

SPECTRA OF LAPLACIANS ON FORMS ON AN INFINITE GRAPH

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SPECTRA OF LAPLACIANS ON FORMS ON AN INFINITE GRAPH

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ABSTRACT. In the context of infinite weighted graphs, we consider the discrete Laplacians on 0-forms and 1-forms. Using Weyl's criterion, we prove the relation between the nonzero spectrum of Δ_0 and that of Δ_1 . Moreover, we give an extension of the work of John Lott to characterize the 0-spectrum of these two Laplacians.

CONTENTS

1. Introduction	I
2. Preliminaries	2
2.1. Definition and notation	2
2.2. Weighted graphs	2
2.3. Functional spaces	3
2.4. Operators and properties	3
2.5. Weyl's criterion	6
3. The relation between the spectrum of Δ_0 and Δ_1	7
3.1. The nonzero spectrum of Δ_0 and Δ_1	7
3.2. The 0-spectrum of Δ_0 and Δ_1	9
4. Examples:	14
References	17

1. INTRODUCTION

In recent years, much attention has been paid to the analysis of discrete Laplacians and elliptic differential operators acting on graphs [13], [5], [6] and [19]. More precisely, authors have intensively studied the spectrum of the discrete Laplacian on an infinite graph in various areas, for example, harmonic analysis on graphs (see [16], [20]), probability theory especially Markov chains (see [8], [12]), potential theory such as electric networks (see [17], [12]), and so on. In this paper, we define two Laplacians, mentioned in [1] and [3], one as an operator acting on functions on vertices denoted by Δ_0 and the other one acting on functions on edges denoted by Δ_1 . So, it is a natural question to characterize the relation between their spectrum in terms of a certain geometric property of the graph and properties of the operators. Especially we show that the nonzero spectrum of Δ_0 and Δ_1 are the same, by using Weyl's criterion. Moreover, with suitable weight conditions we prove that 0 is in the spectrum of Δ_1 , if the operator Δ_0 is invertible. This result is inspired from J. Lott's work [11] (Proposition 9, p. 12) which proves in the case of a simple graph that 0 is either in the spectrum of the Laplacian on 0-forms, or in the spectrum of the Laplacian on 1-forms. In fact, the major interest of J. Lott concerns the zero-in-the-spectrum question for the Laplace-de Rham operator acting on L^2 differential forms of any degree on a complete connected

1

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oriented Riemannian manifold. The article [11] is rather expository and gives some positive answers, in relation with topology, for small dimensions. We finish the paper with examples of constructions of Δ_1 -harmonic nonzero square-integrable functions.

2. Preliminaries

2.1. Definition and notation.

- A graph G is a couple $(\mathcal{V}, \mathcal{E})$ where \mathcal{V} is a set at most countable whose elements are called vertices and \mathcal{E} is a set of oriented edges, considered as a subset of $\mathcal{V} \times \mathcal{V}$.
- If the graph G has a finite set of vertices, it is called a finite graph. Otherwise, G is called an infinite graph.
- We assume that \mathcal{E} has no self-loops and is symmetric:

$$v \in \mathcal{V} \Rightarrow (v, v) \notin \mathcal{E}, \ (v_1, v_2) \in \mathcal{E} \Rightarrow (v_2, v_1) \in \mathcal{E}.$$

• Choosing an orientation of G consists of defining a partition of $\mathcal{E}: \mathcal{E}^+ \sqcup \mathcal{E}^- = \mathcal{E}$

$$(v_1, v_2) \in \mathcal{E}^+ \Leftrightarrow (v_2, v_1) \in \mathcal{E}^-.$$

• For $e = (v_1, v_2)$, we denote

$$e^- = v_1, \ e^+ = v_2 \text{ and } - e = (v_2, v_1).$$

- We write $v_1 \sim v_2$ for $e = (v_1, v_2) \in \mathcal{E}$.
- The graph G is connected if any two vertices x, y in V can be joined by a path of edges γ_{xy}, that means γ_{xy} = {e_k}_{k=1,...,n} such that

 $e_1^- = x, \ e_n^+ = y$ and, if $n \ge 2$, $\forall j \ ; 1 \le j \le (n-1) \Rightarrow e_j^+ = e_{j+1}^-$.

• The degree (or valence) of a vertex x is the number of edges emanating from x. We denote

$$\deg(x) := \sharp \{ e \in \mathcal{E}; \ e^- = x \}.$$

• If $deg(x) < \infty$, $\forall x \in \mathcal{V}$, we say that G is a locally finite graph.

2.2. Weighted graphs.

Definition 2.1. A weighted graph (G, c) is given by a graph $G = (\mathcal{V}, \mathcal{E})$ and weights on the edges $c : \mathcal{E} \to [0, \infty[$ such that

- $c(x, x) = 0, \forall x \in \mathcal{V}.$
- $c(x,y) > 0, \ \forall (x,y) \in \mathcal{E}.$
- $c(x,y) = c(y,x), \ \forall (x,y) \in \mathcal{E}.$

If $\sum_{y \sim x} c(x, y) < \infty$ for each $x \in \mathcal{V}$, we can define a weight on \mathcal{V} by

$$\tilde{c}(x) = \sum_{y \sim x} c(x, y), \ x \in \mathcal{V}.$$

Remark 2.1. If the graph G is locally finite, the weight \tilde{c} on any vertex is well defined.

Examples: - An infinite electrical network is a weighted graph (G, c) where the weight c on the edges are called conductances and their reciprocals are called resistances. This is the convention used in the study of random walks on weighted graphs, see [12] and [16]. Then, $\tilde{c}(x) = \sum_{y \in \mathcal{V}} c(x, y)$ is the weight

associated to the vertex x.

-A graph G is called a simple graph if the edge weights are equal to 1. In this case,

$$\tilde{c}(x) = \deg(x), \ \forall x \in \mathcal{V}.$$

All the graphs we shall consider in the sequel will be connected, locally finite and weights c given in Definition 2.1.

2.3. Functional spaces. We denote the set of real functions on \mathcal{V} by:

$$\mathcal{C}(\mathcal{V}) = \{ f : \mathcal{V} \to \mathbb{R} \}$$

and the set of functions of finite support by $C_0(\mathcal{V})$.

Moreover, we denote the set of real skew-symmetric functions on \mathcal{E} by:

$$\mathcal{C}^{a}(\mathcal{E}) = \{ \varphi : \mathcal{E} \to \mathbb{R} ; \varphi(-e) = -\varphi(e) \}$$

and the set of functions of finite support by $C_0^a(\mathcal{E})$.

We define on the weighted graph (G, c) the following function spaces endowed with the scalar products.

a):

$$l^{2}(\mathcal{V}) := \left\{ f \in \mathcal{C}(\mathcal{V}); \sum_{x \in \mathcal{V}} \tilde{c}(x) f^{2}(x) < \infty \right\},$$

with the inner product

$$\langle f,g\rangle_{\mathcal{V}} = \sum_{x\in\mathcal{V}} \tilde{c}(x)f(x)g(x)$$

and the norm

$$\|f\|_{\mathcal{V}} = \sqrt{\langle f, f \rangle_{\mathcal{V}}}.$$

b):

$$l^{2}(\mathcal{E}) := \left\{ \varphi \in \mathcal{C}^{a}(\mathcal{E}); \ \frac{1}{2} \sum_{e \in \mathcal{E}} c(e) \varphi^{2}(e) < \infty \right\},$$

with the inner product

$$\langle \varphi, \psi \rangle_{\mathcal{E}} = \frac{1}{2} \sum_{e \in \mathcal{E}} c(e) \varphi(e) \psi(e)$$

and the norm

$$\|\varphi\|_{\mathcal{E}} = \sqrt{\langle \varphi, \varphi \rangle_{\mathcal{E}}}.$$

Then, $l^2(\mathcal{V})$ and $l^2(\mathcal{E})$ are separable Hilbert spaces (since \mathcal{V} is countable).

2.4. Operators and properties. The difference operator

$$\mathrm{d}: l^2(\mathcal{V}) \longrightarrow l^2(\mathcal{E}),$$

is given by

$$d(f)(e) = f(e^+) - f(e^-)$$

for all $f \in l^2(\mathcal{V})$ and for all $\varphi \in l^2(\mathcal{E})$.

As consequence, we have the following formula characterizing δ :

Lemma 2.1. The coboundary operator δ is characterized by the formula

$$\delta\varphi(x) = \frac{1}{\tilde{c}(x)} \sum_{e,e^+=x} c(e)\varphi(e),$$

for all $\varphi \in l^2(\mathcal{E})$.

Proof

For $f \in l^2(\mathcal{V})$ and $\varphi \in l^2(\mathcal{E})$, using (2.1), we get

$$\begin{split} \langle \mathrm{d}f, \varphi \rangle_{\mathcal{E}} &= \frac{1}{2} \sum_{e \in \mathcal{E}} c(e) \mathrm{d}f(e) \varphi(e) \\ &= \frac{1}{2} \sum_{e \in \mathcal{E}} c(e) \left(f(e^+) - f(e^-) \right) \varphi(e) \\ &= \frac{1}{2} \sum_{x \in \mathcal{V}} f(x) \left(\sum_{e, e^+ = x} c(e) \varphi(e) - \sum_{e, e^- = x} c(e) \varphi(e) \right). \end{split}$$

But c(-e) = c(e) and φ is skew-symmetric, so we have $\sum_{e,e^+=x} c(e)\varphi(e) = -\sum_{e,e^-=x} c(e)\varphi(e)$.

Then,

$$\begin{split} \langle \mathrm{d} f, \varphi \rangle_{\mathcal{E}} &= \sum_{x \in \mathcal{V}} \tilde{c}(x) f(x) \left(\frac{1}{\tilde{c}(x)} \sum_{e, e^+ = x} c(e) \varphi(e) \right) \\ &= \langle f, \delta \varphi \rangle_{\mathcal{V}} \end{split}$$

and the formula for $\delta \varphi$ follows.

Definition 2.2. The Laplacian on 0-forms Δ_0 defined by δd on $l^2(\mathcal{V})$ is given by

$$\Delta_0 f(x) = \frac{1}{\tilde{c}(x)} \sum_{y \sim x} c(x, y) \left(f(x) - f(y) \right).$$

In fact, we have

$$\begin{split} \Delta_0 f(x) &= \delta(\mathrm{d}f)(x) \\ &= \frac{1}{\tilde{c}(x)} \sum_{e,e^+=x} c(e) \mathrm{d}f(e) \\ &= \frac{1}{\tilde{c}(x)} \sum_{e,e^+=x} c(e) \left(f(e^+) - f(e^-)\right) \\ &= \frac{1}{\tilde{c}(x)} \sum_{y \sim x} c(x,y) \left(f(x) - f(y)\right). \end{split}$$

Definition 2.3. The Laplacian on 1-forms Δ_1 defined by $d\delta$ on $l^2(\mathcal{E})$ is given by

$$\Delta_1 \varphi(e) = \frac{1}{\tilde{c}(e^+)} \sum_{e_1, e_1^+ = e^+} c(e_1)\varphi(e_1) - \frac{1}{\tilde{c}(e^-)} \sum_{e_2, e_2^+ = e^-} c(e_2)\varphi(e_2).$$

In fact, we have

$$\begin{aligned} \Delta_1 \varphi(e) &= \mathrm{d}(\delta \varphi)(e) \\ &= \delta \varphi(e^+) - \delta \varphi(e^-) \\ &= \frac{1}{\tilde{c}(e^+)} \sum_{e_1, e_1^+ = e^+} c(e_1) \varphi(e_1) - \frac{1}{\tilde{c}(e^-)} \sum_{e_2, e_2^+ = e^-} c(e_2) \varphi(e_2). \end{aligned}$$

Proposition 2.1. The operator Δ_0 is bounded and self-adjoint.

Proof

(2.2)

(2.3)

For $f, g \in l^2(\mathcal{V})$, we have

$$\begin{aligned} |\langle \Delta_0 f, g \rangle_{\mathcal{V}}| &= \left| \sum_x \tilde{c}(x) \frac{1}{\tilde{c}(x)} \sum_{y \sim x} c(x, y) \left(f(x) - f(y) \right) g(x) \right| \\ &\leq \left| \sum_x \sum_{y \sim x} c(x, y) \left| (f(x) - f(y)) \right| \left| g(x) \right| \\ &\leq \left| \sum_x \sum_{y \sim x} c(x, y) \left| f(x) \right| \left| g(x) \right| + \sum_x \sum_{y \sim x} c(x, y) \left| f(y) \right| \left| g(x) \right| \\ &= \left| \sum_x \tilde{c}(x) \left| f(x) \right| \left| g(x) \right| + \sum_x \sum_{y \sim x} c(x, y) \left| f(y) \right| \left| g(x) \right| \\ &\leq \left\| f \right\|_{\mathcal{V}} \left\| g \right\|_{\mathcal{V}} + I \end{aligned}$$

where $I := \sum_{x} \sum_{y \sim x} c(x, y) |f(y)| |g(x)|.$

Using the Cauchy-Schwarz inequality, we obtain

$$I \leq \sum_{x} \left(\sum_{y \sim x} c(x, y) |f(y)|^{2} \right)^{\frac{1}{2}} \left(\sum_{y \sim x} c(x, y) \right)^{\frac{1}{2}} |g(x)|$$

$$= \sum_{x} \left(\sum_{y \sim x} c(x, y) f^{2}(y) \right)^{\frac{1}{2}} (\tilde{c}(x))^{\frac{1}{2}} |g(x)|$$

$$\leq \left(\sum_{x} \sum_{y \sim x} c(x, y) f^{2}(y) \right)^{\frac{1}{2}} \left(\sum_{x} \tilde{c}(x) g^{2}(x) \right)^{\frac{1}{2}}$$

$$= \left(\sum_{y} \tilde{c}(y) f^{2}(y) \right)^{\frac{1}{2}} \left(\sum_{x} \tilde{c}(x) g^{2}(x) \right)^{\frac{1}{2}}$$

$$= \|f\|_{\mathcal{V}} \|g\|_{\mathcal{V}}.$$

Therefore, (2.2) and (2.3) gives

$$|\langle \Delta_0 f, g \rangle_{\mathcal{V}}| \le 2 \, \|f\|_{\mathcal{V}} \, \|g\|_{\mathcal{V}} \, .$$

But by the definition of the norm of operator, we have

$$\|\Delta_0\| = \sup_{\|f\|=1} \|\Delta_0 f\|_{\mathcal{V}} = \sup_{\|f\|=1} \sup_{\|g\|=1} \langle \Delta_0 f, g \rangle_{\mathcal{V}}$$

So $\|\Delta_0\| \leq 2$, which shows that Δ_0 is a bounded operator .

Now, we want to prove the selfadjointess of the operator Δ_0 defined on $l^2(\mathcal{V})$. As Δ_0 is a bounded operator on $l^2(\mathcal{V})$, it remains to show that Δ_0 is symmetric.

As we have $\Delta_0 = \delta d$ and δ is the adjoint operator of d, we obtain for f and $g \in l^2(\mathcal{V})$

$$\begin{split} \langle \Delta_0 f, g \rangle_{\mathcal{V}} &= \langle \delta \mathrm{d} f, g \rangle_{\mathcal{V}} \\ &= \langle \mathrm{d} f, \mathrm{d} g \rangle_{\mathcal{E}} \\ &= \langle f, \delta \mathrm{d} g \rangle_{\mathcal{V}} \\ &= \langle f, \Delta_0 g \rangle_{\mathcal{V}}. \end{split}$$

• The operators d and δ are bounded. Indeed, using the inequality $(a - b)^2 \leq$ Remark 2.2. $2(a^2+b^2)$ and the definition of the weights on vertices: $\tilde{c}(x) = \sum_{y \sim x} c(x,y)$, we obtain

$$\begin{split} \|\mathrm{d}f\|_{\mathcal{E}}^{2} &= \frac{1}{2} \sum_{(x,y)\in\mathcal{E}} c(x,y) (\mathrm{d}f(x,y))^{2} \\ &= \frac{1}{2} \sum_{(x,y)\in\mathcal{E}} c(x,y) (f(y) - f(x))^{2} \\ &\leq \sum_{(x,y)\in\mathcal{E}} c(x,y) (f^{2}(y) + f^{2}(x)) \\ &= 2 \sum_{x\in\mathcal{V}} f^{2}(x) \sum_{y\sim x} c(x,y) \\ &= 2 \sum_{x\in\mathcal{V}} f^{2}(x) \tilde{c}(x) \\ &= 2 \|f\|_{\mathcal{V}}^{2}. \end{split}$$

So d is bounded, and the same is true for the adjoint δ .

Notice that since Δ_0 is the composite operator of δ and d; this gives another proof that Δ_0 is bounded.

• It is easy to see that Δ_0 is also positive, since $\langle \Delta_0 f, f \rangle_{\mathcal{V}} = \langle df, df \rangle_{\mathcal{E}} \ge 0$.

Corollary 2.1. As the operator Δ_0 is self-adjoint and positive, its spectrum is real and lies in [0, 2].

2.5. Weyl's criterion. As our operator is bounded and self-adjoint on a Hilbert space, we can use Weyl's criterion [14] to characterize its spectrum.

Weyl's criterion : Let \mathcal{H} be a separable Hilbert space, and let Δ be a bounded self-adjoint operator on \mathcal{H} . Then λ is in the spectrum of Δ if and only if there exists a sequence $(f_n)_{n\in\mathbb{N}}$ so that $||f_n|| = 1$ and $\lim_{n \to \infty} \|(\Delta - \lambda)f_n\| = 0.$ We denote $\sigma(\Delta)$ the spectrum of Δ and we set

- $\sigma_d(\Delta)$ is the set of $\lambda \in \sigma(\Delta)$ which is an isolated point and an eigenvalue with finite multiplicity.
- $\sigma_{ess}(\Delta) := \sigma(\Delta) \setminus \sigma_d(\Delta).$

3.1. The nonzero spectrum of Δ_0 and Δ_1 . In this section we will prove the relation between the spectrum of Δ_0 and that of Δ_1 , by using Weyl's criterion.

Following [15] and [18] we have the next lemma.

Lemma 3.1. Let $\Delta_0 = \delta d$ and $\Delta_1 = d\delta$. Then we have

(1) $d\Delta_0 = \Delta_1 d.$

(2) $\delta \Delta_1 = \Delta_0 \delta$.

Lemma 3.2.

- (1) $\ker \Delta_0 = \ker d.$
- (2) ker $\Delta_1 = \ker \delta$.

Proof

(1) Clearly, we have ker $d \subset \ker \Delta_0$.

On the other hand, if $\Delta_0 f = 0$ for $f \in l^2(\mathcal{V})$ and $f \neq 0$, we have

 $0 = \langle \Delta_0 f, f \rangle_{\mathcal{V}} = \langle \mathrm{d}f, \mathrm{d}f \rangle_{\mathcal{E}}.$

Then df = 0 for $f \in l^2(\mathcal{V})$.

(2) If $\varphi \in \ker \delta$, then $\varphi \in l^2(\mathcal{E})$ and $\delta \varphi = 0$. Thus, $d\delta \varphi = 0$ and we obtain $\varphi \in \ker \Delta_1$.

For the other inclusion, let $\varphi \in l^2(\mathcal{V}), \varphi \neq 0$ such that $\Delta_1 \varphi = 0$. Then

$$0 = \langle \Delta_1 \varphi, \varphi \rangle_{\mathcal{E}} = \langle \delta \varphi, \delta \varphi \rangle_{\mathcal{E}}.$$

We get $\delta \varphi = 0$ and as a result ker $\Delta_1 \subset \ker \delta$.

We arrive at our main result.

Theorem 1.

$$\sigma(\Delta_1) \setminus \{0\} = \sigma(\Delta_0) \setminus \{0\}.$$

Proof

• Let $\lambda \neq 0$ be in the spectrum of Δ_0 . By Weyl's criterion, there exists a sequence $(f_n)_n$ of $l^2(\mathcal{V})$ such that

$$||f_n||_{\mathcal{V}} = 1$$
 and $\lim_{n \to \infty} ||(\Delta_0 - \lambda)f_n||_{\mathcal{V}} = 0.$

We want to find a sequence $(\varphi_n)_n$ of $l^2(\mathcal{E})$ such that

$$\|\varphi_n\|_{\mathcal{E}} = 1 \text{ and } \lim_{n \to \infty} \|(\Delta_1 - \lambda)\varphi_n\|_{\mathcal{E}} = 0.$$

We set

$$\varphi_n := \frac{\mathrm{d}f_n}{\|\mathrm{d}f_n\|_{\mathcal{E}}}.$$

First, let us check that $\|df_n\|_{\mathcal{E}} \neq 0$. We have

$$\begin{aligned} \|\mathrm{d}f_n\|_{\mathcal{E}}^2 &= \langle \Delta_0 f_n, f_n \rangle_{\mathcal{V}} \\ &= \langle (\Delta_0 - \lambda) f_n, f_n \rangle_{\mathcal{V}} + \langle \lambda f_n, f_n \rangle_{\mathcal{V}} \\ &= \underbrace{\langle (\Delta_0 - \lambda) f_n, f_n \rangle_{\mathcal{V}}}_{\text{converges to } 0} + \lambda. \end{aligned}$$

Then, $\lim_{n\to\infty} \|df_n\|_{\mathcal{E}}^2 = \lambda$. Thus, by positivity of Δ_0 , there exists A > 0 and an integer n_0 such that for all $n \ge n_0$, we have $\|df_n\|_{\mathcal{E}} > A$. This implies that the sequence $(\varphi_n)_n$ is well defined.

Now, we verify that $\lim_{n\to\infty} \|(\Delta_1 - \lambda)\varphi_n\|_{\mathcal{E}} = 0$. By the first assertion of Lemma 3.1 and the fact that the operator d is bounded, we obtain for all *n* sufficiently large

$$\begin{aligned} \|(\Delta_1 - \lambda)\varphi_n\|_{\mathcal{E}} &= \left\| (\Delta_1 - \lambda) \frac{\mathrm{d}f_n}{\|\mathrm{d}f_n\|_{\mathcal{E}}} \right\|_{\mathcal{E}} \\ &= \frac{\|(\Delta_1 - \lambda)\mathrm{d}f_n\|_{\mathcal{E}}}{\|\mathrm{d}f_n\|_{\mathcal{E}}} \\ &= \frac{\|\mathrm{d}(\Delta_0 - \lambda)f_n\|_{\mathcal{E}}}{\|\mathrm{d}f_n\|_{\mathcal{E}}} \\ &\leq \frac{\|\mathrm{d}\|}{A} \|(\Delta_0 - \lambda)f_n\|_{\mathcal{V}}. \end{aligned}$$

But $\lim_{n\to\infty} \|(\Delta_0 - \lambda)f_n\|_{\mathcal{V}} = 0$. Therefore, $\lim_{n\to\infty} \|(\Delta_1 - \lambda)\varphi_n\|_{\mathcal{E}} = 0$ and we can conclude that λ is in the spectrum of $\Delta_1 \setminus \{0\}$.

• The second part of the proof follows in the same fashion, with the roles of d and δ swapped.

There is a second method to prove Theorem 1 when 0 is not in the spectrum of Δ_0 .

Lemma 3.3. If 0 is not in the spectrum of Δ_0 , then the operator d defined in $l^2(\mathcal{V})$ has a closed range.

Proof

Let $\varphi \in \overline{\text{Im d}}$, let us check that $\varphi \in \text{Im d}$, that means we look for a function $f \in l^2(\mathcal{V})$ such that $\varphi = df$. We have $\varphi \in \overline{\text{Im d}}$, so there exists a sequence $(\varphi_n)_n$ of Im d such that $\varphi_n = df_n$, for $f_n \in l^2(\mathcal{V})$. Moreover, the sequence $(\varphi_n)_n$ converges to φ in $l^2(\mathcal{E})$. On the other hand, by assumption 0 is not in the spectrum of Δ_0 which implies the existence of a positive constant C such that

$$\|f\|_{\mathcal{V}} \le C \, \|\Delta_0 f\|_{\mathcal{V}}, \, \forall f \in l^2(\mathcal{V}).$$

But by the definition of the operator norm and Remark 2.2, we obtain

$$\|\Delta_0 f\| = \sup_{g, \|g\|_{\mathcal{V}}=1} \langle \Delta_0 f, g \rangle_{\mathcal{V}} \le \|\mathrm{d}f\|_{\mathcal{E}} \sup_{g, \|g\|_{\mathcal{V}}=1} \|\mathrm{d}g\|_{\mathcal{E}} \le \sqrt{2} \|\mathrm{d}f\|_{\mathcal{E}}.$$

Then

$$\|f\|_{\mathcal{V}} \leq \sqrt{2}C \, \|\mathrm{d}f\|_{\mathcal{E}}, \, \forall f \in l^2(\mathcal{V}).$$

Thus

$$\|f_n - f_m\|_{\mathcal{V}} \le \sqrt{2}C \,\|\mathrm{d}f_n - \mathrm{d}f_m\|_{\mathcal{E}}, \ f_n, \ f_m \in l^2(\mathcal{V}).$$

And

$$\|f_n - f_m\|_{\mathcal{V}} \le \sqrt{2}C \,\|\varphi_n - \varphi_m\|_{\mathcal{E}} \,, \, f_n, \, f_m \in l^2(\mathcal{V})$$

As the sequence $(\varphi_n)_n$ converges, so it is a Cauchy sequence and also $(f_n)_n$ is a Cauchy sequence in $l^2(\mathcal{V})$ which is complete. Then, $(f_n)_n$ converges to f. By the boundedness of the operator d, we obtain $df_n = \varphi_n$ converges to df and by uniqueness of the limit, we have $df = \varphi$. So φ is in Im d.

Corollary 3.1. If 0 is not in the spectrum of Δ_0 , then

$$\sigma(\Delta_1|_{\operatorname{Im} d}) = \sigma(\Delta_0).$$

Proof

By the first assertion of Lemma 3.1, we obtain

$$\Delta_1 \mathbf{d} = \mathbf{d} \Delta_0.$$

But by assumption 0 is not in the spectrum of Δ_0 . Then by the first assertion of Lemma 3.1, the operator d is invertible. So we obtain $\Delta_1|_{\text{Im d}} = d\Delta_0 d^{-1}.$

 $\sigma(\Delta_1|_{\operatorname{Im} d}) = \sigma(\Delta_0).$

3.2. The 0-spectrum of Δ_0 and Δ_1 . As the nonzero spectrum of Δ_0 and Δ_1 are the same, we are interested in characterizing the 0-spectrum. We give in the following an extension of a result of John Lott's [11] (Proposition 9, p. 12).

Theorem 2. Let (G, c) be a connected, locally finite and weighted infinite graph such that the weight on edges c is bounded, i.e., there exists a constant $\alpha > 0$ such that $\frac{1}{\alpha} \leq c(x, y) \leq \alpha$, for all $(x, y) \in \mathcal{E}$. Then

$$0 \in \sigma(\Delta_1) \text{ or } 0 \in \sigma(\Delta_0).$$

First, we start with preliminary results.

By [17] (page 44) and [9] (chapter 4) we have the next definition.

Definition 3.1. The graph G verifies the isoperimetric inequality if there exists a constant C > 0 such that for all finite sub-graphs $G_U = (U, \mathcal{E}_U)$ of G, we have

$$\left|\partial \mathcal{E}_U\right| \ge C \left|U\right|,$$

where

$$|\partial \mathcal{E}_U| = \sum_{x \in U} \sum_{y \notin U} c(x, y) \text{ and } |U| = \sum_{x \in U} \tilde{c}(x).$$

Lemma 3.4. If Δ_0 is invertible then the isoperimetric inequality holds.

Proof

Let U a finite sub-graph of G. Let us set $g = \mathbf{1}_U$, meaning that g(x) = 1 if $x \in U$ and g(x) = 0 if $x \notin U$. Then we obtain

$$|U| = \sum_{x \in U} \tilde{c}(x) = ||g||_{\mathcal{V}}^2$$

and

$$|\partial \mathcal{E}_U| = \sum_{x \in U} \sum_{y \notin U} c(x, y) = ||\mathrm{d}g||_{\mathcal{E}}^2$$

By assumption 0 is not in the spectrum of Δ_0 . Then by the first assertion of Lemma 3.1, the operator d is invertible, so there exists a positive constant λ so that

$$\|g\|_{\mathcal{V}} \leq \lambda \|\mathrm{d}g\|_{\mathcal{E}}, \ \forall g \in l^{2}(\mathcal{V}).$$

Thus, it follows that

$$|\partial \mathcal{E}_U| \ge C |U|$$
, with $C = \frac{1}{\lambda^2}$.

Definition 3.2. • A branch B is a finite sequence of vertices $x_0, x_1, ..., x_{m+1}$ such that for all $j; 1 \le j \le m$, we have $\deg(x_j) = 2$.

- The length of a branch B, denoted long(B), is the number of vertices in this branch, here, long(B) = m + 2.
- The interior of the branch B is the set of vertices x_j of B satisfying the following conditions: i) $\deg(x_j) = 2$.

ii)
$$\forall y \in \mathcal{V}; \ y \sim x_j \Rightarrow y \in B.$$

See [5] and [19] for the definition of the interior set of vertices.



FIGURE 1. A branch of length m + 2

Instead of the argument of Lott [11] inspired by Gromov [10] (p. 236-237), we use the following lemma:

Lemma 3.5. We suppose that the following conditions are satisfied:

- The weight on edges c is bounded, i.e., there exists a constant $\alpha > 0$ such that $\frac{1}{\alpha} \leq c(x, y) \leq \alpha$, $\forall (x, y) \in \mathcal{E}$.
- The operator Δ_0 is invertible.
- The operator Δ_1 is injective.

Then the graph (G, c) is a tree which contains branches with uniformly bounded lengths, that means $\exists M > 0, \forall B \text{ branch of } G, \log(B) \leq M.$

Proof

On the one hand, the operator Δ_1 is injective which leads to the absence of cycles in the graph, so that G is a tree.

On the other hand, the operator Δ_0 is invertible, then the isoperimetric inequality is checked, by Lemma 3.4 there is a positive constant C such that for all finite sub-graphs U, we have

$$\left|\partial \mathcal{E}_U\right| \ge C \left|U\right|.$$

Let B be a branch with vertices $x_0, x_1, ..., x_m, x_{m+1}$. We set $U = \{x_1, ..., x_m\}$ the interior of the branch B, then

(3.4)
$$c(x_0, x_1) + c(x_m, x_{m+1}) \ge C \sum_{j=1}^m \tilde{c}(x_j)$$

For the sake of simplicity, we first prove the lemma for the case of the constant weight c = 1, before handling the case of general weights.

• If c = 1, then we have $\tilde{c}(x) = \sum_{y \sim x} c(x, y) = \sum_{y \sim x} 1 = \deg(x)$, $\forall x \in \mathcal{V}$ (this is J. Lott's case [11]). Therefore, the inequality (3.4) and Definition 3.2 gives

$$2 \ge C \sum_{j=1}^{m} \deg(x_j) = C \sum_{j=1}^{m} 2 = 2Cm.$$

We set $M := \frac{1}{C} + 2$ (independent of B), then $long(B) \le M$. Thus the lengths of branches of G are uniformly bounded.

• If $c \neq 1$ but c is bounded, that means there exists $\alpha > 0$ such that $\frac{1}{\alpha} \leq c(x, y) \leq \alpha$, for all $(x, y) \in \mathcal{E}$. And as we have the weight on the vertices is $\tilde{c}(x) = \sum_{y \sim x} c(x, y)$, we obtain that \tilde{c} is also bounded from below by $\frac{1}{\alpha}$.

By the inequality (3.4) we have

$$2\alpha \ge c(x_0, x_1) + c(x_m, x_{m+1}) \ge C \sum_{j=1}^m \tilde{c}(x_j) \ge Cm \frac{1}{\alpha}.$$

Hence,

$$\frac{2\alpha^2}{C} \ge m.$$

We set $M = \frac{2\alpha^2}{C} + 2$ (independent of B), then $\log(B) \le M$. Thus, the lengths of the branches of G are uniformly bounded.

Now, we arrive to the proof of Theorem 2.

Proof

Taking the arguments from [11], we argue by contradiction. Suppose that both operators Δ_0 and Δ_1 are invertible. Then, by Lemma 3.5, the graph G is a tree which contains branches with uniformly bounded lengths; see Figure 2 for an example.



FIGURE 2. A branch tree

But the existence of such tree gives a δ -harmonic nonzero square-integral function φ . Indeed: we consider a part of the branch tree in Figure 2 as an example to simplify the understanding of the construction. For the sake of simplicity, we first prove the theorem for the case of the constant weight c = 1, before handling the case of general weights.

First case: c = 1, we fix a vertex 0 as the origin of the tree and we set 0^- and 1 its different neighbors. Let us take

$$\varphi(0,0^-) = \varphi(0,1) = 1.$$

Then, we obtain $\delta \varphi(0) = 0$ (the tree is oriented).

Afterwards on the branch B_1 , φ is constant, in other words, $\varphi(j, j+1) = 1$, for all j, such that $1 \leq j \leq 3$. And at the point 4, we have $\varphi(4,5) = \varphi(4,5^-) = \frac{1}{2}$. It is claimed that $\delta\varphi(4) = 0$.

And for the points which are in the branch B_2 , the function φ is constant and takes the value $\frac{1}{2}$. And so on to the point 8, we have $\varphi(8,9) = \varphi(8,9^-) = \frac{1}{4}$, to obtain $\delta\varphi(8) = 0$. And for the points which are in the branch B_3 , the function φ is constant and takes the value $\frac{1}{4}$. And we continue in this way...



FIGURE 3. An example of a branch tree



FIGURE 4. Another example of a branch tree

In a general way, G is a tree which contains branches with uniformly bounded lengths and we construct a functions φ in a part of G by selecting always two branches at bifurcation points and at all edges that occur on other branches, φ is set to zero, as in Figure 4. In the Figure 3, the construction of φ is done in the following way: on B_0 the function φ is constant and equals to 1. Then we add a generation, we get two branches $B_{1,1}$ and $B_{1,2}$ such that the function φ takes the value $\frac{1}{2}$. And to the generation m, we have $B_{m,k}$ branches, where $1 \le k \le 2^m$, then the function φ is equal to $\frac{1}{2^m}$. As a result, we show that this construction of φ is in $l^2(\mathcal{E})$. Using the fact that the lengths of the branches of the tree are uniformly bounded by a constant M > 0, we obtain

$$\begin{aligned} \|\varphi\|_{\mathcal{E}}^{2} &= \frac{1}{2} \sum_{m \ge 0} \sum_{k=1}^{2^{m}} \sum_{e \in B_{m,k}} (\varphi(e))^{2} \\ &= \frac{1}{2} \sum_{m \ge 0} \sum_{k=1}^{2^{m}} \sum_{e \in B_{m,k}} \left(\frac{1}{2^{m}}\right)^{2} \\ &\leq \frac{1}{2} \sum_{m \ge 0} 2^{m} M \left(\frac{1}{2^{m}}\right)^{2} \\ &= \frac{M}{2} \sum_{m \ge 0} \frac{1}{2^{m}} < \infty. \end{aligned}$$

Second case: $c \neq 1$ but c bounded by a positive constant. That means there exists $\alpha > 0$ such that $\frac{1}{\alpha} \leq c(x,y) \leq \alpha, \forall (x,y)$ in \mathcal{E} . As in Figure 2, on the branch B_1 , the vertex 0 has two neighbors denoted 0^- and 1. We want $\delta \varphi(0) = 0$, so we choose the function φ in the following way $\varphi(0,1) = 0$.

 $\frac{c(0,0^-)}{c(0,1)}\varphi(0,0^-).$ And in the interior of B_1 , we set $\varphi(j,j+1) = \frac{c(0,0^-)}{c(j,j+1)}\varphi(0,0^-) \quad \forall j, 1 \leq j \leq 3.$ Next, we look at the point 4 which has two neighbors 5 and 5⁻, to obtain $\delta\varphi(4) = 0$ and as we have $\varphi(3,4) = \frac{c(0,0^-)}{c(3,4)}\varphi(0,0^-).$ We choose $\varphi(4,5) = \frac{c(0,0^-)}{2c(4,5)}\varphi(0,0^-)$ and $\varphi(4,5^-) = \frac{c(0,0^-)}{2c(4,5^-)}\varphi(0,0^-).$ Therefore, in the interior of the branch B_2 ,

$$\varphi(j, j+1) = \frac{c(0, 0^-)}{2c(j, j+1)}\varphi(0, 0^-) \; \forall j, \; 5 \le j \le 7.$$

And for the vertex 8, which has two neighbors 9 and 9⁻. To have $\delta\varphi(8) = 0$ and by using that $\varphi(7,8) = \frac{c(0,0^-)}{2c(7,8)}\varphi(0,0^-)$. We choose $\varphi(8,9) = \frac{c(0,0^-)}{4c(8,9)}\varphi(0,0^-)$ and $\varphi(8,9^-) = \frac{c(0,0^-)}{4c(8,9^-)}\varphi(0,0^-)$. And in the interior of the branch B_3 ,

$$\varphi(j, j+1) = \frac{c(0, 0^-)}{4c(j, j+1)}\varphi(0, 0^-)$$
 for $j = 10$.

And so on... In a general way, see Figure 3, on B_0 the function $\varphi(e_0) = \frac{c(0,0^-)}{c(e_0)}\varphi(0,0^-)$, where e_0 is an edge of B_0 . Then we add a generation, we get two branches $B_{1,1}$ and $B_{1,2}$ such that the function φ has a value $\varphi(e_1^k) = \frac{c(0,0^-)}{2c(e_1^k)}\varphi(0,0^-)$, where e_1^k denotes the edges of $B_{1,k}$ for $1 \le k \le 2$. And at generation m, we have $B_{m,k}$ branches, where $1 \le k \le 2^m$, then the function φ equals to $\varphi(e_m^k) = \frac{c(0,0^-)}{2^m c(e_m^k)}\varphi(0,0^-)$, where e_m^k denotes the edges of $B_{m,k}$. And to simplify the formulas, we can suppose that

$$\varphi(0,0^-) = \frac{1}{c(0,0^-)}.$$

Then, we obtain

$$\varphi(e_m^k) = \frac{1}{2^m c(e_m^k)}, \; \forall m \ge 0 \text{ and } 1 \le k \le 2^m.$$

Therefore, this construction gives $\varphi \in l^2(\mathcal{E})$. Using the fact that the lengths of the branches of the tree are uniformly bounded by a constant M > 0 and the weight c on the edges is bounded by a positive constant, we obtain

$$\begin{aligned} \|\varphi\|_{\mathcal{E}}^2 &= \frac{1}{2} \sum_{e} c(e)(\varphi(e))^2 \\ &\leq \sum_{m \ge 0} \sum_{k=1}^{2^m} \sum_{e \in B_{m,k}} c(e)(\varphi(e))^2 \\ &= \sum_{m \ge 0} \sum_{k=1}^{2^m} \sum_{e \in B_{m,k}} c(e) \left(\frac{1}{2^m c(e)}\right)^2 \\ &= \sum_{m \ge 0} \sum_{k=1}^{2^m} \sum_{e \in B_{m,k}} \frac{1}{2^{2m} c(e)} \\ &\leq \alpha M \sum_{m \ge 0} \frac{1}{2^m} \\ &= 2M\alpha < \infty. \end{aligned}$$

Finally, we have φ in $l^2(\mathcal{E})$ and δ -harmonic. So, $0 \in \sigma(\Delta_1)$, which contradicts the assumption that 0 is not in the spectrum of Δ_1 .

Remark 3.1. Any point of the graph can play the role of the first vertex 0 in the previous construction. It is then clear that we can construct an infinite family of independent functions φ which are in $l^2(\mathcal{E})$ and δ -harmonic. Then 0 is an eigenvalue of Δ_1 with infinite multiplicity, so $0 \in \sigma_{ess}(\Delta_1)$.

4. EXAMPLES:

In this section, we will construct a δ -harmonic function φ in different examples of trees.

1) Symmetric tree: Following [7] we introduce the next definition:

Definition 4.1. A tree T_s is symmetric around o with branching numbers $\{m_i\}_{i=0}^{\infty}$ if the degree of each vertex depends only on its distance from o, namely, for each $x \in T_s$, $\deg(x) = m_i$ if d(o, x) = i.

Example of a symmetric tree:

We fix a vertex o as an origin of the tree. We set $S_n = \{x \in T_s; d(o, x) = n\}$. T_s is symmetric around o with branching numbers $\{m_n\}_{n=0}^{\infty}$. In Figure 4, we choose $m_n = 3 + n$ for all $n \in \mathbb{N}$ which is an increasing sequence. So, we have $m_0 = 3$ that means $\deg(o) = 3$. And for $x \in S_1$, we obtain $\deg(x) = m_1 = 4$. In the same way, if $x \in S_2$ we have $m_2 = 5$ and so on.



FIGURE 5. Symmetric tree

Proposition 4.1. If the symmetric tree T_s is simple (the edge weights are equal to 1) with $\deg(x) > 2$ for all $x \in T_s$, then there is a δ -harmonic function $\varphi \in l^2(\mathcal{E})$.

Proof

We fix a vertex x_0 as an origin of the tree T_s , we can find an increasing sequence of finite subgraph $\{S_n\}_n$ such that $S_n = \{x \in T_s; d(x_0, x) = n\}$ and $T_s = \bigcup_n S_n$. By the definition of the symmetric tree, we have for all $n \deg(x_n) = m_n$, $\forall x_n \in S_n$. First, we construct a function φ so that $\delta \varphi = 0$ as follows: Let e_0 and b_0 denote two distinct outward edges connecting to the vertex x_0 . We define φ to be 0 excepted on these edges where $\varphi(e_0) = 1$ and $\varphi(b_0) = -1$ which gives $\delta\varphi(x_0) = 0$. And denote e_n^k , $n \ge 1$, $1 \le k \le \prod_{j=1}^n (m_j - 1)$, resp. b_n^k , $n \ge 1$, $1 \le k \le \prod_{j=1}^n (m_j - 1)$, the outward edges emanating from e_0 , resp. b_0 , of generation n. We define

$$\varphi(e_n^k) = \frac{1}{\prod_{j=1}^n (m_j - 1)} \varphi(e_0),$$
$$\varphi(b_n^k) = \frac{1}{\prod_{j=1}^n (m_j - 1)} \varphi(b_0)$$

and φ takes value 0 on all edges other than e_n^k and b_n^k . Second, through this construction, we look for $\varphi \in l^2(\mathcal{E})$. Using the fact that $\deg(x_n) = m_n \ge 3$, $\forall x_n \in S_n, \forall n$, we obtain

$$\begin{aligned} \|\varphi\|_{\mathcal{E}}^{2} &= \frac{1}{2} \sum_{e \in \mathcal{E}} \varphi^{2}(e) \\ &= \frac{1}{2} \left(2 + \sum_{n \ge 1} \sum_{k=1}^{\prod_{j=1}^{n} (m_{j}-1)} \varphi^{2}(e_{n}^{k}) + \varphi^{2}(b_{n}^{k}) \right) \\ &= 1 + \sum_{n \ge 1} \sum_{k=1}^{\prod_{j=1}^{n} (m_{j}-1)} \left(\frac{1}{(m_{1}-1)(m_{2}-1)...(m_{n}-1)} \right)^{2} \\ &= 1 + \sum_{n \ge 1} \frac{1}{(m_{1}-1)(m_{2}-1)...(m_{n}-1)} \\ &\leq 1 + \sum_{n \ge 1} \frac{1}{2^{n}} \\ &< \infty. \end{aligned}$$

2) Triadic tree with weights bounded from below: As [2] (p. 19), we have the following definition of a triadic tree.

Definition 4.2. A tree is a connected graph containing no cycles. The triadic tree is a tree such that all the vertices have degree 3.

Proposition 4.2. If the triadic tree has weights on the edges bounded from below by a positive constant λ , then there is a δ -harmonic function $\varphi \in l^2(\mathcal{E})$.



FIGURE 6. Triadic tree

Proof

We fix a vertex o as the origin of the tree T. Define the increasing sequence of finite subgraphs $\{G_n\}_n, G_n = \{x \in \mathcal{V}; \ d(o, x) \leq n\}$ and let $G = \bigcup_n G_n$. Denote $S_n = \{x \in T; \ d(o, x) = n\}$. We set x_1^1, x_1^2 and x_1^3 the different neighbors of o which are in S_1 . We suppose that $\varphi(o, x_1^i) = 0$ for all $i \in \{1, 2, 3\}$, so we have $\delta \varphi(o) = 0$.

We fix one vertex of S_1 for example $x_1 := x_1^1$, let e_0 and b_0 be the two outward edges of x_1 and define inductively e_m^k , $m \ge 1$, $1 \le k \le 2^m$, resp. b_m^k , $m \ge 1$, $1 \le k \le 2^m$, to be the outward edges emanating from e_0 , resp. b_0 , of generation m (the edge are oriented outward). For $m \ge 0$, we define φ to be 0 excepted on these edges where

$$\varphi(e_m^k) = \frac{1}{2^m} \frac{1}{c(e_m^k)}, \ \forall k; \ 1 \leqslant k \leqslant 2^m$$

and

$$\varphi(b_m^k) = \frac{-1}{2^m} \frac{1}{c(b_m^k)}, \; \forall k; \; 1 \leqslant k \leqslant 2^m.$$

With this construction, we obtain for each $x_n \in S_n$, $\delta \varphi(x_n) = 0$, $\forall n \ge 1$. Moreover, $\varphi \in l^2(\mathcal{E})$. Indeed: by using the assumption that the weights on the edges are bounded from below by a positive constant λ , we obtain

$$\begin{split} \|\varphi\|_{\mathcal{E}}^{2} &= \frac{1}{2} \sum_{e \in \mathcal{E}} c(e) \varphi^{2}(e) \\ &= \frac{1}{2} \left(\sum_{m \ge 0} \sum_{k=1}^{2^{m}} c(e_{m}^{k}) \varphi^{2}(e_{m}^{k}) + c(b_{m}^{k}) \varphi^{2}(b_{m}^{k}) \right) \\ &= \frac{1}{2} \sum_{m \ge 0} \sum_{k=1}^{2^{m}} c(e_{m}^{k}) \frac{1}{2^{2m}} \frac{1}{c^{2}(e_{m}^{k})} + \frac{1}{2} \sum_{m \ge 0} \sum_{k=1}^{2^{m}} c(b_{m}^{k}) \frac{1}{2^{2m}} \frac{1}{c^{2}(b_{m}^{k})} \\ &= \frac{1}{2} \sum_{m \ge 0} \sum_{k=1}^{2^{m}} \frac{1}{2^{2m}} \frac{1}{c(e_{m}^{k})} + \frac{1}{2} \sum_{m \ge 0} \sum_{k=1}^{2^{m}} \frac{1}{2^{2m}} \frac{1}{c(b_{m}^{k})} \\ &\leq \lambda' \sum_{m \ge 0} \sum_{k=1}^{2^{m}} \frac{1}{2^{2m}} \\ &= \lambda' \sum_{m \ge 0} 2^{m} \frac{1}{2^{2m}} \\ &= \lambda' \sum_{m \ge 0} \frac{1}{2^{m}} \\ &= 2\lambda', \end{split}$$
 where $\lambda' = \frac{1}{2}.$

Remark 4.1. • The construction of a δ -harmonic nonzero square-integral function depends on the edge weights.

• In the simple triadic tree, 0 is both in the spectrum of Δ_0 [4] and in the spectrum of Δ_1 [2].

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REFERENCES

- C. Anné, N. Torki-Hamza, The Gauss-Bonnet Operator of an Infinite Graph, Analysis Mathematical Physics 5 (2015) 137–159.
- [2] H. Ayadi, Semi-Fredholmness of the discrete Gauss-Bonnet operator, to appear in Filomat (2015).
- [3] H. Baloudi, S. Golénia, A. Jeribi, The Adjacency Matrix and the Discrete Laplacian Acting on Forms. Preprint arXiv:1505.06109(2015).
- [4] Y. Colin de Verdière, Spectre de Graphes, Cours spécialisés, volume 4 Société Mathématique de France (1998).
- [5] J. Dodziuk, Elliptic operators on infinite graphs, Analysis, geometry and topology of elliptic operators, World Sci. Publ., Hackensack NJ. (2006) 353–368.
- [6] D. E. Dutkay, P. E.T. Jorgensen, Spectral theory for discrete Laplacians, Complex analysis and operator theory 1 (2010) 1–38.
- [7] K. Fujiwara, The Laplacian on rapidly branching trees, Duke mathematical journal 83 (1996) 191–202.
- [8] A. Grigor'yan, Analytic and geometric background of recurrence and non-explosion of the Brownian motion on Riemannian manifolds, Bulletin of the American Mathematical Society **36** (1999) 135–249.
- [9] A. Grigor'yan, Analysis on graphs, Lecture Notes (2011/2012).
- [10] M. Gromov, Asymptotic Invariants of Infinite Groups, Geometric Group Theory, Vol. 2, ed. by Graham Niblo and Martin Roller, London Math. Soc. Lecture Notes 182, Cambridge University Press, Cambridge (1993).
- [11] J. Lott, The zero-in-the spectrum question, Enseign. Math 42 (1996) 341-376.
- [12] R. Lyons, Y. Peres, Probability on Trees and Networks, Cambridge University Press, New York (2016). Available at http://pages.iu.edu/ rdlyons/.
- [13] B. Mohar, W. Woess, A survey on spectra of infinite graphs Bull. London Math. Soc. 21, (1989) 209–234.
- [14] M. Reed, B. Simon Methods of Modern Mathematical Physics I, Academic Press (1980).
- [15] T. Shirai, The spectrum of infinite regular line graphs, Transactions of the American Mathematical Society 352 (2000) 115–132.
- [16] P. M. Soardi, Potential Theory on Infinite Networks, Lecture Notes in Mathematics 1590 (1994)
- [17] P. M. Soardi, W. Woess, Uniqueness of currents in infinite resistive networks, Discrete Applied Mathematics 31 (1991) 37–49.
- [18] R. S. Strichartz, Transformation of spectra of graph laplacians, Rocky Mountain Journal of Math, 40 (6) (2010).
- [19] N. Torki-Hamza, Laplaciens de graphes infinis I-Graphes métriquement complets, Confluentes Mathematici 2 (2010) 333–350. Translated to: Laplacians of infinite graphs. I: Metrically complete graphs, Arxiv:1201.4644.
- [20] W. Woess, Random walks on infinite graphs and groups, Bulletin of the London Mathematical Society 26 (1994) 1–60.

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