta t=0.01$ we apply the proposed online policy iterations.

The simulation is run for 5 seconds. Fig. 1 shows the online evolution of the state and input with the proposed adaptive law. Fig. 2 shows the online evolution of the set $\mathscr{N}^k$ inside which the estimates are projected. Finally Fig. 3 shows the offline evolution of the cost using the controllers synthesized at every time step: it can be observed that the cost is monotonically decreasing. The online evolution of the Hamilton-Jacobi-Bellman equation is also shown: it can be observed that for the presented example the controller synthesized at every time step are stabilizing not only the estimated plant, but also the actual plant.

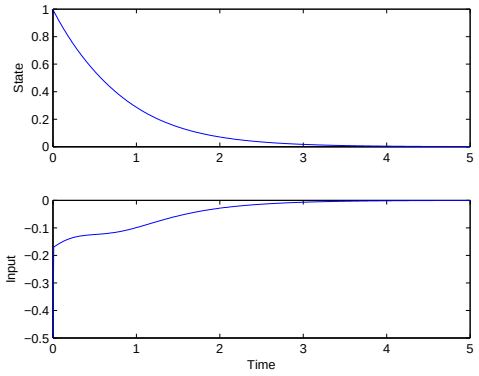

Fig. 1: Online state (upper) and input (lower) evolution.

VIII. Conclusions

This work proposed an online policy iteration procedure for the synthesis of approximately optimal control laws for uncertain Linear Time Invariant (LTI) Asymptotically Null-Controllable with Bounded Inputs (ANCBI) systems. The proposed policy iteration method relies on: a policy evaluation step with a piecewise quadratic Lyapunov function

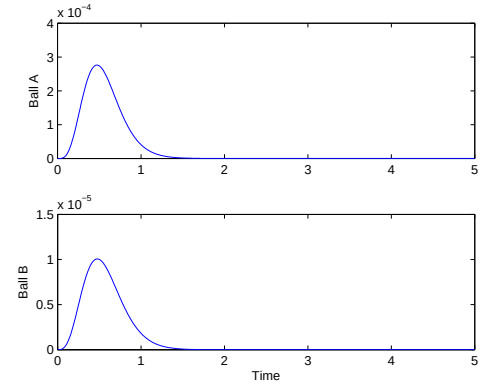

Fig. 2: Online evolution of the set $\mathscr{N}^k$.

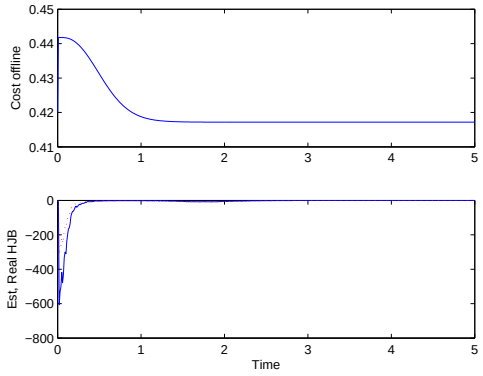

Fig. 3: Offline evolution of the cost using the controllers synthesized at every time step (upper) and online evolution of the Hamilton-Jacobi-Bellman equation (lower). With a solid line is the HJB for the estimated plant.

which is non-differentiable, but continuous, and polynomial in both the state and the deadzone functions of the input signals; a policy improvement step which guarantees at the same time close to optimality (exploitation) and persistence of excitation (exploration). The proposed approach guarantees convergence of the trajectory to a neighborhood around the origin. Besides, the trajectories can be made arbitrarily close to the optimal one provided that the rate at which the the value function and the control policy are updated is fast enough.

Future work includes the extension of the proposed methodology to linear systems with exponentially unstable modes for which only local stability is achievable. Such an extension is under study and will account for generalized sector condition which is instrumental to compute region of attraction estimates. We will also generalize the obtained conditions to systems defined by polynomial vector fields and polynomial input matrices.

APPENDIX

PROOF OF THEOREM 2

Estimating the matrix B creates the following problem: it might happen that we are in the uncontrollable region (defined by $\left|\frac{\partial V_k}{\partial x}B^k\right| \leq \hat{\varepsilon}_1$) but we estimate (via $\left|\frac{\partial V_k}{\partial x}\hat{B}^k\right| \geq \hat{\varepsilon}_1$) that we are not in the uncontrollable region. For this reason we have to define the following four cases for $|x| > \varepsilon_3$:

Case 1: $x(t_k) \notin \mathscr{R}$ and $\left|\frac{\partial V_k}{\partial x}\hat{B}^k\right| \geq \hat{\varepsilon}_1$ This means that we are not in the uncontrollable region, and we estimate correctly that we are not inside that region.

We have that

$$\dot{V}^{ok}_{(\pm j)} = \frac{\partial V^{ok}}{\partial x}\left[Ax(t_k) + B(\pm u^{(j)})\right] \tag{50}$$

where V^o indicates the optimal value function, $V^{ok} = V^o(t_k)$, while A and B are the matrices associated with the real plant (the argument Θ is omitted for easiness of notation). From the fact that we are outside the uncontrollable region, we can verify that

$$\min_{j\in\{1,\ldots,m\}} \frac{\partial V^{ok}}{\partial x} B\left[\pm u^{(j)}\right] < -c\varepsilon_1\alpha_k \tag{51}$$

and

$$\min_{j\in\{1,\ldots,m\}} \dot{V}^{ok}_{(\pm j)} = \frac{\partial V^{ok}}{\partial x}Ax(t_k) - c\varepsilon_1\alpha_k \tag{52}$$

By making the difference between the derivative of V^o with the real plant and the derivative of V with the estimated one, we obtain

$$\begin{aligned} \dot{V}^{ok}_{(\pm j)} - \hat{\dot{V}}^{k}_{(\pm j)} &= \frac{\partial V^{ok}}{\partial x}\left[Ax(t_k) + B(\pm u^{(j)})\right] \\ &\quad - \frac{\partial V^{k}}{\partial x}\left[\hat{A}^k x(t_k) + \hat{B}^k(\pm u^{(j)})\right] \quad (53) \\ &= \mathscr{O}(\left|\tilde{\Theta}^k\right|) + \mathscr{O}(\left|\tilde{\Theta}^k\right|)(\pm u^{(j)}) \quad (54) \end{aligned}$$

where $\tilde{\Theta}^k = \Theta - \hat{\Theta}^k$. The choice of the control input guarantees a persistence of excitation that implies

$$\tilde{\Theta}^k = \mathscr{O}(\frac{1}{k\gamma}). \tag{55}$$

Using (53) and (55) we then obtain

$$\dot{V}^{ok}_{(\pm j)} - \hat{\dot{V}}^{k}_{(\pm j)} = \mathscr{O}(\frac{1}{k\gamma}) + \mathscr{O}(\frac{1}{k\gamma})\alpha_k. \tag{56}$$

Combining (56) and (52) we then obtain that, if

$$\varepsilon_1 > \mathscr{O}(\frac{1}{k\gamma}), \tag{57}$$

then

$$\begin{aligned} &\arg\min_{j\in\{1,\ldots,m\}} \dot{V}^{ok}_{(\pm j)} + Q(x) + (\pm u^{(j)})^T R(\pm u^{(j)}) = \\ &\quad \arg\min_{j\in\{1,\ldots,m\}} \hat{\dot{V}}^{k}_{(\pm j)} + Q(x) + (\pm u^{(j)})^T R(\pm u^{(j)}) \end{aligned} \tag{58}$$

and $\exists\beta_1 > 0 : \alpha_k > \beta_1$ we have $-\bar{\alpha} < \dot{\mathscr{L}}(t_k^+) < 0$.

Case 2: $x(t_k) \in \mathscr{R}$ and $\left|\frac{\partial V^k}{\partial x}\hat{B}_k\right| < \hat{\varepsilon}_1$ This means that we are in the uncontrollable region, and we estimate correctly that we are inside that region.

Inside the uncontrollable region we have

$$\frac{\partial V^{ok}}{\partial x}Ax < \varepsilon_2, \tag{59}$$

which implies, by choosing the input as in the algorithm

$$\dot{V}^{ok}(t_k^+) < -\varepsilon_2 + \sqrt{m}\varepsilon_1\alpha_k = -\bar{\bar{\alpha}}. \tag{60}$$

Case 3: $x(t_k) \notin \mathscr{R}$ and $\left|\frac{\partial V^k}{\partial x}\hat{B}_k\right| < \hat{\varepsilon}_1$ This means that we are not in the uncontrollable region, and we estimate erroneously that we are inside that region.

By using similar reasoning as in Case 1, we obtain

$$\left|\frac{\partial V^{ok}}{\partial x}B - \frac{\partial V^k}{\partial x}\hat{B}^k\right| < \mathscr{O}(\frac{1}{k\gamma}). \tag{61}$$

So, if $\hat{\varepsilon}_1 < \varepsilon_1/2$, and using the reverse triangular inequality, we obtain that if $\mathscr{O}(\frac{1}{k\gamma}) < \varepsilon_1$, then Case 3 is never verified.
Case 4: $x(t_k) \in \mathscr{R}$ and $\left|\frac{\partial V^k}{\partial x}\hat{B}^k\right| \geq \hat{\varepsilon}_1$ This means that we are in the uncontrollable region, and we estimate erroneously that we are not inside that region.

If $\varepsilon_1/4 < \hat{\varepsilon}_1$, and using the reverse triangular inequality we have that

$$\left|\frac{\partial V^k}{\partial x}\hat{B}^k\right| > \varepsilon_1/4. \tag{62}$$

Then the same arguments of Case 1 hold, after replacing ε_1 with $\varepsilon_1/4$.

From the analysis of these 4 cases we have that, if $\mathscr{O}(\frac{1}{k\gamma}) < \varepsilon_1$, then $\dot{V}^o(t_k^+) < 0$. We conclude the proof by noticing that, since Δt is sufficiently small, we have that there exists a positive integer $k^* > \bar{h}+1$ such that $x_k \in \bar{\mathscr{X}}$ and $\mathscr{O}(\frac{1}{k\gamma}) < \varepsilon_1$ holds for all $k \in \{\bar{h}+1,\ldots,k^*\}$ which, in turn implies that either $\dot{V}^o(t_k^+) < -\bar{\alpha}$ (if $x_k \notin \mathscr{R}$) or $\dot{V}^o(t_k^+) < -\bar{\bar{\alpha}}$ (if $x_k \in \mathscr{R}$ and $|x(t_k)| \geq \varepsilon_3$) for all $k \in \{\bar{h}+1,\ldots,k^*\}$. It is then possible to show closed-loop signal boundedness and convergence of $x(t)$ to the subset $\{x : \mathbb{R}^n : |x| < \varepsilon_3\}$.

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