Carleman commutator approach in logarithmic convexity for parabolic equations

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Abstract

In this paper we investigate on a new strategy combining the logarithmic convexity (or frequency function) and the Carleman commutator to obtain an observation estimate at one time for the heat equation in a bounded domain. We also consider the heat equation with an inverse square potential. Moreover, spectral inequality for the associated eigenvalue problem is derived.

1 Introduction and main results

When we mention the logarithmic convexity method for the heat equation in a bounded domain $\Omega \subset \mathbb{R}^n$:

$$\begin{cases} \partial_t u - \Delta u = 0 \ , & \text{in } \Omega \times (0,T) \ , \\ u = 0 \ , & \text{on } \partial\Omega \times (0,T) \ , \\ u(\cdot,0) = u_0 \in L^2(\Omega) \setminus \{0\} \ , & \end{cases}$$

we have in mind that $t \mapsto \ln \|u(\cdot,t)\|^2_{L^2(\Omega)}$ is a convex function by evaluating the sign of the derivative of $t \mapsto \dfrac{\int_\Omega |\nabla u(x,t)|^2 dx}{\int_\Omega |u(x,t)|^2 dx}$ (see [AN], [Pa, p.11], [I, p.43], [Ve]). As a consequence, the following well-known estimate holds. For any $0 \leq t \leq T$,

$$\left\| e^{t\Delta} u_0 \right\|_{L^2(\Omega)} \leq \left\| e^{T\Delta} u_0 \right\|^{t/T}_{L^2(\Omega)} \|u_0\|^{1-t/T}_{L^2(\Omega)} \ .$$

In a series of articles (see [PW1], [PW2], [PWZ], [BP] for parabolic equations) inspired by [Po] and [EFV], we were interested on the function $t \mapsto \int_\Omega |u(x,t)|^2 e^{\Phi(x,t)} dx$ and its frequency function $t \mapsto \dfrac{\int_\Omega |\nabla u(x,t)|^2 e^{\Phi(x,t)} dx}{\int_\Omega |u(x,t)|^2 e^{\Phi(x,t)} dx}$ when $e^{\Phi(x,t)} = \frac{1}{(T-t+\hbar)^{n/2}} e^{-\frac{|x-x_0|^2}{4(T-t+\hbar)}}$

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with $x_0 \in \Omega$, $\hbar > 0$. It provides us with an observation estimate at one point in time: For any $T > 0$ and any ω nonempty open subset of Ω,

$$\left\|e^{T\Delta}u_0\right\|_{L^2(\Omega)} \leq \left(ce^{\frac{K}{T}}\left\|e^{T\Delta}u_0\right\|_{L^2(\omega)}\right)^{\beta}\|u_0\|^{1-\beta}_{L^2(\Omega)} \ .$$

Here $c, K > 0$ and $\beta \in (0,1)$. From the above observation at one time, many applications were derived as bang-bang control [PW2] and impulse control [PWX], fast stabilization [PWX] or local backward reconstruction [Vo]. In particular, we can also deduce the observability estimate for parabolic equations on a positive measurable set in time [PW2]. Recall that observability for parabolic equations have a long history now from the works of [LR] and [FI] based on Carleman inequalities. Furthermore, it was remarked in [AEWZ] that the observation estimate at one point in time is equivalent to the Lebeau-Robbiano spectral inequality on the sum of eigenfunctions of the Dirichlet Laplacian. Recall that the Lebeau-Robbiano spectral inequality, originally derived from Carleman inequalities for elliptic equations (see [JL], [LRL], [Lu]), was used in different contexts as in thermoelasticity (see [LZ], [BN]), for the Stokes operator [CL], in transmission problem and coupled systems (see [Le], [LLR]), for the Bilaplacian (see [Ga], [EMZ], [LRR3]), in Kolmogorov equation (see [LRM], [Z]). We also refer to [M].

In this paper, we study the equation solved by $f(x,t) = u(x,t)\, e^{\frac{1}{2}\Phi(x,t)}$ for a larger set of weight functions $\Phi(x,t)$ and establish a kind of convexity property for $t \mapsto \ln\|f(\cdot,t)\|^2_{L^2(\Omega)}$. By such approach we make appear the Carleman commutator. The link between logarithmic convexity (or frequency function) and Carleman inequality has already appeared in [EKPV1] (see also [EKPV2], [EKPV3]).

Choosing suitable weight functions Φ (not necessary linked to the heat kernel) we obtain the following new results:

Theorem 1.1. *Suppose that $\Omega \subset \mathbb{R}^n$ is a convex domain or a star-shaped domain with respect to $x_0 \in \Omega$ such that $\{x; |x - x_0| < r\} \Subset \Omega$ for some $r \in (0,1)$. Then for any $u_0 \in L^2(\Omega), T > 0$, $(a_j)_{j\geq 1} \in \mathbb{R}$, $\lambda > 0$, $\varepsilon \in (0,1)$, one has*

$$\left\|e^{T\Delta}u_0\right\|_{L^2(\Omega)} \leq K_\varepsilon e^{\frac{K_\varepsilon}{r^\varepsilon}\frac{1}{T}}\int_0^T \left\|e^{t\Delta}u_0\right\|_{L^2(|x-x_0|<r)} dt$$

and

$$\sum_{\lambda_j \leq \lambda} |a_j|^2 \leq 4e^{\frac{K_\varepsilon}{r^\varepsilon}\sqrt{\lambda}}\int_{|x-x_0|<r}\left|\sum_{\lambda_j\leq\lambda} a_j e_j(x)\right|^2 dx$$

where $K_\varepsilon > 0$ is a constant only depending on $\left(\varepsilon, \max\left\{|x - x_0|; x \in \overline{\Omega}\right\}\right)$. Here (λ_j, e_j) denotes the eigenbasis of the Laplace operator with Dirichlet boundary condition.

Theorem 1.1 thus states both the observability for the heat equation and the spectral inequality for the Dirichlet Laplacian in a simple geometry. One can see how fast the constant cost blows up when the observation region ω becomes smaller. Notice that the constant K_ε does not depend on the dimension n (see [BP, Theorem 4.2]).

Theorem 1.2. *Let $n \geq 3$ and consider a C^2 bounded domain $\Omega \subset \mathbb{R}^n$ such that $0 \in \Omega$. Let $\omega \subset \Omega$ be a nonempty open set. Suppose that*

$$\mu \leq \left| \begin{array}{ll} \frac{7}{2 \cdot 3^3} \ , & \text{if } n = 3 \ , \\ \frac{1}{4}(n-1)(n-3) \ , & \text{if } n \geq 4 \ . \end{array} \right.$$

Then, there exist constants $c > 0$, $K > 0$ such that for any $(a_j)_{j \geq 1} \in \mathbb{R}$ and any $\lambda > 0$, we have

$$\sqrt{\sum_{\lambda_j \leq \lambda} |a_j|^2} \leq c e^{K\sqrt{\lambda}} \int_\omega \left| \sum_{\lambda_j \leq \lambda} a_j e_j(x) \right| dx$$

where (λ_j, e_j) denotes the eigenbasis of the Schrödinger operator $-\Delta - \frac{\mu}{|x|^2}$ with Dirichlet boundary condition

$$\left\{ \begin{array}{rll} -\Delta e_j - \frac{\mu}{|x|^2} e_j & = \lambda_j e_j \ , & \text{in } \Omega \ , \\ e_j & = 0 \ , & \text{on } \partial\Omega \ . \end{array} \right.$$

Theorem 1.2 gives a spectral inequality for the Schrödinger operator $-\Delta - \frac{\mu}{|x|^2}$ under a quite strong assumption on $\mu < \mu^*$ where the critical coefficient is $\mu^* = \frac{1}{4}(n-2)^2$. Our first motivation was to be able to choose $0 \notin \overline{\omega}$ by performing localization with annulus. We believe that a similar analysis can be handle with more suitable weight function Φ than those considered here and may considerably improve the results presented here.

We have organized our paper as follows. Section 2 is the important part of this article. We present the strategy to get the observation at one point by studying the equation solved by $f = ue^{\Phi/2}$ for a larger set of weight functions Φ adapting the energy estimates style of computations in [BT] (see also [BP, Section 4]). The Carleman commutator appears naturally here. Section 3 is devoted to check different possibilities for the weight function Φ, and in particular for the localization with annulus. In Section 4, we prove Theorem 1.1. The proof of Theorem 1.2 is given in Section 5. In Appendix, we recall the useful link between the observation at one point and the spectral inequality.

I am happy to dedicate this paper to my friend and colleague Jiongmin Yong on the occasion of his 60th birthday. I am also grateful for his book [LY] in where I often found the answer on my questions.

2 The strategy of logarithmic convexity with the Carleman commutator

We present an approach to get the observation estimate at one point in time for a model heat equation in a bounded domain $\Omega \subset \mathbb{R}^n$ with Dirichlet boundary condition. We shall present this strategy step-by-step. Two different geometric cases are discussed: When Ω is convex or star-shaped, we can used a global weight function; For the more general

C^2 domain Ω, we will use localized weight functions exploiting a covering argument and propagation of interpolation inequalities along a chain of balls (also called propagation of smallness).

2.1 Convex domain

Throughout this subsection, we assume that $\Omega \subset \mathbb{R}^n$ is a convex domain or a star-shaped domain with respect to $x_0 \in \Omega$. Let $\langle \cdot, \cdot \rangle$ denote the usual scalar product in $L^2(\Omega)$ and let $\|\cdot\|$ be its corresponding norm. Here, recall that $u(x,t) = e^{t\Delta} u_0(x) \in C([0,T]; L^2(\Omega)) \cap C((0,T]; H_0^1(\Omega))$ and we aim to check that

$$\|u(\cdot,T)\| \leq \left(c e^{\frac{K}{T}} \|u(\cdot,T)\|_{L^2(\omega)} \right)^{\beta} \|u(\cdot,0)\|^{1-\beta} \ .$$

The strategy to establish the above observation at one time is as follows. We decompose the proof into six steps.

Step 2.1.1. *Symmetric part and antisymmetric part.*

Let Φ be a sufficiently smooth function of $(x,t) \in \mathbb{R}^n \times \mathbb{R}_t$ and define

$$f(x,t) = u(x,t)\, e^{\Phi(x,t)/2} \ .$$

We look for the equation solved by f by computing $e^{\Phi(x,t)/2} (\partial_t - \Delta) \left(e^{-\Phi(x,t)/2} f(x,t) \right)$. We find that

$$\partial_t f - \Delta f - \frac{1}{2} f \left(\partial_t \Phi + \frac{1}{2} |\nabla \Phi|^2 \right) + \nabla \Phi \cdot \nabla f + \frac{1}{2} \Delta \Phi f = 0 \quad \text{in } \Omega \times (0,T) \ ,$$

and furthermore, $f_{|\partial\Omega} = 0$. Introduce

$$\begin{cases} \mathcal{A} f = -\nabla \Phi \cdot \nabla f - \frac{1}{2} \Delta \Phi f \ , \\ \mathcal{S} f = \Delta f + \eta f \text{ where } \eta = \frac{1}{2} \left(\partial_t \Phi + \frac{1}{2} |\nabla \Phi|^2 \right) \ . \end{cases}$$

We can check that

$$\begin{cases} \langle \mathcal{A} f, g \rangle = - \langle \mathcal{A} g, f \rangle \ , \\ \langle \mathcal{S} f, g \rangle = \langle \mathcal{S} g, f \rangle \text{ for any } g \in H_0^1(\Omega) \ . \end{cases}$$

Furthermore, we have

$$\partial_t f - \mathcal{S} f - \mathcal{A} f = 0 \ .$$

Step 2.1.2. *Energy estimates.*

Multiplying by f the above equation, integrating over Ω, we obtain that

$$\frac{1}{2} \frac{d}{dt} \|f\|^2 + \langle -\mathcal{S} f, f \rangle = 0 \ .$$

Introduce the frequency function $t \mapsto \mathbf{N}(t)$ defined by

$$\mathbf{N} = \frac{\langle -\mathcal{S}f, f\rangle}{\|f\|^2} \ .$$

Thus,

$$\frac{1}{2}\frac{d}{dt}\|f\|^2 + \mathbf{N}\|f\|^2 = 0 \ .$$

Now, we compute the derivative of $\mathbf{N}$ and claim that:

$$\frac{d}{dt}\mathbf{N} \leq \frac{1}{\|f\|^2}\langle -(\mathcal{S}' + [\mathcal{S}, \mathcal{A}])f, f\rangle - \frac{1}{\|f\|^2}\int_{\partial\Omega} \partial_\nu f \mathcal{A} f d\sigma \ .$$

Indeed,

$$\begin{aligned}
&\frac{d}{dt}\mathbf{N} \\
&= \frac{1}{\|f\|^4}\left(\frac{d}{dt}\langle -\mathcal{S}f, f\rangle \|f\|^2 + \langle \mathcal{S}f, f\rangle \frac{d}{dt}\|f\|^2\right) \\
&= \frac{1}{\|f\|^2}\left[\langle -\mathcal{S}'f, f\rangle - 2\langle \mathcal{S}f, f'\rangle\right] + \frac{2}{\|f\|^4}\langle \mathcal{S}f, f\rangle^2 \\
&= \frac{1}{\|f\|^2}\left[\langle -\mathcal{S}'f, f\rangle - 2\langle \mathcal{S}f, \mathcal{A}f\rangle\right] + \frac{2}{\|f\|^4}\left[-\|\mathcal{S}f\|^2\|f\|^2 + \langle \mathcal{S}f, f\rangle^2\right] \\
&= \frac{1}{\|f\|^2}\left[\langle -(\mathcal{S}' + [\mathcal{S}, \mathcal{A}])f, f\rangle - \int_{\partial\Omega}\partial_\nu f \mathcal{A} f d\sigma\right] + \frac{2}{\|f\|^4}\left[-\|\mathcal{S}f\|^2\|f\|^2 + \langle \mathcal{S}f, f\rangle^2\right] \\
&\leq \frac{1}{\|f\|^2}\left[\langle -(\mathcal{S}' + [\mathcal{S}, \mathcal{A}])f, f\rangle - \int_{\partial\Omega}\partial_\nu f \mathcal{A} f d\sigma\right] \ .
\end{aligned}$$

In the third line, we used $\frac{1}{2}\frac{d}{dt}\|f\|^2 + \langle -\mathcal{S}f, f\rangle = 0$; In the fourth line, multiplying the equation of f by $\mathcal{S}f$, and integrating over Ω, give $\langle \mathcal{S}f, f'\rangle = \|\mathcal{S}f\|^2 + \langle \mathcal{S}f, \mathcal{A}f\rangle$; In the fifth line, ∂_ν denotes the normal derivative to the boundary, and we used

$$\begin{aligned}
2\langle \mathcal{S}f, \mathcal{A}f\rangle &= \langle \mathcal{S}\mathcal{A}f, f\rangle - \langle \mathcal{A}\mathcal{S}f, f\rangle + \int_{\partial\Omega}\partial_\nu f \mathcal{A} f d\sigma \\
&:= \langle [\mathcal{S}, \mathcal{A}]f, f\rangle + \int_{\partial\Omega}\partial_\nu f \mathcal{A} f d\sigma \ ;
\end{aligned}$$

In the sixth line, we used Cauchy-Schwarz inequality.

Here we have followed the energy estimates style of computations in [BT] (see also [Ph, p.535]) The interested reader may wish here to compare with [EKPV1, Theorem 3].

Step 2.1.3. *Assumption on Carleman commutator.*

Assume that $\displaystyle\int_{\partial\Omega}\partial_\nu f \mathcal{A} f d\sigma \geq 0$ on $(0, T)$ by convexity or star-shaped property of Ω, and suppose that

$$\langle -(\mathcal{S}' + [\mathcal{S}, \mathcal{A}])f, f\rangle \leq \frac{1}{\Upsilon}\langle -\mathcal{S}f, f\rangle$$

on $(0,T)$ where $\Upsilon(t) = T - t + \hbar$ and $\hbar > 0$. Therefore the following differential inequalities hold.

$$\begin{cases} \frac{1}{2}\frac{d}{dt}\|f(\cdot,t)\|^2 + \mathbf{N}(t)\|f(\cdot,t)\|^2 = 0 \ , \\ \frac{d}{dt}\mathbf{N}(t) \leq \frac{1}{\Upsilon(t)}\mathbf{N}(t) \ . \end{cases}$$

By solving such system of differential inequalities, we obtain (see [BP, p.655]): For any $0 < t_1 < t_2 < t_3 \leq T$,

$$\left(\|f(\cdot,t_2)\|^2\right)^{1+M} \leq \left(\|f(\cdot,t_1)\|^2\right)^{M} \|f(\cdot,t_3)\|^2$$

where

$$M = \frac{-\ln(T-t_3+\hbar) + \ln(T-t_2+\hbar)}{-\ln(T-t_2+\hbar) + \ln(T-t_1+\hbar)} \ .$$

In other words, we have

$$\left(\int_\Omega |u(x,t_2)|^2 e^{\Phi(x,t_2)}dx\right)^{1+M} \leq \left(\int_\Omega |u(x,t_1)|^2 e^{\Phi(x,t_1)}dx\right)^{M} \int_\Omega |u(x,t_3)|^2 e^{\Phi(x,t_3)}dx \ .$$

Step 2.1.4.

Let ω be a nonempty open subset of Ω. We take off the weight function Φ from the integrals:

$$\begin{aligned} \left(\int_\Omega |u(x,t_2)|^2 dx\right)^{1+M} &\leq \exp\left[-(1+M)\min_{x\in\overline{\Omega}}\Phi(x,t_2) + M\max_{x\in\overline{\Omega}}\Phi(x,t_1)\right] \\ &\quad \times \left(\int_\Omega |u(x,t_1)|^2 dx\right)^{M} \int_\Omega |u(x,t_3)|^2 e^{\Phi(x,t_3)}dx \end{aligned}$$

and

$$\begin{aligned} \int_\Omega |u(x,t_3)|^2 e^{\Phi(x,t_3)}dx &= \int_\omega |u(x,t_3)|^2 e^{\Phi(x,t_3)}dx + \int_{\Omega\setminus\omega} |u(x,t_3)|^2 e^{\Phi(x,t_3)}dx \\ &\leq \exp\left[\max_{x\in\overline{\omega}}\Phi(x,t_3)\right] \int_\omega |u(x,t_3)|^2 dx \\ &\quad + \exp\left[\max_{x\in\overline{\Omega\setminus\omega}}\Phi(x,t_3)\right] \int_\Omega |u(x,t_3)|^2 dx \ . \end{aligned}$$

Therefore, we obtain that

$$\begin{aligned} &\left(\int_\Omega |u(x,t_2)|^2 dx\right)^{1+M} \\ &\leq \exp\left[-(1+M)\min_{x\in\overline{\Omega}}\Phi(x,t_2) + M\max_{x\in\overline{\Omega}}\Phi(x,t_1) + \max_{x\in\overline{\omega}}\Phi(x,t_3)\right] \\ &\quad \times \left(\int_\Omega |u(x,t_1)|^2 dx\right)^{M} \int_\omega |u(x,t_3)|^2 dx \\ &\quad + \exp\left[-(1+M)\min_{x\in\overline{\Omega}}\Phi(x,t_2) + M\max_{x\in\overline{\Omega}}\Phi(x,t_1) + \max_{x\in\overline{\Omega\setminus\omega}}\Phi(x,t_3)\right] \\ &\quad \times \left(\int_\Omega |u(x,t_1)|^2 dx\right)^{M} \int_\Omega |u(x,t_3)|^2 dx \ . \end{aligned}$$

Using the fact that $\|u(\cdot,T)\| \leq \|u(\cdot,t)\| \leq \|u(\cdot,0)\|\ \forall 0<t<T$, the above inequality becomes

$$\begin{aligned}\left(\|u(\cdot,T)\|^2\right)^{1+M} &\leq \exp\left[-(1+M)\min_{x\in\overline{\Omega}}\Phi(x,t_2)+M\max_{x\in\overline{\Omega}}\Phi(x,t_1)+\max_{x\in\overline{\omega}}\Phi(x,t_3)\right]\\ &\quad\times\left(\|u(\cdot,0)\|^2\right)^M\int_\omega|u(x,t_3)|^2\,dx\\ &\quad+\exp\left[-(1+M)\min_{x\in\overline{\Omega}}\Phi(x,t_2)+M\max_{x\in\overline{\Omega}}\Phi(x,t_1)+\max_{x\in\overline{\Omega\setminus\omega}}\Phi(x,t_3)\right]\\ &\quad\times\left(\|u(\cdot,0)\|^2\right)^{1+M}.\end{aligned}$$

Step 2.1.5. *Special weight function.*

Assume that $\Phi(x,t)=\dfrac{\varphi(x)}{T-t+\hbar}$, we get that

$$\begin{aligned}\|u(\cdot,T)\|^{1+M} &\leq \exp\tfrac{1}{2}\left[-\tfrac{1+M}{T-t_2+\hbar}\min_{x\in\overline{\Omega}}\varphi(x)+\tfrac{M}{T-t_1+\hbar}\max_{x\in\overline{\Omega}}\varphi(x)+\tfrac{1}{T-t_3+\hbar}\max_{x\in\overline{\omega}}\varphi(x)\right]\\ &\quad\times\|u(\cdot,0)\|^M\|u(\cdot,t_3)\|_{L^2(\omega)}\\ &\quad+\exp\tfrac{1}{2}\left[-\tfrac{1+M}{T-t_2+\hbar}\min_{x\in\overline{\Omega}}\varphi(x)+\tfrac{M}{T-t_1+\hbar}\max_{x\in\overline{\Omega}}\varphi(x)+\tfrac{1}{T-t_3+\hbar}\max_{x\in\overline{\Omega\setminus\omega}}\varphi(x)\right]\\ &\quad\times\|u(\cdot,0)\|^{1+M}.\end{aligned}$$

Choose $t_3=T$, $t_2=T-\ell\hbar$, $t_1=T-2\ell\hbar$ with $0<2\ell\hbar<T$ and $\ell>1$, and denote

$$M_\ell=\frac{\ln(\ell+1)}{\ln\left(\frac{2\ell+1}{\ell+1}\right)}.$$

Therefore, we have

$$\begin{aligned}\|u(\cdot,T)\|^{1+M_\ell} &\leq \exp\tfrac{1}{2\hbar}\left[-\tfrac{1+M_\ell}{1+\ell}\min_{x\in\overline{\Omega}}\varphi(x)+\tfrac{M_\ell}{1+2\ell}\max_{x\in\overline{\Omega}}\varphi(x)+\max_{x\in\overline{\omega}}\varphi(x)\right]\\ &\quad\times\|u(\cdot,0)\|^{M_\ell}\|u(\cdot,T)\|_{L^2(\omega)}\\ &\quad+\exp\tfrac{1}{2\hbar}\left[-\tfrac{1+M_\ell}{1+\ell}\min_{x\in\overline{\Omega}}\varphi(x)+\tfrac{M_\ell}{1+2\ell}\max_{x\in\overline{\Omega}}\varphi(x)+\max_{x\in\overline{\Omega\setminus\omega}}\varphi(x)\right]\\ &\quad\times\|u(\cdot,0)\|^{1+M_\ell}.\end{aligned}$$

Step 2.1.6. *Assumption on weight function.*

We construct $\varphi(x)$ and choose $\ell>1$ sufficiently large in order that

$$\left[-\frac{1+M_\ell}{1+\ell}\min_{x\in\overline{\Omega}}\varphi(x)+\frac{M_\ell}{1+2\ell}\max_{x\in\overline{\Omega}}\varphi(x)+\max_{x\in\overline{\Omega\setminus\omega}}\varphi(x)\right]<0.$$

Consequently, there are $C_1>0$ and $C_2>0$ such that for any $\hbar>0$ with $0<2\ell\hbar<T$,

$$\|u(\cdot,T)\|^{1+M_\ell}\leq e^{C_1\frac{1}{\hbar}}\|u(\cdot,0)\|^{M_\ell}\|u(\cdot,T)\|_{L^2(\omega)}+e^{-C_2\frac{1}{\hbar}}\|u(\cdot,0)\|^{1+M_\ell}.$$

Notice that $\|u(\cdot,T)\| \leq \|u(\cdot,0)\|$ and for any $2\ell\hbar \geq T$, $1 \leq e^{C_2\frac{2\ell}{T}}e^{-C_2\frac{1}{\hbar}}$. We deduce that for any $\hbar > 0$,

$$\|u(\cdot,T)\|^{1+M_\ell} \leq e^{C_1\frac{1}{\hbar}}\|u(\cdot,0)\|^{M_\ell}\|u(\cdot,T)\|_{L^2(\omega)} + e^{C_2\frac{2\ell}{T}}e^{-C_2\frac{1}{\hbar}}\|u(\cdot,0)\|^{1+M_\ell} \ .$$

Finally, we choose $\hbar > 0$ such that

$$e^{C_2\frac{2\ell}{T}}e^{-C_2\frac{1}{\hbar}}\|u(\cdot,0)\|^{1+M_\ell} = \frac{1}{2}\|u(\cdot,T)\|^{1+M_\ell} \ ,$$

that is,

$$e^{C_2\frac{1}{\hbar}} := 2e^{C_2\frac{2\ell}{T}}\left(\frac{\|u(\cdot,0)\|}{\|u(\cdot,T)\|}\right)^{1+M_\ell}$$

in order that

$$\|u(\cdot,T)\|^{1+M_\ell} \leq 2\left(2e^{C_2\frac{2\ell}{T}}\left(\frac{\|u(\cdot,0)\|}{\|u(\cdot,T)\|}\right)^{1+M_\ell}\right)^{\frac{C_1}{C_2}}\|u(\cdot,0)\|^{M_\ell}\|u(\cdot,T)\|_{L^2(\omega)} \ ,$$

that is,

$$\|u(\cdot,T)\| \leq 2^{1+\frac{C_1}{C_2}}e^{C_1\frac{2\ell}{T}}\left(\frac{\|u(\cdot,0)\|}{\|u(\cdot,T)\|}\right)^{M_\ell+(1+M_\ell)\frac{C_1}{C_2}}\|u(\cdot,T)\|_{L^2(\omega)} \ .$$

This ends to the desired inequality.

2.2 C^2 bounded domain

For C^2 bounded domain Ω, we will use localized weight functions exploiting a covering argument and propagation of smallness.

Let $0 < r < R$, $x_0 \in \Omega$ and $\delta \in (0,1]$. Denote $R_0 := (1+2\delta)R$ and $B_{x_0,r} := \{x; |x-x_0| < r\}$. Assume that $B_{x_0,r} \Subset \Omega$ and $\Omega \cap B_{x_0,R_0}$ is star-shaped with respect to x_0. Let $\langle\cdot,\cdot\rangle_0$ denote the usual scalar product in $L^2(\Omega \cap B_{x_0,R_0})$ and let $\|\cdot\|_0$ be its corresponding norm.

It suffices to prove the following result to get the desired observation inequality at one point in time for the heat equation with Dirichlet boundary condition in a C^2 bounded domain Ω (see [PWZ, Lemma 4 and Lemma 5 at p.493]).

Lemma 2.1. *There is ω_0 a nonempty open subset of $B_{x_0,r}$ and constants c, $K > 0$ and $\beta \in (0,1)$ such that for any $T > 0$ and $u_0 \in L^2(\Omega)$,*

$$\left\|e^{T\Delta}u_0\right\|_{L^2\left(\Omega\cap B_{x_0,R}\right)} \leq \left(ce^{\frac{K}{T}}\left\|e^{T\Delta}u_0\right\|_{L^2(\omega_0)}\right)^{\beta}\|u_0\|^{1-\beta} \ .$$

The strategy to establish the above Lemma 2.1 is as follows. It will be divided into seven steps.

Step 2.2.1. *Localization, symmetric and antisymmetric parts.*

Let $\chi \in C_0^\infty(B_{x_0,R_0})$, $0 \leq \chi \leq 1$, $\chi = 1$ on $\{x; |x-x_0| \leq (1+3\delta/2)\, R\}$. Introduce $z = \chi u$. It solves

$$\partial_t z - \Delta z = g := -2\nabla\chi \cdot \nabla u - \Delta\chi u \ ,$$

and furthermore, $z_{|\partial(\Omega \cap B_{x_0,R_0})} = 0$. Let Φ be a sufficiently smooth function of $(x,t) \in \mathbb{R}^n \times \mathbb{R}_t$ depending on x_0. Set

$$f(x,t) = z(x,t)\, e^{\Phi(x,t)/2} \ .$$

We look for the equation solved by f by computing $e^{\Phi(x,t)/2}(\partial_t - \Delta)\left(e^{-\Phi(x,t)/2} f(x,t)\right)$. It gives

$$\partial_t f - \Delta f - \frac{1}{2} f\left(\partial_t \Phi + \frac{1}{2}|\nabla\Phi|^2\right) + \nabla\Phi\cdot\nabla f + \frac{1}{2}\Delta\Phi f = e^{\Phi/2} g \quad \text{in } (\Omega \cap B_{x_0,R_0}) \times (0,T) \ ,$$

and furthermore, $f_{|\partial(\Omega \cap B_{x_0,R_0})} = 0$. Introduce

$$\begin{cases} \mathcal{A}f = -\nabla\Phi\cdot\nabla f - \frac{1}{2}\Delta\Phi f \ , \\ \mathcal{S}f = \Delta f + \eta f \text{ where } \eta = \frac{1}{2}\left(\partial_t\Phi + \frac{1}{2}|\nabla\Phi|^2\right) \ . \end{cases}$$

It holds

$$\begin{cases} \langle \mathcal{A}f, v\rangle_0 = -\langle \mathcal{A}v, f\rangle_0 \ , \\ \langle \mathcal{S}f, v\rangle_0 = \langle \mathcal{S}v, f\rangle_0 \text{ for any } v \in H_0^1(\Omega \cap B_{x_0,R_0}) \ . \end{cases}$$

Furthermore, one has

$$\partial_t f - \mathcal{S}f - \mathcal{A}f = e^{\Phi/2} g \ .$$

Step 2.2.2. *Energy estimates.*

Multiplying by f the above equation and integrating over $\Omega \cap B_{x_0,R_0}$, we find that

$$\frac{1}{2}\frac{d}{dt}\|f\|_0^2 + \langle -\mathcal{S}f, f\rangle_0 = \left\langle f, e^{\Phi/2} g\right\rangle_0 \ .$$

Introduce the frequency function $t \mapsto \mathbf{N}(t)$ defined by

$$\mathbf{N} = \frac{\langle -\mathcal{S}f, f\rangle_0}{\|f\|_0^2} \ .$$

Thus,

$$\frac{1}{2}\frac{d}{dt}\|f\|_0^2 + \mathbf{N}\|f\|_0^2 = \left\langle e^{\Phi/2} g, f\right\rangle_0 \ .$$

Now, we compute the derivative of $\mathbf{N}$ and claim that:

$$\frac{d}{dt}\mathbf{N} \leq \frac{1}{\|f\|_0^2}\langle -(\mathcal{S}' + [\mathcal{S},\mathcal{A}])\, f, f\rangle_0 - \frac{1}{\|f\|_0^2}\int_{\partial(\Omega \cap B_{x_0,R_0})} \partial_\nu f \mathcal{A} f d\sigma + \frac{1}{2\|f\|_0^2}\left\|e^{\Phi/2} g\right\|_0^2 \ .$$

Indeed,

$$
\begin{aligned}
&\frac{d}{dt}\mathbf{N}\\
&=\frac{1}{\|f\|_0^4}\left(-\frac{d}{dt}\langle\mathcal{S}f,f\rangle_0\|f\|_0^2+\langle\mathcal{S}f,f\rangle_0\frac{d}{dt}\|f\|_0^2\right)\\
&=\frac{-1}{\|f\|_0^2}\left[\langle\mathcal{S}'f,f\rangle_0+2\langle\mathcal{S}f,f'\rangle_0\right]+\frac{2}{\|f\|_0^4}\left[\langle\mathcal{S}f,f\rangle_0^2+\langle\mathcal{S}f,f\rangle_0\left\langle f,e^{\Phi/2}g\right\rangle_0\right]\\
&=\frac{-1}{\|f\|_0^2}\left[\langle\mathcal{S}'f,f\rangle_0+2\langle\mathcal{S}f,f'\rangle_0\right]+\frac{2}{\|f\|_0^4}\left[\left|\langle\mathcal{S}f,f\rangle_0+\frac{1}{2}\left\langle f,e^{\Phi/2}g\right\rangle_0\right|^2-\left|\frac{1}{2}\left\langle f,e^{\Phi/2}g\right\rangle_0\right|^2\right]\\
&=\frac{-1}{\|f\|_0^2}\left[\langle\mathcal{S}'f,f\rangle_0+2\langle\mathcal{S}f,\mathcal{A}f\rangle_0\right]+\frac{-2}{\|f\|_0^2}\left[\|\mathcal{S}f\|_0^2+\left\langle\mathcal{S}f,e^{\Phi/2}g\right\rangle_0\right]\\
&\quad+\frac{2}{\|f\|_0^4}\left[\left|\left\langle\mathcal{S}f+\frac{1}{2}e^{\Phi/2}g,f\right\rangle_0\right|^2-\left|\frac{1}{2}\left\langle f,e^{\Phi/2}g\right\rangle_0\right|^2\right]\\
&\leq\frac{-1}{\|f\|_0^2}\left[\langle\mathcal{S}'f,f\rangle_0+2\langle\mathcal{S}f,\mathcal{A}f\rangle_0\right]+\frac{-2}{\|f\|_0^2}\left[\|\mathcal{S}f\|_0^2+\left\langle\mathcal{S}f,e^{\Phi/2}g\right\rangle_0\right]\\
&\quad+\frac{2}{\|f\|_0^4}\left\|\mathcal{S}f+\frac{1}{2}e^{\Phi/2}g\right\|_0^2\|f\|_0^2\\
&=\frac{-1}{\|f\|_0^2}\left[\langle\mathcal{S}'f,f\rangle_0+2\langle\mathcal{S}f,\mathcal{A}f\rangle_0\right]+\frac{2}{\|f\|_0^2}\left\|\frac{1}{2}e^{\Phi/2}g\right\|_0^2 .
\end{aligned}
$$

In the third line, we used $\frac{1}{2}\frac{d}{dt}\|f\|_0^2-\langle\mathcal{S}f,f\rangle_0=\left\langle f,e^{\Phi/2}g\right\rangle_0$; In the fifth line, multiplying the equation of f by $\mathcal{S}f$, and integrating over $\Omega\cap B_{x_0,R_0}$, give

$$
\begin{aligned}
\langle\mathcal{S}f,f'\rangle_0&=\left\langle\mathcal{S}f,\left(\mathcal{S}f+\mathcal{A}f+e^{\Phi/2}g\right)\right\rangle_0\\
&=\|\mathcal{S}f\|_0^2+\langle\mathcal{S}f,\mathcal{A}f\rangle_0+\left\langle\mathcal{S}f,e^{\Phi/2}g\right\rangle_0 ;
\end{aligned}
$$

In the sixth line, we used Cauchy-Schwarz inequality. Finally, recall that

$$
2\langle\mathcal{S}f,\mathcal{A}f\rangle_0:=\langle[\mathcal{S},\mathcal{A}]f,f\rangle_0+\int_{\partial\left(\Omega\cap B_{x_0,R_0}\right)}\partial_\nu f\mathcal{A}fd\sigma .
$$

Step 2.2.3. *Assumption on Carleman commutator.*

Assume that $\int_{\partial\left(\Omega\cap B_{x_0,R_0}\right)}\partial_\nu f\mathcal{A}fd\sigma\geq 0$ on $(0,T)$ by the star-shaped property of $\Omega\cap B_{x_0,R_0}$, and suppose that

$$
\langle-\left(\mathcal{S}'+[\mathcal{S},\mathcal{A}]\right)f,f\rangle_0\leq\frac{1}{\Upsilon}\langle-\mathcal{S}f,f\rangle_0
$$

on $(0,T)$ where $\Upsilon(t)=T-t+\hbar$ and $\hbar>0$. Therefore, the following differential inequalities hold.

$$
\begin{cases}
\left|\frac{1}{2}\frac{d}{dt}\|f(\cdot,t)\|_0^2+\mathbf{N}(t)\|f(\cdot,t)\|_0^2\right|\leq\left\|e^{\Phi/2}g(\cdot,t)\right\|_0\|f(\cdot,t)\|_0 ,\\
\frac{d}{dt}\mathbf{N}(t)\leq\frac{1}{\Upsilon(t)}\mathbf{N}(t)+\frac{\left\|e^{\Phi/2}g(\cdot,t)\right\|_0^2}{\|f(\cdot,t)\|_0^2} .
\end{cases}
$$

By solving such system of differential inequalities, we have: For any $0 < t_1 < t_2 < t_3 \leq T$,

$$\left(\|f(\cdot,t_2)\|_0^2\right)^{1+M} \leq \left(\|f(\cdot,t_1)\|_0^2\right)^M \|f(\cdot,t_3)\|_0^2 e^{2D}$$

where

$$M = \frac{\int_{t_2}^{t_3} \frac{1}{T-t+\hbar} dt}{\int_{t_1}^{t_2} \frac{1}{T-t+\hbar} dt} = \frac{-\ln(T-t_3+\hbar)+\ln(T-t_2+\hbar)}{-\ln(T-t_2+\hbar)+\ln(T-t_1+\hbar)}$$

and

$$\begin{aligned} D &= M\left((t_2-t_1)\int_{t_1}^{t_2} \frac{\left\|e^{\Phi/2}g(\cdot,t)\right\|_0^2}{\|f(\cdot,t)\|_0^2} dt + \int_{t_1}^{t_2} \frac{\left\|e^{\Phi/2}g(\cdot,t)\right\|_0}{\|f(\cdot,t)\|_0} dt\right) \\ &\quad + \int_{t_2}^{t_3} \frac{1}{T-t+\hbar} dt \int_{t_2}^{t_3} \frac{\left\|e^{\Phi/2}g(\cdot,t)\right\|_0^2}{\|f(\cdot,t)\|_0^2} dt + \int_{t_2}^{t_3} \frac{\left\|e^{\Phi/2}g(\cdot,t)\right\|_0}{\|f(\cdot,t)\|_0} dt \ . \end{aligned}$$

Indeed, we shall distinguish two cases: $t \in [t_1,t_2]$; $t \in [t_2,t_3]$. For $t_1 \leq t \leq t_2$, we integrate $\left((T-t+\hbar)\mathbf{N}(t)\right)' \leq (T-t+\hbar)\frac{\left\|e^{\Phi/2}g(\cdot,t)\right\|_0^2}{\|f(\cdot,t)\|_0^2}$ over (t,t_2) to get

$$\left(\frac{T-t_2+\hbar}{T-t+\hbar}\right)\mathbf{N}(t_2) - \int_{t_1}^{t_2} \frac{\left\|e^{\Phi/2}g(\cdot,s)\right\|_0^2}{\|f(\cdot,s)\|_0^2} ds \leq \mathbf{N}(t) \ .$$

Then we solve

$$\frac{1}{2}\frac{d}{dt}\|f\|_0^2 + \left[\left(\frac{T-t_2+\hbar}{T-t+\hbar}\right)\mathbf{N}(t_2) - \int_{t_1}^{t_2} \frac{\left\|e^{\Phi/2}g(\cdot,s)\right\|_0^2}{\|f(\cdot,s)\|_0^2} ds - \frac{\left\|e^{\Phi/2}g\right\|_0}{\|f\|_0}\right]\|f\|_0^2 \leq 0$$

and integrate it over (t_1,t_2) to obtain

$$\begin{aligned} e^{2\mathbf{N}(t_2)\int_{t_1}^{t_2} \frac{T-t_2+\hbar}{T-t+\hbar} dt} &\leq \frac{\|f(\cdot,t_1)\|_0^2}{\|f(\cdot,t_2)\|_0^2} \\ &\quad \times e^{2(t_2-t_1)\left(\int_{t_1}^{t_2} \frac{\left\|e^{\Phi/2}g(\cdot,t)\right\|_0^2}{\|f(\cdot,t)\|_0^2} dt\right) + 2\int_{t_1}^{t_2} \frac{\left\|e^{\Phi/2}g(\cdot,t)\right\|_0}{\|f(\cdot,t)\|_0} dt} \ . \end{aligned}$$

For $t_2 \leq t \leq t_3$, we integrate $\left((T-t+\hbar)\mathbf{N}(t)\right)' \leq (T-t+\hbar)\frac{\left\|e^{\Phi/2}g(\cdot,t)\right\|_0^2}{\|f(\cdot,t)\|_0^2}$ over (t_2,t) to get

$$\mathbf{N}(t) \leq \frac{T-t_2+\hbar}{T-t+\hbar}\left(\mathbf{N}(t_2) + \int_{t_2}^{t_3} \frac{\left\|e^{\Phi/2}g(\cdot,s)\right\|_0^2}{\|f(\cdot,s)\|_0^2} ds\right) \ .$$

Then we solve

$$0 \leq \frac{1}{2}\frac{d}{dt}\|f\|_0^2 + \left[\frac{T-t_2+\hbar}{T-t+\hbar}\left(\mathbf{N}(t_2) + \int_{t_2}^{t_3} \frac{\left\|e^{\Phi/2}g(\cdot,s)\right\|_0^2}{\|f(\cdot,s)\|_0^2} ds\right) + \frac{\left\|e^{\Phi/2}g\right\|_0}{\|f\|_0}\right]\|f\|_0^2$$

and integrate it over (t_2, t_3) to obtain

$$\begin{aligned}\|f(\cdot,t_2)\|_0^2 &\leq \|f(\cdot,t_3)\|_0^2 e^{2\mathbf{N}(t_2)\int_{t_2}^{t_3}\frac{T-t_2+\hbar}{T-t+\hbar}dt} \\ &\times e^{2\int_{t_2}^{t_3}\frac{T-t_2+\hbar}{T-t+\hbar}dt\int_{t_2}^{t_3}\frac{\left\|e^{\Phi/2}g(\cdot,t)\right\|_0^2}{\|f(\cdot,t)\|_0^2}dt+2\int_{t_2}^{t_3}\frac{\left\|e^{\Phi/2}g(\cdot,t)\right\|_0}{\|f(\cdot,t)\|_0}dt}.\end{aligned}$$

Finally, combining the case $t_1 \leq t \leq t_2$ and the case $t_2 \leq t \leq t_3$, we have

$$\begin{aligned}\|f(\cdot,t_2)\|_0^2 &\leq \|f(\cdot,t_3)\|_0^2\left(\frac{\|f(\cdot,t_1)\|_0^2}{\|f(\cdot,t_2)\|_0^2}\right)^M \\ &\times e^{2M(t_2-t_1)\int_{t_1}^{t_2}\frac{\left\|e^{\Phi/2}g(\cdot,t)\right\|_0^2}{\|f(\cdot,t)\|_0^2}dt}\ e^{2M\int_{t_1}^{t_2}\frac{\left\|e^{\Phi/2}g(\cdot,t)\right\|_0}{\|f(\cdot,t)\|_0}dt} \\ &\times e^{2\int_{t_2}^{t_3}\frac{T-t_2+\hbar}{T-t+\hbar}dt\int_{t_2}^{t_3}\frac{\left\|e^{\Phi/2}g(\cdot,t)\right\|_0^2}{\|f(\cdot,t)\|_0^2}dt}\ e^{2\int_{t_2}^{t_3}\frac{\left\|e^{\Phi/2}g(\cdot,t)\right\|_0}{\|f(\cdot,t)\|_0}dt}\end{aligned}$$

which implies the desired inequality.

Step 2.2.4. *The rest term.*

We estimate $\dfrac{\left\|e^{\Phi/2}g\right\|_0^2}{\|f\|_0^2}$. We begin by giving the following result. (Recall that we have introduced $0 < r < R$, $x_0 \in \Omega$ and $\delta \in (0,1]$).

Lemma 2.2. *For any $T-\theta \leq t \leq T$, one has*

$$\frac{\|u(\cdot,0)\|^2}{\|u(\cdot,t)\|^2_{L^2\left(\Omega\cap B_{x_0,(1+\delta)R}\right)}} \leq e^{(1+\delta)\delta\frac{R^2}{2\theta}}$$

where

$$\frac{1}{\theta} = \frac{2}{(\delta R)^2}\ln\left(2e^{R^2\left(1+\frac{1}{T}\right)}\frac{\|u(\cdot,0)\|^2}{\|u(\cdot,T)\|^2_{L^2\left(\Omega\cap B_{x_0,R}\right)}}\right),$$

with $0 < \theta \leq \min(1, T/2)$.

Indeed, denote $u(x,t) = e^{t\Delta}u_0(x)$ with $u_0 \in L^2(\Omega)$ non-null initial data. Recall that for any locally Lipschitz function $\xi(x,t)$ such that $\partial_t\xi + \frac{1}{2}|\nabla\xi|^2 \leq 0$, the following integral $\displaystyle\int_\Omega |u(x,t)|^2 e^{\xi(x,t)}dx$ is a decreasing function in t by integral maximun principle (see [Gr]). Choose $\xi(x,t) = -\frac{|x-x_0|^2}{2(T-t+\epsilon)}$, then

$$\int_\Omega |u(x,T)|^2 e^{-\frac{|x-x_0|^2}{2\epsilon}}dx \leq \int_\Omega |u(x,t)|^2 e^{-\frac{|x-x_0|^2}{2(T-t+\epsilon)}}dx\ .$$

It implies that

$$\begin{aligned}\left\|u\left(\cdot,T\right)\right\|^2_{L^2\left(\Omega\cap B_{x_0,R}\right)} &\leq e^{\frac{R^2}{2\epsilon}}\int_{\Omega\cap B_{x_0,R}}\left|u\left(x,T\right)\right|^2 e^{-\frac{|x-x_0|^2}{2\epsilon}}dx\\ &\leq e^{\frac{R^2}{2\epsilon}}\int_{\Omega}\left|u\left(x,t\right)\right|^2 e^{-\frac{|x-x_0|^2}{2(T-t+\epsilon)}}dx\\ &\leq e^{\frac{R^2}{2\epsilon}}\left\|u\left(\cdot,t\right)\right\|^2_{L^2\left(\Omega\cap B_{x_0,(1+\delta)R}\right)}+e^{\frac{R^2}{2\epsilon}}e^{-\frac{R^2(1+\delta)^2}{2(T-t+\epsilon)}}\left\|u\left(\cdot,0\right)\right\|^2\ .\end{aligned}$$

Choose $T/2\leq T-\epsilon\delta\leq t\leq T$ with $0<\epsilon\leq T/2$ and $\delta\in(0,1]$, then we get that

$$\left\|u\left(\cdot,T\right)\right\|^2_{L^2\left(\Omega\cap B_{x_0,R}\right)}\leq e^{\frac{R^2}{2\epsilon}}\left\|u\left(\cdot,t\right)\right\|^2_{L^2\left(\Omega\cap B_{x_0,(1+\delta)R}\right)}+e^{-\frac{\delta R^2}{2\epsilon}}\left\|u\left(\cdot,0\right)\right\|^2\ .$$

Choose

$$\epsilon=\frac{\delta R^2}{2\ln\left(2e^{R^2\left(1+\frac{1}{T}\right)}\frac{\|u(\cdot,0)\|^2}{\|u(\cdot,T)\|^2_{L^2\left(\Omega\cap B_{x_0,R}\right)}}\right)}\leq\min\left(1,T/2\right)\ ,$$

that is,

$$e^{-\frac{\delta R^2}{2\epsilon}}\left\|u\left(\cdot,0\right)\right\|^2=\frac{1}{2}e^{-R^2\left(1+\frac{1}{T}\right)}\left\|u\left(\cdot,T\right)\right\|^2_{L^2\left(\Omega\cap B_{x_0,R}\right)}$$

in order that

$$\left(1-\frac{1}{2}e^{-R^2\left(1+\frac{1}{T}\right)}\right)\left\|u\left(\cdot,T\right)\right\|^2_{L^2\left(\Omega\cap B_{x_0,R}\right)}\leq e^{\frac{R^2}{2\epsilon}}\left\|u\left(\cdot,t\right)\right\|^2_{L^2\left(\Omega\cap B_{x_0,(1+\delta)R}\right)}$$

and

$$e^{-\frac{\delta R^2}{2\epsilon}}\left\|u\left(\cdot,0\right)\right\|^2\leq\frac{1}{2}e^{-R^2\left(1+\frac{1}{T}\right)}\left(1-\frac{1}{2}e^{-R^2\left(1+\frac{1}{T}\right)}\right)^{-1}e^{\frac{R^2}{2\epsilon}}\left\|u\left(\cdot,t\right)\right\|^2_{L^2\left(\Omega\cap B_{x_0,(1+\delta)R}\right)}\ .$$

This above inequality implies

$$\frac{\left\|u\left(\cdot,0\right)\right\|^2}{\left\|u\left(\cdot,t\right)\right\|^2_{L^2\left(\Omega\cap B_{x_0,(1+\delta)R}\right)}}\leq e^{(1+\delta)\frac{R^2}{2\epsilon}}=e^{(1+\delta)\delta\frac{R^2}{2\theta}}\ ,$$

that is,

$$\frac{\left\|u\left(\cdot,0\right)\right\|^2}{\left\|u\left(\cdot,t\right)\right\|^2_{L^2\left(\Omega\cap B_{x_0,(1+\delta)R}\right)}}\leq e^{\left(1+\frac{1}{\delta}\right)R^2\left(1+\frac{1}{T}\right)}\left(\frac{2\left\|u\left(\cdot,0\right)\right\|^2}{\left\|u\left(\cdot,T\right)\right\|^2_{L^2\left(\Omega\cap B_{x_0,R}\right)}}\right)^{1+\frac{1}{\delta}}$$

as long as $T/2\leq T-\theta\leq t\leq T$ with

$$\frac{1}{\theta}=\frac{2}{\left(\delta R\right)^2}\ln\left(2e^{R^2\left(1+\frac{1}{T}\right)}\frac{\left\|u\left(\cdot,0\right)\right\|^2}{\left\|u\left(\cdot,T\right)\right\|^2_{L^2\left(\Omega\cap B_{x_0,R}\right)}}\right)\ .$$

Notice that $\theta\leq\min(1,T/2)$. This completes the proof of Lemma 2.2. The interested reader may wish here to compare this lemma's proof with [BP, p.660] or [EFV, p.216].

Now we can estimate $\frac{\left\|e^{\Phi/2}g\right\|_0^2}{\|f\|_0^2}$, by regularizing effect, as follows.

$$\begin{aligned}
&\frac{\left\|e^{\Phi/2}g\left(\cdot,t\right)\right\|_0^2}{\left\|f\left(\cdot,t\right)\right\|_0^2}\\
&=\frac{\displaystyle\int_{\Omega\cap B_{x_0,(1+2\delta)R}}\left|-2\nabla\chi\cdot\nabla u\left(x,t\right)-\Delta\chi u\left(x,t\right)\right|^2 e^{\Phi(x,t)}dx}{\displaystyle\int_{\Omega\cap B_{x_0,(1+2\delta)R}}\left|\chi u\left(x,t\right)\right|^2 e^{\Phi(x,t)}dx}\\
&\leq\frac{\displaystyle\int_{\Omega\cap\{(1+3\delta/2)R\leq|x-x_0|\leq R_0\}}\left|-2\nabla\chi\cdot\nabla u\left(x,t\right)-\Delta\chi u\left(x,t\right)\right|^2 e^{\Phi(x,t)}dx}{\displaystyle\int_{\Omega\cap B_{x_0,(1+\delta)R}}\left|u\left(x,t\right)\right|^2 e^{\Phi(x,t)}dx}\\
&\leq\exp\left[-\min_{|x-x_0|\leq(1+\delta)R}\Phi\left(x,t\right)+\max_{(1+3\delta/2)R\leq|x-x_0|\leq R_0}\Phi\left(x,t\right)\right]\frac{C\left(1+\frac{1}{t}\right)\left\|u\left(\cdot,0\right)\right\|^2}{\left\|u\left(\cdot,t\right)\right\|^2_{L^2\left(\Omega\cap B_{x_0,(1+\delta)R}\right)}}\\
&\leq\exp\left[-\min_{|x-x_0|\leq(1+\delta)R}\Phi\left(x,t\right)+\max_{(1+3\delta/2)R\leq|x-x_0|\leq R_0}\Phi\left(x,t\right)\right]C\left(1+\frac{1}{t}\right)e^{(1+\delta)\delta\frac{R^2}{2\theta}}
\end{aligned}$$

as long as $T/2\leq T-\theta\leq t\leq T$.

Step 2.2.5. *First assumption on the weight function.*

We choose a weight function $\Phi\left(x,t\right)=\dfrac{\varphi\left(x\right)}{T-t+\hbar}$ such that

$$\max_{(1+3\delta/2)R\leq|x-x_0|\leq R_0}\varphi\left(x\right)-\min_{|x-x_0|\leq(1+\delta)R}\varphi\left(x\right)<0$$

in order that

$$\begin{aligned}
&-\min_{|x-x_0|\leq(1+\delta)R}\Phi\left(x,t\right)+\max_{(1+3\delta/2)R\leq|x-x_0|\leq R_0}\Phi\left(x,t\right)+\left(1+\delta\right)\delta\frac{R^2}{2\theta}\\
&=\frac{-1}{T-t+\hbar}\left|\min_{|x-x_0|\leq(1+\delta)R}\varphi\left(x\right)-\max_{(1+3\delta/2)R\leq|x-x_0|\leq R_0}\varphi\left(x\right)\right|+\left(1+\delta\right)\delta\frac{R^2}{2\theta}\\
&\leq\left(1+\delta\right)\delta\frac{R^2}{2\theta}-\frac{1}{\left(1+2\ell\right)\hbar}\left|\min_{|x-x_0|\leq(1+\delta)R}\varphi\left(x\right)-\max_{(1+3\delta/2)R\leq|x-x_0|\leq R_0}\varphi\left(x\right)\right|\ \text{when } T-2\ell\hbar\leq t\\
&<0
\end{aligned}$$

by taking

$$\hbar\leq\theta\frac{1}{\left(1+2\ell\right)\left(1+\delta\right)\delta R^2}\left|\min_{|x-x_0|\leq(1+\delta)R}\varphi\left(x\right)-\max_{(1+3\delta/2)R\leq|x-x_0|\leq R_0}\varphi\left(x\right)\right|:=\theta C_{(\ell,\varphi)}\ .$$

Here $\ell>1$. Combining with the previous Step 2.2.4, one conclude that for any $\hbar\leq\theta\min\left(C_{(\ell,\varphi)},1/\left(2\ell\right)\right)$ and any $T-2\ell\hbar\leq t$,

$$\frac{\left\|e^{\Phi/2}g\left(\cdot,t\right)\right\|_0^2}{\left\|f\left(\cdot,t\right)\right\|_0^2}\leq C\left(1+\frac{1}{t}\right)\ .$$

Next, we choose $t_3 = T$, $t_2 = T - \ell\hbar$, $t_1 = T - 2\ell\hbar$, with $\hbar \leq \theta \min\big(C_{(\ell,\varphi)}, 1/(2\ell)\big)$. Therefore, the inequality of Step 2.2.3

$$\left(\|f(\cdot,t_2)\|_0^2\right)^{1+M} \leq \left(\|f(\cdot,t_1)\|_0^2\right)^{M} \|f(\cdot,t_3)\|_0^2 e^{2D}$$

becomes

$$\left(\|f(\cdot,T-\ell\hbar)\|_0^2\right)^{1+M_\ell} \leq e^{2C_\ell \frac{1}{T}} \left(\|f(\cdot,T-2\ell\hbar)\|_0^2\right)^{M_\ell} \|f(\cdot,T)\|_0^2$$

as long as $\hbar \leq \theta \min\big(C_{(\ell,\varphi)}, 1/(2\ell)\big)$. Here $C_\ell > 0$ is a constant depending on ℓ and recall that

$$M_\ell = \frac{\ln(\ell+1)}{\ln\left(\frac{2\ell+1}{\ell+1}\right)}\ .$$

Step 2.2.6.

Let ω_0 be a nonempty open subset of $B_{x_0,r}$. Now by taking off the weight function $\Phi(x,t) = \dfrac{\varphi(x)}{T-t+\hbar}$ from the integrals, we have that for any $0 < \hbar \leq \theta \min\big(C_{(\ell,\varphi)}, 1/(2\ell)\big)$,

$$\begin{aligned}
&\|u(\cdot,T-\ell\hbar)\|^{1+M_\ell}_{L^2\left(\Omega\cap B_{x_0,(1+\delta)R}\right)}\\
&\leq \exp\tfrac{1}{2\hbar}\left[-\tfrac{1+M_\ell}{1+\ell}\min_{x\in\overline{\Omega}\cap\overline{B}_{x_0,(1+\delta)R}}\varphi(x) + \tfrac{M_\ell}{1+2\ell}\max_{x\in\overline{\Omega}\cap\overline{B}_{x_0,R_0}}\varphi(x) + \max_{x\in\overline{\omega_0}}\varphi(x)\right]\\
&\quad\times e^{C_\ell\frac{1}{T}}\|u(\cdot,0)\|^{M_\ell}\|u(\cdot,T)\|_{L^2(\omega_0)}\\
&\quad+\exp\tfrac{1}{2\hbar}\left[-\tfrac{1+M_\ell}{1+\ell}\min_{x\in\overline{\Omega}\cap\overline{B}_{x_0,(1+\delta)R}}\varphi(x) + \tfrac{M_\ell}{1+2\ell}\max_{x\in\overline{\Omega}\cap\overline{B}_{x_0,R_0}}\varphi(x) + \max_{x\in\left(\overline{\Omega}\cap\overline{B}_{x_0,R_0}\right)\setminus\omega_0}\varphi(x)\right]\\
&\quad\times e^{C_\ell\frac{1}{T}}\|u(\cdot,0)\|^{1+M_\ell}\ .
\end{aligned}$$

But, by Lemma 2.2, observe that

$$\frac{\|u(\cdot,0)\|^2}{\|u(\cdot,t)\|^2_{L^2\left(\Omega\cap B_{x_0,(1+\delta)R}\right)}} \leq e^{(1+\delta)\delta\frac{R^2}{2\theta}}$$

which gives, with $C_{(\delta,R)} := (1+\delta)\,\delta\frac{R^2}{4}$ and $2\ell\hbar \leq \theta$,

$$\|u(\cdot,0)\| \leq e^{\frac{1}{\theta}C_{(\delta,R)}}\|u(\cdot,T-\ell\hbar)\|_{L^2\left(\Omega\cap B_{x_0,(1+\delta)R}\right)}\ .$$

Since $\|u(\cdot,T)\| \leq \|u(\cdot,0)\|$, we can see that

$$e^{-\frac{1}{\theta}C_{(\delta,R)}}\|u(\cdot,T)\| \leq \|u(\cdot,T-\ell\hbar)\|_{L^2\left(\Omega\cap B_{x_0,(1+\delta)R}\right)}$$

and conclude that

$$\begin{aligned}
&\left(e^{-\frac{1}{\theta}C_{(\delta,R)}}\|u(\cdot,T)\|\right)^{1+M_\ell}\\
&\leq \exp\tfrac{1}{2\hbar}\left[-\tfrac{1+M_\ell}{1+\ell}\min_{x\in\overline{\Omega}\cap\overline{B}_{x_0,(1+\delta)R}}\varphi(x) + \tfrac{M_\ell}{1+2\ell}\max_{x\in\overline{\Omega}\cap\overline{B}_{x_0,R_0}}\varphi(x) + \max_{x\in\overline{\omega_0}}\varphi(x)\right]\\
&\quad\times e^{C_\ell\frac{1}{T}}\|u(\cdot,0)\|^{M_\ell}\|u(\cdot,T)\|_{L^2(\omega_0)}\\
&\quad+\exp\tfrac{1}{2\hbar}\left[-\tfrac{1+M_\ell}{1+\ell}\min_{x\in\overline{\Omega}\cap\overline{B}_{x_0,(1+\delta)R}}\varphi(x) + \tfrac{M_\ell}{1+2\ell}\max_{x\in\overline{\Omega}\cap\overline{B}_{x_0,R_0}}\varphi(x) + \max_{x\in\left(\overline{\Omega}\cap\overline{B}_{x_0,R_0}\right)\setminus\omega_0}\varphi(x)\right]\\
&\quad\times e^{C_\ell\frac{1}{T}}\|u(\cdot,0)\|^{1+M_\ell}\ .
\end{aligned}$$

Step 2.2.7. *Second assumption on the weight function.*

We construct $\varphi(x)$ and choose $\ell > 1$ sufficiently large in order that

$$-\frac{1+M_\ell}{1+\ell}\min_{x\in\overline{\Omega}\cap\overline{B}_{x_0,(1+\delta)R}}\varphi(x)+\frac{M_\ell}{1+2\ell}\max_{x\in\overline{\Omega}\cap\overline{B}_{x_0,R_0}}\varphi(x)+\max_{x\in\left(\overline{\Omega}\cap\overline{B}_{x_0,R_0}\right)\setminus\omega_0}\varphi(x)<0\ .$$

Consequently, there are $C_1 > 0$ and $C_2 > 0$ such that for any $\hbar > 0$ with $\hbar \leq \theta\min\big(C_{(\ell,\varphi)}, 1/(2\ell)\big) := \theta C_3$,

$$\left(e^{-\frac{1}{\theta}C_{(\delta,R)}}\left\|u(\cdot,T)\right\|\right)^{1+M_\ell} \leq e^{C_1\frac{1}{\hbar}}\left\|u(\cdot,0)\right\|^{M_\ell}\left\|u(\cdot,T)\right\|_{L^2(\omega_0)} + e^{-C_2\frac{1}{\hbar}}\left\|u(\cdot,0)\right\|^{1+M_\ell}\ .$$

On the other hand, for any $\hbar \geq \theta C_3$, $1 \leq e^{\frac{C_2}{C_3}\frac{1}{\theta}}e^{-C_2\frac{1}{\hbar}}$. Therefore for any $\hbar > 0$,

$$\left(e^{-\frac{1}{\theta}C_{(\delta,R)}}\left\|u(\cdot,T)\right\|\right)^{1+M_\ell} \leq e^{C_1\frac{1}{\hbar}}\left\|u(\cdot,0)\right\|^{M_\ell}\left\|u(\cdot,T)\right\|_{L^2(\omega_0)} + e^{\frac{C_2}{C_3}\frac{1}{\theta}}e^{-C_2\frac{1}{\hbar}}\left\|u(\cdot,0)\right\|^{1+M_\ell}\ .$$

Finally, we choose $\hbar > 0$ such that

$$e^{\frac{C_2}{C_3}\frac{1}{\theta}}e^{-C_2\frac{1}{\hbar}}\left\|u(\cdot,0)\right\|^{1+M_\ell} = \frac{1}{2}\left(e^{-\frac{1}{\theta}C_{(\delta,R)}}\left\|u(\cdot,T)\right\|\right)^{1+M_\ell}\ ,$$

that is,

$$e^{C_2\frac{1}{\hbar}} := 2e^{\frac{C_2}{C_3}\frac{1}{\theta}}\left(\frac{\left\|u(\cdot,0)\right\|}{e^{-\frac{1}{\theta}C_{(\delta,R)}}\left\|u(\cdot,T)\right\|}\right)^{1+M_\ell}$$

in order that

$$\begin{aligned}\left(e^{-C\frac{1}{\theta}}\left\|u(\cdot,T)\right\|\right)^{1+M_\ell} &\leq 2\left(2e^{\frac{C_2}{C_3}\frac{1}{\theta}}\left(\frac{\|u(\cdot,0)\|}{e^{-\frac{1}{\theta}C_{(\delta,R)}}\|u(\cdot,T)\|}\right)^{1+M_\ell}\right)^{\frac{C_1}{C_2}}\\ &\quad\times\left\|u(\cdot,0)\right\|^{M_\ell}\left\|u(\cdot,T)\right\|_{L^2(\omega_0)}\ ,\end{aligned}$$

that is,

$$e^{-C\frac{1}{\theta}}\left\|u(\cdot,T)\right\| \leq 2^{1+\frac{C_1}{C_2}}e^{\frac{C_2}{C_3}\frac{1}{\theta}}\left(\frac{\left\|u(\cdot,0)\right\|}{e^{-\frac{1}{\theta}C_{(\delta,R)}}\left\|u(\cdot,T)\right\|}\right)^{M_\ell+(1+M_\ell)\frac{C_1}{C_2}}\left\|u(\cdot,T)\right\|_{L^2(\omega_0)}\ .$$

As a consequence, we obtain that for some $c > 0$,

$$\left\|u(\cdot,T)\right\|^{1+c} \leq ce^{c\frac{1}{\theta}}\left\|u(\cdot,0)\right\|^{c}\left\|u(\cdot,T)\right\|_{L^2(\omega_0)}\ .$$

But recall the definition of θ in Lemma 2.2 saying that

$$\frac{1}{\theta} = \frac{2}{(\delta R)^2}\ln\left(2e^{R^2\left(1+\frac{1}{T}\right)}\frac{\left\|u(\cdot,0)\right\|^2}{\left\|u(\cdot,T)\right\|^2_{L^2\left(\Omega\cap B_{x_0,R}\right)}}\right)\ .$$

Therefore,

$$\left\|u(\cdot,T)\right\|^{1+c} \leq \left[2e^{R^2\left(1+\frac{1}{T}\right)}\frac{\left\|u(\cdot,0)\right\|^2}{\left\|u(\cdot,T)\right\|^2_{L^2\left(\Omega\cap B_{x_0,R}\right)}}\right]^c c\left\|u(\cdot,0)\right\|^c\left\|u(\cdot,T)\right\|_{L^2(\omega_0)}\ .$$

which gives for some $K > 0$, the following inequality

$$\|u(\cdot,T)\|^{1+K}_{L^2\left(\Omega\cap B_{x_0,R}\right)} \leq e^{K\left(1+\frac{1}{T}\right)} \|u(\cdot,0)\|^K \|u(\cdot,T)\|_{L^2(\omega_0)}$$

and yields to the desired conclusion of Lemma 2.1.

3 The weight function

In the previous section, the observation estimate at one time was derived by using appropriate assumptions on the weight function Φ and by solving a system of differential inequalities. Now, our goal is to explore different explicit choices of weight function Φ.

The weight function $\Phi_\hbar$ used in a series of results for the doubling property or frequency function for heat equations was based on the backward heat kernel (we also refer to [BP] for parabolic equations where the Euclidian distance is replaced by the geodesic distance). Precisely,

$$e^{\Phi_\hbar(x,t)} = G_\hbar(x,t) = \frac{1}{(T-t+\hbar)^{n/2}} e^{\frac{-|x-x_0|^2}{4(T-t+\hbar)}}$$

or simply

$$\Phi_\hbar(x,t) = \frac{-|x-x_0|^2}{4(T-t+\hbar)} - \frac{n}{2}\ln(T-t+\hbar) \ .$$

It leads to the following differential inequalities (see [PWZ, Lemma 2 at p.487]):

Define for $z \in H^1(0,T;L^2(\Omega\cap B_{x_0,R_0})) \cap L^2(0,T;H^2\cap H_0^1(\Omega\cap B_{x_0,R_0}))$ and $t \in (0,T]$,

$$\mathbf{N}_\hbar(t) = \frac{\displaystyle\int_{\Omega\cap B_{x_0,R_0}} |\nabla z(x,t)|^2 G_\hbar(x,t)\,dx}{\displaystyle\int_{\Omega\cap B_{x_0,R_0}} |z(x,t)|^2 G_\hbar(x,t)\,dx}, \text{ whenever } \int_{\Omega\cap B_{x_0,R_0}} |z(x,t)|^2\,dx \neq 0 \ .$$

The following two properties hold.

i)

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\Omega\cap B_{x_0,R_0}} |z(x,t)|^2 G_\hbar(x,t)\,dx + \int_{\Omega\cap B_{R_0}} |\nabla z(x,t)|^2 G_\hbar(x,t)\,dx\\ &= \int_{\Omega\cap B_{x_0,R_0}} z(x,t)(\partial_t-\Delta)z(x,t)\,G_\hbar(x,t)\,dx \ .\end{aligned}$$

ii) When $\Omega\cap B_{x_0,R_0}$ is star-shaped with respect to x_0,

$$\frac{d}{dt}\mathbf{N}_\hbar(t) \leq \frac{1}{T-t+\hbar}\mathbf{N}_\hbar(t) + \frac{\displaystyle\int_{\Omega\cap B_{x_0,R_0}} |(\partial_t-\Delta)z(x,t)|^2 G_\hbar(x,t)\,dx}{\displaystyle\int_{\Omega\cap B_{x_0,R_0}} |z(x,t)|^2 G_\hbar(x,t)\,dx} \ .$$

The differential inequalities obtained with the Carleman commutator are given in Step 2.1.2 and Step 2.2.2 of the previous Section 2:

Define for $f \in H^1(0,T;L^2(\Omega\cap B_{x_0,R_0}))\cap L^2(0,T;H^2\cap H_0^1(\Omega\cap B_{x_0,R_0}))$ and $t\in(0,T]$,

$$\left\{\begin{array}{l}\mathcal{A}f=-\nabla\Phi\cdot\nabla f-\frac{1}{2}\Delta\Phi f\ ,\\ \mathcal{S}f=\Delta f+\eta f \text{ where } \eta=\frac{1}{2}\left(\partial_t\Phi+\frac{1}{2}\left|\nabla\Phi\right|^2\right)\ ,\end{array}\right.$$

and

$$\mathbf{N}=\frac{\langle -\mathcal{S}f,f\rangle_0}{\|f\|_0^2}\ .$$

The following two properties hold.

i)

$$\frac{1}{2}\frac{d}{dt}\|f\|_0^2+\mathbf{N}\|f\|_0^2=\langle\partial_t f-\mathcal{S}f-\mathcal{A}f,f\rangle_0\ .$$

ii)

$$\begin{aligned}\frac{d}{dt}\mathbf{N}\ &\leq\frac{1}{\|f\|_0^2}\langle-\left(\mathcal{S}'+[\mathcal{S},\mathcal{A}]\right)f,f\rangle_0-\frac{1}{\|f\|_0^2}\int_{\partial\left(\Omega\cap B_{x_0,R_0}\right)}\partial_\nu f\mathcal{A}f d\sigma\\ &\quad+\frac{1}{\|f\|_0^2}\|\partial_t f-\mathcal{S}f-\mathcal{A}f\|_0^2\ .\end{aligned}$$

We will assume that $\displaystyle\int_{\partial\left(\Omega\cap B_{x_0,R_0}\right)}\partial_\nu f\mathcal{A}f d\sigma\geq 0$ by the star-shaped property of $\Omega\cap B_{x_0,R_0}$. Now we focus our attention on the term $\langle-\left(\mathcal{S}'+[\mathcal{S},\mathcal{A}]\right)f,f\rangle_0$. We decompose our presentation into three parts.

Part 3.1. *Key formula.*

We claim that:

$$\begin{aligned}\langle-\left(\mathcal{S}'+[\mathcal{S},\mathcal{A}]\right)f,f\rangle_0\ &=-2\int_{\Omega\cap B_{x_0,R_0}}\nabla f\cdot\nabla^2\Phi\nabla f dx\\ &\quad+\frac{1}{2}\int_{\Omega\cap B_{x_0,R_0}}\Delta^2\Phi\left|f\right|^2dx-\int_{\Omega\cap B_{x_0,R_0}}\left(\partial_t\eta+\nabla\Phi\cdot\nabla\eta\right)\left|f\right|^2dx\end{aligned}$$

which is, with the computation of $\partial_t\eta+\nabla\Phi\cdot\nabla\eta$,

$$\begin{aligned}&\langle-\left(\mathcal{S}'+[\mathcal{S},\mathcal{A}]\right)f,f\rangle_0\\ &=-2\int_{\Omega\cap B_{x_0,R_0}}\nabla f\cdot\nabla^2\Phi\nabla f dx\\ &\quad+\frac{1}{2}\int_{\Omega\cap B_{x_0,R_0}}\left(\Delta^2\Phi-\partial_t^2\Phi-2\nabla\Phi\cdot\nabla\partial_t\Phi-\nabla\Phi\cdot\nabla^2\Phi\nabla\Phi\right)\left|f\right|^2dx\ .\end{aligned}$$

Proof of the claim .- First, $\mathcal{S}'f=\partial_t\eta f$. Next, we compute $[\mathcal{S},\mathcal{A}]f:=\mathcal{S}\mathcal{A}f-\mathcal{A}\mathcal{S}f$. Precisely, with standard summation notations,

$$\begin{aligned}\mathcal{S}\mathcal{A}f\ &=\Delta\left(-\nabla\Phi\cdot\nabla f-\tfrac{1}{2}\Delta\Phi f\right)+\eta\left(-\nabla\Phi\cdot\nabla f-\tfrac{1}{2}\Delta\Phi f\right)\\ &=-\Delta\nabla\Phi\cdot\nabla f-2\partial_i\nabla\Phi\cdot\partial_i\nabla f-\nabla\Phi\cdot\Delta\nabla f-\tfrac{1}{2}\Delta^2\Phi f-\nabla\Delta\Phi\cdot\nabla f-\tfrac{1}{2}\Delta\Phi\Delta f\\ &\quad-\eta\nabla\Phi\cdot\nabla f-\tfrac{1}{2}\eta\Delta\Phi f\ ,\\ \mathcal{A}\mathcal{S}f\ &=-\nabla\Phi\cdot\nabla\left(\Delta f+\eta f\right)-\tfrac{1}{2}\Delta\Phi\left(\Delta f+\eta f\right)\\ &=-\nabla\Phi\cdot\nabla\Delta f-\nabla\Phi\cdot\nabla\eta f-\eta\nabla\Phi\cdot\nabla f-\tfrac{1}{2}\Delta\Phi\Delta f-\tfrac{1}{2}\Delta\Phi\eta f\ .\end{aligned}$$

This implies that

$$[\mathcal{S},\mathcal{A}]\, f = -2\partial_i \nabla\Phi\cdot\partial_i\nabla f + \nabla\Phi\cdot\nabla\eta f - \frac{1}{2}\Delta^2\Phi f - 2\Delta\nabla\Phi\cdot\nabla f\ .$$

Therefore, we obtain that

$$-\left(\mathcal{S}' + [\mathcal{S},\mathcal{A}]\right) f = 2\partial_i\nabla\Phi\cdot\partial_i\nabla f - \left(\partial_t\eta + \nabla\Phi\cdot\nabla\eta\right) f + \frac{1}{2}\Delta^2\Phi f + 2\Delta\nabla\Phi\cdot\nabla f\ .$$

Furthermore, by one integration by parts we have

$$\langle \partial_i\nabla\Phi\cdot\partial_i\nabla f, f\rangle_0 = \frac{1}{2}\int_{\Omega\cap B_{x_0,R_0}} \Delta^2\Phi\left|f\right|^2 dx - \int_{\Omega\cap B_{x_0,R_0}} \nabla f\cdot\nabla^2\Phi\nabla f dx$$

and

$$\langle \Delta\nabla\Phi\cdot\nabla f, f\rangle_0 = -\frac{1}{2}\int_{\Omega\cap B_{x_0,R_0}} \Delta^2\Phi\left|f\right|^2 dx\ .$$

Combining the above equalities yields the desired formula. Then the claim follows.

Example linked with the heat kernel .- If

$$\Phi\left(x,t\right) = \frac{-\left|x-x_0\right|^2}{4\left(T-t+\hbar\right)} - \frac{n}{2}\ln\left(T-t+\hbar\right)\ ,$$

then we have

$$\langle -\left(\mathcal{S}' + [\mathcal{S},\mathcal{A}]\right) f, f\rangle_0 = \frac{1}{\Upsilon}\left\langle -\mathcal{S}f, f\right\rangle_0\ ,$$

and

$$\int_{\partial\left(\Omega\cap B_{x_0,R_0}\right)} \partial_\nu f \mathcal{A} f d\sigma = \frac{1}{2\Upsilon}\int_{\partial\left(\Omega\cap B_{x_0,R_0}\right)} \left|\partial_\nu f\right|^2 \left(x-x_0\right)\cdot\overrightarrow{\nu} d\sigma \geq 0$$

by the star-shaped property of $\Omega\cap B_{x_0,R_0}$. Here and from now, $\Upsilon\left(t\right) := T-t+\hbar$ and $\overrightarrow{\nu}$ is the outward unit normal vector to $\partial\left(\Omega\cap B_{x_0,R_0}\right)$.

Part 3.2. *A particular form of the weight function.*

Assume that $\Phi\left(x,t\right) = \dfrac{\varphi\left(x\right)}{T-t+\hbar}$. Then, we can see that

$$\begin{aligned}
&\langle -\left(\mathcal{S}' + [\mathcal{S},\mathcal{A}]\right) f, f\rangle_0 - \frac{1}{\Upsilon}\left\langle -\mathcal{S}f, f\right\rangle_0 \\
&= -\frac{1}{\Upsilon}\int_{\Omega\cap B_{x_0,R_0}} \nabla f\cdot\left(2\nabla^2\varphi + I_d\right)\nabla f dx + \frac{1}{2\Upsilon}\int_{\Omega\cap B_{x_0,R_0}} \Delta^2\varphi\left|f\right|^2 dx \\
&\quad -\frac{1}{2\Upsilon^3}\int_{\Omega\cap B_{x_0,R_0}} \left(\varphi + \left|\nabla\varphi\right|^2 + \frac{1}{2}\nabla\varphi\cdot\left(2\nabla^2\varphi + I_d\right)\nabla\varphi\right)\left|f\right|^2 dx\ .
\end{aligned}$$

Indeed,

$$\begin{aligned}
\langle -\left(\mathcal{S}' + [\mathcal{S},\mathcal{A}]\right) f, f\rangle_0 \ &= -\frac{2}{\Upsilon}\int_{\Omega\cap B_{x_0,R_0}} \nabla f\cdot\nabla^2\varphi\nabla f dx + \frac{1}{2\Upsilon}\int_{\Omega\cap B_{x_0,R_0}} \Delta^2\varphi\left|f\right|^2 dx \\
&\quad -\frac{1}{\Upsilon^3}\int_{\Omega\cap B_{x_0,R_0}} \left(\varphi + \left|\nabla\varphi\right|^2 + \frac{1}{2}\nabla\varphi\cdot\nabla^2\varphi\nabla\varphi\right)\left|f\right|^2 dx\ ,
\end{aligned}$$

and

$$\frac{1}{\Upsilon}\left\langle -\mathcal{S}f,f\right\rangle_0=\frac{1}{\Upsilon}\int_{\Omega\cap B_{x_0,R_0}}\left|\nabla f\right|^2dx-\frac{1}{\Upsilon^3}\int_{\Omega\cap B_{x_0,R_0}}\left(\frac{1}{2}\varphi+\frac{1}{4}\left|\nabla\varphi\right|^2\right)\left|f\right|^2dx\ .$$

Example of a weight function for localization with balls .- If

$$\Phi\left(x,t\right)=\frac{-\left|x-x_0\right|^2}{4\left(T-t+\hbar\right)}\text{ that is, }\varphi\left(x\right)=-\frac{1}{4}\left|x-x_0\right|^2\ ,$$

then we have

$$\left\langle -\left(\mathcal{S}'+\left[\mathcal{S},\mathcal{A}\right]\right)f,f\right\rangle_0=\frac{1}{\Upsilon}\left\langle -\mathcal{S}f,f\right\rangle_0\ ,$$

and

$$\int_{\partial\left(\Omega\cap B_{x_0,R_0}\right)}\partial_\nu f\mathcal{A}fd\sigma=\frac{1}{2\Upsilon}\int_{\partial\left(\Omega\cap B_{x_0,R_0}\right)}\left|\partial_\nu f\right|^2\left(x-x_0\right)\cdot\overrightarrow{\nu}d\sigma\geq 0$$

by the star-shaped property of $\Omega\cap B_{x_0,R_0}$. One conclude that, with such weight function Φ, the assumptions of Step 2.2.3 of the previous Section 2 are satisfied and therefore

$$\begin{cases}\left|\frac{1}{2}\frac{d}{dt}\left\|f\left(\cdot,t\right)\right\|_0^2+\mathbf{N}\left(t\right)\left\|f\left(\cdot,t\right)\right\|_0^2\right|\leq\left\|e^{\Phi/2}g\left(\cdot,t\right)\right\|_0\left\|f\left(\cdot,t\right)\right\|_0\ ,\\ \frac{d}{dt}\mathbf{N}\left(t\right)\leq\frac{1}{T-t+\hbar}\mathbf{N}\left(t\right)+\frac{\left\|e^{\Phi/2}g\left(\cdot,t\right)\right\|_0^2}{\left\|f\left(\cdot,t\right)\right\|_0^2}\ .\end{cases}$$

Now we check the assumptions on $\varphi\left(x\right)=-\frac{1}{4}\left|x-x_0\right|^2$ at Step 2.2.5 and Step 2.2.7 of the previous Section 2. We observe that

$$\begin{array}{c}\max\limits_{(1+3\delta/2)R\leq\left|x-x_0\right|\leq R_0}\varphi\left(x\right)-\min\limits_{\left|x-x_0\right|\leq(1+\delta)R}\varphi\left(x\right)\\ =-\frac{1}{4}\left(1+3\delta/2\right)^2R^2+\frac{1}{4}\left(1+\delta\right)^2R^2<0\end{array}$$

and

$$\begin{array}{l}-\frac{1+M_\ell}{1+\ell}\min\limits_{x\in\overline{\Omega}\cap\overline{B}_{x_0,(1+\delta)R}}\varphi\left(x\right)+\frac{M_\ell}{1+2\ell}\max\limits_{x\in\overline{\Omega}\cap\overline{B}_{x_0,R_0}}\varphi\left(x\right)+\max\limits_{x\in\left(\overline{\Omega}\cap\overline{B}_{x_0,R_0}\right)\setminus\omega_0}\varphi\left(x\right)\\ \leq\left(1+\frac{\ln(\ell+1)}{\ln\left(\frac{2\ell+1}{\ell+1}\right)}\right)\frac{1}{1+\ell}\frac{1}{4}\left(1+\delta\right)^2R^2-\frac{1}{4}r^2<0\end{array}$$

by choosing $\omega_0=B_{x_0,r}\Subset\Omega$ with $0<r<R$ and by taking $\ell>1$ sufficiently large.

Part 3.3. *The weight function for localization with annulus.*

Assume that $\varphi\left(x\right)=-a\left|x-x_0\right|^2+b\left|x-x_0\right|^s-c$ for some $a,b,c>0$ and $1\leq s<2$. We would like to check the assumptions of the previous Section 2 and find the adequate parameters a,b,c,s. First, we observe that the formula in the previous Part 3.2

$$\begin{array}{l}\left\langle -\left(\mathcal{S}'+\left[\mathcal{S},\mathcal{A}\right]\right)f,f\right\rangle_0-\frac{1}{\Upsilon}\left\langle -\mathcal{S}f,f\right\rangle_0\\ =-\frac{1}{\Upsilon}\int_{\Omega\cap B_{x_0,R_0}}\nabla f\cdot\left(2\nabla^2\varphi+I_d\right)\nabla fdx+\frac{1}{2\Upsilon}\int_{\Omega\cap B_{x_0,R_0}}\Delta^2\varphi\left|f\right|^2dx\\ -\frac{1}{2\Upsilon^3}\int_{\Omega\cap B_{x_0,R_0}}\left(\varphi+\left|\nabla\varphi\right|^2+\frac{1}{2}\nabla\varphi\cdot\left(2\nabla^2\varphi+I_d\right)\nabla\varphi\right)\left|f\right|^2dx\end{array}$$

gives

$$
\begin{aligned}
&\langle -\left(\mathcal{S}' + [\mathcal{S}, \mathcal{A}]\right) f, f\rangle_0 - \frac{1}{\Upsilon}\langle -\mathcal{S}f, f\rangle_0 \\
=& -\frac{1}{\Upsilon}(-4a+1)\int_{\Omega\cap B_{x_0,R_0}} |\nabla f|^2\, dx \\
&-\frac{2bs}{\Upsilon}\left[\int_{\Omega\cap B_{x_0,R_0}} |x-x_0|^{s-2}|\nabla f|^2\, dx\right. \\
&\quad \left. -(2-s)\int_{\Omega\cap B_{x_0,R_0}} |x-x_0|^{s-4}|(x-x_0)\cdot\nabla f|^2\, dx\right] \\
&-\frac{1}{2\Upsilon}bs(2-s)(n+s-2)(n+s-4)\int_{\Omega\cap B_{x_0,R_0}} |x-x_0|^{s-4}|f|^2\, dx \\
&+\frac{1}{2\Upsilon^3}c\int_{\Omega\cap B_{x_0,R_0}} |f|^2\, dx \\
&+\frac{1}{2\Upsilon^3}a(1-2a)(1-4a)\int_{\Omega\cap B_{x_0,R_0}} |x-x_0|^2|f|^2\, dx \\
&+\frac{1}{2\Upsilon^3}b\left(-1+6as-4a^2s-4a^2s^2\right)\int_{\Omega\cap B_{x_0,R_0}} |x-x_0|^s|f|^2\, dx \\
&-\frac{1}{2\Upsilon^3}(bs)^2\left(\frac{3}{2}+2a-4as\right)\int_{\Omega\cap B_{x_0,R_0}} |x-x_0|^{2s-2}|f|^2\, dx \\
&-\frac{1}{2\Upsilon^3}(bs)^3(s-1)\int_{\Omega\cap B_{x_0,R_0}} |x-x_0|^{3s-4}|f|^2\, dx\ .
\end{aligned}
$$

We start to choose $a=\frac{1}{4}$. Next we treat the third line of the above formula by using Cauchy-Schwarz inequality, we find that

$$
\begin{aligned}
&\langle -\left(\mathcal{S}' + [\mathcal{S}, \mathcal{A}]\right) f, f\rangle_0 - \frac{1}{\Upsilon}\langle -\mathcal{S}f, f\rangle_0 \\
\leq& -\frac{1}{2\Upsilon}bs(2-s)(n+s-2)(n+s-4)\int_{\Omega\cap B_{x_0,R_0}} |x-x_0|^{s-4}|f|^2\, dx \\
&+\frac{1}{2\Upsilon^3}c\int_{\Omega\cap B_{x_0,R_0}} |f|^2\, dx \\
&+\frac{1}{2\Upsilon^3}b\left(-1+\frac{5}{4}s-\frac{1}{4}s^2\right)\int_{\Omega\cap B_{x_0,R_0}} |x-x_0|^s|f|^2\, dx \\
&-\frac{1}{2\Upsilon^3}(bs)^2(2-s)\int_{\Omega\cap B_{x_0,R_0}} |x-x_0|^{2s-2}|f|^2\, dx \\
&-\frac{1}{2\Upsilon^3}(bs)^3(s-1)\int_{\Omega\cap B_{x_0,R_0}} |x-x_0|^{3s-4}|f|^2\, dx\ .
\end{aligned}
$$

In order that $\langle -\left(\mathcal{S}' + [\mathcal{S}, \mathcal{A}]\right) f, f\rangle_0 - \frac{1}{\Upsilon}\langle -\mathcal{S}f, f\rangle_0 \leq 0$, we can take $n \geq 3$ and $s = 1$

with $c \leq (bs)^2 (2-s) = b^2$. Another choice is $n \geq 3$ and $s = \frac{4}{3}$ which gives

$$\begin{aligned}
& \langle -(\mathcal{S}' + [\mathcal{S}, \mathcal{A}]) f, f \rangle_0 - \frac{1}{\Upsilon} \langle -\mathcal{S} f, f \rangle_0 \\
\leq \quad & \frac{1}{\Upsilon^3} \frac{c}{2} \int_{\Omega \cap B_{x_0, R_0}} |f|^2 \, dx \\
& + \frac{1}{\Upsilon^3} \frac{b}{9} \int_{\Omega \cap B_{x_0, R_0}} |x - x_0|^{4/3} |f|^2 \, dx - \frac{1}{\Upsilon^3} \frac{1}{3} \left(\frac{4}{3} b \right)^2 \int_{\Omega \cap B_{x_0, R_0}} |x - x_0|^{2/3} |f|^2 \, dx \\
& - \frac{1}{\Upsilon^3} \frac{1}{6} \left(\frac{4}{3} b \right)^3 \int_{\Omega \cap B_{x_0, R_0}} |f|^2 \, dx \ ,
\end{aligned}$$

and finally, we can choose $\frac{c}{2} \leq \frac{1}{6} \left(\frac{4}{3} b \right)^3$ and $\frac{b}{9} R_0^{2/3} \leq \frac{1}{3} \left(\frac{4}{3} b \right)^2$.

Now, set $\varphi(x) = -\frac{1}{4} |x - x_0|^2 + \frac{1}{4} |x - x_0|^{4/3} - \left(\frac{1}{3} \right)^4$ and $R = 1$, $R_0 = \left(\frac{4}{3} \right)^{3/2} \approx 1.53$ and $\delta = \frac{R_0 - 1}{2} \approx 0.26$. Here $1 < (1 + 3\delta/2) R \approx 1.40 < R_0 = (1 + 2\delta) R$. We can see that for $n \geq 3$, $\langle -(\mathcal{S}' + [\mathcal{S}, \mathcal{A}]) f, f \rangle_0 - \frac{1}{\Upsilon} \langle -\mathcal{S} f, f \rangle_0 \leq 0$.

We write $\varphi(x) = W(|x - x_0|)$ with $W(\rho) = -\frac{1}{4} \rho^2 + \frac{1}{4} \rho^{4/3} - \left(\frac{1}{3} \right)^4$. We have $W(0) = W(1) = -\left(\frac{1}{3} \right)^4$, $W' \left((2/3)^{3/2} \right) = 0$ and $\rho \mapsto W(\rho)$ is strictly decreasing for $\rho \geq 1$.

Finally, we check the assumptions on φ at Step 2.2.5 and Step 2.2.7 of the previous Section 2: We observe that

$$\begin{aligned}
& \max_{(1+3\delta/2) R \leq |x - x_0| \leq R_0} \varphi(x) - \min_{|x - x_0| \leq (1+\delta) R} \varphi(x) \\
& \leq W((1 + 3\delta/2) R) - W((1 + \delta) R) < 0
\end{aligned}$$

(because $\rho \mapsto W(\rho)$ is strictly decreasing for $\rho \geq 1 = R$) and

$$\begin{aligned}
& -\tfrac{1 + M_\ell}{1 + \ell} \min_{x \in \overline{\Omega} \cap \overline{B}_{x_0, (1+\delta) R}} \varphi(x) + \tfrac{M_\ell}{1 + 2\ell} \max_{x \in \overline{\Omega} \cap \overline{B}_{x_0, R_0}} \varphi(x) + \max_{x \in \left(\overline{\Omega} \cap \overline{B}_{x_0, R_0} \right) \setminus \omega_0} \varphi(x) \\
& \leq - \left(1 + \tfrac{\ln(\ell+1)}{\ln\left(\frac{2\ell+1}{\ell+1} \right)} \right) \tfrac{1}{1+\ell} W((1 + \delta) R) + \tfrac{\ln(\ell+1)}{\ln\left(\frac{2\ell+1}{\ell+1} \right)} \tfrac{1}{1+2\ell} W \left((2/3)^{3/2} \right) + W(r_0) < 0
\end{aligned}$$

by choosing $\omega_0 = \{x; r_0 < |x - x_0| < r\} \Subset \Omega$ with $0 < r_0 < r < 1$, $W(r_0) = W(r) \in \left(-\left(\frac{1}{3} \right)^4, 0 \right)$ and by taking $\ell > 1$ sufficiently large.

4 Proof of Theorem 1.1

The observability estimate in Theorem 1.1 can be deduced from the observation inequality at one time (see [PW2] or the following Lemma 4.1). It was noticed in [AEWZ] that the spectral inequality in Theorem 1.1 is a consequence of the observation inequality at one time (see Lemma A in Appendix (see page 33)).

Lemma 4.1. *Let ω be a nonempty open subset of Ω. Let $p \in [1,2]$, $\gamma > 0$, $\beta \in (0,1)$, $C_\beta = \frac{1+\beta}{\beta\left[(1+\beta)^{1/(2\gamma)}-1\right]^\gamma}$ and c, $K > 0$. Suppose that for any $u_0 \in L^2(\Omega)$ and any $T > 0$,*

$$\left\|e^{T\Delta}u_0\right\|_{L^2(\Omega)} \leq \left(ce^{\frac{1}{T^\gamma}K}\left\|e^{T\Delta}u_0\right\|_{L^p(\omega)}\right)^\beta \|u_0\|^{1-\beta}_{L^2(\Omega)} \ .$$

Then for any $u_0 \in L^2(\Omega)$ and any$T > 0$, one has

$$\left\|e^{T\Delta}u_0\right\|_{L^2(\Omega)} \leq \frac{c}{K^{1/\gamma}}e^{\frac{1}{T^\gamma}(1+1/\gamma)KC_\beta}\int_0^T \left\|e^{t\Delta}u_0\right\|_{L^p(\omega)}dt \ .$$

The above lemma is somehow standard, but we still give the proof here to make a self-contained discussion.

Proof of Lemma 4.1 .- First, by Young inequality, the following interpolation estimate

$$\|u(\cdot,T)\|_{L^2(\Omega)} \leq \left(ce^{\frac{K}{T^\gamma}}\|u(\cdot,T)\|_{L^p(\omega)}\right)^\beta \|u(\cdot,0)\|^{1-\beta}_{L^2(\Omega)}$$

implies that for any $\varepsilon > 0$, we have

$$\|u(\cdot,T)\|_{L^2(\Omega)} \leq \frac{1}{\varepsilon^{\frac{1-\beta}{\beta}}}ce^{\frac{K}{T^\gamma}}\|u(\cdot,T)\|_{L^p(\omega)} + \varepsilon\|u(\cdot,0)\|_{L^2(\Omega)} \ .$$

Next introduce a decreasing sequence $(T_m)_{m\geq 0}$ of positive real numbers defined by

$$T_m = \frac{T}{z^m} \text{ with } z > 1 \ .$$

Take $0 < T_{m+2} < T_{m+1} \leq t < T_m < \cdots < T$ and apply the observation estimate at one time t with initial time T_{m+2}. We find that

$$\|u(\cdot,t)\|_{L^2(\Omega)} \leq \frac{1}{\varepsilon^{\frac{1-\beta}{\beta}}}ce^{\frac{K}{(t-T_{m+2})^\gamma}}\|u(\cdot,t)\|_{L^p(\omega)} + \varepsilon\|u(\cdot,T_{m+2})\|_{L^2(\Omega)} \ .$$

Since $\|u(\cdot,T_m)\| \leq \|u(\cdot,t)\|$, we deduce that

$$\|u(\cdot,T_m)\|_{L^2(\Omega)} \leq \frac{1}{\varepsilon^{\frac{1-\beta}{\beta}}}ce^{\frac{K}{(t-T_{m+2})^\gamma}}\|u(\cdot,t)\|_{L^p(\omega)} + \varepsilon\|u(\cdot,T_{m+2})\|_{L^2(\Omega)} \ .$$

Now, integrate the above inequality over (T_{m+1},T_m), it yields that

$$\|u(\cdot,T_m)\|_{L^2(\Omega)} \leq \frac{1}{\varepsilon^{\frac{1-\beta}{\beta}}}\frac{c}{T_m-T_{m+1}}e^{\frac{K}{(T_{m+1}-T_{m+2})^\gamma}}\int_{T_{m+1}}^{T_m}\|u(\cdot,t)\|_{L^p(\omega)}dt + \varepsilon\|u(\cdot,T_{m+2})\|_{L^2(\Omega)}$$

which implies, since $\frac{c}{T_m-T_{m+1}}e^{\frac{K}{(T_{m+1}-T_{m+2})^\gamma}} = \frac{c}{z}\frac{z^{m+2}}{(z-1)T}e^{\left(K^{1/\gamma}\frac{z^{m+2}}{(z-1)T}\right)^\gamma}$,

$$\|u(\cdot,T_m)\|_{L^2(\Omega)} \leq \frac{1}{\varepsilon^{\frac{1-\beta}{\beta}}}\frac{c}{zK^{1/\gamma}}e^{\frac{(1+1/\gamma)K}{(z-1)^\gamma T^\gamma}z^{\gamma(m+2)}}\int_{T_{m+1}}^{T_m}\|u(\cdot,t)\|_{L^p(\omega)}dt + \varepsilon\|u(\cdot,T_{m+2})\|_{L^2(\Omega)} \ ,$$

that is,

$$\begin{aligned}&\varepsilon^{\frac{1-\beta}{\beta}} e^{-\frac{(1+1/\gamma)K}{(z-1)^\gamma T^\gamma} z^{\gamma(m+2)}} \left\|u\left(\cdot,T_m\right)\right\|_{L^2(\Omega)} - \varepsilon^{\frac{1}{\beta}} e^{-\frac{(1+1/\gamma)K}{(z-1)^\gamma T^\gamma} z^{\gamma(m+2)}} \left\|u\left(\cdot,T_{m+2}\right)\right\|_{L^2(\Omega)} \\ &\leq \frac{c}{zK^{1/\gamma}} \int_{T_{m+1}}^{T_m} \left\|u\left(\cdot,t\right)\right\|_{L^p(\omega)} dt \ .\end{aligned}$$

Replacing m by $2m$, we can see that

$$\begin{aligned}&\varepsilon^{\frac{1-\beta}{\beta}} e^{-\frac{(1+1/\gamma)K}{(z-1)^\gamma T^\gamma} z^{\gamma(2m+2)}} \left\|u\left(\cdot,T_{2m}\right)\right\|_{L^2(\Omega)} - \varepsilon^{\frac{1}{\beta}} e^{-\frac{(1+1/\gamma)K}{(z-1)^\gamma T^\gamma} z^{\gamma(2m+2)}} \left\|u\left(\cdot,T_{2m+2}\right)\right\|_{L^2(\Omega)} \\ &\leq \frac{c}{zK^{1/\gamma}} \int_{T_{2m+1}}^{T_{2m}} \left\|u\left(\cdot,t\right)\right\|_{L^p(\omega)} dt \ .\end{aligned}$$

We write $A_m = e^{-\frac{(1+1/\gamma)K}{(z-1)^\gamma T^\gamma} z^{\gamma(2m+2)}}$ and choose $\varepsilon = A_m$, in order to get

$$A_m^{\frac{1}{\beta}} \left\|u\left(\cdot,T_{2m}\right)\right\|_{L^2(\Omega)} - A_m^{1+\frac{1}{\beta}} \left\|u\left(\cdot,T_{2m+2}\right)\right\|_{L^2(\Omega)} \leq \frac{c}{zK^{1/\gamma}} \int_{T_{2m+1}}^{T_{2m}} \left\|u\left(\cdot,t\right)\right\|_{L^p(\omega)} dt \ .$$

Our task is to have

$$A_m^{1+\frac{1}{\beta}} = A_{m+1}^{\frac{1}{\beta}} \text{ , that is, } e^{-\frac{(1+1/\gamma)K}{(z-1)^\gamma T^\gamma} z^{2\gamma m} z^{2\gamma}\left(1+\frac{1}{\beta}\right)} = e^{-\frac{(1+1/\gamma)K}{(z-1)^\gamma T^\gamma} z^{2\gamma m} z^{4\gamma} \frac{1}{\beta}} \ ,$$

in order to get, with $X_m = A_m^{\frac{1}{\beta}} \left\|u\left(\cdot,T_{2m}\right)\right\|_{L^2(\Omega)}$,

$$X_m - X_{m+1} \leq \frac{c}{zK^{1/\gamma}} \int_{T_{2m+1}}^{T_{2m}} \left\|u\left(\cdot,t\right)\right\|_{L^p(\omega)} dt \ .$$

To this end, we take $z^{2\gamma}\frac{1}{\beta} = 1 + \frac{1}{\beta}$. It remains to sum the telescoping series from $m = 0$ to $+\infty$ to complete the proof of Lemma 4.1 and to find that

$$\left\|u\left(\cdot,T\right)\right\|_{L^2(\Omega)} \leq \frac{c}{zK^{1/\gamma}} e^{\frac{(1+1/\gamma)K}{T^\gamma} C_\beta} \int_0^T \left\|u\left(\cdot,t\right)\right\|_{L^p(\omega)} dt$$

with $C_\beta = \frac{\beta+1}{\beta\left[(\beta+1)^{\frac{1}{2\gamma}}-1\right]^\gamma}$.

With the help of Lemma 4.1 and the analysis done in Section 2 for a convex domain $\Omega \subset \mathbb{R}^n$ or a star-shaped domain with respect to $x_0 \in \Omega$, we are ready to show Theorem 1.1. It suffices to prove the observation at one point of Lemma 4.1 with $\gamma = 1$ and $p = 2$.

Let $\hbar > 0$. Set

$$\Phi\left(x,t\right) = -\frac{\left|x - x_0\right|^2}{4\left(T - t + \hbar\right)} \ .$$

The differential inequalities are (see Part 3.2 of Section 3 and its example):

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \left|u\left(x,t\right)\right|^2 e^{\Phi(x,t)} dx + \mathbf{N}\left(t\right) \int_\Omega \left|u\left(x,t\right)\right|^2 e^{\Phi(x,t)} dx = 0 \ ;$$

Since Ω is convex or star-shaped w.r.t. x_0,

$$\frac{d}{dt}\mathbf{N}(t) \leq \frac{1}{T-t+\hbar}\mathbf{N}(t) \ .$$

By solving such differential inequalities, we have: For any $0 < t_1 < t_2 < t_3 \leq T$,

$$\left(\int_\Omega |u(x,t_2)|^2 e^{\Phi(x,t_2)}dx\right)^{1+M} \leq \int_\Omega |u(x,t_3)|^2 e^{\Phi(x,t_3)}dx \left(\int_\Omega |u(x,t_1)|^2 e^{\Phi(x,t_1)}dx\right)^M$$

where

$$M = \frac{-\ln(T-t_3+\hbar)+\ln(T-t_2+\hbar)}{-\ln(T-t_2+\hbar)+\ln(T-t_1+\hbar)} \ .$$

Choose $t_3 = T$, $t_2 = T-\ell\hbar$, $t_1 = T-2\ell\hbar$ with $0 < 2\ell\hbar < T$ and $\ell > 1$, and denote

$$M_\ell = \frac{\ln(\ell+1)}{\ln\left(\frac{2\ell+1}{\ell+1}\right)} \ ,$$

then

$$\left(\int_\Omega |u(x,T-\ell\hbar)|^2 e^{-\frac{|x-x_0|^2}{4(\ell+1)\hbar}}dx\right)^{1+M_\ell} \leq \left(\int_\Omega |u(x,0)|^2 dx\right)^{M_\ell} \int_\Omega |u(x,T)|^2 e^{-\frac{|x-x_0|^2}{4\hbar}}dx$$

which implies

$$\left(\int_\Omega |u(x,T)|^2 dx\right)^{1+M_\ell} \leq e^{\frac{R^2(1+M_\ell)}{4(\ell+1)\hbar}} \left(\int_\Omega |u(x,0)|^2 dx\right)^{M_\ell} \int_\Omega |u(x,T)|^2 e^{-\frac{|x-x_0|^2}{4\hbar}}dx \ .$$

Here and throughout the proof of Theorem 1.1, $R := \max\limits_{x\in\overline{\Omega}} |x-x_0|$. Next, we split $\int_\Omega |u(x,T)|^2 e^{-\frac{|x-x_0|^2}{4\hbar}}dx$ into two parts: With $B_{x_0,r} := \{x; |x-x_0| < r\} \Subset \Omega$ where $r < R$, we can see that

$$\int_\Omega |u(x,T)|^2 e^{-\frac{|x-x_0|^2}{4\hbar}}dx \leq \int_{B_{x_0,r}} |u(x,T)|^2 dx + e^{-\frac{r^2}{4\hbar}} \int_\Omega |u(x,0)|^2 dx \ .$$

Therefore, taking the above estimates into consideration yields that

$$\begin{aligned}
&\left(\int_\Omega |u(x,T)|^2 dx\right)^{1+M_\ell} \\
&\leq \left(\int_\Omega |u(x,0)|^2 dx\right)^{M_\ell} \\
&\quad \times \left(e^{\frac{R^2(1+M_\ell)}{4(\ell+1)\hbar}} \int_{B_{x_0,r}} |u(x,T)|^2 dx + e^{\frac{R^2(1+M_\ell)}{4(\ell+1)\hbar}} e^{-\frac{r^2}{4\hbar}} \int_\Omega |u(x,0)|^2 dx\right) \ .
\end{aligned}$$

But for $\ell > 1$, $\frac{(1+M_\ell)}{4(\ell+1)} \leq \frac{1}{2(\ell+1)} \frac{\ln(\ell+1)}{\ln\left(\frac{2\ell+1}{\ell+1}\right)} \leq \frac{1}{2\ln(3/2)} \frac{\ln(2\ell)}{\ell} \leq \frac{2^\varepsilon}{2\varepsilon\ln(3/2)} \frac{1}{\ell^{1-\varepsilon}}$ $\forall \varepsilon \in (0,1)$. Our choice of ℓ:

$$\ell := \left(\frac{R^2}{r^2}\right)^{1/(1-\varepsilon)} \left(\frac{2^{2+\varepsilon}}{\varepsilon\ln(3/2)}\right)^{1/(1-\varepsilon)} \quad \forall \varepsilon \in (0,1)$$

gives

$$\frac{R^2\left(1+M_\ell\right)}{4\left(\ell+1\right)\hbar} \leq \frac{r^2}{8\hbar} \ .$$

One the one hand, it implies that for any $2\ell\hbar < T$

$$\begin{aligned}\left(\int_\Omega \left|u\left(x,T\right)\right|^2 dx\right)^{1+M_\ell} &\leq \left(\int_\Omega \left|u\left(x,0\right)\right|^2 dx\right)^{M_\ell} \\ &\times \left(e^{\frac{r^2}{8\hbar}}\int_{B_{x_0,r}} \left|u\left(x,T\right)\right|^2 dx + e^{-\frac{r^2}{8\hbar}}\int_\Omega \left|u\left(x,0\right)\right|^2 dx\right) \ .\end{aligned}$$

On the other hand, $\left\|u\left(\cdot,T\right)\right\| \leq \left\|u\left(\cdot,0\right)\right\|$ and for any $2\ell\hbar \geq T$, $1 \leq e^{\frac{r^2\ell}{4T}}e^{-\frac{r^2}{8\hbar}}$. Therefore we conclude that for any $\hbar > 0$,

$$\begin{aligned}\left(\int_\Omega \left|u\left(x,T\right)\right|^2 dx\right)^{1+M_\ell} &\leq \left(\int_\Omega \left|u\left(x,0\right)\right|^2 dx\right)^{M_\ell} \\ &\times \left(e^{\frac{r^2}{8\hbar}}\int_{B_{x_0,r}} \left|u\left(x,T\right)\right|^2 dx + e^{\frac{r^2\ell}{4T}}e^{-\frac{r^2}{8\hbar}}\int_\Omega \left|u\left(x,0\right)\right|^2 dx\right) \ .\end{aligned}$$

Finally, we choose $\hbar > 0$ such that

$$e^{\frac{r^2}{8\hbar}} := 2e^{\frac{r^2\ell}{4T}}\left(\frac{\displaystyle\int_\Omega \left|u\left(x,0\right)\right|^2 dx}{\displaystyle\int_\Omega \left|u\left(x,T\right)\right|^2 dx}\right)^{1+M_\ell}$$

in order that

$$\int_\Omega \left|u\left(x,T\right)\right|^2 dx \leq \left(4e^{\frac{r^2\ell}{4T}}\int_{B_{x_0,r}} \left|u\left(x,T\right)\right|^2 dx\right)^{\frac{1}{2\left(1+M_\ell\right)}}\left(\int_\Omega \left|u\left(x,0\right)\right|^2 dx\right)^{\frac{1+2M_\ell}{2\left(1+M_\ell\right)}} \ ,$$

that is,

$$\left\|u\left(x,T\right)\right\|_{L^2(\Omega)} \leq \left(2e^{\frac{r^2\ell}{8T}}\left\|u\left(x,T\right)\right\|_{L^2\left(B_{x_0,r}\right)}\right)^{\frac{1}{2\left(1+M_\ell\right)}}\left(\left\|u\left(x,0\right)\right\|_{L^2(\Omega)}\right)^{\frac{1+2M_\ell}{2\left(1+M_\ell\right)}} \ .$$

Now, we can apply Lemma 4.1 with $\gamma = 1$, $p = 2$ and Lemma A in Appendix (see page 33) with $c = 2$, $K = \frac{r^2\ell}{8}$, $\beta = \frac{1}{2(1+M_\ell)}$. Consequently, we obtain that

$$\left\|e^{T\Delta}u_0\right\|_{L^2(\Omega)} \leq \frac{16}{r^2\ell}e^{\frac{r^2\ell}{4T}C_\beta}\int_0^T \left\|e^{t\Delta}u_0\right\|_{L^2\left(B_{x_0,r}\right)} dt$$

and

$$\sum_{\lambda_i\leq\lambda} \left|a_i\right|^2 \leq 4e^{4\sqrt{\lambda(1+2M_\ell)\frac{r^2\ell}{8}}}\int_{B_{x_0,r}} \left|\sum_{\lambda_i\leq\lambda} a_i e_i\left(x\right)\right|^2 dx \ .$$

We can see that $C_\beta \leq \text{constant}(M_\ell)^2$. By the definition of M_ℓ and of ℓ, we have $M_\ell \leq \frac{\ln(\ell+1)}{\ln(3/2)} \leq \text{constant}\frac{1}{r^\varepsilon}$. Therefore, we conclude that

$$\left\|e^{T\Delta}u_0\right\|_{L^2(\Omega)} \leq K_\varepsilon e^{\frac{K_\varepsilon}{r^\varepsilon}\frac{1}{T}} \int_0^T \left\|e^{t\Delta}u_0\right\|_{L^2\left(B_{x_0,r}\right)} dt$$

and

$$\sum_{\lambda_i\leq\lambda} |a_i|^2 \leq 4e^{\frac{K_\varepsilon}{r^\varepsilon}\sqrt{\lambda}} \int_{B_{x_0,r}} \left|\sum_{\lambda_i\leq\lambda} a_i e_i(x)\right|^2 dx \ .$$

This completes the proof of Theorem 1.1.

5 Proof of Theorem 1.2

Let $n \geq 3$ and consider a C^2 bounded domain $\Omega \subset \mathbb{R}^n$ such that $0 \in \Omega$, and let $\omega \subset \Omega$ be a nonempty open set. To simplify the presentation, we assume that $0 \notin \overline{\omega}$, that can always be done, taking if necessary a smaller set. Let $R_0 = \left(\frac{4}{3}\right)^{3/2} \approx 1.53$. We also assume that the unit ball $\overline{B}_{0,R_0}$ is included in Ω and $\overline{B}_{0,R_0} \cap \overline{\omega}$ is empty. This can always be done by a scaling argument.

We are interested in the following heat equation with an inverse square potential

$$\begin{cases} \partial_t u - \Delta u - \frac{\mu}{|x|^2}u = 0 \ , & \text{in } \Omega\times(0,T) \ , \\ u = 0 \ , & \text{on } \partial\Omega\times(0,T) \ , \\ u(\cdot,0) = u_0 \ , & \text{in } \Omega \ , \end{cases}$$

where $u_0 \in L^2(\Omega)$, $T > 0$ and $\mu < \mu^*(n) := \frac{(n-2)^2}{4}$. It is well-known that this is a well-posed problem [VZ]. In particular, $u \in C([0,T]; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega))$ and for any $t \in (0,T]$, we have

$$\int_\Omega |u(x,t)|^2 dx \leq \int_\Omega |u_0(x)|^2 dx \ ,$$

and the regularizing effect

$$\int_\Omega |\nabla u(x,t)|^2 dx \leq \frac{C}{t}\int_\Omega |u_0(x)|^2 dx \ .$$

Applying Lemma A in Appendix (see page 33), we obtain the following result.

Lemma 5.1. *Let $\beta \in (0,1)$ and c, $K > 0$. Suppose that for any $u_0 \in L^2(\Omega)$ and any $T > 0$,*

$$\|u(\cdot,T)\|_{L^2(\Omega)} \leq \left(ce^{\frac{K}{T}} \|u(\cdot,T)\|_{L^1(\omega)}\right)^{\beta} \left(\|u_0\|_{L^2(\Omega)}\right)^{1-\beta} .$$

Then for any $(a_j)_{j\geq 1} \in \mathbb{R}$ and any $\lambda > 0$, one has

$$\sqrt{\sum_{\lambda_j \leq \lambda} |a_j|^2} \leq ce^{2\sqrt{\frac{1-\beta}{\beta}K\lambda}} \left\| \sum_{\lambda_j \leq \lambda} a_j e_j \right\|_{L^1(\omega)} .$$

Here (λ_j, e_j) denotes the eigenbasis of the Schrödinger operator $-\Delta - \frac{\mu}{|x|^2}$ with Dirichlet boundary condition

$$\begin{cases} -\Delta e_j - \frac{\mu}{|x|^2} e_j &= \lambda_j e_j , \quad \text{in } \Omega , \\ e_j &= 0 , \quad \text{on } \partial\Omega . \end{cases}$$

Now we are ready to prove Theorem 1.2. It suffices to check the assumption of the above Lemma 5.1 when

$$\mu \leq \begin{cases} \frac{7}{2\cdot 3^3} := \mu^*(3) - \frac{13}{4\cdot 3^3} , & \text{if } n = 3 , \\ \frac{1}{4}(n-1)(n-3) := \mu^*(n) - \frac{1}{4} , & \text{if } n \geq 4 . \end{cases}$$

Recall that $R_0 = \left(\frac{4}{3}\right)^{3/2} \approx 1.53$ and let $\delta = \frac{R_0 - 1}{2} \approx 0.26$. We have assumed that the unit ball B_{0,R_0} is included in Ω. Let $\chi \in C_0^\infty(B_{0,R_0})$ be a cut-off function satisfying $0 \leq \chi \leq 1$ and $\chi = 1$ on $\{x; |x| \leq 1 + 3\delta/2\}$. Introduce $z = \chi u$. It solves

$$\partial_t z - \Delta z - \frac{\mu}{|x|^2} z = g := -2\nabla\chi \cdot \nabla u - \Delta\chi u ,$$

and furthermore, $z_{|\partial B_{0,R_0}} = \partial_\nu z_{|\partial B_{0,R_0}} = 0$. Let Φ be a sufficiently smooth function of $(x,t) \in \mathbb{R}^n \times \mathbb{R}_t$ and set

$$f(x,t) = z(x,t) e^{\Phi(x,t)/2} .$$

We look for the equation solved by f by computing $e^{\Phi(x,t)/2}(\partial_t - \Delta)\left(e^{-\Phi(x,t)/2} f(x,t)\right)$. It gives

$$\partial_t f - \Delta f - \frac{1}{2} f \left(\partial_t \Phi + \frac{1}{2}|\nabla\Phi|^2 - \frac{2\mu}{|x|^2}\right) + \nabla\Phi \cdot \nabla f + \frac{1}{2}\Delta\Phi f = e^{\Phi/2} g \quad \text{in } B_{0,R_0} \times (0,T) ,$$

and furthermore, $f_{|\partial B_{x_0,R_0}} = \partial_\nu f_{|\partial B_{x_0,R_0}} = 0$. Introduce

$$\begin{cases} \mathcal{A} f = -\nabla\Phi \cdot \nabla f - \frac{1}{2}\Delta\Phi f , \\ \mathcal{S} f = \Delta f + \left(\frac{1}{2}\partial_t\Phi + \frac{1}{4}|\nabla\Phi|^2 + \frac{\mu}{|x|^2}\right) f . \end{cases}$$

Then, it holds

$$\begin{cases} \langle \mathcal{A} f, v\rangle_0 = -\langle \mathcal{A} v, f\rangle_0 , \\ \langle \mathcal{S} f, v\rangle_0 = \langle \mathcal{S} v, f\rangle_0 \text{ for any } v \in H_0^1(B_{0,R_0}) , \end{cases}$$

where $\langle \cdot, \cdot \rangle_0$ is the usual scalar product in $L^2(B_{0,R_0})$ and $\|\cdot\|_0$ will denote the corresponding norm. Furthermore, we have

$$\partial_t f - \mathcal{S} f - \mathcal{A} f = e^{\Phi/2} g \ .$$

Multiplying by f the above equation, integrating over B_{0,R_0}, it follows that

$$\frac{1}{2}\frac{d}{dt}\|f\|_0^2 + \langle -\mathcal{S} f, f\rangle_0 = \left\langle f, e^{\Phi/2} g\right\rangle_0 \ .$$

Introduce the frequency function $t \mapsto \mathbf{N}(t)$ defined by

$$\mathbf{N} = \frac{\langle -\mathcal{S} f, f\rangle_0}{\|f\|_0^2} \ .$$

Then, we have

$$\frac{1}{2}\frac{d}{dt}\|f\|_0^2 + \mathbf{N}\|f\|_0^2 = \left\langle e^{\Phi/2} g, f\right\rangle_0 \ ,$$

and the derivative of $\mathbf{N}$ satisfies (see Step 2.2.2 in Section 2):

$$\frac{d}{dt}\mathbf{N} \leq \frac{1}{\|f\|_0^2}\langle -\left(\mathcal{S}' + [\mathcal{S}, \mathcal{A}]\right) f, f\rangle_0 + \frac{1}{\|f\|_0^2}\left\|e^{\Phi/2} g\right\|_0^2 \ .$$

Notice that the boundary terms have vanished since $f_{\left|\partial B_{x_0,R_0}\right.} = \partial_\nu f_{\left|\partial B_{x_0,R_0}\right.} = 0$.

The estimate of $\dfrac{\left\|e^{\Phi/2} g\right\|_0^2}{\|f\|_0^2}$ can be obtained in a similar way than in Step 2.2.4 and Step 2.2.5 of Section 2. Indeed, first, we check that

$$\frac{d}{dt}\int_\Omega |u(x,t)|^2 e^{\xi(x,t)} dx \leq 0 \quad \text{ with } \xi(x,t) = -\frac{|x|^2}{2(T-t+\epsilon)} \ ,$$

as follows:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2 e^\xi dx \\
&= \int_\Omega u \partial_t u e^\xi dx + \frac{1}{2}\int_\Omega |u|^2 \partial_t \xi e^\xi dx \\
&= \int_\Omega u\left(\Delta u + \frac{\mu}{|x|^2} u\right) e^\xi dx + \frac{1}{2}\int_\Omega |u|^2 \partial_t \xi e^\xi dx \\
&= -\int_\Omega |\nabla u|^2 e^\xi dx - \int_\Omega u \nabla u \cdot \nabla \xi e^\xi dx + \int_\Omega \frac{\mu}{|x|^2}\left|u e^{\xi/2}\right|^2 dx + \frac{1}{2}\int_\Omega |u|^2 \partial_t \xi e^\xi dx \\
&\leq -\int_\Omega |\nabla u|^2 e^\xi dx - \int_\Omega u \nabla u \cdot \nabla \xi e^\xi dx + \int_\Omega \left|\nabla\left(u e^{\xi/2}\right)\right|^2 dx + \frac{1}{2}\int_\Omega |u|^2 \partial_t \xi e^\xi dx \\
&= -\int_\Omega |\nabla u|^2 e^\xi dx - \int_\Omega u \nabla u \cdot \nabla \xi e^\xi dx + \int_\Omega \left|\nabla u e^{\xi/2} + u\frac{1}{2}\nabla \xi e^{\xi/2}\right|^2 dx + \frac{1}{2}\int_\Omega |u|^2 \partial_t \xi e^\xi dx \\
&= \frac{1}{4}\int_\Omega |u|^2 |\nabla \xi|^2 e^\xi dx + \frac{1}{2}\int_\Omega |u|^2 \partial_t \xi e^\xi dx = 0 \ ,
\end{aligned}$$

where in the fifth line we used Hardy inequality. Therefore, we have

$$\int_\Omega |u(x,T)|^2 e^{-\frac{|x-x_0|^2}{2\epsilon}} dx \leq \int_\Omega |u(x,t)|^2 e^{-\frac{|x-x_0|^2}{2(T-t+\epsilon)}} dx \ ,$$

which implies that Lemma 2.2 is still true for any u solution of the heat equation with an inverse square potential.

Next we can estimate $\dfrac{\left\|e^{\Phi/2}g\right\|_0^2}{\|f\|_0^2}$ as follows.

$$\begin{aligned}
&\frac{\left\|e^{\Phi/2}g(\cdot,t)\right\|_0^2}{\|f(\cdot,t)\|_0^2} \\
&\leq \frac{\displaystyle\int_{\Omega\cap\{(1+3\delta/2)R\leq|x|\leq R_0\}} |-2\nabla\chi\cdot\nabla u(x,t) - \Delta\chi u(x,t)|^2 e^{\Phi(x,t)} dx}{\displaystyle\int_{\Omega\cap B_{0,(1+\delta)R}} |u(x,t)|^2 e^{\Phi(x,t)} dx} \\
&\leq \exp\left[-\min_{|x|\leq(1+\delta)R}\Phi(x,t) + \max_{(1+3\delta/2)R\leq|x|\leq R_0}\Phi(x,t)\right] \frac{C\left(1+\frac{1}{t}\right)\|u(\cdot,0)\|^2}{\|u(\cdot,t)\|^2_{L^2\left(\Omega\cap B_{0,(1+\delta)R}\right)}} \\
&\leq \exp\left[-\min_{|x|\leq(1+\delta)R}\Phi(x,t) + \max_{(1+3\delta/2)R\leq|x|\leq R_0}\Phi(x,t)\right] C\left(1+\tfrac{1}{t}\right) e^{(1+\delta)\delta\frac{R^2}{2\theta}}
\end{aligned}$$

as long as $T/2 \leq T-\theta \leq t \leq T$, where in the third line we used the regularizing effect of a gradient term for the solution u of the heat equation with an inverse square potential. Therefore, the conclusion of Step 2.2.5 still holds: Under the assumptions of Step 2.2.5, we have

$$\frac{\left\|e^{\Phi/2}g(\cdot,t)\right\|_0^2}{\|f(\cdot,t)\|_0^2} \leq C\left(1+\frac{1}{t}\right) \ .$$

The difficulty with the heat equation with an inverse square potential comes with the estimate of $\langle -(\mathcal{S}'+[\mathcal{S},\mathcal{A}])f, f\rangle_0 - \dfrac{1}{\Upsilon}\langle -\mathcal{S}f, f\rangle_0$. Notice also that the treatment far from the point $0\in\Omega$ where the inverse square potential have its singularities can be done in the same way than for the heat equation with a potential in $L^\infty(\Omega\times(0,T))$ (see [PWZ]). Our main task is to treat the assumptions of Step 2.2.5 and Step 2.2.7 of Section 2, carefully with a suitable choice of Φ (see also Part 3.3 of Section 3).

We claim that:

$$\begin{aligned}
&\langle -(\mathcal{S}'+[\mathcal{S},\mathcal{A}])f, f\rangle_0 \\
&= -2\int_{\Omega\cap B_{0,R_0}} \nabla f\cdot\nabla^2\Phi\nabla f dx + 2\mu\int_{\Omega\cap B_{0,R_0}} \nabla\Phi\cdot\frac{x}{|x|^4}|f|^2 dx \\
&\quad +\frac{1}{2}\int_{\Omega\cap B_{0,R_0}} \left(\Delta^2\Phi - \partial_t^2\Phi - 2\nabla\Phi\cdot\nabla\partial_t\Phi - \nabla\Phi\cdot\nabla^2\Phi\nabla\Phi\right)|f|^2 dx \ .
\end{aligned}$$

Proof of the claim .- First, $\mathcal{S}'f = \left(\frac{1}{2}\partial_t^2\Phi + \frac{1}{2}\nabla\Phi\cdot\nabla\partial_t\Phi\right)f$. Next, we compute $[\mathcal{S},\mathcal{A}]f = \mathcal{S}\mathcal{A}f - \mathcal{A}\mathcal{S}f$ and get

$$[\mathcal{S},\mathcal{A}]f = -2\partial_i\nabla\Phi\cdot\partial_i\nabla f + \nabla\Phi\cdot\nabla\left(\frac{1}{2}\partial_t\Phi + \frac{1}{4}|\nabla\Phi|^2 + \frac{\mu}{|x|^2}\right)f - \frac{1}{2}\Delta^2\Phi f - 2\Delta\nabla\Phi\cdot\nabla f$$

(which corresponds to the formula in the claim of Part 3.1 in Section 3 with $\eta = \frac{1}{2}\partial_t\Phi + \frac{1}{4}\left|\nabla\Phi\right|^2 + \frac{\mu}{|x|^2}$). Therefore,

$$\begin{aligned}-\left(\mathcal{S}' + [\mathcal{S},\mathcal{A}]\right) f &= 2\partial_i\nabla\Phi\cdot\partial_i\nabla f + 2\Delta\nabla\Phi\cdot\nabla f + \tfrac{1}{2}\Delta^2\Phi f \\ &\quad - \left(\tfrac{1}{2}\partial_t^2\Phi + \tfrac{1}{2}\nabla\Phi\cdot\nabla\partial_t\Phi + \nabla\Phi\cdot\nabla\left(\tfrac{1}{2}\partial_t\Phi + \tfrac{1}{4}\left|\nabla\Phi\right|^2 + \tfrac{\mu}{|x|^2}\right)\right) f \\ &= 2\partial_i\nabla\Phi\cdot\partial_i\nabla f + 2\Delta\nabla\Phi\cdot\nabla f + \tfrac{1}{2}\Delta^2\Phi f \\ &\quad - \left(\tfrac{1}{2}\partial_t^2\Phi + \nabla\Phi\cdot\nabla\partial_t\Phi + \tfrac{1}{2}\nabla\Phi\cdot\nabla^2\Phi\nabla\Phi + \nabla\Phi\cdot\nabla\left(\tfrac{\mu}{|x|^2}\right)\right) f\ .\end{aligned}$$

Furthermore, by one integration by parts we have

$$2\left\langle\partial_i\nabla\Phi\cdot\partial_i\nabla f, f\right\rangle_0 = \int_{\Omega\cap B_{0,R_0}}\Delta^2\Phi\left|f\right|^2 dx - 2\int_{\Omega\cap B_{0,R_0}}\nabla f\cdot\nabla^2\Phi\nabla f dx$$

and

$$2\left\langle\Delta\nabla\Phi\cdot\nabla f, f\right\rangle_0 = -\int_{\Omega\cap B_{0,R_0}}\Delta^2\Phi\left|f\right|^2 dx\ .$$

Combining the above equalities yields the desired claim.

Assume that $\Phi(x,t) = \dfrac{\varphi(x)}{T-t+\hbar}$ and recall that $\Upsilon(t) := T - t + \hbar$. Then, we can see that

$$\begin{aligned}&\left\langle-\left(\mathcal{S}' + [\mathcal{S},\mathcal{A}]\right) f, f\right\rangle_0 - \frac{1}{\Upsilon}\left\langle-\mathcal{S}f, f\right\rangle_0 \\ =& -\frac{1}{\Upsilon}\int_{\Omega\cap B_{0,R_0}}\nabla f\cdot\left(2\nabla^2\varphi + I_d\right)\nabla f dx \\ &+\frac{1}{\Upsilon}\int_{\Omega\cap B_{0,R_0}}\left(\frac{1}{2}\Delta^2\varphi + 2\mu\nabla\varphi\cdot\frac{x}{|x|^4} + \frac{\mu}{|x|^2}\right)\left|f\right|^2 dx \\ &-\frac{1}{2\Upsilon^3}\int_{\Omega\cap B_{0,R_0}}\left(\varphi + \left|\nabla\varphi\right|^2 + \frac{1}{2}\nabla\varphi\cdot\left(2\nabla^2\varphi + I_d\right)\nabla\varphi\right)\left|f\right|^2 dx\ .\end{aligned}$$

Indeed,

$$\begin{aligned}&\left\langle-\left(\mathcal{S}' + [\mathcal{S},\mathcal{A}]\right) f, f\right\rangle_0 \\ =& -\frac{2}{\Upsilon}\int_{\Omega\cap B_{x_0,R_0}}\nabla f\cdot\nabla^2\varphi\nabla f dx + \frac{1}{2\Upsilon}\int_{\Omega\cap B_{x_0,R_0}}\Delta^2\varphi\left|f\right|^2 dx \\ &+\frac{1}{\Upsilon}\int_{\Omega\cap B_{0,R_0}}2\mu\nabla\varphi\cdot\frac{x}{|x|^4}\left|f\right|^2 dx \\ &-\frac{1}{\Upsilon^3}\int_{\Omega\cap B_{x_0,R_0}}\left(\varphi + \left|\nabla\varphi\right|^2 + \frac{1}{2}\nabla\varphi\cdot\nabla^2\varphi\nabla\varphi\right)\left|f\right|^2 dx\ ,\end{aligned}$$

and

$$\begin{aligned}\frac{1}{\Upsilon}\left\langle-\mathcal{S}f, f\right\rangle_0 &= \frac{1}{\Upsilon}\int_{\Omega\cap B_{0,R_0}}\left(\left|\nabla f\right|^2 - \frac{\mu}{|x|^2}\left|f\right|^2\right) dx \\ &\quad -\frac{1}{\Upsilon^3}\int_{\Omega\cap B_{0,R_0}}\left(\frac{1}{2}\varphi + \frac{1}{4}\left|\nabla\varphi\right|^2\right)\left|f\right|^2 dx\ .\end{aligned}$$

Assume that $\varphi(x) = -a\left|x\right|^2 + b\left|x\right|^s - c$ for some $a, b, c > 0$ and $1 \le s < 2$. We would like to check the assumptions of Section 2 and find the adequat parameters a, b, c, s.

First, we observe that the identity

$$\begin{aligned}
&\langle -\left(\mathcal{S}'+[\mathcal{S},\mathcal{A}]\right)f,f\rangle_0-\frac{1}{\Upsilon}\langle -\mathcal{S}f,f\rangle_0\\
&=-\frac{1}{\Upsilon}\int_{\Omega\cap B_{0,R_0}}\nabla f\cdot\left(2\nabla^2\varphi+I_d\right)\nabla f dx\\
&+\frac{1}{\Upsilon}\int_{\Omega\cap B_{0,R_0}}\left(\frac{1}{2}\Delta^2\varphi+2\mu\nabla\varphi\cdot\frac{x}{|x|^4}+\frac{\mu}{|x|^2}\right)|f|^2\,dx\\
&-\frac{1}{2\Upsilon^3}\int_{\Omega\cap B_{0,R_0}}\left(\varphi+|\nabla\varphi|^2+\frac{1}{2}\nabla\varphi\cdot\left(2\nabla^2\varphi+I_d\right)\nabla\varphi\right)|f|^2\,dx
\end{aligned}$$

gives

$$\begin{aligned}
&\langle -\left(\mathcal{S}'+[\mathcal{S},\mathcal{A}]\right)f,f\rangle_0-\frac{1}{\Upsilon}\langle -\mathcal{S}f,f\rangle_0\\
&=-\frac{1}{\Upsilon}\left(-4a+1\right)\int_{\Omega\cap B_{0,R_0}}\left(|\nabla f|^2-\frac{\mu}{|x|^2}|f|^2\right)dx\\
&-\frac{2bs}{\Upsilon}\left[\int_{\Omega\cap B_{0,R_0}}|x|^{s-2}|\nabla f|^2\,dx-(2-s)\int_{\Omega\cap B_{0,R_0}}|x|^{s-4}|x\cdot\nabla f|^2\,dx\right]\\
&+\frac{1}{2\Upsilon}bs\left[4\mu-(2-s)(n+s-2)(n+s-4)\right]\int_{\Omega\cap B_{0,R_0}}|x|^{s-4}|f|^2\,dx\\
&+\frac{1}{2\Upsilon^3}c\int_{\Omega\cap B_{0,R_0}}|f|^2\,dx\\
&+\frac{1}{2\Upsilon^3}a(1-2a)(1-4a)\int_{\Omega\cap B_{0,R_0}}|x|^2|f|^2\,dx\\
&+\frac{1}{2\Upsilon^3}b\left(-1+6as-4a^2s-4a^2s^2\right)\int_{\Omega\cap B_{0,R_0}}|x|^s|f|^2\,dx\\
&-\frac{1}{2\Upsilon^3}(bs)^2\left(\frac{3}{2}+2a-4as\right)\int_{\Omega\cap B_{0,R_0}}|x|^{2s-2}|f|^2\,dx\\
&-\frac{1}{2\Upsilon^3}(bs)^3(s-1)\int_{\Omega\cap B_{0,R_0}}|x|^{3s-4}|f|^2\,dx\ .
\end{aligned}$$

When $n=3$, $a=\frac{1}{4}$, $b=\frac{1}{4}$, $c=\left(\frac{1}{3}\right)^4$ and $s=\frac{4}{3}$, since $R_0=\left(\frac{4}{3}\right)^{3/2}$, we have

$$\begin{aligned}
&\langle -\left(\mathcal{S}'+[\mathcal{S},\mathcal{A}]\right)f,f\rangle_0-\frac{1}{T-t+\hbar}\langle -\mathcal{S}f,f\rangle_0\\
&\leq\frac{2bs}{\Upsilon}\left(\mu-\frac{7}{2\cdot 3^3}\right)\int_{\Omega\cap B_{0,R_0}}|x|^{-8/3}|f|^2\,dx\\
&\leq 0
\end{aligned}$$

with our assumption on μ. Since $\varphi(x)=W(|x|)$ with $W(\rho)=-\frac{1}{4}\rho^2+\frac{1}{4}\rho^{4/3}-\left(\frac{1}{3}\right)^4$, we have $W(0)=W(1)=-\left(\frac{1}{3}\right)^4$, $W'\left((2/3)^{3/2}\right)=0$ and $\rho\mapsto W(\rho)$ is strictly decreasing for $\rho\geq 1$ and the assumptions on φ at Step 2.2.5 and Step 2.2.7 of Section 2 hold by choosing $\omega_0=\{x;r_0<|x|<r\}$ with $0<r_0<r<1$, $W(r_0)=W(r)\in\left(-\left(\frac{1}{3}\right)^4,0\right)$ and by taking $\ell>1$ sufficiently large.

When $n \geq 4$, $a = \frac{1}{4}$, $b = \frac{1}{4}$, $c = b^2$ and $s = 1$, we have

$$\begin{aligned} & \langle -\left(\mathcal{S}' + [\mathcal{S}, \mathcal{A}]\right) f, f \rangle_0 - \frac{1}{T - t + \hbar} \langle -\mathcal{S} f, f \rangle_0 \\ & \leq \frac{2bs}{\Upsilon} \left(\mu - \frac{1}{4} (n-1)(n-3) \right) \int_{\Omega \cap B_{0,R_0}} |x|^{-3} |f|^2 \, dx \\ & \leq 0 \end{aligned}$$

with our assumption on μ. Since $\rho \mapsto -\frac{1}{4}\rho^2 + \frac{1}{4}\rho - \frac{1}{16}$ is a non positive function and is strictly decreasing for $\rho \geq 1$, the assumptions on φ at Step 2.2.5 and Step 2.2.7 of Section 2 hold by choosing $\omega_0 = \{x; r_0 < |x| < r\}$ with $0 < r_0 < r < 1$, and by taking $\ell > 1$ sufficiently large.

One conclude that for any $u_0 \in L^2(\Omega)$ and any $T > 0$,

$$\|u(\cdot, T)\|_{L^2\left(B_{0,R}\right)} \leq \left(c e^{\frac{K}{T}} \|u(\cdot, T)\|_{L^2(\omega_0)} \right)^{\beta} \|u_0\|^{1-\beta}_{L^2(\Omega)} \ .$$

Since $0 \notin \omega_0 \Subset \Omega$, we can replace ω_0 by any nonempty open subset $\widetilde{\omega}$ of Ω by propagation of smallness. The treatment far from the point $0 \in \Omega$ where the inverse square potential have its singularities can be done in the same way than for the heat equation with a potential in $L^\infty(\Omega \times (0,T))$ (see [PWZ]) and we also have

$$\|u(\cdot, T)\|_{L^2\left(\Omega \setminus B_{0,R}\right)} \leq \left(c e^{\frac{K}{T}} \|u(\cdot, T)\|_{L^2(\widetilde{\omega})} \right)^{\beta} \|u_0\|^{1-\beta}_{L^2(\Omega)} \ .$$

Finally, we will replace $\|u(\cdot, T)\|_{L^2(\widetilde{\omega})}$ by $\|u(\cdot, T)\|_{L^1(\omega)}$ thanks to Nash inequality: Here $\widetilde{\omega} \Subset \omega$. Let $\phi \in C_0^\infty(\omega)$ be such that $0 \leq \phi \leq 1$ and $\phi = 1$ on $\widetilde{\omega}$. Then we have

$$\begin{aligned} \|u(\cdot, T)\|_{L^2(\widetilde{\omega})} & \leq \|\phi u(\cdot, T)\|_{L^2(\omega)} \\ & \leq \left(e \|\phi u(\cdot, T)\|_{L^1(\omega)} \right)^{\frac{2}{2+n}} \left(\tfrac{1}{2\sqrt{\pi}} \|\phi u(\cdot, T)\|_{H_0^1(\omega)} \right)^{\frac{n}{2+n}} \\ & \leq \left(e \|u(\cdot, T)\|_{L^1(\omega)} \right)^{\frac{2}{2+n}} \left(\tfrac{1}{2\sqrt{\pi}} \|\phi u(\cdot, T)\|_{H_0^1(\Omega)} \right)^{\frac{n}{2+n}} \\ & \leq C_n \|u(\cdot, T)\|^{\frac{2}{2+n}}_{L^1(\omega)} \|u(\cdot, T)\|^{\frac{n}{2+n}}_{H_0^1(\Omega)} \\ & \leq C_n \|u(\cdot, T)\|^{\frac{2}{2+n}}_{L^1(\omega)} \left(\tfrac{C}{\sqrt{T}} \|u_0\|_{L^2(\Omega)} \right)^{\frac{n}{2+n}} \ . \end{aligned}$$

This completes the proof.

Appendix

Let H be a real Hilbert space endowed with an inner product $\langle \cdot, \cdot \rangle$, and A be a linear self-adjoint operator from $D(A)$ into H, where $D(A)$ being the domain of A is a subspace of H. We assume that A is an isomorphism from $D(A)$ (equipped with the graph norm)

onto H, that A^{-1} is a linear compact operator in H and that $\langle Av, v\rangle > 0\ \forall v \in D(A)$, $v \neq 0$. Introduce the set $\{\lambda_j\}_{j=1}^{\infty}$ for the family of all eigenvalues of A so that

$$0 < \lambda_1 \leq \lambda_2 \leq \cdot\cdot \leq \lambda_m \leq \lambda_{m+1} \leq \cdots \text{ and } \lim_{j\to\infty} \lambda_j = \infty\ ,$$

and let $\{e_j\}_{j=1}^{\infty}$ be the family of the corresponding orthogonal normalized eigenfunctions: $Ae_j = \lambda_j e_j$, $e_j \in D(A)$ and $\langle e_j, e_i\rangle = \delta_{i,j}$.

By Lumer-Phillips theorem, $-A$ generates on H a strongly continuous semigroup $S : t \mapsto S(t) = e^{-tA}$. For any $t \geq 0$ and any $u_0 \in H$, we have that $S(t)\, u_0 = \sum_{j\geq 1} \langle u_0, e_j\rangle\, e^{-\lambda_j t} e_j := u(\cdot, t)$ and $u \in C([0, +\infty); H) \cap C^1((0, +\infty); D(A))$ is the unique solution of $\partial_t u + Au = 0$ with $u(\cdot, 0) = u_0$.

Below, $H := L^2(\Omega)$ where Ω is a bounded open set of $\mathbb{R}^n$.

Lemma A. *Let ω be a nonempty open subset of Ω. Let $p \in [1, 2]$, $\beta \in (0, 1)$ and c, K, $\gamma > 0$. Suppose that for any $u_0 \in L^2(\Omega)$ and any $T > 0$,*

$$\|u(\cdot, T)\|_{L^2(\Omega)} \leq \left(ce^{\frac{1}{T^\gamma}K} \|u(\cdot, T)\|_{L^p(\omega)}\right)^{\beta} \|u_0\|^{1-\beta}_{L^2(\Omega)}\ .$$

Then for any $(a_j)_{j\geq 1} \in \mathbb{R}$ and any $\lambda > 0$, one has

$$\sqrt{\sum_{\lambda_j\leq\lambda} |a_j|^2} \leq ce^{\lambda^{\frac{\gamma}{1+\gamma}} 2\left(\frac{1-\beta}{\beta}\right)^{\frac{\gamma}{1+\gamma}} K^{\frac{1}{1+\gamma}}} \left\|\sum_{\lambda_j\leq\lambda} a_j e_j\right\|_{L^p(\omega)}\ .$$

Indeed, we choose $u_0 = \sum\limits_{\lambda_j\leq\lambda} a_j e^{\lambda_j T} e_j$ and apply

$$\|u(\cdot, T)\|_{L^2(\Omega)} \leq \left(ce^{K\frac{1}{T^\gamma}} \|u(\cdot, T)\|_{L^p(\omega)}\right)^{\beta} \|u_0\|^{1-\beta}_{L^2(\Omega)}\ ,$$

to get

$$\begin{aligned}\sqrt{\sum_{\lambda_j\leq\lambda} |a_j|^2} &\leq \left(ce^{K\frac{1}{T^\gamma}} \left\|\sum_{\lambda_j\leq\lambda} a_j e_j\right\|_{L^p(\omega)}\right)^{\beta} \left(\sqrt{\sum_{\lambda_j\leq\lambda} |a_j|^2 e^{2\lambda_j T}}\right)^{1-\beta} \\ &\leq \left(ce^{K\frac{1}{T^\gamma}} \left\|\sum_{\lambda_j\leq\lambda} a_j e_j\right\|_{L^p(\omega)}\right)^{\beta} e^{\lambda T(1-\beta)} \left(\sqrt{\sum_{\lambda_j\leq\lambda} |a_j|^2}\right)^{1-\beta}\ .\end{aligned}$$

Therefore,

$$\sqrt{\sum_{\lambda_j\leq\lambda} |a_j|^2} \leq ce^{K\frac{1}{T^\gamma} + \lambda T\left(\frac{1-\beta}{\beta}\right)} \left\|\sum_{\lambda_j\leq\lambda} a_j e_j\right\|_{L^p(\omega)}\ .$$

We conclude by choosing

$$T = \left[\left(\frac{\beta}{1-\beta} \right) \frac{K}{\lambda} \right]^{\frac{1}{1+\gamma}} .$$

Remark .- Conversely, suppose that there are constants $p \in [1,2]$ and D_1, D_2, $\gamma > 0$ such that any $(a_j)_{j\geq 0} \in \ell^2$ and any $\lambda > \lambda_1$,

$$\sqrt{\sum_{\lambda_j \leq \lambda} |a_j|^2} \leq D_1 e^{\lambda^{\frac{\gamma}{1+\gamma}} D_2} \left\| \sum_{\lambda_j \leq \lambda} a_j e_j \right\|_{L^p(\omega)} .$$

Then for any $\beta \in (0,1)$ and any $T > 0$,

$$\|u(\cdot,T)\|_{L^2(\Omega)} \leq D_3 e^{\frac{1}{T^\gamma} D_4} \|u(\cdot,T)\|^{\beta}_{L^p(\omega)} \|u(\cdot,0)\|^{1-\beta}_{L^2(\Omega)}$$

with

$$D_3 = 2\left(1 + \max\left(1, |\omega|^{\frac{1}{p}-\frac{1}{2}}\right) D_1\right) \text{ and } D_4 = (D_2)^{1+\gamma} \frac{1}{(1-\beta)^\gamma} .$$

Indeed, let $\alpha := \frac{\gamma}{1+\gamma}$ and $u_0 := u(\cdot,0) = \sum_{j\geq 1} a_j e_j$. First, we have

$$\|u(\cdot,T)\|^2_{L^2(\Omega)} = \sum_{j\geq 1} \left|a_j e^{-\lambda_j T}\right|^2 = \sum_{\lambda_j \leq \lambda} \left|a_j e^{-\lambda_j T}\right|^2 + \sum_{\lambda_j > \lambda} \left|a_j e^{-\lambda_j T}\right|^2$$

and

$$\|u(\cdot,T)\|_{L^2(\Omega)} \leq \sqrt{\sum_{\lambda_j \leq \lambda} \left|a_j e^{-\lambda_j T}\right|^2} + \sqrt{\sum_{\lambda_j > \lambda} \left|a_j e^{-\lambda_j T}\right|^2} .$$

Next, we apply the estimate on the sum of eigenfunctions with a_j replaced by $a_j e^{-\lambda_j T}$ in order to get

$$\begin{aligned}
\|u(\cdot,T)\|_{L^2(\Omega)} &\leq D_1 e^{D_2\lambda^\alpha} \left\| \sum_{\lambda_j \leq \lambda} a_j e^{-\lambda_j T} e_j \right\|_{L^p(\omega)} + \sqrt{\sum_{\lambda_j > \lambda} \left|a_j e^{-\lambda_j T}\right|^2} \\
&\leq D_1 e^{D_2\lambda^\alpha} \left\| \sum_{\lambda_j \leq \lambda} a_j e^{-\lambda_j T} e_j + \sum_{\lambda_j > \lambda} a_j e^{-\lambda_j T} e_j \right\|_{L^p(\omega)} \\
&\quad + D_1 e^{D_2\lambda^\alpha} \left\| \sum_{\lambda_j > \lambda} a_j e^{-\lambda_j T} e_j \right\|_{L^p(\omega)} + \sqrt{\sum_{\lambda_j > \lambda} \left|a_j e^{-\lambda_j T}\right|^2} \\
&\leq D_1 e^{D_2\lambda^\alpha} \|u(\cdot,T)\|_{L^p(\omega)} + \left(1 + |\omega|^{\frac{1}{p}-\frac{1}{2}} D_1\right) e^{D_2\lambda^\alpha} e^{-\lambda T} \sqrt{\sum_{\lambda_j > \lambda} |a_j|^2} \\
&\leq D_1 e^{D_2\lambda^\alpha} \|u(\cdot,T)\|_{L^p(\omega)} + \left(1 + |\omega|^{\frac{1}{p}-\frac{1}{2}} D_1\right) e^{D_2\lambda^\alpha} e^{-\lambda T} \|u(\cdot,0)\|_{L^2(\Omega)} .
\end{aligned}$$

Now, by Young inequality, for any $\epsilon > 0$,

$$D_2\lambda^\alpha = \frac{D_2}{(\epsilon T)^\alpha} (\epsilon\lambda T)^\alpha \leq \epsilon\lambda T + \left(\frac{D_2}{(\epsilon T)^\alpha}\right)^{\frac{1}{1-\alpha}} .$$

Therefore,

$$\begin{aligned}\|u(\cdot,T)\|_{L^2(\Omega)} &\leq \left(1+\max\left(1,|\omega|^{\frac{1}{p}-\frac{1}{2}}\right)D_1\right)e^{\left(\frac{D_2}{(\epsilon T)^\alpha}\right)^{\frac{1}{1-\alpha}}} \\ &\quad\times\left(e^{\epsilon\lambda T}\|u(\cdot,T)\|_{L^p(\omega)}+e^{(\epsilon-1)\lambda T}\|u(\cdot,0)\|_{L^2(\Omega)}\right)\ .\end{aligned}$$

Choosing $0<\epsilon<1$ and optimizing with respect to λT by taking

$$\lambda T=\ln\left(\frac{\|u(\cdot,0)\|_{L^2(\Omega)}}{\|u(\cdot,T)\|_{L^p(\omega)}}\right)\ ,$$

yield

$$\begin{aligned}\|u(\cdot,T)\|_{L^2(\Omega)} &\leq \left(1+\max\left(1,|\omega|^{\frac{1}{p}-\frac{1}{2}}\right)D_1\right)e^{\left(\frac{D_2}{(\epsilon T)^\alpha}\right)^{\frac{1}{1-\alpha}}} \\ &\quad\times\left(2\left[\frac{\|u(\cdot,0)\|_{L^2(\Omega)}}{\|u(\cdot,T)\|_{L^p(\omega)}}\right]^{\epsilon}\|u(\cdot,T)\|_{L^p(\omega)}\right)\ .\end{aligned}$$

Setting $\beta=1-\epsilon$, we finally have the desired observation estimate at one time.

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