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Ye Wang[1], Sorin Olaru[2], Giorgio Valmorbida[2], Vicenç Puig[1] and Gabriela Cembrano[1,3]

***Abstract*— In this paper, the computation of robust invariant sets for discrete-time uncertain descriptor systems are investigated. The studied descriptor systems are assumed to be regular and stable subject to unknown but bounded disturbances. The robust invariant sets of both causal and non-causal descriptor systems are studied. It is recalled that a descriptor system can be transformed into an equivalent system by using a suitable linear coordinate transformation. Transformations for causal and non-causal descriptor systems are used in the characterization of the effect of the disturbances. For causal descriptor systems, two robust positively invariant (RPI) sets are computed separately. For non-causal descriptor systems, an RPI set and a robust negatively invariant (RNI) set can be found. As a result, the general RPI set of a descriptor system can be obtained from a linear projection image of these two sets. Besides, active mode detection method is proposed based on set invariance theory.**

I. Introduction

Set invariance has played an essential role in control theory with a variety of applications to constrained standard dynamic systems, uncertain control systems as well as their related different control designs [1]. It is instrumental for control strategies such as reference governor [2], fault diagnosis [3], fault-tolerant control [4] and robust model predictive control [5], [6]. The robust positively invariant (RPI) set and minimal RPI (mRPI) set have been well studied and the approximation of the mRPI set has been investigated and summarized in [7], [8]. Alternatively, ultimate bounds of dynamic systems [9] can lead to RPI set and the computations of such ultimate bounds can be of relative low complexity. An iterative strategy is proposed in [4] and leads to the mRPI estimation based on the computation of ultimate bounds.

In different scenarios, due to mass, volume or energy conservation laws, the differential equations describing a dynamic system are coupled with a set of algebraic equations. This class of systems described by differential and algebraic equations is called *descriptor systems* or *singular systems*. Instances of such systems are water distribution networks [10], chemical processes [11], electrical circuits [12], aircrafts [13], biological systems [14] and economic models [15]. A well-posed description, for which a solution exists and is unique, is said to be *regular*. Regularity, however, does not imply *causality* and models of interest in economy are non-causal, see e.g. the Leontief model [15], [16].

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A. Contribution

The main contribution of this paper is to present the computation of robust invariant (RI) sets for discrete-time regular descriptor systems. For the case of non-causal descriptor systems, we first decouple the set of states into causal and anti-causal states. The computation of the mRPI set of the causal states is obtained via an iterative method with ultimate bounds. On the other hand, for the anti-causal states, we introduce robust negatively invariant (RNI) sets. Then, we apply the obtained computational results for active mode detection. Finally, some numerical examples illustrate the proposed methods.

B. Outline

The remainder of this paper is structured as follows: In Section II, the problem statement and some preliminary results are expressed. In Section III, main results on the computation of mRPI sets of causal and non-causal descriptor systems are presented. In Section IV, active mode detection methods are proposed based on set invariance theory. In Section V, numerical examples are provided to illustrate the proposed methods. In Section VI, some conclusions are drawn.

C. Notation

The Minkowski addition operator is denoted by $\oplus$. I_j denotes the identity matrix of dimension j. Let $\mathcal{X}$ and $\mathcal{Y}$ be two non-empty sets, the Hausdorff distance is defined by

$$d_H(\mathcal{X},\mathcal{Y}) = \max\left\{\sup_{x\in\mathcal{X}}\inf_{y\in\mathcal{Y}} d(x,y), \sup_{y\in\mathcal{Y}}\inf_{x\in\mathcal{X}} d(x,y)\right\},$$

with $d(x,y)$ the distance between two points x and y.

II. Problem Statement and Preliminary Results

In this section, we recall some definitions for discrete-time descriptor systems and results on set invariance.

A. Discrete-time Descriptor Systems

Consider the discrete-time linear time invariant (LTI) descriptor system:

$$Ex(k+1) = Ax(k) + B_w w(k), \tag{1}$$

where $x(k) \in \mathbb{R}^n$ denotes state vector, $w(k) \in \mathbb{R}^q$ denotes the disturbance vector, $A \in \mathbb{R}^{n\times n}$, $B_w \in \mathbb{R}^{n\times q}$ and $E \in \mathbb{R}^{n\times n}$ with $\mathrm{rank}(E) = r \leq n$. For a descriptor system (1), under the disturbance-free condition, we have the following definitions:

Definition 1 (Regularity): The descriptor system (1) is said to be regular if it has a unique solution.

If (1) is regular, we also say the matrix pair (E, A) is regular.

Definition 2 (Causality): A regular descriptor system (or a regular matrix pair (E, A)) is said to be causal if its state $x(k)$ at any time instant k is determined completely by the initial condition $x(0)$. Otherwise, it is called non-causal.

Definition 3 (Asymptotic Stability): A regular descriptor system is said to be asymptotically stable if $\lim_{k\to\infty} x(k) = 0$.

Definition 4 (Admissibility): The descriptor system (1) is said to be admissible if it is regular, causal and asymptotically stable.

Lemma 1 (Regularity, causality and asymptotic stability): The following conditions can be used to verify [15]

- Regularity: the matrix pair (E, A) is regular if $\det(zE - A)$ is not identically zero.
- Causality: the matrix pair (E, A) is causal if $\deg(\det(zE - A)) = \mathrm{rank}(E)$.
- Stability: the matrix pair (E, A) is asymptotically stable if $|\lambda(E, A)| < 1$ for $\forall\lambda(E, A) \in \{z \mid \det(zE - A) = 0\}$.

In this paper, we study the descriptor systems that are not necessarily admissible, i.e. we consider descriptor systems which involves anti-causal states.

Assumption 1: The descriptor system (1) (the matrix pair (E, A)) is regular and asymptotically stable.

We now establish transformations that decompose the descriptor systems in subsystems suitable for analysis.

Definition 5 (Equivalence of descriptor systems): The descriptor system (1) with (E, A, B_w) is said to be restrictively equivalent to the descriptor system with $(\tilde{E}, \tilde{A}, \tilde{B}_w)$ under the transformation (P, Q) if there exists a pair of non-singular matrices $P \in \mathbb{R}^{n\times n}$ and $Q \in \mathbb{R}^{n\times n}$ satisfying

$$QEP = \tilde{E},\ QAP = \tilde{A},\ QB_w = \tilde{B}_w. \tag{2}$$

For a given descriptor system (1), we now present two standard equivalent forms that are of interest.

1) Dynamic Decomposition Form: Consider the descriptor system (1) with $rank(E) = r$, there always exists a transformation (M, N) yielding

$$MEN = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, MAN = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, MB_w = \begin{bmatrix} B_{w1} \\ B_{w2} \end{bmatrix}, \tag{3}$$

with $A_1 \in \mathbb{R}^{r\times r}$, $A_2 \in \mathbb{R}^{r\times(n-r)}$, $A_3 \in \mathbb{R}^{(n-r)\times r}$, $A_4 \in \mathbb{R}^{(n-r)\times(n-r)}$, $B_{w1} \in \mathbb{R}^{r\times q}$ and $B_{w2} \in \mathbb{R}^{(n-r)\times q}$.

Lemma 2 (Dynamic decomposition form): The descriptor system (1) is causal if and only if there exists a transformation in the sense of *Definition 5*, giving (3) with a non-singular block A_4 [17, Ch.2].

Lemma 3 (Equivalent causal descriptor system): A causal descriptor system defined in (1) with $rank(E) = r$ can be transformed in a standard dynamic form as

$$\tilde{x}(k+1) = \tilde{A}\tilde{x}(k) + \tilde{B}_w\tilde{w}(k), \tag{4}$$

with

$$\tilde{A} = \begin{bmatrix} A_1 - A_2A_4^{-1}A_3 & 0 \\ -A_4^{-1}A_3\left(A_1 - A_2A_4^{-1}A_3\right) & 0 \end{bmatrix}, \tag{5a}$$

$$\tilde{B}_w = \begin{bmatrix} B_{w1} - A_2A_4^{-1}B_{w2} & 0 \\ -A_4^{-1}A_3\left(B_{w1} - A_2A_4^{-1}B_{w2}\right) & -A_4^{-1}B_{w2} \end{bmatrix}. \tag{5b}$$

and

$$\tilde{x}(k) = N^{-1}x(k) = \begin{bmatrix} \tilde{x}_1(k) \\ \tilde{x}_2(k) \end{bmatrix}, \tilde{w}(k) = \begin{bmatrix} w(k) \\ w(k+1) \end{bmatrix}, \tag{6}$$

with $\tilde{x}_1(k) \in \mathbb{R}^r$, $\tilde{x}_2(k) \in \mathbb{R}^{n-r}$.

Proof: From (1), the transformation (M, N) gives

$$MENN^{-1}x(k+1) = MANN^{-1}x(k) + MB_w w(k).$$

Using (3) and defining $\tilde{x}(k)$ as in (6), we have

$$\tilde{x}_1(k+1) = A_1\tilde{x}_1(k) + A_2\tilde{x}_2(k) + B_{w1}w(k), \tag{7a}$$

$$0 = A_3\tilde{x}_1(k) + A_4\tilde{x}_2(k) + B_{w2}w(k). \tag{7b}$$

Since the descriptor system is causal, A_4 is invertible based on *Lemma 2*. Then, from (7b), it comes

$$\tilde{x}_2(k) = -A_4^{-1}A_3\tilde{x}_1(k) - A_4^{-1}B_{w2}w(k). \tag{8}$$

Substituting $\tilde{x}_2(k)$ in (7a) by (8), it gives

$$\tilde{x}_1(k+1) = \left(A_1 - A_2A_4^{-1}A_3\right)\tilde{x}_1(k) + \left(B_{w1} - A_2A_4^{-1}B_{w2}\right)w(k). \tag{9}$$

From (8), we also have $\tilde{x}_2(k+1) = -A_4^{-1}A_3\tilde{x}_1(k+1) - A_4^{-1}B_{w2}w(k+1)$ and use (9) to obtain

$$\begin{aligned}\tilde{x}_2(k+1) =& -A_4^{-1}A_3\left(A_1 - A_2A_4^{-1}A_3\right)\tilde{x}_1(k) - A_4^{-1}B_{w2}w(k+1) \\ & -A_4^{-1}A_3\left(B_{w1} - A_2A_4^{-1}B_{w2}\right)w(k)\end{aligned} \tag{10}$$

Hence, we obtain (4) and the proof is completed. ■

2) Kronecker Canonical Form:

Lemma 4 (Kronecker canonical form): The descriptor system (1) is regular if and only if there exists a transformation $(\hat{M}, \hat{N})$ yielding

$$\hat{M}E\hat{N} = \begin{bmatrix} I_p & 0 \\ 0 & \bar{N} \end{bmatrix}, \hat{M}A\hat{N} = \begin{bmatrix} \bar{A} & 0 \\ 0 & I_{(n-p)} \end{bmatrix}, \tag{11}$$

where $\bar{N}$ is a nilpotent matrix and $p \leq r = rank(E)$.

Lemma 5 (Causality): Let the matrix pair (E, A) be given in the regular form (11), then it is causal if and only if $\bar{N} = 0$.

B. Set Invariance

For the regular and stable descriptor system (1), the additive disturbance vector $w(k)$ belongs to a compact set $\mathcal{W}$, $\forall k \in \mathbb{Z}$. As a consequence of the boundedness of the disturbances and the stability, the system trajectories are expected to be confined in a set within the state space [18]. Given an initial state $x(0)$ and unique solution of (1) denoted by $x(k)$, $\forall k \in \mathbb{Z}$ (the time instant k can be negative for the backward propagation), the following definitions are introduced in view of the set-theoretic analysis:

Definition 6 (Robust invariant set): A set $\Omega \in \mathbb{R}^n$ is said to be *robust invariant* with respect to the descriptor system (1) if $x(0) \in \Omega$ implies $x(k) \in \Omega$ $\forall w(k) \in \mathcal{W}$ and $\forall k \in \mathbb{Z}$.

Definition 7 (Robust positively invariant set): A set $\Omega \in \mathbb{R}^n$ is said to be *robust positively invariant* with respect to (1) if $x(0) \in \Omega$ implies $x(k) \in \Omega$ $\forall w(k) \in \mathcal{W}$ and $\forall k \in \mathbb{Z}_{[>0]}$.

Definition 8 (Minimal RPI set): A RPI set $\Omega_\infty \in \mathbb{R}^n$ is said to be *minimal RPI* with respect to (1) if it is contained in every closed RPI set of (1).

Definition 9 (L-step robust negatively invariant set): The set $\Omega \in \mathbb{R}^n$ is L-step *robust negatively invariant* with respect to the descriptor system (1) if the state $x(L) \in \Omega$ implies $x(L+k) \in \Omega$ for $\forall w(k) \in \mathcal{W}$ and $\forall k \in \mathbb{Z}_{[-L,0]}$.

For standard LTI systems (corresponding to (1) with $E = I_n$), the mRPI sets are known to be not finitely determined [18] and it is characterized as a limit set. A number of strategies to approximate the mRPI sets have been proposed [4], [7], [8], [9]. The iterative method proposed in [4] yields a polytopic approximation of the mRPI set and will be used in the next section as the main constructive method.

III. Main Results on Set Analysis for Descriptor Systems

A. Minimal Robust Positively Invariant Set of Causal Descriptor Systems

For a regular and stable descriptor system (1), which is also causal, the set analysis can be made based on (4) into two sub-spaces as in the following theorem.

Theorem 1 (mRPI set of admissible descriptor systems): Consider the admissible descriptor system (1) and two non-singular matrices M and $N = \begin{bmatrix} N_1 & N_2 \end{bmatrix}$ satisfying (3), the mRPI set Ω^c of the admissible descriptor system (1) is given by the Minkowski addition of the image of two mRPI sets for two subcomponents as follows:

$$\Omega^c = N_1 \Phi_1 \oplus N_2 \Phi_2, \tag{12}$$

with

$$\Phi_1 = \bigoplus_{i=0}^{\infty} \tilde{A}_1^i \tilde{B}_{w1} \mathcal{W}, \tag{13a}$$

$$\Phi_2 = \left(-A_4^{-1} A_3 \Phi_1\right) \oplus \left(-A_4^{-1} B_{w2} \mathcal{W}\right). \tag{13b}$$

Proof: From (5), the descriptor system (1) is equivalent to the following two subsystems:

$$\tilde{x}_1(k+1) = \tilde{A}_1 \tilde{x}_1(k) + \tilde{B}_{w1} w(k), \tag{14a}$$

$$\tilde{x}_2(k+1) = \tilde{A}_2 \tilde{x}_1(k) + \tilde{B}_{w2} w(k) + \tilde{B}_{w3} w(k+1), \tag{14b}$$

with $\tilde{A}_1 = A_1 - A_2 A_4^{-1} A_3$, $\tilde{A}_2 = -A_4^{-1} A_3 (A_1 - A_2 A_4^{-1} A_3)$, $\tilde{B}_{w1} = B_{w1} - A_2 A_4^{-1} B_{w2}$, $\tilde{B}_{w2} = -A_4^{-1} A_3 (B_{w1} - A_2 A_4^{-1} B_{w2})$, $\tilde{B}_{w3} = -A_4^{-1} B_{w2}$.

At the same time, the stability implies the matrix $\tilde{A}_1$ is Schur. Then, the characterization of the mRPI set of $\tilde{x}_1$ can be obtained as in (13a) using the classical LTI notions [18]. On the other hand, the mRPI set Φ_2 of $\tilde{x}_2$ can be found by a linear projection image of the set Φ_1 in (13a), which bounds the signal $\tilde{x}_1(k)$. This leads to

$$\begin{aligned} \Phi_2 &= \tilde{A}_2 \Phi_1 \oplus \tilde{B}_{w1} \mathcal{W} \oplus \tilde{B}_{w2} \mathcal{W}, \\ &= \left(-A_4^{-1} A_3 \Phi_1\right) \oplus \left(-A_4^{-1} B_{w2} \mathcal{W}\right). \end{aligned}$$

Subsequently, by means of the Minkowski addition of the sets obtained via the images defined by the matrices N_1 and N_2, the RPI set of the admissible descriptor system (1) can be obtained. Hence, the proof is completed. ■

Lemma 6 (Ultimate bounds [9]): Given a standard LTI system (1) with $E = I_n$ and the Schur matrix $A \in \mathbb{R}^{n \times n}$, the Jordan decomposition form of $A = V \Lambda V^{-1}$ and the compact disturbance set $w(k) \in \mathcal{W} = \{w \in \mathbb{R}^q \mid |w| \leq \overline{w}\}$ for $\forall k \in \mathbb{N}_+$, the set

$$\Phi = \left\{ x \in \mathbb{R}^n \mid \left|V^{-1} x\right| \leq (I - |\Lambda|)^{-1} \left|V^{-1} B_w\right| \overline{w} + \theta \right\} \tag{15}$$

is RPI and attractive for all the trajectories of this system, where θ is an arbitrary small vector with positive elements.

Corollary 1 (Approximation of the mRPI set): Consider the admissible descriptor system (1), an RPI approximation of the mRPI set (12) for the admissible descriptor system (1) is given by

$$\Omega_0^c = N_1 \hat{\Phi}_{1,0} \oplus N_2 \hat{\Phi}_{2,0}, \tag{16}$$

with

$$\hat{\Phi}_{1,0} = \left\{ \tilde{x}_1 \in \mathbb{R}^{n_{\tilde{x}_1}} \mid \left|V_1^{-1} \tilde{x}_1\right| \leq (I - |\Lambda_1|)^{-1} \left|V_1^{-1} \tilde{B}_{w1}\right| \overline{w} + \theta_1 \right\}, \tag{17a}$$

$$\hat{\Phi}_{2,0} = \left(-A_4^{-1} A_3 \hat{\Phi}_{1,0}\right) \oplus \left(-A_4^{-1} B_{w2} \mathcal{W}\right), \tag{17b}$$

where $\tilde{A}_1 = V_1 \Lambda_1 V_1^{-1}$, θ_1 is an arbitrary small vector with all positive elements.

Furthermore, the sets $\Omega_i^c = N_1 \hat{\Phi}_{1,i} \oplus N_2 \hat{\Phi}_{2,i}$ with $i \in \mathbb{N}_+$ can be constructed based on the recurrence:

$$\hat{\Phi}_{1,i} = \tilde{A}_1 \Phi_{1,i-1} \oplus \tilde{B}_{w1} \mathcal{W}, \tag{18a}$$

$$\hat{\Phi}_{2,i} = \left(-A_4^{-1} A_3 \hat{\Phi}_{1,i}\right) \oplus \left(-A_4^{-1} B_{w2} \mathcal{W}\right), \tag{18b}$$

and lead to RPI approximation of the mRPI set (12) with

$$\Omega_0^c \supseteq \Omega_1^c \supseteq \cdots \supseteq \Omega_i^c \supseteq \cdots \supseteq \Omega^c, \tag{19}$$

for any $\epsilon > 0$, it exists an $i \in \mathbb{N}_+$ such that $d_H\left(\Omega_i^c, \Omega^c\right) < \epsilon$.

Proof: By *Lemma 6*, the RPI set Φ_1 of $\tilde{x}_1$ from the subsystem (14a) can be approximated by ultimate bounds as $\hat{\Phi}_{1,0}$ in (17a). Therefore, the mRPI set $\hat{\Phi}_{2,0}$ for $\tilde{x}_2$ can be obtained through a linear image as follows:

$$\hat{\Phi}_{2,0} = \left(-A_4^{-1} A_3 \hat{\Phi}_{1,0}\right) \oplus \left(-A_4^{-1} B_{w2} \mathcal{W}\right). \tag{20}$$

As introduced in [4] (see *Algorithm 1*), the iterative approximation of the mRPI set (18) can be obtained through

i-step propagation. Ultimately, given a constant ϵ, the constrained $d_H\left(\Omega_i^c, \Omega^c\right) < \epsilon$ can be achieved for a finite index i by exploiting the convergence of the sequence Ω_i^c to Ω^c. ■

In this subsection, we have proved that an admissible descriptor system can be transformed into an standard dynamic system with two subsystems of particular structure. The mRPI sets of these two subsystems can be computed separately. As a result, the mRPI set of the admissible descriptor system is a linear projection image of two mRPI sets from its transformed equivalent subsystems.

B. Minimal Robust Positively Invariant Set of Non-causal Descriptor Systems

We now extend the results of the previous subsection to the non-causal descriptor systems. If the descriptor system (1) is regular and stable but not causal, there still exists a unique solution at each sampling time instant and thus the set characterization of the dynamics can be investigated. For a regular descriptor system, the Kronecker canonical form in (11) is used in this subsection.

From *Lemmas 4* and *5*, a non-causal descriptor system can be transformed in (11) with a nilpotent matrix $\bar{N}$ satisfying $\bar{N} \neq 0$. As introduced in [15, Ch.8], it is always possible for a regular arbitrary matrix pair (E, A) to find a suitable transformation $(\hat{M}, \hat{N})$ yielding (11). Computationally efficient methods exist to obtain these transformations as e.g. [19].

By using *Lemma 4*, we set the new state variable with the partitioned form as

$$\check{x}(k) = \hat{N}x(k) = \begin{bmatrix} \check{x}_1(k) \\ \check{x}_2(k) \end{bmatrix}, \hat{M}B_w = \begin{bmatrix} \check{B}_{w1} \\ \check{B}_{w2} \end{bmatrix}, \tag{21}$$

with $\check{x}_1(k) \in \mathbb{R}^{\check{n}_1}$, $\check{x}_2(k) \in \mathbb{R}^{\check{n}_2}$ and $n = \check{n}_1 + \check{n}_2$.

Therefore, from (11) and (21), an equivalent system of a non-causal descriptor system (1) can be expressed into the following two subsystems:

$$\check{x}_1(k+1) = \bar{A}\check{x}_1(k) + \check{B}_{w1}w(k), \tag{22a}$$

$$\bar{N}\check{x}_2(k+1) = \check{x}_2(k) + \check{B}_{w2}w(k). \tag{22b}$$

As shown in (22), the non-causal descriptor system (1) is stable if and only if the matrix $\bar{A}$ is Schur. Hence, the mRPI set of a non-causal descriptor system can be formulated in the following theorem.

Theorem 2 (mRPI set of non-causal descriptor systems): Consider the non-causal descriptor system (1) in the equivalent form (11) and its solutions $x(k), \forall k \in \mathbb{Z}_{>0}$. The mRPI set of the non-causal descriptor system (1) can be obtained by the Minkowski addition of the image of two invariant sets with $\hat{N}_1$ and $\hat{N}_2$ as follows:

$$\Omega^n = \hat{N}_1\Theta_1 \oplus \hat{N}_2\Theta_2, \tag{23}$$

with

$$\Theta_1 = \bigoplus_{i=0}^{\infty} \bar{A}^i\check{B}_{w1}\mathcal{W}, \tag{24a}$$

$$\Theta_2 = \bigoplus_{i=0}^{s-1} \bar{N}^i\check{B}_{w2}\mathcal{W}, \tag{24b}$$

where $s \in \mathbb{N}_+$ is scalar satisfying $\bar{N}^{s-1} \neq 0$ and $\bar{N}^s = 0$. Besides, $\hat{N} = \begin{bmatrix} \hat{N}_1 & \hat{N}_2 \end{bmatrix}$.

Proof: The non-causal descriptor system can be decomposed in two subsystems, where (22a) is a standard dynamic difference equation. Hence, the mRPI set of $\check{x}_1$ can be constructed as in (24a). On the other hand, from (22b), the anti-causal state $\check{x}_2(k)$ can be propagated as follows:

$$\begin{aligned} \check{x}_2(k) &= \bar{N}\check{x}_2(k+1) - \check{B}_{w2}w(k), \\ \check{x}_2(k+1) &= \bar{N}\check{x}_2(k+2) - \check{B}_{w2}w(k+1), \\ &\vdots \end{aligned}$$

and after the L-step iterations, it becomes

$$\check{x}_2(k) = \bar{N}^L\check{x}_2(k+L) - \sum_{i=0}^{L-1} \bar{N}^i\check{B}_{w2}w(k+i). \tag{25}$$

Since $\bar{N}$ is a nilpotent matrix, there exists a finite index s such that the following conditions holds:

$$\begin{aligned} \bar{N}^{s-1} &\neq 0, \\ \bar{N}^s &= 0. \end{aligned}$$

Therefore, considering the structural relations $s \leq n_{\check{x}_2} \leq L$, we have that the solution $\check{x}_2 \in \mathbb{R}^{n_{\check{x}_2}}$ is well defined as a solution of (22b) and (25) can be written as

$$\check{x}_2(k) = \sum_{i=0}^{s-1} \bar{N}^i\check{B}_{w2}w(k+i). \tag{27}$$

With the compact set $\mathcal{W}$, the RNI set of $\check{x}_2$ can be explicitly computed as follows:

$$\begin{aligned} \Theta_2 &= \bigoplus_{i=0}^{s-1} \bar{N}^i\check{B}_{w2}\mathcal{W} \\ &= \check{B}_{w2}\mathcal{W} \oplus \bar{N}\check{B}_{w2}\mathcal{W} \oplus \cdots \oplus \bar{N}^{s-1}\check{B}_{w2}\mathcal{W}. \end{aligned} \tag{28}$$

Finally, the mRPI set Ω^n of the non-causal descriptor system (1) can be obtained by the linear image and Minkowski addition of Θ_1 and Θ_2. Hence, the proof is completed. ■

It is worth to be mentioned that *Theorem 2* is built under the assumption that the solution of the system (1) is defined for any $k \in \mathbb{Z}_{[>0]}$. The properties of the infinite-time trajectory lead to a positive invariance concept although the system is not causal. *Theorem 2* need to be reconsidered whenever the trajectories are defined only for a finite-time interval.

Corollary 2 (Approximation of the non-causal mRPI set): As defined in Theorem 2, the mRPI set (23) of the non-causal descriptor system (1) can be approximated by

$$\Omega_0^n = \hat{N}_1\hat{\Theta}_{1,0} \oplus \hat{N}_2\Theta_2, \tag{29}$$

with

$$\hat{\Theta}_{1,0} = \left\{ \check{x}_1 \in \mathbb{R}^{n_{\check{x}_1}} \mid \left|\bar{V}_1^{-1}\check{x}_1\right| \leq \left(I - \left|\bar{\Lambda}_1\right|\right)^{-1} \left|\bar{V}_1^{-1}\check{B}_{w1}\right| \overline{w} + \bar{\theta}_1 \right\}, \tag{30a}$$

$$\Theta_2 = \check{B}_{w2}\mathcal{W} \oplus \bar{N}\check{B}_{w2}\mathcal{W} \oplus \cdots \oplus \bar{N}^{s-1}\check{B}_{w2}\mathcal{W}, \tag{30b}$$

where $\bar{A} = \bar{V}_1\bar{\Lambda}_1\bar{V}_1^{-1}$, $\bar{\theta}$ is an arbitrary small vector with positive elements and $\hat{N} = \begin{bmatrix} \hat{N}_1 & \hat{N}_2 \end{bmatrix}$.

Furthermore, any set $\Omega_j^n = \hat{N}_1\hat{\Theta}_{1,j} \oplus \hat{N}_2\Theta_2$ with $j \in \mathbb{N}_+$ where

$$\hat{\Theta}_{1,j} = \tilde{A}_1\Phi_{1,j-1} \oplus \tilde{B}_{w1}\mathcal{W}, \tag{31}$$

is also an RPI approximation of the mRPI set (23) with

$$\Omega_0^n \supseteq \Omega_1^n \supseteq \cdots \supseteq \Omega_j^n \supseteq \cdots \supseteq \Omega^n. \tag{32}$$

Moreover, for any $\epsilon > 0$, it exists an $i \in \mathbb{N}_+$ such that $d_H\left(\Omega_j^n, \Omega^n\right) < \epsilon$.

Proof: By *Lemma 6*, the mRPI set Θ_1 of $\check{x}_1$ can be approximated by $\Theta_1 \subseteq \hat{\Theta}_1$ in (30a). As in *Corollary 1*, by the iterative algorithm, Ω_j^n can be obtained. ■

C. Robust Negatively Invariant Set for Finite-time Trajectories of Non-causal Descriptor Systems

We consider the non-causal descriptor system (1) focused on trajectories defined only on a finite-time window $x(k)$, $k \in \mathbb{Z}_{[0,L]}$ with $L > 0$. The dynamics obey to the equivalent subsystems in (22) but the set-theoretic characterization need to be relaxed in order to consider the finite number of dynamical constraints as well as the non-causal particularities.

The difficulties are related to the mixture of causal and anti-causal dynamics in (22a) and (22b). For (22a), the positive invariance will be the appropriate concept while for (22b), the negative invariance offers the suitable framework.

Theorem 3 (L-step RNI set): Consider the anti-causal subsystem (22b). The set Υ^L is L-step negatively invariant if

$$\begin{aligned}\Upsilon^L &\supseteq \bar{N}\Upsilon^L \oplus \left\{-\check{B}_{w2}\mathcal{W}\right\} \\ &\supseteq \cdots \supseteq \bar{N}^L\Upsilon^L \bigoplus_{i=0}^{L-1} \left\{-\bar{N}^i\check{B}_{w2}\mathcal{W}\right\}.\end{aligned} \tag{33}$$

Proof: For $\check{x}_2(k+L) \in \Upsilon^L$, $k \in \mathbb{Z}_{[-L,0]}$, from (25), (33) can be obtained by the backward propagations. ■

Remark 1: Given two sets Υ^{L_1} and Υ^{L_2} which are L_1- and L_2-step RNI with $L_1 > L_2$, we have $\Upsilon^{L_1} \supseteq \Upsilon^{L_2}$.

Remark 2: The set Θ_2 in (24b) is a L-step RNI for $\forall L \geq 0$.

Remark 3: Let the set Θ_2 constructed as in (24b). A L-step RNI set Υ^L for (22b) can be constructed iteratively starting from $\Upsilon^0 = \Theta_2$ and using for $i \in \mathbb{Z}_{[1,L]}$ the recursive construction:

$$\Upsilon^i = \left\{\check{x}_2 \in \mathcal{X}_2 \mid \exists w \in \mathcal{W}, \bar{N}x - \check{B}_{w2}w \in \Upsilon^{i-1}\right\}, \tag{34}$$

and $\mathcal{X}_2 \subseteq \mathbb{R}^{\check{n}_2}$ is the physical constraint of the state $\check{x}_2$.

Theorem 4 (L-step RI set): Consider the non-causal descriptor system (1) in the equivalent form (11). The set

$$\Omega^L = \hat{N}_1\Theta_1 \oplus \hat{N}_2\Upsilon^L \tag{35}$$

guarantees that $x(k) \in \Omega^L$ for $\forall k \in \mathbb{Z}_{[0,L]}$ if $\check{x}_1(0) \in \Theta_1$ and $\check{x}_2(L) \in \Upsilon^L$.

Proof: $x(k)$ has two equivalent subcomponents $\check{x}_1(k)$ and $\check{x}_2(k)$. From (24a), Θ_1 is RPI for $\check{x}_1(k)$. If $\check{x}_1(0) \in \Theta_1$, then it follows $\check{x}_1(k) \in \Theta_1$ for $\forall k \in \mathbb{Z}_{[0,L]}$.

At the same time, Υ^L is the L-step RNI set for $\check{x}_2(k)$ as discussed in *Theorem 3*. If $\check{x}_2(L) \in \Upsilon^L$, then it follows $\check{x}_2(k) \in \Upsilon^L$ for $\forall k \in \mathbb{Z}_{[0,L]}$.

Therefore, by the Minkowski addition of the linear projection image, (35) can be obtained. ■

Proposition 1: Consider the non-causal descriptor system (1) in the equivalent form (11) and define a finite-time trajectories $x(k)$, $k \in \mathbb{Z}_{[0,L]}$ with $L > 0$. If $x(0) \in \Omega^0$ for $L > s$ with $\bar{N}^s = 0$ and $\bar{N}^{s-1} \neq 0$, then $x(k) \in \Omega^0$ for $k \in \mathbb{Z}_{[0,L-s]}$ and $x(k) \in \Omega^{k-(L-s)}$ for $k \in \mathbb{Z}_{[L-s,L]}$, where $\Omega^i = \hat{N}_1\Theta_1 \oplus \hat{N}_2\Upsilon^i$ with Υ^i in (34).

Proof: For $k \in \mathbb{Z}_{[0,L-s]}$, from (25), $x(k)$ is contained in the RI set $\Omega^0 = \Theta_2$ as defined in (35). On the other hand, for $k \in \mathbb{Z}_{[L-s,L]}$, the anti-causal component is contained in Υ^i, which can be propagated by using (34) leading to the confinement of the finite time trajectories for $L-s < k < L$. ■

IV. A Set-based Active Mode Detection for Descriptor Systems

As an application of the obtained mRPI sets, the identification of different modes in a descriptor system (1) can be performed. In this subsection, we propose two solutions to the problems of mode detection for a causal and a switched non-causal descriptor systems based on the characterization of RPI sets. In safety critical systems, a fault can be interpreted as a change in the mode of operation. Therefore, active mode detection can be seen an important step towards fault-tolerant control.

A. Two-mode Descriptor Systems

Given the causal descriptor system:

$$E_{\sigma(k)}x(k+1) = A_{\sigma(k)}x(k) + D_{\sigma(k)}w(k) + B_{\sigma(k)}u(k), \tag{36}$$

where $\sigma(k)$ is a constant function $\sigma : \mathbb{N} \to \{1, 2\}$. The signal $u(k) \in \mathbb{R}^m$ denotes an input vector that can be used for set separation at time instant k and $B_{\sigma(k)}$ is the associate input matrix.

Practically, the descriptor system (36) has two modes of functioning:

(i) Mode 1:

$$E_1x(k+1) = A_1x(k) + D_1w(k) + B_1u(k), \tag{37}$$

(ii) Mode 2:

$$E_2x(k+1) = A_2x(k) + D_2w(k) + B_2u(k). \tag{38}$$

Assumption 2: The matrix pairs (E_1, A_1) and (E_2, A_2) are admissible.

Active Mode Detection Problem for Causal Descriptor Systems: Considering that $\mathcal{W}$ is a compact polyhedral set containing the origin in its interior, the mRPI sets with respect to (37) and (38) can be approximated by applying *Corollary 1* assuming $u(k) = 0$ for $\forall k \in \mathbb{N}_+$. Therefore, active mode detection using $u(k) \neq 0$ is summarized in the following steps.

1) Compute the mRPI sets Ω_1 and Ω_2 for (E_1, A_1, D_1) and (E_2, A_2, D_2) respectively for $u = 0$.
2) Consider the nominal systems

$$E_1\bar{x}^{m_1}(k+1) = A_1\bar{x}^{m_1}(k) + B_1u(k), \tag{39a}$$
$$E_2\bar{x}^{m_2}(k+1) = A_2\bar{x}^{m_2}(k) + B_2u(k), \tag{39b}$$

and the states $x^{m_1}(k)$ and $x^{m_2}(k)$ of (37) and (38) satisfy the following conditions for $\forall k \in \mathbb{N}_+$:

$$x^{m_1}(k) \in \bar{x}^{m_1}(k) \oplus \Omega_1, \text{ if } x^{m_1}(0) \in \Omega_1, \quad (40a)$$
$$x^{m_2}(k) \in \bar{x}^{m_2}(k) \oplus \Omega_2, \text{ if } x^{m_2}(0) \in \Omega_2. \quad (40b)$$

3) Find a suitable $u(k)$ for $\forall k \in \mathbb{N}_+$ such that $\{\bar{x}^{m_1}(k) \oplus \Omega_1\} \cap \{\bar{x}^{m_2}(k) \oplus \Omega_2\} = \varnothing$.
4) **On-line monitoring:** Consider the sequence of states $x(k)$ from (36). If for some time index k it holds $x(k) \in \bar{x}^{m_1}(k) \oplus \Omega_1$, then the trajectory is governed by the Mode 1. Alternatively, if $x(k) \in \bar{x}^{m_2}(k) \oplus \Omega_2$, then the system is governed by Mode 2.

B. Switched Non-causal Descriptor Systems

As discussed in Section III-B, the mRPI set of non-causal descriptor system can be computed by a linear projection image of a RPI set of causal states and a finite-step RNI set of anti-causal states. With respect to the non-causal descriptor system, it is important to notice that the RNI set of non-causal states will grow for a finite number of steps before the mode switch. Hence, this property can be used for identifying the mode change of a switch non-causal descriptor system.

The switched non-causal descriptor system (36) is redefined with $\sigma(k) = 1$ for $\forall k \in \mathbb{Z}_{\leq k_0}$ and $\sigma(k) = 2$ for $\forall k \in \mathbb{Z}_{>k_0}$ as the following two modes:

(i) Mode 1:

$$\hat{E}_1 x(k+1) = \hat{A}_1 x(k) + \hat{D}_1 w(k) + \hat{B}_1 u(k), \quad (41)$$

with $\hat{E}_1 = \begin{bmatrix} I_{n_1} & 0 \\ 0 & \bar{N}_1 \end{bmatrix}$, $\hat{A}_1 = \begin{bmatrix} \bar{A}_1 & 0 \\ 0 & I_{n_2} \end{bmatrix}$, $\forall k \in \mathbb{Z}_{\leq k_0}$,

(ii) Mode 2:

$$\hat{E}_2 x(k+1) = \hat{A}_2 x(k) + \hat{D}_2 w(k) + \hat{B}_2 u(k), \quad (42)$$

with $\hat{E}_2 = \begin{bmatrix} \bar{N}_2 & 0 \\ 0 & I_{n_1} \end{bmatrix}$, $\hat{A}_2 = \begin{bmatrix} I_{n_2} & 0 \\ 0 & \bar{A}_2 \end{bmatrix}$, $\forall k \in \mathbb{Z}_{>k_0}$,

where $\bar{A}_1 \in \mathbb{R}^{n_1 \times n_1}$ and $\bar{A}_2 \in \mathbb{R}^{n_2 \times n_2}$ are assumed to be Schur matrices, $\bar{N}_1$ and $\bar{N}_2$ are nilpotent matrices and not identically zero and the finite step of $\bar{N}_1$ and $\bar{N}_2$ are denoted as s_1 and s_2 such that $\bar{N}_1^{s_1} = 0$ with $\bar{N}_1^{s_1-1} \neq 0$ and $\bar{N}_2^{s_2} = 0$ with $\bar{N}_2^{s_2-1} \neq 0$.

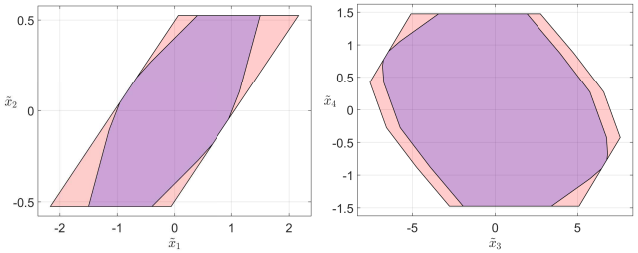
The non-causal descriptor system switches from Mode 1 to Mode 2 at some moment k_0. The objective of the following procedure is to detect in finite time in which mode the switched non-causal system is operating for a finite number of steps before the mode switch.

Active Mode Detection Problem for Switched Non-Causal Descriptor Systems: The problem is summarized in the following off-line and on-line procedures:

Off-line procedure:

1) Compute the mRI sets Ω_1^n and Ω_2^n for $(\hat{E}_1, \hat{A}_1, \hat{D}_1)$ and $(\hat{E}_2, \hat{A}_2, \hat{D}_2)$ by using *Corollary 2*.
2) Compute the L-step RI set Ω_i^L for $(\hat{E}_1, \hat{A}_1, \hat{D}_1)$ with the corresponding index s of the nilpotent matrix.
3) Find the test signal $\exists u(k)$ such that the nominal states $\bar{x}^{m_1}(k)$ and $\bar{x}^{m_2}(k)$ satisfy

$$\{\bar{x}^{m_1}(k) \oplus \Omega_1^n\} \cap \{\bar{x}^{m_2}(k) \oplus \Omega_2^n\} = \varnothing, \quad (43a)$$
$$\left\{\bar{x}^{m_1}(k) \oplus \Omega_i^L\right\} \cap \{\bar{x}^{m_2}(k) \oplus \Omega_2^n\} = \varnothing, \quad (43b)$$



(a) Subsystem 1 (b) Subsystem 2

Fig. 1. The RPI and mRPI sets of equivalent causal descriptor subsystems: RPI sets (ultimate bounds) are plotted in purple region; mRPI sets are plotted in red region.

4) Consider the initial state $x(0) \in \{\bar{x}^{m_1}(0) \oplus \Omega_1^n\}$.

On-line procedure:

Consider a sequence of the states $x(k)$ of the switched non-causal descriptor system and check the following conditions:

1) If $x(k) \notin \{\bar{x}^{m_1}(k) \oplus \Omega_1^n\}$ and $x(k) \in \{\bar{x}^{m_2}(k) \oplus \Omega_2^n\}$, then the non-causal system is switched from Mode 1 to Mode 2.
2) If $x(k) \notin \{\bar{x}^{m_1}(k) \oplus \Omega_1^n\}$ and $x(k) \in \left\{\bar{x}^{m_1}(k) \oplus \Omega_i^L\right\}$, then a switch from Mode 1 to Mode 2 will occur at most $(k+s-i)$.
3) If $x(k-1) \in \left\{\bar{x}^{m_1}(k) \oplus \Omega_i^L\right\}$ and $x(k) \notin \left\{\bar{x}^{m_1}(k) \oplus \Omega_{i+1}^L\right\}$, then this switch from Mode 1 to Mode 2 occurs at time instant k.

V. Illustrative Examples

A. A Causal Descriptor System

Consider a causal descriptor system (1) with

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, A = \begin{bmatrix} 0.5 & 1 & 0.2 & 0 \\ 0 & 0.6 & 0 & 0 \\ 1 & 0 & 0.2 & 0.8 \\ 0 & 1 & 0 & 0.6 \end{bmatrix}, D = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.7 \\ 0.2 & 0.2 \\ 0.9 & 0.6 \end{bmatrix}.$$

Let $w(k) \in \mathcal{W}_1 = \left\{w \in \mathbb{R}^{q_1} \mid |w(k)| \leq \begin{bmatrix} 0.2 & 0.3 \end{bmatrix}^T\right\}$. By applying *Corollary 1*, approximations of the mRPI sets for two subsystems are shown in Fig. 1.

B. Non-causal Descriptor System: Dynamic Leontief Model

Given an uncertain dynamic Leontief model with describes the time pattern of production sectors by [15],[16]:

$$x(k) = Fx(k) + G\left(x(k+1) - x(k)\right) + Bu(k) + Dw(k), \quad (44)$$

with

$$F = \begin{bmatrix} 1.25 & 0.5 & 1.5 \\ 0.75 & 0.5 & 1.1 \\ 0.25 & 0 & 1.5 \end{bmatrix}, G = \begin{bmatrix} 1 & 0.5 & 0.75 \\ 0.25 & 0 & 0.5 \\ 0 & 0 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} -1 & -1 & -1 \end{bmatrix}^T.$$

As in [16], a state feedback law K with $u(k) = Kx(k)$ is used as follows:

$$K = \begin{bmatrix} 0.3828 & 0.1427 & 0.4087 \end{bmatrix}.$$

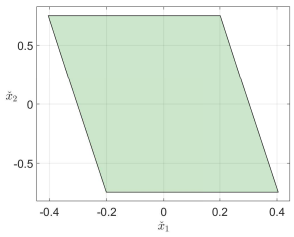


Fig. 2. The RNI set of $\tilde{x}_2$ in the non-causal subsystem

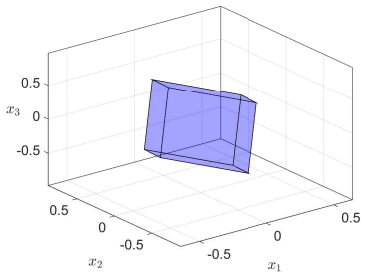


Fig. 3. The mRPI set of non-causal descriptor system

Then, the resulting closed-loop descriptor system can be rewritten as follows:

$$Ex(k+1) = Ax(k) + Dw(k),$$

with

$$E = G, A = \begin{bmatrix} 1.1328 & 0.1427 & -0.3413 \\ -0.1172 & 0.6427 & -0.1913 \\ 0.1328 & 0.1427 & -0.0913 \end{bmatrix}, D = BK.$$

Let

$$w(k) \in \mathcal{W}_2 = \left\{ w \in \mathbb{R}^{q_2} \mid |w| \leq \begin{bmatrix} 0.2 & 0.3 & 0.1 \end{bmatrix}^T \right\}.$$

By using the transformation $\left(\hat{M}, \hat{N}\right)$:

$$\hat{M} = \begin{bmatrix} -0.5524 & -0.7530 & 3.8890 \\ 0.5393 & -1.1540 & -0.0193 \\ 0 & 0 & 4.6456 \end{bmatrix},$$
$$\hat{N} = \begin{bmatrix} -0.2576 & 0.6414 & 0.5385 \\ -0.3391 & -0.8020 & 0.5596 \\ -0.9048 & -0.3206 & -0.6998 \end{bmatrix},$$

this dynamic Leontief model can be transformed into the form in (11). Hence, the mRPI set of $\tilde{x}_1$ can be computed:

$$\tilde{x}_1 \in \hat{\Theta}_1 = \{x \in \mathbb{R}^{n_{\tilde{x}_1}} \mid |x| \leq 0.4261\}.$$

And the RNI set Θ_2 of $\tilde{x}_2$ can be computed through a backward propagation with $\bar{N}^2 = 0$ as shown in Fig. 2.

As shown in Fig. 3, the mRPI set Ω^n of this dynamic Leontief model can be approximated in the following set:

$$\Omega^n = \{x \in \mathbb{R}^n \mid Hx \leq b\},$$

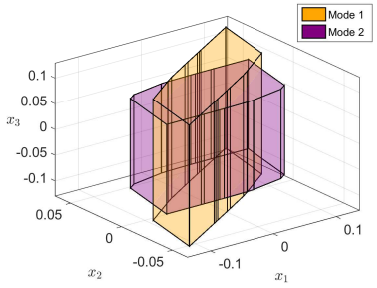


Fig. 4. The mRPI sets of two-mode descriptor system

with

$$H = \begin{bmatrix} -0.7856 & 0.5476 & 0.0184 \\ 0.4949 & 0.5318 & -0.3402 \\ -0.6190 & -0.2308 & -0.6609 \\ 0.6190 & 0.2308 & 0.6609 \\ 0.7856 & -0.5476 & -0.0184 \\ -0.4949 & -0.5318 & 0.3402 \end{bmatrix}, b = \begin{bmatrix} 0.2873 \\ 0.5971 \\ 0.3561 \\ 0.3561 \\ 0.2873 \\ 0.5971 \end{bmatrix}.$$

C. Active Mode Detection: A Two-mode Descriptor System

Given a two-mode causal descriptor system (36) with

$$E_1 = E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, D_1 = D_2 = \begin{bmatrix} 0.5 & 0.6 \\ 1.2 & 0.2 \\ 0 & 0.8 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} 0.42 & 0.95 & 0 \\ 0.1 & 0.35 & 0 \\ 0.5 & 0.8 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} 0.65 & 1.2 & 0 \\ 0 & 0.01 & 0 \\ 0.5 & 0.8 & 1 \end{bmatrix}.$$

From these two matrix pairs (E_1, A_1) and (E_2, A_2), it can be verified that both are admissible. Let

$$w(k) \in \mathcal{W}_3 = \left\{ w \in \mathbb{R}^{q_3} \mid |w| \leq \begin{bmatrix} 0.02 & 0.01 \end{bmatrix}^T \right\}.$$

Therefore, the mRPI sets of two modes can be obtained by applying *Corollary 1*. The computation results of two mRPI sets are shown in Fig. 4. An input signal and its distribution matrices are chosen as follows:

$$B_1 = B_2 = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^T,$$

and $u(k) = 0.5$ for $\forall k \in \mathbb{N}_+$. The set separation result is shown in Fig. 5. This figure shows two state sets for two-mode descriptor system are separated by the selection of the input signal. Besides, the system states are propagating in the blue dashed line. From Fig. 5, this state is staying in the orange set. Hence, it implies that the system is in Mode 1.

D. Active Mode Detection: A Switched Non-causal Descriptor System

Given a switched non-causal descriptor system including two modes (41) and (42) can be transformed to

$$\hat{E}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0.75 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \hat{A}_1 = \begin{bmatrix} 0.42 & 0.95 & 0 & 0 \\ 0.1 & 0.35 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \hat{D}_1 = \begin{bmatrix} 0.2 & 0.6 \\ 0.9 & 0.5 \\ 0.06 & 0.96 \\ 1.5 & 0.15 \end{bmatrix},$$

$$\hat{E}_2 = \begin{bmatrix} 0 & 0.48 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \hat{A}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0.65 & 0.8 \\ 0 & 0 & 0 & 0.01 \end{bmatrix}, \hat{D}_2 = \begin{bmatrix} 0.06 & 0.96 \\ 1.5 & 0.15 \\ 0.2 & 0.6 \\ 0.9 & 0.5 \end{bmatrix},$$

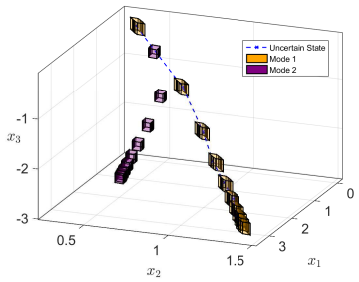
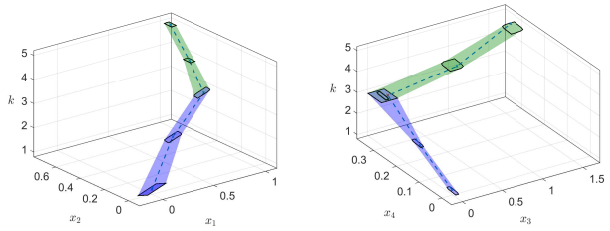


Fig. 5. Mode detection result of two-mode descriptor system



(a) (x_1, x_2) space (b) (x_3, x_4) space

Fig. 6. Mode detection results of switched non-causal descriptor system

and let

$$x(k) \in \mathcal{X} = \left\{ x \in \mathbb{R}^n \mid |x| \leq \begin{bmatrix} 0.1 & 0.1 & 0.1 & 0.1 \end{bmatrix}^T \right\},$$
$$w(k) \in \mathcal{W}_4 = \left\{ w \in \mathbb{R}^{q_4} \mid |w| \leq \begin{bmatrix} 0.02 & 0.01 \end{bmatrix}^T \right\}.$$

Matrices $\hat{B}_1$ and $\hat{B}_2$ are the following:

$$\hat{B}_1 = \begin{bmatrix} 2 & 1 & 0.1 & 0.65 \end{bmatrix}^T, \hat{B}_2 = \begin{bmatrix} 0.1 & 0.65 & 2 & 1 \end{bmatrix}^T,$$

and the input signal is set as $u(k) = 0.25$ for $\forall k \in \mathbb{N}_+$. As shown in Fig. 6, the system states of the non-causal system are switched between two modes. From $k = 1$ to $k = 2$, the state $x(k)$ is located in the set $\{\bar{x}^{m_1}(k) \oplus \Omega_1^n\}$. Hence, the switched non-causal system (36) is Mode 1. At $k = 3$, $x(k) \notin \{\bar{x}^{m_1}(k) \oplus \Omega_1^n\}$ and $x(k) \in \{\bar{x}^{m_1}(k) \oplus \Omega_i^L\}$ implies there is a switch forthcoming. And at $k = 4$, the conditions $x(k-1) \in \{\bar{x}^{m_1}(k) \oplus \Omega_i^L\}$ and $x(k) \notin \{\bar{x}^{m_1}(k) \oplus \Omega_{i+1}^L\}$ are satisfied. Hence, the switch occurs at $k = 4$.

VI. Conclusion

The RPI sets of causal and non-causal descriptor systems have been computed in this paper. The coordinate transformation for descriptor system are used to find a suitable equivalent system in order to compute the RPI sets for each subsystem. In particular, it is shown that the non-causal descriptor system can be transformed into two subsystems including causal and anti-causal states. In order to compute the RPI set of non-causal descriptor systems, we have introduced the RNI set with a finite horizon. The RPI and RNI sets can be obtained through forward and backward propagations. Finally, the mode detection based on the RPI set is discussed. A descriptor system with different modes can be detected and identified by implementing the RPI set separation. As a future work, the control design for the set separation in the framework of descriptor systems will be explored.

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