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Walking on two legs: Growth accounting with labor-saving and capital-saving technical change*

Preliminary version, all comments welcome

Pierre Barral[†] Mehdi Senouci[‡]

February 15, 2018

Abstract

We present an alternative to growth accounting *à la* Solow, on the same set of variables, that provides a metric for labor-saving technical change (' λ ') and capital-saving technical change (' μ '). These two components are identified through the variations of the factor shares, which we assume to reflect marginal productivities. We run our algorithm using BEA data from 1948 to 2015, and compare the predictive power of our time series of (λ_t, μ_t) with the one of the Solow residual. Through simple regressions, we find: (i) that λ and μ are as good predictors of the growth rate of GDP per capita as the Solow residual, and (ii) that λ and μ , together with capital accumulation, are strong predictors of the variation of the factor shares, while the Solow residual is not. We conclude that a bi-dimensional representation of productivity has a stronger empirical relevance than the usual linear representation; however the former carries some different theoretical properties than the latter – notably on the consequences of capital accumulation.

Keywords: Productivity, factor-saving technical change, capital accumulation.

JEL codes: E25, O33, O40, O47.

Introduction

Although the issue of factor-saving technical change has received considerable attention by industry analysts as well as by macroeconomic historians, no consensus has emerged on the proper way to represent it nor to estimate it. This paper aims at partially filling the latter gap¹ by introducing a new – and, to the best of our knowledge, the first – method to assess labor-saving technical change and capital-saving technical change from the very same set of observables than the ones used in

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¹The former gap is addressed in Senouci (2014).

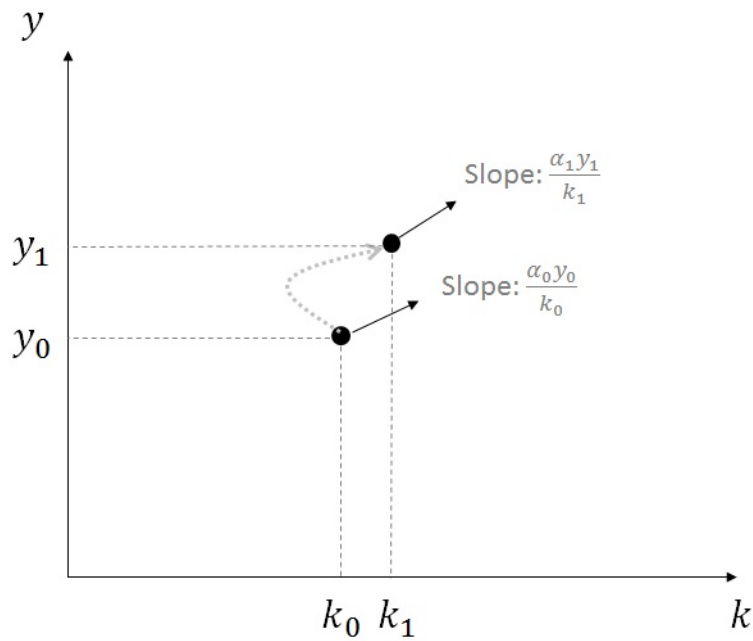


Figure 1: *The classical growth accounting problem.*

the seminal paper by Solow (1957): the output-labor ratio (y_t), the capital-labor ratio (k_t) and the labor share ($1 - \alpha_t$).

As one of us has argued in a previous article,² in a framework with two inputs A and B , ‘ A -saving technical change’ is best defined as the transition to a production function that yields more output than the original one for lower values of relative inputs A/B . ‘ B -saving technical change’ then denotes the transition to a production function that yields more output than the original one for higher values of relative inputs A/B .

The basic problem of growth accounting can be stated as follows: how can we infer the changes in the production function between dates 0 and 1 with data only on output, inputs and input share of income at each date? To address this question, one has to make some assumptions.

Figure 1 represents the problem of growth accounting with a reproducible factor of production (capital) and a non-reproducible one (labor). Observable data consists of the capital-labor ratio k , the output-labor ratio y and the labor share of income ($1 - \alpha$) at dates 0 and 1.

The classical answer to that problem is Solow’s (1957) method, which is summarized on figure 2, which consists in inferring that both production functions f_0 and f_1 are both Cobb-Douglas, and then measure the gap between both functions which is called the Solow residual. With the additional assumption that inputs are paid their marginal product, the Solow residual measures the contribution of total factor productivity (TFP) growth to output growth, as opposed to the contribution of capital accumulation.

One reason for the success of Solow’s method probably lies in the result that the residual is

²Senouci (2014).

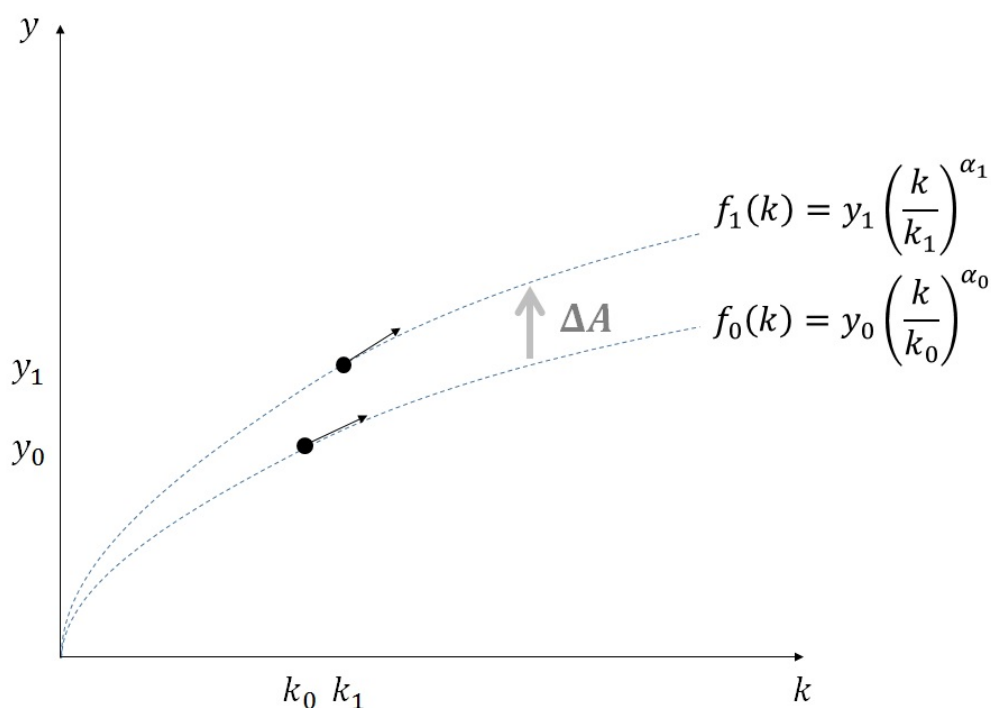


Figure 2: *Growth accounting à la Solow (1957).*

responsible for the bulk of growth. This empirical regularity is in line with the key prediction of neoclassical growth theory: growth can be sustained only by the accumulation of a factor that is not subject to diminishing marginal returns. Another reason is the great literature that followed which split up the Solow residual in different components, reducing our ignorance on the causes of growth.³

By limiting the manifestations of technical change to variations of TFP, Solow's accounting method does not aim at explaining the variations of factor shares. The theoretical assumption of orthogonality between TFP and factor shares has been put into serious doubt by the global decline of the labor share starting in the early 1980's and which is suspected to be driven by technological change itself.⁴

Our growth accounting method, in contrast, takes variations of the labor share very seriously.⁵ It yields two series, rather than one. Labor-saving technical change (LSTC) tends to decrease the labor share, while capital-saving technical change (KSTC) tends to increase the labor share. Both LSTC and KSTC tend to increase GDP per capita. Running our algorithm on Solow data and on US data from the Bureau of Economic Analysis (BEA) and between 1948 and 2015, it comes out that:

³For instance, Mankiw et al. (1992) assessed the contribution of human capital to TFP growth, while Greenwood et al. (1997) assessed the contribution of TFP growth in the investment-goods sector to overall TFP growth.

⁴Karabarbounis and Neiman (2014) document the global decline of the labor share. Bengtsson and Waldenström (2017) investigate the long-run links between the capital share and top income shares. Acemoglu and Retrespo (2017) present evidence supporting the view that industrial robots tend to decrease employment and earnings of competing workers.

⁵Another growth accounting method interpreting changes in the labor share as the reflection of some bias of technical change has been developed by Sturgill (2014) and Sturgill and Zuleta (2017), but their method is plagued by an index number problem that makes the estimates dependent on the unit of measure of capital.

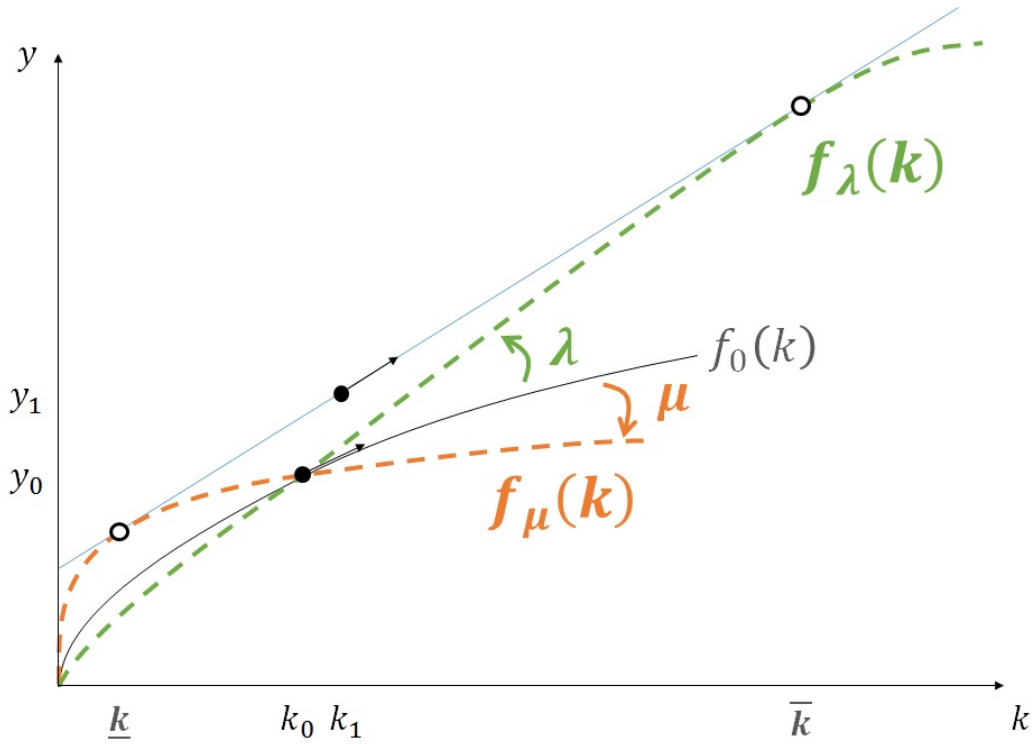


Figure 3: Our growth accounting method. λ represents labor-saving technical change and μ represents capital-saving technical change.

- The rates of LSTC and KSTC are extremely well correlated with each other, as well as with the Solow residual;
- λ and μ taken together are as strong predictors of the growth rate of GDP per capita as the Solow residual;
- LSTC and KSTC are strong predictors of the variations of the labor share while the Solow residual is not.

Section 1 presents our growth accounting procedure and the underlying model. Sections 2 sets up the algorithm with Cobb-Douglas and constant elasticity of substitution (CES) production functions. Section 4 presents our estimates of LSTC and KSTC in US data and investigate their empirical content. Section 4 concludes.

1 The model and the growth accounting procedure

Our method is depicted on figure 3. The observations set consists of (k_0, y_0, α_0) and (k_1, y_1, α_1) . We interpret the passage from the situation at $t = 0$ to the situation at $t = 1$ as the passage from production function f_0 to *two* production functions, f_λ and f_μ .

Function f_λ results from a anti-clockwise rotation of f_0 . This means that:

$$\forall k > 0, \begin{cases} f_\lambda(k) < f_0(k) & \Leftrightarrow k < k_0 \\ f_\lambda(k) > f_0(k) & \Leftrightarrow k > k_0 \end{cases}$$

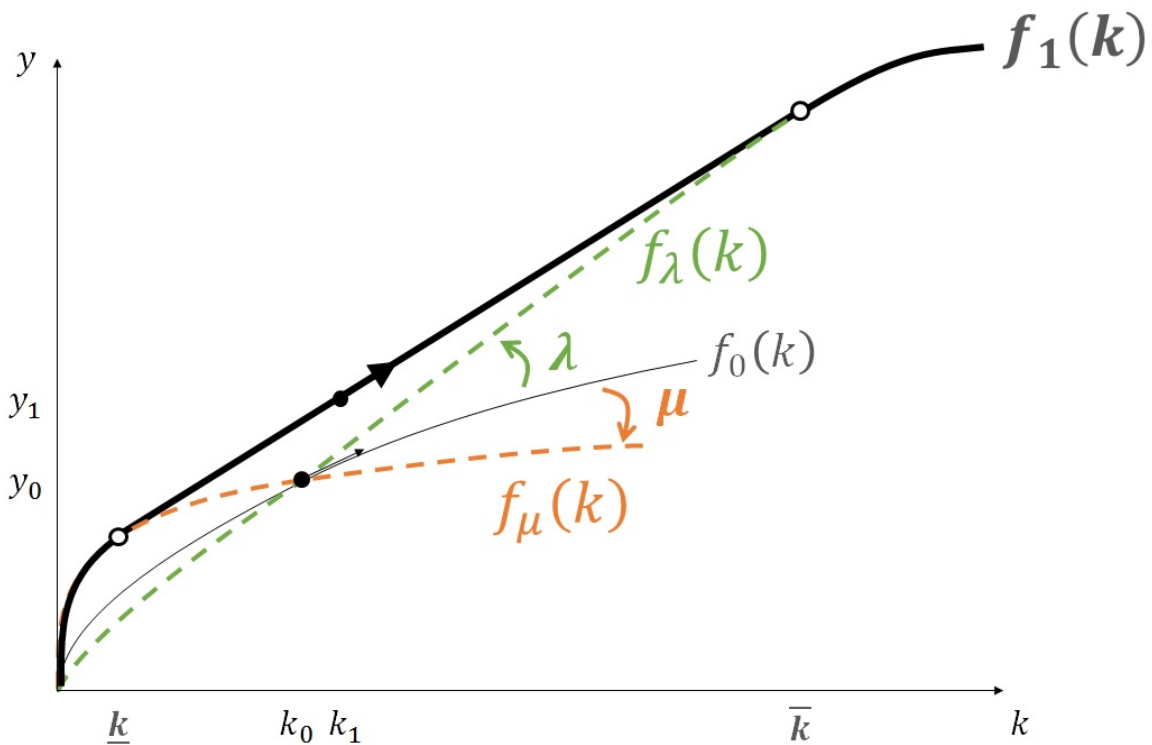


Figure 4: f_1 as the convex envelope of f_λ and f_μ .

So f_λ yields more output than the original production function if and only if the capital-labor ratio increases. Consequently, the passage from f_0 to f_λ stands for LSTC.

Function f_μ is a clockwise rotation of f_0 , meaning:

$$\forall k > 0, \begin{cases} f_\mu(k) > f_0(k) & \Leftrightarrow k < k_0 \\ f_\mu(k) < f_0(k) & \Leftrightarrow k > k_0 \end{cases}$$

f_μ is more efficient than f_0 for capital-labor ratios lower than k_0 , so the passage from f_0 to f_μ stands for KSTC.

We assume that at date $t = 1$, the economy uses both production functions f_λ and f_μ . Our growth accounting procedure consists in inferring the ‘amounts of rotation’ λ and μ from the set of observables.

What are the efficient production plans when two production functions f_λ and f_μ are available? The answer is represented on figure 4. As shown in Jones (2005), the aggregate production function is the convex envelope of functions f_λ and f_μ . For low capital-labor ratios ($k < \underline{k}$), the efficient production plan consists in putting all inputs in production function f_μ . For high capital-labor ratios ($k > \bar{k}$), the efficient production consists in putting all inputs in production function f_λ . For intermediate capital-labor ratios ($\underline{k} < k < \bar{k}$), both technologies are used. The capital-labor ratio in the μ -industry is then \underline{k} and the capital-labor ratio in the λ -industry is \bar{k} , where \underline{k} and \bar{k} are the

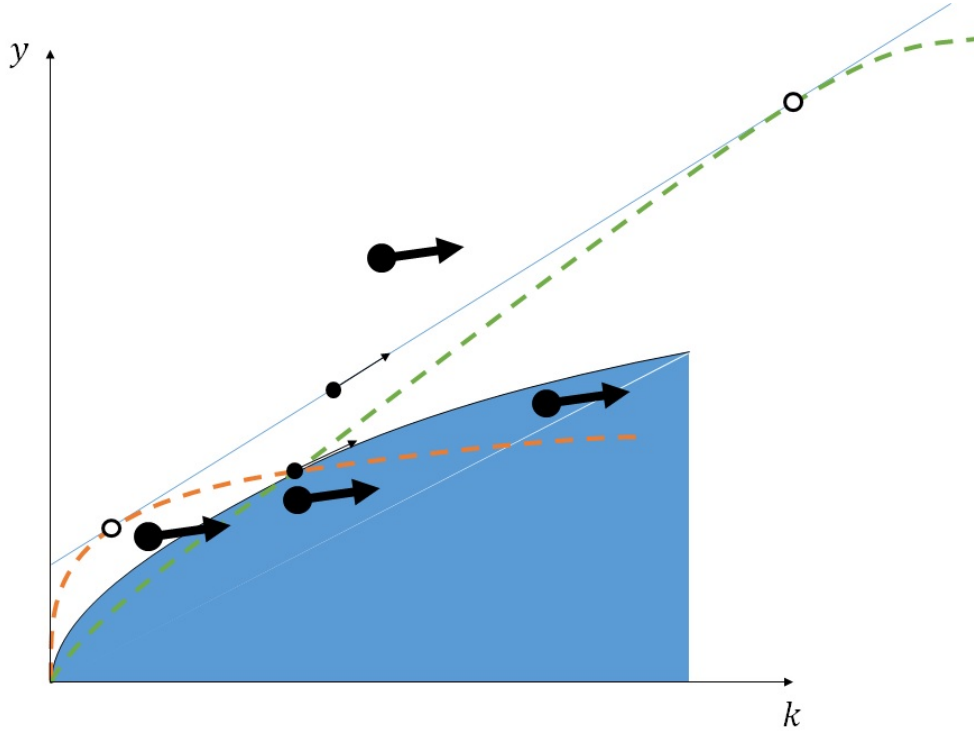


Figure 5: Some cases where our method cannot account for the passage from $t = 0$ observations to $t = 1$ observations. Large dots and arrows represent examples of such $t = 1$ observables.

solutions to the system:

$$\begin{cases} f'_\mu(\underline{k}) = f'_\lambda(\bar{k}) \\ f_\mu(\underline{k}) - \underline{k}f'_\mu(\underline{k}) = f_\lambda(\bar{k}) - \bar{k}f'_\lambda(\bar{k}) \end{cases} \quad (1)$$

We call f_1 the convex envelope of f_λ and f_μ . To account for the passage from f_0 to f_1 , we need to estimate f_λ and f_μ . To do so, we need to restrict ourselves to some subset of production functions – which is what we do in the next two sections. But before that, some remarks are worth making.

First, our algorithm to estimate f_λ and f_μ can be understood graphically in a simple way in the (k, y) plane. We describe it in steps:

- 1– Locate the point (k_0, y_0) and draw production function f_0 that has slope $\alpha_0 y_0 / k_0$ (the shape of f_0 is part of the assumptions);
- 2– Locate the point (k_1, y_1) and draw the line of slope $\alpha_1 y_1 / k_1$ that passes through (k_1, y_1) ; call this line D_1 ;
- 3– Rotate f_0 anti-clockwise around (k_0, y_0) until the rotated function touches D_1 . This amount of rotation is λ ;
- 4– Rotate f_0 clockwise around (k_0, y_0) until the rotated function touches D_1 . This amount of rotation is μ .

Secondly, not all sets of observables $(k_0, y_0, \alpha_0; k_1, y_1, \alpha_1)$ are interpretable this way. For instance, if $k_1 = k_0$ and $y_1 < y_0$, we could not find some λ and some μ such that the data could be interpreted

as the passage from one production function to the convex envelope of two rotated production functions (let's call this process 'rotation-convexification'). Figure 5 represents cases where the above graphical method would fail. Indeed, the above method works under two assumptions:

A1 – The D_1 line cuts the vertical axis below y_0 , or:

$$(1 - \alpha_1)y_1 < y_0$$

A2 – The D_1 line is above y_0 at k_0 , or:

$$y_1 + \frac{k_0 - k_1}{k_1} \alpha_1 y_1 > y_0$$

Although assumption A1 is always realized in practice, assumption A2 is not. We set up a method for the cases where A2 does not hold in subsection 1.1.

Thirdly and finally, it should be noted that there is some theoretical inconsistency in our growth accounting procedure. Indeed, when performing our algorithm between dates $t = 0$ and 1, we assume that technology switches from one to two production functions. However, when performing the same algorithm between dates $t = 1$ and 2, we assume that date-1 technology consists of only one production function, while date-2 technology consists of two production functions. We do not assume that date-2 technology consists of four production functions, as the reader might expect. The merit of our method, as we shall argue in section 3, lies in the information contained in our estimated series for (λ_t, μ_t) .

1.1 The identification of λ and μ under $\neg A2$

Whenever A2 does not hold, we use a *time-inversion method*. The idea is that if we cannot account for the transition from $t = 0$ to $t = 1$ through some (λ, μ) , then we can account for a hypothetical transition from $t = 1$ to $t = 0$ for some (λ', μ') . We then define $\lambda = -\lambda' < 0$ and $\mu = -\mu' \in (-1, 0)$.

We can account for the transition from $t = 1$ to $t = 0$ through some (λ', μ') if and only if:

$$A1' - (1 - \alpha_0)y_0 < y_1$$

$$A2' - y_0 + \frac{k_1 - k_0}{k_0} \alpha_0 y_0 > y_1$$

Indeed, in the data, assumption A1' is always valid and assumption A2' holds exactly when assumption A2 does not hold.⁶

1.2 Properties of λ and μ

Before implementing our growth accounting procedure, we state below how $t = 1$ observations influence the estimates of λ and μ .

Proposition 1.

$$\begin{aligned} \frac{\partial \lambda}{\partial k_1} < 0 & , & \frac{\partial \mu}{\partial k_1} < 0 \\ \frac{\partial \lambda}{\partial y_1} > 0 & , & \frac{\partial \mu}{\partial y_1} > 0 \\ \frac{\partial \lambda}{\partial \alpha_1} > 0 & , & \frac{\partial \mu}{\partial \alpha_1} < 0 \end{aligned}$$

Proof. Obvious from graphical analysis. □

⁶We could have some observations for which neither A2 nor A2' would be valid, but this does not happen in practice.

2 The implementation of the growth accounting method

We now see how to implement numerically our procedure when production functions belong to specified families.

2.1 The method with Cobb-Douglas production functions

2.1.1 Under A2

If f_0 , f_λ and f_μ are Cobb-Douglas, then:

$$\forall k > 0, \begin{cases} f_0(k) = y_0 \left(\frac{k}{k_0}\right)^{\alpha_0} \\ f_\lambda(k) = y_0 \left(\frac{k}{k_0}\right)^{\alpha_\lambda} \\ f_\mu(k) = y_0 \left(\frac{k}{k_0}\right)^{\alpha_\mu} \end{cases} \quad (2)$$

with $\alpha_\lambda = \alpha_0^{1-\lambda}$ and $\alpha_\mu = \alpha_0^{1+\mu}$. We are looking for $\lambda \in (0, 1)$ and $\mu > 0$ such that there exist some \bar{k} and \underline{k} such that:

$$\begin{cases} f'_\mu(\underline{k}) = f'_\lambda(\bar{k}) = \frac{\alpha_1 y_1}{k_1} \\ f_\mu(\underline{k}) - \underline{k} f'_\mu(\underline{k}) = f_\lambda(\bar{k}) - \bar{k} f'_\lambda(\bar{k}) = (1 - \alpha_1) y_1 \end{cases} \quad (3)$$

System (3) has four equations and four unknown ($\alpha_\lambda, \alpha_\mu, \bar{k}, \underline{k}$). The following proposition reduces the system to one equation with two solutions:

Proposition 2. *Under A1 and A2, α_λ and α_μ are the solutions of equation:*

$$(1 - X) \left(\frac{1}{\alpha_1} \frac{k_1}{k_0} X \right)^{\frac{X}{1-X}} = (1 - \alpha_1) \left(\frac{y_1}{y_0} \right)^{\frac{1}{1-X}} \quad (4)$$

and: $\lambda = 1 - \frac{\ln \alpha_\lambda}{\ln \alpha_0}$ and $\mu = \frac{\ln \alpha_\mu}{\ln \alpha_0} - 1$.

Proof. See appendix A. □

2.1.2 Under $\neg A2$

When assumption A2 is not valid, we apply our time-inversion method. When assumptions A1' and A2' are valid, we apply the above method to find some $\mu' \in (0, 1)$ and λ' that account for the transition from $t = 1$ observations to $t = 0$ observations.

Proposition 3. *Under A1' and A2', $\alpha_{\lambda'}$ and $\alpha_{\mu'}$ are the solutions of equation:*

$$(1 - X) \left(\frac{1}{\alpha_0} \frac{k_0}{k_1} X \right)^{\frac{X}{1-X}} = (1 - \alpha_0) \left(\frac{y_0}{y_1} \right)^{\frac{1}{1-X}} \quad (5)$$

and $\lambda' = 1 - \frac{\ln \alpha_{\lambda'}}{\ln \alpha_1}$ and $\mu' = \frac{\ln \alpha_{\mu'}}{\ln \alpha_1} - 1$.

Proof. Proposition 2 applied to observations (k_1, y_1, α_1) and (k_0, y_0, α_0) , i.e. to time-inverted data. □

We then take $\lambda = -\lambda'$ and $\mu = -\mu'$ as respectively the measures of LSTC and KSTC between dates $t = 0$ and $t = 1$.

2.1.3 Summary

To summarize, we obtain our metric of LSTC (λ) and KSTC (μ) with Cobb-Douglas productions function with the following method, that we apply for each couple of consecutive dates ($t, t + 1$):

1. If assumptions A1 and A2 are jointly met, then solve equation (4):

$$(1 - X) \left(\frac{1}{\alpha_{t+1}} \frac{k_{t+1}}{k_t} X \right)^{\frac{X}{1-X}} = (1 - \alpha_{t+1}) \left(\frac{y_{t+1}}{y_t} \right)^{\frac{1}{1-X}}$$

Roots are $\alpha_{\lambda,t} > \alpha_{\mu,t}$. Then, we define:

$$\lambda_t = 1 - \frac{\ln \alpha_{\lambda,t}}{\ln \alpha_t}$$

$$\mu_t = \frac{\ln \alpha_{\mu,t}}{\ln \alpha_t} - 1$$

2. If assumptions A1 and A2 are not jointly met, then if A1' and A2' are jointly met; then solve equation 5:

$$(1 - X) \left(\frac{1}{\alpha_t} \frac{k_t}{k_{t+1}} X \right)^{\frac{X}{1-X}} = (1 - \alpha_t) \left(\frac{y_t}{y_{t+1}} \right)^{\frac{1}{1-X}}$$

Roots are $\alpha_{\lambda',t} > \alpha_{\mu',t}$. Then, we define:

$$\lambda_t = \frac{\ln \alpha_{\lambda',t}}{\ln \alpha_{t+1}} - 1$$

$$\mu_t = 1 - \frac{\ln \alpha_{\mu',t}}{\ln \alpha_{t+1}}$$

3. If neither (A1 and A2) nor (A1' and A2') are jointly met, then our method does not apply.

We implement this procedure using Python. The program is available in appendix C.

2.2 The method with CES production functions

We implement a similar procedure under the assumption that the production functions are CES rather than Cobb-Douglas.

Moysan and Senouci (2016) proved that for any $k^* > 0$, $y^* > 0$, and $\alpha^* \in (0, 1)$, the only CES production function f of parameter σ such that $f(k^*) = y^*$ and $k^* f'(k^*) / f(k^*) = \alpha^*$ is defined by:

$$\forall k > 0, f(k) = y^* \left(\alpha^* \left(\frac{k}{k^*} \right)^{\frac{\sigma-1}{\sigma}} + 1 - \alpha^* \right)^{\frac{\sigma}{\sigma-1}} \quad (6)$$

We invoke this result below.

2.2.1 Under A2

We now assume that f_0 , f_λ and f_μ are CES production functions with the same elasticity of substitution $\sigma \in (0, 1)$. Specifically:

$$\forall k > 0, \begin{cases} f_0(k) = y_0 \left(\alpha_0 \left(\frac{k}{k_0} \right)^{\frac{\sigma-1}{\sigma}} + 1 - \alpha_0 \right)^{\frac{\sigma}{\sigma-1}} \\ f_\lambda(k) = y_0 \left(\alpha_\lambda \left(\frac{k}{k_0} \right)^{\frac{\sigma-1}{\sigma}} + 1 - \alpha_\lambda \right)^{\frac{\sigma}{\sigma-1}} \\ f_\mu(k) = y_0 \left(\alpha_\mu \left(\frac{k}{k_0} \right)^{\frac{\sigma-1}{\sigma}} + 1 - \alpha_\mu \right)^{\frac{\sigma}{\sigma-1}} \end{cases} \quad (7)$$

with $0 < \alpha_\mu < \alpha_0 < \alpha_\lambda < 1$ and $\alpha_\lambda = \alpha_0^{1-\lambda}$ and $\alpha_\mu = \alpha_0^{1+\mu}$.

We are looking for the values of λ and μ such that there exists some \underline{k} and \bar{k} such that:

$$\begin{cases} f'_\mu(\underline{k}) = f'_\lambda(\bar{k}) = \frac{\alpha_1 y_1}{k_1} \\ f_\mu(\underline{k}) - \underline{k} f'_\mu(\underline{k}) = f_\lambda(\bar{k}) - \bar{k} f'_\lambda(\bar{k}) = (1 - \alpha_1) y_1 \end{cases} \quad (8)$$

where f_0 , f_λ and f_μ are defined in (7).

Proposition 4. *Under A1 and A2, assuming CES production functions of parameter $\sigma \in (0, 1)$, α_λ and α_μ are the solutions of equation:*

$$(1 - \alpha_1)^{\sigma-1} \left(\frac{y_1}{y_0} \right)^{\sigma-1} = X^\sigma \left(\frac{1 - \alpha_1}{\alpha_1} \right)^{\sigma-1} \left(\frac{k_1}{k_0} \right)^{\sigma-1} + (1 - X)^\sigma \quad (9)$$

and: $\lambda = 1 - \frac{\ln \alpha_\lambda}{\ln \alpha_0}$ and $\mu = \frac{\ln \alpha_\mu}{\ln \alpha_0} - 1$.

Proof. See appendix B. □

2.2.2 Under $\neg A2$

When assumption A2 does not hold, we apply again our time-inversion method. If A1' and A2' hold, then there exist some $\lambda' > 0$ and $\mu' \in (0, 1)$ that account for the transition from $t = 1$ observations to $t = 0$ observations. We then take $\lambda = -\lambda'$ and $\mu = -\mu'$.

Proposition 5. *Under A1' and A2', assuming CES production functions of parameter $\sigma \in (0, 1)$, α'_λ and α'_μ are the solutions of equation:*

$$(1 - \alpha_0)^{\sigma-1} \left(\frac{y_0}{y_1} \right)^{\sigma-1} = X^\sigma \left(\frac{1 - \alpha_0}{\alpha_0} \right)^{\sigma-1} \left(\frac{k_0}{k_1} \right)^{\sigma-1} + (1 - X)^\sigma \quad (10)$$

and: $\lambda' = 1 - \frac{\ln \alpha_{\lambda'}}{\ln \alpha_1}$ and $\mu' = \frac{\ln \alpha_{\mu'}}{\ln \alpha_1} - 1$.

Proof. Proposition 4 applied to observations (k_1, y_1, α_1) and (k_0, y_0, α_0) , i.e. to time-inverted data. □

Like in the Cobb-Douglas case, we then take $\lambda = -\lambda' < 0$ and $\mu = -\mu' \in (-1, 0)$ as respectively the measures of LSTC and KSTC between dates $t = 0$ and $t = 1$.

2.2.3 Summary

The procedure with CES production functions of parameter $\sigma \in (0, 1)$ is:

1. If assumptions A1 and A2 are jointly met, then solve equation (9):

$$(1 - \alpha_{t+1})^{\sigma-1} \left(\frac{y_{t+1}}{y_t} \right)^{\sigma-1} = X^\sigma \left(\frac{1 - \alpha_{t+1}}{\alpha_{t+1}} \right)^{\sigma-1} \left(\frac{k_{t+1}}{k_t} \right)^{\sigma-1} + (1 - X)^\sigma$$

Roots are $\alpha_{\lambda,t} > \alpha_{\mu,t}$. Then, we define:

$$\lambda_t = 1 - \frac{\ln \alpha_{\lambda,t}}{\ln \alpha_t}$$

$$\mu_t = \frac{\ln \alpha_{\mu,t}}{\ln \alpha_t} - 1$$

2. If assumptions $A1$ and $A2$ are not jointly met, then if $A1'$ and $A2'$ are jointly met; then solve equation (10):

$$(1 - \alpha_t)^{\sigma-1} \left(\frac{y_t}{y_{t+1}} \right)^{\sigma-1} = X^\sigma \left(\frac{1 - \alpha_t}{\alpha_t} \right)^{\sigma-1} \left(\frac{k_t}{k_{t+1}} \right)^{\sigma-1} + (1 - X)^\sigma$$

Roots are $\alpha_{\lambda',t} > \alpha_{\mu',t}$. Then, we define:

$$\lambda_t = \frac{\ln \alpha_{\lambda',t}}{\ln \alpha_{t+1}} - 1$$

$$\mu_t = 1 - \frac{\ln \alpha_{\mu',t}}{\ln \alpha_{t+1}}$$

3. If neither ($A1$ and $A2$) nor ($A1'$ and $A2'$) are jointly met, then our method does not apply.

We also implement this procedure using Python. The program is available in appendix D.

3 λ and μ in US data and their empirical content

3.1 Data

We run our algorithm on two datasets: the first is simply Solow's (1957) dataset which extends from 1909 to 1949, the second is built on Bureau of Economic Analysis and Federal Reserve data, and covers the time period from 1948 to 2015.

Solow's (1957) dataset From the seminal Solow (1957) paper we take the variables "Private non-farm GDP per manhour," "Employed capital per manhour" and "Share of property in income" as measures, respectively, of y , k and α . See Solow (1957, Table I, p. 315) for data sources.

1948–2015 dataset Post-WWII real GDP (' Y '), aggregate labor share (' β ') are taken from BEA National Income and Product Accounts database.⁷

The capital stock data (' K ') – the one with the greatest conceptual and practical measurement problems – is taken from the BEA Fixed Assets database.⁸

Aggregate worked hours (' L ') data is taken from the Federal Reserve Economic Data (FRED) database.⁹

Then, we define $y = Y/L$ as real GDP per manhour, $k = K/L$ as the capital-labor ratio and $\alpha = 1 - \beta$ as the capital share.

3.2 Results

We run our algorithm (Cobb-Douglas version) and the traditional Solow (1957) algorithm on both datasets. The results are displayed on figures 6 and 7. It comes out that, for all dates ($t, t + 1$), either

⁷https://www.bea.gov/iTable/index_nipa.cfm. Y is from "Table 1.1.3. Real Gross Domestic Product, Quantity Indexes," line 1 "Gross domestic product"; β is from "Table 1.11. Percentage Shares of Gross Domestic Income," line 2 "Compensation of employees, paid."

⁸https://www.bea.gov/iTable/index_FA.cfm. K is from "Table 1.2. Chain-Type Quantity Indexes for Net Stock of Fixed Assets and Consumer Durable Goods," line 2 "Fixed assets."

⁹<https://fred.stlouisfed.org>. The variable we use here is "Hours worked by full-time and part-time employees (B4701C0A222NBEA)."

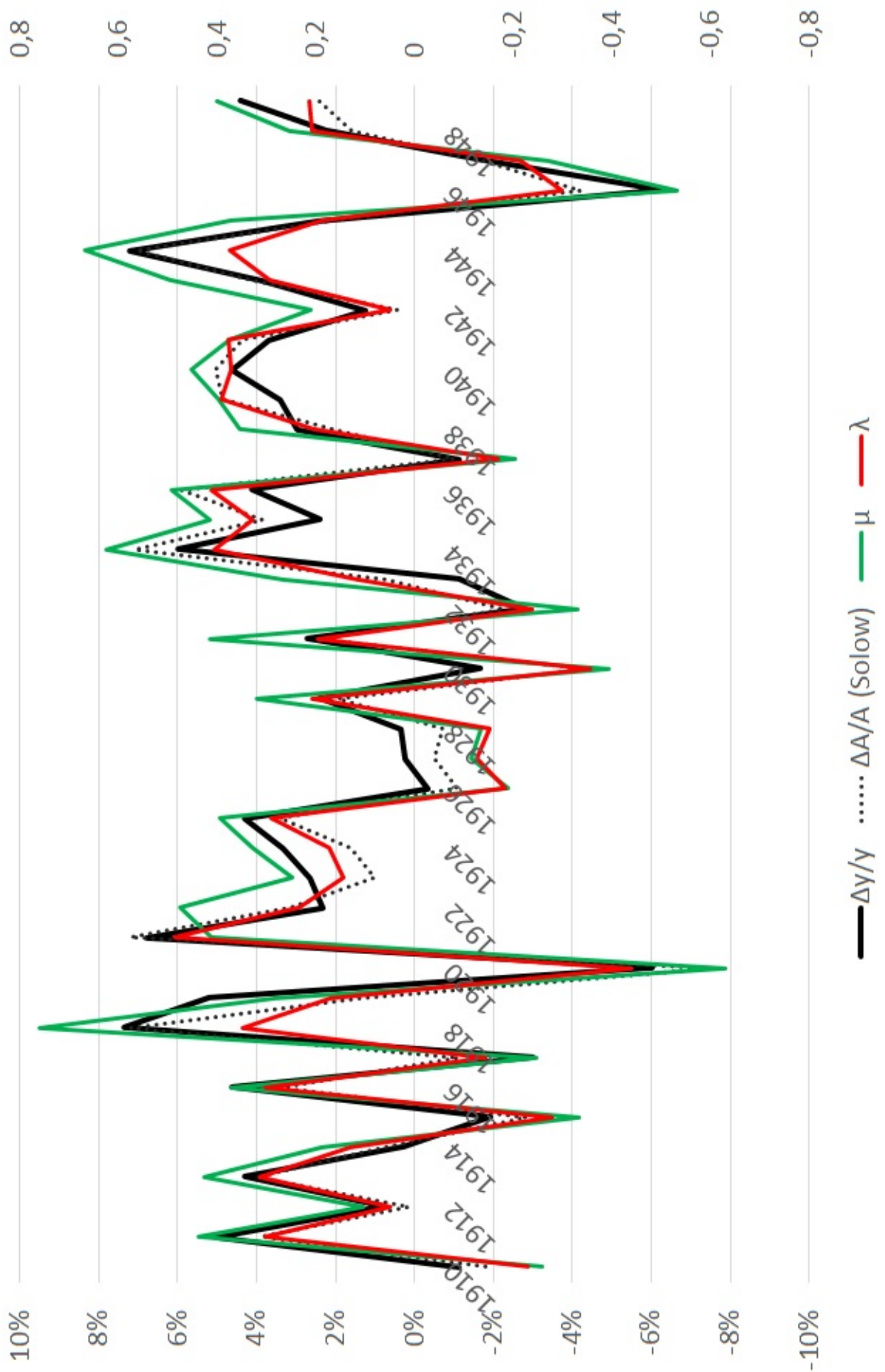


Figure 6: Estimates of LSTC (λ) and KSTC (μ) on right-hand scale, growth rate of real GDP and Solow residual on left-hand scale, 1909–1949, assuming Cobb-Douglas core production functions. Data from Solow (1957).

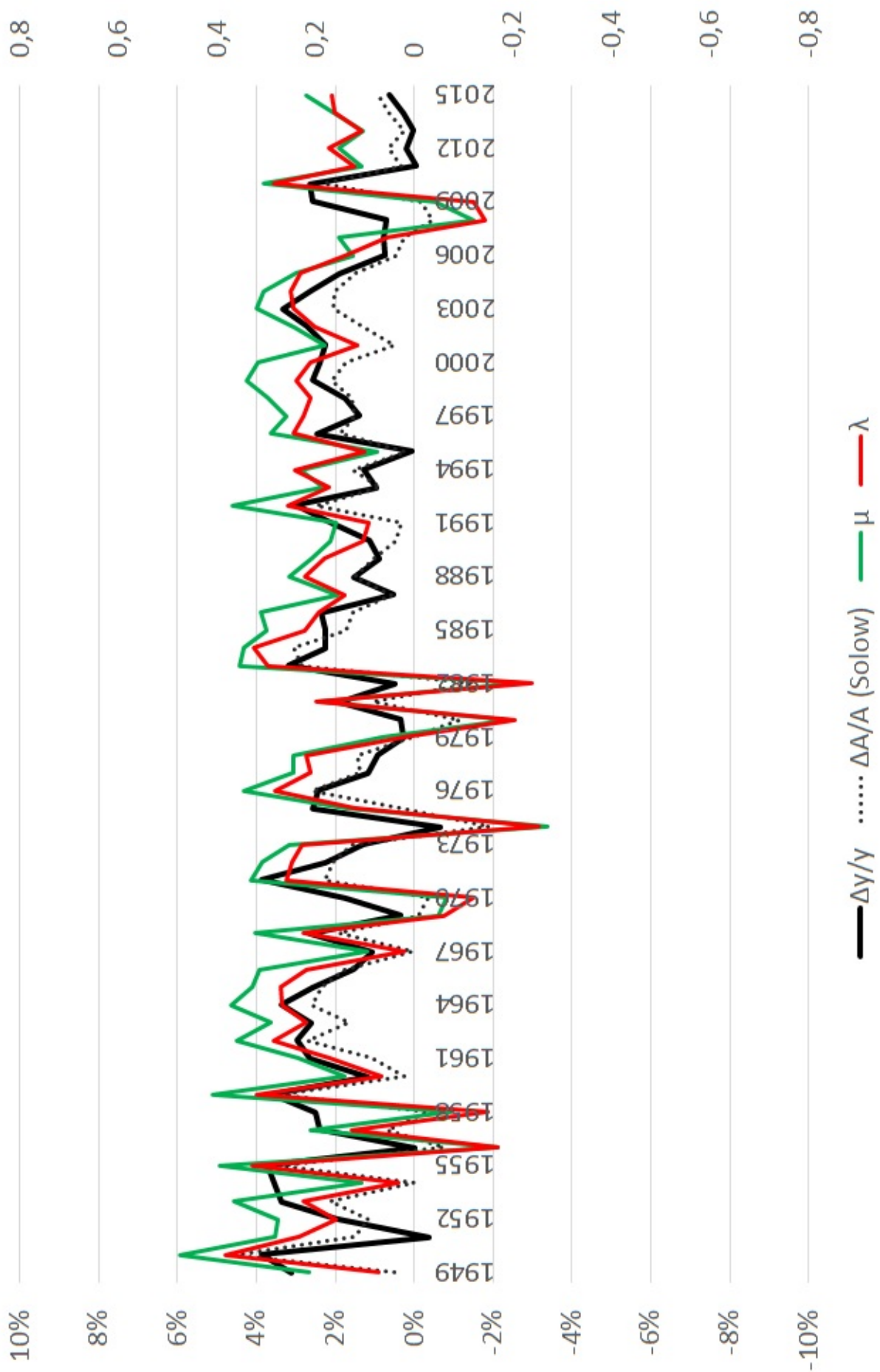


Figure 7: Estimates of LSTC (λ) and KSTC (μ) on right-hand scale, growth rate of real GDP and Solow residual on left-hand scale, 1948–2015, assuming Cobb-Douglas core production functions. Data from BEA-NIPA and FED-FRED datasets (see subsection 3.1).

Table 1: The explanatory power of our measures of LSTC and KSTC (λ, μ) compared to the explanatory power of the Solow residual ($\Delta A/A$) over the growth rate of real GDP per capita ($\Delta y/y$) in the Solow (1957) dataset.

		<i>Dependent variable: $\Delta y/y$</i>		
<i>Dataset</i> →	Solow (1957) [$N = 40$]			
<i>Constant</i>	0.004045*	0.00303565	0.00139902	
	(1.849)	(1.319)	(0.8312)	
$\Delta A/A$	0.906542*** (15.42)			
λ		0.0254126	0.0738151***	
		(0.8744)	(3.280)	
μ		0.0696811***	0.0388880**	
		(3.330)	(2.436)	
$\Delta k/k$			0.250630*** (0.141)	
R^2	0.862	0.862	0.930	
\bar{R}^2	0.859	0.855	0.924	

OLS estimates. t -statistics in parentheses.

* $p < 0.1$, ** $p < 0.05$, *** $p < 0.01$

assumptions ($A1, A2$) or assumptions ($A1', A2'$) hold, and they never hold jointly, so our algorithm yields uninterrupted and unambiguous series for λ and μ .

The most striking feature of the output of our algorithm is that our measures of technical change λ and μ are extremely well-correlated with each other, as well as with the growth rate of real GDP per capita ($\Delta y/y$) and with the Solow residual ($\Delta A/A$), reflecting the predominantly neutral nature of technical change, i.e. the low variance of α in both datasets.

3.3 Explanatory power of λ and μ

First, we assess the explanatory power of (λ, μ) over ($\Delta y/y$). We perform three OLS regressions on each of the dataset we use. As table 1 shows for the Solow (1957) dataset, (λ, μ) are as good predictors of the growth rate of real GDP per capita as the Solow residual. When the growth rate of capital per manhour ($\Delta k/k$) is added to the set of explanatory variables, the R^2 reaches 93%. Table 2 performs the very regressions on the 1948-2015 dataset. Here too λ and μ appear as strong predictors of the growth rate of real GDP per capita.

Secondly, we run a few regressions with the variations of the capital share ($\Delta\alpha$) as the dependent variable. Results are displayed in tables 3 and 4, respectively for the Solow (1957) dataset and the 1948–2015 dataset. It comes out that the Solow residual is at best a weak predictor of the variations of the capital share.

In the Solow (1957) dataset (table 3), the Solow residual is not significant in all the regressions

Table 2: Idem table 1, 1948–2015 dataset.

<i>Dependent variable: $\Delta y/y$</i>			
<i>Dataset</i> →	1948–2015 [$N = 67$]		
<i>Constant</i>	0.0104530*** (6.996)	0.00515756*** (3.086)	-0.00134955 (1.284)
$\Delta A/A$	0.650314*** (7.294)		
λ		-0.134619*** (5.093)	0.0109270 (0.5876)
μ		0.164362*** (6.879)	0.0593085*** (3.817)
$\Delta k/k$			0.415178*** (12.36)
R^2	0.450	0.578	0.877
\bar{R}^2	0.442	0.565	0.871

OLS estimates. t -statistics in parentheses.

* $p < 0.1$, ** $p < 0.05$, *** $p < 0.01$

Table 3: The explanatory power of our measures of LSTC and KSTC (λ, μ) compared to the explanatory power of the Solow residual ($\Delta A/A$) over the variations of the capital share ($\Delta \alpha$) in the Solow (1957) dataset.

<i>Dependent variable: $\Delta \alpha$</i>					
<i>Dataset</i> →	Solow (1957) [$N = 40$]				
<i>Constant</i>	-0.000401293 (0.1097)	0.00429165 (1.3925)	-0.000419504 (0.1084)	0.0035149 (1.1457)	-0.000217105 (0.0645)
$\Delta A/A$	0.0109813 (0.1116)		0.011402 (0.1107)		
λ		0.168398*** (4.3261)		0.19137*** (4.6653)	
μ		-0.127436*** (4.5473)		-0.14205*** (4.8810)	
$\Delta k/k$			0.00153661 (0.0164)	0.11895 (0.1317)	-0.00105888 (0.0118)
R^2	0.000	0.359	0.000	0.399	0.000
\bar{R}^2	-0.026	0.324	-0.054	0.348	-0.026

OLS estimates. t -statistics in parentheses.

* $p < 0.1$, ** $p < 0.05$, *** $p < 0.01$

Table 4: Idem table 3, 1948–2015 dataset.

Dataset →	Dependent variable: $\Delta\alpha$				
	1948–2015 [$N = 67$]				
Constant	-0.00180884*	0.000966788	-0.00322667**	-0.00308333***	-0.000275419
	(1.8370)	(0.9394)	(2.5095)	(4.8916)	(0.3111)
$\Delta A/A$	0.147028**		0.19446***		
	(2.5023)		(3.0177)		
λ		0.0977628***		0.188353***	
		(6.0052)		(16.8930)	
μ		-0.0777473***		-0.143134***	
		(5.2840)		(15.3625)	
$\Delta k/k$			0.0602948*	0.258413***	0.0129829
			(1.6825)	(12.8336)	(0.3799)
R^2	0.088	0.389	0.127	0.831	0.002
\bar{R}^2	0.074	0.370	0.099	0.823	-0.013

OLS estimates. t -statistics in parentheses.

* $p < 0.1$, ** $p < 0.05$, *** $p < 0.01$

where it is included while, λ and μ taken together are strong predictors of $\Delta\alpha$.

In the 1948–2015 dataset (table 4), the Solow residual is a significant predictor of $\Delta\alpha$, but the explanatory power of λ , μ and $\Delta k/k$ far exceeds that of $\Delta A/A$. Moreover, when the regression is augmented with the growth rate of the capital-labor ratio, it has a positive and significant effect – consistently with the model that underlies our growth accounting method – and the regression reaches an adjusted R^2 of 82%.

4 Conclusion

We attempted to measure LSTC and KSTC through a method that treat both directions of technical change symmetrically. We have come to invoke *rotations* of the production function – rather than shifts – as the manifestation of technical change, in the vein of some theoretical literature on economic growth.¹⁰ Our estimates of LSTC and KSTC are very strong predictors of the growth rate of real GDP per capita and of the variations of the factor shares.

In future research, we plan to investigate technical change at the sector level through input-output matrices.

¹⁰See, for instance, Glachant (2003), Zuleta (2008) and Zuleta and Young (2013).

A Proof of proposition 2

For a set of observations $(k_0, y_0, \alpha_0, k_1, y_1, \alpha_1)$ that satisfies assumptions A1 and A2, $\alpha_\lambda, \alpha_\mu, \underline{k}$ and \bar{k} are the solutions of the system:

$$\begin{aligned}
 (\mathcal{S}) : & \begin{cases} f'_\mu(\underline{k}) = f'_\lambda(\bar{k}) = \frac{\alpha_1 y_1}{k_1} \\ f_\mu(\underline{k}) - \underline{k} f'_\mu(\underline{k}) = f_\lambda(\bar{k}) - \bar{k} f'_\lambda(\bar{k}) = (1 - \alpha_1) y_1 \end{cases} \\
 (\mathcal{S}) \iff & \begin{cases} \frac{y_0}{\bar{k}_0} \alpha_\mu \left(\frac{\underline{k}}{\bar{k}_0}\right)^{\alpha_\mu - 1} = \frac{y_0}{\bar{k}_0} \alpha_\lambda \left(\frac{\bar{k}}{\bar{k}_0}\right)^{\alpha_\lambda - 1} = \frac{\alpha_1 y_1}{k_1} \\ (1 - \alpha_\mu) y_0 \left(\frac{\underline{k}}{\bar{k}_0}\right)^{\alpha_\mu} = (1 - \alpha_\lambda) y_0 \left(\frac{\bar{k}}{\bar{k}_0}\right)^{\alpha_\lambda} = (1 - \alpha_1) y_1 \end{cases} \\
 (\mathcal{S}) \implies & \begin{cases} (1 - \alpha_\mu) \left(\frac{1}{\alpha_1} \frac{k_1}{\bar{k}_0} \alpha_\mu\right)^{\frac{\alpha_\mu}{1 - \alpha_\mu}} = (1 - \alpha_1) \left(\frac{y_1}{y_0}\right)^{\frac{1}{1 - \alpha_\mu}} \\ (1 - \alpha_\lambda) \left(\frac{1}{\alpha_1} \frac{k_1}{\bar{k}_0} \alpha_\lambda\right)^{\frac{\alpha_\lambda}{1 - \alpha_\lambda}} = (1 - \alpha_1) \left(\frac{y_1}{y_0}\right)^{\frac{1}{1 - \alpha_\lambda}} \end{cases}
 \end{aligned}$$

Then, $\alpha_\lambda = \alpha_0^{1 - \lambda} \Rightarrow \lambda = 1 - \ln \alpha_\lambda / \ln \alpha_0$ and $\alpha_\mu = \alpha_0^{1 + \mu} \Rightarrow \mu = \ln \alpha_\mu / \ln \alpha_0 - 1$.

B Proof of proposition 4

By definition of functions f_λ and f_μ :

$$\forall k > 0, \begin{cases} f'_\mu(k) = y_0 \frac{\alpha_\mu}{\bar{k}_0} \left(\frac{k}{\bar{k}_0}\right)^{-\frac{1}{\sigma}} \left(\alpha_\mu \left(\frac{k}{\bar{k}_0}\right)^{\frac{\sigma-1}{\sigma}} + 1 - \alpha_\mu\right)^{\frac{1}{\sigma-1}} \\ f'_\lambda(k) = y_0 \frac{\alpha_\lambda}{\bar{k}_0} \left(\frac{k}{\bar{k}_0}\right)^{-\frac{1}{\sigma}} \left(\alpha_\lambda \left(\frac{k}{\bar{k}_0}\right)^{\frac{\sigma-1}{\sigma}} + 1 - \alpha_\lambda\right)^{\frac{1}{\sigma-1}} \\ f_\mu(k) - k f'_\mu(k) = (1 - \alpha_\mu) y_0 \left(\alpha_\mu \left(\frac{k}{\bar{k}_0}\right)^{\frac{\sigma-1}{\sigma}} + 1 - \alpha_\mu\right)^{\frac{1}{\sigma-1}} \\ f_\lambda(k) - k f'_\lambda(k) = (1 - \alpha_\lambda) y_0 \left(\alpha_\lambda \left(\frac{k}{\bar{k}_0}\right)^{\frac{\sigma-1}{\sigma}} + 1 - \alpha_\lambda\right)^{\frac{1}{\sigma-1}} \end{cases}$$

System (8) is, then, equivalent to:

$$\begin{cases} \frac{\alpha_\mu y_0}{\bar{k}_0} \left(\frac{k}{\bar{k}_0}\right)^{-\frac{1}{\sigma}} \left(\alpha_\mu \left(\frac{k}{\bar{k}_0}\right)^{\frac{\sigma-1}{\sigma}} + 1 - \alpha_\mu\right)^{\frac{1}{\sigma-1}} = \frac{\alpha_1 y_1}{k_1} \\ \frac{\alpha_\lambda y_0}{\bar{k}_0} \left(\frac{\bar{k}}{\bar{k}_0}\right)^{-\frac{1}{\sigma}} \left(\alpha_\lambda \left(\frac{\bar{k}}{\bar{k}_0}\right)^{\frac{\sigma-1}{\sigma}} + 1 - \alpha_\lambda\right)^{\frac{1}{\sigma-1}} = \frac{\alpha_1 y_1}{k_1} \\ y_0 (1 - \alpha_\mu) \left(\alpha_\mu \left(\frac{k}{\bar{k}_0}\right)^{\frac{\sigma-1}{\sigma}} + 1 - \alpha_\mu\right)^{\frac{1}{\sigma-1}} = (1 - \alpha_1) y_1 \\ y_0 (1 - \alpha_\lambda) \left(\alpha_\lambda \left(\frac{\bar{k}}{\bar{k}_0}\right)^{\frac{\sigma-1}{\sigma}} + 1 - \alpha_\lambda\right)^{\frac{1}{\sigma-1}} = (1 - \alpha_1) y_1 \end{cases}$$

Combining the first and third equations, as well as the second and fourth equations above, we get rid of unknown \underline{k} and \bar{k} :

$$\begin{cases} (1 - \alpha_1)^{\sigma-1} \left(\frac{y_1}{y_0}\right)^{\sigma-1} = \alpha_\mu^\sigma \left(\frac{1 - \alpha_1}{\alpha_1}\right)^{\sigma-1} \left(\frac{k_1}{\bar{k}_0}\right)^{\sigma-1} + (1 - \alpha_\mu)^\sigma \\ (1 - \alpha_1)^{\sigma-1} \left(\frac{y_1}{y_0}\right)^{\sigma-1} = \alpha_\lambda^\sigma \left(\frac{1 - \alpha_1}{\alpha_1}\right)^{\sigma-1} \left(\frac{k_1}{\bar{k}_0}\right)^{\sigma-1} + (1 - \alpha_\lambda)^\sigma \end{cases}$$

which proves proposition 4.

C The program with Cobb-Douglas production functions

```
import pandas as pd
import numpy as np
import matplotlib.pyplot as plt
import openpyxl
from scipy.optimize import fsolve
import matplotlib.patches as mpatches

df = pd.read_excel(open('Classeur3.xlsx', 'rb'))

dfadd = pd.DataFrame({
    'y_shifted':[],
    'k_shifted':[],
    'a_shifted':[],
    'a_lambda':[],
    'a_mu':[],
    'time_regular':[],
})

df = df.join(dfadd)
print(df.head())

y1 = df['y'].shift(-1)
k1 = df['k'].shift(-1)
a1 = df['a'].shift(-1)

df['y_shifted'] = y1
df['k_shifted'] = k1
df['a_shifted'] = a1

y0 = df['y']
k0 = df['k']

time_regular = y0 < y1+a1*y1*(k0-k1)/k1
df['time_regular'] = time_regular

df = df.drop(len(df)-1,0) # supprime la derniere ligne car le calcul est impossible dessus

def function_sol(x,L): # L = [y0,y1,k0,k1,alpha1] ou [y1,y0,k1,k0,alpha0]
    return (1-x)*L[0]*(L[0]*x/(L[4]*L[1]*L[2]/L[3]))**(x/(1-x))-(1-L[4])*L[1]

def parameter_list(df,n): #rend la liste L bien ordonnee
    if df['time_regular'][n]:
        return [df['y'][n],df['y_shifted'][n],df['k'][n],df['k_shifted'][n],df['a_shifted'][n]]
    else:
        return [df['y_shifted'][n],df['y'][n],df['k_shifted'][n],df['k'][n],df['a'][n]]

for i in range(len(df)):

    def function(x):
        return function_sol(x,parameter_list(df,i))

    df.loc[i,'a_mu'] = fsolve(function,0)[0]
    df.loc[i,'a_lambda'] = fsolve(function,0.99)[0]

df_out = df.loc[:,('annee','y','k','a','a_mu','a_lambda','a_mu','a_lambda','time_regular')]
df_out.columns = ['annee','y','k','a','a_mu','a_lambda','a_mu','a_lambda','time_regular']

for i in range(len(df_out)):
    if df_out.loc[i,'time_regular']:
        df_out.loc[i,'a_mu'] = np.nan
        df_out.loc[i,'a_lambda'] = np.nan
    else:
        df_out.loc[i,'a_mu'] = np.nan
        df_out.loc[i,'a_lambda'] = np.nan

df_out.to_csv('data_out.csv',sep=';')

for i in range(len(df)-1):
    def f_0(x):
        return df.loc[i,'y']*(x/df.loc[i,'k'])**df.loc[i,'a']
    def f_1(x):
        return df.loc[i+1,'y']*(x/df.loc[i+1,'k'])**df.loc[i+1,'a']
    def f_a_mu(x):
        if df.loc[i,'time_regular']:
            return df.loc[i,'y']*(x/df.loc[i,'k'])**df.loc[i,'a_mu']
        else:
            return df.loc[i+1,'y']*(x/df.loc[i+1,'k'])**df.loc[i,'a_mu']
    def f_a_lambda(x):
        if df.loc[i,'time_regular']:
            return df.loc[i,'y']*(x/df.loc[i,'k'])**df.loc[i,'a_lambda']
        else:
            return df.loc[i+1,'y']*(x/df.loc[i+1,'k'])**df.loc[i,'a_lambda']

    x = np.linspace(0,4,500)
    y0 = f_0(x)
    y1 = f_1(x)
    ym = f_a_mu(x)
    yl = f_a_lambda(x)

    plt.figure()
```

```

plt.ylim(0,1.5)
plt.xlim(1.5,3.5)
plt.plot([df.loc[i,'k']], [df.loc[i,'y']], 'ro', label='(y0,k0)')
plt.plot([df.loc[i+1,'k']], [df.loc[i+1,'y']], 'bo', label='(y1,k1)')
plt.plot(x,y0,'r-', label='f0')
plt.plot(x,y1,'b-', label='f1')
plt.plot(x,ym,'y', label='f_a_mu')
plt.plot(x,y1,'g', label='f_a_lambda')
plt.legend(loc='lower right')
plt.xlabel('k')
plt.ylabel('y')
plt.title('annee '+str(1909+i))
plt.savefig('annee_zoom{:03}.png'.format(i))
plt.close()

```

```

plt.figure()
plt.ylim(0,1.5)
plt.plot([df.loc[i,'k']], [df.loc[i,'y']], 'ro')
plt.plot([df.loc[i+1,'k']], [df.loc[i+1,'y']], 'bo')
plt.plot(x,y0,'r-', label='f0')
plt.plot(x,y1,'b-', label='f1')
plt.plot(x,ym,'y', label='f_a_mu')
plt.plot(x,y1,'g', label='f_a_lambda')
plt.legend(loc='lower right')
plt.xlabel('k')
plt.ylabel('y')
plt.title('annee '+str(1909+i))
plt.savefig('annee{:03}.png'.format(i))
plt.close()

```

D The program with CES production functions

```

import pandas as pd
import numpy as np
import matplotlib.pyplot as plt
import openpyxl
from scipy.optimize import fsolve
import matplotlib.patches as mpatches

df = pd.read_excel(open('BEA_data_comp_employ.xlsx', 'rb'))
sigma = 0.5

dfadd = pd.DataFrame({
    'y_shifted':[],
    'k_shifted':[],
    'a_shifted':[],
    'a_lambda':[],
    'a_mu':[],
    'time_regular':[],
    'mu_lin':[],
    'lambda_lin':[]
})

df = df.join(dfadd) y1 = df['y'].shift(-1)
k1 = df['k'].shift(-1)
a1 = df['a'].shift(-1)

df['y_shifted'] = y1
df['k_shifted'] = k1
df['a_shifted'] = a1

y0 = df['y']
k0 = df['k']

time_regular = y0 < y1+a1*y1*(k0-k1)/k1
df['time_regular'] = time_regular

print(df.head())
#ici on a tout le dataframe correct

df = df.drop(len(df)-1,0) # supprime la derniere ligne car le calcul est impossible dessus

def fonction_sol_sigma(x,L): # L = [y0,y1,k0,k1,alpha1] ou [y1,y0,k1,k0,alpha0]
    # fonction a resoudre
    y0 = L[0]
    y1 = L[1]
    k0 = L[2]
    k1 = L[3]
    alpha = L[4]
    return (1-alpha)**(sigma-1)*(y1/y0)**(sigma-1)-x**sigma*((1-alpha)/alpha)**(sigma-1)*(k1/k0)**(sigma-1)-(1-x)**sigma

def fonction_sol_k(x,L,alpha_mu):
    k0 = L[2]
    k1 = L[3]
    alpha = L[4]
    return (x/k0)**(1/sigma)-((1-alpha)/alpha)*(k1/k0)*(alpha_mu/(1-alpha_mu))

def parameter_list(df,n): #rend la liste L bien ordonnee
    if df['time_regular'][n]:
        return [df['y'][n],df['y_shifted'][n],df['k'][n],df['k_shifted'][n],df['a_shifted'][n]]
    else:
        return [df['y_shifted'][n],df['y'][n],df['k_shifted'][n],df['k'][n],df['a'][n]]

```

```

for i in range(len(df)):

    def function_sigma(x):
        return function_sol_sigma(x, parameter_list(df, i)) # cree la fonction de resolution de l'annee i

    df.loc[i, 'a_mu'] = fsolve(function_sigma, 0)[0] # cherche la premiere racine
    df.loc[i, 'a_lambda'] = fsolve(function_sigma, 0.99)[0] # cherche la seconde racine
    alpha_mu = 1
    def function_k(x):
        return function_sol_k(x, parameter_list(df, i), alpha_mu)

df_out = df.loc[:, ('annee', 'y', 'k', 'a', 'a_mu', 'a_lambda', 'a_mu', 'a_lambda', 'mu_lin', 'lambda_lin', 'time_regular')] # retire les colonnes inutiles (shift)
df_out.columns = ['annee', 'y', 'k', 'a', 'a_mu', 'a_lambda', 'a_mu', 'a_lambda', 'mu_lin', 'lambda_lin', 'time_regular'] # renome

for i in range(len(df_out)): # met des nan dans les colonnes qui n'ont pas servies a cause du time_regular
    if df_out.loc[i, 'time_regular']:
        df_out.loc[i, 'a_mu'] = np.nan
        df_out.loc[i, 'a_lambda'] = np.nan
        df_out.loc[i, 'mu_lin'] = df_out.loc[i, 'a'] - df_out.loc[i, 'a_mu']
        df_out.loc[i, 'lambda_lin'] = - df_out.loc[i, 'a'] + df_out.loc[i, 'a_lambda']
    else:
        df_out.loc[i, 'a_mu'] = np.nan
        df_out.loc[i, 'a_lambda'] = np.nan
        df_out.loc[i, 'mu_lin'] = df_out.loc[i, 'a_mu'] - df_out.loc[i+1, 'a']
        df_out.loc[i, 'lambda_lin'] = - df_out.loc[i, 'a_lambda'] + df_out.loc[i+1, 'a']

df_out.to_csv('data_out_BEA_data_comp_employ.csv', sep=';')

ymin = df['y'].min()
ymax = df['y'].max()
kmin = df['k'].min()
kmax = df['k'].max()

for i in range(len(df)-1):
    def f_0(x): # fonction de production annee n
        return df.loc[i, 'y']*(x/df.loc[i, 'k'])**df.loc[i, 'a']
    def f_1(x): # fonction de production annee n+1
        return df.loc[i+1, 'y']*(x/df.loc[i+1, 'k'])**df.loc[i+1, 'a']
    def f_a_mu(x): # fonction de production mu
        if df.loc[i, 'time_regular']:
            return df.loc[i, 'y']*(x/df.loc[i, 'k'])**df.loc[i, 'a_mu']
        else:
            return df.loc[i+1, 'y']*(x/df.loc[i+1, 'k'])**df.loc[i, 'a_mu']
    def f_a_lambda(x): # fonction de production lambda
        if df.loc[i, 'time_regular']:
            return df.loc[i, 'y']*(x/df.loc[i, 'k'])**df.loc[i, 'a_lambda']
        else:
            return df.loc[i+1, 'y']*(x/df.loc[i+1, 'k'])**df.loc[i, 'a_lambda']

    x = np.linspace(0, int(1.2*kmax)+1, 500)
    y0 = f_0(x)
    y1 = f_1(x)
    ym = f_a_mu(x)
    yl = f_a_lambda(x)

    plt.figure()
    plt.ylim(0.8*ymin, ymax+0.2*ymin)
    plt.xlim(0.8*kmin, kmax+0.2*kmin)
    plt.plot([df.loc[i, 'k']], [df.loc[i, 'y']], 'ro', label='(y0, k0)')
    plt.plot([df.loc[i+1, 'k']], [df.loc[i+1, 'y']], 'bo', label='(y1, k1)')
    plt.plot(x, y0, 'r-', label='f0')
    plt.plot(x, y1, 'b-', label='f1')
    plt.plot(x, ym, 'y', label='f_a_mu')
    plt.plot(x, yl, 'g', label='f_a_lambda')
    plt.legend(loc='lower right')
    plt.xlabel('k')
    plt.ylabel('y')
    plt.title('annee '+str(1909+i))
    plt.savefig('annee_zoom{:03}.png'.format(i))
    plt.close()

    plt.figure()
    plt.ylim(0, ymax*1.2)
    plt.xlim(0, kmax*1.2)
    plt.plot([df.loc[i, 'k']], [df.loc[i, 'y']], 'ro')
    plt.plot([df.loc[i+1, 'k']], [df.loc[i+1, 'y']], 'bo')
    plt.plot(x, y0, 'r-', label='f0')
    plt.plot(x, y1, 'b-', label='f1')
    plt.plot(x, ym, 'y', label='f_a_mu')
    plt.plot(x, yl, 'g', label='f_a_lambda')
    plt.legend(loc='lower right')
    plt.xlabel('k')
    plt.ylabel('y')
    plt.title('annee '+str(1909+i))
    plt.savefig('annee{:03}.png'.format(i))
    plt.close()

```

References

- [1] D. ACEMOGLU and P. RESTREPO (2017) “Robots and Jobs: Evidence from US Labor Markets,” *NBER WP No. 23285* (Mar.).
- [2] E. BENGTSSON and D. WALDENSTRÖM (2017) “Capital Shares and Income Inequality: Evidence from the Long Run,” *Journal of Economic History*, forthcoming.
- [3] J. GLACHANT (2003) “Distributional Consequences of Competitive Structural Change,” *Journal of Economics*, Vol. 80, No. 3 (Nov.), pp. 231–248.
- [4] J. GREENWOOD, Z. HERCOWITZ and P. KRUSELL (1997) “Long-Run Implications of Investment-Specific Technological Change,” *American Economic Review*, Vol. 87, No. 3 (Jun.), pp. 342–362.
- [5] C. I. JONES (2005) “The Shape of the Production Function and the Direction of Technical Change,” *Quarterly Journal of Economics*, Vol. 120, No. 2 (May), pp. 517–549.
- [6] L. KARABARBOUNIS and B. NEIMAN (2014) “The Global Decline of the Labor Share,” *Quarterly Journal of Economics*, Vol. 129 No. 1 (Feb.), pp. 61–103.
- [7] N. G. MANKIW, D. ROMER and D. N. WEIL (1992) “A Contribution to the Empirics of Economic Growth,” *Quarterly Journal of Economics*, Vol. 107, No. 2 (May), pp. 407–437.
- [8] G. MOYSAN and M. SENOUCI (2016) “A Note on 2-Input Neoclassical Production Functions,” *Journal of Mathematical Economics*, No. 67 (Dec.), 80–86.
- [9] M. SENOUCI (2014) “The Habakkuk hypothesis in a neoclassical framework,” *HAL working paper No. 01206032*.
- [10] R. M. SOLOW (1957) “Technical Change and the Aggregate Production Function,” *Review of Economics and Statistics*, Vol. 39, No. 3 (Aug.), pp. 312–320.
- [11] B. STURGILL (2014) “Back to the basics: Revisiting the development accounting methodology,” *Journal of Macroeconomics*, Vol. 42 (Dec.), pp. 52–68.
- [12] H. ZULETA (2008) “Factor saving innovations and factor income shares,” *Review of Economic Dynamics*, Vol. 11, No. 4 (Oct.), pp. 836–851.
- [13] H. ZULETA and A. T. YOUNG (2013) “Labor shares in a model of induced innovation,” *Structural Change and Economic Dynamics*, Vol. 24 (Mar.), pp. 112–122.
- [14] B. STURGILL and H. ZULETA (2017) “Variable factor shares and the index number problem: a generalization,” *Economics Bulletin*, Vol. 37, No.1, pp. 30–37.