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Abstract

We consider the binomial approximation of the American put price in the Black-Scholes model (with continuous dividend yield). Our main result is that the error of approximation is $O((\ln n)^\alpha/n)$, where n is the number of time periods and the exponent α is a positive number, the value of which may differ according to the respective levels of the interest rate and the dividend yield.

1 The binomial approximation

Consider the Black-Scholes model, in which the stock price at time t is given by

$$S_t = S_0 e^{(r-d-\frac{\sigma^2}{2})t+\sigma B_t},$$

where, under the risk-neutral probability measure, $(B_t)_{t\geq 0}$ is a standard Brownian motion. Here, r is the instantaneous interest rate, and d is the dividend rate (or the foreign interest rate in the case of forex options). We assume $r > 0$ and $d \geq 0$.

Denote by P the price function of the American put with maturity T and strike price K, so that

$$P(t,x) = \sup_{\tau\in\mathcal{T}_{0,T-t}} \mathbb{E}_x\left(e^{-r\tau} f(S_\tau)\right), \quad 0 \leq t \leq T, \quad x \in [0,+\infty),$$

with $f(x) = (K-x)^+$, and $\mathbb{E}_x = \mathbb{E}(\cdot \mid S_0 = x)$. Here, $\mathcal{T}_{0,t}$ denotes the set of all stopping times with respect to the Brownian filtration, with values in the interval $[0,t]$.

For technical reasons (especially for the derivation of regularity estimates for the second time derivative of the price function), it is more convenient to use the log-stock price. So, we introduce

$$X_t^x = x + \mu t + \sigma B_t, \quad \text{with } \mu = r - d - \frac{\sigma^2}{2},$$

and

$$U(T,x) = \sup_{\tau\in\mathcal{T}_{0,T}} \mathbb{E}\left(e^{-r\tau}\varphi(X_\tau^x)\right),$$

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with $\varphi(x) = (K - e^x)^+$. We then have

$$P(t,x) = U(T-t, \ln(x)), \quad t > 0, x > 0.$$

Note that $U(t,x)$ satisfies the following parabolic variational inequality

$$\max\left[-\frac{\partial U}{\partial t} + (A-r)U, \varphi - U\right] = 0,$$

with the initial condition $U(0,.) = \varphi$.

Here, A is the infinitesimal generator of X, namely

$$A = \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} + \mu\frac{\partial}{\partial x}.$$

Recall that, for each $T > 0$, there is a real number $\tilde{b}(T) \leq \ln(K)$ such that

$$U(T,x) > \varphi(x) \Leftrightarrow x > \tilde{b}(T).$$

In fact, if $(b(t), 0 \leq t \leq T)$ is the exercise boundary of the American put with maturity T, we have $\tilde{b}(t) = \ln(b(T-t))$. We will also need the European value function, defined by

$$\bar{U}(T,x) = \mathbb{E}\left(e^{-rT}\varphi(X_T^x)\right).$$

Note that $\bar{U}(0,.) = \varphi$ and

$$-\frac{\partial \bar{U}}{\partial t} + (A-r)\bar{U} = 0.$$

Note that, in Section 3, the function $\bar{U}$ will be denoted by u_φ.

We now introduce the random walk approximation of Brownian motion. To be more precise, assume $(X_n)_{n\geq 1}$ is a sequence of i.i.d. real random variables satisfying $\mathbb{E}X_n^2 = 1$ and $\mathbb{E}X_n = 0$, and define, for any positive integer n, the process $B^{(n)}$ by

$$B_t^{(n)} = \sqrt{T/n}\sum_{k=1}^{[nt/T]} X_k, \quad 0 \leq t \leq T,$$

where $[nt/T]$ denotes the greatest integer in nt/T.

We will assume the following about the common distribution of the X_n's (cf. hypothesis (H4) of [6]). Note that, in the binomial case, X_1 takes its values in $\{-1, +1\}$.

(H4) The random variable X_1 is bounded and satisfies $\mathbb{E}X_1^2 = 1$ and $\mathbb{E}X_1 = \mathbb{E}X_1^3 = 0$.

In the following, we fix S_0 and set

$$P_0 = P(0, S_0) = U(T, \ln S_0).$$

Note that, if we introduce the notation $g(x) = (K - S_0 e^{\sigma x})^+$, we have

$$P_0 = \sup_{\tau \in \mathcal{T}_{0,T}} \mathbb{E}\left(e^{-r\tau} g(\mu_0 \tau + B_\tau)\right),$$

with $\mu_0 = \mu/\sigma$. We now have a natural approximation of P_0, given by

$$P_0^{(n)} = \sup_{\tau \in \mathcal{T}_{0,T}^{(n)}} \mathbb{E}\left(e^{-r\tau} g(\mu_0 \tau + B_\tau^{(n)})\right),$$

where $\mathcal{T}_{0,T}^{(n)}$ denotes the set of all stopping times (with respect to the natural filtration of $B^{(n)}$), with values in $[0,T] \cap \{0, T/n, 2T/n, \ldots, (n-1)T/n, T\}$. Our main result is the following.

Theorem 1.1. *There exists a positive constant C such that, for all positive integers n,*

$$-C\frac{(\ln n)^{\bar{\alpha}}}{n} \leq P_0^{(n)} - P_0 \leq C\frac{(\ln n)^{\alpha}}{n},$$

where $\bar{\alpha} = \alpha = 1$ if $d > r$, and $\bar{\alpha} = 3/2, \alpha = 5/4$ if $d \leq r$.

The above estimates improve our previous results (see [6], Theorem 5.6) which gave an upper bound of the form $C\left(\frac{\sqrt{\ln n}}{n}\right)^{4/5}$. Note that, for European options, the error estimate is $O(1/n)$ (see [2], [11]). We also mention the results of [8] about finite difference schemes, which give the rate $O(1/\sqrt{n})$, but their estimate is uniform over the time interval, while we concentrate on the error estimate for a fixed time. The paper [8] also has results about the approximation of the exercise boundary. We also refer to [9] and its references for a review of recent results on the approximation of American option prices.

Our approach remains the same as in [6]: we relate the error estimates to the regularity of the value function. The improvement comes from a refinement of the quadratic estimates for the second order time derivative, in the spirit of Friedman and Kinderlehrer (see [3] and [5]). We also exploit the smoothness of the exercise boundary and its asymptotic properties close to maturity.

The constant C in Theorem 1.1 is related to the Berry-Esseen estimate and to the regularity of the value function. Although it is hard to keep track of the constants in the regularity estimates, it may be worth mentioning that they remain uniform with respect to μ and σ as long as (μ, σ) remains in a compact subset of $\mathbb{R} \times (0, \infty)$. A consequence of this observation is that the bounds in Theorem 1.1 are also valid for variants of the approximation in which the process approximating $\ln(S_t/S_0)$, instead of being $\mu t + \sigma B_t^{(n)}$, is given by $\mu_n t + \sigma_n B_t^{(n)}$ at discrete times t, with $\mu_n = \mu + O(1/n)$ and $\sigma_n^2 = \sigma^2 + O(1/n)$, as occurs in the classical risk-neutral approximation. Indeed, standard arguments show that the value function is locally Lipschitz-continuous with respect to σ^2 (away from 0) and μ.

The paper is organized as follows. In the next Section we recall some results of [6]. Section 3 is devoted to estimates for the derivatives of the value function. The estimates are then used in Sections 4 and 5 to prove Theorem 1.1: in Section 4, we give an upper bound for $P_0^{(n)} - P_0$ and in Section 5, we derive the lower bound.

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2 The value function and the approximating process

As in [6], we introduce the modified value function

$$u(t,x) = e^{-rt}U(T-t, \ln(S_0) + \mu t + \sigma x), \quad t \geq 0, \quad x \in \mathbb{R}.$$

We have $P_0 = u(0,0)$ and $u(T,x) = e^{-rT}U(0, \ln(S_0) + \mu T + \sigma x) = e^{-rT}(K - S_0 e^{\mu T + \sigma x})^+$ and, for $t \in [0,T]$,

$$u(t,x) \geq e^{-rt}(K - S_0 e^{\mu t + \sigma x})^+ = e^{-rt} g(\mu_0 t + x). \tag{1}$$

We will need the European analogue of u, namely

$$\bar{u}(t,x) = e^{-rt}\bar{U}(T-t, \ln(S_0) + \mu t + \sigma x) = e^{-rT}\mathbb{E}\left(g(\mu_0 T + x + B_{T-t})\right), \quad t \geq 0, x \in \mathbb{R}.$$

We will also use the notation:

$$h = \frac{T}{n}\,.$$

With this notation, we have

$$B_t^{(n)} = \sqrt{h}\sum_{k=1}^{[t/h]} X_k, \quad 0 \leq t \leq T.$$

We have, for all $t \in \{0, h, 2h, \ldots, (n-1)h, nh = T\}$ (cf. Proposition 3.1 of [6]),

$$u(t, B_t^{(n)}) = u(0,0) + M_t + \sum_{j=1}^{t/h} \mathcal{D}u((j-1)h, B_{(j-1)h}^{(n)}),$$

where $(M_t)_{0 \leq t \leq T}$ is a martingale (with respect to the natural filtration of $B^{(n)}$), such that $M_0 = 0$, and

$$\mathcal{D}u(t,x) = \mathbb{E}\left(u\left(t+h, x + \sqrt{h}X_1\right)\right) - u(t,x), \quad 0 \leq t \leq T-h, \quad x \in \mathbb{R}.$$

The above decomposition of $u(t, B_t^{(n)})$ (which is in fact Doob's decomposition) can be viewed as a discrete version of Itô's formula, which, for a smooth function $v : [0,T] \times \mathbb{R} \to \mathbb{R}$, implies that $v(t, B_t) - \int_0^t \delta v(s, B_s) ds$ is a (local) martingale, where

$$\delta v = \frac{\partial v}{\partial t} + \frac{1}{2}\frac{\partial^2 v}{\partial x^2}.$$

It is also easy to check that, if v is smooth and $\mathcal{D}v(t,x) = \mathbb{E}\left(v\left(t+h, x+\sqrt{h}X_1\right)\right) - v(t,x)$, we have

$$(1/h) \times \mathcal{D}v(t,x) = \delta v(t,x) + O(h).$$

The main technical difficulty that we have to deal with is the lack of smoothness of the modfied value function u.

Remark 2.1. The derivatives of u are related to those of U by the following formulas. We have

$$\frac{\partial u}{\partial t}(t,x) = e^{-rt}\left(-\frac{\partial U}{\partial t}+\mu\frac{\partial U}{\partial x}-rU\right)(T-t,\ln(S_0)+\mu t+\sigma x)$$

and

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2}(t,x) &= e^{-rt}\left(\frac{\partial^2 U}{\partial t^2}-2\mu\frac{\partial^2 U}{\partial t\partial x}+\mu^2\frac{\partial^2 U}{\partial x^2}\right.\\ &\quad\left.+2r\frac{\partial U}{\partial t}-2r\mu\frac{\partial U}{\partial x}+r^2U\right)(T-t,\ln(S_0)+\mu t+\sigma x).\end{aligned}$$

We also have

$$\begin{aligned}\delta u(t,x)=\frac{\partial u}{\partial t}(t,x)+\frac{1}{2}\frac{\partial^2 u}{\partial x^2}(t,x) &= e^{-rt}\left(-\frac{\partial U}{\partial t}+(A-r)U\right)(T-t,\ln(S_0)+\mu t+\sigma x)\\ &= e^{-rt}(A-r)\varphi(\ln(S_0)+\mu t+\sigma x)\mathbb{1}_{\{\ln(S_0)+\mu t+\sigma x\le\tilde{b}(T-t)\}},\end{aligned}$$

where the last equality follows from regularity results (see, for instance, [4]).

We will need a more precise description of the operator $\mathcal{D}$, given by the following proposition (see Proposition 3.4 of [6]). For convenience, we denote by X a random variable with the same distribution as X_1, which is independent of the sequence $(X_n)_{n\ge1}$.

Proposition 2.1. *Assume that (H4) is satisfied and that v is a function of class C^3 on $[0,T]\times\mathbb{R}$. For $0\le t\le T-h$ and $x\in\mathbb{R}$, define*

$$\tilde{\mathcal{D}}v(t,x)=2\int_0^{\sqrt{h}}d\xi\int_0^{\xi}dz\mathbb{E}\left[X\left(\xi-X^2(\xi-z)\right)\frac{\partial^2 v}{\partial t\partial x}(t+\xi^2,x+zX)\right].$$

We have

$$\mathcal{D}v(t,x)=\tilde{\mathcal{D}}v(t,x)+2\int_0^{\sqrt{h}}d\xi\int_0^{\xi}dz\mathbb{E}\left(X^2\delta v(t+\xi^2,x+zX)\right),$$

with the notation $\delta v=\frac{\partial v}{\partial t}+\frac{1}{2}\frac{\partial^2 v}{\partial x^2}$, and

$$\tilde{\mathcal{D}}v(t,x)=2\int_0^{\sqrt{h}}d\xi\int_0^{\xi}dz(\xi-z)\mathbb{E}\left[X^2\left(\xi-X^2\frac{(\xi-z)}{2}\right)\frac{\partial^3 v}{\partial t\partial x^2}(t+\xi^2,x+zX)\right].$$

Remark 2.2. Note that, if $\delta v(s,x+zX)=0$ for all $s\in[t,t+h]$ and $z\in[0,\sqrt{h}]$, we have $v(t,x)=\mathcal{D}v(t,x)$.

From the last equality in Proposition 2.1, we derive the following estimates.

$$\begin{aligned}\left|\tilde{\mathcal{D}}v(t,x)\right| &\le 2\int_0^{\sqrt{h}}\xi^2d\xi\int_0^{\xi}dz\mathbb{E}\left[\left(X^2+\frac{X^4}{2}\right)\left|\frac{\partial^3 v}{\partial t\partial x^2}(t+\xi^2,x+zX)\right|\right]\\ &\le \sqrt{h}\int_0^{\sqrt{h}}2\xi d\xi\mathbb{E}\left[\int_0^{\xi}dz\left(X^2+\frac{X^4}{2}\right)\left|\frac{\partial^3 v}{\partial t\partial x^2}(t+\xi^2,x+zX)\right|\right]\\ &\le \sqrt{h}\int_t^{t+h}ds\mathbb{E}\left(\int dy\mathbb{1}_{\{|y-x|\le\sqrt{h}|X|\}}\left(|X|+\frac{|X|^3}{2}\right)\left|\frac{\partial^3 v}{\partial t\partial x^2}(s,y)\right|\right)\\ &= \sqrt{h}\int_t^{t+h}ds\int dy\mathbb{E}\left(\mathbb{1}_{\{|y-x|\le\sqrt{h}|X|\}}\left(|X|+\frac{|X|^3}{2}\right)\right)\left|\frac{\partial^3 v}{\partial t\partial x^2}(s,y)\right|\end{aligned}$$

We know from Proposition 3.2 of [6] (based on Berry-Esseen estimates) that, for every $k \in (1,3]$, there exists a positive constant C_k (which does not depend on X), such that, for all $y \in \mathbb{R}$, $n \geq 1$ and $j \in \{1, 2, \ldots, n\}$,

$$\mathbb{E}\left(\left(|X| + \frac{|X|^3}{2}\right) \mathbb{1}_{\left\{\left|B_{jh}^{(n)} - y\right| \leq \sqrt{h}|X|\right\}}\right) \leq \frac{C_k}{\sqrt{j}} \frac{\mathbb{E}\left(|X|^3\right)\left(1 + \mathbb{E}\,|X|^{3+k}\right)}{1 + |y|^k}.$$

Hence, for $j = 1, \ldots, n-1$,

$$\begin{aligned}
\mathbb{E}\left(\left|\tilde{\mathcal{D}} v(jh, B_{jh}^{(n)})\right|\right) &\leq \frac{C_{k,X}}{\sqrt{j}} \sqrt{h} \int_{jh}^{jh+h} ds \int \frac{dy}{1+|y|^k} \left|\frac{\partial^3 v}{\partial t \partial x^2}(s,y)\right| \\
&\leq C_{k,X} h \sqrt{2} \int_{jh}^{jh+h} \frac{ds}{\sqrt{s}} \int \frac{dy}{1+|y|^k} \left|\frac{\partial^3 v}{\partial t \partial x^2}(s,y)\right|, \qquad (2)
\end{aligned}$$

where, for the last inequality, we used the inequality $jh \geq (j+1)h/2$.

3 Estimates for the second order time derivative

In this section, we refine the regularity results that we used in [6]. We first establish some elementary L_1-estimates. Then, we obtain a quadratic estimate for the second order time derivative of the difference $\tilde{U} = U - \bar{U}$. For the definition of the relevant weighted Sobolev spaces, we will use the notation

$$\nu_j(dx) = \frac{dx}{(1+x^2)^{j/2}}, \quad j > 1.$$

3.1 Some elementary L_1-estimates

Proposition 3.1. *Assume that the function φ is continuous and satisfies $\varphi \in L_1(\nu_j)$, $\varphi' \in L_1(\nu_j)$ and the second derivative φ'' is a Radon measure on $\mathbb{R}$, with $\int_{\mathbb{R}} \frac{|\varphi''(dz)|}{(1+z^2)^{j/2}} < \infty$. Let*

$$u_\varphi(t,x) = e^{-rt}\mathbb{E}(\varphi(X_t^x)), \quad t \geq 0, \quad x \in \mathbb{R}.$$

Then, for all $T > 0$, there exists a constant $C_T > 0$, such that

$$\forall t \in (0,T], \quad \left|\left|\frac{\partial^2 u_\varphi}{\partial t^2}(t,.)\right|\right|_{L_1(\nu_j)} \leq \frac{C_T}{t}.$$

We will easily deduce this proposition from the following lemma.

Lemma 3.1. *If ρ is a Radon measure on $\mathbb{R}$ and q a nonnegative integrable function on $\mathbb{R}$, we have*

$$||\rho * q||_{L_1(\nu_j)} \leq 2^{j/2} \int_{\mathbb{R}} \frac{|\rho(dz)|}{(1+z^2)^{j/2}} \int_{-\infty}^{\infty} q(x)(1+x^2)^{j/2} dx.$$

We also have, for any measurable function f on $\mathbb{R}$,

$$\forall y \in \mathbb{R}, \quad ||f(.-y)||_{L_1(\nu_j)} \leq 2^{j/2}(1+y^2)^{j/2}||f||_{L_1(\nu_j)}.$$

Proof: We have

$$\begin{aligned} ||\rho * q||_{L_1(\nu_j)} &\leq \int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^{j/2}} \int_{\mathbb{R}} |\rho(dz)| q(x-z) \\ &= \int_{\mathbb{R}} \frac{|\rho(dz)|}{(1+z^2)^{j/2}} \int_{-\infty}^{\infty} q(x-z) \frac{(1+z^2)^{j/2}}{(1+x^2)^{j/2}} dx \\ &= \int_{\mathbb{R}} \frac{|\rho(dz)|}{(1+z^2)^{j/2}} \int_{-\infty}^{\infty} q(x) \frac{(1+z^2)^{j/2}}{(1+(x+z)^2)^{j/2}} dx. \end{aligned}$$

Note that

$$z^2 \leq 2((x+z)^2 + x^2),$$

so that we deduce

$$\begin{aligned} \frac{1+z^2}{1+(x+z)^2} &\leq \frac{1+2(x+z)^2+2x^2}{1+(x+z)^2} \\ &\leq 2(1+x^2). \end{aligned}$$

Hence

$$||\rho * q||_{L_1(\nu_j)} \leq 2^{j/2} \int_{\mathbb{R}} \frac{|\rho(dz)|}{(1+z^2)^{j/2}} \int_{-\infty}^{\infty} q(x)(1+x^2)^{j/2} dx.$$

Similarly, we have, for any measurable function f and $y \in \mathbb{R}$,

$$\begin{aligned} ||f(.-y)||_{L_1(\nu_j)} &= \int |f(x-y)| \frac{dx}{(1+x^2)^{j/2}} \\ &= \int |f(x)| \frac{dx}{(1+(x+y)^2)^{j/2}} \\ &= \int |f(x)| \left(\frac{1+x^2}{1+(x+y)^2} \right)^{j/2} \frac{dx}{(1+x^2)^{j/2}} \\ &\leq \int |f(x)| \left(\frac{1+2(x+y)^2+2y^2}{1+(x+y)^2} \right)^{j/2} \frac{dx}{(1+x^2)^{j/2}} \\ &\leq 2^{j/2}(1+y^2)^{j/2} ||f||_{L_1(\nu_j)}. \end{aligned}$$

◇

Proof of Proposition 3.1: We have

$$\begin{aligned} u_\varphi(t,x) &= e^{-rt} \int_{-\infty}^{\infty} \varphi(x+y) \exp\left(-\frac{(y-\mu t)^2}{2\sigma^2 t} \right) \frac{dy}{\sigma\sqrt{2\pi t}} \\ &= e^{-rt} p_t * \varphi(x), \end{aligned}$$

with

$$p_t(x) = \frac{1}{\sigma\sqrt{2\pi t}} \exp\left(-\frac{(x+\mu t)^2}{2\sigma^2 t} \right) = \frac{1}{\sigma\sqrt{t}} n\left(\frac{x+\mu t}{\sigma\sqrt{t}} \right).$$

Here, n denotes the standard normal density function.

On the other hand, we know that u_φ satisfies the equation

$$\frac{\partial u_\varphi}{\partial t} = (A-r)u_\varphi, \tag{3}$$

so that

$$\begin{aligned}\frac{\partial u_\varphi}{\partial t}(t,.) &= e^{-rt}(A-r)p_t * \varphi \\ &= e^{-rt}p_t * [(A-r)\varphi].\end{aligned}$$

It follows from our assumptions that $(A-r)\varphi$ is a Radon measure satisfying

$$\int_{\mathbb{R}} |(A-r)\varphi(dz)| \frac{1}{(1+z^2)^{j/2}} < \infty.$$

So that, using Lemma 3.1,

$$\begin{aligned}\left\| \frac{\partial u_\varphi}{\partial t}(t,.) \right\|_{L_1(\nu_j)} &\leq C_j \int_{-\infty}^{\infty} p_t(x)(1+|x|^j)dx \\ &= C_j \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{t}} n\left(\frac{x+\mu t}{\sigma\sqrt{t}}\right)(1+|x|^j)dx \\ &= C_j \int_{-\infty}^{\infty} n(y)\,(1+|y\sigma\sqrt{t}-\mu t|^j)dy \\ &\leq C_j\left(1+t^j\right).\end{aligned}$$

On the other hand, by differentiating (3), we have

$$\frac{\partial^2 u_\varphi}{\partial t^2} = (A-r)\frac{\partial u_\varphi}{\partial t} = e^{-rt}\left((A-r)p_t\right) * (A-r)\varphi.$$

Hence, using Lemma 3.1, and the definition of p_t,

$$\begin{aligned}\left\| \frac{\partial^2 u_\varphi}{\partial t^2}(t,.) \right\|_{L_1(\nu_j)} &\leq C_j \int_{-\infty}^{\infty} |(A-r)p_t(x)|(1+|x|^j)dx \\ &\leq \frac{C_j}{t}\left(1+t^j\right).\end{aligned}$$

◇

3.2 Quadratic estimates

Recall the notation:

$$U(t,x) = \sup_{\tau \in \mathcal{T}_{0,t}} \mathbb{E}\left(e^{-r\tau}\varphi(X_\tau^x)\right), \quad u_\varphi(t,x) = e^{-rt}\mathbb{E}(\varphi(X_t^x)), \quad t \geq 0, \quad x \in \mathbb{R},$$

with $\varphi(x) = (K-e^x)^+$. We now introduce the difference $\tilde{U} = U - u_\varphi$ (which corresponds to the early exercise premium). We have the following L_2-estimate for the second time derivative of $\tilde{U} = U - u_\varphi$.

Theorem 3.1. *Fix $T > 0$ and $j > 1$. There exists a constant $C > 0$ such that, for all $\xi \in (0, T]$,*

$$\int_\xi^T (t-\xi) \left|\left| \frac{\partial^2 \tilde{U}}{\partial t^2}(t,.) \right|\right|^2_{L_2(\nu_j)} dt \leq C \left(1 + |\ln \xi|^\beta\right), \quad \text{with } \beta = \begin{cases} 3/2, & \text{if } d \leq r, \\ 1, & \text{if } d > r. \end{cases}$$

This estimate is closely related to Theorem 2.4 of [6], a variant of results due to Friedman and Kinderlehrer (see [3], Lemma 4.1, and [5], Chapter VIII). Note that by considering the difference $\tilde{U} = U - u_\varphi$, we are able to derive a logarithmic upper bound, instead of a power of ξ, which would come up by considering U (see Theorem 2.4 of [6]). For the proof of Theorem 3.1, we need some preliminary estimates on the derivatives $\frac{\partial^2 \tilde{U}}{\partial x^2}$ and $\frac{\partial^2 \tilde{U}}{\partial t \partial x}$.

Lemma 3.2. *Fix $T > 0$ and $j > 1$. For any $\varepsilon \in (0, 1/4)$, there exists a constant $C > 0$ such that, for all $t \in (0, T]$,*

$$\left|\left| \frac{\partial \tilde{U}}{\partial x}(t,.) \right|\right|_{L_2(\nu_j)} \leq C\sqrt{t} \quad \text{and} \quad \left|\left| \frac{\partial^2 \tilde{U}}{\partial x^2}(t,.) \right|\right|_{L_2(\nu_j)} \leq C t^\varepsilon.$$

Proof: We know that $\tilde{U}$ solves the equation

$$-\frac{\partial \tilde{U}}{\partial t} + (A - r)\tilde{U} = \tilde{h},$$

with initial condition $\tilde{U}(0,.) = 0$, where the function $\tilde{h}$ is given by

$$\tilde{h}(t,x) = (A-r)\varphi(x) \mathbb{1}_{\{x \leq \tilde{b}(t)\}}, \quad t > 0, \quad x \in \mathbb{R}.$$

We have the following identity (which can be viewed as a form of the early exercise premium formula).

$$\tilde{U}(t,.) = -\int_0^t e^{-r(t-s)} p_{t-s} * \tilde{h}(s,.) ds,$$

where

$$p_t(x) = \frac{1}{\sigma\sqrt{2\pi t}} \exp\left(-\frac{(x+\mu t)^2}{2\sigma^2 t}\right) = \frac{1}{\sigma\sqrt{t}} n\left(\frac{x+\mu t}{\sigma\sqrt{t}}\right),$$

with n denoting the standard normal density function. It is straightforward to check that

$$\frac{\partial \tilde{U}}{\partial x}(t,.) = -\int_0^t e^{-r(t-s)} p_{t-s} * \frac{\partial \tilde{h}}{\partial x}(s,.) ds,$$

and, with the notation δ_z for the Dirac measure at a point z,

$$\begin{aligned} \frac{\partial \tilde{h}}{\partial x}(t,x) &= (A-r)\varphi'(x)\mathbb{1}_{\{x \leq \tilde{b}(t)\}} - (A-r)\varphi(x)\delta_{\tilde{b}(t)}(x) \\ &= -\kappa(t,x) + \gamma(t)\delta_{\tilde{b}(t)}(x), \end{aligned} \tag{4}$$

with $\kappa(t,x) = -(A-r)\varphi'(x)\mathbb{1}_{\{x \leq \tilde{b}(t)\}}$ and $\gamma(t) = -(A-r)\varphi(\tilde{b}(t))$. Note that κ is a bounded function on $(0,\infty) \times \mathbb{R}$ and γ is a continuous, nonnegative and bounded function on $(0, +\infty)$. At this stage, it is clear that $||p_{t-s} * \frac{\partial \tilde{h}}{\partial x}(s,.)||_\infty \leq C/\sqrt{t-s}$, so that

$$\left|\left| \frac{\partial \tilde{U}}{\partial x}(t,.) \right|\right|_{L_2(\nu_j)} \leq C\sqrt{t}.$$

On the other hand, we have

$$\left|\left|\frac{\partial^2 \tilde{U}}{\partial x^2}(t,.)\right|\right|_{L_2(\nu_j)} \leq \int_0^t e^{-r(t-s)} \left|\left|p'_{t-s} * \kappa(s,.)\right|\right|_{L_2(\nu_j)} ds + ||\zeta(t,.)||_{L_2(\nu_j)},$$

with

$$\begin{aligned}\zeta(t,.) &= \int_0^t e^{-r(t-s)}\gamma(s)p'_{t-s} * \delta_{\tilde{b}(s)} ds \\ &= \int_0^t e^{-r(t-s)}\gamma(s)p'_{t-s}(. - \tilde{b}(s))ds.\end{aligned}$$

We have, using Lemma 3.1,

$$\begin{aligned}\left|\left|p'_{t-s} * \kappa(s,.)\right|\right|_{L_2(\nu_j)} &= \left|\left|\int p'_{t-s}(y)\kappa(s,. - y)dy\right|\right|_{L_2(\nu_j)} \\ &\leq \int |p'_{t-s}(y)|\, ||\kappa(s,. - y)||_{L_2(\nu_j)}\, dy \\ &\leq 2^{j/4}\, ||\kappa(s,.)||_{L_2(\nu_j)} \int |p'_{t-s}(y)|(1+y^2)^{j/4}dy.\end{aligned}$$

Note that, since κ is bounded and $j > 1$, $\sup_{s>0} ||\kappa(s,.)||_{L_2(\nu_j)} < \infty$, so that, for some constant $C > 0$ (which may vary from line to line)

$$\begin{aligned}\left|\left|p'_{t-s} * \kappa(s,.)\right|\right|_{L_2(\nu_j)} &\leq C\int |p'_{t-s}(y)|(1+y^2)^{j/4}dy \\ &= C\int \frac{1}{\sigma^2(t-s)}\left|n'\left(\frac{y+\mu(t-s)}{\sigma\sqrt{t-s}}\right)\right|(1+y^2)^{j/4}dy \\ &= C\int \frac{1}{\sigma\sqrt{t-s}}\left|n'(z)\right|(1+(-\mu(t-s)+\sigma\sqrt{t-s}z)^2)^{j/4}dz \\ &\leq \frac{C}{\sqrt{t-s}}\left(1+(t-s)^{j/2}\right).\end{aligned}$$

Hence, if $0 < t < T$,

$$\int_0^t e^{-r(t-s)} \left|\left|p'_{t-s} * \kappa(s,.)\right|\right|_{L_2(\nu_j)} ds \leq C\int_0^t \frac{ds}{\sqrt{t-s}} = 2C\sqrt{t}.$$

We now estimate $||\zeta(t,.)||_{L_2(\nu_j)}$.

We have, using the boundedness of γ,

$$\begin{aligned}|\zeta(t,x)| &= \left|\int_0^t e^{-r(t-s)}\gamma(s)p'_{t-s}(x - \tilde{b}(s))ds\right| \\ &\leq C\int_0^t \frac{1}{\sigma^2(t-s)}\left|n'\left(\frac{x-\tilde{b}(s)+\mu(t-s)}{\sigma\sqrt{t-s}}\right)\right|ds\end{aligned}$$

Recall that $n'(x) = -xn(x)$. Therefore

$$\begin{aligned}|\zeta(t,x)| &\leq C\int_0^t \frac{|x-\tilde{b}(s)+\mu(t-s)|}{(t-s)^{3/2}} n\left(\frac{x-\tilde{b}(s)+\mu(t-s)}{\sigma\sqrt{t-s}}\right)ds \\ &\leq C\int_0^t \frac{ds}{\sqrt{t-s}} + C\int_0^t \frac{|x-\tilde{b}(s)|}{(t-s)^{3/2}} n\left(\frac{x-\tilde{b}(s)+\mu(t-s)}{\sigma\sqrt{t-s}}\right)ds.\end{aligned}$$

Note that

$$n(x_1+x_2) = n(x_1)\exp\left(-\frac{x_2^2}{2}-x_1x_2\right) \le n(x_1)\exp(-x_1x_2)$$
$$\le n(x_1)\exp\left(\frac{x_1^2}{4}+x_2^2\right) = n(x_1/\sqrt{2})e^{x_2^2}.$$

Hence, for $t\in(0,T)$,

$$|\zeta(t,x)| \;\le\; C\int_0^t \frac{ds}{\sqrt{t-s}} + C_T\int_0^t \frac{|x-\tilde b(s)|}{(t-s)^{3/2}} n\left(\frac{x-\tilde b(s)}{\sqrt{2}\sigma\sqrt{t-s}}\right) ds.$$

Note that, for all $\alpha>0$, there exists $C_\alpha>0$, such that, for all $y\in\mathbb{R}$, $n(y/\sqrt{2})\le C_\alpha/|y|^{2\alpha}$. Hence, for $t\in(0,T)$,

$$\begin{aligned}|\zeta(t,x)| &\le C\sqrt{t} + C_\alpha\int_0^t \frac{|x-\tilde b(s)|}{(t-s)^{3/2}}\frac{(t-s)^\alpha}{|x-\tilde b(s)|^{2\alpha}}ds\\ &= C\sqrt{t} + C_\alpha\int_0^t \frac{(t-s)^{\alpha-\frac32}}{|x-\tilde b(s)|^{2\alpha-1}}ds\\ &= C\sqrt{t} + C_\alpha t^{\alpha-\frac12}\int_0^1 \frac{1}{(1-u)^{\frac32-\alpha}|x-\tilde b(tu)|^{2\alpha-1}}du.\end{aligned}$$

Now, take $\alpha=\frac12+\varepsilon$ (with $0<\varepsilon<1/4$) and put $\beta(t,x)=|x-\tilde b(t)|^{1-2\alpha}=|x-\tilde b(t)|^{-2\varepsilon}$. We get

$$||\zeta(t,.)||_{L_2(\nu_j)} \;\le\; C\sqrt{t} + Ct^\varepsilon\int_0^1 \frac{1}{(1-u)^{1-\varepsilon}}\,||\beta(tu,.)||_{L_2(\nu_j)}\,du.$$

Using Lemma 3.1, we have

$$||\beta(tu,.)||^2_{L_2(\nu_j)} \le 2^{j/2}\left(1+\tilde b(tu)^2\right)^{j/2}\int\frac{1}{|x|^{4\varepsilon}}\frac{dx}{(1+x^2)^{j/2}}.$$

Since $\varepsilon<1/4$, the integral on the righthand side is finite, and the lemma easily follows. $\diamond$

We now turn to the study of $\frac{\partial^2\tilde U}{\partial t\partial x}$. Recall that $\partial U/\partial t$ solves the parabolic equation $-\partial v/\partial t+(A-r)v=0$ in the set $\{(t,x)\mid t>0, x>\tilde b(t)\}$. Since the exercise boundary is differentiable and $\partial U/\partial t$ is continuous and vanishes on the exercise boundary, it follows that $\frac{\partial^2 U}{\partial t\partial x}$ is continuous "up to the boundary", i.e. on the set $\{(t,x)\mid t>0, x\ge\tilde b(t)\}$ (see [3], Lemma 4.5). We first show that $\frac{\partial^2 U}{\partial t\partial x}$ is nonnegative along the exercise boundary.

Lemma 3.3. *We have, for any $t>0$,*

$$\frac{\partial^2 U}{\partial t\partial x}(t,\tilde b(t))\ge 0.$$

Proof: We have, for all $t>0$, due to the *smooth fit* property,

$$\frac{\partial U}{\partial x}(t,\tilde b(t)) = \varphi'(\tilde b(t)),$$

so that, by differentiating with respect to t,

$$\frac{\partial^2 U}{\partial t \partial x}(t, \tilde{b}(t)) + \frac{\partial^2 U}{\partial x^2}(t, \tilde{b}(t))\tilde{b}'(t) = \varphi''(\tilde{b}(t))\tilde{b}'(t)$$

and

$$\frac{\partial^2 U}{\partial t \partial x}(t, \tilde{b}(t)) = -\left(\frac{\partial^2 U}{\partial x^2}(t, \tilde{b}(t)) - \varphi''(\tilde{b}(t))\right)\tilde{b}'(t).$$

Observe that, for each $t > 0$, the function $x \mapsto U(t,x) - \varphi(x)$ is C^2 on the interval $[\tilde{b}(t), \infty)$ and has a minimum at $\tilde{b}(t)$. Therefore, its second derivative must be nonnegative at this point. Since $\tilde{b}'(t) \leq 0$, the lemma is proved. $\diamond$

Lemma 3.4. *Fix $T > 0$ and $j > 1$. There exists a constant $C > 0$ such that, for all $t_1 \in (0, T \wedge 1]$,*

$$\int_{t_1}^{T} \left|\left| \frac{\partial^2 \tilde{U}}{\partial t \partial x}(t, .) \right|\right|^2_{L_2(\nu_j)} dt \leq C \ln(1/t_1).$$

For the proof of Lemma 3.4, we will need the bilinear form associated with the operator $A - r$.

We first introduce the relevant weighted Sobolev spaces. For $j > 1$, let $H_j = L^2(\mathbb{R}, \nu_j)$ and $V_j = \{f \in H_j \mid f' \in H_j\}$. The inner product on H_j will be denoted by $(\cdot, \cdot)_j$ and the associated norm by $|\cdot|_j$. The natural norm on V_j will be denoted by $||\cdot||_j$. Thus, we have

$$|f|_j^2 = \int_{-\infty}^{+\infty} f^2(x) \frac{dx}{(1+x^2)^{j/2}},$$

and $||f||_j^2 = |f|_j^2 + |f'|_j^2$.

Recall that the partial differential operator A is defined by

$$A = \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} + \mu \frac{\partial}{\partial x}.$$

We associate with the operator $A - r$ a bilinear functional on V_j, defined by

$$\begin{aligned} a_j(f,g) &= \frac{\sigma^2}{2}\int_{-\infty}^{\infty} f'(x)g'(x)\frac{dx}{(1+x^2)^{j/2}} - \frac{j\sigma^2}{2}\int_{-\infty}^{\infty} f'(x)g(x)\frac{x}{(1+x^2)^{(j/2)+1}}dx \\ &\quad -\mu\int_{-\infty}^{\infty} f'(x)g(x)\frac{dx}{(1+x^2)^{j/2}} + r\int_{-\infty}^{\infty} f(x)g(x)\frac{dx}{(1+x^2)^{j/2}}, \end{aligned}$$

so that, if $f' \in V_j$,

$$a_j(f,g) = -((A-r)f, g)_j.$$

It will be convenient to write $a_j(f,g)$ as $a_j(f,g) = \tilde{a}_j(f,g) + \bar{a}_j(f,g)$, with

$$\tilde{a}_j(f,g) = \frac{\sigma^2}{2}\left[(f', g')_j + (f,g)_j\right] \quad \text{and} \quad \bar{a}_j(f,g) = a_j(f,g) - \tilde{a}_j(f,g). \tag{5}$$

With these notations, it is easy to check that $|\bar{a}_j(f,g)| \leq C||f||_j|g|_j$ and $|\bar{a}_j(f,g)| \leq C||g||_j|f|_j$, for some constant C which does not depend on f nor g.

Proof of Lemma 3.4: In order to rule out regularity issues, we introduce a C^∞, nonnegative function ρ on $\mathbb{R}\times\mathbb{R}$, with $\int \rho(t,x)dtdx = 1$ and supp $\rho \subset [-1,0]\times[-1,+1]$ and set, for any positive integer m, $\rho_m(t,x) = m^2\rho(mt,mx)$.

Now, let $W = \frac{\partial \tilde{U}}{\partial x}$, $W_m = W * \rho_m$, and $h_m = \tilde{h} * \rho_m$. For each $m > 0$, the functions W_m, h_m are C^∞ with bounded derivatives and we have

$$-\frac{\partial W_m}{\partial t} + (A-r)W_m = \frac{\partial h_m}{\partial x}.$$

Multiply by $\partial W_m/\partial t$ and integrate with respect to ν_j to get, for any fixed $t > 0$,

$$-\left(\frac{\partial W_m}{\partial t}(t,.), \frac{\partial W_m}{\partial t}(t,.)\right)_j - a_j\left(W_m(t,.), \frac{\partial W_m}{\partial t}(t,.)\right) = \int \frac{\partial h_m}{\partial x}(t,x)\frac{\partial W_m}{\partial t}(t,x)\nu_j(dx).$$

Note that

$$\begin{aligned} a_j\left(W_m(t,.), \frac{\partial W_m}{\partial t}(t,.)\right) &= \tilde{a}_j\left(W_m(t,.), \frac{\partial W_m}{\partial t}(t,.)\right) + \bar{a}_j\left(W_m(t,.), \frac{\partial W_m}{\partial t}(t,.)\right) \\ &= \frac{1}{2}\frac{d}{dt}\left(\tilde{a}_j\left(W_m(t,.), W_m(t,.)\right)\right) + \bar{a}_j\left(W_m(t,.), \frac{\partial W_m}{\partial t}(t,.)\right). \end{aligned}$$

By integrating with respect to time, we get, if $0 < t_1 < T$,

$$\begin{aligned} -\int_{t_1}^T \left|\frac{\partial W_m}{\partial t}(t,.)\right|_j^2 dt + \frac{1}{2}\left(\tilde{a}_j\left(W_m(t_1,.), W_m(t_1,.)\right) - \tilde{a}_j\left(W_m(T,.), W_m(T,.)\right)\right) = \\ \int_{t_1}^T \bar{a}_j\left(W_m(t,.), \frac{\partial W_m}{\partial t}(t,.)\right)dt + \int_{t_1}^T \left(\frac{\partial h_m}{\partial x}(t,.), \frac{\partial W_m}{\partial t}(t,.)\right)_j dt. \end{aligned}$$

Hence

$$\begin{aligned} \int_{t_1}^T \left|\frac{\partial W_m}{\partial t}(t,.)\right|_j^2 dt &\leq \frac{1}{2}\tilde{a}_j\left(W_m(t_1,.), W_m(t_1,.)\right) - \int_{t_1}^T \bar{a}_j\left(W_m(t,.), \frac{\partial W_m}{\partial t}(t,.)\right)dt \\ &\quad - \int_{t_1}^T \left(\frac{\partial h_m}{\partial x}(t,.), \frac{\partial W_m}{\partial t}(t,.)\right)_j dt \\ &\leq \frac{1}{2}\tilde{a}_j\left(W_m(t_1,.), W_m(t_1,.)\right) + C\int_{t_1}^T ||W_m(t,.)||_j \left|\frac{\partial W_m}{\partial t}(t,.)\right|_j dt \\ &\quad - \int_{t_1}^T \left(\frac{\partial h_m}{\partial x}(t,.), \frac{\partial W_m}{\partial t}(t,.)\right)_j dt. \end{aligned}$$

Using the inequality

$$2\,||W_m(t,.)||_j \left|\frac{\partial W_m}{\partial t}(t,.)\right|_j \leq \varepsilon\left|\frac{\partial W_m}{\partial t}(t,.)\right|_j^2 + \frac{1}{\varepsilon}||W_m(t,.)||_j^2,$$

we get

$$\begin{aligned} \frac{1}{2}\int_{t_1}^T \left|\frac{\partial W_m}{\partial t}(t,.)\right|_j^2 dt &\leq \frac{1}{2}\tilde{a}_j\left(W_m(t_1,.), W_m(t_1,.)\right) + C\int_{t_1}^T ||W_m(t,.)||_j^2\,dt \\ &\quad - \int_{t_1}^T \left(\frac{\partial h_m}{\partial x}(t,.), \frac{\partial W_m}{\partial t}(t,.)\right)_j dt. \end{aligned}$$

We have

$$\tilde{a}_j\left(W_m(t_1,.),W_m(t_1,.)\right) \;\leq\; C\left|\left|W_m(t_1,.)\right|\right|_j^2$$

and, using Lemma 3.1,

$$\begin{aligned}\left|\left|W_m(t_1,.)\right|\right|_j &\leq \int \rho_m(t_1-t,y)\left|\left|W(t,.-y)\right|\right|_j dtdy\\ &\leq \int \rho_m(t_1-t,y)2^{j/4}(1+y^2)^{j/4}\left|\left|W(t,.)\right|\right|_j dtdy.\end{aligned}$$

Using Lemma 3.2, we have (for $\varepsilon \in (0,1/4)$) $\left|\left|W(t,.)\right|\right|_j \leq Ct^{\varepsilon}$. Hence

$$\begin{aligned}\tilde{a}_j\left(W_m(t_1,.),W_m(t_1,.)\right) &\leq C\int \rho_m(t_1-t,y)2^{j/2}(1+y^2)^{j/2}t^{2\varepsilon}dtdy\\ &= C\int \rho_m(t,y)2^{j/2}(1+y^2)^{j/2}|t_1-t|^{2\varepsilon}dtdy\\ &= C\int \rho(t,y)2^{j/2}\left(1+\frac{y^2}{m^2}\right)^{j/2}\left|t_1-\frac{t}{m}\right|^{2\varepsilon}dtdy\\ &\leq C(1+t_1^{2\varepsilon}).\end{aligned}$$

We also have

$$\begin{aligned}\int_{t_1}^T \left|\left|W_m(t,.)\right|\right|_j^2 dt &= \int_{t_1}^T \left|\left|\int dsdy\rho_m(t-s,y)W(s,.-y)\right|\right|_j^2 dt\\ &\leq \int_{t_1}^T\int dsdy\rho_m(t-s,y)\left|\left|W(s,.-y)\right|\right|_j^2 dt\\ &\leq \int_{t_1}^T dt\int dsdy\rho_m(t-s,y)2^{j/2}(1+y^2)^{j/2}\left|\left|W(s,.)\right|\right|_j^2\\ &\leq \int_{t_1}^{T+\frac{1}{m}} ds\left|\left|W(s,.)\right|\right|_j^2\int dtdy\rho_m(t-s,y)2^{j/2}(1+y^2)^{j/2}.\end{aligned}$$

Since $\int_0^{T+1}\left|\left|W(s,.)\right|\right|_j^2 ds<\infty$, we deduce that

$$\frac{1}{2}\int_{t_1}^T\left|\frac{\partial W_m}{\partial t}(t,.)\right|_j^2 dt \;\leq\; C\left(1+t_1^{2\varepsilon}\right)-\int_{t_1}^T\left(\frac{\partial h_m}{\partial x}(t,.),\frac{\partial W_m}{\partial t}(t,.)\right)_j dt.$$

It follows from the proof of Lemma 3.2 (see (4)) that

$$\frac{\partial h_m}{\partial x}(t,x)=-\kappa_m(t,x)+\gamma_m(t,x),$$

where $\kappa_m=\kappa*\rho_m$, and κ is a bounded function, and

$$\gamma_m(t,x)=\int \rho_m(t-\tau,x-\tilde{b}(\tau))\gamma(\tau)d\tau.$$

Hence

$$\begin{aligned}-\int_{t_1}^T\left(\frac{\partial h_m}{\partial x}(t,.),\frac{\partial W_m}{\partial t}(t,.)\right)_j dt &\leq C\int_{t_1}^T\left|\frac{\partial W_m}{\partial t}(t,.)\right|_j dt-\int_{t_1}^T\left(\gamma_m(t,.),\frac{\partial W_m}{\partial t}(t,.)\right)_j dt\\ &\leq \frac{1}{4}\int_{t_1}^T\left|\frac{\partial W_m}{\partial t}(t,.)\right|_j^2 dt+C^2T-\int_{t_1}^T\left(\gamma_m(t,.),\frac{\partial W_m}{\partial t}(t,.)\right)_j dt,\end{aligned}$$

where we have used $C\left|\frac{\partial W_m}{\partial t}(t,.)\right|_j \leq \frac{1}{4}\left|\frac{\partial W_m}{\partial t}(t,.)\right|_j^2 + C^2$. Therefore

$$\frac{1}{4}\int_{t_1}^T \left|\frac{\partial W_m}{\partial t}(t,.)\right|_j^2 dt \;\leq\; C\left(1+t_1^{2\varepsilon}\right) - \int_{t_1}^T \left(\gamma_m(t,.), \frac{\partial W_m}{\partial t}(t,.)\right)_j dt. \tag{6}$$

Note that

$$\frac{\partial W_m}{\partial t} = \frac{\partial^2 U_m}{\partial t \partial x} - \frac{\partial^2 u_m}{\partial t \partial x},$$

where

$$U_m = U * \rho_m \quad \text{and} \quad u_m = u_\varphi * \rho_m,$$

so that

$$-\int_{t_1}^T \left(\gamma_m(t,.), \frac{\partial W_m}{\partial t}(t,.)\right)_j dt = J_m^{(1)} + J_m^{(2)},$$

with

$$J_m^{(1)} = -\int_{t_1}^T \left(\gamma_m(t,.), \frac{\partial^2 U_m}{\partial t \partial x}(t,.)\right)_j dt \quad \text{and} \quad J_m^{(2)} = \int_{t_1}^T \left(\gamma_m(t,.), \frac{\partial^2 u_m}{\partial t \partial x}(t,.)\right)_j dt.$$

We have

$$\begin{aligned}
J_m^{(1)} &= -\int_{t_1}^T dt \int \frac{dx}{(1+x^2)^{j/2}} \gamma_m(t,x) \frac{\partial^2 U_m}{\partial t \partial x}(t,x) \\
&= -\int_{t_1}^T dt \int \frac{dx}{(1+x^2)^{j/2}} \int d\tau \gamma(\tau) \rho_m(t-\tau, x-\tilde{b}(\tau)) \frac{\partial^2 U_m}{\partial t \partial x}(t,x) \\
&= -\int d\tau \int ds \int dy \mathbb{1}_{\{t_1<\tau+s<T\}} \gamma(\tau) \rho_m(s,y) \frac{\partial^2 U_m}{\partial t \partial x}(\tau+s, \tilde{b}(\tau)+y) \frac{1}{(1+(y+\tilde{b}(\tau))^2)^{j/2}} \\
&= -\int_{t_1}^{T+\frac{1}{m}} \gamma(\tau)\eta_m(\tau) d\tau,
\end{aligned}$$

where

$$\begin{aligned}
\eta_m(\tau) &= \int_{t_1-\tau}^{T-\tau} ds \int \frac{dy}{(1+(y+\tilde{b}(\tau))^2)^{j/2}} \rho_m(s,y) \frac{\partial^2 U_m}{\partial t \partial x}(\tau+s, \tilde{b}(\tau)+y) \\
&= \int_{t_1-\tau}^{T-\tau} ds \int \frac{\rho_m(s,y)dy}{(1+(y+\tilde{b}(\tau))^2)^{j/2}} \int\int ds'dy' \rho_m(s',y') \frac{\partial^2 U}{\partial t \partial x}(\tau+s-s', \tilde{b}(\tau)+y-y').
\end{aligned}$$

Note that $\frac{\partial^2 U}{\partial t \partial x} = 0$ on the open set $S = \{(t,x) \mid t>0, x<\tilde{b}(t)\}$ (which is the interior set of the stopping region), so that

$$\eta_m(\tau) = \int_{t_1-\tau}^{T-\tau} ds \int \frac{\rho_m(s,y)dy}{(1+(y+\tilde{b}(\tau))^2)^{j/2}} \int\int ds'dy' \rho_m(s',y') \bar{W}(\tau, s-s', y-y'),$$

where

$$\bar{W}(\tau,\theta,z) = \frac{\partial^2 U}{\partial t \partial x}(\tau+\theta, \tilde{b}(\tau)+z) \mathbb{1}_{\{\tilde{b}(\tau)+z \geq \tilde{b}(\tau+\theta)\}}.$$

Since $\frac{\partial^2 U}{\partial t \partial x}(\tau, \tilde{b}(\tau)) \geq 0$, for $\tau > 0$, we have

$$\eta_m(\tau) \;\geq\; \int_{t_1-\tau}^{T-\tau} ds \int\int\int \frac{\rho_m(s,y)dy\rho_m(s',y')ds'dy'}{\left(1+(y+\tilde{b}(\tau))^2\right)^{j/2}} D(\tau, s-s', y-y'),$$

where

$$D(\tau,\theta,z) = \left(\frac{\partial^2 U}{\partial t\partial x}(\tau+\theta, \tilde{b}(\tau)+z) - \frac{\partial^2 U}{\partial t\partial x}(\tau,\tilde{b}(\tau))\right) \mathbb{1}_{\{\tilde{b}(\tau)+z \geq \tilde{b}(\tau+\theta)\}}.$$

Hence (since $\gamma \geq 0$)

$$J_m^{(1)} \;\leq\; -\int_{t_1}^{T+\frac{1}{m}} \gamma(\tau)\varepsilon_m(\tau)d\tau,$$

with

$$\varepsilon_m(\tau) = \int_{t_1-\tau}^{T-\tau} ds \int\int\int \frac{\rho_m(s,y)dy\rho_m(s',y')ds'dy'}{(1+(y+\tilde{b}(\tau))^2)^{j/2}} D(\tau, s-s', y-y').$$

We have

$$|\varepsilon_m(\tau)| \leq \int_{t_1-\tau}^{T-\tau} D_m(\tau)ds = (T-t_1)D_m(\tau),$$

where

$$D_m(\tau) = \sup_{|\theta|\leq 1/m, |z|\leq 2/m} \left(|D(\tau,\theta,z)|\, \mathbb{1}_{\{\tilde{b}(\tau)+z\geq\tilde{b}(\tau+\theta)\}}\right).$$

Due to the continuity properties of $\frac{\partial^2 U}{\partial t\partial x}$, as $m \to \infty$, the function D_m converges to 0, uniformly on the interval $[t_1, T+1]$. Therefore, we have

$$\limsup_{m\to\infty} J_m^{(1)} \leq 0. \tag{7}$$

We now examine $J_m^{(2)}$. We have, using the boundedness of γ,

$$\begin{aligned} |J_m^{(2)}| \;&\leq\; \int_{t_1}^{T} dt \int \frac{dx}{(1+x^2)^{j/2}} |\gamma_m(t,x)| \left|\frac{\partial^2 u_m}{\partial t\partial x}(t,x)\right| \\ &\leq\; C\int_{t_1}^{T} dt \int \frac{dx}{(1+x^2)^{j/2}} \int d\tau \rho_m(t-\tau, x-\tilde{b}(\tau)) \left|\frac{\partial^2 u_m}{\partial t\partial x}(t,x)\right|. \end{aligned}$$

Note that, since φ is Lipschitz, we have $\left|\left|\frac{\partial^2 u_\varphi}{\partial t\partial x}(t,.)\right|\right|_\infty \leq \frac{C}{t}$ and, since supp $\rho \subset [-1,0]\times[-1,+1]$,

$$\left|\left|\frac{\partial^2 u_m}{\partial t\partial x}(t,.)\right|\right|_\infty \leq \int\int d\tau dy \rho_m(\tau,y) \left|\left|\frac{\partial^2 u_\varphi}{\partial t\partial x}(t-\tau,.)\right|\right|_\infty \leq \frac{C}{t}.$$

Hence

$$|J_m^{(2)}| \;\leq\; C\int_{t_1}^{T} \frac{dt}{t} \int dx \int d\tau \rho_m(t-\tau, x-\tilde{b}(\tau)) = C\ln\frac{T}{t_1}. \tag{8}$$

It follows from (6), (7) and (8) that

$$\limsup_{m\to\infty} \int_{t_1}^{T} \left|\frac{\partial W_m}{\partial t}(t,.)\right|_j^2 dt \;\leq\; C\left(1+t_1^{2\varepsilon}+\ln\frac{T}{t_1}\right),$$

which proves the lemma. ◇

3.3 Proof of Theorem 3.1

For the proof of Theorem 3.1, we will work on the equation satisfied by $\partial\tilde{U}/\partial t$. Let

$$V = \frac{\partial \tilde{U}}{\partial t}.$$

We have

$$-\frac{\partial V}{\partial t} + (A-r)V = \frac{\partial \tilde{h}}{\partial t},$$

where

$$\tilde{h}(t,x) = (A-r)\varphi(x)\mathbb{1}_{\{x\leq \tilde{b}(t)\}}, \quad t>0, \quad x\in\mathbb{R}.$$

The following lemma will clarify the computation of the derivative $\partial\tilde{h}/\partial t$ in the sense of distributions.

Lemma 3.5. *Define the function I on $(0,+\infty)\times\mathbb{R}$ by*

$$I(t,x) = \mathbb{1}_{\{x\leq \tilde{b}(t)\}}, \quad t>0, \quad x\in\mathbb{R}.$$

The distribution $\partial I/\partial t$ applied to a compactly supported C^∞ function ρ on $(0,+\infty)\times\mathbb{R}$ is given by

$$\langle \frac{\partial I}{\partial t}, \rho\rangle = \int \tilde{b}'(t)\rho(t,\tilde{b}(t))dt.$$

This can be written (less precisely): $\frac{\partial I}{\partial t}(t,.) = \tilde{b}'(t)\delta_{\tilde{b}(t)}$.

Proof: We have

$$\begin{aligned}\langle \frac{\partial I}{\partial t}, \rho\rangle &= -\langle I, \frac{\partial \rho}{\partial t}\rangle \\ &= -\int dt \int dx I(t,x)\frac{\partial \rho}{\partial t}(t,x)\end{aligned}$$

Let J be the range of $\tilde{b}$. We have $J = (\tilde{b}(\infty), \tilde{b}(0))$. Note that, if $x\leq \tilde{b}(\infty)$, $I(t,x)=1$ for all $t>0$ and if $x\geq \tilde{b}(0)$ $I(t,x)=0$ for all $t>0$, so that, in both cases, $\int I(t,x)\frac{\partial\rho}{\partial t}(t,x)dt = 0$. Therefore

$$\begin{aligned}\langle \frac{\partial I}{\partial t}, \rho\rangle &= -\int_J dx \int dt \mathbb{1}_{\{x\leq \tilde{b}(t)\}}\frac{\partial \rho}{\partial t}(t,x) \\ &= -\int_J dx \int dt \mathbb{1}_{\{t\leq \tilde{b}^{-1}(x)\}}\frac{\partial \rho}{\partial t}(t,x) \\ &= -\int_J dx \rho(\tilde{b}^{-1}(x),x) \\ &= \int \tilde{b}'(t)\rho(t,\tilde{b}(t))dt.\end{aligned}$$

Here, we have used the fact that $\tilde{b}$ is strictly decreasing (which is proved in [10]), but we can also approximate $\tilde{b}$ by the strictly decreasing functions $\tilde{b}_\varepsilon(t) = -\varepsilon t + \tilde{b}(t)$ to derive the formula. In fact, we only need $\tilde{b}$ to be C^1: indeed, we can replace $\tilde{b}(t)$ by $\tilde{b}_\mu(t) = -\mu t + \tilde{b}(t)$

and choose μ so that $\tilde{b}_\mu$ is strictly increasing in a neighborhood of the time projection of the support of ρ. $\diamond$

We now proceed with the proof of Theorem 3.1. As in the proof of Lemma 3.4, we introduce a regularizing sequence ρ_m, and set

$$V_m = V * \rho_m \quad \text{and} \quad \chi_m = \frac{\partial \tilde{h}}{\partial t} * \rho_m,$$

so that

$$-\frac{\partial V_m}{\partial t} + (A - r)V_m = \chi_m$$

Note that the functions V_m, χ_m are C^∞, with bounded derivatives on any subset $[t_1, T] \times \mathbb{R}$, with $0 < t_1 < T$. This is due to the fact that V is bounded on such subsets. For any fixed $t > 0$, multiply by $\partial V_m/\partial t$ and integrate with respect to ν_j to get

$$-\left|\frac{\partial V_m}{\partial t}(t,.)\right|_j^2 - a_j\left(V_m(t,.), \frac{\partial V_m}{\partial t}(t,.)\right) = \int \chi_m(t,x)\frac{\partial V_m}{\partial t}(t,x)\nu_j(dx).$$

We have

$$a_j\left(V_m(t,.), \frac{\partial V_m}{\partial t}(t,.)\right) = \frac{1}{2}\frac{d}{dt}\left(\tilde{a}_j\left(V_m(t,.), V_m(t,.)\right)\right) + \bar{a}_j\left(V_m(t,.), \frac{\partial V_m}{\partial t}(t,.)\right).$$

By integrating with respect to time, we get, if $0 < t_1 < T$,

$$-\int_{t_1}^T \left|\frac{\partial V_m}{\partial t}(t,.)\right|_j^2 dt + \frac{1}{2}\left[\tilde{a}_j\left(V_m(t_1,.), V_m(t_1,.)\right) - \tilde{a}_j\left(V_m(T,.), V_m(T,.)\right)\right] =$$
$$\int_{t_1}^T \bar{a}_j\left(V_m(t,.), \frac{\partial V_m}{\partial t}(t,.)\right)dt + \int_{t_1}^T \left(\chi_m(t,.), \frac{\partial V_m}{\partial t}(t,.)\right)_j dt.$$

Hence

$$\begin{aligned}\int_{t_1}^T \left|\frac{\partial V_m}{\partial t}(t,.)\right|_j^2 dt &\leq C\left|\left|V_m(t_1,.)\right|\right|_j^2 - \int_{t_1}^T \bar{a}_j\left(V_m(t,.), \frac{\partial V_m}{\partial t}(t,.)\right)dt \\ &\quad - \int_{t_1}^T \left(\chi_m(t,.), \frac{\partial V_m}{\partial t}(t,.)\right)_j dt \\ &\leq C\left(\left|\left|V_m(t_1,.)\right|\right|_j^2 + \int_{t_1}^T \left|\left|V_m(t,.)\right|\right|_j \left|\frac{\partial V_m}{\partial t}(t,.)\right|_j dt\right) + J_m(t_1,T),\end{aligned}$$

with

$$J_m(t_1,T) = -\int_{t_1}^T \left(\chi_m(t,.), \frac{\partial V_m}{\partial t}(t,.)\right)_j dt.$$

Using the inequality

$$2\left|\left|V_m(t,.)\right|\right|_j \left|\frac{\partial V_m}{\partial t}(t,.)\right|_j \leq \varepsilon\left|\frac{\partial V_m}{\partial t}(t,.)\right|_j^2 + \frac{1}{\varepsilon}\left|\left|V_m(t,.)\right|\right|_j^2,$$

we derive

$$\frac{1}{2}\int_{t_1}^{T}\left|\frac{\partial V_m}{\partial t}(t,.)\right|_j^2 dt \;\leq\; C\left(||V_m(t_1,.)||_j^2+\int_{t_1}^{T}||V_m(t,.)||_j^2\,dt\right)+J_m(t_1,T). \qquad (9)$$

We now study $J_m(t_1,T)$. Note that, for any fixed $t>0$,

$$\left(\chi_m(t,.),\frac{\partial V_m}{\partial t}(t,.)\right)_j \;=\; \int \nu_j(dx)\frac{\partial V_m}{\partial t}(t,x)\frac{\partial \tilde{h}}{\partial t}*\rho_m(t,x)$$

We have

$$\frac{\partial \tilde{h}}{\partial t}(t,.)=(A-r)\varphi\frac{\partial I}{\partial t}(t,.),$$

so that, using Lemma 3.5, and the notation $\gamma(t)=-(A-r)\varphi(\tilde{b}(t))$

$$\frac{\partial \tilde{h}}{\partial t}*\rho_m(t,x)=-\int d\tau\rho_m(t-\tau,x-\tilde{b}(\tau))\tilde{b}'(\tau)\gamma(\tau),$$

Recall that $\gamma(\tau)\geq 0$. Hence

$$\begin{aligned}\left(\chi_m(t,.),\frac{\partial V_m}{\partial t}(t,.)\right)_j &= -\int \nu_j(dx)\frac{\partial V_m}{\partial t}(t,x)\int d\tau\rho_m(t-\tau,x-\tilde{b}(\tau))\tilde{b}'(\tau)\gamma(\tau)\\ &= -\int d\tau\int\frac{dx}{(1+x^2)^{j/2}}\frac{\partial V_m}{\partial t}(t,x)\rho_m(t-\tau,x-\tilde{b}(\tau))\tilde{b}'(\tau)\gamma(\tau)\\ &= -\int d\tau\int\frac{dy}{(1+(y+\tilde{b}(\tau))^2)^{j/2}}\frac{\partial V_m}{\partial t}(t,y+\tilde{b}(\tau))\rho_m(t-\tau,y)\tilde{b}'(\tau)\gamma(\tau).\end{aligned}$$

Going back to $J_m(t_1,T)$, we have

$$J_m(t_1,T) \;=\; -\int_{t_1}^{T}dt\int d\tau\int dy\frac{\partial V_m}{\partial t}(t,y+\tilde{b}(\tau))\rho_m(t-\tau,y)\bar{\gamma}_j(\tau,y),$$

with

$$\bar{\gamma}_j(\tau,y)=-\frac{1}{(1+(y+\tilde{b}(\tau))^2)^{j/2}}\tilde{b}'(\tau)\gamma(\tau).$$

Note that $\bar{\gamma}_j(\tau,y)\geq 0$. We have

$$\begin{aligned}J_m(t_1,T) &= -\int d\tau\int dt\int dy\mathbb{1}_{\{t_1<t<T\}}\frac{\partial V_m}{\partial t}(t,y+\tilde{b}(\tau))\rho_m(t-\tau,y)\bar{\gamma}_j(\tau,y)\\ &= -\int d\tau\int ds\int dy\mathbb{1}_{\{t_1<\tau+s<T\}}\frac{\partial V_m}{\partial t}(\tau+s,y+\tilde{b}(\tau))\rho_m(s,y)\bar{\gamma}_j(\tau,y).\end{aligned}$$

Observe that

$$\frac{d}{d\tau}\left(V_m(\tau+s,y+\tilde{b}(\tau))\right)=\frac{\partial V_m}{\partial t}(\tau+s,y+\tilde{b}(\tau))+\frac{\partial V_m}{\partial x}(\tau+s,y+\tilde{b}(\tau))\tilde{b}'(\tau),$$

so that

$$\frac{\partial V_m}{\partial t}(\tau+s,y+\tilde{b}(\tau))=\frac{d}{d\tau}\left(V_m(\tau+s,y+\tilde{b}(\tau))\right)-\frac{\partial V_m}{\partial x}(\tau+s,y+\tilde{b}(\tau))\tilde{b}'(\tau).$$

Hence

$$J_m(t_1,T) = \hat{J}_m(t_1,T) + \bar{J}_m(t_1,T),$$

with

$$\hat{J}_m(t_1,T) = -\int d\tau \int ds \int dy \mathbb{1}_{\{t_1<\tau+s<T\}} \frac{d}{d\tau}\left(V_m(\tau+s, y+\tilde{b}(\tau))\right) \rho_m(s,y)\bar{\gamma}_j(\tau,y)$$

and

$$\bar{J}_m(t_1,T) = +\int d\tau \int ds \int dy \mathbb{1}_{\{t_1<\tau+s<T\}} \frac{\partial V_m}{\partial x}(\tau+s, y+\tilde{b}(\tau))\tilde{b}'(\tau)\rho_m(s,y)\bar{\gamma}_j(\tau,y).$$

We have, using integration by parts,

$$\begin{aligned}
\hat{J}_m(t_1,T) &= -\int ds \int dy \rho_m(s,y) \left(\int_{t_1-s}^{T-s} \frac{d}{d\tau}\left(V_m(\tau+s, y+\tilde{b}(\tau))\right) \bar{\gamma}_j(\tau,y) d\tau \right) \\
&= -\int ds \int dy \rho_m(s,y) V_m(T, y+\tilde{b}(T-s))\bar{\gamma}_j(T-s,y) \\
&\quad + \int ds \int dy \rho_m(s,y) V_m(t_1, y+\tilde{b}(t_1-s))\bar{\gamma}_j(t_1-s,y) \\
&\quad + \int ds \int dy \rho_m(s,y) \int_{t_1-s}^{T-s} V_m(s+\tau, y+\tilde{b}(\tau)) \frac{\partial \bar{\gamma}_j}{\partial \tau}(\tau,y) d\tau.
\end{aligned}$$

Note that, due to the continuity of $V(=\partial\tilde{U}/\partial t)$ on $(0,\infty)\times\mathbb{R}$, the sequence V_m converges uniformly to V on compact sets. We also have the continuity of $\bar{\gamma}_j$ and $\partial\bar{\gamma}_j/\partial\tau$ (due to the fact that $\tilde{b}$ is C^2). We easily deduce thereof that

$$\lim_{m\to\infty} \hat{J}_m(t_1,T) = -V(T,\tilde{b}(T))\bar{\gamma}_j(T,0) + V(t_1,\tilde{b}(t_1))\bar{\gamma}_j(t_1,0) + \int_{t_1}^{T} V(\tau,\tilde{b}(\tau))\frac{\partial\bar{\gamma}_j}{\partial\tau}(\tau,0)d\tau, \quad (10)$$

and the convergence is uniform with respect to t_1, as long as t_1 remains in a compact set of the form $[\xi,T]$, where $0<\xi<T$. For $\bar{J}_m(t_1,T)$, we have

$$\bar{J}_m(t_1,T) = \bar{J}_m^{(1)}(t_1,T) + \bar{J}_m^{(2)}(t_1,T),$$

with

$$\bar{J}_m^{(1)}(t_1,T) = +\int d\tau \int ds \int dy \mathbb{1}_{\{t_1<\tau+s<T\}} \frac{\partial^2 U_m}{\partial t\partial x}(\tau+s, y+\tilde{b}(\tau))\tilde{b}'(\tau)\rho_m(s,y)\bar{\gamma}_j(\tau,y)$$

and

$$\bar{J}_m^{(2)}(t_1,T) = -\int d\tau \int ds \int dy \mathbb{1}_{\{t_1<\tau+s<T\}} \frac{\partial^2 u_m}{\partial t\partial x}(\tau+s, y+\tilde{b}(\tau))\tilde{b}'(\tau)\rho_m(s,y)\bar{\gamma}_j(\tau,y).$$

We deal with $\bar{J}_m^{(1)}(t_1,T)$ in the same way as for the proof of (7). Using the fact that $\tilde{b}'(\tau)\bar{\gamma}_j(\tau,y)\le 0$, we have $\bar{J}_m^{(1)}(t_1,T)\le\tilde{J}_m^{(1)}(t_1,T)$, with

$$\tilde{J}_m^{(1)}(t_1,T) = \int d\tau \int ds \int dy \mathbb{1}_{\{t_1<\tau+s<T\}} \int\int ds'dy' \rho_m(s',y') D(\tau,s-s',y-y')\tilde{b}'(\tau)\rho_m(s,y)\bar{\gamma}_j(\tau,y),$$

where

$$D(\tau,\theta,z) = \left(\frac{\partial^2 U}{\partial t\partial x}(\tau+\theta,\tilde{b}(\tau)+z) - \frac{\partial^2 U}{\partial t\partial x}(\tau,\tilde{b}(\tau))\right)\mathbb{1}_{\{\tilde{b}(\tau)+z\ge\tilde{b}(\tau+\theta)\}}.$$

Due to the continuity properties of $\frac{\partial^2 U}{\partial t \partial x}$, we have

$$\lim_{m\to\infty} \tilde{J}_m^{(1)}(t_1,T) = 0,$$

and the convergence is uniform with respect to t_1, as long as t_1 remains in $[\xi,T]$. On the other hand, due to the continuity of $\frac{\partial^2 u_\varphi}{\partial t \partial x}$, we have

$$\lim_{m\to\infty} \bar{J}_m^{(2)}(t_1,T) = -\int_{t_1}^T d\tau \frac{\partial^2 u_\varphi}{\partial t \partial x}(\tau,\tilde{b}(\tau))\tilde{b}'(\tau)\bar{\gamma}_j(\tau,0),$$

uniformly with respect to $t_1 \in [\xi,T]$. At this stage, we can state that $J_m(t_1,T) \leq \tilde{J}_m(t_1,T)$, with $\tilde{J}_m(t_1,T) = \hat{J}_m(t_1,T) + \tilde{J}_m^{(1)}(t_1,T) + \bar{J}_m^{(2)}(t_1,T)$, and

$$\lim_{m\to\infty} \sup_{t_1\in[\xi,T]} \left|\tilde{J}_m(t_1,T) - \tilde{J}(t_1,T)\right| = 0,$$

where

$$\begin{aligned}\tilde{J}(t_1,T) &= -V(T,\tilde{b}(T))\bar{\gamma}_j(T,0) + V(t_1,\tilde{b}(t_1))\bar{\gamma}_j(t_1,0) + \int_{t_1}^T V(\tau,\tilde{b}(\tau))\frac{\partial\bar{\gamma}_j}{\partial\tau}(\tau,0)d\tau \\ &\quad - \int_{t_1}^T d\tau \frac{\partial^2 u}{\partial t\partial x}(\tau,\tilde{b}(\tau))\tilde{b}'(\tau)\bar{\gamma}_j(\tau,0).\end{aligned}$$

Since $\partial U/\partial t$ vanishes along the exercise boundary, we have $V(t,\tilde{b}(t)) = -\frac{\partial u_\varphi}{\partial t}(t,\tilde{b}(t))$, so that

$$\begin{aligned}&-V(T,\tilde{b}(T))\bar{\gamma}_j(T,0) + V(t_1,\tilde{b}(t_1))\bar{\gamma}_j(t_1,0) + \int_{t_1}^T V(\tau,\tilde{b}(\tau))\frac{\partial\bar{\gamma}_j}{\partial\tau}(\tau,0)d\tau = \\ &\quad \frac{\partial u_\varphi}{\partial t}(T,\tilde{b}(T))\bar{\gamma}_j(T,0) - \frac{\partial u_\varphi}{\partial t}(t_1,\tilde{b}(t_1))\bar{\gamma}_j(t_1,0) - \int_{t_1}^T \frac{\partial u_\varphi}{\partial t}(\tau,\tilde{b}(\tau))\frac{\partial\bar{\gamma}_j}{\partial\tau}(\tau,0)d\tau \\ &= \int_{t_1}^T \frac{d}{d\tau}\left(\frac{\partial u_\varphi}{\partial t}(\tau,\tilde{b}(\tau))\right)\bar{\gamma}_j(\tau,0)d\tau,\end{aligned}$$

so that

$$\begin{aligned}\tilde{J}(t_1,T) &= \int_{t_1}^T \left[\frac{d}{d\tau}\left(\frac{\partial u_\varphi}{\partial t}(\tau,\tilde{b}(\tau))\right) - \frac{\partial^2 u_\varphi}{\partial t\partial x}(\tau,\tilde{b}(\tau))\tilde{b}'(\tau)\right]\bar{\gamma}_j(\tau,0)d\tau \\ &= \int_{t_1}^T \frac{\partial^2 u_\varphi}{\partial t^2}(\tau,\tilde{b}(\tau))\bar{\gamma}_j(\tau,0)d\tau.\end{aligned}$$

We now go back to (9) and integrate with respect to t_1 to derive

$$\begin{aligned}\frac{1}{2}\int_\xi^T dt_1 \int_{t_1}^T \left|\frac{\partial^2\tilde{U}_m}{\partial t^2}(t,.)\right|_j^2 dt &\leq C\left(\int_\xi^T ||V_m(t_1,.)||_j^2\, dt_1 + \int_\xi^T \left(\int_{t_1}^T ||V_m(t,.)||_j^2\, dt\right) dt_1\right) \\ &\quad + \int_\xi^T dt_1 \tilde{J}_m(t_1,T).\end{aligned}$$

Hence

$$\frac{1}{2}\int_\xi^T (t-\xi)\left|\frac{\partial^2\tilde{U}_m}{\partial t^2}(t,.)\right|_j^2 dt \leq C\int_\xi^T ||V_m(t,.)||_j^2\, dt + \int_\xi^T dt_1 \tilde{J}_m(t_1,T).$$

Note that

$$\begin{aligned}\lim_{m\to\infty}\int_\xi^T ||V_m(t,.)||_j^2\,dt &= \int_\xi^T ||V(t,.)||_j^2\,dt \\ &\le C\left(1+\ln\frac{T}{\xi}\right),\end{aligned}$$

where the last inequality follows from Lemma 3.4. Moreover,

$$\begin{aligned}\lim_{m\to\infty}\int_\xi^T dt_1\tilde{J}_m(t_1,T) &= \int_\xi^T dt_1\tilde{J}(t_1,T) \\ &= \int_\xi^T (t-\xi)\frac{\partial^2 u_\varphi}{\partial t^2}(t,\tilde{b}(t))\bar{\gamma}_j(t,0)dt.\end{aligned}$$

Hence

$$\frac{1}{2}\int_\xi^T (t-\xi)\left|\frac{\partial^2\tilde{U}}{\partial t^2}(t,.)\right|_j^2 dt \le C\left(1+\ln\frac{T}{\xi}\right)+\int_\xi^T (t-\xi)\frac{\partial^2 u_\varphi}{\partial t^2}(t,\tilde{b}(t))\bar{\gamma}_j(t,0)dt.$$

Theorem 3.1 now follows from the following lemma, which relies on the asymptotic behavior of the exercice boundary near maturity (see [1], [7]).

Lemma 3.6. *We have*

$$\int_\xi^T (t-\xi)\left|\frac{\partial^2 u_\varphi}{\partial t^2}(t,\tilde{b}(t))\right||\tilde{b}'(t)|dt \le C\left(1+|\ln\xi|^\beta\right), \quad \text{with } \beta=\begin{cases} 3/2, & \text{if } d\le r,\\ 1, & \text{if } d>r.\end{cases}$$

Proof: We first note that, since φ is bounded and Lipschitz continuous, we have

$$\left\|\frac{\partial^2 u_\varphi}{\partial t^2}(t,.)\right\|_\infty \le \frac{C}{t^{3/2}}.$$

This can be seen by arguing, as in the proof of Proposition 3.1, that $u_\varphi(t,.)=e^{-rt}p_t*\varphi$, so that and $\frac{\partial^2 u_\varphi}{\partial t^2}(t,.)=(A-r)^2u_\varphi$. In order to estimate the x-derivatives of u_φ up to the order 4, we may differentiate p_t three times and use the boundedness of φ'.

We then have

$$\begin{aligned}\int_\xi^T (t-\xi)\left|\frac{\partial^2 u_\varphi}{\partial t^2}(t,\tilde{b}(t))\right||\tilde{b}'(t)|dt &\le C\int_\xi^T \frac{t-\xi}{t^{3/2}}|\tilde{b}'(t)|dt \\ &\le C\int_\xi^T \frac{1}{\sqrt{t}}|\tilde{b}'(t)|dt.\end{aligned}$$

Now, since $\tilde{b}'(t)\le 0$, we have

$$\begin{aligned}\int_\xi^T \frac{1}{\sqrt{t}}|\tilde{b}'(t)|dt &= -\int_\xi^T \frac{1}{\sqrt{t}}\tilde{b}'(t)dt \\ &= -\left(\frac{\tilde{b}(T)-\tilde{b}(0)}{\sqrt{T}}-\frac{\tilde{b}(\xi)-\tilde{b}(0)}{\sqrt{\xi}}\right)-\frac{1}{2}\int_\xi^T \frac{1}{t^{3/2}}(\tilde{b}(t)-\tilde{b}(0))dt \\ &\le \frac{\tilde{b}(0)-\tilde{b}(T)}{\sqrt{T}}+\frac{1}{2}\int_\xi^T \frac{1}{t^{3/2}}(\tilde{b}(0)-\tilde{b}(t))dt\end{aligned}$$

If $d \leq r$, we have $\tilde{b}(0) - \tilde{b}(t) \leq C\sqrt{t|\ln t|}$ for t close to 0, so that

$$\int_\xi^T \frac{1}{t^{3/2}}(\tilde{b}(0) - \tilde{b}(t))dt \leq C\int_\xi^T \frac{1}{t}\sqrt{|\ln t|}dt \leq C(1 + |\ln \xi|^{3/2}).$$

If $d > r$, we have $\tilde{b}(0) - \tilde{b}(t) \leq C\sqrt{t}$ for t close to 0, so that

$$\int_\xi^T \frac{1}{t^{3/2}}(\tilde{b}(0) - \tilde{b}(t))dt \leq C\int_\xi^T \frac{1}{t}dt = C\ln(T/\xi).$$

$\diamond$

4 Upper bound for $P_0^{(n)} - P_0$

In order to derive an upper bound for $P_0^{(n)} - P_0$, we relate this quantity to the modified value function u using (1) as follows:

$$\begin{aligned} P_0^{(n)} - P_0 &= \sup_{\tau \in \mathcal{T}_{0,T}^{(n)}} \mathbb{E}\left(e^{-r\tau} g(\mu_0 \tau + B_\tau^{(n)})\right) - u(0,0) \\ &\leq \sup_{\tau \in \mathcal{T}_{0,T}^{(n)}} \mathbb{E}\left(u(\tau, B_\tau^{(n)}) - u(0,0)\right) \\ &= \sup_{\tau \in \mathcal{T}_{0,T}^{(n)}} \mathbb{E}\left(\sum_{j=1}^{\tau/h} \mathcal{D}u((j-1)h, B_{(j-1)h}^{(n)})\right). \end{aligned}$$

We observe that $\mathcal{D}u \leq \tilde{\mathcal{D}}u$, and recall from [6] (Lemma 4.1) that $\sup_{0\leq j\leq n-1} \mathbb{E}\left|\mathcal{D}u(jh, B_{jh}^{(n)})\right| \leq Ch$, so that

$$\begin{aligned} P_0^{(n)} - P_0 &\leq \mathbb{E}\left(\sum_{j=1}^{n-2} \left|\tilde{\mathcal{D}}u(jh, B_{jh}^{(n)})\right|\right) + O(h) \\ &\leq C_{k,X} h\sqrt{2} \int_h^{T-h} \frac{ds}{\sqrt{s}} \int \frac{dy}{1+|y|^k} \left|\frac{\partial^3 u}{\partial t \partial x^2}(s,y)\right| + O(h). \end{aligned} \tag{11}$$

Here, we have a regularity problem, since u is not C^3. This problem can be fixed as follows. By convolution, one can approximate u by a sequence u_m which is smooth, uniformly bounded and satisfies $\delta u_m \leq 0$, and $\mathcal{D}u_m \leq \tilde{\mathcal{D}}u_m$. We need the following variant of Lemma 3.1.

Lemma 4.1. *If ρ is a Radon measure on $(0,T)\times\mathbb{R}$ and q a nonnegative integrable function on $(0,T)\times\mathbb{R}$, with $q(t,x) = 0$ for $t \notin (0,a)$, where a satisfies $0 < a < h$, we have*

$$\int_h^{T-h} \frac{ds}{\sqrt{s}} \int \frac{|\rho * q(s,y)|}{(1+|y|^2)^{k/2}} dy \leq 2^{k/2} \int_{h-a}^{T-h} \int_{\mathbb{R}} \frac{|\rho(dt,dz)|}{\sqrt{t}(1+z^2)^{k/2}} \int_0^a ds \int_{-\infty}^{\infty} q(s,x)(1+x^2)^{k/2} dx.$$

Proof: We have

$$\int_h^{T-h} \frac{ds}{\sqrt{s}} \int \frac{dy}{(1+|y|^2)^{k/2}} \left|\rho * q(s,y)\right| \leq$$

$$\begin{aligned}
&\int_h^{T-h} \frac{ds}{\sqrt{s}} \int \frac{dy}{(1+|y|^2)^{k/2}} \int\int |\rho(dt,dz)|\, q(s-t,y-z) \\
&= \int\int |\rho(dt,dz)| \int_h^{T-h} \frac{ds}{\sqrt{s}} \int \frac{dy}{(1+|y|^2)^{k/2}} q(s-t,y-z) \\
&\leq \int_{h-a}^{T-h} \int |\rho(dt,dz)| \int_0^a \frac{d\theta}{\sqrt{t+\theta}} \int \frac{dy}{(1+|y|^2)^{k/2}} q(\theta,y-z) \\
&\leq \int_{h-a}^{T-h} \int \frac{|\rho(dt,dz)|}{\sqrt{t}} \int_0^a d\theta \int \frac{dy}{(1+|y|^2)^{k/2}} q(\theta,y-z) \\
&= \int_{h-a}^{T-h} \int \frac{|\rho(dt,dz)|}{\sqrt{t}(1+z^2)^{k/2}} \int_0^a d\theta \int \frac{(1+z^2)^{k/2}dx}{(1+|x+z|^2)^{k/2}} q(\theta,x) \\
&\leq 2^{k/2} \int_{h-a}^{T-h} \int \frac{|\rho(dt,dz)|}{\sqrt{t}(1+z^2)^{k/2}} \int_0^a d\theta \int dx (1+x^2)^{k/2} q(\theta,x),
\end{aligned}$$

where the last inequality follows from $\frac{1+z^2}{1+(x+z)^2} \leq 2(1+x^2)$. ◇

Using Lemma 4.1 and the fact that $\partial^3 u/(\partial t \partial x^2)$ is a Radon measure (see (13) and the comment below), we derive the correct version of (11), namely

$$P_0^{(n)} - P_0 \;\leq\; C_{k,X} h\sqrt{2} \int_h^{T-h} \frac{1}{\sqrt{s}} \int \frac{1}{1+|y|^k} \left|\frac{\partial^3 u}{\partial t \partial x^2}(ds,dy)\right| + O(h). \tag{12}$$

If we introduce the function $\tilde{u} := u - \bar{u}$, we have, using the fact that $\delta \bar{u} = 0$,

$$\begin{aligned}
\int_h^{T-h} \frac{1}{\sqrt{s}} \int \frac{1}{1+|y|^k} \left|\frac{\partial^3 u}{\partial t \partial x^2}(ds,dy)\right| \;&\leq\; \int_h^{T-h} \frac{1}{\sqrt{s}} \int \frac{1}{1+|y|^k} \left|\frac{\partial^3 \tilde{u}}{\partial t \partial x^2}(ds,dy)\right| \\
&\quad + 2\int_h^{T-h} \frac{ds}{\sqrt{s}} \int \frac{dy}{1+|y|^k} \left|\frac{\partial^2 \bar{u}}{\partial t^2}(s,y)\right| \\
&\leq\; \int_h^{T-h} \frac{1}{\sqrt{s}} \int \frac{1}{1+|y|^k} \left|\frac{\partial^3 \tilde{u}}{\partial t \partial x^2}(ds,dy)\right| \\
&\quad + 2C_T \int_h^{T-h} \frac{ds}{\sqrt{s}(T-s)} \\
&\leq\; \int_h^{T-h} \frac{1}{\sqrt{s}} \int \frac{1}{1+|y|^k} \left|\frac{\partial^3 \tilde{u}}{\partial t \partial x^2}(ds,dy)\right| + C_T |\ln h|,
\end{aligned}$$

where we have used Proposition 3.1.

We now need to estimate $\int_h^{T-h} \frac{1}{\sqrt{s}} \int \frac{1}{1+|y|^k} \left|\frac{\partial^3 \tilde{u}}{\partial t \partial x^2}(ds,dy)\right|$. Recall from Remark 2.1 that

$$\frac{\partial \tilde{u}}{\partial t}(t,x) + \frac{1}{2}\frac{\partial^2 \tilde{u}}{\partial x^2}(t,x) \;=\; \zeta(t,x) \mathbb{1}_{\{x \leq \hat{b}(t)\}},$$

where $\zeta(t,x) = e^{-rt}(A-r)\varphi(\ln(S_0) + \mu t + \sigma x)$ and $\hat{b}(t) = \left(\tilde{b}(T-t) - \mu t - \ln(S_0)\right)/\sigma$. By differentiating wit respect to t, we derive the following expression

$$\frac{\partial^2 \tilde{u}}{\partial t^2}(t,x) + \frac{1}{2}\frac{\partial^3 \tilde{u}}{\partial t \partial x^2}(t,x) \;=\; \frac{\partial \zeta}{\partial t}(t,x) \mathbb{1}_{\{x \leq \hat{b}(t)\}} + \zeta(t,\hat{b}(t))\hat{b}'(t)\delta_{\hat{b}(t)}, \tag{13}$$

where we have used Lemma 3.5. Note that

$$\sup_{x\leq\hat{b}(t)}\left|\frac{\partial\zeta}{\partial t}(t,x)\right|<\infty \text{ and } \sup_{0<t<T}\left|\zeta(t,\hat{b}(t))\right|<\infty.$$

Moreover, $|\hat{b}'(t)|\leq|\tilde{b}'(T-t)|+|\mu/\sigma|$, so that

$$\begin{aligned}\int_0^T\frac{dt}{\sqrt{t}}|\hat{b}'(t)| &\leq \int_0^{T/2}\frac{dt}{\sqrt{t}}|\tilde{b}'(T-t)|+\int_{T/2}^T\frac{dt}{\sqrt{t}}|\tilde{b}'(T-t)|+2|\mu/\sigma|\sqrt{T}\\ &\leq \sup_{0\leq t\leq T/2}|\tilde{b}'(T-t)|\int_0^{T/2}\frac{dt}{\sqrt{t}}++\sqrt{\frac{2}{T}}\int_{T/2}^T|\tilde{b}'(T-t)|dt+2|\mu/\sigma|\sqrt{T}\\ &= \sup_{0\leq t\leq T/2}|\tilde{b}'(T-t)|\int_0^{T/2}\frac{dt}{\sqrt{t}}+\sqrt{\frac{2}{T}}\left(\tilde{b}(0)-\tilde{b}(T/2)\right)+2|\mu/\sigma|\sqrt{T}<\infty.\end{aligned}$$

Therefore, the righthand side of (13) is a Radon measure and since, due to Theorem 3.1, $\partial^2\tilde{u}/\partial t^2$ is locally integrable, it follows that $\partial^3\tilde{u}/\partial t\partial x^2$ is a Radon measure.

Moreover, we have

$$\int_h^{T-h}\frac{1}{\sqrt{s}}\int\frac{1}{1+|y|^k}\left|\frac{\partial^3\tilde{u}}{\partial t\partial x^2}(ds,dy)\right| \leq C_T+2\int_h^{T-h}\frac{ds}{\sqrt{s}}\int\frac{1}{1+|y|^k}\left|\frac{\partial^2\tilde{u}}{\partial t^2}(s,y)\right|.$$

Now, using the Cauchy-Schwarz inequality and Theorem 3.1, we have

$$\begin{aligned}&\int_h^{T-h}\frac{ds}{\sqrt{s}}\int\frac{dy}{1+|y|^k}\left|\frac{\partial^2\tilde{u}}{\partial t^2}(s,y)\right|\leq\\ &\quad C\left(\int_h^{T-h}\frac{ds}{s(T-s-\frac{h}{2})}\right)^{1/2}\left(\int_h^{T-h}ds(T-s-\frac{h}{2})\left|\frac{\partial^2\tilde{u}}{\partial t^2}(s,.)\right|_k^2\right)^{1/2}\\ &\qquad\leq C\sqrt{|\ln h|}\left(\int_h^{T-\frac{h}{2}}ds(T-s-\frac{h}{2})\left|\frac{\partial^2\tilde{u}}{\partial t^2}(s,.)\right|_k^2\right)^{1/2}\\ &\qquad= C\sqrt{|\ln h|}\left(\int_{h/2}^{T-h}dt(t-\frac{h}{2})\left|\frac{\partial^2\tilde{u}}{\partial t^2}(T-t,.)\right|_k^2\right)^{1/2}\\ &\qquad\leq C\sqrt{|\ln h|^{1+\beta}},\end{aligned}$$

with $\beta=1$ if $d>r$ and $\beta=3/2$ if $d\leq r$. The last inequality follows from Theorem 3.1 and Lemma 3.4, and the connection between the derivatives of the functions $\tilde{U}$ and $\tilde{u}$ (see Remark 2.1; we also use the classical bounds $||\partial U/\partial t(t,.)||_\infty+||\partial^2U/\partial x^2(t,.)||_\infty\leq C/\sqrt{t}$). We conclude that

$$P_0^{(n)}-P_0\leq C\frac{(\ln n)^\alpha}{n},$$

with $\alpha=1$ if $d>r$ and $\alpha=5/4$ if $d\leq r$.

5 Lower bound for $P_0^{(n)}-P_0$

For the derivation of the lower bound, we use the stopping time introduced in [6] (see the proof of Theorem 5.6). Namely

$$\tau=\tau_1\mathbb{1}_{\{\tau_1<T-h\}}+T\mathbb{1}_{\{\tau_1=T-h\}},$$

where

$$\tau_1 = \inf\left\{t \in [0, T-h] \;\mid\; t/h \in \mathbb{N} \quad \text{and} \quad d(B_t^{(n)}, I_{t+h}) \leq \sqrt{h}||X||_\infty + |\mu_0|h\right\}.$$

Here, $I_t = \{x \in \mathbb{R} \mid u(t,x) = g(t, x+\mu_0 t)\}$. Note that, if $t < T$, $I_t = (-\infty, \hat{b}(t)]$.

The modification of τ_1 into τ is motivated by the unboundedness of $\partial u/\partial t$ near T. We have, due to the definition of τ_1,

$$\begin{aligned} P^{(n)} - P &\geq \mathbb{E}\left(e^{-r\tau} g(\mu_0\tau + B_\tau^{(n)}) - u(0,0)\right) \\ &= \mathbb{E}\left(e^{-r\tau} g(\mu_0\tau + B_\tau^{(n)}) - u(\tau, B_\tau^{(n)}) + u(\tau, B_\tau^{(n)}) - u(0,0)\right). \end{aligned}$$

We have

$$\begin{aligned} \mathbb{E}\left(u(\tau, B_\tau^{(n)}) - u(0,0)\right) &= \mathbb{E}\left(\sum_{j=1}^{\tau/h} \mathcal{D}u((j-1)h, B_{(j-1)h}^{(n)})\right) \\ &= \mathbb{E}\left(\sum_{j=1}^{(\tau/h)\wedge(n-2)} \mathcal{D}u(jh-h, B_{jh-h}^{(n)})\right) \\ &\quad + \mathbb{E}\left(\mathbb{1}_{\{\tau=T\}} \sum_{j=n-1}^{n} \mathcal{D}u(jh-h, B_{jh-h}^{(n)})\right) \\ &= \mathbb{E}\left(\sum_{j=1}^{(\tau/h)\wedge(n-2)} \mathcal{D}u(jh-h, B_{jh-h}^{(n)})\right) + O(h), \end{aligned}$$

where the last equality follows from Lemma 4.1 of [6]. Now, if $j < (\tau/h) \wedge (n-2)$, we have $d(B_{jh}^{(n)}, I_{jh+h}) > \sqrt{h}||X||_\infty + |\mu_0|h$, so that

$$B_{jh}^{(n)} > \hat{b}(jh+h) + \sqrt{h}||X||_\infty + |\mu_0|h.$$

We then have, for $s \in [jh, jh+h]$ and $z \in [0, \sqrt{h}]$

$$\begin{aligned} B_{jh}^{(n)} + zX &> \hat{b}(jh+h) + zX + \sqrt{h}||X||_\infty + |\mu_0|h \\ &\geq \hat{b}(jh+h) + |\mu_0|h \\ &= \hat{b}(s) + \hat{b}(jh+h) + \mu_0(jh+h) - (\hat{b}(s) + \mu_0 s) - \mu_0(jh+h-s) + |\mu_0|h \\ &\geq \hat{b}(s), \end{aligned}$$

the last inequality coming from the fact that $t \mapsto \hat{b}(t) + \mu_0 t$ is increasing. We can now assert that, for $j < (\tau/h) \wedge (n-2)$, $\mathcal{D}u(B_{jh}^{(n)}, jh) = \tilde{\mathcal{D}}u(B_{jh}^{(n)}, jh)$, so that

$$\begin{aligned} \mathbb{E}\left(u(\tau, B_\tau^{(n)}) - u(0,0)\right) &\leq \mathbb{E}\left(\sum_{j=1}^{n-2} \left|\tilde{\mathcal{D}}u(jh, B_{jh}^{(n)})\right|\right) + O(h) \\ &\leq C\frac{(\ln n)^\alpha}{n}, \end{aligned}$$

with $\alpha = 1$ if $d > r$ and $\alpha = 5/4$ if $d \leq r$, as follows from the discussion in the previous section.

We now want a lower bound for

$$\mathbb{E}\left(e^{-r\tau}g(\mu_0\tau + B_\tau^{(n)}) - u(\tau, B_\tau^{(n)})\right).$$

We have, using the equality $\{\tau \geq T-h\} = \{\tau = T\}$,

$$\begin{aligned} u(\tau, B_\tau^{(n)}) - e^{-r\tau}g(\mu_0\tau + B_\tau^{(n)}) &= \left(u(\tau, B_\tau^{(n)}) - e^{-r\tau}g(\mu_0\tau + B_\tau^{(n)})\right)\mathbb{1}_{\{\tau<T-h\}} \\ &= \left(u(\tau+h, B_\tau^{(n)}) - e^{-r\tau}g(\mu_0\tau + B_\tau^{(n)})\right)\mathbb{1}_{\{\tau<T-h\}} \\ &\quad + \left(u(\tau, B_\tau^{(n)}) - u(\tau+h, B_\tau^{(n)})\right)\mathbb{1}_{\{\tau<T-h\}}. \end{aligned}$$

On the set $\{\tau < T-h\}$, we have $d(B_\tau^{(n)}, I_{\tau+h}) \leq \sqrt{h}||X||_\infty + |\mu|h$. It follows from Proposition 2.6 of [6] that

$$u(\tau+h, B_\tau^{(n)}) - e^{-r\tau}g(\mu_0\tau + B_\tau^{(n)}) \leq C\frac{\left(\sqrt{h}||X||_\infty + |\mu|h\right)^2}{\sqrt{T-\tau-h}}.$$

Using the estimate $\left|\left|\frac{\partial u}{\partial t}(t,.)\right|\right|_\infty < C/\sqrt{T-t}$, we obtain

$$\mathbb{E}\left(u(\tau, B_\tau^{(n)}) - e^{-r\tau}g(\mu_0\tau + B_\tau^{(n)})\right) \leq Ch\mathbb{E}\left(\frac{1}{\sqrt{T-\tau-h}}\mathbb{1}_{\{\tau\leq T-2h\}}\right).$$

The estimate $P^{(n)} - P \geq -C\frac{(\ln n)^{\tilde{\alpha}}}{n}$ is now an easy consequence of Lemma 5.7 and Remark 5.8 of [6], which can be summarized in the following statement.

Lemma 5.1. *There exists a positive constant C such that*

$$\mathbb{E}\left(\frac{1}{\sqrt{T-\tau-h}}\mathbb{1}_{\{\tau\leq T-2h\}}\right) \leq C\left(\ln h\right)^\beta,$$

with

$$\beta = \begin{cases} 3/2, \ \textit{if } d \leq r, \\ \\ 1, \ \textit{if } d > r. \end{cases}$$

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