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An implicit sweeping process approach to quasistatic evolution variational inequalities.

Samir ADLY* and Tahar HADDAD†

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Abstract

In this paper, we study a new variant of Moreau's sweeping process with velocity constraint. Based on an adapted version of Moreau's catching-up algorithm, we show the well-posedness (in the sense existence and uniqueness) of this problem in a general framework. We show the equivalence between this implicit sweeping process and a quasistatic evolution variational inequality. It is well known that the variational formulations of many mechanical problems with unilateral contact and friction lead to an evolution variational inequality. As an application, we reformulate the quasistatic antiplane frictional contact problem for linear elastic materials with short memory as an implicit sweeping process with velocity constraint. The link between the implicit sweeping process and the quasistatic evolution variational inequality is possible thanks to some standard tools from convex analysis and is new in the literature.

Keywords Moreau's sweeping process, evolution variational inequalities, unilateral constraints, quasistatic frictional contact problems.

AMS subject classifications 49J40, 47J20, 47J22, 34G25, 58E35, 74M15, 74M10, 74G25.

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1 Introduction

The notion of the so-called sweeping process was introduced by Jean Jacques Moreau in the 1970s. Jean Jacques Moreau wrote more than 25 papers devoted to the treatment of both theoretical and numerical aspects of the sweeping process as well as its applications in unilateral mechanics [13, 14, 15, 16, 17]. It was first considered for modeling the quasistatic evolution of elastoplastic systems. The sweeping process consists of finding a trajectory $t \in [0, T] \mapsto u(t) \in C(t)$ satisfying the following generalized Cauchy problem

$$-\dot{u}(t) \in N_{C(t)}(u(t)) \quad \text{a.e. on } [0, T], u(0) = u_0 \in C(0), \quad (1)$$

where $C : [0, T] \rightrightarrows H$ is a set-valued mapping defined from $[0, T]$ ($T > 0$) to a Hilbert space H with convex and closed values, and $N_{C(t)}(u(t))$ denotes the outward normal cone, in the sense of convex analysis, to the set $C(t)$ at the point $u(t)$. Translating inclusion (1) to a mechanical language, we obtain the following interpretation:

- If the position $u(t)$ of a particule lies in the interior of the moving set $C(t)$, then the normal cone is reduced to the singleton $\{0\}$ and hence $\dot{u}(t) = 0$, which means that the particule remains at rest.

- When the boundary of $C(t)$ catches up with the particle, then this latter is pushed in an inward normal direction by the boundary of $C(t)$ to stay inside $C(t)$ and satisfies the viability constraint $u(t) \in C(t)$. This mechanical visualization led Moreau to call this problem the *sweeping process*: the particle is swept by the moving set.

Using the definition of the normal cone, it is easy to see that (1) is equivalent to the following evolution variational inequality:

$$\begin{cases} \text{Find } u(t) \in C(t) \text{ such that} \\ \langle \dot{u}(t), v - u(t) \rangle \geq 0, \quad \forall v \in C(t) \text{ and for a. e. } t \in [0, T]. \end{cases}$$

In nonsmooth mechanics, the moving set is usually expressed in inequalities form, corresponding to the so-called unilateral constraints,

$$C(t) := \bigcap_{i=1}^m \{x \in H : f_i(t, x) \leq 0\}, \quad (2)$$

where $f_i : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$, ($i = 1, 2, \dots, m$) are some given regular convex functions.

Several extensions of the sweeping process in diverse ways have been studied in the literature (see e.g. [1, 10] and references therein). A natural generalization of the sweeping process is the differential inclusion

$$\begin{cases} -\dot{u}(t) \in N_{C(t)}(u(t)) + f(t, u(t)) + \mathcal{F}(t, u(t)) \\ u(0) = u_0 \in C(0) \\ u(t) \in C(t), \quad \forall t \in [0, T], \end{cases} \quad (3)$$

where $f : [0, T] \times H \rightarrow H$ is a Lipschitz mapping and \mathcal{F} is a set-valued mapping from $[0, T] \times H$ into weakly compact convex sets of a Hilbert space H .

In this paper we are interested in a new variant of the sweeping process of the following form

$$\begin{cases} -\dot{u}(t) \in N_{C(t)}(A\dot{u}(t) + Bu(t)) & \text{a.e. } t \in [0, T]. \\ u(0) = u_0 \in H. \end{cases} \quad (4)$$

We assume that the following assumptions hold:

(\mathcal{SP}_1) $A, B : H \longrightarrow H$ are two linear, bounded and symmetric operators satisfying:

$$\begin{cases} \langle Ax, x \rangle \geq \beta \|x\|^2, \text{ for all } x \in H \text{ for some constant } \beta > 0 \\ \langle Bx, x \rangle \geq 0, \text{ for all } x \in H. \end{cases}$$

(\mathcal{SP}_2) For every $t \in [0, T]$, $C(t) \subset H$ is a closed convex and nonempty set such that $t \mapsto C(t)$ is absolutely continuous, in the sense that there exists a nondecreasing absolutely continuous function $v : [0, T] \rightarrow \mathbb{R}^+$ with $v(0) = 0$ such that

$$d_H(C(t), C(s)) \leq v(t) - v(s), \text{ for all } 0 \leq s \leq t \leq T,$$

where d_H denotes the Hausdorff distance defined in (9).

It is worth mentioning that in the particular case where $A := I$ and $B := 0$, problem (4) has been studied in [5] by assuming that the set $C(t)$ is prox-regular and satisfying a compactness condition.

The main goal of this paper is to prove a general existence and uniqueness result for the implicit differential inclusion described by (4) by assuming that the set-valued mapping $t \mapsto C(t)$ moves in an absolutely continuous way with respect to the Hausdorff distance. By using an implicit time discretization, we solve at each iteration a variational inequality. The limit of a sequence of functions, constructed via linear interpolation, is showed to be a solution of (4). For the particular case when the moving set $C(t) = C - f(t)$ (with $f \in W^{1,1}([0, T]; H)$ and C a fixed closed convex subset of H), we give an application to the quasistatic frictional contact problem involving viscoelastic materials with short memory [7, 11].

The paper is organized as follows. In section 2, we introduce some notations and state some preliminary results which will be used to establish the existence of discretization and to prove the convergence of the approximants. In section 3, we present an existence and uniqueness theorem related to the new variant of the sweeping process problem (4). In section 4, we give an application to the quasistatic frictional contact problem.

2 Notation and preliminaries

Let H be a real separable Hilbert space endowed with the inner product $\langle \cdot, \cdot \rangle$ and the associated norm $\| \cdot \|$. For any $x \in H$ and $r \geq 0$, the closed ball centered at x with radius r will be denoted by $\overline{\mathbb{B}}(x, r)$. For $x = 0$ and $r = 1$, we will set $\overline{\mathbb{B}}$ instead of $\overline{\mathbb{B}}(0, 1)$. Given a set-valued map $A : H \rightrightarrows H$, we denote by $D(A)$, $G(A)$ and $R(A)$ respectively the domain,

the graph and the range of A , defined by

$$D(A) = \{x \in H : A(x) \neq \emptyset\}, \quad G(A) = \{(x, y) \in D(A) \times H : y \in A(x)\}$$

and $R(A) = \bigcup_{x \in D(A)} A(x).$

We define the *inverse* of A , A^{-1} by

$$y \in A(x) \iff x \in A^{-1}(y), \text{ i.e. } (x, y) \in G(A) \iff (y, x) \in G(A^{-1}).$$

We say that $A : H \rightrightarrows H$ is *monotone* iff

$$\langle x^* - y^*, x - y \rangle \geq 0, \quad \forall x^* \in A(x), \forall y^* \in A(y).$$

We say that $A : H \rightrightarrows H$ is *maximal monotone* iff it is monotone and its graph is maximal in the sense of the inclusion, i.e., $G(A)$ is not properly contained in the graph of any other monotone operator.

Let $J : H \rightarrow]-\infty, +\infty]$ be a lower semicontinuous, convex and proper function, i.e. $J \in \Gamma_0(H)$. The effective domain of J , denoted by $\text{Dom}(J)$ is defined by

$$\text{Dom}(J) = \{x \in H : J(x) < +\infty\}.$$

For any $x \in \text{Dom}(J)$, the subdifferential of J at x is defined by

$$\partial J(x) = \{\xi \in H : \langle \xi, y - x \rangle \leq J(y) - J(x), \forall y \in H\}. \quad (5)$$

We recall that for $x \notin \text{Dom}(J)$, $\partial J(x) = \emptyset$ and that if J is of class C^1 at x , then $\partial J(x) = \{\nabla J(x)\}$.

For the above function J , its *Legendre-Fenchel conjugate* is defined as

$$J^* : H \rightarrow \mathbb{R} \cup \{-\infty, +\infty\} \quad \text{with } J^*(x^*) := \sup_{x \in H} (\langle x^*, x \rangle - J(x)).$$

The Legendre-Fenchel conjugate is also related to the subdifferential. Indeed, for $J(x)$ finite, one has

$$x^* \in \partial J(x) \iff J^*(x^*) + J(x) = \langle x^*, x \rangle \iff x \in \partial J^*(x^*),$$

which means that $(\partial J)^{-1} = \partial J^*$, for every $J \in \Gamma_0(H)$.

Given a nonempty closed convex subset C of H , those functions corresponding to the *indicator* I_C , to the *support function* $\sigma(C, \cdot)$ of C , and to the *distance function* d_C from the set C , are defined by

$$I_C : H \rightarrow \mathbb{R} \cup \{+\infty\} \quad \text{with } I_C(x) = 0 \text{ if } x \in C \text{ and } I_C(x) = +\infty \text{ if } x \notin C,$$

$$\sigma(C, \cdot) : H \rightarrow \mathbb{R} \cup \{+\infty\} \quad \text{with } \sigma(C, x^*) := \sup_{x \in C} \langle x^*, x \rangle,$$

$$d_C : H \rightarrow \mathbb{R} \quad \text{with } d_C(x) := \inf_{y \in C} \|x - y\|.$$

From the definition of $\sigma(C, \cdot)$, we deduce that $\sigma(C, \cdot)$ coincides with the Legendre-Fenchel conjugate of I_C , that is, $\sigma(C, \cdot) = (I_C)^*$.

When $J = I_C$ and $x \in C$, we have

$$x^* \in \partial I_C(x) \text{ if and only if } \langle x^*, y - x \rangle \leq 0, \text{ for all } y \in C,$$

so $\partial I_C(x)$ is the set $N_C(x)$ of *outward normals* of the convex set C at the point $x \in C$, defined by

$$N_C(x) = \{x^* \in H : \langle x^*, y - x \rangle \leq 0, \forall y \in C\}.$$

We have also,

$$x^* \in N_C(x) \text{ if and only if } \sigma(C, x^*) = \langle x^*, x \rangle \text{ and } x \in C. \quad (6)$$

It is also clear from the inequality characterization above that

$$x - P_C(x) \in N_C(P_C(x)), \quad \text{for all } x \in H, \quad (7)$$

where $P_C(y)$ denotes the metric projection onto C .

It is easy to check that

$$N_C(-x) = -N_{-C}(x), \quad N_C(y + z) = N_{C-z}(y), \quad (8)$$

for any $x \in -C$ and y, z such that $y + z \in C$.

The Hausdorff distance between two subsets C_1 and C_2 of H , denoted by $d_H(C_1, C_2)$, is defined by

$$d_H(C_1, C_2) = \max \left\{ \sup_{x \in C_2} d_{C_1}(x), \sup_{x \in C_1} d_{C_2}(x) \right\}. \quad (9)$$

The following lemma will be useful.

Lemma 2.1 *Let C_1 and C_2 be two subsets of a Hilbert space H , and $z \in H$. Assuming that $d = d_H(C_1, C_2) < +\infty$ (i.e., C_1 and C_2 are non-empty), then we have*

$$|\sigma(C_1, z) - \sigma(C_2, z)| \leq \|z\| d_H(C_1, C_2). \quad (10)$$

Proof. From the definition of the Hausdorff distance, we have

$$\sup_{x \in C_1} d_{C_2}(x) \leq d = d_H(C_1, C_2).$$

Hence, $C_1 \subset C_2 + d\overline{\mathbb{B}}$. On the other hand, we have,

$$\sigma(C_1, z) = \sup_{x \in C_1} \langle z, x \rangle \leq \sup_{x \in C_2 + d\overline{\mathbb{B}}} \langle z, x \rangle = \sup_{x \in C_2} \langle z, x \rangle + d \sup_{x \in \overline{\mathbb{B}}} \langle z, x \rangle = \sigma(C_2, z) + d \|z\|.$$

Therefore,

$$\sigma(C_1, z) - \sigma(C_2, z) \leq \|z\| d_H(C_1, C_2). \quad (11)$$

Since C_1 and C_2 play a symmetric role, we obtain (10). ■

We collect below some classical results, that will be useful later, concerning maximal monotone operators (see e.g. [3]).

Lemma 2.2 (i) *If A is a maximal monotone operator with bounded domain, then A is onto.*

(ii) *Let $A : H \rightarrow H$ be a linear, bounded and symmetric operator satisfying:*

$$\langle Ax, x \rangle \geq 0, \text{ for all } x \in H.$$

Then $A = \nabla\varphi$ for the continuous convex function $\varphi(x) = \frac{1}{2}\langle Ax, x \rangle; \forall x \in H$.

(iii) *Let A be a maximal monotone operator and B be a maximal Lipschitz single-valued operator from H into H . Then $A + B$ is maximal monotone.*

We end this section with the following lemma on the approximation of unbounded $C(t)$.

Lemma 2.3 [10, 18] *Let $C : [0, T] \rightrightarrows H$, $t \mapsto C(t)$ satisfy (\mathcal{SP}_2) . Then there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $C_n(t) := C(t) \cap \overline{\mathbb{B}}(0, n) \neq \emptyset$ for $t \in [0, T]$, and*

$$d_H(C_n(t), C_n(s)) \leq 8d_H(C(t), C(s)) \leq 8(v(t) - v(s)), \text{ for all } 0 \leq s \leq t \leq T.$$

3 Main result

The following theorem establishes the well-posedness (existence and uniqueness result) of the evolution problem (4).

Theorem 3.1 *Assume that (\mathcal{SP}_1) and (\mathcal{SP}_2) are satisfied. Then for any initial point $u_0 \in H$, with $Bu_0 \in C(0)$ there exists a unique Lipschitz continuous mapping $u : [0, T] \rightarrow H$ satisfying (4), i.e.*

$$-\dot{u}(t) \in N_{C(t)}(Au(t) + Bu(t)) \quad \text{a.e. } t \in [0, T], \quad u(0) = u_0.$$

Proof. We proceed by discretization of the evolution problem (4): a sequence of continuous mappings $(u_n(\cdot))_{n \in \mathbb{N}}$ in $C([0, T], H)$ will be defined such that the limit of a convergent subsequence is a solution of (4). The sequence is defined via an implicit algorithm. The proof will be divided into five steps.

Step 1. Construction of approximants u_i^n . Consider for each $n \in \mathbb{N}^*$ the following partition of the interval $I := [0, T]$

$$\begin{aligned} t_i^n &:= i \frac{T}{n} && \text{for } 0 \leq i \leq n, \\ I_i^n &:=]t_i^n, t_{i+1}^n] && \text{for } 0 \leq i \leq n-1, \\ I_0^n &:= \{t_0^n\}. \end{aligned} \tag{12}$$

Lemma 2.3 ensures the existence of $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $C_n(t) := C(t) \cap \overline{\mathbb{B}}(0, n) \neq \emptyset$ for $t \in [0, T]$, and

$$d_H(C_n(t), C_n(s)) \leq 8d_H(C(t), C(s)) \leq 8(v(t) - v(s)), \text{ for all } 0 \leq s \leq t \leq T. \tag{13}$$

We propose the following numerical method based on the discretization of (4):

Set $\mu_n := \frac{T}{n}$. Fix $n \geq n_0$. We choose by induction :

• $u_0^n = u_0$,

For $i = 0, 1, \dots, n-1$:

• Find z_{i+1}^n by solving the following variational inclusion

$$-z_{i+1}^n \in N_{C_n(t_{i+1}^n)}(Az_{i+1}^n + Bu_i^n). \quad (14)$$

We show later that the following estimation holds:

$$\|z_{i+1}^n\| \leq \frac{8v(T)}{\beta} \exp\left(\frac{1}{\beta}\|B\|T\right) := M. \quad (15)$$

• Set: $u_{i+1}^n = u_i^n + \mu_n z_{i+1}^n$

The numerical method proposed above is well defined. Indeed, for $i = 0$, we have

$$-z_1^n \in N_{C_n(t_1^n)}(Az_1^n + Bu_0^n) \quad (16)$$

or equivalently,

$$-z_1^n \in N_{C_n(t_1^n) - Bu_0}(Az_1^n). \quad (17)$$

Assumption (\mathcal{SP}_1) implies that $A^{-1} : H \rightarrow H$ is monotone and $\frac{1}{\beta}$ Lipschitz. Hence, (17) can be rewritten as

$$0 \in [A^{-1} + N_{C_n(t_1^n) - Bu_0}](Az_1^n). \quad (18)$$

The lemma 2.2 ensures that the operator

$$A^{-1} + N_{C_n(t_1^n) - Bu_0} : C_n(t_1^n) - Bu_0 \rightrightarrows R(A^{-1} + N_{C_n(t_1^n) - Bu_0})$$

is also maximal monotone with domain $C_n(t_1^n) - Bu_0$.

As the operator B is bounded and all sets $C_n(t)$ are bounded, it follows from Lemma 2.2 that $[A^{-1} + N_{C_n(t_1^n) - Bu_0}]$ is onto.

Consequently,

$$R(A^{-1} + N_{C_n(t_1^n) - Bu_0}) = H,$$

i.e. there exists z_1^n solution of (16) or (18) such that $Az_1^n + Bu_0 \in C_n(t_1^n)$.

We set then, $u_1^n = u_0^n + \mu_n z_1^n$.

By (16) we have

$$\langle Az_1^n + Bu_0^n - v, z_1^n \rangle \leq 0, \text{ for all } v \in C_n(t_1^n). \quad (19)$$

Using (\mathcal{SP}_1) and (19), we have

$$\begin{aligned} \beta \|z_1^n\|^2 &\leq \langle Az_1^n, z_1^n \rangle \\ &= \langle Az_1^n + Bu_0^n - v + v - Bu_0^n, z_1^n \rangle \\ &= \langle Az_1^n + Bu_0^n - v, z_1^n \rangle + \langle v - Bu_0^n, z_1^n \rangle \\ &\leq \langle v - Bu_0^n, z_1^n \rangle \\ &\leq \|v - Bu_0^n\| \|z_1^n\|, \text{ for all } v \in C_n(t_1^n). \end{aligned}$$

Hence,

$$\|z_1^n\| \leq \frac{1}{\beta} \inf_{v \in C_n(t_1^n)} \|v - Bu_0^n\| = \frac{1}{\beta} d(Bu_0^n, C_n(t_1^n)).$$

Using the fact that $Bu_0^n \in C_n(0)$ (since $Bu_0^n \in C(0)$ and B is bounded), we get

$$\begin{aligned} \|z_1^n\| &\leq \frac{1}{\beta} d_H(C_n(t_1^n), C_n(0)) \\ &\leq \frac{8}{\beta} d_H(C(t_1^n), C(0)) \\ &\leq \frac{8}{\beta} (v(t_1^n) - v(0)) \\ &\leq \frac{8v(T)}{\beta}. \end{aligned}$$

Now suppose that $u_0^n, u_1^n, \dots, u_i^n, z_1^n, z_2^n, \dots, z_i^n$ have been constructed. Observe that the operator $A^{-1} + N_{C_n(t_{i+1}^n) - Bu_i^n} : C_n(t_{i+1}^n) - Bu_i^n \rightrightarrows R(A^{-1} + N_{C_n(t_{i+1}^n) - Bu_i^n})$ is maximal monotone with bounded domain $C_n(t_{i+1}^n) - Bu_i^n$. Then Lemma 2.2 ensures that $R(A^{-1} + N_{C_n(t_{i+1}^n) - Bu_i^n}) = H$. Therefore, there exists z_{i+1}^n such that $Az_{i+1}^n + Bu_i^n \in C_n(t_{i+1}^n)$ solution of

$$-z_{i+1}^n \in N_{C_n(t_{i+1}^n)}(Az_{i+1}^n + Bu_i^n), \quad (20)$$

which allows us to set $u_{i+1}^n := u_i^n + \mu_n z_{i+1}^n$.

Also (20), $u_i^n = u_0^n + \mu_n \sum_{k=1}^i z_k^n$, $Bu_0^n \in C_n(0)$ and (\mathcal{SP}_1) imply that for all $v \in C_n(t_{i+1}^n)$

$$\begin{aligned} \|z_{i+1}^n\| &\leq \frac{1}{\beta} \|v - Bu_i^n\| = \frac{1}{\beta} \|v - Bu_0 - \mu_n \sum_{k=1}^i Bz_k^n\| \\ &\leq \frac{1}{\beta} d(Bu_0, C_n(t_{i+1}^n)) + \frac{1}{\beta} \|B\| \mu_n \sum_{k=1}^i \|z_k^n\| \\ &\leq \frac{1}{\beta} d_H(C_n(0), C_n(t_{i+1}^n)) + \frac{1}{\beta} \|B\| \mu_n \sum_{k=1}^i \|z_k^n\| \\ &\leq \frac{8}{\beta} d_H(C(0), C(t_{i+1}^n)) + \frac{1}{\beta} \|B\| \mu_n \sum_{k=1}^i \|z_k^n\| \\ &\leq \frac{8}{\beta} (v(t_{i+1}^n) - v(0)) + \frac{1}{\beta} \|B\| \mu_n \sum_{k=1}^i \|z_k^n\|. \end{aligned}$$

Hence,

$$\|z_{i+1}^n\| \leq \frac{8v(T)}{\beta} + \frac{1}{\beta} \|B\| \mu_n \sum_{k=1}^i \|z_k^n\|.$$

Using a discrete version of Gronwall's inequality, we obtain

$$\|z_{i+1}^n\| \leq \frac{8v(T)}{\beta} \exp\left(\frac{1}{\beta}\|B\|i\mu_n\right)$$

or equivalently,

$$\|z_{i+1}^n\| \leq \frac{8v(T)}{\beta} \exp\left(\frac{1}{\beta}\|B\|T\right) := M,$$

which means that (15) is satisfied and the numerical method is therefore well defined.

Step 2. Construction of the sequence $(u_n(\cdot))$.

Using the sequences u_i^n and z_i^n , we construct the sequence of mapping $u_n : [0, T] \rightarrow H$, $t \mapsto u_n(t)$ by defining their restrictions to each interval I_i^n as follows:

$$u_n(t) = \begin{cases} u_0^n & \text{if } t = 0 \\ u_i^n + \frac{(t-t_i^n)}{\mu_n}(u_{i+1}^n - u_i^n) & \text{if } t \in I_i^n, i = 0, 1, \dots, n-1. \end{cases}$$

Clearly the mapping $u_n(\cdot)$ is Lipschitz on $[0, T]$, and M is a Lipschitz constant of $u_n(\cdot)$ on $[0, T]$ since for every $t \in]t_i^n, t_{i+1}^n[$, we have

$$\dot{u}_n(t) = \frac{(u_{i+1}^n - u_i^n)}{\mu_n} = z_{i+1}^n \text{ with } Az_{i+1}^n + Bu_i^n \in C_n(t_{i+1}^n).$$

Furthermore, for every $t \in [0, T]$ one has $u_n(t) = u_0 + \int_0^t \dot{u}_n(s)ds$.

Hence,

$$\|u_n(t)\| \leq \|u_0\| + MT.$$

By (14) we have

$$-z_{i+1}^n \in N_{C_n(t_{i+1}^n)}(Az_{i+1}^n + Bu_i^n). \quad (21)$$

Define the functions θ_n and δ_n from $[0, T]$ to $[0, T]$ by $\theta_n(t) = t_{i+1}^n$ and $\delta_n(t) = t_i^n$ for any $t \in I_i^n$. Inclusion (21) becomes

$$-\dot{u}_n(t) \in N_{C_n(\theta_n(t))}(A\dot{u}_n(t) + Bu_n(\delta_n(t))) \text{ a.e. } t \in [0, T]. \quad (22)$$

Step 3. Convergence of $(u_n(\cdot))$. First, We note that

$$\sup_{t \in [0, T]} |\theta_n(t) - t| \rightarrow 0 \text{ as } n \rightarrow \infty \text{ and } \sup_{t \in [0, T]} |\delta_n(t) - t| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Now, let us prove the convergence of sequences (u_n) and (\dot{u}_n) . We have for all $n \geq n_0$

$$\begin{cases} \|u_n(t)\| \leq \|u_0\| + TM, & \text{for all } t \in [0, T] \text{ and} \\ \|\dot{u}_n(t)\| \leq M & \text{for almost all } t \in [0, T]. \end{cases}$$

We deduce that the sequence (u_n) is uniformly bounded in norm and variation. Using Theorem 0.2.1 in [12], there exists a function $u : [0, T] \rightarrow H$ of bounded variation and a subsequence, still denoted (u_n) , such that

$$u_n(t) \rightharpoonup u(t) \text{ weakly in } H \text{ for all } t \in [0, T], \quad (23)$$

$$u_n \rightharpoonup u \text{ in the weak-star topology of } L^\infty([0, T], H), \quad (24)$$

and, for some $v_* \in L^2([0, T], H)$

$$\dot{u}_n \rightharpoonup v_* \text{ in the weak topology of } L^2([0, T], H). \quad (25)$$

In particular, $u(0) = u_0$. The Lipschitz continuity of u_n and the weak lower semicontinuity of the norm give

$$\|u(t) - u(s)\| \leq \liminf_{n \rightarrow +\infty} \|u_n(t) - u_n(s)\| \leq M|t - s| \text{ for all } t, s \in [0, T],$$

which shows that $u(\cdot)$ is Lipschitz continuous on $[0, T]$, and hence its derivative $\dot{u}(\cdot)$ exists for almost every $t \in [0, T]$.

Fix any $t \in [0, T]$. For each $w \in H$ with $\|w\| \leq 1$, we can write

$$\begin{aligned} \left| \langle w, u_n(\theta_n(t)) - u(t) \rangle \right| &\leq \left| \langle w, u_n(\theta_n(t)) - u_n(t) \rangle \right| + \left| \langle w, u_n(t) - u(t) \rangle \right| \\ &\leq M \left| \theta_n(t) - t \right| + \left| \langle w, u_n(t) - u(t) \rangle \right|. \end{aligned} \quad (26)$$

Taking into account (23), we get $u_n(\theta_n(t)) \rightharpoonup u(t)$ weakly in H as $n \rightarrow \infty$. On the other hand, we have

$$\langle w, u_n(t) \rangle = \langle w, u_0 \rangle + \int_0^t \langle \mathbf{1}_{[0,t]}(s)w, \dot{u}_n(s) \rangle ds.$$

Using (25) and taking the limit as $n \rightarrow \infty$, we obtain

$$\langle w, u(t) \rangle = \langle w, u_0 \rangle + \int_0^t \langle \mathbf{1}_{[0,t]}(s)w, v_*(s) \rangle ds = \left\langle w, u_0 + \int_0^t v_*(s) ds \right\rangle.$$

The latter equality being true for all $w \in H$, we deduce that $u(t) = u_0 + \int_0^t v_*(s) ds$, and this guarantees that $\dot{u}(\cdot) = v_*(\cdot)$ almost everywhere.

Consequently,

$$\dot{u}_n(\cdot) \rightharpoonup \dot{u}(\cdot) \text{ weakly in } L^2([0, T], H). \quad (27)$$

Using (26), we deduce that $u_n(\theta_n(t)) \rightharpoonup u(t)$, as $n \rightarrow +\infty$, weakly in H , for all $t \in [0, T]$.

Step 4. We show that $u(\cdot)$ is a solution of (4).

Let us prove first the following viability condition:

$$A\dot{u}(t) + Bu(t) \in C(t), \text{ a.e. on } [0, T]. \quad (28)$$

Fix any $t \in [0, T]$ such that $\dot{u}(t)$ exists. For each $z \in H$ with $\|z\| \leq 1$, we can write

$$\begin{aligned} \langle z, A\dot{u}(t) + Bu(t) \rangle &= \langle z, A\dot{u}(t) + Bu(t) - A\dot{u}_n(t) - Bu_n(\delta_n(t)) + A\dot{u}_n(t) + Bu_n(\delta_n(t)) \rangle \\ &= \langle z, A\dot{u}_n(t) + Bu_n(\delta_n(t)) \rangle + \langle z, A\dot{u}(t) - A\dot{u}_n(t) + Bu(t) - Bu_n(\delta_n(t)) \rangle. \end{aligned}$$

as $A\dot{u}_n(t) + Bu_n(\delta_n(t)) \in C_n(\theta_n(t)) \subset C(\theta_n(t))$, the last inequality becomes

$$\langle z, A\dot{u}(t) + Bu(t) \rangle \leq \sigma(C(\theta_n(t)), z) + \langle z, A\dot{u}(t) - A\dot{u}_n(t) + Bu(t) - Bu_n(\delta_n(t)) \rangle. \quad (29)$$

From the property of the support function (see Lemma 2.1), we derive

$$|\sigma(C(\theta_n(t)), z) - \sigma(C(t), z)| \leq \|z\|d_H(C(\theta_n(t)), C(t)).$$

Hence,

$$\sigma(C(\theta_n(t)), z) \leq \sigma(C(t), z) + |v(\theta_n(t)) - v(t)| \quad (30)$$

Combining (29) and (30) we obtain

$$\begin{aligned} \langle z, A\dot{u}(t) + Bu(t) \rangle &\leq \sigma(C(t), z) + |v(\theta_n(t)) - v(t)| + \langle z, A\dot{u}(t) - A\dot{u}_n(t) \rangle + \\ &\quad \langle z, Bu(t) - Bu_n(\delta_n(t)) \rangle. \end{aligned} \quad (31)$$

Integrating (31) with $\tau > 0$ small, we obtain

$$\begin{aligned} \int_{t-\tau}^{t+\tau} \langle z, A\dot{u}(s) + Bu(s) \rangle ds &\leq \int_{t-\tau}^{t+\tau} \sigma(C(s), z) ds + \int_{t-\tau}^{t+\tau} \langle z, A\dot{u}(s) - A\dot{u}_n(s) \rangle ds + \\ &\quad \int_{t-\tau}^{t+\tau} \langle z, Bu(s) - Bu_n(\delta_n(s)) \rangle ds + \int_0^T |v(\theta_n(t)) - v(t)| dt. \end{aligned}$$

It is easily seen that $\int_0^T |v(\theta_n(t)) - v(t)| dt \rightarrow 0$ as $n \rightarrow \infty$. Also the weak convergence of u_n and \dot{u}_n to u and \dot{u} in $L^2([0, T]; H)$ respectively and the properties of A and B yield as $n \rightarrow \infty$

$$\int_{t-\tau}^{t+\tau} \langle z, A\dot{u}(s) + Bu(s) \rangle ds \leq \int_{t-\tau}^{t+\tau} \sigma(C(s), z) ds.$$

Dividing by 2τ and letting τ tend to zero, the Lebesgue differentiation theorem gives

$$\langle z, A\dot{u}(t) + Bu(t) \rangle \leq \sigma(C(t), z).$$

The latter inequality being true for all $z \in H$, we deduce according to the closeness and the convexity of $C(t)$ that $A\dot{u}(t) + Bu(t) \in C(t)$, which means that (28) is proved.

Finally we show that $u(\cdot)$ satisfies the differential inclusion in (4). By (18) and the definition of the normal cone, we have

$$\langle -\dot{u}_n(t), v - A\dot{u}_n(t) - Bu_n(\delta_n(t)) \rangle \leq 0, \quad \forall v \in C_n(\theta_n(t)), \quad \text{a.e. } t \in [0, T]. \quad (32)$$

We claim that for all $t \in [0, T]$ for which (32) holds and $v \in C_n(t)$ we have

$$\langle -\dot{u}_n(t), v - A\dot{u}_n(t) - Bu_n(\delta_n(t)) \rangle \leq \epsilon_n(t), \quad (33)$$

with $\epsilon_n(t) = 8M|v(\theta_n(t)) - v(t)|$.

Indeed, by (13) we have $v \in C_n(t) \subset C_n(\theta_n(t)) + 8|v(\theta_n(t)) - v(t)|\mathbb{B}$. So, there exists

$\tilde{v} \in C_n(\theta_n(t))$ with $\|v - \tilde{v}\| \leq 8|v(\theta_n(t)) - v(t)|$.

By (32), we obtain

$$\begin{aligned} \langle -\dot{u}_n(t), v - A\dot{u}_n(t) - Bu_n(\delta_n(t)) \rangle &= \langle -\dot{u}_n(t), v - \tilde{v} + \tilde{v} - A\dot{u}_n(t) - Bu_n(\delta_n(t)) \rangle \\ &= \langle -\dot{u}_n(t), v - \tilde{v} \rangle + \\ &\quad \langle -\dot{u}_n(t), \tilde{v} - A\dot{u}_n(t) - Bu_n(\delta_n(t)) \rangle \\ &\leq 8M|v(\theta_n(t)) - v(t)| = \epsilon_n(t), \end{aligned}$$

the claim (33) follows.

Choose arbitrary $t_0 \in [0, T]$ and $v_0 \in C(t_0)$. Suppose that $\tau > 0$ is such that for all most all $t \in [0, T]$, there exists a unique projection

$$v(t) = \begin{cases} P_{C(t)}(v_0) \in C(t), & \text{if } t \in [t_0 - \tau, t_0 + \tau], \\ P_{C(t)}(A\dot{u}(t) + Bu(t)) = A\dot{u}(t) + Bu(t) \in C(t), & \text{otherwise.} \end{cases}$$

Hence, $v(\cdot)$ is a $L^\infty([0, T]; H)$ selection of the absolutely continuous set-valued map $C(\cdot)$ on $[0, T]$. The boundedness of $v(\cdot)$ on the compact interval $[0, T]$ implies that $v(t) \in C_n(t)$ for all $n \in \mathbb{N}$ sufficiently large. Using (33), we have

$$\int_0^T \langle -\dot{u}_n(t), v(t) - A\dot{u}_n(t) - Bu_n(\delta_n(t)) \rangle dt \leq \int_0^T \epsilon_n(t) dt, \quad (34)$$

or equivalently,

$$\int_0^T \langle \dot{u}_n(t), A\dot{u}_n(t) \rangle dt + \int_0^T \langle \dot{u}_n(t), Bu_n(\delta_n(t)) \rangle dt + \int_0^T \langle \dot{u}_n(t), -v(t) \rangle dt \leq \int_0^T \epsilon_n(t) dt. \quad (35)$$

From the properties of A , we note that the function $x(t) \mapsto \int_0^T \langle x(t), Ax(t) \rangle dt$ is convex

and weakly lower semicontinuous on $L^2([0, T]; H)$.

So we have,

$$\int_0^T \langle \dot{u}(t), A\dot{u}(t) \rangle dt \leq \liminf_{n \rightarrow \infty} \int_0^T \langle \dot{u}_n(t), A\dot{u}_n(t) \rangle dt, \quad (36)$$

and

$$\int_0^T \langle \dot{u}(t), -v(t) \rangle dt = \lim_{n \rightarrow \infty} \int_0^T \langle \dot{u}_n(t), -v(t) \rangle dt. \quad (37)$$

On the other hand, we have $B = \nabla\varphi_B$, for the continuous convex function $\varphi_B(x) = \frac{1}{2}\langle Bx, x \rangle$. Therefore, the absolute continuity of $\varphi_B \circ u$ and $\varphi_B \circ u_n$ gives

$$\begin{aligned}
\int_0^T \langle Bu(t), \dot{u}(t) \rangle dt &= \int_0^T \frac{d}{dt} \varphi_B(u(t)) dt = \varphi_B(u(T)) - \varphi_B(u(0)) \\
&\leq \liminf_{n \rightarrow \infty} \left(\varphi_B(u_n(T)) - \varphi_B(u_n(0)) \right) \\
&= \liminf_{n \rightarrow \infty} \left(\int_0^T \frac{d}{dt} \varphi_B(u_n(t)) dt \right) \\
&= \liminf_{n \rightarrow \infty} \int_0^T \langle Bu_n(t), \dot{u}_n(t) \rangle dt,
\end{aligned} \tag{38}$$

where the inequality is due to the weak lower semicontinuity of φ_B on H and to the fact that $u_n(T) \rightharpoonup u(T)$ weakly in H as $n \rightarrow \infty$.

Since,

$$\int_0^T \left| \langle Bu_n(t) - Bu_n(\delta_n(t)), \dot{u}_n(t) \rangle \right| dt \leq M^2 \|B\| \int_0^T |t - \delta_n(t)| dt,$$

we deduce that

$$\liminf_{n \rightarrow \infty} \int_0^T \langle Bu_n(t), \dot{u}_n(t) \rangle dt = \liminf_{n \rightarrow \infty} \int_0^T \langle Bu_n(\delta_n(t)), \dot{u}_n(t) \rangle dt. \tag{39}$$

Since $\int_0^T \epsilon_n(t) dt \rightarrow 0$ as $n \rightarrow \infty$, inequalities (35), (36), (37), (38) and (39) yield as $n \rightarrow \infty$

$$\int_0^T \langle -\dot{u}(t), v(t) - A\dot{u}(t) - Bu(t) \rangle dt \leq 0. \tag{40}$$

Using the definition of $v(t)$ above, we get

$$\int_{t_0-\tau}^{t_0+\tau} \langle -\dot{u}(t), v(t) - A\dot{u}(t) - Bu(t) \rangle dt \leq 0. \tag{41}$$

Dividing (41) by 2τ , letting τ goes to zero and using the Lebesgue differentiation theorem, we get

$$\langle -\dot{u}(t_0), v(t_0) - A\dot{u}(t_0) - Bu(t_0) \rangle \leq 0, \tag{42}$$

or equivalently,

$$\langle -\dot{u}(t_0), v_0 - A\dot{u}(t_0) - Bu(t_0) \rangle \leq 0, \tag{43}$$

for all $t_0 \in [0, T]$, outside a fixed set of measure zero $\{t_i^n, i = 0, 1, \dots, n; n \in \mathbb{N}\}$, and all $v_0 \in C(t_0)$. This means that $u(\cdot)$ is a solution of the inclusion (4).

Step 5. Uniqueness of the solution. Suppose that (u_1, u_2) are two solutions satisfying (4) such that $u_1(0) = u_2(0) = u_0$. Then for almost every $t \in [0, T]$, we have for $i = 1, 2$

$$\langle -\dot{u}_i(t), v - A\dot{u}_i(t) - Bu_i(t) \rangle \leq 0, \text{ for all } v \in C(t). \quad (44)$$

Using the fact that $A\dot{u}_i(t) + Bu_i(t) \in C(t)$ a.e., we obtain, for a.e. $t \in [0, T]$,

$$\begin{cases} \langle \dot{u}_1(t), A\dot{u}_1(t) + Bu_1(t) - A\dot{u}_2(t) - Bu_2(t) \rangle \leq 0, \\ \langle -\dot{u}_2(t), A\dot{u}_1(t) + Bu_1(t) - A\dot{u}_2(t) - Bu_2(t) \rangle \leq 0. \end{cases}$$

By adding the last two inequalities, we get

$$\left\langle \dot{u}_1(t) - \dot{u}_2(t), A(\dot{u}_1(t) - \dot{u}_2(t)) + B(u_1(t) - u_2(t)) \right\rangle \leq 0.$$

Since A is coercive, we obtain

$$\beta \|\dot{u}_1(t) - \dot{u}_2(t)\|^2 \leq \|B\| \|\dot{u}_1(t) - \dot{u}_2(t)\| \|u_1(t) - u_2(t)\|.$$

As $u_1(0) = u_2(0) = u_0$, we get

$$\|\dot{u}_1(t) - \dot{u}_2(t)\| \leq \frac{\|B\|}{\beta} \int_0^t \|\dot{u}_1(\tau) - \dot{u}_2(\tau)\| d\tau,$$

which means by Gronwall's inequality that $\dot{u}_1(t) = \dot{u}_2(t)$ for a.e. $t \in [0, T]$. Therefore $u_1(t) = u_2(t)$ for all $t \in [0, T]$. The proof of Theorem 3.1 is thereby completed. ■

4 Application to quasistatic frictional contact problem

As an application of the sweeping process problem (4), we consider the following evolution variational inequality

$$\begin{cases} \text{Find } u : [0, T] \longrightarrow H \text{ such that } \dot{u}(t) = \frac{du(t)}{dt} \in \mathcal{K} \text{ a.e. } t \in [0, T] \text{ and} \\ a(\dot{u}(t), v - \dot{u}(t)) + b(u(t), v - \dot{u}(t)) + j(v) - j(\dot{u}(t)) \geq \langle f(t), v - \dot{u}(t) \rangle, \forall v \in \mathcal{K}. \\ u(0) = u_0 \in H, \end{cases} \quad (45)$$

Assume that the following assumptions are satisfied:

($\mathcal{V}\mathcal{I}_1$) $\mathcal{K} \subset H$ is a nonempty, closed and convex cone (hence $0 \in \mathcal{K}$).

($\mathcal{V}\mathcal{I}_2$) $a(\cdot, \cdot), b(\cdot, \cdot) : H \times H \rightarrow \mathbb{R}$ are two real continuous bilinear and symmetric forms satisfying for all $u \in H$

$$a(u, u) \geq \alpha_0 \|u\|^2,$$

for some positive constant $\alpha_0 > 0$ and $b(u, u) \geq 0$.

(\mathcal{VT}_3) $j : \mathcal{K} \rightarrow \mathbb{R}$ is a convex, positively homogeneous of degree 1 (i.e. $j(\lambda x) = \lambda j(x)$, $\forall \lambda > 0$) and Lipschitz continuous with $j(0) = 0$.

(\mathcal{VT}_4) $f \in W^{1,1}([0, T]; H)$ with $b(u_0, v) + j(v) \geq \langle f(0), v \rangle$, for all $v \in \mathcal{K}$.

Remark 4.1 The compatibility condition on the initial data

$$b(u_0, v) + j(v) \geq \langle f(0), v \rangle, \quad \forall v \in \mathcal{K},$$

ensures that initially the state is in equilibrium and that $Bu_0 \in C(0)$ (see (53)).

The evolution variational inequality (45) is of great interest in the modeling of the quasistatic frictional contact problems (see [7, 8, 11]). In a mechanical language, the bilinear form $a(\cdot, \cdot)$ represents the viscosity term, the bilinear form $b(\cdot, \cdot)$ represents the elasticity term, the functional j represents the friction functional of Tresca type.

For our purpose of motivation, the main concern is to prove that the variational inequality (45) is of type (4). In other words, we will convert the quasistatic variational inequality (45) to the problem of finding a solution of the sweeping process (4).

Let us first extend the function j from \mathcal{K} to the whole space H by introducing the functional $J : H \rightarrow \mathbb{R} \cup \{+\infty\}$, $z \mapsto J(z)$ defined by

$$J(z) = \begin{cases} j(z), & z \in \mathcal{K}, \\ +\infty, & z \notin \mathcal{K}. \end{cases} \quad (46)$$

Since \mathcal{K} is a nonempty, closed and convex cone, and j is convex, positively homogeneous of degree 1 and Lipschitz continuous on \mathcal{K} , we deduce that the extended functional $J : H \rightarrow \mathbb{R} \cup \{+\infty\}$ is proper, positively homogeneous of degree 1, convex and lower semicontinuous with $J(0) = 0$.

With this extension, (45) is equivalent to

$$\begin{cases} \text{Find } u : [0, T] \rightarrow H \text{ such that for a.e. } t \in [0, T] \text{ we have} \\ a(\dot{u}(t), v - \dot{u}(t)) + b(u(t), v - \dot{u}(t)) + J(v) - J(\dot{u}(t)) \geq \langle f(t), v - \dot{u}(t) \rangle \quad \forall v \in H. \\ u(0) = u_0 \in H, \end{cases} \quad (47)$$

Let A and B be the linear bounded and symmetric operators associated respectively to the bilinear forms $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$, that is,

$$\langle Au, v \rangle = a(u, v) \text{ and } \langle Bu, v \rangle = b(u, v), \text{ for all } u, v \in H. \quad (48)$$

Using the definition of the subdifferential given in (5), we can rewrite (47) in the following form

$$\begin{cases} f(t) - A\dot{u}(t) - Bu(t) \in \partial J(\dot{u}(t)) \quad \text{a.e. } t \in [0, T], \\ u(0) = u_0 \in H. \end{cases} \quad (49)$$

The following Proposition shows the equivalence between the variant of the sweeping process introduced in (4) and the quasistatic variational inequality (47).

Proposition 4.2 *Assume that assumptions (\mathcal{VI}_1) - (\mathcal{VI}_4) are satisfied. The function $u : [0, T] \rightarrow H$ is a solution of (45) if and only if it is a solution of the sweeping process (4), where A and B are the linear bounded and symmetric operators associated with $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$, and $C(t) := f(t) - \partial J(0)$, $t \in [0, T]$ with J defined in (46).*

Proof. It is easy to see that u is a solution of the variational inequality (45) if and only if it is a solution of the differential inclusion (49).

From the properties of the subdifferential of ∂J and since $J(0) = 0$, we deduce that the subset

$$C := \partial J(0) = \{\xi \in H : \langle \xi, v \rangle \leq J(v), \forall v \in H\}, \quad (50)$$

is a closed convex subset in H .

Since J is positively homogeneous of degree 1 with $J(0) = 0$, from a standard result in convex analysis, we have

$$J(z) = \sigma(C, z) = \mathbf{I}_C^*(z).$$

Hence,

$$\partial J(\cdot) = \partial \mathbf{I}_C^*(\cdot) \text{ and } J^*(\cdot) = \mathbf{I}_C^{**}(\cdot) = \mathbf{I}_C(\cdot).$$

On the other hand, we have

$$p \in \partial J(z) \iff z \in \partial J^*(p).$$

Therefore,

$$p \in \partial J(z) \iff z \in \partial \mathbf{I}_C(p) \iff z \in \mathbf{N}_C(p), \text{ with } C = \partial J(0). \quad (51)$$

Applying (51) to (49) and using (8), we get for a.e. $t \in [0, T]$,

$$\begin{aligned} f(t) - A\dot{u}(t) - Bu(t) \in \partial J(\dot{u}(t)) &\iff \dot{u}(t) \in \mathbf{N}_C(f(t) - A\dot{u}(t) - Bu(t)) \\ &\iff \dot{u}(t) \in \mathbf{N}_{C-f(t)}(-A\dot{u}(t) - Bu(t)) \\ &\iff -\dot{u}(t) \in \mathbf{N}_{C(t)}(A\dot{u}(t) + Bu(t)), \end{aligned}$$

with $C(t) := f(t) - C = f(t) - \partial J(0)$.

Hence, problem (49) is equivalent to

$$\begin{cases} -\dot{u}(t) \in \mathbf{N}_{C(t)}(A\dot{u}(t) + Bu(t)) & \text{a.e. } t \in [0, T], \\ u(0) = u_0 \in H, \end{cases} \quad (52)$$

which is exactly of the form of the variant of the sweeping process introduced in (45). \blacksquare

As a consequence of Theorem 3.1, we have the following existence and uniqueness result for the quasistatic variational inequality (45).

Corollary 4.3 *Assume that assumptions $(\mathcal{V}\mathcal{I}_1)$ - $(\mathcal{V}\mathcal{I}_4)$ are satisfied. Then for each $u_0 \in H$, the evolution variational inequality (45) has a unique solution u .*

Proof. Let us check that all assumptions of Theorem 3.1 are satisfied. It is clear that assumptions $(\mathcal{V}\mathcal{I}_2)$ are equivalent to $(\mathcal{S}\mathcal{P}_1)$. Let us check now that $(\mathcal{S}\mathcal{P}_2)$ is verified. For every $t \in [0, T]$, we have $C(t) = f(t) - C = f(t) - \partial J(0)$. It is clear that $C(t)$ is a closed and convex set of H . On the hand, we have,

$$\begin{aligned} Bu_0 \in C(0) &\iff f(0) - Bu_0 \in C \\ &\iff f(0) - Bu_0 \in \partial J(0) \\ &\iff \langle f(0), v \rangle \leq j(v) + b(u_0, v), \forall v \in \mathcal{K}. \end{aligned} \quad (53)$$

Let us show now that the set-valued map $t \mapsto C(t)$ moves in an absolute continuous way. In fact, for all $0 \leq s \leq t \leq T$, we have

$$\begin{aligned} d_H(C(t), C(s)) &\leq \|f(t) - f(s)\| \\ &= \left\| \int_s^t \dot{f}(\tau) d\tau \right\| \\ &\leq \int_s^t \|\dot{f}(\tau)\| d\tau \\ &= v(t) - v(s) \text{ with } v(t) := \int_0^t \|\dot{f}(\tau)\| d\tau. \end{aligned}$$

which means that $C(\cdot)$ varies in an absolutely continuous way.

Hence, all assumptions of Theorem 3.1 are satisfied. The existence and uniqueness of a solution to problem (45) is simply a consequence of Theorem 3.1. ■

Example 4.4 Quasistatic frictional contact problem involving viscoelastic materials with short memory [11]. Let $\Omega \subset \mathbb{R}^2$ be the section of a tube with infinity length $\Omega \times]-\infty, +\infty[$ (see Figure 1). We assume that Ω is an open bounded connected set with a regular boundary $\Gamma = \partial\Omega$, (Γ is a one-dimensional manifold of class C^m ($m \geq 1$)) and Ω is located on one side of $\partial\Omega$). We suppose that Γ is composed of three parts $\Gamma = \bar{\Gamma}_1 \cup \bar{\Gamma}_2 \cup \bar{\Gamma}_3$, with Γ_i three open subsets for $i = 1, 2, 3$ and $\text{meas}(\Gamma_1) > 0$. In this case the well-known Korn's inequality is satisfied. We assume that the cylinder is clamped on Γ_1 and in contact with a rigid foundation on Γ_3 . The cylinder deforms under the action of a surface density force f_0 on Ω acting in the axle-direction and traction forces of density f_2 on Γ_2 (for more details about the mathematical modeling of the antiplane shear, we refer to [11], Chapter 8). For simplification, we will omit the dependence of functions with respect to the space variable $x \in \Omega \cup \Gamma$ and the time variable $t \in [0, T]$. Moreover, the dot represents the time derivative i.e. $\dot{u} = \frac{du}{dt}$.

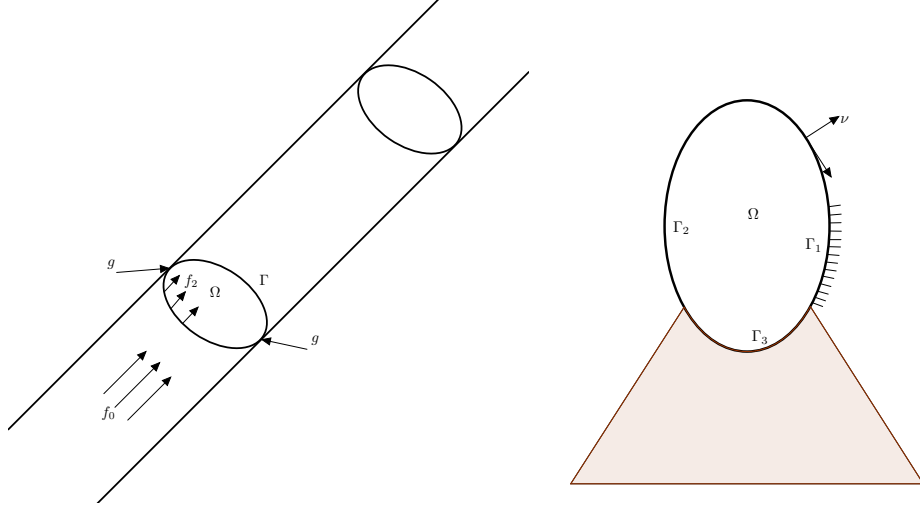


Figure 1: Cross section of the cylinder in contact with a foundation.

The displacement of the cylinder is governed by the following quasistatic variational inequality:

$$\left\{ \begin{array}{l} \text{Find } u : [0, T] \longrightarrow H \text{ such that for a.e. } t \in [0, T] \text{ we have} \\ a(\dot{u}(t), v - \dot{u}(t)) + b(u(t), v - \dot{u}(t)) + J(v) - J(\dot{u}(t)) \geq \langle f(t), v - \dot{u}(t) \rangle \quad \forall v \in H, \\ u(0) = u_0 \in H, \end{array} \right. \quad (54)$$

where $H = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_1\}$, the bilinear forms: $a(\cdot, \cdot)$, $b(\cdot, \cdot) : H \times H \rightarrow \mathbb{R}$, $(u, v) \mapsto a(u, v)$, $b(u, v)$, the frictional functional $J : H \rightarrow \mathbb{R}$, $v \mapsto J(v)$ and the function $f : [0, T] \rightarrow H$, $t \mapsto f(t)$ are defined respectively by

$$a(u, v) = \int_{\Omega} \eta \nabla u \cdot \nabla v \, dx \quad (55a)$$

$$b(u, v) = \int_{\Omega} \kappa \nabla u \cdot \nabla v \, dx \quad (55b)$$

$$J(v) = \int_{\Gamma_3} g|v| \, d\Gamma. \quad (55c)$$

$$\langle f(t), v \rangle = \int_{\Omega} f_0(t)v \, dx + \int_{\Gamma_2} f_2(t)v \, d\Gamma. \quad (55d)$$

Remark 4.5 The variational formulation (54) derived from the following problem:

Find a displacement field $u : [0, T] \times \Omega \rightarrow \mathbb{R}$, $(t, x) \mapsto u(t, x)$ such that

$$\begin{cases}
 \operatorname{div}(\eta(x)\nabla\dot{u}(t, x) + \kappa(x)\nabla u(t, x)) + f_0 = 0 & \text{in }]0, T[\times \Omega; & (56a) \\
 u(t, \cdot) = 0 & \text{on }]0, T[\times \Gamma_1; & (56b) \\
 \eta\partial_\nu\dot{u} + \kappa\partial_\nu u = f_2 & \text{on }]0, T[\times \Gamma_2; & (56c) \\
 |\eta\partial_\nu\dot{u} + \kappa\partial_\nu u| \leq g & \text{on }]0, T[\times \Gamma_3; & (56d) \\
 \eta\partial_\nu\dot{u} + \kappa\partial_\nu u = -g\frac{\dot{u}}{|\dot{u}|} \text{ if } \dot{u} \neq 0, & \text{on }]0, T[\times \Gamma_3; & (56e) \\
 u(0, \cdot) = u_0(\cdot) & \text{in } \Omega. & (56f)
 \end{cases}$$

Here ν denotes the unit outer normal on the boundary Γ . Equation (56a) is the equilibrium state equation where a viscoelastic constitutive law with short memory is assumed, (56b) is the Dirichlet boundary condition on Γ_1 , (56c) is the traction boundary condition on Γ_2 , (56d)-(56e) are the frictional conditions and (56f) is the initial condition. For more details we refer to [11] page 191. If the contact is modeled with a nonmonotone normal compliance condition and a unilateral constraint, then it is possible to study the problem in the framework of variational-hemivariational inequalities (see e.g. the recent papers [2, 9] and references therein).

We suppose that the viscosity coefficient η , the Lamé coefficient κ , the forces f_0 , f_2 and the friction function g satisfy the following conditions

$$\kappa \in L^\infty(\Omega) \quad (57a)$$

$$\eta \in L^\infty(\Omega) \text{ with } \eta(x) \geq \eta^* \text{ a.e. } x \in \Omega \text{ (for some } \eta^* > 0). \quad (57b)$$

$$f_0 \in W^{1,1}([0, T]; L^2(\Omega)), \quad f_2 \in W^{1,1}([0, T]; L^2(\Gamma_2)) \quad (57c)$$

$$g(x) \geq 0 \text{ a.e. } x \in \Gamma_3 \text{ and } g \in L^2(\Gamma_3). \quad (57d)$$

$$b(u_0, v) + \int_{\Gamma_3} g|v| \, d\Gamma \geq \int_{\Omega} f_0(0)v \, dx + \int_{\Gamma_2} f_2(0)v \, d\Gamma, \quad \forall v \in H. \quad (57e)$$

As a direct consequence of Corollary 4.3, we show that problem (54) is well-posed.

Corollary 4.6 *Assume (57a)-(57d). Then for each $u_0 \in H$ satisfying (57e), problem (54)-(55) has a unique solution.*

Proof. We have

$$|a(u, v)| \leq \|\eta\|_\infty \|u\| \|v\| \text{ and } |b(u, v)| \leq \|\kappa\|_\infty \|u\| \|v\|.$$

The coercivity of $a(\cdot, \cdot)$ follows from (57b)

$$a(v, v) \geq \eta^* \|v\|^2, \quad \forall v \in H.$$

Assumption (57e) implies the compatibility condition $(\mathcal{V}\mathcal{I}_4)$. All assumptions $(\mathcal{V}\mathcal{I}_1)$ - $(\mathcal{V}\mathcal{I}_4)$ are satisfied. The conclusion follows by Corollary 4.3. ■

Remark 4.7 We note that the existence of a unique solution to problem (54)-(55) was obtained in [11] without the compatibility condition (57e). This condition was used in [11] for the study of quasistatic frictional problems with elastic materials (see Section 9.3 page 184 and (11.37) page 208 in [11]). We note that the compatibility condition (57e) is necessary in many quasistatic problems, it guarantees that the initial state is in equilibrium otherwise the inertial terms $\ddot{u}(t)$ cannot be neglected and the problem is no longer quasistatic (it will be a dynamic of second-order). For the implicit sweeping process studied in this paper, condition (57e) is equivalent to the viability condition $Bu_0 \in C(0)$ (necessary to start the algorithm since outside this set the normal cone would be empty).

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