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# SPECTRA OF LAPLACIANS ON FORMS ON AN INFINITE GRAPH

HÈLA AYADI

ABSTRACT. In the context of infinite weighted graphs, we consider the discrete Laplacians on 0-forms and 1-forms. Using Weyl's criterion, we prove the relation between the nonzero spectrum of  $\Delta_0$  and that of  $\Delta_1$ . Moreover, we give an extension of the work of John Lott to characterize the 0-spectrum of these two Laplacians.

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## 1. INTRODUCTION

In recent years, much attention has been paid to the analysis of discrete Laplacians and elliptic differential operators acting on graphs [13], [5], [6] and [19]. More precisely, authors have intensively studied the spectrum of the discrete Laplacian on an infinite graph in various areas, for example, harmonic analysis on graphs (see [16], [20]), probability theory especially Markov chains (see [8], [12]), potential theory such as electric networks (see [17], [12]), and so on. In this paper, we define two Laplacians, mentioned in [1] and [3], one as an operator acting on functions on vertices denoted by  $\Delta_0$  and the other one acting on functions on edges denoted by  $\Delta_1$ . So, it is a natural question to characterize the relation between their spectrum in terms of a certain geometric property of the graph and properties of the operators. Especially we show that the nonzero spectrum of  $\Delta_0$  and  $\Delta_1$  are the same, by using Weyl's criterion. Moreover, with suitable weight conditions we prove that 0 is in the spectrum of  $\Delta_1$ , if the operator  $\Delta_0$  is invertible. This result is inspired from J. Lott's work [11] (Proposition 9, p. 12) which proves in the case of a simple graph that 0 is either in the spectrum of the Laplacian on 0-forms, or in the spectrum of the Laplacian on 1-forms. In fact, the major interest of J. Lott concerns the zero-in-the-spectrum question for the Laplace-de Rham operator acting on  $L^2$  differential forms of any degree on a complete connected

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oriented Riemannian manifold. The article [11] is rather expository and gives some positive answers, in relation with topology, for small dimensions. We finish the paper with examples of constructions of  $\Delta_1$ -harmonic nonzero square-integrable functions.

## 2. PRELIMINARIES

### 2.1. Definition and notation.

- A graph  $G$  is a couple  $(\mathcal{V}, \mathcal{E})$  where  $\mathcal{V}$  is a set at most countable whose elements are called vertices and  $\mathcal{E}$  is a set of oriented edges, considered as a subset of  $\mathcal{V} \times \mathcal{V}$ .
- If the graph  $G$  has a finite set of vertices, it is called a finite graph. Otherwise,  $G$  is called an infinite graph.
- We assume that  $\mathcal{E}$  has no self-loops and is symmetric:

$$v \in \mathcal{V} \Rightarrow (v, v) \notin \mathcal{E}, \quad (v_1, v_2) \in \mathcal{E} \Rightarrow (v_2, v_1) \in \mathcal{E}.$$

- Choosing an orientation of  $G$  consists of defining a partition of  $\mathcal{E}$ :  $\mathcal{E}^+ \sqcup \mathcal{E}^- = \mathcal{E}$

$$(v_1, v_2) \in \mathcal{E}^+ \Leftrightarrow (v_2, v_1) \in \mathcal{E}^-.$$

- For  $e = (v_1, v_2)$ , we denote

$$e^- = v_1, \quad e^+ = v_2 \quad \text{and} \quad -e = (v_2, v_1).$$

- We write  $v_1 \sim v_2$  for  $e = (v_1, v_2) \in \mathcal{E}$ .
- The graph  $G$  is connected if any two vertices  $x, y$  in  $\mathcal{V}$  can be joined by a path of edges  $\gamma_{xy}$ , that means  $\gamma_{xy} = \{e_k\}_{k=1, \dots, n}$  such that

$$e_1^- = x, \quad e_n^+ = y \quad \text{and, if } n \geq 2, \quad \forall j; 1 \leq j \leq (n-1) \Rightarrow e_j^+ = e_{j+1}^-.$$

- The degree (or valence) of a vertex  $x$  is the number of edges emanating from  $x$ . We denote

$$\deg(x) := \#\{e \in \mathcal{E}; e^- = x\}.$$

- If  $\deg(x) < \infty, \forall x \in \mathcal{V}$ , we say that  $G$  is a locally finite graph.

### 2.2. Weighted graphs.

**Definition 2.1.** A weighted graph  $(G, c)$  is given by a graph  $G = (\mathcal{V}, \mathcal{E})$  and weights on the edges  $c : \mathcal{E} \rightarrow [0, \infty[$  such that

- $c(x, x) = 0, \forall x \in \mathcal{V}$ .
- $c(x, y) > 0, \forall (x, y) \in \mathcal{E}$ .
- $c(x, y) = c(y, x), \forall (x, y) \in \mathcal{E}$ .

If  $\sum_{y \sim x} c(x, y) < \infty$  for each  $x \in \mathcal{V}$ , we can define a weight on  $\mathcal{V}$  by

$$\tilde{c}(x) = \sum_{y \sim x} c(x, y), \quad x \in \mathcal{V}.$$

**Remark 2.1.** If the graph  $G$  is locally finite, the weight  $\tilde{c}$  on any vertex is well defined.

**Examples:** - An infinite electrical network is a weighted graph  $(G, c)$  where the weight  $c$  on the edges are called conductances and their reciprocals are called resistances. This is the convention used in the study of random walks on weighted graphs, see [12] and [16]. Then,  $\tilde{c}(x) = \sum_{y \in \mathcal{V}} c(x, y)$  is the weight

associated to the vertex  $x$ .

-A graph  $G$  is called a simple graph if the edge weights are equal to 1. In this case,

$$\tilde{c}(x) = \deg(x), \quad \forall x \in \mathcal{V}.$$

**All the graphs we shall consider in the sequel will be connected, locally finite and weights  $c$  given in Definition 2.1.**

**2.3. Functional spaces.** We denote the set of real functions on  $\mathcal{V}$  by:

$$\mathcal{C}(\mathcal{V}) = \{f : \mathcal{V} \rightarrow \mathbb{R}\}$$

and the set of functions of finite support by  $\mathcal{C}_0(\mathcal{V})$ .

Moreover, we denote the set of real skew-symmetric functions on  $\mathcal{E}$  by:

$$\mathcal{C}^a(\mathcal{E}) = \{\varphi : \mathcal{E} \rightarrow \mathbb{R}; \varphi(-e) = -\varphi(e)\}$$

and the set of functions of finite support by  $\mathcal{C}_0^a(\mathcal{E})$ .

We define on the weighted graph  $(G, c)$  the following function spaces endowed with the scalar products.

**a):**

$$l^2(\mathcal{V}) := \left\{ f \in \mathcal{C}(\mathcal{V}); \sum_{x \in \mathcal{V}} \tilde{c}(x) f^2(x) < \infty \right\},$$

with the inner product

$$\langle f, g \rangle_{\mathcal{V}} = \sum_{x \in \mathcal{V}} \tilde{c}(x) f(x) g(x)$$

and the norm

$$\|f\|_{\mathcal{V}} = \sqrt{\langle f, f \rangle_{\mathcal{V}}}.$$

**b):**

$$l^2(\mathcal{E}) := \left\{ \varphi \in \mathcal{C}^a(\mathcal{E}); \frac{1}{2} \sum_{e \in \mathcal{E}} c(e) \varphi^2(e) < \infty \right\},$$

with the inner product

$$\langle \varphi, \psi \rangle_{\mathcal{E}} = \frac{1}{2} \sum_{e \in \mathcal{E}} c(e) \varphi(e) \psi(e)$$

and the norm

$$\|\varphi\|_{\mathcal{E}} = \sqrt{\langle \varphi, \varphi \rangle_{\mathcal{E}}}.$$

Then,  $l^2(\mathcal{V})$  and  $l^2(\mathcal{E})$  are separable Hilbert spaces (since  $\mathcal{V}$  is countable).

**2.4. Operators and properties.** The difference operator

$$d : l^2(\mathcal{V}) \longrightarrow l^2(\mathcal{E}),$$

is given by

$$d(f)(e) = f(e^+) - f(e^-).$$

The coboundary operator is  $\delta$ , the formal adjoint of  $d$ . Thus it satisfies

$$(2.1) \quad \langle df, \varphi \rangle_{\mathcal{E}} = \langle f, \delta\varphi \rangle_{\mathcal{V}}$$

for all  $f \in l^2(\mathcal{V})$  and for all  $\varphi \in l^2(\mathcal{E})$ .

As consequence, we have the following formula characterizing  $\delta$  :

**Lemma 2.1.** *The coboundary operator  $\delta$  is characterized by the formula*

$$\delta\varphi(x) = \frac{1}{\tilde{c}(x)} \sum_{e, e^+=x} c(e)\varphi(e),$$

for all  $\varphi \in l^2(\mathcal{E})$ .

**Proof**

For  $f \in l^2(\mathcal{V})$  and  $\varphi \in l^2(\mathcal{E})$ , using (2.1), we get

$$\begin{aligned} \langle df, \varphi \rangle_{\mathcal{E}} &= \frac{1}{2} \sum_{e \in \mathcal{E}} c(e) df(e) \varphi(e) \\ &= \frac{1}{2} \sum_{e \in \mathcal{E}} c(e) (f(e^+) - f(e^-)) \varphi(e) \\ &= \frac{1}{2} \sum_{x \in \mathcal{V}} f(x) \left( \sum_{e, e^+=x} c(e) \varphi(e) - \sum_{e, e^-=x} c(e) \varphi(e) \right). \end{aligned}$$

But  $c(-e) = c(e)$  and  $\varphi$  is skew-symmetric, so we have  $\sum_{e, e^+=x} c(e) \varphi(e) = - \sum_{e, e^-=x} c(e) \varphi(e)$ .

Then,

$$\begin{aligned} \langle df, \varphi \rangle_{\mathcal{E}} &= \sum_{x \in \mathcal{V}} \tilde{c}(x) f(x) \left( \frac{1}{\tilde{c}(x)} \sum_{e, e^+=x} c(e) \varphi(e) \right) \\ &= \langle f, \delta\varphi \rangle_{\mathcal{V}} \end{aligned}$$

and the formula for  $\delta\varphi$  follows. □

**Definition 2.2.** *The Laplacian on 0-forms  $\Delta_0$  defined by  $\delta d$  on  $l^2(\mathcal{V})$  is given by*

$$\Delta_0 f(x) = \frac{1}{\tilde{c}(x)} \sum_{y \sim x} c(x, y) (f(x) - f(y)).$$

In fact, we have

$$\begin{aligned} \Delta_0 f(x) &= \delta(df)(x) \\ &= \frac{1}{\tilde{c}(x)} \sum_{e, e^+=x} c(e) df(e) \\ &= \frac{1}{\tilde{c}(x)} \sum_{e, e^+=x} c(e) (f(e^+) - f(e^-)) \\ &= \frac{1}{\tilde{c}(x)} \sum_{y \sim x} c(x, y) (f(x) - f(y)). \end{aligned}$$

**Definition 2.3.** *The Laplacian on 1-forms  $\Delta_1$  defined by  $d\delta$  on  $l^2(\mathcal{E})$  is given by*

$$\Delta_1 \varphi(e) = \frac{1}{\tilde{c}(e^+)} \sum_{e_1, e_1^+=e^+} c(e_1) \varphi(e_1) - \frac{1}{\tilde{c}(e^-)} \sum_{e_2, e_2^+=e^-} c(e_2) \varphi(e_2).$$

In fact, we have

$$\begin{aligned}\Delta_1\varphi(e) &= d(\delta\varphi)(e) \\ &= \delta\varphi(e^+) - \delta\varphi(e^-) \\ &= \frac{1}{\tilde{c}(e^+)} \sum_{e_1, e_1^+ = e^+} c(e_1)\varphi(e_1) - \frac{1}{\tilde{c}(e^-)} \sum_{e_2, e_2^+ = e^-} c(e_2)\varphi(e_2).\end{aligned}$$

**Proposition 2.1.** *The operator  $\Delta_0$  is bounded and self-adjoint.*

**Proof**

For  $f, g \in l^2(\mathcal{V})$ , we have

$$\begin{aligned}|\langle \Delta_0 f, g \rangle_{\mathcal{V}}| &= \left| \sum_x \tilde{c}(x) \frac{1}{\tilde{c}(x)} \sum_{y \sim x} c(x, y) (f(x) - f(y)) g(x) \right| \\ &\leq \sum_x \sum_{y \sim x} c(x, y) |(f(x) - f(y))| |g(x)| \\ &\leq \sum_x \sum_{y \sim x} c(x, y) |f(x)| |g(x)| + \sum_x \sum_{y \sim x} c(x, y) |f(y)| |g(x)| \\ &= \sum_x \tilde{c}(x) |f(x)| |g(x)| + \sum_x \sum_{y \sim x} c(x, y) |f(y)| |g(x)| \\ (2.2) \quad &\leq \|f\|_{\mathcal{V}} \|g\|_{\mathcal{V}} + I\end{aligned}$$

where  $I := \sum_x \sum_{y \sim x} c(x, y) |f(y)| |g(x)|$ .

Using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}I &\leq \sum_x \left( \sum_{y \sim x} c(x, y) |f(y)|^2 \right)^{\frac{1}{2}} \left( \sum_{y \sim x} c(x, y) \right)^{\frac{1}{2}} |g(x)| \\ &= \sum_x \left( \sum_{y \sim x} c(x, y) f^2(y) \right)^{\frac{1}{2}} (\tilde{c}(x))^{\frac{1}{2}} |g(x)| \\ &\leq \left( \sum_x \sum_{y \sim x} c(x, y) f^2(y) \right)^{\frac{1}{2}} \left( \sum_x \tilde{c}(x) g^2(x) \right)^{\frac{1}{2}} \\ &= \left( \sum_y \tilde{c}(y) f^2(y) \right)^{\frac{1}{2}} \left( \sum_x \tilde{c}(x) g^2(x) \right)^{\frac{1}{2}} \\ (2.3) \quad &= \|f\|_{\mathcal{V}} \|g\|_{\mathcal{V}}.\end{aligned}$$

Therefore, (2.2) and (2.3) gives

$$|\langle \Delta_0 f, g \rangle_{\mathcal{V}}| \leq 2 \|f\|_{\mathcal{V}} \|g\|_{\mathcal{V}}.$$

But by the definition of the norm of operator, we have

$$\|\Delta_0\| = \sup_{\|f\|=1} \|\Delta_0 f\|_{\mathcal{V}} = \sup_{\|f\|=1} \sup_{\|g\|=1} \langle \Delta_0 f, g \rangle_{\mathcal{V}}$$

So  $\|\Delta_0\| \leq 2$ , which shows that  $\Delta_0$  is a bounded operator .

Now, we want to prove the selfadjointness of the operator  $\Delta_0$  defined on  $l^2(\mathcal{V})$ . As  $\Delta_0$  is a bounded operator on  $l^2(\mathcal{V})$ , it remains to show that  $\Delta_0$  is symmetric.

As we have  $\Delta_0 = \delta d$  and  $\delta$  is the adjoint operator of  $d$ , we obtain for  $f$  and  $g \in l^2(\mathcal{V})$

$$\begin{aligned} \langle \Delta_0 f, g \rangle_{\mathcal{V}} &= \langle \delta d f, g \rangle_{\mathcal{V}} \\ &= \langle d f, d g \rangle_{\mathcal{E}} \\ &= \langle f, \delta d g \rangle_{\mathcal{V}} \\ &= \langle f, \Delta_0 g \rangle_{\mathcal{V}}. \end{aligned}$$

□

**Remark 2.2.** • *The operators  $d$  and  $\delta$  are bounded. Indeed, using the inequality  $(a - b)^2 \leq 2(a^2 + b^2)$  and the definition of the weights on vertices:  $\tilde{c}(x) = \sum_{y \sim x} c(x, y)$ , we obtain*

$$\begin{aligned} \|d f\|_{\mathcal{E}}^2 &= \frac{1}{2} \sum_{(x,y) \in \mathcal{E}} c(x, y) (d f(x, y))^2 \\ &= \frac{1}{2} \sum_{(x,y) \in \mathcal{E}} c(x, y) (f(y) - f(x))^2 \\ &\leq \sum_{(x,y) \in \mathcal{E}} c(x, y) (f^2(y) + f^2(x)) \\ &= 2 \sum_{x \in \mathcal{V}} f^2(x) \sum_{y \sim x} c(x, y) \\ &= 2 \sum_{x \in \mathcal{V}} f^2(x) \tilde{c}(x) \\ &= 2 \|f\|_{\mathcal{V}}^2. \end{aligned}$$

*So  $d$  is bounded, and the same is true for the adjoint  $\delta$ .*

*Notice that since  $\Delta_0$  is the composite operator of  $\delta$  and  $d$ ; this gives another proof that  $\Delta_0$  is bounded.*

- *It is easy to see that  $\Delta_0$  is also positive, since  $\langle \Delta_0 f, f \rangle_{\mathcal{V}} = \langle d f, d f \rangle_{\mathcal{E}} \geq 0$ .*

**Corollary 2.1.** *As the operator  $\Delta_0$  is self-adjoint and positive, its spectrum is real and lies in  $[0, 2]$ .*

**2.5. Weyl's criterion.** As our operator is bounded and self-adjoint on a Hilbert space, we can use Weyl's criterion [14] to characterize its spectrum.

Weyl's criterion : Let  $\mathcal{H}$  be a separable Hilbert space, and let  $\Delta$  be a bounded self-adjoint operator on  $\mathcal{H}$ . Then  $\lambda$  is in the spectrum of  $\Delta$  if and only if there exists a sequence  $(f_n)_{n \in \mathbb{N}}$  so that  $\|f_n\| = 1$  and  $\lim_{n \rightarrow \infty} \|(\Delta - \lambda)f_n\| = 0$ .

We denote  $\sigma(\Delta)$  the spectrum of  $\Delta$  and we set

- $\sigma_d(\Delta)$  is the set of  $\lambda \in \sigma(\Delta)$  which is an isolated point and an eigenvalue with finite multiplicity.
- $\sigma_{ess}(\Delta) := \sigma(\Delta) \setminus \sigma_d(\Delta)$ .

3. THE RELATION BETWEEN THE SPECTRUM OF  $\Delta_0$  AND  $\Delta_1$ 

3.1. **The nonzero spectrum of  $\Delta_0$  and  $\Delta_1$ .** In this section we will prove the relation between the spectrum of  $\Delta_0$  and that of  $\Delta_1$ , by using Weyl's criterion.

Following [15] and [18] we have the next lemma.

**Lemma 3.1.** *Let  $\Delta_0 = \delta d$  and  $\Delta_1 = d\delta$ . Then we have*

$$(1) \quad d\Delta_0 = \Delta_1 d.$$

$$(2) \quad \delta\Delta_1 = \Delta_0 \delta.$$

**Lemma 3.2.**

$$(1) \quad \ker \Delta_0 = \ker d.$$

$$(2) \quad \ker \Delta_1 = \ker \delta.$$

**Proof**

(1) Clearly, we have  $\ker d \subset \ker \Delta_0$ .

On the other hand, if  $\Delta_0 f = 0$  for  $f \in l^2(\mathcal{V})$  and  $f \neq 0$ , we have

$$0 = \langle \Delta_0 f, f \rangle_{\mathcal{V}} = \langle df, df \rangle_{\mathcal{E}}.$$

Then  $df = 0$  for  $f \in l^2(\mathcal{V})$ .

(2) If  $\varphi \in \ker \delta$ , then  $\varphi \in l^2(\mathcal{E})$  and  $\delta\varphi = 0$ . Thus,  $d\delta\varphi = 0$  and we obtain  $\varphi \in \ker \Delta_1$ .

For the other inclusion, let  $\varphi \in l^2(\mathcal{V})$ ,  $\varphi \neq 0$  such that  $\Delta_1\varphi = 0$ . Then

$$0 = \langle \Delta_1\varphi, \varphi \rangle_{\mathcal{E}} = \langle \delta\varphi, \delta\varphi \rangle_{\mathcal{E}}.$$

We get  $\delta\varphi = 0$  and as a result  $\ker \Delta_1 \subset \ker \delta$ .

□

We arrive at our main result.

**Theorem 1.**

$$\sigma(\Delta_1) \setminus \{0\} = \sigma(\Delta_0) \setminus \{0\}.$$

**Proof**

- Let  $\lambda \neq 0$  be in the spectrum of  $\Delta_0$ . By Weyl's criterion, there exists a sequence  $(f_n)_n$  of  $l^2(\mathcal{V})$  such that

$$\|f_n\|_{\mathcal{V}} = 1 \text{ and } \lim_{n \rightarrow \infty} \|(\Delta_0 - \lambda)f_n\|_{\mathcal{V}} = 0.$$

We want to find a sequence  $(\varphi_n)_n$  of  $l^2(\mathcal{E})$  such that

$$\|\varphi_n\|_{\mathcal{E}} = 1 \text{ and } \lim_{n \rightarrow \infty} \|(\Delta_1 - \lambda)\varphi_n\|_{\mathcal{E}} = 0.$$

We set

$$\varphi_n := \frac{df_n}{\|df_n\|_{\mathcal{E}}}.$$



First, let us check that  $\|df_n\|_{\mathcal{E}} \neq 0$ . We have

$$\begin{aligned} \|df_n\|_{\mathcal{E}}^2 &= \langle \Delta_0 f_n, f_n \rangle_{\mathcal{V}} \\ &= \langle (\Delta_0 - \lambda) f_n, f_n \rangle_{\mathcal{V}} + \langle \lambda f_n, f_n \rangle_{\mathcal{V}} \\ &= \underbrace{\langle (\Delta_0 - \lambda) f_n, f_n \rangle_{\mathcal{V}}}_{\text{converges to 0}} + \lambda. \end{aligned}$$

Then,  $\lim_{n \rightarrow \infty} \|df_n\|_{\mathcal{E}}^2 = \lambda$ . Thus, by positivity of  $\Delta_0$ , there exists  $A > 0$  and an integer  $n_0$  such that for all  $n \geq n_0$ , we have  $\|df_n\|_{\mathcal{E}} > A$ . This implies that the sequence  $(\varphi_n)_n$  is well defined.

Now, we verify that  $\lim_{n \rightarrow \infty} \|(\Delta_1 - \lambda)\varphi_n\|_{\mathcal{E}} = 0$ . By the first assertion of Lemma 3.1 and the fact that the operator  $d$  is bounded, we obtain for all  $n$  sufficiently large

$$\begin{aligned} \|(\Delta_1 - \lambda)\varphi_n\|_{\mathcal{E}} &= \left\| (\Delta_1 - \lambda) \frac{df_n}{\|df_n\|_{\mathcal{E}}} \right\|_{\mathcal{E}} \\ &= \frac{\|(\Delta_1 - \lambda)df_n\|_{\mathcal{E}}}{\|df_n\|_{\mathcal{E}}} \\ &= \frac{\|d(\Delta_0 - \lambda)f_n\|_{\mathcal{E}}}{\|df_n\|_{\mathcal{E}}} \\ &\leq \frac{\|d\|}{A} \|(\Delta_0 - \lambda)f_n\|_{\mathcal{V}}. \end{aligned}$$

But  $\lim_{n \rightarrow \infty} \|(\Delta_0 - \lambda)f_n\|_{\mathcal{V}} = 0$ . Therefore,  $\lim_{n \rightarrow \infty} \|(\Delta_1 - \lambda)\varphi_n\|_{\mathcal{E}} = 0$  and we can conclude that  $\lambda$  is in the spectrum of  $\Delta_1 \setminus \{0\}$ .

- The second part of the proof follows in the same fashion, with the roles of  $d$  and  $\delta$  swapped. □

There is a second method to prove Theorem 1 when 0 is not in the spectrum of  $\Delta_0$ .

**Lemma 3.3.** *If 0 is not in the spectrum of  $\Delta_0$ , then the operator  $d$  defined in  $l^2(\mathcal{V})$  has a closed range.*

**Proof**

Let  $\varphi \in \overline{\text{Im } d}$ , let us check that  $\varphi \in \text{Im } d$ , that means we look for a function  $f \in l^2(\mathcal{V})$  such that  $\varphi = df$ . We have  $\varphi \in \overline{\text{Im } d}$ , so there exists a sequence  $(\varphi_n)_n$  of  $\text{Im } d$  such that  $\varphi_n = df_n$ , for  $f_n \in l^2(\mathcal{V})$ . Moreover, the sequence  $(\varphi_n)_n$  converges to  $\varphi$  in  $l^2(\mathcal{E})$ . On the other hand, by assumption 0 is not in the spectrum of  $\Delta_0$  which implies the existence of a positive constant  $C$  such that

$$\|f\|_{\mathcal{V}} \leq C \|\Delta_0 f\|_{\mathcal{V}}, \quad \forall f \in l^2(\mathcal{V}).$$

But by the definition of the operator norm and Remark 2.2, we obtain

$$\|\Delta_0 f\| = \sup_{g, \|g\|_{\mathcal{V}}=1} \langle \Delta_0 f, g \rangle_{\mathcal{V}} \leq \|df\|_{\mathcal{E}} \sup_{g, \|g\|_{\mathcal{V}}=1} \|dg\|_{\mathcal{E}} \leq \sqrt{2} \|df\|_{\mathcal{E}}.$$

Then

$$\|f\|_{\mathcal{V}} \leq \sqrt{2} C \|df\|_{\mathcal{E}}, \quad \forall f \in l^2(\mathcal{V}).$$

Thus

$$\|f_n - f_m\|_{\mathcal{V}} \leq \sqrt{2} C \|df_n - df_m\|_{\mathcal{E}}, \quad f_n, f_m \in l^2(\mathcal{V}).$$

And

$$\|f_n - f_m\|_{\mathcal{V}} \leq \sqrt{2} C \|\varphi_n - \varphi_m\|_{\mathcal{E}}, \quad f_n, f_m \in l^2(\mathcal{V}).$$

As the sequence  $(\varphi_n)_n$  converges, so it is a Cauchy sequence and also  $(f_n)_n$  is a Cauchy sequence in  $l^2(\mathcal{V})$  which is complete. Then,  $(f_n)_n$  converges to  $f$ . By the boundedness of the operator  $d$ , we obtain  $df_n = \varphi_n$  converges to  $df$  and by uniqueness of the limit, we have  $df = \varphi$ . So  $\varphi$  is in  $\text{Im } d$ .  $\square$

**Corollary 3.1.** *If 0 is not in the spectrum of  $\Delta_0$ , then*

$$\sigma(\Delta_1|_{\text{Im } d}) = \sigma(\Delta_0).$$

**Proof**

By the first assertion of Lemma 3.1, we obtain

$$\Delta_1 d = d \Delta_0.$$

But by assumption 0 is not in the spectrum of  $\Delta_0$ . Then by the first assertion of Lemma 3.1, the operator  $d$  is invertible. So we obtain

$$\Delta_1|_{\text{Im } d} = d \Delta_0 d^{-1}.$$

Thus,

$$\sigma(\Delta_1|_{\text{Im } d}) = \sigma(\Delta_0).$$

$\square$

**3.2. The 0-spectrum of  $\Delta_0$  and  $\Delta_1$ .** As the nonzero spectrum of  $\Delta_0$  and  $\Delta_1$  are the same, we are interested in characterizing the 0-spectrum. We give in the following an extension of a result of John Lott's [11] (Proposition 9, p. 12).

**Theorem 2.** *Let  $(G, c)$  be a connected, locally finite and weighted infinite graph such that the weight on edges  $c$  is bounded, i.e., there exists a constant  $\alpha > 0$  such that  $\frac{1}{\alpha} \leq c(x, y) \leq \alpha$ , for all  $(x, y) \in \mathcal{E}$ . Then*

$$0 \in \sigma(\Delta_1) \text{ or } 0 \in \sigma(\Delta_0).$$

First, we start with preliminary results.

By [17] (page 44) and [9] (chapter 4) we have the next definition.

**Definition 3.1.** *The graph  $G$  verifies the isoperimetric inequality if there exists a constant  $C > 0$  such that for all finite sub-graphs  $G_U = (U, \mathcal{E}_U)$  of  $G$ , we have*

$$|\partial \mathcal{E}_U| \geq C |U|,$$

where

$$|\partial \mathcal{E}_U| = \sum_{x \in U} \sum_{y \notin U} c(x, y) \text{ and } |U| = \sum_{x \in U} \tilde{c}(x).$$

**Lemma 3.4.** *If  $\Delta_0$  is invertible then the isoperimetric inequality holds.*

**Proof**

Let  $U$  a finite sub-graph of  $G$ . Let us set  $g = \mathbf{1}_U$ , meaning that  $g(x) = 1$  if  $x \in U$  and  $g(x) = 0$  if  $x \notin U$ . Then we obtain

$$|U| = \sum_{x \in U} \tilde{c}(x) = \|g\|_{\mathcal{V}}^2$$

and

$$|\partial \mathcal{E}_U| = \sum_{x \in U} \sum_{y \notin U} c(x, y) = \|dg\|_{\mathcal{E}}^2.$$

By assumption 0 is not in the spectrum of  $\Delta_0$ . Then by the first assertion of Lemma 3.1, the operator  $d$  is invertible, so there exists a positive constant  $\lambda$  so that

$$\|g\|_{\mathcal{V}} \leq \lambda \|dg\|_{\mathcal{E}}, \quad \forall g \in l^2(\mathcal{V}).$$

Thus, it follows that

$$|\partial\mathcal{E}_U| \geq C |U|, \quad \text{with } C = \frac{1}{\lambda^2}.$$

□

**Definition 3.2.** • A branch  $B$  is a finite sequence of vertices  $x_0, x_1, \dots, x_{m+1}$  such that for all  $j; 1 \leq j \leq m$ , we have  $\deg(x_j) = 2$ .

• The length of a branch  $B$ , denoted  $\text{long}(B)$ , is the number of vertices in this branch, here,  $\text{long}(B) = m + 2$ .

• The interior of the branch  $B$  is the set of vertices  $x_j$  of  $B$  satisfying the following conditions:

i)  $\deg(x_j) = 2$ .

ii)  $\forall y \in \mathcal{V}; y \sim x_j \Rightarrow y \in B$ .

See [5] and [19] for the definition of the interior set of vertices.

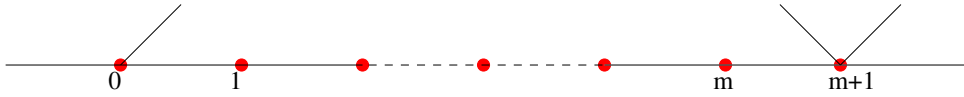


FIGURE 1. A branch of length  $m + 2$

Instead of the argument of Lott [11] inspired by Gromov [10] (p. 236-237), we use the following lemma:

**Lemma 3.5.** We suppose that the following conditions are satisfied:

- The weight on edges  $c$  is bounded, i.e., there exists a constant  $\alpha > 0$  such that  $\frac{1}{\alpha} \leq c(x, y) \leq \alpha$ ,  $\forall (x, y) \in \mathcal{E}$ .
- The operator  $\Delta_0$  is invertible.
- The operator  $\Delta_1$  is injective.

Then the graph  $(G, c)$  is a tree which contains branches with uniformly bounded lengths, that means  $\exists M > 0, \forall B$  branch of  $G, \text{long}(B) \leq M$ .

**Proof**

On the one hand, the operator  $\Delta_1$  is injective which leads to the absence of cycles in the graph, so that  $G$  is a tree.

On the other hand, the operator  $\Delta_0$  is invertible, then the isoperimetric inequality is checked, by Lemma 3.4 there is a positive constant  $C$  such that for all finite sub-graphs  $U$ , we have

$$|\partial\mathcal{E}_U| \geq C |U|.$$

Let  $B$  be a branch with vertices  $x_0, x_1, \dots, x_m, x_{m+1}$ . We set  $U = \{x_1, \dots, x_m\}$  the interior of the branch  $B$ , then

$$(3.4) \quad c(x_0, x_1) + c(x_m, x_{m+1}) \geq C \sum_{j=1}^m \tilde{c}(x_j).$$

For the sake of simplicity, we first prove the lemma for the case of the constant weight  $c = 1$ , before handling the case of general weights.

- If  $c = 1$ , then we have  $\tilde{c}(x) = \sum_{y \sim x} c(x, y) = \sum_{y \sim x} 1 = \deg(x)$ ,  $\forall x \in \mathcal{V}$  (this is J. Lott's case [11]). Therefore, the inequality (3.4) and Definition 3.2 gives

$$2 \geq C \sum_{j=1}^m \deg(x_j) = C \sum_{j=1}^m 2 = 2Cm.$$

We set  $M := \frac{1}{C} + 2$  (independent of  $B$ ), then  $\text{long}(B) \leq M$ . Thus the lengths of branches of  $G$  are uniformly bounded.

- If  $c \neq 1$  but  $c$  is bounded, that means there exists  $\alpha > 0$  such that  $\frac{1}{\alpha} \leq c(x, y) \leq \alpha$ , for all  $(x, y) \in \mathcal{E}$ . And as we have the weight on the vertices is  $\tilde{c}(x) = \sum_{y \sim x} c(x, y)$ , we obtain that  $\tilde{c}$  is also bounded from below by  $\frac{1}{\alpha}$ .

By the inequality (3.4) we have

$$2\alpha \geq c(x_0, x_1) + c(x_m, x_{m+1}) \geq C \sum_{j=1}^m \tilde{c}(x_j) \geq Cm \frac{1}{\alpha}.$$

Hence,

$$\frac{2\alpha^2}{C} \geq m.$$

We set  $M = \frac{2\alpha^2}{C} + 2$  (independent of  $B$ ), then  $\text{long}(B) \leq M$ . Thus, the lengths of the branches of  $G$  are uniformly bounded. □

Now, we arrive to the proof of Theorem 2.

### Proof

Taking the arguments from [11], we argue by contradiction. Suppose that both operators  $\Delta_0$  and  $\Delta_1$  are invertible. Then, by Lemma 3.5, the graph  $G$  is a tree which contains branches with uniformly bounded lengths; see Figure 2 for an example.

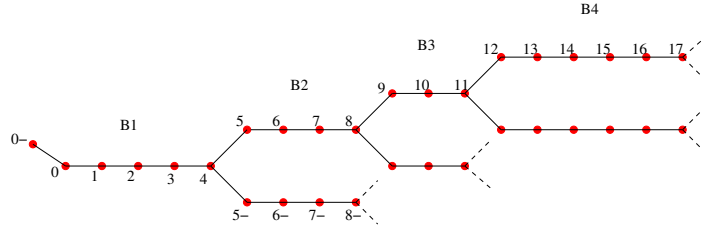


FIGURE 2. A branch tree

But the existence of such tree gives a  $\delta$ -harmonic nonzero square-integral function  $\varphi$ . Indeed: we consider a part of the branch tree in Figure 2 as an example to simplify the understanding of the construction. For the sake of simplicity, we first prove the theorem for the case of the constant weight  $c = 1$ , before handling the case of general weights.

**First case:**  $c = 1$ , we fix a vertex  $0$  as the origin of the tree and we set  $0^-$  and  $1$  its different neighbors. Let us take

$$\varphi(0, 0^-) = \varphi(0, 1) = 1.$$

Then, we obtain  $\delta\varphi(0) = 0$  (the tree is oriented).

Afterwards on the branch  $B_1$ ,  $\varphi$  is constant, in other words,  $\varphi(j, j+1) = 1$ , for all  $j$ , such that  $1 \leq j \leq 3$ . And at the point  $4$ , we have  $\varphi(4, 5) = \varphi(4, 5^-) = \frac{1}{2}$ . It is claimed that  $\delta\varphi(4) = 0$ .

And for the points which are in the branch  $B_2$ , the function  $\varphi$  is constant and takes the value  $\frac{1}{2}$ . And so on to the point 8, we have  $\varphi(8, 9) = \varphi(8, 9^-) = \frac{1}{4}$ , to obtain  $\delta\varphi(8) = 0$ . And for the points which are in the branch  $B_3$ , the function  $\varphi$  is constant and takes the value  $\frac{1}{4}$ . And we continue in this way...

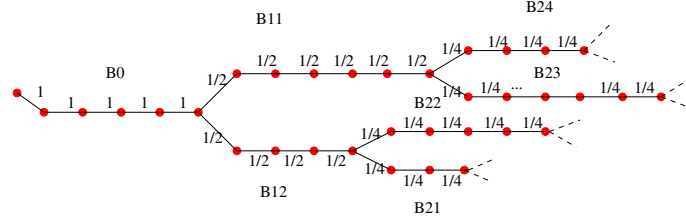


FIGURE 3. An example of a branch tree

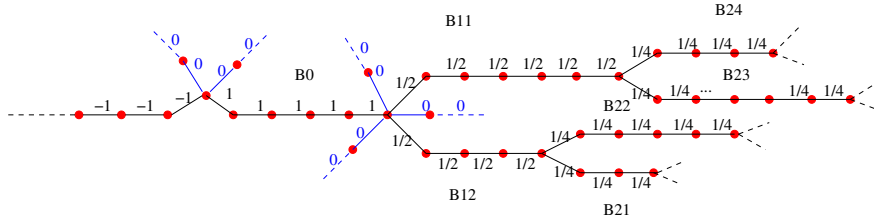


FIGURE 4. Another example of a branch tree

In a general way,  $G$  is a tree which contains branches with uniformly bounded lengths and we construct a functions  $\varphi$  in a part of  $G$  by selecting always two branches at bifurcation points and at all edges that occur on other branches,  $\varphi$  is set to zero, as in Figure 4. In the Figure 3, the construction of  $\varphi$  is done in the following way: on  $B_0$  the function  $\varphi$  is constant and equals to 1. Then we add a generation, we get two branches  $B_{1,1}$  and  $B_{1,2}$  such that the function  $\varphi$  takes the value  $\frac{1}{2}$ . And to the generation  $m$ , we have  $B_{m,k}$  branches, where  $1 \leq k \leq 2^m$ , then the function  $\varphi$  is equal to  $\frac{1}{2^m}$ . As a result, we show that this construction of  $\varphi$  is in  $l^2(\mathcal{E})$ . Using the fact that the lengths of the branches of the tree are uniformly bounded by a constant  $M > 0$ , we obtain

$$\begin{aligned}
 \|\varphi\|_{\mathcal{E}}^2 &= \frac{1}{2} \sum_{m \geq 0} \sum_{k=1}^{2^m} \sum_{e \in B_{m,k}} (\varphi(e))^2 \\
 &= \frac{1}{2} \sum_{m \geq 0} \sum_{k=1}^{2^m} \sum_{e \in B_{m,k}} \left(\frac{1}{2^m}\right)^2 \\
 &\leq \frac{1}{2} \sum_{m \geq 0} 2^m M \left(\frac{1}{2^m}\right)^2 \\
 &= \frac{M}{2} \sum_{m \geq 0} \frac{1}{2^m} < \infty.
 \end{aligned}$$

**Second case:**  $c \neq 1$  but  $c$  bounded by a positive constant. That means there exists  $\alpha > 0$  such that  $\frac{1}{\alpha} \leq c(x, y) \leq \alpha$ ,  $\forall (x, y)$  in  $\mathcal{E}$ . As in Figure 2, on the branch  $B_1$ , the vertex 0 has two neighbors denoted  $0^-$  and 1. We want  $\delta\varphi(0) = 0$ , so we choose the function  $\varphi$  in the following way  $\varphi(0, 1) =$

$\frac{c(0,0^-)}{c(0,1)}\varphi(0,0^-)$ . And in the interior of  $B_1$ , we set  $\varphi(j, j+1) = \frac{c(0,0^-)}{c(j,j+1)}\varphi(0,0^-) \forall j, 1 \leq j \leq 3$ . Next, we look at the point 4 which has two neighbors 5 and  $5^-$ , to obtain  $\delta\varphi(4) = 0$  and as we have  $\varphi(3,4) = \frac{c(0,0^-)}{c(3,4)}\varphi(0,0^-)$ . We choose  $\varphi(4,5) = \frac{c(0,0^-)}{2c(4,5)}\varphi(0,0^-)$  and  $\varphi(4,5^-) = \frac{c(0,0^-)}{2c(4,5^-)}\varphi(0,0^-)$ . Therefore, in the interior of the branch  $B_2$ ,

$$\varphi(j, j+1) = \frac{c(0,0^-)}{2c(j,j+1)}\varphi(0,0^-) \forall j, 5 \leq j \leq 7.$$

And for the vertex 8, which has two neighbors 9 and  $9^-$ . To have  $\delta\varphi(8) = 0$  and by using that  $\varphi(7,8) = \frac{c(0,0^-)}{2c(7,8)}\varphi(0,0^-)$ . We choose  $\varphi(8,9) = \frac{c(0,0^-)}{4c(8,9)}\varphi(0,0^-)$  and  $\varphi(8,9^-) = \frac{c(0,0^-)}{4c(8,9^-)}\varphi(0,0^-)$ . And in the interior of the branch  $B_3$ ,

$$\varphi(j, j+1) = \frac{c(0,0^-)}{4c(j,j+1)}\varphi(0,0^-) \text{ for } j = 10.$$

And so on... In a general way, see Figure 3, on  $B_0$  the function  $\varphi(e_0) = \frac{c(0,0^-)}{c(e_0)}\varphi(0,0^-)$ , where  $e_0$  is an edge of  $B_0$ . Then we add a generation, we get two branches  $B_{1,1}$  and  $B_{1,2}$  such that the function  $\varphi$  has a value  $\varphi(e_1^k) = \frac{c(0,0^-)}{2c(e_1^k)}\varphi(0,0^-)$ , where  $e_1^k$  denotes the edges of  $B_{1,k}$  for  $1 \leq k \leq 2$ . And at generation  $m$ , we have  $B_{m,k}$  branches, where  $1 \leq k \leq 2^m$ , then the function  $\varphi$  equals to  $\varphi(e_m^k) = \frac{c(0,0^-)}{2^m c(e_m^k)}\varphi(0,0^-)$ , where  $e_m^k$  denotes the edges of  $B_{m,k}$ . And to simplify the formulas, we can suppose that

$$\varphi(0,0^-) = \frac{1}{c(0,0^-)}.$$

Then, we obtain

$$\varphi(e_m^k) = \frac{1}{2^m c(e_m^k)}, \forall m \geq 0 \text{ and } 1 \leq k \leq 2^m.$$

Therefore, this construction gives  $\varphi \in l^2(\mathcal{E})$ . Using the fact that the lengths of the branches of the tree are uniformly bounded by a constant  $M > 0$  and the weight  $c$  on the edges is bounded by a positive constant, we obtain

$$\begin{aligned} \|\varphi\|_{\mathcal{E}}^2 &= \frac{1}{2} \sum_e c(e)(\varphi(e))^2 \\ &\leq \sum_{m \geq 0} \sum_{k=1}^{2^m} \sum_{e \in B_{m,k}} c(e)(\varphi(e))^2 \\ &= \sum_{m \geq 0} \sum_{k=1}^{2^m} \sum_{e \in B_{m,k}} c(e) \left( \frac{1}{2^m c(e)} \right)^2 \\ &= \sum_{m \geq 0} \sum_{k=1}^{2^m} \sum_{e \in B_{m,k}} \frac{1}{2^{2m} c(e)} \\ &\leq \alpha M \sum_{m \geq 0} \frac{1}{2^m} \\ &= 2M\alpha < \infty. \end{aligned}$$

Finally, we have  $\varphi$  in  $l^2(\mathcal{E})$  and  $\delta$ -harmonic. So,  $0 \in \sigma(\Delta_1)$ , which contradicts the assumption that 0 is not in the spectrum of  $\Delta_1$ .  $\square$

**Remark 3.1.** Any point of the graph can play the role of the first vertex 0 in the previous construction. It is then clear that we can construct an infinite family of independent functions  $\varphi$  which are in  $l^2(\mathcal{E})$  and  $\delta$ -harmonic. Then 0 is an eigenvalue of  $\Delta_1$  with infinite multiplicity, so  $0 \in \sigma_{ess}(\Delta_1)$ .

#### 4. EXAMPLES:

In this section, we will construct a  $\delta$ -harmonic function  $\varphi$  in different examples of trees.

1) **Symmetric tree:** Following [7] we introduce the next definition:

**Definition 4.1.** A tree  $T_s$  is symmetric around  $o$  with branching numbers  $\{m_i\}_{i=0}^{\infty}$  if the degree of each vertex depends only on its distance from  $o$ , namely, for each  $x \in T_s$ ,  $\deg(x) = m_i$  if  $d(o, x) = i$ .

##### Example of a symmetric tree:

We fix a vertex  $o$  as an origin of the tree. We set  $S_n = \{x \in T_s; d(o, x) = n\}$ .  $T_s$  is symmetric around  $o$  with branching numbers  $\{m_n\}_{n=0}^{\infty}$ . In Figure 4, we choose  $m_n = 3 + n$  for all  $n \in \mathbb{N}$  which is an increasing sequence. So, we have  $m_0 = 3$  that means  $\deg(o) = 3$ . And for  $x \in S_1$ , we obtain  $\deg(x) = m_1 = 4$ . In the same way, if  $x \in S_2$  we have  $m_2 = 5$  and so on.

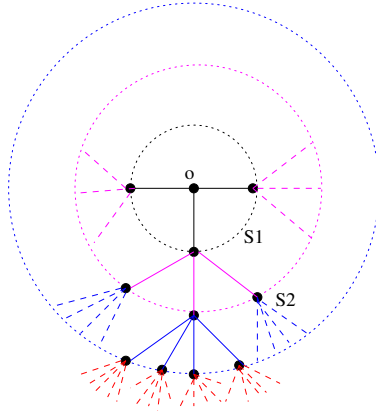


FIGURE 5. Symmetric tree

**Proposition 4.1.** If the symmetric tree  $T_s$  is simple ( the edge weights are equal to 1) with  $\deg(x) > 2$  for all  $x \in T_s$ , then there is a  $\delta$ -harmonic function  $\varphi \in l^2(\mathcal{E})$ .

##### Proof

We fix a vertex  $x_0$  as an origin of the tree  $T_s$ , we can find an increasing sequence of finite subgraph  $\{S_n\}_n$  such that  $S_n = \{x \in T_s; d(x_0, x) = n\}$  and  $T_s = \cup_n S_n$ . By the definition of the symmetric tree, we have for all  $n$   $\deg(x_n) = m_n, \forall x_n \in S_n$ . First, we construct a function  $\varphi$  so that  $\delta\varphi = 0$  as follows: Let  $e_0$  and  $b_0$  denote two distinct outward edges connecting to the vertex  $x_0$ . We define  $\varphi$  to be 0 excepted on these edges where  $\varphi(e_0) = 1$  and  $\varphi(b_0) = -1$  which gives  $\delta\varphi(x_0) = 0$ . And denote  $e_n^k, n \geq 1, 1 \leq k \leq \prod_{j=1}^n (m_j - 1)$ , resp.  $b_n^k, n \geq 1, 1 \leq k \leq \prod_{j=1}^n (m_j - 1)$ , the outward edges emanating from  $e_0$ , resp.  $b_0$ , of generation  $n$ . We define

$$\varphi(e_n^k) = \frac{1}{\prod_{j=1}^n (m_j - 1)} \varphi(e_0),$$

$$\varphi(b_n^k) = \frac{1}{\prod_{j=1}^n (m_j - 1)} \varphi(b_0)$$

and  $\varphi$  takes value 0 on all edges other than  $e_n^k$  and  $b_n^k$ .

Second, through this construction, we look for  $\varphi \in l^2(\mathcal{E})$ . Using the fact that  $\deg(x_n) = m_n \geq 3$ ,  $\forall x_n \in S_n, \forall n$ , we obtain

$$\begin{aligned}
\|\varphi\|_{\mathcal{E}}^2 &= \frac{1}{2} \sum_{e \in \mathcal{E}} \varphi^2(e) \\
&= \frac{1}{2} \left( 2 + \sum_{n \geq 1} \sum_{k=1}^{\prod_{j=1}^n (m_j - 1)} \varphi^2(e_n^k) + \varphi^2(b_n^k) \right) \\
&= 1 + \sum_{n \geq 1} \sum_{k=1}^{\prod_{j=1}^n (m_j - 1)} \left( \frac{1}{(m_1 - 1)(m_2 - 1) \dots (m_n - 1)} \right)^2 \\
&= 1 + \sum_{n \geq 1} \frac{1}{(m_1 - 1)(m_2 - 1) \dots (m_n - 1)} \\
&\leq 1 + \sum_{n \geq 1} \frac{1}{2^n} \\
&< \infty.
\end{aligned}$$

□

2) **Triadic tree with weights bounded from below:** As [2] (p. 19), we have the following definition of a triadic tree.

**Definition 4.2.** A tree is a connected graph containing no cycles. The **triadic tree** is a tree such that all the vertices have degree 3.

**Proposition 4.2.** If the triadic tree has weights on the edges bounded from below by a positive constant  $\lambda$ , then there is a  $\delta$ -harmonic function  $\varphi \in l^2(\mathcal{E})$ .

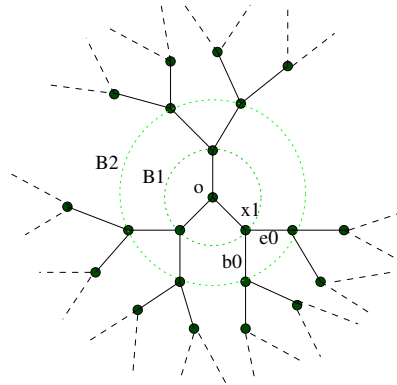


FIGURE 6. Triadic tree

### Proof

We fix a vertex  $o$  as the origin of the tree  $T$ . Define the increasing sequence of finite subgraphs  $\{G_n\}_n$ ,  $G_n = \{x \in \mathcal{V}; d(o, x) \leq n\}$  and let  $G = \bigcup_n G_n$ . Denote  $S_n = \{x \in T; d(o, x) = n\}$ . We set  $x_1^1, x_1^2$  and  $x_1^3$  the different neighbors of  $o$  which are in  $S_1$ . We suppose that  $\varphi(o, x_1^i) = 0$  for all  $i \in \{1, 2, 3\}$ , so we have  $\delta\varphi(o) = 0$ .



We fix one vertex of  $S_1$  for example  $x_1 := x_1^1$ , let  $e_0$  and  $b_0$  be the two outward edges of  $x_1$  and define inductively  $e_m^k$ ,  $m \geq 1$ ,  $1 \leq k \leq 2^m$ , resp.  $b_m^k$ ,  $m \geq 1$ ,  $1 \leq k \leq 2^m$ , to be the outward edges emanating from  $e_0$ , resp.  $b_0$ , of generation  $m$  (the edge are oriented outward). For  $m \geq 0$ , we define  $\varphi$  to be 0 excepted on these edges where

$$\varphi(e_m^k) = \frac{1}{2^m} \frac{1}{c(e_m^k)}, \quad \forall k; 1 \leq k \leq 2^m$$

and

$$\varphi(b_m^k) = \frac{-1}{2^m} \frac{1}{c(b_m^k)}, \quad \forall k; 1 \leq k \leq 2^m.$$

With this construction, we obtain for each  $x_n \in S_n$ ,  $\delta\varphi(x_n) = 0$ ,  $\forall n \geq 1$ . Moreover,  $\varphi \in l^2(\mathcal{E})$ . Indeed: by using the assumption that the weights on the edges are bounded from below by a positive constant  $\lambda$ , we obtain

$$\begin{aligned} \|\varphi\|_{\mathcal{E}}^2 &= \frac{1}{2} \sum_{e \in \mathcal{E}} c(e) \varphi^2(e) \\ &= \frac{1}{2} \left( \sum_{m \geq 0} \sum_{k=1}^{2^m} c(e_m^k) \varphi^2(e_m^k) + c(b_m^k) \varphi^2(b_m^k) \right) \\ &= \frac{1}{2} \sum_{m \geq 0} \sum_{k=1}^{2^m} c(e_m^k) \frac{1}{2^{2m}} \frac{1}{c^2(e_m^k)} + \frac{1}{2} \sum_{m \geq 0} \sum_{k=1}^{2^m} c(b_m^k) \frac{1}{2^{2m}} \frac{1}{c^2(b_m^k)} \\ &= \frac{1}{2} \sum_{m \geq 0} \sum_{k=1}^{2^m} \frac{1}{2^{2m}} \frac{1}{c(e_m^k)} + \frac{1}{2} \sum_{m \geq 0} \sum_{k=1}^{2^m} \frac{1}{2^{2m}} \frac{1}{c(b_m^k)} \\ &\leq \lambda' \sum_{m \geq 0} \sum_{k=1}^{2^m} \frac{1}{2^{2m}} \\ &= \lambda' \sum_{m \geq 0} 2^m \frac{1}{2^{2m}} \\ &= \lambda' \sum_{m \geq 0} \frac{1}{2^m} \\ &= 2\lambda', \end{aligned}$$

where  $\lambda' = \frac{1}{\lambda}$ .

□

**Remark 4.1.** • *The construction of a  $\delta$ -harmonic nonzero square-integral function depends on the edge weights.*

- *In the simple triadic tree, 0 is both in the spectrum of  $\Delta_0$  [4] and in the spectrum of  $\Delta_1$  [2].*

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