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Elastomer Biaxial Characterization Using Bubble Inflation Technique. II: Numerical Investigation of Some Constitutive Models

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An elastomeric material was investigated with a bubble inflation rheometer, and its mechanical behavior was modeled as a rubber-like solid. Classical strain energy functions were considered and the hyper-elastic constants were calculated by a direct identification procedure from simple uniaxial and equibiaxial extension test data, and the results are compared against those obtained by an inverse method from bubble inflation test data. The latter amounted to minimizing a cost function and matching the measured response to a finite element analysis solution, which depended on the unknown material parameters. The optimization employed the Levenberg-Marquardt algorithm and Abaqus software to compute the cost function and its gradients. The constants so obtained were further used in finite element analysis, and the numerical results were compared with experiments. This study showed that the inverse method, used to estimate the material parameters, is a good alternative to the direct identification, especially since the latter often requires homogeneous strain state, which is very difficult to obtain.

1. INTRODUCTION

For theoretical convenience, elastomeric materials are modeled using rubber-like solid model. With this model, the material is assumed to be isotropic and perfectly elastic for large strains; see Ogden (1). In addition, the incompressibility assumption is often made. Such a material is called hyper-elastic, and its constitutive equation can be stated by means of a strain energy function W. For isotropic behavior, W is a state function of deformation tensor invariant and material rheological parameters. During recent decades, several forms of W were proposed for rubber-like materials and various hyper-elastic models were integrated in computer software, intended for non-linear calculations, like Abaqus or Marc.

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For a given form of W to be of a practical use, the material parameters have to be determined. In the standard method for material parameter estimations, a specimen is subjected to homogeneous strains (uniaxial, biaxial extension or pure shear tests) and the stress-strain measurements are used to fit hyper-elastic constants to the measured data. The effectiveness of such a method requires a homogeneous state of deformation during the tests.

In earlier works on non-homogeneous large elastic deformations, Green and Adkins (2) used the Mooney form of W to predict the inflation of a circular plane sheet. The profiles of the inflated membrane were compared with the profiles measured by Treloar (3). This membrane inflation technique was also used by Winman *et al.* (4) in biomechanics for material identification of soft tissue. The authors proposed an original approach to investigate the domain of definition of W.

During the last decade, many research works were devoted to the material parameter estimation using inverse methods. This approach based on mixed experimental numerical methods does not require the homogeneity of deformation that is needed for the so-called standard method. The inverse identification uses the straight comparison between measured response and the numerical results from the considered constitutive model. This method was successfully used to estimate material parameters of many complex constitutive equations in the inelastic range. In this work, we will investigate some fitting procedures and various strain energy potentials for mechanical behavior characterization of the elastomer studied in part I of this paper. The investigated material is a natural rubber/SBR compound, used by SNECMA-SEP for mechanical applications (flexible parts). The used inverse method is based on the comparison between the measured and the calculated responses for the inflation of a circular membrane. The experimental data are obtained by means of the bubble inflation rheometer previously described.

2. HYPER-ELASTIC MODEL

In this section, we will briefly recall the fundamental equations of an hyper-elastic model. The mechanical behavior of hyper-elastic materials is completely characterized by means of a strain energy potential that is a function of the deformation tensor invariant:

$$\begin{cases} W = W(I_1, I_2, I_3) \\ I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \\ I_2 = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_1^2\lambda_3^2 \\ I_3 = \lambda_1^2\lambda_2^2\lambda_3^2 \end{cases} \tag{1}$$

where λ_1, λ_2 and λ_3 are principal stretches.

Hyper-elastic materials are often nearly or fully incompressible; hence they undergo volume-preserving deformation, which can be expressed as:

$$I_3 = 1 \tag{2}$$

Among available techniques for taking into account the incompressibility constraint, the Lagrange multiplier is the most convenient one; see Sussman *et al.*(5) for a complete review on the subject. In its simplest form, the strain energy function of *Eq 1* is replaced by:

$$W = W(I_1, I_2, I_3) = \hat{W}(I_1, I_2) - \frac{P}{2}(I_3 - 1) \tag{3}$$

where P is the Lagrange multiplier, equivalent to the hydrostatic pressure.

3. IDENTIFICATION OF MATERIAL PARAMETERS

The mechanical characterization of materials is essential in the vast majority of engineering applications. Once the constitutive model is established, it is necessary to determine its material constants. In the so-called direct identification method, the material parameters are determined using measurement on some test specimen with simple boundary condition (uniaxial extension, torsion etc.). The main drawback of this method is the strain homogeneity requirement, which limits the choice of test specimen and boundary conditions; Michino *et al.* (6). These difficulties can be overcome using the inverse method for which the strain homogeneity is not required. When used in material parameter identification, the inverse method consists in finding the unknown parameters in such a way that the response calculated with the constitutive model matches the measured response. The method combines optimization techniques with numerical analysis based on finite element or boundary element method to obtain the computed response; Den Camp *et al.* (7). The inverse technique was successfully used by Aoki *et al.* (8) to identify constitutive model including damage phenomena, and also by Tillier *et al.* (9) to characterize time dependent constitutive equations. In this section, we compare material constants obtained by direct identification method from uniaxial and equibiaxial extension test data with those obtained by an inverse method from the bubble inflation test data. First, we will briefly recall the standard (direct) identification procedure and then we will describe the inverse method.

3. 1. Direct Identification Method

The mechanical behavior of a rubber-like solid (incompressible) material can be determined by means of experiments involving simple deformation modes like uniaxial extension, equibiaxial extension and pure shear. Once the test data are available, the material constants in the strain energy function can be fitted to these experimental data by means of least square fit procedure. Given "n" measured stress-strain pairs, the best set of constants is the one that minimizes the following error:

$$E = \sum_{i=1}^{n}\left(1 - \frac{S_i^{th}}{S_i^{test}}\right)^2 \tag{4}$$

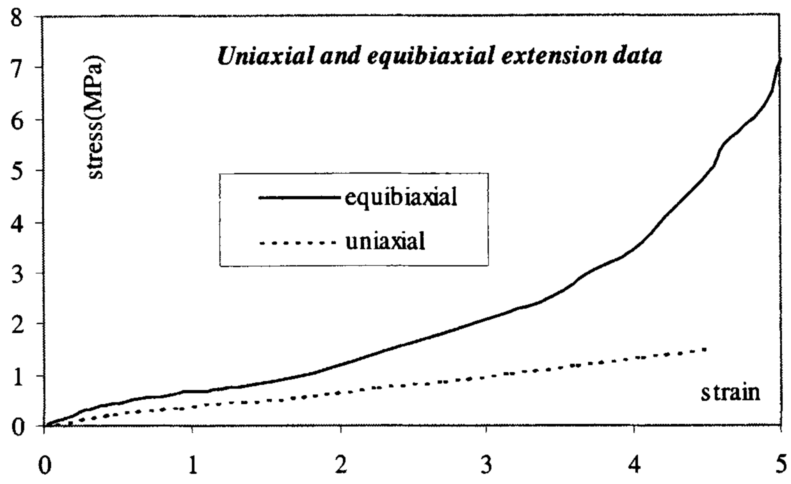


Fig. 1. Nominal strain-nominal stress curves obtained from uniaxial and equibiaxial extension tests.

S_i^{test}: measured nominal stress

S_i^{th}: theoretical stress derived from a specific strain energy potential

For the investigated material, uniaxial and equibiaxial extension test data are shown in *Fig. 1*.

Depending on the constitutive model, *Eq 4* can be non-linear in material parameters. Consequently, a non-linear least squares fit is needed.

3.2. Inverse Identification Method

In this work the hyper-elastic constants fitted to uniaxial and biaxial extension test data were used in finite element analysis of the bubble inflation test. We have noticed a substantial difference between the obtained results and the measurements. This discrepancy has motivated the development of an inverse method, which combines optimization technique with finite element analysis of the bubble inflation test described in part I of this paper.

To reduce the errors on cost function and its gradient, several finite element models were compared (2 and 3 node axisymmetric shell elements, 4 and 8 node axisymmetric solid elements with hybrid formulation, reduced integration and incompatible modes). Since the obtained results are quite similar, the 2 node axisymetric shell element was adopted for its computational cost effectiveness. A complete description of the axisymetric finite element model is given in *Fig. 2*.

The material parameter estimation consists in adjusting the parameters in the finite element model until the calculated displacements u^* (bubble height *Fig. 3*) match the measured displacement $\hat{u}$, Gelin *et al.* (10). It should be pointed out that the comparison between the computed and the measured displacements is performed at the center of the circular membrane. The cost function to minimize with respect to material parameters is given by:

$$\Phi(p) = \frac{1}{N_m}\sum_{i=1}^{N_m} r^T Qr \qquad (5)$$

$$r = r(p) = u^* - \hat{u}$$

$<p> = <p_1\ p_2 \ldots p_n>$ material parameters

N_m is the number of measured bubble heights.

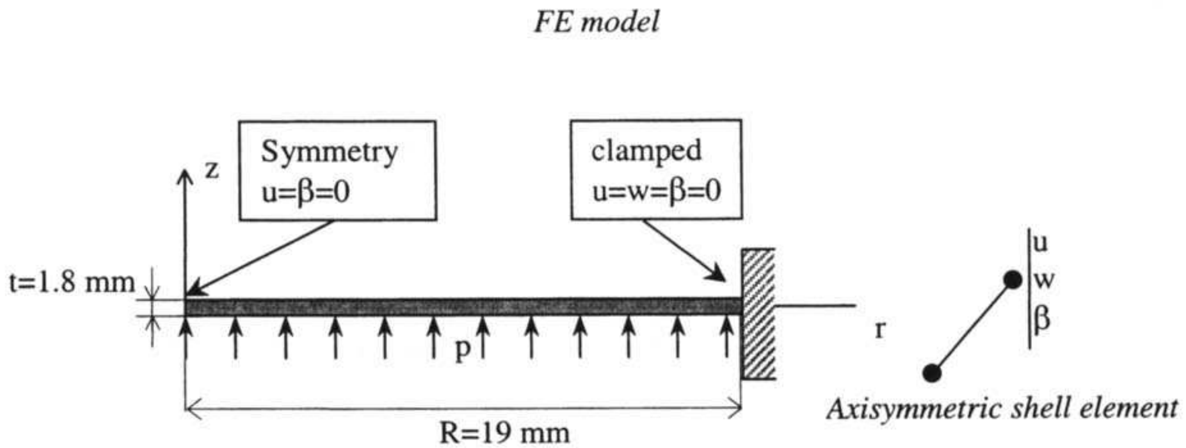
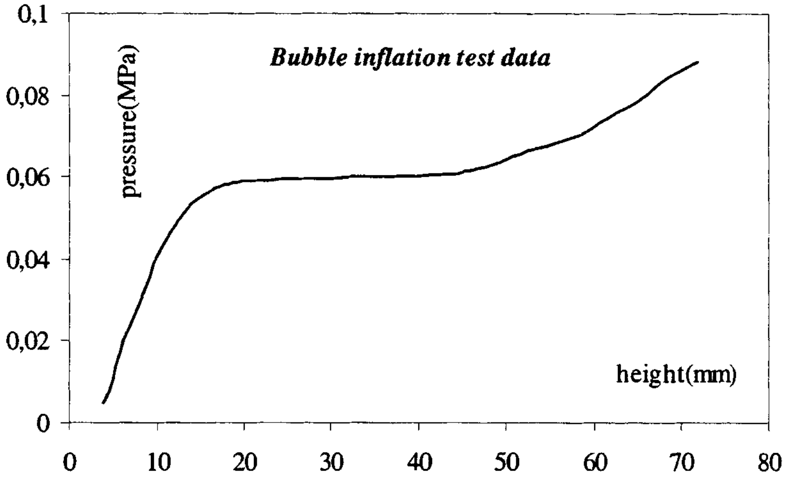


Fig. 2. Axisymetric finite element model of bubble inflation test used with inverse identification method.

Fig. 3. *Pressure-bubble height curve obtained by the bubble inflation rheometer and used for inverse identification.*



Q is a weighting matrix used to improve the result accuracy when more than one test data are available (bubble height, thickness...)

For physical considerations like Drucker stability, the material parameters must be constrained, Schnur *et al.* (11), which can be written in the form:

$$C_i(p) \leq 0 \quad i = 1, m \tag{6}$$

These constraints can be taken into account by means of the penalty method, and the modified cost function is then given by:

$$\Phi^*(p) = \frac{1}{N_m}\sum_{i=1}^{N_m} r^T Qr + \sum_{j=1}^{m}\{\alpha_j\, Max(0, C_j(p))\}^2 \tag{7}$$

The cost function $\Phi^*(p)$ is minimized by means of the Levenberg-Marquardt algorithm. A detailed description of the used inverse method was given by Pilvin (12).

4. IDENTIFICATION OF SOME HYPER-ELASTIC MODELS

Since elastomeric materials are increasingly used in industry, great research effort was devoted to the development of constitutive models that describe realistically their behavior. Hence several strain energy functions were proposed. In this work, the material parameters of some popular strain energy functions are estimated using the direct identification method and the inverse method. With the direct method, the hyper-elastic constants are fitted to uniaxial and equibiaxial extension test data, whereas with the inverse method, the constants are obtained from the bubble inflation test data. For each constitutive model, the following investigations are performed:

- Material parameter estimation using direct identification method and uniaxial/equibiaxial extension test data.
- Material parameter estimation using inverse method and bubble inflation test data.
- Check of correlation between the numerical results from the hyper-elastic model and bubble inflation test data.
- Check of correlation between the numerical results from the hyper-elastic model and the equibiaxial extension test data.

The finite element calculations are performed using the material parameters obtained by the direct method and those obtained by the inverse method.

4.1 Polynomial Strain Energy Function and Its Variant

This model is based on phenomenological theory first initiated by Rivlin, who proposed a polynomial function as the general form of the strain energy potential:

$$\hat{W} = \sum_{i+j=1}^{n} C_{ij}(I_1 - 3)^i (I_2 - 3)^j \tag{8}$$

where C_{ij} are hyper-elastic constants.
This general form includes some popular energy functions like the Mooney-Rivlin model and the Neo-Hookean one. The Mooney-Rivlin function is obtained by setting n = 1 and $C_{ij} = 0$ except C_{10} and C_{01}:

$$\hat{W} = C_{10}(I_1 - 3) + C_{01}(I_2 - 3) \tag{9}$$

In this work, only the Mooney-Rivlin model (polynomial model with n = 1) is identified. For high order models, oscillations and material instability occur because the available test data are not sufficient.

The material parameter values obtained by direct method, those obtained by the inverse methods, and the associated constraints are given in *Table 1*.

In *Fig. 4*, experimental and calculated pressure-bubble height curves are represented, along with a comparison between direct identification and inverse identification when the obtained material parameters are used in finite element analysis of the bubble inflation test. It should be noticed that for large strains,

Table. 1. Material Parameter Values for Different Models.

Model	Parameter	Direct identification	Inverse identification	Constraints
polynomial (n = 1)	C_{10} (Mpa) C_{01} (Mpa)	0.101732 1.243427×10^{-2}	0.1571831 1.010127×10^{-2}	$C_{10} + C_{01} > 0$
Yeoh	C_{10} (Mpa) C_{20} (Mpa) C_{30} (Mpa)	0.137183 1.415×10^{-3} 1.5568×10^{-5}	0.19405 0.93719×10^{-5} 0.33713×10^{-4}	$C_{10} > 0$ $C_{20} > 0$ $C_{30} > 0$
Arruda-Boyce	μ (Mpa) λ_m	0.28191 4.695	0.327993 4.4	$\mu > 0$ $\lambda_m > 0$
Van der waals	μ (Mpa) λ_m a	0.244314 11.2545 -7.9445×10^{-02}	0.384766 10.25569 0.189963	$\mu > 0$ $\lambda_m > 0$
Ogden (n = 2)	μ_1 (Mpa) μ_2 (Mpa) α_1 α_2	0.219472 1.79949×10^{-3} −0.7487 −2.82844	0.270533 0.131425×10^{-2} −0.7266249 −3.588489	$\mu_1 + \mu_2 > 0$

the correlation between numerical results from the constitutive model and experimental results is equally poor for the inverse method as for the direct method. But for small strains the inverse method seems to be more effective. The observed discrepancy is due to the low order of the constitutive model (n = 1).

When the material parameters obtained are used in an equibiaxial test simulation, both the inverse method and the direct method give satisfactory correlation between numerical and experimental results (*Fig. 5*).

4.2. Yeoh Model

It has been shown experimentally that the sensitivity of the strain energy potential to changes in the first invariant I_1 is largely dominant, and the model prediction can be enhanced when neglecting the second invariant contribution that is, in addition, difficult to measure.

Following these considerations, Yeoh (13) proposed a reduced form of the polynomial strain energy function where the second invariant contribution is completely omitted:

$$\hat{W} = \sum_{i=1}^{3} C_{i0} (I_1 - 3)^i \tag{10}$$

The material parameters obtained by the direct method, those obtained by the inverse methods, and the associated constraints are summarized in *Table 1*. These material parameters are used in numerical simulation of bubble inflation test. In *Fig. 6*, experimental and computed pressure-bubble height curves are presented. It is clearly shown that the inverse identification is better than the direct one. We can also notice that the numerical results are in good agreement with experiments for the simulation of the equibiaxial extension test. A comparison between the direct identification and the inverse one is given in *Fig. 7*.

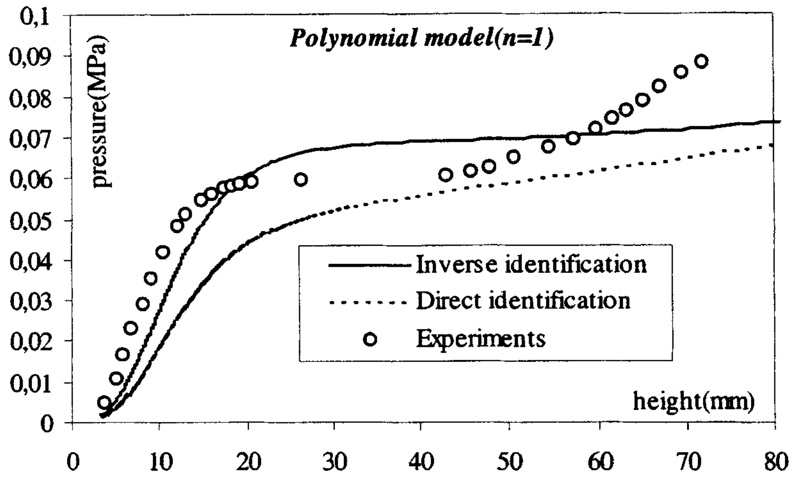


Fig. 4. Comparison of numerical results from Mooney-Rivlin constitutive model (direct and inverse identifications) and bubble inflation test data.

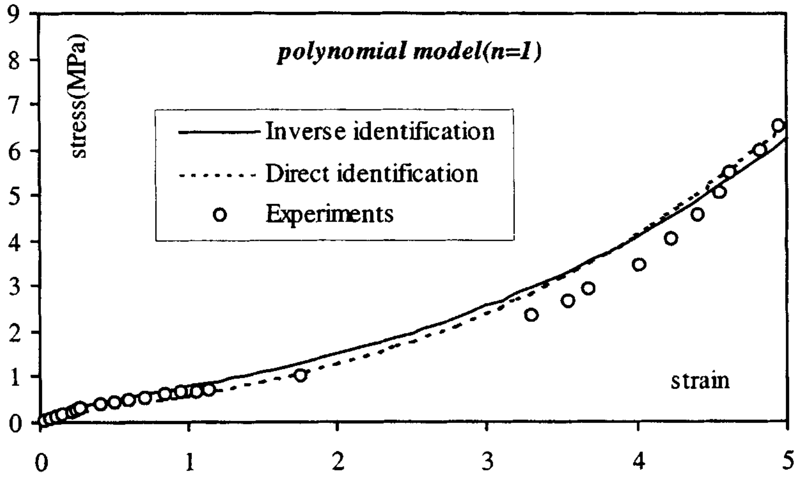


Fig. 5. Comparison of numerical results from Mooney-Rivlin constitutive model (direct and inverse identifications) and equibiaxial extension test data.

Compared to the polynomial model, this model is more effective because it contains higher powers of the first invariant and the contribution of the second invariant to the strain energy is neglected.

4. 3. Arruda-Boyce Model

This strain energy function, which depends on the first invariant only, was developed by means of the mechanical statistical theory, Arruda *et al.* (14):

$$\hat{W} = \mu \sum_{i=1}^{5} \frac{C_i}{\lambda_m^{2i-2}}(I_1^i - 3^i)$$

$$C_1 = 0.5, \quad C_2 = 0.05, \quad C_3 = \frac{1}{1050}, \quad C_4 = \frac{19}{7050}, \quad C_5 = \frac{514}{673750} \tag{11}$$

where μ and λ_m are material parameters to be estimated.

The identified parameter values obtained by the direct method, those obtained by the inverse methods, and the associated constraints are given in *Table 1*.

The numerical results from this constitutive model are in good agreement with experiments for the simulation of bubble inflation test in *Fig. 8*, as well as for the simulation of equibiaxial extension test in *Fig. 9*. Once again, the inverse identification method seems to be more effective than the direct one when the obtained material parameters are used in numerical simulation of the bubble inflation test.

4.4. Van der Waals Model

In our work, this model fitted in its general form leads to material instability. Thus a particular form, where the second invariant contribution is omitted, was used:

$$\hat{W} = \mu\left\{-(\lambda_m^2 - 3)(\mathrm{Log}(1-\eta)+\eta) - \frac{2}{3}a\left(\frac{I_1-3}{2}\right)^{\frac{3}{2}}\right\}$$

$$\eta = \left(\frac{I_1-3}{\lambda_m-3}\right)^{\frac{1}{2}} \tag{12}$$

where λ_m, μ and a are hyper-elastic constants

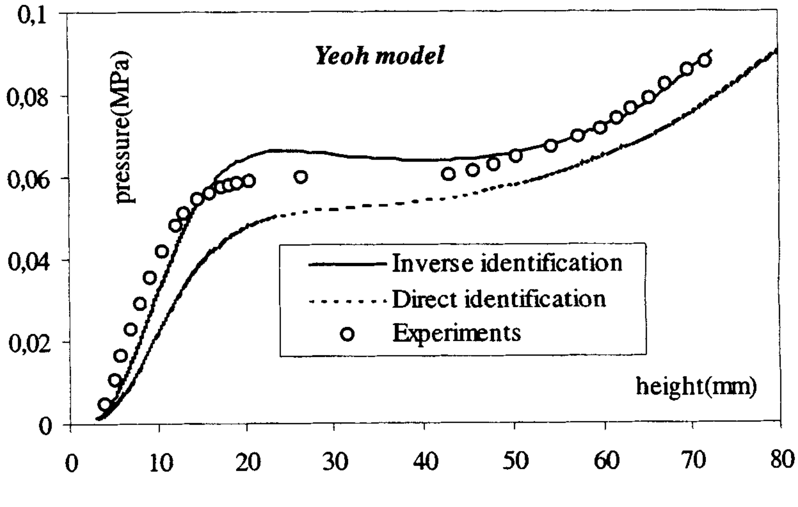


Fig. 6. Comparison of numerical results from Yeoh constitutive model (direct and inverse identifications) and bubble inflation test data.

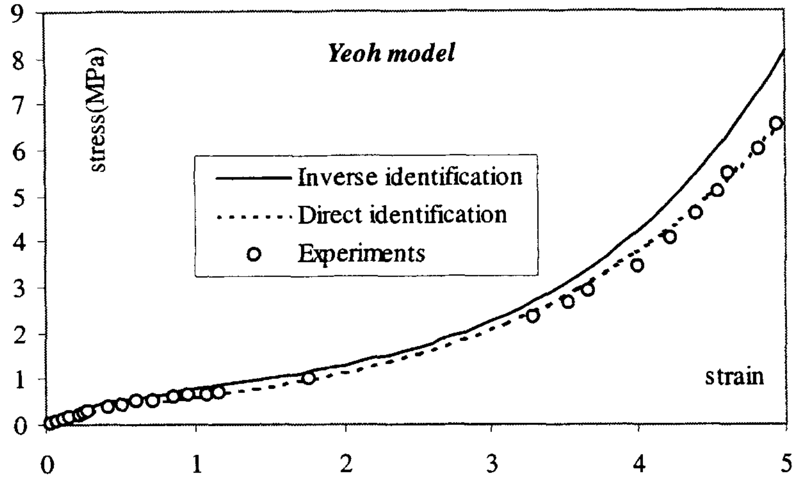


Fig. 7. Comparison of numerical results from Yeoh constitutive model (direct and inverse identifications) and equibiaxial extension test data.

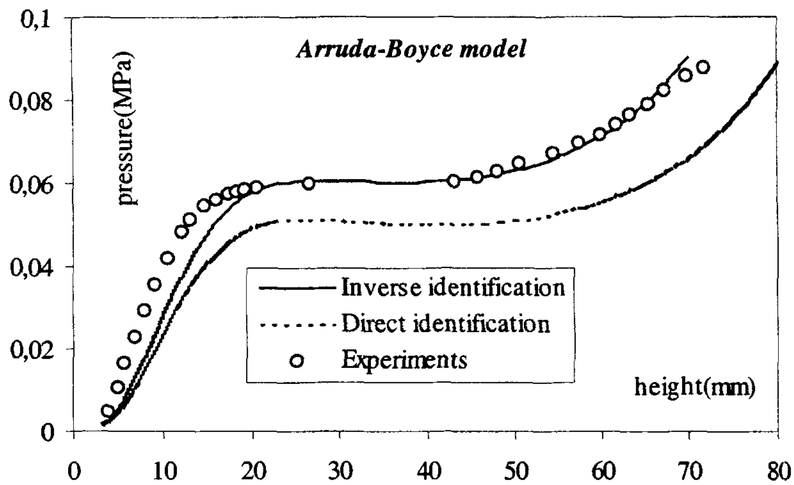


Fig. 8. Comparison of numerical results from Arruda-Boyce constitutive model (direct and inverse identifications) and bubble inflation test data.

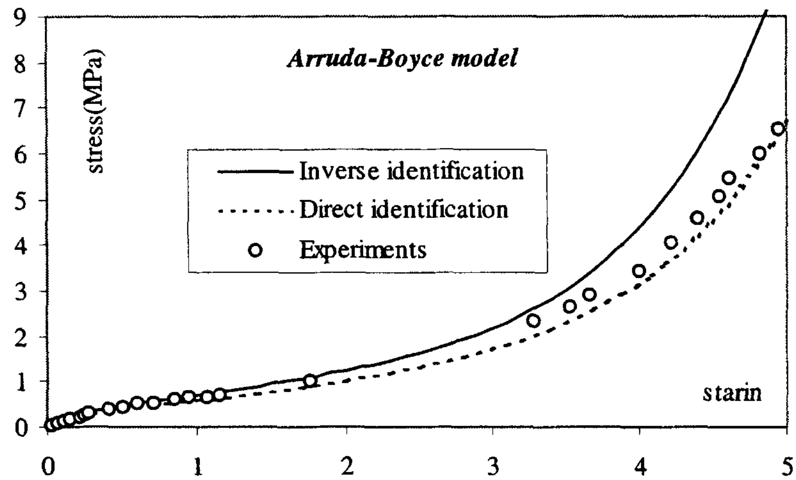


Fig. 9. Comparison of numerical results from Arruda-Boyce constitutive model (direct and inverse identifications) and equibiaxial extension test data.

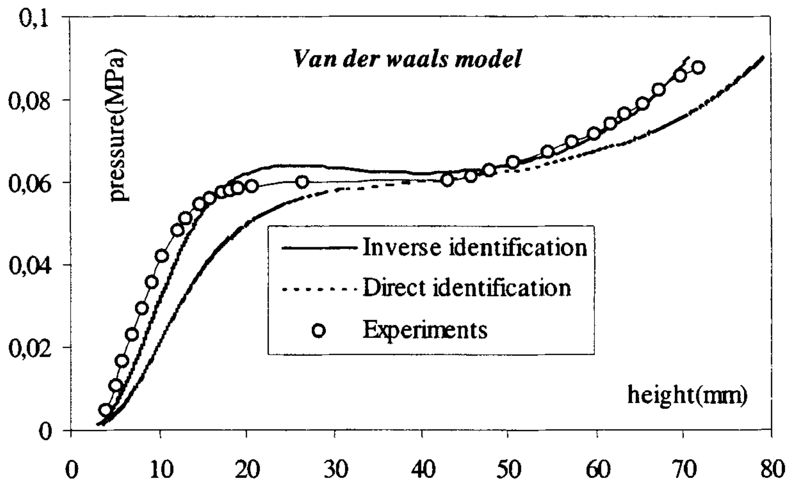


Fig. 10. Comparison of numerical results from Van der Waals constitutive model (direct and inverse identifications) and bubble inflation test data.

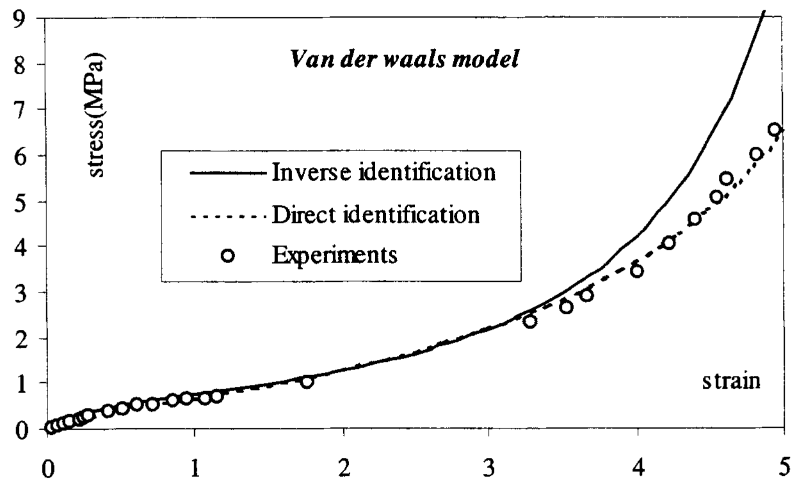


Fig. 11. Comparison of numerical results from Van der Waals constitutive model (direct and inverse identifications) and equibiaxial extension test data

The material parameters obtained by the direct method and those obtained by the inverse methods are summarized in *Table 1*. They are used in numerical simulation of the bubble inflation test in *Fig. 10* and the equibiaxial extension test in *Fig. 11*. For both tests, the numerical results are in good agreement with experiments. With the bubble inflation test, the inverse method is more effective than the direct one, but with the equibiaxial extension test, a discrepancy is observed for large strain.

4.5. Ogden Model

Among the strain energy functions investigated, the Ogden model holds a particular position since the strain energy function is expressed in terms of the principal stretches λ_1, λ_2 and λ_3:

$$\hat{W} = \sum_{i=1}^{n} \frac{\mu_i}{\alpha_i} (\lambda_1^{\alpha_i} + \lambda_2^{\alpha_i} + \lambda_3^{\alpha_i} - 3) \qquad (13)$$

where α_i and μ_i are material parameters.

The identified parameter values obtained by the direct method, those obtained by the inverse methods, and the associated constraints are summarized in *Table 1*. When these material parameters are used for numerical simulation of the bubble inflation test in *Fig. 12*, we can clearly see that the model fails in the membrane behavior prediction when the direct identification is used. The discrepancy becomes significant in the instability zone (27 < bubble height < 43).

For the equibiaxial extension test, the numerical results obtained from the constitutive model are in good agreement with experiments in *Fig. 13*.

5. CONCLUSION

Experimental data obtained by means of a bubble inflation rheometer are used to characterize the mechanical behavior of a natural rubber. Several hyperelastic constitutive models are considered. Among the strain energy functions investigated in this work, the best fit is obtained when the second invariant contribution is omitted. Except with the Ogden and the Mooney-Rivlin models, the correlation between numerical result from the constitutive model and experiments is satisfactory for the bubble inflation test as

Fig. 12. Comparison of numerical results from Ogden (n = 2) constitutive model (direct and inverse identifications) and bubble inflation test data.

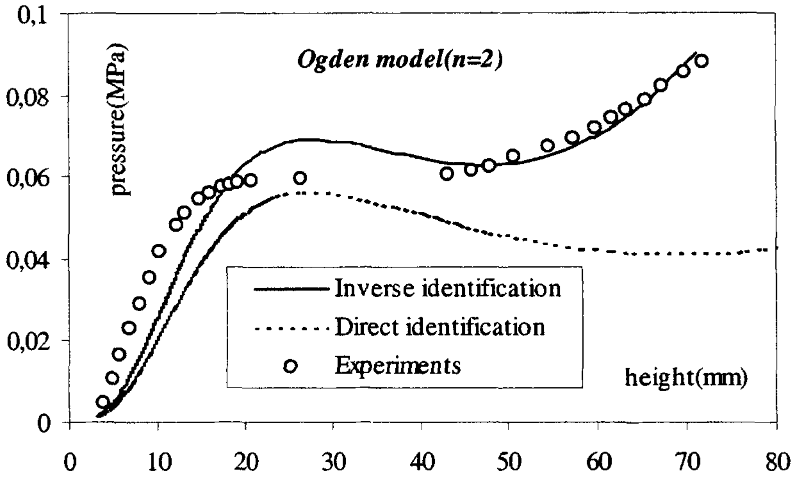


Fig. 13. Comparison of numerical results from Ogden (n = 2) constitutive model (direct and inverse identifications) and equibiaxial extension test data.

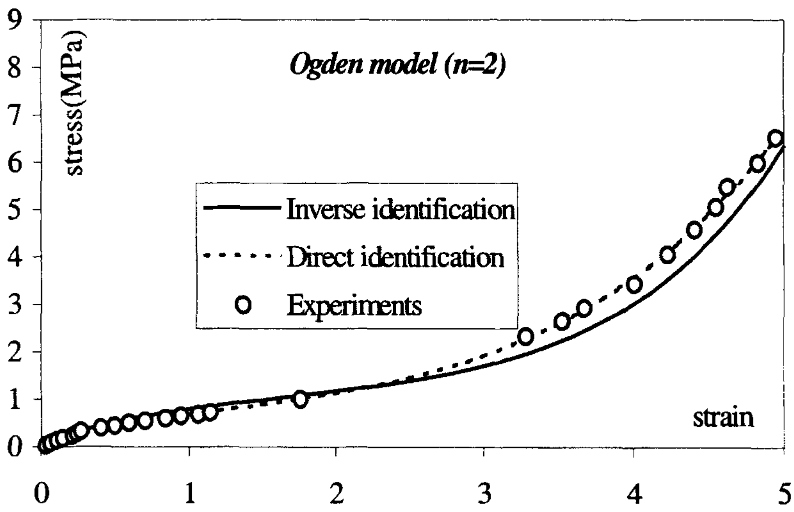
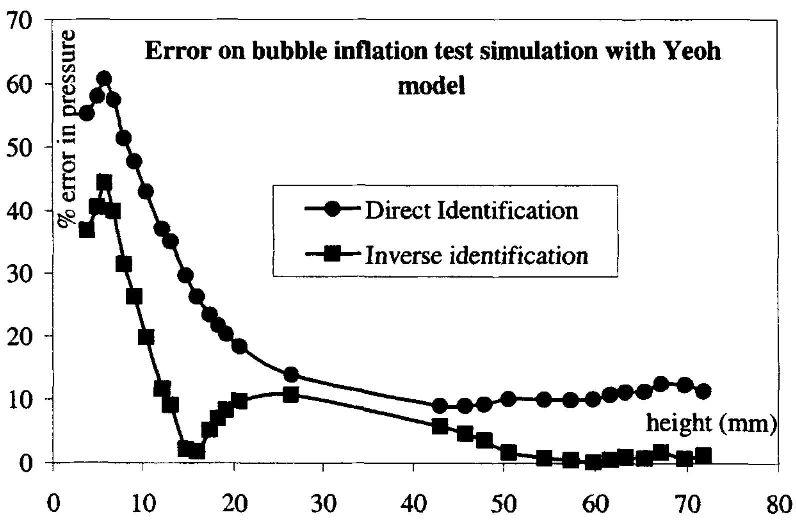


Fig. 14. Error in numerical results of bubble inflation test simulation with Yeoh model (comparison between direct and inverse identifications).



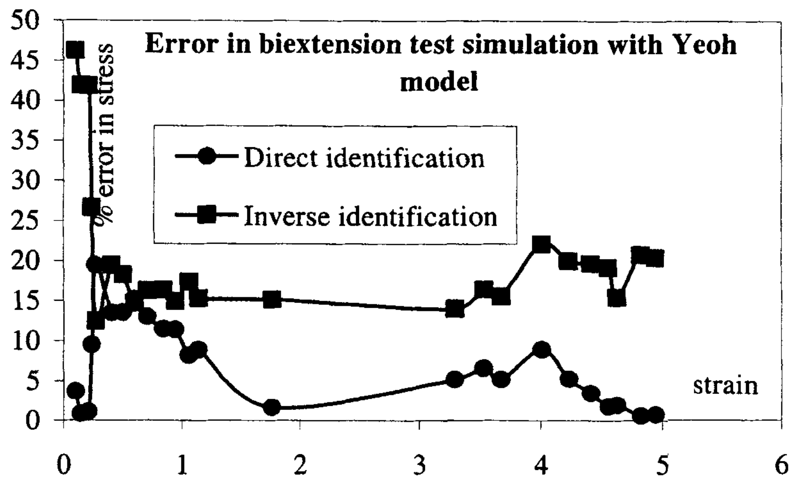


Fig. 15. Error on numerical results of equibiaxial test simulation with Yeoh model (comparison between direct and inverse identifications).

well as for the equibiaxial extension test. In addition, it can be noticed that the model prediction is enhanced when the second invariant contribution is neglected.

An inverse identification method, which combines optimization technique and finite element analysis, is used to estimate material parameters. The parameter values obtained are compared with those obtained by the standard direct method. With regard to the error analysis, we observed the same tendency for the various constitutive models. To illustrate this, we present the error in numerical results from the simulation of the bubble inflation test and those from the equibiaxial extension test simulation with the Yeoh model. In particular concerning the bubble inflation test, the evolution of the error in the pressure with respect to the bubble height, for both the direct identification and the inverse one, is given in *Fig. 14*. The material parameters obtained by the direct identification seem to fail in the description of non-homogeneous state of deformation.

The same investigations were carried out for the equibiaxial extension test. As shown in *Fig. 15*, presenting the error in nominal stress with respect to the nominal strain, the comparison between the two methods shows that the error remains limited in the case of the direct identification, whereas it tends to grow in the case of the inverse identification.

In light of the previous observations, the inverse method seems to be a good tool for the material parameter estimation. Although such a method does not need the restrictive requirement of the strain state homogeneity, it can be further improved or calibrated when associated with simple homogeneous test data, like extension or equibiaxial extension.

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