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Abstract

We consider the binomial approximation of the American put price in the Black-Scholes model (with continuous dividend yield). Our main result is that the error of approximation is $O((\ln n)^\alpha/n)$ where n is the number of time periods and the exponent α is a positive number, the value of which may differ according to the respective levels of the interest rate and the dividend yield.

1 The binomial approximation

Consider the Black-Scholes model in which the stock price at time t is given by

$$S_t = S_0 e^{(r-d-\frac{\sigma^2}{2})t+\sigma B_t},$$

where, under the risk-neutral probability measure, $(B_t)_{t\geq 0}$ is a standard Brownian motion. Here, r is the instantaneous interest rate, and d is the dividend rate (or the foreign interest rate in the case of forex options). We assume $r > 0$ and $d \geq 0$.

Denote by P the price function of the American put with maturity T and strike price K, so that

$$P(t,x) = \sup_{\tau\in\mathcal{T}_{0,T-t}} \mathbb{E}_x\left(e^{-r\tau} f(S_\tau)\right), \quad 0 \leq t \leq T, \quad x \in [0,+\infty),$$

with $f(x) = (K-x)^+$, and $\mathbb{E}_x = \mathbb{E}(\cdot \mid S_0 = x)$. Here $\mathcal{T}_{0,t}$ denotes the set of all stopping times with respect to the Brownian filtration, with values in the interval $[0,t]$.

For technical reasons (especially for the derivation of regularity estimates for the second time derivative of the price function), it is more convenient to use the log-stock price. So, we introduce

$$X_t^x = x + \mu t + \sigma B_t, \quad \text{with } \mu = r - d - \frac{\sigma^2}{2},$$

and

$$U(T,x) = \sup_{\tau\in\mathcal{T}_{0,T}} \mathbb{E}\left(e^{-r\tau}\varphi(X_\tau^x)\right),$$

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with $\varphi(x) = (K - e^x)^+$. We then have

$$P(t,x) = U(T-t, \ln(x)), \quad t > 0, x > 0.$$

Note that $U(t,x)$ satisfies the following parabolic variational inequality

$$\max\left[-\frac{\partial U}{\partial t} + (A-r)U, \varphi - U\right] = 0,$$

with the initial condition $U(0,.) = \varphi$.

Here, A is the infinitesimal generator of X, namely

$$A = \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} + \mu\frac{\partial}{\partial x}.$$

Recall that, for each $T > 0$, there is a real number $\tilde{b}(T) \leq \ln(K)$ such that

$$U(T,x) > \varphi(x) \Leftrightarrow x > \tilde{b}(T).$$

In fact, if $(b(t), 0 \leq t \leq T)$ is the exercise boundary of the American put with maturity T, we have $\tilde{b}(t) = \ln(b(T-t))$. We will also need the European value function, defined by

$$\bar{U}(T,x) = \mathbb{E}\left(e^{-rT}\varphi(X_T^x)\right).$$

Note that $\bar{U}(0,.) = \varphi$ and

$$-\frac{\partial \bar{U}}{\partial t} + (A-r)\bar{U} = 0.$$

Note that, in Section 3, the function $\bar{U}$ will be denoted by u_φ.

We now introduce the random walk approximation of Brownian motion. To be more precise, assume $(X_n)_{n\geq 1}$ is a sequence of i.i.d. real random variables satisfying $\mathbb{E}X_n^2 = 1$ and $\mathbb{E}X_n = 0$, and define, for any positive integer n, the process $B^{(n)}$ by

$$B_t^{(n)} = \sqrt{T/n}\sum_{k=1}^{[nt/T]} X_k, \quad 0 \leq t \leq T,$$

where $[nt/T]$ denotes the greatest integer in nt/T.

We will assume the following about the common distribution of the X_n's (cf. hypothesis (H4) of [6]).

(H4) The random variable X_1 is bounded and satisfies $\mathbb{E}X_1^2 = 1$ and $\mathbb{E}X_1 = \mathbb{E}X_1^3 = 0$.

In the following, we fix S_0 and set

$$P_0 = P(0, S_0) = U(T, \ln S_0).$$

Note that, if we introduce the notation $g(x) = (K - S_0 e^{\sigma x})^+$, we have

$$P_0 = \sup_{\tau \in \mathcal{T}_{0,T}} \mathbb{E}\left(e^{-r\tau} g(\mu_0 \tau + B_\tau)\right)$$

with $\mu_0 = \mu/\sigma$. We now have a natural approximation of P_0, given by

$$P_0^{(n)} = \sup_{\tau \in \mathcal{T}_{0,T}^{(n)}} \mathbb{E}\left(e^{-r\tau} g(\mu_0 \tau + B_\tau^{(n)})\right),$$

where $\mathcal{T}_{0,T}^{(n)}$ denotes the set of all stopping times (with respect to the natural filtration of $B^{(n)}$), with values in $[0,T] \cap \{0, T/n, 2T/n, \ldots, (n-1)T/n, T\}$. Our main result is the following.

Theorem 1.1. *There exists a positive constant C such that, for all positive integers n,*

$$-C\frac{(\ln n)^{\bar{\alpha}}}{n} \leq P_0^{(n)} - P_0 \leq C\frac{(\ln n)^{\alpha}}{n},$$

where $\bar{\alpha} = \alpha = 1$ if $d > r$, and $\bar{\alpha} = 3/2, \alpha = 5/4$ if $d \leq r$.

The above estimates improve our previous results (see [6], Theorem 5.6) which gave an upper bound of the form $C\left(\frac{\sqrt{\ln n}}{n}\right)^{4/5}$. Note that, for European options, the error estimate is $O(1/n)$ (see [2], [9]).

Our approach remains the same as in [6]: we relate the error estimates to the regularity of the value function. The improvement comes from a refinement of the quadratic estimates for the second order time derivative, in the spirit of Friedman and Kinderlehrer (see [3] and [5]). We also exploit the smoothness of the exercise boundary and its asymptotic properties close to maturity.

The paper is organized as follows. In the next Section we recall some results of [6]. Section 3 is devoted to estimates for the derivatives of the value function. The estimates are then used in Sections 4 and 5 to prove Theorem 1.1: in Section 4, we give an upper bound for $P_0^{(n)} - P_0$ and in Section 5, we derive the lower bound.

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2 The value function and the approximating process

As in [6], we introduce the modified value function

$$u(t,x) = e^{-rt}U(T-t, \ln(S_0) + \mu t + \sigma x), \quad t \geq 0, \quad x \in \mathbb{R}.$$

We have $P_0 = u(0,0)$ and $u(T,x) = e^{-rT}U(0, \ln(S_0) + \mu T + \sigma x) = e^{-rT}(K - S_0 e^{\mu T + \sigma x})^+$ and, for $t \in [0,T]$,

$$u(t,x) \geq e^{-rt}(K - S_0 e^{\mu t + \sigma x})^+ = e^{-rt} g(\mu_0 t + x). \tag{1}$$

We will need the European analogue of u, namely

$$\bar{u}(t,x) = e^{-rt}\bar{U}(T-t, \ln(S_0) + \mu t + \sigma x) = e^{-rT}\mathbb{E}\left(g(\mu_0 T + x + B_{T-t})\right), \quad t \geq 0, x \in \mathbb{R}.$$

We will also use the notation:

$$h = \frac{T}{n}.$$

With this notation, we have

$$B_t^{(n)} = \sqrt{h} \sum_{k=1}^{[t/h]} X_k, \quad 0 \le t \le T.$$

We have, for all $t \in \{0, h, 2h, \ldots, (n-1)h, nh = T\}$ (cf. Proposition 3.1 of [6]),

$$u(t, B_t^{(n)}) = u(0,0) + M_t + \sum_{j=1}^{t/h} \mathcal{D}u((j-1)h, B_{(j-1)h}^{(n)}),$$

where $(M_t)_{0 \le t \le T}$ is a martingale (with respect to the natural filtration of $B^{(n)}$), such that $M_0 = 0$, and

$$\mathcal{D}u(t,x) = \mathbb{E}\left(u\left(t+h, x+\sqrt{h}X_1\right)\right) - u(t,x), \quad 0 \le t \le T-h, \quad x \in \mathbb{R}.$$

For convenience, we introduce the following notation, for any function $u : [0,T] \times \mathbb{R} \to \mathbb{R}$,

$$\delta u = \frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}.$$

It is easy to check that, if u is smooth and X_1 bounded,

$$(1/h) \times \mathcal{D}u(t,x) = \delta u(t,x) + O(h).$$

Remark 2.1. The derivatives of u are related to those of U by the following formulas. We have

$$\frac{\partial u}{\partial t}(t,x) = e^{-rt}\left(-\frac{\partial U}{\partial t} + \mu\frac{\partial U}{\partial x} - rU\right)(T-t, \ln(S_0) + \mu t + \sigma x)$$

and

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2}(t,x) &= e^{-rt}\left(\frac{\partial^2 U}{\partial t^2} - 2\mu\frac{\partial^2 U}{\partial t \partial x} + \mu^2\frac{\partial^2 U}{\partial x^2}\right.\\ &\quad \left. + 2r\frac{\partial U}{\partial t} - 2r\mu\frac{\partial U}{\partial x} + r^2 U\right)(T-t, \ln(S_0) + \mu t + \sigma x).\end{aligned}$$

We also have

$$\begin{aligned}\delta u(t,x) = \frac{\partial u}{\partial t}(t,x) + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(t,x) &= e^{-rt}\left(-\frac{\partial U}{\partial t} + (A-r)U\right)(T-t, \ln(S_0) + \mu t + \sigma x)\\ &= e^{-rt}(A-r)\varphi(\ln(S_0) + \mu t + \sigma x)\mathbb{1}_{\{\ln(S_0)+\mu t+\sigma x \le \tilde{b}(T-t)\}},\end{aligned}$$

where the last equality follows from regularity results (see, for instance, [4]).

We will need a more precise description of $\mathcal{D}$, given by the following proposition (see Proposition 3.4 of [6]). For convenience, we denote by X a random variable with the same distribution as X_1, which is independent of the sequence $(X_n)_{n \ge 1}$.

Proposition 2.1. *Assume that (H4) is satisfied and that v is a function of class C^3 on $[0,T]\times\mathbb{R}$. For $0\le t\le T-h$ and $x\in\mathbb{R}$, define*

$$\tilde{\mathcal{D}}v(t,x)=2\int_0^{\sqrt{h}}d\xi\int_0^{\xi}dz\mathbb{E}\left[X\left(\xi-X^2(\xi-z)\right)\frac{\partial^2 v}{\partial t\partial x}(t+\xi^2,x+zX)\right].$$

We have

$$\mathcal{D}v(t,x)=\tilde{\mathcal{D}}v(t,x)+2\int_0^{\sqrt{h}}d\xi\int_0^{\xi}dz\mathbb{E}\left(X^2\delta v(t+\xi^2,x+zX)\right).$$

and

$$\tilde{\mathcal{D}}v(t,x)=2\int_0^{\sqrt{h}}d\xi\int_0^{\xi}dz(\xi-z)\mathbb{E}\left[X^2\left(\xi-X^2\frac{(\xi-z)}{2}\right)\frac{\partial^3 v}{\partial t\partial x^2}(t+\xi^2,x+zX)\right].$$

Remark 2.2. Note that, if $\delta v(s,x+zX)=0$ for all $s\in[t,t+h]$ and $z\in[0,\sqrt{h}]$, we have $v(t,x)=\mathcal{D}v(t,x)$.

From the last equality in Proposition 2.1, we derive the following estimates.

$$\begin{aligned}
\left|\tilde{\mathcal{D}}v(t,x)\right| &\le 2\int_0^{\sqrt{h}}\xi^2 d\xi\int_0^{\xi}dz\mathbb{E}\left[\left(X^2+\frac{X^4}{2}\right)\left|\frac{\partial^3 v}{\partial t\partial x^2}(t+\xi^2,x+zX)\right|\right]\\
&\le \sqrt{h}\int_0^{\sqrt{h}}2\xi d\xi\mathbb{E}\left[\int_0^{\xi}dz\left(X^2+\frac{X^4}{2}\right)\left|\frac{\partial^3 v}{\partial t\partial x^2}(t+\xi^2,x+zX)\right|\right]\\
&\le \sqrt{h}\int_t^{t+h}ds\mathbb{E}\left(\int dy\mathbb{1}_{\{|y-x|\le\sqrt{h}|X|\}}\left(|X|+\frac{|X|^3}{2}\right)\left|\frac{\partial^3 v}{\partial t\partial x^2}(s,y)\right|\right)\\
&= \sqrt{h}\int_t^{t+h}ds\int dy\mathbb{E}\left(\mathbb{1}_{\{|y-x|\le\sqrt{h}|X|\}}\left(|X|+\frac{|X|^3}{2}\right)\right)\left|\frac{\partial^3 v}{\partial t\partial x^2}(s,y)\right|
\end{aligned}$$

We know from Proposition 3.2 of [6] that, for every $k\in(1,3]$, there exists a positive constant C_k (which does not depend on X), such that, for all $y\in\mathbb{R}$, $n\ge 1$ and $j\in\{1,2,\dots,n\}$

$$\mathbb{E}\left(\left(|X|+\frac{|X|^3}{2}\right)\mathbb{1}_{\left\{\left|B_{jh}^{(n)}-y\right|\le\sqrt{h}|X|\right\}}\right) \le \frac{C_k}{\sqrt{j}}\frac{\mathbb{E}\left(|X|^3\right)\left(1+\mathbb{E}\,|X|^{3+k}\right)}{1+|y|^k}.$$

Hence, for $j=1,\dots,n-1$,

$$\begin{aligned}
\mathbb{E}\left(\left|\tilde{\mathcal{D}}v(jh,B_{jh}^{(n)})\right|\right) &\le \frac{C_{k,X}}{\sqrt{j}}\sqrt{h}\int_{jh}^{jh+h}ds\int\frac{dy}{1+|y|^k}\left|\frac{\partial^3 v}{\partial t\partial x^2}(s,y)\right|\\
&\le C_{k,X}h\sqrt{2}\int_{jh}^{jh+h}\frac{ds}{\sqrt{s}}\int\frac{dy}{1+|y|^k}\left|\frac{\partial^3 v}{\partial t\partial x^2}(s,y)\right|, \qquad (2)
\end{aligned}$$

where, for the last inequality, we used the inequality $jh\ge(j+1)h/2$.

3 Estimates for the second order time derivative

In this section, we refine the regularity results that we used in [6]. We first establish some elementary L_1-estimates. Then, we obtain a quadratic estimate for the second order time derivative of the difference $\tilde{U} = U - \bar{U}$. For the definition of the relevant weighted Sobolev spaces, we will use the notation

$$\nu_j(dx) = \frac{dx}{(1+x^2)^{j/2}}, \quad j > 1.$$

3.1 Some elementary L_1-estimates

Proposition 3.1. *Assume that the function φ is continuous and satisfies $\varphi \in L_1(\nu_j)$, $\varphi' \in L_1(\nu_j)$ and the second derivative φ'' is a Radon measure on $\mathbb{R}$, with $\int_{\mathbb{R}} \frac{|\varphi''(dz)|}{(1+z^2)^{j/2}} < \infty$. Let*

$$u_\varphi(t,x) = e^{-rt}\mathbb{E}_x(\varphi(X_t)), \quad t \geq 0, \quad x \in \mathbb{R}.$$

Then, for all $T > 0$, there exists a constant $C_T > 0$, such that

$$\forall t \in (0,T], \quad \left|\left|\frac{\partial^2 u_\varphi}{\partial t^2}(t,.)\right|\right|_{L_1(\nu_j)} \leq \frac{C_T}{t}.$$

We will easily deduce this proposition from the following lemma.

Lemma 3.1. *If ρ is a Radon measure on $\mathbb{R}$ and q a nonnegative integrable function on $\mathbb{R}$, we have*

$$||\rho * q||_{L_1(\nu_j)} \leq 2^{j/2} \int_{\mathbb{R}} \frac{|\rho(dz)|}{(1+z^2)^{j/2}} \int_{-\infty}^{\infty} q(x)(1+x^2)^{j/2} dx.$$

We also have, for any measurable function f on $\mathbb{R}$,

$$\forall y \in \mathbb{R}, \quad ||f(.-y)||_{L_1(\nu_j)} \leq 2^{j/2}(1+y^2)^{j/2}||f||_{L_1(\nu_j)}.$$

Proof: We have

$$\begin{aligned} ||\rho * q||_{L_1(\nu_j)} &\leq \int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^{j/2}} \int_{\mathbb{R}} |\rho(dz)| q(x-z) \\ &= \int_{\mathbb{R}} \frac{|\rho(dz)|}{(1+z^2)^{j/2}} \int_{-\infty}^{\infty} q(x-z) \frac{(1+z^2)^{j/2}}{(1+x^2)^{j/2}} dx \\ &= \int_{\mathbb{R}} \frac{|\rho(dz)|}{(1+z^2)^{j/2}} \int_{-\infty}^{\infty} q(x) \frac{(1+z^2)^{j/2}}{(1+(x+z)^2)^{j/2}} dx. \end{aligned}$$

Note that

$$z^2 \leq 2((x+z)^2 + x^2),$$

so that we deduce

$$\begin{aligned} \frac{1+z^2}{1+(x+z)^2} &\leq \frac{1+2(x+z)^2+2x^2}{1+(x+z)^2} \\ &\leq 2(1+x^2). \end{aligned}$$

Hence

$$||\rho * q||_{L_1(\nu_j)} \leq 2^{j/2} \int_{\mathbb{R}} \frac{|\rho(dz)|}{(1+z^2)^{j/2}} \int_{-\infty}^{\infty} q(x)(1+x^2)^{j/2} dx.$$

Similarly, we have, for any measurable function f and $y \in \mathbb{R}$,

$$\begin{aligned}
||f(.-y)||_{L_1(\nu_j)} &= \int |f(x-y)| \frac{dx}{(1+x^2)^{j/2}} \\
&= \int |f(x)| \frac{dx}{(1+(x+y)^2)^{j/2}} \\
&= \int |f(x)| \left(\frac{1+x^2}{1+(x+y)^2} \right)^{j/2} \frac{dx}{(1+x^2)^{j/2}} \\
&\leq \int |f(x)| \left(\frac{1+2(x+y)^2+2y^2}{1+(x+y)^2} \right)^{j/2} \frac{dx}{(1+x^2)^{j/2}} \\
&\leq 2^{j/2}(1+y^2)^{j/2} \, ||f||_{L_1(\nu_j)} .
\end{aligned}$$

◇

Proof of Proposition 3.1: We have

$$\begin{aligned}
u_\varphi(t,x) &= e^{-rt} \int_{-\infty}^{\infty} \varphi(x+y) \exp\left(-\frac{(y-\mu t)^2}{2\sigma^2 t} \right) \frac{dy}{\sigma\sqrt{2\pi t}} \\
&= e^{-rt} p_t * \varphi(x),
\end{aligned}$$

with

$$p_t(x) = \frac{1}{\sigma\sqrt{2\pi t}} \exp\left(-\frac{(x+\mu t)^2}{2\sigma^2 t} \right) = \frac{1}{\sigma\sqrt{t}} n\left(\frac{x+\mu t}{\sigma\sqrt{t}} \right).$$

Here, n denotes the standard normal density function.

On the other hand, we know that u_φ satisfies the equation

$$\frac{\partial u_\varphi}{\partial t} = (A-r)u_\varphi, \tag{3}$$

so that

$$\begin{aligned}
\frac{\partial u_\varphi}{\partial t}(t,.) &= e^{-rt}(A-r)p_t * \varphi \\
&= e^{-rt} p_t * [(A-r)\varphi].
\end{aligned}$$

It follows from our assumptions that $(A-r)\varphi$ is a Radon measure satisfying

$$\int_{\mathbb{R}} |(A-r)\varphi(dz)| \frac{1}{(1+z^2)^{j/2}} < \infty.$$

So that, using Lemma 3.1,

$$\begin{aligned}
\left|\left| \frac{\partial u_\varphi}{\partial t}(t,.) \right|\right|_{L_1(\nu_j)} &\leq C_j \int_{-\infty}^{\infty} p_t(x)(1+|x|^j)dx \\
&= C_j \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{t}} n\left(\frac{x+\mu t}{\sigma\sqrt{t}} \right)(1+|x|^j)dx \\
&= C_j \int_{-\infty}^{\infty} n(y)\,(1+|y\sigma\sqrt{t}-\mu t|^j)dy \\
&\leq C_j \left(1+t^j\right).
\end{aligned}$$

On the other hand, by differentiating (3), we have

$$\frac{\partial^2 u_\varphi}{\partial t^2} = (A-r)\frac{\partial u_\varphi}{\partial t} = e^{-rt}\left((A-r)p_t\right) * (A-r)\varphi.$$

Hence, using Lemma 3.1, and the definition of p_t,

$$\begin{aligned}\left\|\frac{\partial^2 u_\varphi}{\partial t^2}(t,.)\right\|_{L_1(\nu_j)} &\leq C_j \int_{-\infty}^{\infty} |(A-r)p_t(x)|(1+|x|^j)dx \\ &\leq \frac{C_j}{t}\left(1+t^j\right).\end{aligned}$$

◇

3.2 Quadratic estimates

Recall the notation:

$$U(t,x) = \sup_{\tau\in\mathcal{T}_{0,t}} \mathbb{E}_x\left(e^{-r\tau}\varphi(X_\tau)\right), \quad u_\varphi(t,x) = e^{-rt}\mathbb{E}_x(\varphi(X_t)), \quad t\geq 0, x\in\mathbb{R}.$$

with $\varphi(x) = (K-e^x)^+$. We now introduce the difference $\tilde{U} = U - u_\varphi$ (which corresponds to the early exercise premium). We have the following L_2-estimate for the second time derivative of $\tilde{U} = U - u_\varphi$.

Theorem 3.1. *Fix $T>0$ and $j>1$. There exists a constant $C>0$ such that, for all $\xi\in(0,T]$,*

$$\int_\xi^T (t-\xi)\left\|\frac{\partial^2\tilde{U}}{\partial t^2}(t,.)\right\|^2_{L_2(\nu_j)} dt \leq C\left(1+|\ln\xi|^\beta\right), \quad \textit{with } \beta = \begin{cases} 3/2, \textit{ if } d\leq r, \\ 1, \textit{ if } d>r. \end{cases}$$

For the proof of Theorem 3.1, we need some preliminary estimates on the derivatives $\frac{\partial^2\tilde{U}}{\partial x^2}$ and $\frac{\partial^2\tilde{U}}{\partial t\partial x}$.

Lemma 3.2. *Fix $T>0$ and $j>1$. For any $\varepsilon\in(0,1/4)$, there exists a constant $C>0$ such that, for all $t\in(0,T]$,*

$$\left\|\frac{\partial\tilde{U}}{\partial x}(t,.)\right\|_{L_2(\nu_j)} \leq C\sqrt{t} \quad \textit{and} \quad \left\|\frac{\partial^2\tilde{U}}{\partial x^2}(t,.)\right\|_{L_2(\nu_j)} \leq Ct^\varepsilon.$$

Proof: We know that $\tilde{U}$ solves the equation

$$-\frac{\partial\tilde{U}}{\partial t} + (A-r)\tilde{U} = h,$$

with initial condition $\tilde{U}(0,.)=0$, where h is given by

$$h(t,x) = (A-r)\varphi(x)\mathbb{1}_{\{x\leq\tilde{b}(t)\}}, \quad t>0, \quad x\in\mathbb{R}.$$

We have the following identity (which can be viewed as a form of the early exercise premium formula).

$$\tilde{U}(t,.) = -\int_0^t e^{-r(t-s)} p_{t-s} * h(s,.)ds,$$

where

$$p_t(x) = \frac{1}{\sigma\sqrt{2\pi t}} \exp\left(-\frac{(x+\mu t)^2}{2\sigma^2 t}\right) = \frac{1}{\sigma\sqrt{t}} n\left(\frac{x+\mu t}{\sigma\sqrt{t}}\right),$$

with n denoting the standard normal density function. It is straightforward to check that

$$\frac{\partial \tilde{U}}{\partial x}(t,.) = -\int_0^t e^{-r(t-s)} p_{t-s} * \frac{\partial h}{\partial x}(s,.)ds,$$

and, with the notation δ_z for the Dirac measure at a point z,

$$\begin{aligned} \frac{\partial h}{\partial x}(t,x) &= (A-r)\varphi'(x)\mathbb{1}_{\{x\leq \tilde{b}(t)\}} - (A-r)\varphi(x)\delta_{\tilde{b}(t)}(x) \\ &= -\kappa(t,x) + \gamma(t)\delta_{\tilde{b}(t)}(x), \end{aligned} \tag{4}$$

with $\kappa(t,x) = -(A-r)\varphi'(x)\mathbb{1}_{\{x\leq \tilde{b}(t)\}}$ and $\gamma(t) = -(A-r)\varphi(\tilde{b}(t))$. Note that κ is a bounded function on $(0,\infty)\times\mathbb{R}$ and γ is a continuous, nonnegative and bounded function on $(0,+\infty)$. At this stage, it is clear that $||p_{t-s} * \frac{\partial h}{\partial x}(s,.)||_\infty \leq C/\sqrt{t-s}$, so that

$$\left|\left|\frac{\partial \tilde{U}}{\partial x}(t,.)\right|\right|_{L_2(\nu_j)} \leq C\sqrt{t}.$$

On the other hand, we have

$$\left|\left|\frac{\partial^2 \tilde{U}}{\partial x^2}(t,.)\right|\right|_{L_2(\nu_j)} \leq \int_0^t e^{-r(t-s)} \left|\left|p'_{t-s} * \kappa(s,.)\right|\right|_{L_2(\nu_j)} ds + ||\zeta(t,.)||_{L_2(\nu_j)},$$

with

$$\begin{aligned} \zeta(t,.) &= \int_0^t e^{-r(t-s)}\gamma(s) p'_{t-s} * \delta_{\tilde{b}(s)} ds \\ &= \int_0^t e^{-r(t-s)}\gamma(s) p'_{t-s}(. - \tilde{b}(s))ds. \end{aligned}$$

We have, using Lemma 3.1,

$$\begin{aligned} \left|\left|p'_{t-s} * \kappa(s,.)\right|\right|_{L_2(\nu_j)} &= \left|\left|\int p'_{t-s}(y)\kappa(s,.-y)dy\right|\right|_{L_2(\nu_j)} \\ &\leq \int |p'_{t-s}(y)|\, ||\kappa(s,.-y)||_{L_2(\nu_j)}\, dy \\ &\leq 2^{j/4}\, ||\kappa(s,.)||_{L_2(\nu_j)} \int |p'_{t-s}(y)|(1+y^2)^{j/4} dy. \end{aligned}$$

Note that, since κ is bounded and $j > 1$, $\sup_{s>0} ||\kappa(s,.)||_{L_2(\nu_j)} < \infty$, so that, for some constant $C > 0$ (which may vary from line to line)

$$\begin{aligned}
\left|\left|p'_{t-s} * \kappa(s,.)\right|\right|_{L_2(\nu_j)} &\leq C \int |p'_{t-s}(y)|(1+y^2)^{j/4} dy \\
&= C \int \frac{1}{\sigma^2(t-s)} \left| n'\left(\frac{y+\mu(t-s)}{\sigma\sqrt{t-s}}\right)\right| (1+y^2)^{j/4} dy \\
&= C \int \frac{1}{\sigma\sqrt{t-s}} \left|n'(z)\right| (1+(-\mu(t-s)+\sigma\sqrt{t-s}z)^2)^{j/4} dz \\
&\leq \frac{C}{\sqrt{t-s}} \left(1+(t-s)^{j/2}\right).
\end{aligned}$$

Hence, if $0 < t < T$,

$$\int_0^t e^{-r(t-s)} \left|\left|p'_{t-s} * \kappa(s,.)\right|\right|_{L_2(\nu_j)} ds \leq C \int_0^t \frac{ds}{\sqrt{t-s}} = 2C\sqrt{t}.$$

We now estimate $||\zeta(t,.)||_{L_2(\nu_j)}$.

We have, using the boundedness of γ,

$$\begin{aligned}
|\zeta(t,x)| &= \left| \int_0^t e^{-r(t-s)} \gamma(s) p'_{t-s}(x - \tilde{b}(s)) ds \right| \\
&\leq C \int_0^t \frac{1}{\sigma^2(t-s)} \left| n'\left(\frac{x-\tilde{b}(s)+\mu(t-s)}{\sigma\sqrt{t-s}}\right)\right| ds
\end{aligned}$$

Recall that $n'(x) = -xn(x)$. Therefore

$$\begin{aligned}
|\zeta(t,x)| &\leq C \int_0^t \frac{|x-\tilde{b}(s)+\mu(t-s)|}{(t-s)^{3/2}} n\left(\frac{x-\tilde{b}(s)+\mu(t-s)}{\sigma\sqrt{t-s}}\right) ds \\
&\leq C \int_0^t \frac{ds}{\sqrt{t-s}} + C \int_0^t \frac{|x-\tilde{b}(s)|}{(t-s)^{3/2}} n\left(\frac{x-\tilde{b}(s)+\mu(t-s)}{\sigma\sqrt{t-s}}\right) ds.
\end{aligned}$$

Note that

$$\begin{aligned}
n(x_1+x_2) = n(x_1)\exp\left(-\frac{x_2^2}{2} - x_1x_2\right) &\leq n(x_1)\exp(-x_1x_2) \\
&\leq n(x_1)\exp\left(\frac{x_1^2}{4} + x_2^2\right) = n(x_1/\sqrt{2})e^{x_2^2}.
\end{aligned}$$

Hence, for $t \in (0,T)$,

$$|\zeta(t,x)| \leq C \int_0^t \frac{ds}{\sqrt{t-s}} + C_T \int_0^t \frac{|x-\tilde{b}(s)|}{(t-s)^{3/2}} n\left(\frac{x-\tilde{b}(s)}{\sqrt{2}\sigma\sqrt{t-s}}\right) ds.$$

Note that, for all $\alpha > 0$, there exists $C_\alpha > 0$, such that, for all $y \in \mathbb{R}$, $n(y/\sqrt{2}) \leq C_\alpha/|y|^{2\alpha}$. Hence, for $t \in (0,T)$,

$$\begin{aligned}
|\zeta(t,x)| &\leq C\sqrt{t} + C_\alpha \int_0^t \frac{|x-\tilde{b}(s)|}{(t-s)^{3/2}} \frac{(t-s)^\alpha}{|x-\tilde{b}(s)|^{2\alpha}} ds \\
&= C\sqrt{t} + C_\alpha \int_0^t \frac{(t-s)^{\alpha-\frac{3}{2}}}{|x-\tilde{b}(s)|^{2\alpha-1}} ds \\
&= C\sqrt{t} + C_\alpha t^{\alpha-\frac{1}{2}} \int_0^1 \frac{1}{(1-u)^{\frac{3}{2}-\alpha}|x-\tilde{b}(tu)|^{2\alpha-1}} du.
\end{aligned}$$

Now, take $\alpha = \frac{1}{2} + \varepsilon$ (with $0 < \varepsilon < 1/4$) and put $\beta(t,x) = |x - \tilde{b}(t)|^{1-2\alpha} = |x - \tilde{b}(t)|^{-2\varepsilon}$. We get

$$||\zeta(t,.)||_{L_2(\nu_j)} \leq C\sqrt{t} + Ct^{\varepsilon} \int_0^1 \frac{1}{(1-u)^{1-\varepsilon}} ||\beta(tu,.)||_{L_2(\nu_j)} \, du.$$

Using Lemma 3.1, we have

$$||\beta(tu,.)||^2_{L_2(\nu_j)} \leq 2^{j/2} \left(1 + \tilde{b}(tu)^2\right)^{j/2} \int \frac{1}{|x|^{4\varepsilon}} \frac{dx}{(1+x^2)^{j/2}}.$$

Since $\varepsilon < 1/4$, the integral on the righthand side is finite, and the lemma easily follows. $\diamond$

We now turn to the study of $\frac{\partial^2 \tilde{U}}{\partial t \partial x}$. Recall that $\partial U/\partial t$ solves the parabolic equation $-\partial v/\partial t + (A - r)v = 0$ in the set $\{(t,x) \mid t > 0, x > \tilde{b}(t)\}$. Since the exercise boundary is differentiable and $\partial U/\partial t$ is continuous and vanishes on the exercise boundary, it follows that $\frac{\partial^2 U}{\partial t \partial x}$ is continuous "up to the boundary", i.e. on the set $\{(t,x) \mid t > 0, x \geq \tilde{b}(t)\}$ (see [3], Lemma 4.5). We first show that $\frac{\partial^2 U}{\partial t \partial x}$ is nonnegative along the exercise boundary.

Lemma 3.3. *We have, for any $t > 0$,*

$$\frac{\partial^2 U}{\partial t \partial x}(t, \tilde{b}(t)) \geq 0.$$

Proof: We have, for all $t > 0$, due to the *smooth fit* property,

$$\frac{\partial U}{\partial x}(t, \tilde{b}(t)) = \varphi'(\tilde{b}(t)),$$

so that, by differentiating with respect to t,

$$\frac{\partial^2 U}{\partial t \partial x}(t, \tilde{b}(t)) + \frac{\partial^2 U}{\partial x^2}(t, \tilde{b}(t))\tilde{b}'(t) = \varphi''(\tilde{b}(t))\tilde{b}'(t)$$

and

$$\frac{\partial^2 U}{\partial t \partial x}(t, \tilde{b}(t)) = -\left(\frac{\partial^2 U}{\partial x^2}(t, \tilde{b}(t)) - \varphi''(\tilde{b}(t))\right)\tilde{b}'(t).$$

Observe that, for each $t > 0$, the function $x \mapsto U(t,x) - \varphi(x)$ is C^2 on the interval $[\tilde{b}(t), \infty)$ and has a minimum at $\tilde{b}(t)$. Therefore, its second derivative must be nonnegative at this point. Since $\tilde{b}'(t) \leq 0$, the lemma is proved. $\diamond$

Lemma 3.4. *Fix $T > 0$ and $j > 1$. There exists a constant $C > 0$ such that, for all $t_1 \in (0, T \wedge 1]$,*

$$\int_{t_1}^T \left|\left| \frac{\partial^2 \tilde{U}}{\partial t \partial x}(t,.) \right|\right|^2_{L_2(\nu_j)} dt \leq C \ln(1/t_1).$$

For the proof of Lemma 3.4, we will need the bilinear form associated with the operator $A - r$.

We first introduce the relevant weighted Sobolev spaces. For $j > 1$, let $H_j = L^2(\mathbb{R}, \nu_j)$ and $V_j = \{f \in H_j \mid f' \in H_j\}$. The inner product on H_j will be denoted by $(\cdot, \cdot)_j$ and the associated norm by $|\cdot|_j$. The natural norm on V_j will be denoted by $||\cdot||_j$. Thus, we have

$$|f|_j^2 = \int_{-\infty}^{+\infty} f^2(x) \frac{dx}{(1+x^2)^{j/2}},$$

and $||f||_j^2 = |f|_j^2 + |f'|_j^2$.

Recall that the partial differential operator A is defined by

$$A = \frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} + \mu \frac{\partial}{\partial x}.$$

We associate with the operator $A - r$ a bilinear functional on V_j, defined by

$$\begin{aligned} a_j(f,g) &= \frac{\sigma^2}{2} \int_{-\infty}^{\infty} f'(x) g'(x) \frac{dx}{(1+x^2)^{j/2}} - \frac{j\sigma^2}{2} \int_{-\infty}^{\infty} f'(x) g(x) \frac{x}{(1+x^2)^{(j/2)+1}} dx \\ &\quad -\mu \int_{-\infty}^{\infty} f'(x) g(x) \frac{dx}{(1+x^2)^{j/2}} + r \int_{-\infty}^{\infty} f(x) g(x) \frac{dx}{(1+x^2)^{j/2}}, \end{aligned}$$

so that, if $f' \in V_j$,

$$a_j(f,g) = -((A-r)f, g)_j.$$

It will be convenient to write $a_j(f,g)$ as $a_j(f,g) = \tilde{a}_j(f,g) + \bar{a}_j(f,g)$, with

$$\tilde{a}_j(f,g) = \frac{\sigma^2}{2} \left[(f', g')_j + (f,g)_j \right] \quad \text{and} \quad \bar{a}_j(f,g) = a_j(f,g) - \tilde{a}_j(f,g). \tag{5}$$

With these notations, it is easy to check that $|\bar{a}_j(f,g)| \leq C||f||_j |g|_j$ and $|\bar{a}_j(f,g)| \leq C||g||_j |f|_j$, for some constant C which does not depend on f nor g.

Proof of Lemma 3.4: In order to rule out regularity issues, we introduce a C^∞, nonnegative function ρ on $\mathbb{R} \times \mathbb{R}$, with $\int \rho(t,x) dt dx = 1$ and supp $\rho \subset [-1,0] \times [-1,+1]$ and set, for any positive integer m, $\rho_m(t,x) = m^2 \rho(mt, mx)$.

Now, let $W = \frac{\partial \tilde{U}}{\partial x}$, $W_m = W * \rho_m$, and $h_m = h * \rho_m$. For each $m > 0$, the functions W_m, h_m are C^∞ with bounded derivatives and we have

$$-\frac{\partial W_m}{\partial t} + (A-r) W_m = \frac{\partial h_m}{\partial x}.$$

Multiply by $\partial W_m / \partial t$ and integrate with respect to ν_j to get, for any fixed $t > 0$,

$$-\left(\frac{\partial W_m}{\partial t}(t,.), \frac{\partial W_m}{\partial t}(t,.) \right)_j - a_j \left(W_m(t,.), \frac{\partial W_m}{\partial t}(t,.) \right) = \int \frac{\partial h_m}{\partial x}(t,x) \frac{\partial W_m}{\partial t}(t,x) \nu_j(dx).$$

Note that

$$\begin{aligned} a_j \left(W_m(t,.), \frac{\partial W_m}{\partial t}(t,.) \right) &= \tilde{a}_j \left(W_m(t,.), \frac{\partial W_m}{\partial t}(t,.) \right) + \bar{a}_j \left(W_m(t,.), \frac{\partial W_m}{\partial t}(t,.) \right) \\ &= \frac{1}{2} \frac{d}{dt} \left(\tilde{a}_j \left(W_m(t,.), W_m(t,.) \right) \right) + \bar{a}_j \left(W_m(t,.), \frac{\partial W_m}{\partial t}(t,.) \right). \end{aligned}$$

By integrating with respect to time, we get, if $0 < t_1 < T$,

$$-\int_{t_1}^{T} \left| \frac{\partial W_m}{\partial t}(t,.) \right|_j^2 dt + \frac{1}{2} \left(\tilde{a}_j \left(W_m(t_1,.), W_m(t_1,.) \right) - \tilde{a}_j \left(W_m(T,.), W_m(T,.) \right) \right) =$$
$$\int_{t_1}^{T} \bar{a}_j \left(W_m(t,.), \frac{\partial W_m}{\partial t}(t,.) \right) dt + \int_{t_1}^{T} \left(\frac{\partial h_m}{\partial x}(t,.), \frac{\partial W_m}{\partial t}(t,.) \right)_j dt.$$

Hence

$$\begin{aligned}
\int_{t_1}^{T} \left| \frac{\partial W_m}{\partial t}(t,.) \right|_j^2 dt &\leq \frac{1}{2} \tilde{a}_j \left(W_m(t_1,.), W_m(t_1,.) \right) - \int_{t_1}^{T} \bar{a}_j \left(W_m(t,.), \frac{\partial W_m}{\partial t}(t,.) \right) dt \\
&\quad - \int_{t_1}^{T} \left(\frac{\partial h_m}{\partial x}(t,.), \frac{\partial W_m}{\partial t}(t,.) \right)_j dt \\
&\leq \frac{1}{2} \tilde{a}_j \left(W_m(t_1,.), W_m(t_1,.) \right) + C \int_{t_1}^{T} ||W_m(t,.)||_j \left| \frac{\partial W_m}{\partial t}(t,.) \right|_j dt \\
&\quad - \int_{t_1}^{T} \left(\frac{\partial h_m}{\partial x}(t,.), \frac{\partial W_m}{\partial t}(t,.) \right)_j dt.
\end{aligned}$$

Using the inequality

$$2 ||W_m(t,.)||_j \left| \frac{\partial W_m}{\partial t}(t,.) \right|_j \leq \varepsilon \left| \frac{\partial W_m}{\partial t}(t,.) \right|_j^2 + \frac{1}{\varepsilon} ||W_m(t,.)||_j^2,$$

we get

$$\begin{aligned}
\frac{1}{2} \int_{t_1}^{T} \left| \frac{\partial W_m}{\partial t}(t,.) \right|_j^2 dt &\leq \frac{1}{2} \tilde{a}_j \left(W_m(t_1,.), W_m(t_1,.) \right) + C \int_{t_1}^{T} ||W_m(t,.)||_j^2 dt \\
&\quad - \int_{t_1}^{T} \left(\frac{\partial h_m}{\partial x}(t,.), \frac{\partial W_m}{\partial t}(t,.) \right)_j dt.
\end{aligned}$$

We have

$$\tilde{a}_j \left(W_m(t_1,.), W_m(t_1,.) \right) \leq C ||W_m(t_1,.)||_j^2$$

and, using Lemma 3.1,

$$\begin{aligned}
||W_m(t_1,.)||_j &\leq \int \rho_m(t_1 - t, y) ||W(t, . - y)||_j \, dtdy \\
&\leq \int \rho_m(t_1 - t, y) 2^{j/4} (1 + y^2)^{j/4} ||W(t,.)||_j \, dtdy.
\end{aligned}$$

Using Lemma 3.2, we have (for $\varepsilon \in (0, 1/4)$) $||W(t,.)||_j \leq C t^{\varepsilon}$. Hence

$$\begin{aligned}
\tilde{a}_j \left(W_m(t_1,.), W_m(t_1,.) \right) &\leq C \int \rho_m(t_1 - t, y) 2^{j/2} (1 + y^2)^{j/2} t^{2\varepsilon} dtdy \\
&= C \int \rho_m(t, y) 2^{j/2} (1 + y^2)^{j/2} |t_1 - t|^{2\varepsilon} dtdy \\
&= C \int \rho(t, y) 2^{j/2} \left(1 + \frac{y^2}{m^2} \right)^{j/2} \left| t_1 - \frac{t}{m} \right|^{2\varepsilon} dtdy \\
&\leq C(1 + t_1^{2\varepsilon}).
\end{aligned}$$

We also have

$$\begin{aligned}\int_{t_1}^T ||W_m(t,.)||_j^2\,dt &= \int_{t_1}^T \left|\left|\int dsdy\rho_m(t-s,y)W(s,.-y)\right|\right|_j^2 dt\\ &\le \int_{t_1}^T \int dsdy\rho_m(t-s,y)\,||W(s,.-y)||_j^2\,dt\\ &\le \int_{t_1}^T dt\int dsdy\rho_m(t-s,y)2^{j/2}(1+y^2)^{j/2}\,||W(s,.)||_j^2\\ &\le \int_{t_1}^{T+\frac{1}{m}} ds\,||W(s,.)||_j^2\int dtdy\rho_m(t-s,y)2^{j/2}(1+y^2)^{j/2}.\end{aligned}$$

Since $\int_0^{T+1}||W(s,.)||_j^2\,ds<\infty$, we deduce that

$$\frac{1}{2}\int_{t_1}^T\left|\frac{\partial W_m}{\partial t}(t,.)\right|_j^2 dt \le C\left(1+t_1^{2\varepsilon}\right)-\int_{t_1}^T\left(\frac{\partial h_m}{\partial x}(t,.),\frac{\partial W_m}{\partial t}(t,.)\right)_j dt.$$

It follows from the proof of Lemma 3.2 (see (4)) that

$$\frac{\partial h_m}{\partial x}(t,x)=-\kappa_m(t,x)+\gamma_m(t,x),$$

where $\kappa_m=\kappa*\rho_m$, and κ is a bounded function, and

$$\gamma_m(t,x)=\int\rho_m(t-\tau,x-\tilde{b}(\tau))\gamma(\tau)d\tau.$$

Hence

$$\begin{aligned}-\int_{t_1}^T\left(\frac{\partial h_m}{\partial x}(t,.),\frac{\partial W_m}{\partial t}(t,.)\right)_j dt &\le C\int_{t_1}^T\left|\frac{\partial W_m}{\partial t}(t,.)\right|_j dt-\int_{t_1}^T\left(\gamma_m(t,.),\frac{\partial W_m}{\partial t}(t,.)\right)_j dt\\ &\le \frac{1}{4}\int_{t_1}^T\left|\frac{\partial W_m}{\partial t}(t,.)\right|_j^2 dt+C^2T-\int_{t_1}^T\left(\gamma_m(t,.),\frac{\partial W_m}{\partial t}(t,.)\right)_j dt,\end{aligned}$$

where we have used $C\left|\frac{\partial W_m}{\partial t}(t,.)\right|_j\le\frac{1}{4}\left|\frac{\partial W_m}{\partial t}(t,.)\right|_j^2+C^2$. Therefore

$$\frac{1}{4}\int_{t_1}^T\left|\frac{\partial W_m}{\partial t}(t,.)\right|_j^2 dt \le C\left(1+t_1^{2\varepsilon}\right)-\int_{t_1}^T\left(\gamma_m(t,.),\frac{\partial W_m}{\partial t}(t,.)\right)_j dt. \tag{6}$$

Note that

$$\frac{\partial W_m}{\partial t}=\frac{\partial^2 U_m}{\partial t\partial x}-\frac{\partial^2 u_m}{\partial t\partial x},$$

where

$$U_m=U*\rho_m \quad\text{and}\quad u_m=u_\varphi*\rho_m,$$

so that

$$-\int_{t_1}^T\left(\gamma_m(t,.),\frac{\partial W_m}{\partial t}(t,.)\right)_j dt=J_m^{(1)}+J_m^{(2)},$$

with

$$J_m^{(1)} = -\int_{t_1}^{T} \left(\gamma_m(t,.), \frac{\partial^2 U_m}{\partial t \partial x}(t,.)\right)_j dt \quad \text{and} \quad J_m^{(2)} = \int_{t_1}^{T} \left(\gamma_m(t,.), \frac{\partial^2 u_m}{\partial t \partial x}(t,.)\right)_j dt.$$

We have

$$\begin{aligned}
J_m^{(1)} &= -\int_{t_1}^{T} dt \int \frac{dx}{(1+x^2)^{j/2}} \gamma_m(t,x) \frac{\partial^2 U_m}{\partial t \partial x}(t,x) \\
&= -\int_{t_1}^{T} dt \int \frac{dx}{(1+x^2)^{j/2}} \int d\tau \gamma(\tau) \rho_m(t-\tau, x-\tilde{b}(\tau)) \frac{\partial^2 U_m}{\partial t \partial x}(t,x) \\
&= -\int d\tau \int ds \int dy \mathbb{1}_{\{t_1<\tau+s<T\}} \gamma(\tau) \rho_m(s,y) \frac{\partial^2 U_m}{\partial t \partial x}(\tau+s, \tilde{b}(\tau)+y) \frac{1}{(1+(y+\tilde{b}(\tau))^2)^{j/2}} \\
&= -\int_{t_1}^{T+\frac{1}{m}} \gamma(\tau) \eta_m(\tau) d\tau,
\end{aligned}$$

where

$$\begin{aligned}
\eta_m(\tau) &= \int_{t_1-\tau}^{T-\tau} ds \int \frac{dy}{(1+(y+\tilde{b}(\tau))^2)^{j/2}} \rho_m(s,y) \frac{\partial^2 U_m}{\partial t \partial x}(\tau+s, \tilde{b}(\tau)+y) \\
&= \int_{t_1-\tau}^{T-\tau} ds \int \frac{\rho_m(s,y)dy}{(1+(y+\tilde{b}(\tau))^2)^{j/2}} \int\int ds'dy' \rho_m(s',y') \frac{\partial^2 U}{\partial t \partial x}(\tau+s-s', \tilde{b}(\tau)+y-y').
\end{aligned}$$

Note that $\frac{\partial^2 U}{\partial t \partial x} = 0$ on the open set $S = \{(t,x) \mid t > 0, x < \tilde{b}(t)\}$ (which is the interior set of the stopping region), so that

$$\eta_m(\tau) = \int_{t_1-\tau}^{T-\tau} ds \int \frac{\rho_m(s,y)dy}{(1+(y+\tilde{b}(\tau))^2)^{j/2}} \int\int ds'dy' \rho_m(s',y') \bar{W}(\tau, s-s', y-y'),$$

where

$$\bar{W}(\tau,\theta,z) = \frac{\partial^2 U}{\partial t \partial x}(\tau+\theta, \tilde{b}(\tau)+z) \mathbb{1}_{\{\tilde{b}(\tau)+z \geq \tilde{b}(\tau+\theta)\}}.$$

Since $\frac{\partial^2 U}{\partial t \partial x}(\tau, \tilde{b}(\tau)) \geq 0$, for $\tau > 0$, we have

$$\eta_m(\tau) \geq \int_{t_1-\tau}^{T-\tau} ds \int\int\int \frac{\rho_m(s,y)dy \rho_m(s',y')ds'dy'}{\left(1+(y+\tilde{b}(\tau))^2\right)^{j/2}} D(\tau, s-s', y-y'),$$

where

$$D(\tau,\theta,z) = \left(\frac{\partial^2 U}{\partial t \partial x}(\tau+\theta, \tilde{b}(\tau)+z) - \frac{\partial^2 U}{\partial t \partial x}(\tau, \tilde{b}(\tau))\right) \mathbb{1}_{\{\tilde{b}(\tau)+z \geq \tilde{b}(\tau+\theta)\}}.$$

Hence (since $\gamma \geq 0$)

$$J_m^{(1)} \leq -\int_{t_1}^{T+\frac{1}{m}} \gamma(\tau) \varepsilon_m(\tau) d\tau,$$

with

$$\varepsilon_m(\tau) = \int_{t_1-\tau}^{T-\tau} ds \int\int\int \frac{\rho_m(s,y)dy \rho_m(s',y')ds'dy'}{(1+(y+\tilde{b}(\tau))^2)^{j/2}} D(\tau, s-s', y-y').$$

We have

$$|\varepsilon_m(\tau)| \leq \int_{t_1-\tau}^{T-\tau} D_m(\tau)ds = (T-t_1)D_m(\tau),$$

where

$$D_m(\tau) = \sup_{|\theta|\leq 1/m, |z|\leq 2/m} \left(|D(\tau,\theta,z)| \, \mathbb{1}_{\{\tilde{b}(\tau)+z \geq \tilde{b}(\tau+\theta)\}} \right).$$

Due to the continuity properties of $\frac{\partial^2 U}{\partial t \partial x}$, as $m \to \infty$, the function D_m converges to 0, uniformly on the interval $[t_1, T+1]$. Therefore, we have

$$\limsup_{m\to\infty} J_m^{(1)} \leq 0. \tag{7}$$

We now examine $J_m^{(2)}$. We have, using the boundedness of γ,

$$\begin{aligned} |J_m^{(2)}| &\leq \int_{t_1}^T dt \int \frac{dx}{(1+x^2)^{j/2}} |\gamma_m(t,x)| \left| \frac{\partial^2 u_m}{\partial t \partial x}(t,x) \right| \\ &\leq C \int_{t_1}^T dt \int \frac{dx}{(1+x^2)^{j/2}} \int d\tau \rho_m(t-\tau, x-\tilde{b}(\tau)) \left| \frac{\partial^2 u_m}{\partial t \partial x}(t,x) \right|. \end{aligned}$$

Note that

$$\left\| \frac{\partial^2 u_m}{\partial t \partial x}(t,.) \right\|_\infty \leq \left\| \frac{\partial^2 u_\varphi}{\partial t \partial x}(t,.) \right\|_\infty \leq \frac{C}{t}.$$

Hence

$$|J_m^{(2)}| \leq C \int_{t_1}^T \frac{dt}{t} \int dx \int d\tau \rho_m(t-\tau, x-\tilde{b}(\tau)) = C \ln \frac{T}{t_1}. \tag{8}$$

It follows from (6), (7) and (8) that

$$\limsup_{m\to\infty} \int_{t_1}^T \left| \frac{\partial W_m}{\partial t}(t,.) \right|_j^2 dt \leq C \left(1 + t_1^{2\varepsilon} + \ln \frac{T}{t_1} \right),$$

which proves the lemma. ◇

3.3 Proof of Theorem 3.1

For the proof of Theorem 3.1, we will work on the equation satisfied by $\partial \tilde{U}/\partial t$. Let

$$V = \frac{\partial \tilde{U}}{\partial t}.$$

We have

$$-\frac{\partial V}{\partial t} + (A-r)V = \frac{\partial h}{\partial t},$$

where

$$h(t,x) = (A-r)\varphi(x)\mathbb{1}_{\{x \leq \tilde{b}(t)\}}, \quad t > 0, \quad x \in \mathbb{R}.$$

The following lemma will clarify the computation of the derivative $\partial h/\partial t$ in the sense of distributions.

Lemma 3.5. *Define the function I on $(0,+\infty)\times\mathbb{R}$ by*

$$I(t,x) = \mathbb{1}_{\{x\leq \tilde{b}(t)\}}, \quad t>0, \quad x\in\mathbb{R}.$$

The distribution $\partial I/\partial t$ applied to a compactly supported C^∞ function ρ on $(0,+\infty)\times\mathbb{R}$ is given by

$$\langle \frac{\partial I}{\partial t}, \rho\rangle = \int \tilde{b}'(t)\rho(t,\tilde{b}(t))dt.$$

This can be written (less precisely): $\frac{\partial I}{\partial t}(t,.) = \tilde{b}'(t)\delta_{\tilde{b}(t)}$.

Proof: We have

$$\begin{aligned}\langle \frac{\partial I}{\partial t}, \rho\rangle &= -\langle I, \frac{\partial \rho}{\partial t}\rangle \\ &= -\int dt \int dx I(t,x)\frac{\partial \rho}{\partial t}(t,x)\end{aligned}$$

Let J be the range of $\tilde{b}$. We have $J=(\tilde{b}(\infty),\tilde{b}(0))$. Note that, if $x\leq\tilde{b}(\infty)$, $I(t,x)=1$ for all $t>0$ and if $x\geq\tilde{b}(0)$ $I(t,x)=0$ for all $t>0$, so that, in both cases, $\int I(t,x)\frac{\partial\rho}{\partial t}(t,x)dt=0$. Therefore

$$\begin{aligned}\langle \frac{\partial I}{\partial t}, \rho\rangle &= -\int_J dx\int dt \mathbb{1}_{\{x\leq\tilde{b}(t)\}}\frac{\partial\rho}{\partial t}(t,x) \\ &= -\int_J dx\int dt \mathbb{1}_{\{t\leq\tilde{b}^{-1}(x)\}}\frac{\partial\rho}{\partial t}(t,x) \\ &= -\int_J dx \rho(\tilde{b}^{-1}(x),x) \\ &= \int \tilde{b}'(t)\rho(t,\tilde{b}(t))dt.\end{aligned}$$

Here, we have used the fact that $\tilde{b}$ is strictly decreasing (which is proved in [8]), but we can also approximate $\tilde{b}$ by the strictly decreasing functions $\tilde{b}_\varepsilon(t)=-\varepsilon t+\tilde{b}(t)$ to derive the formula. In fact, we only need $\tilde{b}$ to be C^1: indeed, we can replace $\tilde{b}(t)$ by $\tilde{b}_\mu(t)=-\mu t+\tilde{b}(t)$ and choose μ so that $\tilde{b}_\mu$ is strictly increasing in a neighborhood of the time projection of the support of ρ. $\diamond$

We now proceed with the proof of Theorem 3.1. As in the proof of Lemma 3.4, we introduce a regularizing sequence ρ_m, and set

$$V_m = V*\rho_m \quad \text{and} \quad \chi_m = \frac{\partial h}{\partial t}*\rho_m,$$

so that

$$-\frac{\partial V_m}{\partial t} + (A-r)V_m = \chi_m$$

Note that the functions V_m, χ_m are C^∞ with bounded derivatives. For any fixed $t>0$, multiply by $\partial V_m/\partial t$ and integrate with respect to ν_j to get

$$-\left|\frac{\partial V_m}{\partial t}(t,.)\right|_j^2 - a_j\left(V_m(t,.),\frac{\partial V_m}{\partial t}(t,.)\right) = \int \chi_m(t,x)\frac{\partial V_m}{\partial t}(t,x)\nu_j(dx).$$

We have

$$a_j\left(V_m(t,.),\frac{\partial V_m}{\partial t}(t,.)\right) \;=\; \frac{1}{2}\frac{d}{dt}\left(\tilde{a}_j\left(V_m(t,.),V_m(t,.)\right)\right)+\bar{a}_j\left(V_m(t,.),\frac{\partial V_m}{\partial t}(t,.)\right).$$

By integrating with respect to time, we get, if $0<t_1<T$,

$$-\int_{t_1}^{T}\left|\frac{\partial V_m}{\partial t}(t,.)\right|_j^2 dt+\frac{1}{2}\left[\tilde{a}_j\left(V_m(t_1,.),V_m(t_1,.)\right)-\tilde{a}_j\left(V_m(T,.),V_m(T,.)\right)\right]=$$
$$\int_{t_1}^{T}\bar{a}_j\left(V_m(t,.),\frac{\partial V_m}{\partial t}(t,.)\right)dt+\int_{t_1}^{T}\left(\chi_m(t,.),\frac{\partial V_m}{\partial t}(t,.)\right)_j dt.$$

Hence

$$\begin{aligned}\int_{t_1}^{T}\left|\frac{\partial V_m}{\partial t}(t,.)\right|_j^2 dt &\le C\left|\left|V_m(t_1,.)\right|\right|_j^2-\int_{t_1}^{T}\bar{a}_j\left(V_m(t,.),\frac{\partial V_m}{\partial t}(t,.)\right)dt\\ &\quad-\int_{t_1}^{T}\left(\chi_m(t,.),\frac{\partial V_m}{\partial t}(t,.)\right)_j dt\\ &\le C\left(\left|\left|V_m(t_1,.)\right|\right|_j^2+\int_{t_1}^{T}\left|\left|V_m(t,.)\right|\right|_j\left|\frac{\partial V_m}{\partial t}(t,.)\right|_j dt\right)+J_m(t_1,T),\end{aligned}$$

with

$$J_m(t_1,T)=-\int_{t_1}^{T}\left(\chi_m(t,.),\frac{\partial V_m}{\partial t}(t,.)\right)_j dt.$$

Using the inequality

$$2\left|\left|V_m(t,.)\right|\right|_j\left|\frac{\partial V_m}{\partial t}(t,.)\right|_j\le\varepsilon\left|\frac{\partial V_m}{\partial t}(t,.)\right|_j^2+\frac{1}{\varepsilon}\left|\left|V_m(t,.)\right|\right|_j^2,$$

we derive

$$\frac{1}{2}\int_{t_1}^{T}\left|\frac{\partial V_m}{\partial t}(t,.)\right|_j^2 dt \;\le\; C\left(\left|\left|V_m(t_1,.)\right|\right|_j^2+\int_{t_1}^{T}\left|\left|V_m(t,.)\right|\right|_j^2 dt\right)+J_m(t_1,T). \tag{9}$$

We now study $J_m(t_1,T)$. Note that, for any fixed $t>0$,

$$\left(\chi_m(t,.),\frac{\partial V_m}{\partial t}(t,.)\right)_j \;=\; \int\nu_j(dx)\frac{\partial V_m}{\partial t}(t,x)\frac{\partial h}{\partial t}*\rho_m(t,x)$$

We have

$$\frac{\partial h}{\partial t}(t,.)=(A-r)\varphi\frac{\partial I}{\partial t}(t,.),$$

so that, using Lemma 3.5, and the notation $\gamma(t)=-(A-r)\varphi(\tilde{b}(t))$

$$\frac{\partial h}{\partial t}*\rho_m(t,x)=-\int d\tau\rho_m(t-\tau,x-\tilde{b}(\tau))\tilde{b}'(\tau)\gamma(\tau),$$

Recall that $\gamma(\tau) \geq 0$. Hence

$$\begin{aligned}\left(\chi_m(t,.), \frac{\partial V_m}{\partial t}(t,.)\right)_j &= -\int \nu_j(dx)\frac{\partial V_m}{\partial t}(t,x)\int d\tau \rho_m(t-\tau, x-\tilde{b}(\tau))\tilde{b}'(\tau)\gamma(\tau) \\ &= -\int d\tau \int \frac{dx}{(1+x^2)^{j/2}}\frac{\partial V_m}{\partial t}(t,x)\rho_m(t-\tau, x-\tilde{b}(\tau))\tilde{b}'(\tau)\gamma(\tau) \\ &= -\int d\tau \int \frac{dy}{(1+(y+\tilde{b}(\tau))^2)^{j/2}}\frac{\partial V_m}{\partial t}(t,y+\tilde{b}(\tau))\rho_m(t-\tau, y)\tilde{b}'(\tau)\gamma(\tau).\end{aligned}$$

Going back to $J_m(t_1, T)$, we have

$$J_m(t_1,T) = -\int_{t_1}^T dt \int d\tau \int dy \frac{\partial V_m}{\partial t}(t, y+\tilde{b}(\tau))\rho_m(t-\tau, y)\bar{\gamma}_j(\tau, y),$$

with

$$\bar{\gamma}_j(\tau, y) = -\frac{1}{(1+(y+\tilde{b}(\tau))^2)^{j/2}}\tilde{b}'(\tau)\gamma(\tau).$$

Note that $\bar{\gamma}_j(\tau, y) \geq 0$. We have

$$\begin{aligned}J_m(t_1,T) &= -\int d\tau \int dt \int dy \mathbb{1}_{\{t_1<t<T\}}\frac{\partial V_m}{\partial t}(t, y+\tilde{b}(\tau))\rho_m(t-\tau, y)\bar{\gamma}_j(\tau, y) \\ &= -\int d\tau \int ds \int dy \mathbb{1}_{\{t_1<\tau+s<T\}}\frac{\partial V_m}{\partial t}(\tau+s, y+\tilde{b}(\tau))\rho_m(s, y)\bar{\gamma}_j(\tau, y).\end{aligned}$$

Observe that

$$\frac{d}{d\tau}\left(V_m(\tau+s, y+\tilde{b}(\tau))\right) = \frac{\partial V_m}{\partial t}(\tau+s, y+\tilde{b}(\tau)) + \frac{\partial V_m}{\partial x}(\tau+s, y+\tilde{b}(\tau))\tilde{b}'(\tau),$$

so that

$$\frac{\partial V_m}{\partial t}(\tau+s, y+\tilde{b}(\tau)) = \frac{d}{d\tau}\left(V_m(\tau+s, y+\tilde{b}(\tau))\right) - \frac{\partial V_m}{\partial x}(\tau+s, y+\tilde{b}(\tau))\tilde{b}'(\tau).$$

Hence

$$J_m(t_1,T) = \hat{J}_m(t_1,T) + \bar{J}_m(t_1,T),$$

with

$$\hat{J}_m(t_1,T) = -\int d\tau \int ds \int dy \mathbb{1}_{\{t_1<\tau+s<T\}}\frac{d}{d\tau}\left(V_m(\tau+s, y+\tilde{b}(\tau))\right)\rho_m(s, y)\bar{\gamma}_j(\tau, y)$$

and

$$\bar{J}_m(t_1,T) = +\int d\tau \int ds \int dy \mathbb{1}_{\{t_1<\tau+s<T\}}\frac{\partial V_m}{\partial x}(\tau+s, y+\tilde{b}(\tau))\tilde{b}'(\tau)\rho_m(s, y)\bar{\gamma}_j(\tau, y).$$

We have, using integration by parts,

$$\begin{aligned}\hat{J}_m(t_1,T) &= -\int ds \int dy \rho_m(s,y)\left(\int_{t_1-s}^{T-s}\frac{d}{d\tau}\left(V_m(\tau+s, y+\tilde{b}(\tau))\right)\bar{\gamma}_j(\tau, y)d\tau\right) \\ &= -\int ds \int dy \rho_m(s,y)V_m(T, y+\tilde{b}(T-s))\bar{\gamma}_j(T-s, y) \\ &\quad + \int ds \int dy \rho_m(s,y)V_m(t_1, y+\tilde{b}(t_1-s))\bar{\gamma}_j(t_1-s, y) \\ &\quad + \int ds \int dy \rho_m(s,y)\int_{t_1-s}^{T-s} V_m(s+\tau, y+\tilde{b}(\tau))\frac{\partial \bar{\gamma}_j}{\partial \tau}(\tau, y)d\tau.\end{aligned}$$

Note that, due to the continuity of $V(=\partial\tilde{U}/\partial t)$ on $(0,\infty)\times\mathbb{R}$, the sequence V_m converges uniformly to V on compact sets. We also have the continuity of $\bar{\gamma}_j$ and $\partial\bar{\gamma}_j/\partial\tau$ (due to the fact that $\tilde{b}$ is C^2). We easily deduce thereof that

$$\lim_{m\to\infty}\hat{J}_m(t_1,T) = -V(T,\tilde{b}(T))\bar{\gamma}_j(T,0) + V(t_1,\tilde{b}(t_1))\bar{\gamma}_j(t_1,0) + \int_{t_1}^{T} V(\tau,\tilde{b}(\tau))\frac{\partial\bar{\gamma}_j}{\partial\tau}(\tau,0)d\tau, \quad (10)$$

and the convergence is uniform with respect to t_1, as long as t_1 remains in a compact set of the form $[\xi,T]$, where $0<\xi<T$. For $\bar{J}_m(t_1,T)$, we have

$$\bar{J}_m(t_1,T) \;=\; \bar{J}_m^{(1)}(t_1,T) + \bar{J}_m^{(2)}(t_1,T),$$

with

$$\bar{J}_m^{(1)}(t_1,T) = +\int d\tau\int ds\int dy \mathbb{1}_{\{t_1<\tau+s<T\}}\frac{\partial^2 U_m}{\partial t\partial x}(\tau+s,y+\tilde{b}(\tau))\tilde{b}'(\tau)\rho_m(s,y)\bar{\gamma}_j(\tau,y)$$

and

$$\bar{J}_m^{(2)}(t_1,T) = -\int d\tau\int ds\int dy \mathbb{1}_{\{t_1<\tau+s<T\}}\frac{\partial^2 u_m}{\partial t\partial x}(\tau+s,y+\tilde{b}(\tau))\tilde{b}'(\tau)\rho_m(s,y)\bar{\gamma}_j(\tau,y).$$

We deal with $\bar{J}_m^{(1)}(t_1,T)$ in the same way as for the proof of (7). Using the fact that $\tilde{b}'(\tau)\bar{\gamma}_j(\tau,y)\le 0$, we have $\bar{J}_m^{(1)}(t_1,T)\le\tilde{J}_m^{(1)}(t_1,T)$, with

$$\tilde{J}_m^{(1)}(t_1,T) \;=\; \int d\tau\int ds\int dy \mathbb{1}_{\{t_1<\tau+s<T\}}\int\int ds'dy'\rho_m(s',y')D(\tau,s-s',y-y')\tilde{b}'(\tau)\rho_m(s,y)\bar{\gamma}_j(\tau,y),$$

where

$$D(\tau,\theta,z) = \left(\frac{\partial^2 U}{\partial t\partial x}(\tau+\theta,\tilde{b}(\tau)+z) - \frac{\partial^2 U}{\partial t\partial x}(\tau,\tilde{b}(\tau))\right)\mathbb{1}_{\{\tilde{b}(\tau)+z\ge\tilde{b}(\tau+\theta)\}}.$$

Due to the continuity properties of $\frac{\partial^2 U}{\partial t\partial x}$, we have

$$\lim_{m\to\infty}\tilde{J}_m^{(1)}(t_1,T) = 0,$$

and the convergence is uniform with respect to t_1, as long as t_1 remains in $[\xi,T]$. On the other hand, due to the continuity of $\frac{\partial^2 u}{\partial t\partial x}$, we have

$$\lim_{m\to\infty}\bar{J}_m^{(2)}(t_1,T) = -\int_{t_1}^{T} d\tau\frac{\partial^2 u}{\partial t\partial x}(\tau,\tilde{b}(\tau))\tilde{b}'(\tau)\bar{\gamma}_j(\tau,0),$$

uniformly with respect to $t_1\in[\xi,T]$. At this stage, we can state that $J_m(t_1,T)\le\tilde{J}_m(t_1,T)$, with $\tilde{J}_m(t_1,T) = \hat{J}_m(t_1,T) + \tilde{J}_m^{(1)}(t_1,T) + \bar{J}_m^{(2)}(t_1,T)$, and

$$\lim_{m\to\infty}\sup_{t_1\in[\xi,T]}\left|\tilde{J}_m(t_1,T) - \tilde{J}(t_1,T)\right| = 0,$$

where

$$\begin{aligned}\tilde{J}(t_1,T) \;=\;& -V(T,\tilde{b}(T))\bar{\gamma}_j(T,0) + V(t_1,\tilde{b}(t_1))\bar{\gamma}_j(t_1,0) + \int_{t_1}^{T} V(\tau,\tilde{b}(\tau))\frac{\partial\bar{\gamma}_j}{\partial\tau}(\tau,0)d\tau\\ &- \int_{t_1}^{T} d\tau\frac{\partial^2 u}{\partial t\partial x}(\tau,\tilde{b}(\tau))\tilde{b}'(\tau)\bar{\gamma}_j(\tau,0).\end{aligned}$$

Since $\partial U/\partial t$ vanishes along the exercise boundary, we have $V(t,\tilde{b}(t)) = -\frac{\partial u_\varphi}{\partial t}(t,\tilde{b}(t))$, so that

$$\begin{aligned}
-V(T,\tilde{b}(T))\bar{\gamma}_j(T,0) + V(t_1,\tilde{b}(t_1))\bar{\gamma}_j(t_1,0) + \int_{t_1}^T V(\tau,\tilde{b}(\tau))\frac{\partial\bar{\gamma}_j}{\partial\tau}(\tau,0)d\tau = \\
\frac{\partial u_\varphi}{\partial t}(T,\tilde{b}(T))\bar{\gamma}_j(T,0) - \frac{\partial u_\varphi}{\partial t}(t_1,\tilde{b}(t_1))\bar{\gamma}_j(t_1,0) - \int_{t_1}^T \frac{\partial u_\varphi}{\partial t}(\tau,\tilde{b}(\tau))\frac{\partial\bar{\gamma}_j}{\partial\tau}(\tau,0)d\tau \\
= \int_{t_1}^T \frac{d}{d\tau}\left(\frac{\partial u_\varphi}{\partial t}(\tau,\tilde{b}(\tau))\right)\bar{\gamma}_j(\tau,0)d\tau,
\end{aligned}$$

so that

$$\begin{aligned}
\tilde{J}(t_1,T) &= \int_{t_1}^T \left[\frac{d}{d\tau}\left(\frac{\partial u_\varphi}{\partial t}(\tau,\tilde{b}(\tau))\right) - \frac{\partial^2 u_\varphi}{\partial t\partial x}(\tau,\tilde{b}(\tau))\tilde{b}'(\tau)\right]\bar{\gamma}_j(\tau,0)d\tau \\
&= \int_{t_1}^T \frac{\partial^2 u_\varphi}{\partial t^2}(\tau,\tilde{b}(\tau))\bar{\gamma}_j(\tau,0)d\tau.
\end{aligned}$$

We now go back to (9) and integrate with respect to t_1 to derive

$$\begin{aligned}
\frac{1}{2}\int_\xi^T dt_1\int_{t_1}^T \left|\frac{\partial^2\tilde{U}_m}{\partial t^2}(t,.)\right|_j^2 dt &\leq C\left(\int_\xi^T ||V_m(t_1,.)||_j^2\,dt_1 + \int_\xi^T\left(\int_{t_1}^T ||V_m(t,.)||_j^2\,dt\right)dt_1\right) \\
&\quad + \int_\xi^T dt_1\tilde{J}_m(t_1,T).
\end{aligned}$$

Hence

$$\frac{1}{2}\int_\xi^T (t-\xi)\left|\frac{\partial^2\tilde{U}_m}{\partial t^2}(t,.)\right|_j^2 dt \leq C\int_\xi^T ||V_m(t,.)||_j^2\,dt + \int_\xi^T dt_1\tilde{J}_m(t_1,T).$$

Note that

$$\begin{aligned}
\lim_{m\to\infty}\int_\xi^T ||V_m(t,.)||_j^2\,dt &= \int_\xi^T ||V(t,.)||_j^2\,dt \\
&\leq C\left(1+\ln\frac{T}{\xi}\right),
\end{aligned}$$

where the last inequality follows from Lemma 3.4. Moreover,

$$\begin{aligned}
\lim_{m\to\infty}\int_\xi^T dt_1\tilde{J}_m(t_1,T) &= \int_\xi^T dt_1\tilde{J}(t_1,T) \\
&= \int_\xi^T (t-\xi)\frac{\partial^2 u_\varphi}{\partial t^2}(t,\tilde{b}(t))\bar{\gamma}_j(t,0)dt.
\end{aligned}$$

Hence

$$\frac{1}{2}\int_\xi^T (t-\xi)\left|\frac{\partial^2\tilde{U}}{\partial t^2}(t,.)\right|_j^2 dt \leq C\left(1+\ln\frac{T}{\xi}\right) + \int_\xi^T (t-\xi)\frac{\partial^2 u_\varphi}{\partial t^2}(t,\tilde{b}(t))\bar{\gamma}_j(t,0)dt.$$

Theorem 3.1 now follows from the following lemma, which relies on the asymptotic behavior of the exercice boundaty near maturity (see [1], [7]).

Lemma 3.6. *We have*

$$\int_\xi^T (t-\xi)\left|\frac{\partial^2 u_\varphi}{\partial t^2}(t,\tilde b(t))\right||\tilde b'(t)|dt \;\leq\; C\left(1+|\ln\xi|^\beta\right), \quad \textit{with } \beta=\begin{cases} 3/2, & \textit{if } d\leq r,\\ 1, & \textit{if } d>r.\end{cases}$$

Proof: We have

$$\left\|\frac{\partial^2 u_\varphi}{\partial t^2}(t,.)\right\|_\infty \leq \frac{C}{t}\left\|\frac{\partial u_\varphi}{\partial t}(t,.)\right\|_\infty \leq \frac{C}{t^{3/2}}$$

Hence

$$\begin{aligned}\int_\xi^T (t-\xi)\left|\frac{\partial^2 u_\varphi}{\partial t^2}(t,\tilde b(t))\right||\tilde b'(t)|dt &\leq C\int_\xi^T \frac{t-\xi}{t^{3/2}}|\tilde b'(t)|dt\\ &\leq C\int_\xi^T \frac{1}{\sqrt{t}}|\tilde b'(t)|dt.\end{aligned}$$

Now, since $\tilde b'(t)\leq 0$, we have

$$\begin{aligned}\int_\xi^T \frac{1}{\sqrt{t}}|\tilde b'(t)|dt &= -\int_\xi^T \frac{1}{\sqrt{t}}\tilde b'(t)dt\\ &= -\left(\frac{\tilde b(T)-\tilde b(0)}{\sqrt{T}}-\frac{\tilde b(\xi)-\tilde b(0)}{\sqrt{\xi}}\right)-\frac{1}{2}\int_\xi^T \frac{1}{t^{3/2}}(\tilde b(t)-\tilde b(0))dt\\ &\leq \frac{\tilde b(0)-\tilde b(T)}{\sqrt{T}}+\frac{1}{2}\int_\xi^T \frac{1}{t^{3/2}}(\tilde b(0)-\tilde b(t))dt\end{aligned}$$

If $d\leq r$, we have $\tilde b(0)-\tilde b(t)\leq C\sqrt{t|\ln t|}$ for t close to 0, so that

$$\int_\xi^T \frac{1}{t^{3/2}}(\tilde b(0)-\tilde b(t))dt \leq C\int_\xi^T \frac{1}{t}\sqrt{|\ln t|}dt \leq C(1+|\ln\xi|^{3/2}).$$

If $d>r$, we have $\tilde b(0)-\tilde b(t)\leq C\sqrt{t}$ for t close to 0, so that

$$\int_\xi^T \frac{1}{t^{3/2}}(\tilde b(0)-\tilde b(t))dt \leq C\int_\xi^T \frac{1}{t}dt = C\ln(T/\xi).$$

◇

4 Upper bound for $P_0^{(n)}-P_0$

In order to derive an upper bound for $P_0^{(n)}-P_0$, we relate this quantity to u using (1) as follows:

$$\begin{aligned}P_0^{(n)}-P_0 &= \sup_{\tau\in\mathcal{T}_{0,T}^{(n)}}\mathbb{E}\left(e^{-r\tau}g(\mu_0\tau+B_\tau^{(n)})\right)-u(0,0)\\ &\leq \sup_{\tau\in\mathcal{T}_{0,T}^{(n)}}\mathbb{E}\left(u(\tau,B_\tau^{(n)})-u(0,0)\right)\\ &= \sup_{\tau\in\mathcal{T}_{0,T}^{(n)}}\mathbb{E}\left(\sum_{j=1}^{\tau/h}\mathcal{D}u((j-1)h,B_{(j-1)h}^{(n)})\right).\end{aligned}$$

We observe that $\mathcal{D}u \leq \tilde{\mathcal{D}}u$, and recall from [6] (Lemma 4.1) that $\sup_{0\leq j\leq n-1} \mathbb{E}\left|\mathcal{D}u(jh, B_{jh}^{(n)})\right| \leq Ch$, so that

$$\begin{aligned} P_0^{(n)} - P_0 &\leq \mathbb{E}\left(\sum_{j=1}^{n-2}\left|\tilde{\mathcal{D}}u(jh, B_{jh}^{(n)})\right|\right) + O(h) \\ &\leq C_{k,X} h\sqrt{2}\int_h^{T-h}\frac{ds}{\sqrt{s}}\int\frac{dy}{1+|y|^k}\left|\frac{\partial^3 u}{\partial t\partial x^2}(s,y)\right| + O(h). \end{aligned} \tag{11}$$

Here, we have a regularity problem, since u is not C^3. This problem can be fixed as follows. By convolution, one can approximate u by a sequence u_m which is smooth, uniformly bounded and satisfies $\delta u_m \leq 0$, and $\mathcal{D}u_m \leq \tilde{\mathcal{D}}u_m$. We need the following variant of Lemma 3.1.

Lemma 4.1. *If ρ is a Radon measure on $(0,T)\times\mathbb{R}$ and q a nonnegative integrable function on $(0,T)\times\mathbb{R}$, with $q(t,x) = 0$ for $t \notin (0,a)$, where a satisfies $0 < a < h$, we have*

$$\int_h^{T-h}\frac{ds}{\sqrt{s}}\int\frac{dy}{(1+|y|^2)^{k/2}}\left|\rho * q(s,y)\right| \leq 2^{k/2}\int_{h-a}^{T-h}\int_{\mathbb{R}}\frac{|\rho(dt,dz)|}{\sqrt{t}(1+z^2)^{k/2}}\int_0^a ds\int_{-\infty}^{\infty} q(s,x)(1+x^2)^{k/2}dx.$$

Proof: We have

$$\begin{aligned} &\int_h^{T-h}\frac{ds}{\sqrt{s}}\int\frac{dy}{(1+|y|^2)^{k/2}}\left|\rho * q(s,y)\right| \leq \\ &\qquad \int_h^{T-h}\frac{ds}{\sqrt{s}}\int\frac{dy}{(1+|y|^2)^{k/2}}\int\int|\rho(dt,dz)|\, q(s-t,y-z) \\ &\qquad = \int\int|\rho(dt,dz)|\int_h^{T-h}\frac{ds}{\sqrt{s}}\int\frac{dy}{(1+|y|^2)^{k/2}} q(s-t,y-z) \\ &\qquad \leq \int_{h-a}^{T-h}\int|\rho(dt,dz)|\int_0^a\frac{d\theta}{\sqrt{t+\theta}}\int\frac{dy}{(1+|y|^2)^{k/2}} q(\theta,y-z) \\ &\qquad \leq \int_{h-a}^{T-h}\int\frac{|\rho(dt,dz)|}{\sqrt{t}}\int_0^a d\theta\int\frac{dy}{(1+|y|^2)^{k/2}} q(\theta,y-z) \\ &\qquad = \int_{h-a}^{T-h}\int\frac{|\rho(dt,dz)|}{\sqrt{t}(1+z^2)^{k/2}}\int_0^a d\theta\int\frac{(1+z^2)^{k/2}dx}{(1+|x+z|^2)^{k/2}} q(\theta,x) \\ &\qquad \leq 2^{k/2}\int_{h-a}^{T-h}\int\frac{|\rho(dt,dz)|}{\sqrt{t}(1+z^2)^{k/2}}\int_0^a d\theta\int dx(1+x^2)^{k/2} q(\theta,x), \end{aligned}$$

where the last inequality follows from $\frac{1+z^2}{1+(x+z)^2} \leq 2(1+x^2)$. ◇

Using Lemma 4.1 and the fact that $\partial^3 u/(\partial t\partial x^2)$ is a Radon measure, we derive the correct version of (11), namely

$$P_0^{(n)} - P_0 \leq C_{k,X} h\sqrt{2}\int_h^{T-h}\frac{1}{\sqrt{s}}\int\frac{1}{1+|y|^k}\left|\frac{\partial^3 u}{\partial t\partial x^2}(ds,dy)\right| + O(h). \tag{12}$$

If we introduce the function $\tilde{u} := u - \bar{u}$, we have, using the fact that $\delta\bar{u} = 0$,

$$\begin{aligned}
\int_h^{T-h} \frac{1}{\sqrt{s}} \int \frac{1}{1+|y|^k} \left| \frac{\partial^3 u}{\partial t \partial x^2}(ds,dy) \right| &\leq \int_h^{T-h} \frac{1}{\sqrt{s}} \int \frac{1}{1+|y|^k} \left| \frac{\partial^3 \tilde{u}}{\partial t \partial x^2}(ds,dy) \right| \\
&\quad + 2 \int_h^{T-h} \frac{ds}{\sqrt{s}} \int \frac{dy}{1+|y|^k} \left| \frac{\partial^2 \bar{u}}{\partial t^2}(s,y) \right| \\
&\leq \int_h^{T-h} \frac{1}{\sqrt{s}} \int \frac{1}{1+|y|^k} \left| \frac{\partial^3 \tilde{u}}{\partial t \partial x^2}(ds,dy) \right| \\
&\quad + 2C_T \int_h^{T-h} \frac{ds}{\sqrt{s}(T-s)} \\
&\leq \int_h^{T-h} \frac{1}{\sqrt{s}} \int \frac{1}{1+|y|^k} \left| \frac{\partial^3 \tilde{u}}{\partial t \partial x^2}(ds,dy) \right| + C_T |\ln h|,
\end{aligned}$$

where we have used Proposition 3.1.

We now need to estimate $\int_h^{T-h} \frac{1}{\sqrt{s}} \int \frac{1}{1+|y|^k} \left| \frac{\partial^3 \tilde{u}}{\partial t \partial x^2}(ds,dy) \right|$. Recall from Remark 2.1 that

$$\frac{\partial \tilde{u}}{\partial t}(t,x) + \frac{1}{2} \frac{\partial^2 \tilde{u}}{\partial x^2}(t,x) = \zeta(t,x) \mathbb{1}_{\{x \leq \hat{b}(t)\}},$$

where $\zeta(t,x) = e^{-rt}(A-r)\varphi(\ln(S_0) + \mu t + \sigma x)$ and $\hat{b}(t) = \left(\tilde{b}(T-t) - \mu t - \ln(S_0)\right)/\sigma$. By differentiating wit respect to t, we derive the following expression

$$\frac{\partial^2 \tilde{u}}{\partial t^2}(t,x) + \frac{1}{2} \frac{\partial^3 \tilde{u}}{\partial t \partial x^2}(t,x) = \frac{\partial \zeta}{\partial t}(t,x) \mathbb{1}_{\{x \leq \hat{b}(t)\}} + \zeta(t, \hat{b}(t)) \hat{b}'(t) \delta_{\hat{b}(t)}, \tag{13}$$

where we have used Lemma 3.5. Note that

$$\sup_{x \leq \hat{b}(t)} \left| \frac{\partial \zeta}{\partial t}(t,x) \right| < \infty \text{ and } \sup_{0<t<T} \left| \zeta(t, \hat{b}(t)) \right| < \infty.$$

Moreover, $|\hat{b}'(t)| \leq |\tilde{b}'(T-t)| + |\mu/\sigma|$, so that

$$\begin{aligned}
\int_0^T \frac{dt}{\sqrt{t}} |\hat{b}'(t)| &\leq \int_0^{T/2} \frac{dt}{\sqrt{t}} |\tilde{b}'(T-t)| + \int_{T/2}^T \frac{dt}{\sqrt{t}} |\tilde{b}'(T-t)| + 2|\mu/\sigma|\sqrt{T} \\
&\leq \sup_{0 \leq t \leq T/2} |\tilde{b}'(T-t)| \int_0^{T/2} \frac{dt}{\sqrt{t}} + + \sqrt{\frac{2}{T}} \int_{T/2}^T |\tilde{b}'(T-t)| dt + 2|\mu/\sigma|\sqrt{T} \\
&= \sup_{0 \leq t \leq T/2} |\tilde{b}'(T-t)| \int_0^{T/2} \frac{dt}{\sqrt{t}} + \sqrt{\frac{2}{T}} \left(\tilde{b}(0) - \tilde{b}(T/2) \right) + 2|\mu/\sigma|\sqrt{T} < \infty.
\end{aligned}$$

Hence we have

$$\int_h^{T-h} \frac{1}{\sqrt{s}} \int \frac{1}{1+|y|^k} \left| \frac{\partial^3 \tilde{u}}{\partial t \partial x^2}(ds,dy) \right| \leq C_T + 2 \int_h^{T-h} \frac{ds}{\sqrt{s}} \int \frac{1}{1+|y|^k} \left| \frac{\partial^2 \tilde{u}}{\partial t^2}(s,y) \right|.$$

Now, using the Cauchy-Schwarz inequality and Theorem 3.1, we have

$$\begin{aligned}
\int_h^{T-h} \frac{ds}{\sqrt{s}} \int \frac{dy}{1+|y|^k} \left| \frac{\partial^2 \tilde{u}}{\partial t^2}(s,y) \right| &\leq C \left(\int_h^{T-h} \frac{ds}{s(T-s-\frac{h}{2})} \right)^{1/2} \left(\int_h^{T-h} ds (T-s-\frac{h}{2}) \left| \frac{\partial^2 \tilde{u}}{\partial t^2}(s,.) \right|_k^2 \right)^{1/2} \\
&\leq C\sqrt{|\ln h|} \left(\int_h^{T-\frac{h}{2}} ds (T-s-\frac{h}{2}) \left| \frac{\partial^2 \tilde{u}}{\partial t^2}(s,.) \right|_k^2 \right)^{1/2} \\
&= C\sqrt{|\ln h|} \left(\int_{h/2}^{T-h} dt (t-\frac{h}{2}) \left| \frac{\partial^2 \tilde{u}}{\partial t^2}(T-t,.) \right|_k^2 \right)^{1/2} \\
&\leq C\sqrt{|\ln h|^{1+\beta}},
\end{aligned}$$

with $\beta = 1$ if $d > r$ and $\beta = 3/2$ if $d \leq r$. The last inequality follows from Theorem 3.1 and Lemma 3.4, and the connection between the derivatives of the functions $\tilde{U}$ and $\tilde{u}$ (see Remark 2.1; we also use the classical bounds $||\partial U/\partial t(t,.)||_\infty + ||\partial^2 U/\partial x^2(t,.)||_\infty \leq C/\sqrt{t}$). We conclude that

$$P_0^{(n)} - P_0 \leq C \frac{(\ln n)^\alpha}{n},$$

with $\alpha = 1$ if $d > r$ and $\alpha = 5/4$ if $d \leq r$.

5 Lower bound for $P_0^{(n)} - P_0$

For the derivation of the lower bound, we use the stopping time introduced in [6] (see the proof of Theorem 5.6). Namely

$$\tau = \tau_1 \mathbb{1}_{\{\tau_1 < T-h\}} + T \mathbb{1}_{\{\tau_1 = T-h\}},$$

where

$$\tau_1 = \inf \left\{ t \in [0, T-h] \;\mid\; t/h \in \mathbb{N} \quad \text{and} \quad d(B_t^{(n)}, I_{t+h}) \leq \sqrt{h} ||X||_\infty + |\mu_0| h \right\}.$$

Here, $I_t = \{x \in \mathbb{R} \mid u(t,x) = g(t, x+\mu_0 t)\}$. Note that, if $t < T$, $I_t = (-\infty, \hat{b}(t)]$.

The modification of τ_1 into τ is motivated by the unboundedness of $\partial u/\partial t$ near T. We have, due to the definition of τ_1,

$$\begin{aligned}
P^{(n)} - P &\geq \mathbb{E}\left(e^{-r\tau} g(\mu_0 \tau + B_\tau^{(n)}) - u(0,0) \right) \\
&= \mathbb{E}\left(e^{-r\tau} g(\mu_0 \tau + B_\tau^{(n)}) - u(\tau, B_\tau^{(n)}) + u(\tau, B_\tau^{(n)}) - u(0,0) \right).
\end{aligned}$$

We have

$$\begin{aligned}\mathbb{E}\left(u(\tau,B_\tau^{(n)})-u(0,0)\right) &= \mathbb{E}\left(\sum_{j=1}^{\tau/h}\mathcal{D}u((j-1)h,B_{(j-1)h}^{(n)})\right)\\ &= \mathbb{E}\left(\sum_{j=1}^{(\tau/h)\wedge(n-2)}\mathcal{D}u(jh-h,B_{jh-h}^{(n)})\right)\\ &\quad+\mathbb{E}\left(\mathbb{1}_{\{\tau=T\}}\sum_{j=n-1}^{n}\mathcal{D}u(jh-h,B_{jh-h}^{(n)})\right)\\ &= \mathbb{E}\left(\sum_{j=1}^{(\tau/h)\wedge(n-2)}\mathcal{D}u(jh-h,B_{jh-h}^{(n)})\right)+O(h),\end{aligned}$$

where the last equality follows from Lemma 4.1 of [6]. Now, if $j<(\tau/h)\wedge(n-2)$, we have $d(B_{jh}^{(n)},I_{jh+h})>\sqrt{h}||X||_\infty+|\mu_0|h$, so that

$$B_{jh}^{(n)}>\hat{b}(jh+h)+\sqrt{h}||X||_\infty+|\mu_0|h.$$

We then have, for $s\in[jh,jh+h]$ and $z\in[0,\sqrt{h}]$

$$\begin{aligned}B_{jh}^{(n)}+zX &> \hat{b}(jh+h)+zX+\sqrt{h}||X||_\infty+|\mu_0|h\\ &\geq \hat{b}(jh+h)+|\mu_0|h\\ &= \hat{b}(s)+\hat{b}(jh+h)+\mu_0(jh+h)-(\hat{b}(s)+\mu_0 s)-\mu_0(jh+h-s)+|\mu_0|h\\ &\geq \hat{b}(s),\end{aligned}$$

the last inequality coming from the fact that $t\mapsto\hat{b}(t)+\mu_0 t$ is increasing. We can now assert that, for $j<(\tau/h)\wedge(n-2)$, $\mathcal{D}u(B_{jh}^{(n)},jh)=\tilde{\mathcal{D}}u(B_{jh}^{(n)},jh)$, so that

$$\begin{aligned}\mathbb{E}\left(u(\tau,B_\tau^{(n)})-u(0,0)\right) &\leq \mathbb{E}\left(\sum_{j=1}^{n-2}\left|\tilde{\mathcal{D}}u(jh,B_{jh}^{(n)})\right|\right)+O(h)\\ &\leq C\frac{(\ln n)^\alpha}{n},\end{aligned}$$

with $\alpha=1$ if $d>r$ and $\alpha=5/4$ if $d\leq r$, as follows from the discussion in the previous section.

We now want a lower bound for

$$\mathbb{E}\left(e^{-r\tau}g(\mu_0\tau+B_\tau^{(n)})-u(\tau,B_\tau^{(n)})\right).$$

We have, using the equality $\{\tau\geq T-h\}=\{\tau=T\}$,

$$\begin{aligned}u(\tau,B_\tau^{(n)})-e^{-r\tau}g(\mu_0\tau+B_\tau^{(n)}) &= \left(u(\tau,B_\tau^{(n)})-e^{-r\tau}g(\mu_0\tau+B_\tau^{(n)})\right)\mathbb{1}_{\{\tau<T-h\}}\\ &= \left(u(\tau+h,B_\tau^{(n)})-e^{-r\tau}g(\mu_0\tau+B_\tau^{(n)})\right)\mathbb{1}_{\{\tau<T-h\}}\\ &\quad+\left(u(\tau,B_\tau^{(n)})-u(\tau+h,B_\tau^{(n)})\right)\mathbb{1}_{\{\tau<T-h\}}.\end{aligned}$$

On the set $\{\tau < T-h\}$, we have $d(B_\tau^{(n)}, I_{\tau+h}) \leq \sqrt{h}||X||_\infty + |\mu|h$. It follows from Proposition 2.6 of [6] that

$$u(\tau+h, B_\tau^{(n)}) - e^{-r\tau} g(\mu_0\tau + B_\tau^{(n)}) \leq C \frac{\left(\sqrt{h}||X||_\infty + |\mu|h\right)^2}{\sqrt{T-\tau-h}}.$$

Using the estimate $\left|\left|\frac{\partial u}{\partial t}(t,.)\right|\right|_\infty < C/\sqrt{T-t}$, we obtain

$$\mathbb{E}\left(u(\tau, B_\tau^{(n)}) - e^{-r\tau} g(\mu_0\tau + B_\tau^{(n)})\right) \leq Ch\mathbb{E}\left(\frac{1}{\sqrt{T-\tau-h}} \mathbb{1}_{\{\tau \leq T-2h\}}\right).$$

The estimate $P^{(n)} - P \geq -C\frac{(\ln n)^{\tilde{\alpha}}}{n}$ is now an easy consequence of Lemma 5.7 and Remark 5.8 of [6], which can be summarized in the following statement.

Lemma 5.1. *There exists a positive constant C such that*

$$\mathbb{E}\left(\frac{1}{\sqrt{T-\tau-h}} \mathbb{1}_{\{\tau \leq T-2h\}}\right) \leq C\left(\ln h\right)^\beta,$$

with

$$\beta = \begin{cases} 3/2, \ \textit{if } d \leq r, \\ \\ 1, \ \textit{if } d > r. \end{cases}$$

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