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On the binomial approximation of the American put

DAMIEN LAMBERTON*

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Abstract

We consider the binomial approximation of the American put price in the Black-Scholes model (with continuous dividend yield). Our main result is that the error of approximation is $O((\ln n)^\alpha/n)$ where n is the number of time periods and the exponent α is a positive number, the value of which may differ according to the respective levels of the interest rate and the dividend yield.

1 The binomial approximation

Consider the Black-Scholes model in which the stock price at time t is given by

$$S_t = S_0 e^{(r-d-\frac{\sigma^2}{2})t + \sigma B_t},$$

where, under the risk-neutral probability measure, $(B_t)_{t \geq 0}$ is a standard Brownian motion. Here, r is the instantaneous interest rate, and d is the dividend rate (or the foreign interest rate in the case of forex options). We assume $r > 0$ and $d \geq 0$.

Denote by P the price function of the American put with maturity T and strike price K , so that

$$P(t, x) = \sup_{\tau \in \mathcal{T}_{0, T-t}} \mathbb{E}_x \left(e^{-r\tau} f(S_\tau) \right), \quad 0 \leq t \leq T, \quad x \in [0, +\infty),$$

with $f(x) = (K - x)^+$, and $\mathbb{E}_x = \mathbb{E}(\cdot | S_0 = x)$. Here $\mathcal{T}_{0,t}$ denotes the set of all stopping times with respect to the Brownian filtration, with values in the interval $[0, t]$.

For technical reasons (especially for the derivation of regularity estimates for the second time derivative of the price function), it is more convenient to use the log-stock price. So, we introduce

$$X_t^x = x + \mu t + \sigma B_t, \quad \text{with } \mu = r - d - \frac{\sigma^2}{2},$$

and

$$U(T, x) = \sup_{\tau \in \mathcal{T}_{0, T}} \mathbb{E} \left(e^{-r\tau} \varphi(X_\tau^x) \right),$$

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with $\varphi(x) = (K - e^x)^+$. We then have

$$P(t, x) = U(T - t, \ln(x)), \quad t > 0, x > 0.$$

Note that $U(t, x)$ satisfies the following parabolic variational inequality

$$\max \left[-\frac{\partial U}{\partial t} + (A - r)U, \varphi - U \right] = 0,$$

with the initial condition $U(0, \cdot) = \varphi$.

Here, A is the infinitesimal generator of X , namely

$$A = \frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} + \mu \frac{\partial}{\partial x}.$$

Recall that, for each $T > 0$, there is a real number $\tilde{b}(T) \leq \ln(K)$ such that

$$U(T, x) > \varphi(x) \Leftrightarrow x > \tilde{b}(T).$$

In fact, if $(b(t), 0 \leq t \leq T)$ is the exercise boundary of the American put with maturity T , we have $\tilde{b}(t) = \ln(b(T - t))$. We will also need the European value function, defined by

$$\bar{U}(T, x) = \mathbb{E} \left(e^{-rT} \varphi(X_T^x) \right).$$

Note that $\bar{U}(0, \cdot) = \varphi$ and

$$-\frac{\partial \bar{U}}{\partial t} + (A - r)\bar{U} = 0.$$

Note that, in Section 3, the function \bar{U} will be denoted by u_φ .

We now introduce the random walk approximation of Brownian motion. To be more precise, assume $(X_n)_{n \geq 1}$ is a sequence of i.i.d. real random variables satisfying $\mathbb{E}X_n^2 = 1$ and $\mathbb{E}X_n = 0$, and define, for any positive integer n , the process $B^{(n)}$ by

$$B_t^{(n)} = \sqrt{T/n} \sum_{k=1}^{\lfloor nt/T \rfloor} X_k, \quad 0 \leq t \leq T,$$

where $\lfloor nt/T \rfloor$ denotes the greatest integer in nt/T .

We will assume the following about the common distribution of the X_n 's (cf. hypothesis (H4) of [6]).

(H4) The random variable X_1 is bounded and satisfies $\mathbb{E}X_1^2 = 1$ and $\mathbb{E}X_1 = \mathbb{E}X_1^3 = 0$.

In the following, we fix S_0 and set

$$P_0 = P(0, S_0) = U(T, \ln S_0).$$

Note that, if we introduce the notation $g(x) = (K - S_0 e^{\sigma x})^+$, we have

$$P_0 = \sup_{\tau \in \mathcal{T}_{0,T}} \mathbb{E} \left(e^{-r\tau} g(\mu_0 \tau + B_\tau) \right)$$

with $\mu_0 = \mu/\sigma$. We now have a natural approximation of P_0 , given by

$$P_0^{(n)} = \sup_{\tau \in \mathcal{T}_{0,T}^{(n)}} \mathbb{E} \left(e^{-r\tau} g(\mu_0\tau + B_\tau^{(n)}) \right),$$

where $\mathcal{T}_{0,T}^{(n)}$ denotes the set of all stopping times (with respect to the natural filtration of $B^{(n)}$), with values in $[0, T] \cap \{0, T/n, 2T/n, \dots, (n-1)T/n, T\}$. Our main result is the following.

Theorem 1.1. *There exists a positive constant C such that, for all positive integers n ,*

$$-C \frac{(\ln n)^{\bar{\alpha}}}{n} \leq P_0^{(n)} - P_0 \leq C \frac{(\ln n)^\alpha}{n},$$

where $\bar{\alpha} = \alpha = 1$ if $d > r$, and $\bar{\alpha} = 3/2, \alpha = 5/4$ if $d \leq r$.

The above estimates improve our previous results (see [6], Theorem 5.6) which gave an upper bound of the form $C \left(\frac{\sqrt{\ln n}}{n} \right)^{4/5}$. Note that, for European options, the error estimate is $O(1/n)$ (see [2], [9]).

Our approach remains the same as in [6]: we relate the error estimates to the regularity of the value function. The improvement comes from a refinement of the quadratic estimates for the second order time derivative, in the spirit of Friedman and Kinderlehrer (see [3] and [5]). We also exploit the smoothness of the exercise boundary and its asymptotic properties close to maturity.

The paper is organized as follows. In the next Section we recall some results of [6]. Section 3 is devoted to estimates for the derivatives of the value function. The estimates are then used in Sections 4 and 5 to prove Theorem 1.1: in Section 4, we give an upper bound for $P_0^{(n)} - P_0$ and in Section 5, we derive the lower bound.

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2 The value function and the approximating process

As in [6], we introduce the modified value function

$$u(t, x) = e^{-rt} U(T - t, \ln(S_0) + \mu t + \sigma x), \quad t \geq 0, \quad x \in \mathbb{R}.$$

We have $P_0 = u(0, 0)$ and $u(T, x) = e^{-rT} U(0, \ln(S_0) + \mu T + \sigma x) = e^{-rT} (K - S_0 e^{\mu T + \sigma x})^+$ and, for $t \in [0, T]$,

$$u(t, x) \geq e^{-rt} (K - S_0 e^{\mu t + \sigma x})^+ = e^{-rt} g(\mu_0 t + x). \quad (1)$$

We will need the European analogue of u , namely

$$\bar{u}(t, x) = e^{-rt} \bar{U}(T - t, \ln(S_0) + \mu t + \sigma x) = e^{-rT} \mathbb{E} (g(\mu_0 T + x + B_{T-t})), \quad t \geq 0, x \in \mathbb{R}.$$

We will also use the notation:

$$h = \frac{T}{3^n}.$$

With this notation, we have

$$B_t^{(n)} = \sqrt{h} \sum_{k=1}^{\lfloor t/h \rfloor} X_k, \quad 0 \leq t \leq T.$$

We have, for all $t \in \{0, h, 2h, \dots, (n-1)h, nh = T\}$ (cf. Proposition 3.1 of [6]),

$$u(t, B_t^{(n)}) = u(0, 0) + M_t + \sum_{j=1}^{t/h} \mathcal{D}u((j-1)h, B_{(j-1)h}^{(n)}),$$

where $(M_t)_{0 \leq t \leq T}$ is a martingale (with respect to the natural filtration of $B^{(n)}$), such that $M_0 = 0$, and

$$\mathcal{D}u(t, x) = \mathbb{E} \left(u(t+h, x + \sqrt{h}X_1) \right) - u(t, x), \quad 0 \leq t \leq T-h, \quad x \in \mathbb{R}.$$

For convenience, we introduce the following notation, for any function $u : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$,

$$\delta u = \frac{\partial u}{\partial t} + \frac{1}{2} \frac{\partial^2 u}{\partial x^2}.$$

It is easy to check that, if u is smooth and X_1 bounded,

$$(1/h) \times \mathcal{D}u(t, x) = \delta u(t, x) + O(h).$$

Remark 2.1. The derivatives of u are related to those of U by the following formulas. We have

$$\frac{\partial u}{\partial t}(t, x) = e^{-rt} \left(-\frac{\partial U}{\partial t} + \mu \frac{\partial U}{\partial x} - rU \right) (T-t, \ln(S_0) + \mu t + \sigma x)$$

and

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2}(t, x) &= e^{-rt} \left(\frac{\partial^2 U}{\partial t^2} - 2\mu \frac{\partial^2 U}{\partial t \partial x} + \mu^2 \frac{\partial^2 U}{\partial x^2} \right. \\ &\quad \left. + 2r \frac{\partial U}{\partial t} - 2r\mu \frac{\partial U}{\partial x} + r^2 U \right) (T-t, \ln(S_0) + \mu t + \sigma x). \end{aligned}$$

We also have

$$\begin{aligned} \delta u(t, x) &= \frac{\partial u}{\partial t}(t, x) + \frac{1}{2} \frac{\partial^2 u}{\partial x^2}(t, x) = e^{-rt} \left(-\frac{\partial U}{\partial t} + (A-r)U \right) (T-t, \ln(S_0) + \mu t + \sigma x) \\ &= e^{-rt} (A-r) \varphi(\ln(S_0) + \mu t + \sigma x) \mathbb{1}_{\{\ln(S_0) + \mu t + \sigma x \leq b(T-t)\}}, \end{aligned}$$

where the last equality follows from regularity results (see, for instance, [4]).

We will need a more precise description of \mathcal{D} , given by the following proposition (see Proposition 3.4 of [6]). For convenience, we denote by X a random variable with the same distribution as X_1 , which is independent of the sequence $(X_n)_{n \geq 1}$.

Proposition 2.1. Assume that (H4) is satisfied and that v is a function of class C^3 on $[0, T] \times \mathbb{R}$. For $0 \leq t \leq T - h$ and $x \in \mathbb{R}$, define

$$\tilde{\mathcal{D}}v(t, x) = 2 \int_0^{\sqrt{h}} d\xi \int_0^\xi dz \mathbb{E} \left[X (\xi - X^2(\xi - z)) \frac{\partial^2 v}{\partial t \partial x}(t + \xi^2, x + zX) \right].$$

We have

$$\mathcal{D}v(t, x) = \tilde{\mathcal{D}}v(t, x) + 2 \int_0^{\sqrt{h}} d\xi \int_0^\xi dz \mathbb{E} \left(X^2 \delta v(t + \xi^2, x + zX) \right).$$

and

$$\tilde{\mathcal{D}}v(t, x) = 2 \int_0^{\sqrt{h}} d\xi \int_0^\xi dz (\xi - z) \mathbb{E} \left[X^2 \left(\xi - X^2 \frac{(\xi - z)}{2} \right) \frac{\partial^3 v}{\partial t \partial x^2}(t + \xi^2, x + zX) \right].$$

Remark 2.2. Note that, if $\delta v(s, x + zX) = 0$ for all $s \in [t, t + h]$ and $z \in [0, \sqrt{h}]$, we have $v(t, x) = \mathcal{D}v(t, x)$.

From the last equality in Proposition 2.1, we derive the following estimates.

$$\begin{aligned} |\tilde{\mathcal{D}}v(t, x)| &\leq 2 \int_0^{\sqrt{h}} \xi^2 d\xi \int_0^\xi dz \mathbb{E} \left[\left(X^2 + \frac{X^4}{2} \right) \left| \frac{\partial^3 v}{\partial t \partial x^2}(t + \xi^2, x + zX) \right| \right] \\ &\leq \sqrt{h} \int_0^{\sqrt{h}} 2\xi d\xi \mathbb{E} \left[\int_0^\xi dz \left(X^2 + \frac{X^4}{2} \right) \left| \frac{\partial^3 v}{\partial t \partial x^2}(t + \xi^2, x + zX) \right| \right] \\ &\leq \sqrt{h} \int_t^{t+h} ds \mathbb{E} \left(\int dy \mathbf{1}_{\{|y-x| \leq \sqrt{h}|X|\}} \left(|X| + \frac{|X|^3}{2} \right) \left| \frac{\partial^3 v}{\partial t \partial x^2}(s, y) \right| \right) \\ &= \sqrt{h} \int_t^{t+h} ds \int dy \mathbb{E} \left(\mathbf{1}_{\{|y-x| \leq \sqrt{h}|X|\}} \left(|X| + \frac{|X|^3}{2} \right) \right) \left| \frac{\partial^3 v}{\partial t \partial x^2}(s, y) \right| \end{aligned}$$

We know from Proposition 3.2 of [6] that, for every $k \in (1, 3]$, there exists a positive constant C_k (which does not depend on X), such that, for all $y \in \mathbb{R}$, $n \geq 1$ and $j \in \{1, 2, \dots, n\}$

$$\mathbb{E} \left(\left(|X| + \frac{|X|^3}{2} \right) \mathbf{1}_{\{|B_{jh}^{(n)} - y| \leq \sqrt{h}|X|\}} \right) \leq \frac{C_k \mathbb{E}(|X|^3) (1 + \mathbb{E}|X|^{3+k})}{\sqrt{j} (1 + |y|^k)}.$$

Hence, for $j = 1, \dots, n - 1$,

$$\begin{aligned} \mathbb{E} \left(|\tilde{\mathcal{D}}v(jh, B_{jh}^{(n)})| \right) &\leq \frac{C_{k,X}}{\sqrt{j}} \sqrt{h} \int_{jh}^{jh+h} ds \int \frac{dy}{1 + |y|^k} \left| \frac{\partial^3 v}{\partial t \partial x^2}(s, y) \right| \\ &\leq C_{k,X} h \sqrt{2} \int_{jh}^{jh+h} \frac{ds}{\sqrt{s}} \int \frac{dy}{1 + |y|^k} \left| \frac{\partial^3 v}{\partial t \partial x^2}(s, y) \right|, \end{aligned} \quad (2)$$

where, for the last inequality, we used the inequality $jh \geq (j + 1)h/2$.

3 Estimates for the second order time derivative

In this section, we refine the regularity results that we used in [6]. We first establish some elementary L_1 -estimates. Then, we obtain a quadratic estimate for the second order time derivative of the difference $\tilde{U} = U - \bar{U}$. For the definition of the relevant weighted Sobolev spaces, we will use the notation

$$\nu_j(dx) = \frac{dx}{(1+x^2)^{j/2}}, \quad j > 1.$$

3.1 Some elementary L_1 -estimates

Proposition 3.1. *Assume that the function φ is continuous and satisfies $\varphi \in L_1(\nu_j)$, $\varphi' \in L_1(\nu_j)$ and the second derivative φ'' is a Radon measure on \mathbb{R} , with $\int_{\mathbb{R}} \frac{|\varphi''(dz)|}{(1+z^2)^{j/2}} < \infty$.*

Let

$$u_\varphi(t, x) = e^{-rt} \mathbb{E}_x(\varphi(X_t)), \quad t \geq 0, \quad x \in \mathbb{R}.$$

Then, for all $T > 0$, there exists a constant $C_T > 0$, such that

$$\forall t \in (0, T], \quad \left\| \frac{\partial^2 u_\varphi}{\partial t^2}(t, \cdot) \right\|_{L_1(\nu_j)} \leq \frac{C_T}{t}.$$

We will easily deduce this proposition from the following lemma.

Lemma 3.1. *If ρ is a Radon measure on \mathbb{R} and q a nonnegative integrable function on \mathbb{R} , we have*

$$\|\rho * q\|_{L_1(\nu_j)} \leq 2^{j/2} \int_{\mathbb{R}} \frac{|\rho(dz)|}{(1+z^2)^{j/2}} \int_{-\infty}^{\infty} q(x)(1+x^2)^{j/2} dx.$$

We also have, for any measurable function f on \mathbb{R} ,

$$\forall y \in \mathbb{R}, \quad \|f(\cdot - y)\|_{L_1(\nu_j)} \leq 2^{j/2} (1+y^2)^{j/2} \|f\|_{L_1(\nu_j)}.$$

Proof: We have

$$\begin{aligned} \|\rho * q\|_{L_1(\nu_j)} &\leq \int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^{j/2}} \int_{\mathbb{R}} |\rho(dz)| q(x-z) \\ &= \int_{\mathbb{R}} \frac{|\rho(dz)|}{(1+z^2)^{j/2}} \int_{-\infty}^{\infty} q(x-z) \frac{(1+z^2)^{j/2}}{(1+x^2)^{j/2}} dx \\ &= \int_{\mathbb{R}} \frac{|\rho(dz)|}{(1+z^2)^{j/2}} \int_{-\infty}^{\infty} q(x) \frac{(1+z^2)^{j/2}}{(1+(x+z)^2)^{j/2}} dx. \end{aligned}$$

Note that

$$z^2 \leq 2((x+z)^2 + x^2),$$

so that we deduce

$$\begin{aligned} \frac{1+z^2}{1+(x+z)^2} &\leq \frac{1+2(x+z)^2+2x^2}{1+(x+z)^2} \\ &\leq 2(1+x^2). \end{aligned}$$

Hence

$$\|\rho * q\|_{L_1(\nu_j)} \leq 2^{j/2} \int_{\mathbb{R}} \frac{|\rho(dz)|}{(1+z^2)^{j/2}} \int_{-\infty}^{\infty} q(x)(1+x^2)^{j/2} dx.$$

Similarly, we have, for any measurable function f and $y \in \mathbb{R}$,

$$\begin{aligned} \|f(\cdot - y)\|_{L_1(\nu_j)} &= \int |f(x - y)| \frac{dx}{(1+x^2)^{j/2}} \\ &= \int |f(x)| \frac{dx}{(1+(x+y)^2)^{j/2}} \\ &= \int |f(x)| \left(\frac{1+x^2}{1+(x+y)^2} \right)^{j/2} \frac{dx}{(1+x^2)^{j/2}} \\ &\leq \int |f(x)| \left(\frac{1+2(x+y)^2+2y^2}{1+(x+y)^2} \right)^{j/2} \frac{dx}{(1+x^2)^{j/2}} \\ &\leq 2^{j/2}(1+y^2)^{j/2} \|f\|_{L_1(\nu_j)}. \end{aligned}$$

◇

Proof of Proposition 3.1: We have

$$\begin{aligned} u_\varphi(t, x) &= e^{-rt} \int_{-\infty}^{\infty} \varphi(x+y) \exp\left(-\frac{(y-\mu t)^2}{2\sigma^2 t}\right) \frac{dy}{\sigma\sqrt{2\pi t}} \\ &= e^{-rt} p_t * \varphi(x), \end{aligned}$$

with

$$p_t(x) = \frac{1}{\sigma\sqrt{2\pi t}} \exp\left(-\frac{(x+\mu t)^2}{2\sigma^2 t}\right) = \frac{1}{\sigma\sqrt{t}} n\left(\frac{x+\mu t}{\sigma\sqrt{t}}\right).$$

Here, n denotes the standard normal density function.

On the other hand, we know that u_φ satisfies the equation

$$\frac{\partial u_\varphi}{\partial t} = (A-r)u_\varphi, \tag{3}$$

so that

$$\begin{aligned} \frac{\partial u_\varphi}{\partial t}(t, \cdot) &= e^{-rt}(A-r)p_t * \varphi \\ &= e^{-rt}p_t * [(A-r)\varphi]. \end{aligned}$$

It follows from our assumptions that $(A-r)\varphi$ is a Radon measure satisfying

$$\int_{\mathbb{R}} |(A-r)\varphi(dz)| \frac{1}{(1+z^2)^{j/2}} < \infty.$$

So that, using Lemma 3.1,

$$\begin{aligned} \left\| \frac{\partial u_\varphi}{\partial t}(t, \cdot) \right\|_{L_1(\nu_j)} &\leq C_j \int_{-\infty}^{\infty} p_t(x)(1+|x|^j) dx \\ &= C_j \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{t}} n\left(\frac{x+\mu t}{\sigma\sqrt{t}}\right) (1+|x|^j) dx \\ &= C_j \int_{-\infty}^{\infty} n(y) (1+|y\sigma\sqrt{t}-\mu t|^j) dy \\ &\leq C_j (1+t^j). \end{aligned}$$

On the other hand, by differentiating (3), we have

$$\frac{\partial^2 u_\varphi}{\partial t^2} = (A - r) \frac{\partial u_\varphi}{\partial t} = e^{-rt} ((A - r)p_t) * (A - r)\varphi.$$

Hence, using Lemma 3.1, and the definition of p_t ,

$$\begin{aligned} \left\| \frac{\partial^2 u_\varphi}{\partial t^2}(t, \cdot) \right\|_{L_1(\nu_j)} &\leq C_j \int_{-\infty}^{\infty} |(A - r)p_t(x)|(1 + |x|^j) dx \\ &\leq \frac{C_j}{t} (1 + t^j). \end{aligned}$$

◇

3.2 Quadratic estimates

Recall the notation:

$$U(t, x) = \sup_{\tau \in \mathcal{T}_{0,t}} \mathbb{E}_x \left(e^{-r\tau} \varphi(X_\tau) \right), \quad u_\varphi(t, x) = e^{-rt} \mathbb{E}_x(\varphi(X_t)), \quad t \geq 0, x \in \mathbb{R}.$$

with $\varphi(x) = (K - e^x)^+$. We now introduce the difference $\tilde{U} = U - u_\varphi$ (which corresponds to the early exercise premium). We have the following L_2 -estimate for the second time derivative of $\tilde{U} = U - u_\varphi$.

Theorem 3.1. *Fix $T > 0$ and $j > 1$. There exists a constant $C > 0$ such that, for all $\xi \in (0, T]$,*

$$\int_\xi^T (t - \xi) \left\| \frac{\partial^2 \tilde{U}}{\partial t^2}(t, \cdot) \right\|_{L_2(\nu_j)}^2 dt \leq C \left(1 + |\ln \xi|^\beta \right), \quad \text{with } \beta = \begin{cases} 3/2, & \text{if } d \leq r, \\ 1, & \text{if } d > r. \end{cases}$$

For the proof of Theorem 3.1, we need some preliminary estimates on the derivatives $\frac{\partial^2 \tilde{U}}{\partial x^2}$ and $\frac{\partial^2 \tilde{U}}{\partial t \partial x}$.

Lemma 3.2. *Fix $T > 0$ and $j > 1$. For any $\varepsilon \in (0, 1/4)$, there exists a constant $C > 0$ such that, for all $t \in (0, T]$,*

$$\left\| \frac{\partial \tilde{U}}{\partial x}(t, \cdot) \right\|_{L_2(\nu_j)} \leq C\sqrt{t} \quad \text{and} \quad \left\| \frac{\partial^2 \tilde{U}}{\partial x^2}(t, \cdot) \right\|_{L_2(\nu_j)} \leq Ct^\varepsilon.$$

Proof: We know that \tilde{U} solves the equation

$$-\frac{\partial \tilde{U}}{\partial t} + (A - r)\tilde{U} = h,$$

with initial condition $\tilde{U}(0, \cdot) = 0$, where h is given by

$$h(t, x) = (A - r)\varphi(x) \mathbf{1}_{\{x \leq \tilde{b}(t)\}}, \quad t > 0, \quad x \in \mathbb{R}.$$

We have the following identity (which can be viewed as a form of the early exercise premium formula).

$$\tilde{U}(t, \cdot) = - \int_0^t e^{-r(t-s)} p_{t-s} * h(s, \cdot) ds,$$

where

$$p_t(x) = \frac{1}{\sigma\sqrt{2\pi t}} \exp\left(-\frac{(x + \mu t)^2}{2\sigma^2 t}\right) = \frac{1}{\sigma\sqrt{t}} n\left(\frac{x + \mu t}{\sigma\sqrt{t}}\right),$$

with n denoting the standard normal density function. It is straightforward to check that

$$\frac{\partial \tilde{U}}{\partial x}(t, \cdot) = - \int_0^t e^{-r(t-s)} p_{t-s} * \frac{\partial h}{\partial x}(s, \cdot) ds,$$

and, with the notation δ_z for the Dirac measure at a point z ,

$$\begin{aligned} \frac{\partial h}{\partial x}(t, x) &= (A - r)\varphi'(x)\mathbb{1}_{\{x \leq \tilde{b}(t)\}} - (A - r)\varphi(x)\delta_{\tilde{b}(t)}(x) \\ &= -\kappa(t, x) + \gamma(t)\delta_{\tilde{b}(t)}(x), \end{aligned} \tag{4}$$

with $\kappa(t, x) = -(A - r)\varphi'(x)\mathbb{1}_{\{x \leq \tilde{b}(t)\}}$ and $\gamma(t) = -(A - r)\varphi(\tilde{b}(t))$. Note that κ is a bounded function on $(0, \infty) \times \mathbb{R}$ and γ is a continuous, nonnegative and bounded function on $(0, +\infty)$. At this stage, it is clear that $\|p_{t-s} * \frac{\partial h}{\partial x}(s, \cdot)\|_\infty \leq C/\sqrt{t-s}$, so that

$$\left\| \frac{\partial \tilde{U}}{\partial x}(t, \cdot) \right\|_{L_2(\nu_j)} \leq C\sqrt{t}.$$

On the other hand, we have

$$\left\| \frac{\partial^2 \tilde{U}}{\partial x^2}(t, \cdot) \right\|_{L_2(\nu_j)} \leq \int_0^t e^{-r(t-s)} \left\| p'_{t-s} * \kappa(s, \cdot) \right\|_{L_2(\nu_j)} ds + \|\zeta(t, \cdot)\|_{L_2(\nu_j)},$$

with

$$\begin{aligned} \zeta(t, \cdot) &= \int_0^t e^{-r(t-s)} \gamma(s) p'_{t-s} * \delta_{\tilde{b}(s)} ds \\ &= \int_0^t e^{-r(t-s)} \gamma(s) p'_{t-s}(\cdot - \tilde{b}(s)) ds. \end{aligned}$$

We have, using Lemma 3.1,

$$\begin{aligned} \left\| p'_{t-s} * \kappa(s, \cdot) \right\|_{L_2(\nu_j)} &= \left\| \int p'_{t-s}(y) \kappa(s, \cdot - y) dy \right\|_{L_2(\nu_j)} \\ &\leq \int |p'_{t-s}(y)| \|\kappa(s, \cdot - y)\|_{L_2(\nu_j)} dy \\ &\leq 2^{j/4} \|\kappa(s, \cdot)\|_{L_2(\nu_j)} \int |p'_{t-s}(y)| (1 + y^2)^{j/4} dy. \end{aligned}$$

Note that, since κ is bounded and $j > 1$, $\sup_{s>0} \|\kappa(s, \cdot)\|_{L_2(\nu_j)} < \infty$, so that, for some constant $C > 0$ (which may vary from line to line)

$$\begin{aligned} \left\| p'_{t-s} * \kappa(s, \cdot) \right\|_{L_2(\nu_j)} &\leq C \int |p'_{t-s}(y)| (1+y^2)^{j/4} dy \\ &= C \int \frac{1}{\sigma^2(t-s)} \left| n' \left(\frac{y + \mu(t-s)}{\sigma\sqrt{t-s}} \right) \right| (1+y^2)^{j/4} dy \\ &= C \int \frac{1}{\sigma\sqrt{t-s}} |n'(z)| (1 + (-\mu(t-s) + \sigma\sqrt{t-s}z)^2)^{j/4} dz \\ &\leq \frac{C}{\sqrt{t-s}} (1 + (t-s)^{j/2}). \end{aligned}$$

Hence, if $0 < t < T$,

$$\int_0^t e^{-r(t-s)} \left\| p'_{t-s} * \kappa(s, \cdot) \right\|_{L_2(\nu_j)} ds \leq C \int_0^t \frac{ds}{\sqrt{t-s}} = 2C\sqrt{t}.$$

We now estimate $\|\zeta(t, \cdot)\|_{L_2(\nu_j)}$.

We have, using the boundedness of γ ,

$$\begin{aligned} |\zeta(t, x)| &= \left| \int_0^t e^{-r(t-s)} \gamma(s) p'_{t-s}(x - \tilde{b}(s)) ds \right| \\ &\leq C \int_0^t \frac{1}{\sigma^2(t-s)} \left| n' \left(\frac{x - \tilde{b}(s) + \mu(t-s)}{\sigma\sqrt{t-s}} \right) \right| ds \end{aligned}$$

Recall that $n'(x) = -xn(x)$. Therefore

$$\begin{aligned} |\zeta(t, x)| &\leq C \int_0^t \frac{|x - \tilde{b}(s) + \mu(t-s)|}{(t-s)^{3/2}} n \left(\frac{x - \tilde{b}(s) + \mu(t-s)}{\sigma\sqrt{t-s}} \right) ds \\ &\leq C \int_0^t \frac{ds}{\sqrt{t-s}} + C \int_0^t \frac{|x - \tilde{b}(s)|}{(t-s)^{3/2}} n \left(\frac{x - \tilde{b}(s) + \mu(t-s)}{\sigma\sqrt{t-s}} \right) ds. \end{aligned}$$

Note that

$$\begin{aligned} n(x_1 + x_2) &= n(x_1) \exp \left(-\frac{x_2^2}{2} - x_1 x_2 \right) \leq n(x_1) \exp(-x_1 x_2) \\ &\leq n(x_1) \exp \left(\frac{x_1^2}{4} + x_2^2 \right) = n(x_1/\sqrt{2}) e^{x_2^2}. \end{aligned}$$

Hence, for $t \in (0, T)$,

$$|\zeta(t, x)| \leq C \int_0^t \frac{ds}{\sqrt{t-s}} + C_T \int_0^t \frac{|x - \tilde{b}(s)|}{(t-s)^{3/2}} n \left(\frac{x - \tilde{b}(s)}{\sqrt{2}\sigma\sqrt{t-s}} \right) ds.$$

Note that, for all $\alpha > 0$, there exists $C_\alpha > 0$, such that, for all $y \in \mathbb{R}$, $n(y/\sqrt{2}) \leq C_\alpha/|y|^{2\alpha}$.

Hence, for $t \in (0, T)$,

$$\begin{aligned} |\zeta(t, x)| &\leq C\sqrt{t} + C_\alpha \int_0^t \frac{|x - \tilde{b}(s)|}{(t-s)^{3/2}} \frac{(t-s)^\alpha}{|x - \tilde{b}(s)|^{2\alpha}} ds \\ &= C\sqrt{t} + C_\alpha \int_0^t \frac{(t-s)^{\alpha-\frac{3}{2}}}{|x - \tilde{b}(s)|^{2\alpha-1}} ds \\ &= C\sqrt{t} + C_\alpha t^{\alpha-\frac{1}{2}} \int_0^1 \frac{1}{(1-u)^{\frac{3}{2}-\alpha} |x - \tilde{b}(tu)|^{2\alpha-1}} du. \end{aligned}$$

Now, take $\alpha = \frac{1}{2} + \varepsilon$ (with $0 < \varepsilon < 1/4$) and put $\beta(t, x) = |x - \tilde{b}(t)|^{1-2\alpha} = |x - \tilde{b}(t)|^{-2\varepsilon}$. We get

$$\|\zeta(t, \cdot)\|_{L_2(\nu_j)} \leq C\sqrt{t} + Ct^\varepsilon \int_0^1 \frac{1}{(1-u)^{1-\varepsilon}} \|\beta(tu, \cdot)\|_{L_2(\nu_j)} du.$$

Using Lemma 3.1, we have

$$\|\beta(tu, \cdot)\|_{L_2(\nu_j)}^2 \leq 2^{j/2} (1 + \tilde{b}(tu)^2)^{j/2} \int \frac{1}{|x|^{4\varepsilon}} \frac{dx}{(1+x^2)^{j/2}}.$$

Since $\varepsilon < 1/4$, the integral on the righthand side is finite, and the lemma easily follows. \diamond

We now turn to the study of $\frac{\partial^2 \tilde{U}}{\partial t \partial x}$. Recall that $\partial U / \partial t$ solves the parabolic equation $-\partial v / \partial t + (A - r)v = 0$ in the set $\{(t, x) \mid t > 0, x > \tilde{b}(t)\}$. Since the exercise boundary is differentiable and $\partial U / \partial t$ is continuous and vanishes on the exercise boundary, it follows that $\frac{\partial^2 U}{\partial t \partial x}$ is continuous “up to the boundary”, i.e. on the set $\{(t, x) \mid t > 0, x \geq \tilde{b}(t)\}$ (see [3], Lemma 4.5). We first show that $\frac{\partial^2 U}{\partial t \partial x}$ is nonnegative along the exercise boundary.

Lemma 3.3. *We have, for any $t > 0$,*

$$\frac{\partial^2 U}{\partial t \partial x}(t, \tilde{b}(t)) \geq 0.$$

Proof: We have, for all $t > 0$, due to the *smooth fit* property,

$$\frac{\partial U}{\partial x}(t, \tilde{b}(t)) = \varphi'(\tilde{b}(t)),$$

so that, by differentiating with respect to t ,

$$\frac{\partial^2 U}{\partial t \partial x}(t, \tilde{b}(t)) + \frac{\partial^2 U}{\partial x^2}(t, \tilde{b}(t)) \tilde{b}'(t) = \varphi''(\tilde{b}(t)) \tilde{b}'(t)$$

and

$$\frac{\partial^2 U}{\partial t \partial x}(t, \tilde{b}(t)) = - \left(\frac{\partial^2 U}{\partial x^2}(t, \tilde{b}(t)) - \varphi''(\tilde{b}(t)) \right) \tilde{b}'(t).$$

Observe that, for each $t > 0$, the function $x \mapsto U(t, x) - \varphi(x)$ is C^2 on the interval $[\tilde{b}(t), \infty)$ and has a minimum at $\tilde{b}(t)$. Therefore, its second derivative must be nonnegative at this point. Since $\tilde{b}'(t) \leq 0$, the lemma is proved. \diamond

Lemma 3.4. *Fix $T > 0$ and $j > 1$. There exists a constant $C > 0$ such that, for all $t_1 \in (0, T \wedge 1]$,*

$$\int_{t_1}^T \left\| \frac{\partial^2 \tilde{U}}{\partial t \partial x}(t, \cdot) \right\|_{L_2(\nu_j)}^2 dt \leq C \ln(1/t_1).$$

For the proof of Lemma 3.4, we will need the bilinear form associated with the operator $A - r$.

We first introduce the relevant weighted Sobolev spaces. For $j > 1$, let $H_j = L^2(\mathbb{R}, \nu_j)$ and $V_j = \{f \in H_j \mid f' \in H_j\}$. The inner product on H_j will be denoted by $(\cdot, \cdot)_j$ and the associated norm by $\|\cdot\|_j$. The natural norm on V_j will be denoted by $\|\cdot\|_j$. Thus, we have

$$\|f\|_j^2 = \int_{-\infty}^{+\infty} f^2(x) \frac{dx}{(1+x^2)^{j/2}},$$

and $\|f\|_j^2 = \|f\|_j^2 + \|f'\|_j^2$.

Recall that the partial differential operator A is defined by

$$A = \frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} + \mu \frac{\partial}{\partial x}.$$

We associate with the operator $A - r$ a bilinear functional on V_j , defined by

$$\begin{aligned} a_j(f, g) &= \frac{\sigma^2}{2} \int_{-\infty}^{\infty} f'(x)g'(x) \frac{dx}{(1+x^2)^{j/2}} - \frac{j\sigma^2}{2} \int_{-\infty}^{\infty} f'(x)g(x) \frac{x}{(1+x^2)^{(j/2)+1}} dx \\ &\quad - \mu \int_{-\infty}^{\infty} f'(x)g(x) \frac{dx}{(1+x^2)^{j/2}} + r \int_{-\infty}^{\infty} f(x)g(x) \frac{dx}{(1+x^2)^{j/2}}, \end{aligned}$$

so that, if $f' \in V_j$,

$$a_j(f, g) = -((A - r)f, g)_j.$$

It will be convenient to write $a_j(f, g)$ as $a_j(f, g) = \tilde{a}_j(f, g) + \bar{a}_j(f, g)$, with

$$\tilde{a}_j(f, g) = \frac{\sigma^2}{2} [(f', g')_j + (f, g)_j] \quad \text{and} \quad \bar{a}_j(f, g) = a_j(f, g) - \tilde{a}_j(f, g). \quad (5)$$

With these notations, it is easy to check that $|\bar{a}_j(f, g)| \leq C\|f\|_j\|g\|_j$ and $|\tilde{a}_j(f, g)| \leq C\|g\|_j\|f\|_j$, for some constant C which does not depend on f nor g .

Proof of Lemma 3.4: In order to rule out regularity issues, we introduce a C^∞ , nonnegative function ρ on $\mathbb{R} \times \mathbb{R}$, with $\int \rho(t, x) dt dx = 1$ and $\text{supp } \rho \subset [-1, 0] \times [-1, +1]$ and set, for any positive integer m , $\rho_m(t, x) = m^2 \rho(mt, mx)$.

Now, let $W = \frac{\partial \tilde{U}}{\partial x}$, $W_m = W * \rho_m$, and $h_m = h * \rho_m$. For each $m > 0$, the functions W_m , h_m are C^∞ with bounded derivatives and we have

$$-\frac{\partial W_m}{\partial t} + (A - r)W_m = \frac{\partial h_m}{\partial x}.$$

Multiply by $\partial W_m / \partial t$ and integrate with respect to ν_j to get, for any fixed $t > 0$,

$$-\left(\frac{\partial W_m}{\partial t}(t, \cdot), \frac{\partial W_m}{\partial t}(t, \cdot) \right)_j - a_j \left(W_m(t, \cdot), \frac{\partial W_m}{\partial t}(t, \cdot) \right) = \int \frac{\partial h_m}{\partial x}(t, x) \frac{\partial W_m}{\partial t}(t, x) \nu_j(dx).$$

Note that

$$\begin{aligned} a_j \left(W_m(t, \cdot), \frac{\partial W_m}{\partial t}(t, \cdot) \right) &= \tilde{a}_j \left(W_m(t, \cdot), \frac{\partial W_m}{\partial t}(t, \cdot) \right) + \bar{a}_j \left(W_m(t, \cdot), \frac{\partial W_m}{\partial t}(t, \cdot) \right) \\ &= \frac{1}{2} \frac{d}{dt} (\tilde{a}_j(W_m(t, \cdot), W_m(t, \cdot))) + \bar{a}_j \left(W_m(t, \cdot), \frac{\partial W_m}{\partial t}(t, \cdot) \right). \end{aligned}$$

By integrating with respect to time, we get, if $0 < t_1 < T$,

$$\begin{aligned} & - \int_{t_1}^T \left| \frac{\partial W_m}{\partial t}(t, \cdot) \right|_j^2 dt + \frac{1}{2} (\tilde{a}_j(W_m(t_1, \cdot), W_m(t_1, \cdot)) - \tilde{a}_j(W_m(T, \cdot), W_m(T, \cdot))) = \\ & \int_{t_1}^T \tilde{a}_j \left(W_m(t, \cdot), \frac{\partial W_m}{\partial t}(t, \cdot) \right) dt + \int_{t_1}^T \left(\frac{\partial h_m}{\partial x}(t, \cdot), \frac{\partial W_m}{\partial t}(t, \cdot) \right)_j dt. \end{aligned}$$

Hence

$$\begin{aligned} \int_{t_1}^T \left| \frac{\partial W_m}{\partial t}(t, \cdot) \right|_j^2 dt & \leq \frac{1}{2} \tilde{a}_j(W_m(t_1, \cdot), W_m(t_1, \cdot)) - \int_{t_1}^T \tilde{a}_j \left(W_m(t, \cdot), \frac{\partial W_m}{\partial t}(t, \cdot) \right) dt \\ & \quad - \int_{t_1}^T \left(\frac{\partial h_m}{\partial x}(t, \cdot), \frac{\partial W_m}{\partial t}(t, \cdot) \right)_j dt \\ & \leq \frac{1}{2} \tilde{a}_j(W_m(t_1, \cdot), W_m(t_1, \cdot)) + C \int_{t_1}^T \|W_m(t, \cdot)\|_j \left| \frac{\partial W_m}{\partial t}(t, \cdot) \right|_j dt \\ & \quad - \int_{t_1}^T \left(\frac{\partial h_m}{\partial x}(t, \cdot), \frac{\partial W_m}{\partial t}(t, \cdot) \right)_j dt. \end{aligned}$$

Using the inequality

$$2 \|W_m(t, \cdot)\|_j \left| \frac{\partial W_m}{\partial t}(t, \cdot) \right|_j \leq \varepsilon \left| \frac{\partial W_m}{\partial t}(t, \cdot) \right|_j^2 + \frac{1}{\varepsilon} \|W_m(t, \cdot)\|_j^2,$$

we get

$$\begin{aligned} \frac{1}{2} \int_{t_1}^T \left| \frac{\partial W_m}{\partial t}(t, \cdot) \right|_j^2 dt & \leq \frac{1}{2} \tilde{a}_j(W_m(t_1, \cdot), W_m(t_1, \cdot)) + C \int_{t_1}^T \|W_m(t, \cdot)\|_j^2 dt \\ & \quad - \int_{t_1}^T \left(\frac{\partial h_m}{\partial x}(t, \cdot), \frac{\partial W_m}{\partial t}(t, \cdot) \right)_j dt. \end{aligned}$$

We have

$$\tilde{a}_j(W_m(t_1, \cdot), W_m(t_1, \cdot)) \leq C \|W_m(t_1, \cdot)\|_j^2$$

and, using Lemma 3.1,

$$\begin{aligned} \|W_m(t_1, \cdot)\|_j & \leq \int \rho_m(t_1 - t, y) \|W(t, \cdot - y)\|_j dt dy \\ & \leq \int \rho_m(t_1 - t, y) 2^{j/4} (1 + y^2)^{j/4} \|W(t, \cdot)\|_j dt dy. \end{aligned}$$

Using Lemma 3.2, we have (for $\varepsilon \in (0, 1/4)$) $\|W(t, \cdot)\|_j \leq Ct^\varepsilon$. Hence

$$\begin{aligned} \tilde{a}_j(W_m(t_1, \cdot), W_m(t_1, \cdot)) & \leq C \int \rho_m(t_1 - t, y) 2^{j/2} (1 + y^2)^{j/2} t^{2\varepsilon} dt dy \\ & = C \int \rho_m(t, y) 2^{j/2} (1 + y^2)^{j/2} |t_1 - t|^{2\varepsilon} dt dy \\ & = C \int \rho(t, y) 2^{j/2} \left(1 + \frac{y^2}{m^2}\right)^{j/2} \left|t_1 - \frac{t}{m}\right|^{2\varepsilon} dt dy \\ & \leq C(1 + t_1^{2\varepsilon}). \end{aligned}$$

We also have

$$\begin{aligned}
\int_{t_1}^T \|W_m(t, \cdot)\|_j^2 dt &= \int_{t_1}^T \left\| \int ds dy \rho_m(t-s, y) W(s, \cdot - y) \right\|_j^2 dt \\
&\leq \int_{t_1}^T \int ds dy \rho_m(t-s, y) \|W(s, \cdot - y)\|_j^2 dt \\
&\leq \int_{t_1}^T dt \int ds dy \rho_m(t-s, y) 2^{j/2} (1+y^2)^{j/2} \|W(s, \cdot)\|_j^2 \\
&\leq \int_{t_1}^{T+\frac{1}{m}} ds \|W(s, \cdot)\|_j^2 \int dt dy \rho_m(t-s, y) 2^{j/2} (1+y^2)^{j/2}.
\end{aligned}$$

Since $\int_0^{T+1} \|W(s, \cdot)\|_j^2 ds < \infty$, we deduce that

$$\frac{1}{2} \int_{t_1}^T \left| \frac{\partial W_m}{\partial t}(t, \cdot) \right|_j^2 dt \leq C(1+t_1^{2\varepsilon}) - \int_{t_1}^T \left(\frac{\partial h_m}{\partial x}(t, \cdot), \frac{\partial W_m}{\partial t}(t, \cdot) \right)_j dt.$$

It follows from the proof of Lemma 3.2 (see (4)) that

$$\frac{\partial h_m}{\partial x}(t, x) = -\kappa_m(t, x) + \gamma_m(t, x),$$

where $\kappa_m = \kappa * \rho_m$, and κ is a bounded function, and

$$\gamma_m(t, x) = \int \rho_m(t-\tau, x-\tilde{b}(\tau)) \gamma(\tau) d\tau.$$

Hence

$$\begin{aligned}
- \int_{t_1}^T \left(\frac{\partial h_m}{\partial x}(t, \cdot), \frac{\partial W_m}{\partial t}(t, \cdot) \right)_j dt &\leq C \int_{t_1}^T \left| \frac{\partial W_m}{\partial t}(t, \cdot) \right|_j^2 dt - \int_{t_1}^T \left(\gamma_m(t, \cdot), \frac{\partial W_m}{\partial t}(t, \cdot) \right)_j dt \\
&\leq \frac{1}{4} \int_{t_1}^T \left| \frac{\partial W_m}{\partial t}(t, \cdot) \right|_j^2 dt + C^2 T - \int_{t_1}^T \left(\gamma_m(t, \cdot), \frac{\partial W_m}{\partial t}(t, \cdot) \right)_j dt,
\end{aligned}$$

where we have used $C \left| \frac{\partial W_m}{\partial t}(t, \cdot) \right|_j \leq \frac{1}{4} \left| \frac{\partial W_m}{\partial t}(t, \cdot) \right|_j^2 + C^2$. Therefore

$$\frac{1}{4} \int_{t_1}^T \left| \frac{\partial W_m}{\partial t}(t, \cdot) \right|_j^2 dt \leq C(1+t_1^{2\varepsilon}) - \int_{t_1}^T \left(\gamma_m(t, \cdot), \frac{\partial W_m}{\partial t}(t, \cdot) \right)_j dt. \quad (6)$$

Note that

$$\frac{\partial W_m}{\partial t} = \frac{\partial^2 U_m}{\partial t \partial x} - \frac{\partial^2 u_m}{\partial t \partial x},$$

where

$$U_m = U * \rho_m \quad \text{and} \quad u_m = u_\varphi * \rho_m,$$

so that

$$- \int_{t_1}^T \left(\gamma_m(t, \cdot), \frac{\partial W_m}{\partial t}(t, \cdot) \right)_j dt = J_m^{(1)} + J_m^{(2)},$$

with

$$J_m^{(1)} = - \int_{t_1}^T \left(\gamma_m(t, \cdot), \frac{\partial^2 U_m}{\partial t \partial x}(t, \cdot) \right)_j dt \quad \text{and} \quad J_m^{(2)} = \int_{t_1}^T \left(\gamma_m(t, \cdot), \frac{\partial^2 u_m}{\partial t \partial x}(t, \cdot) \right)_j dt.$$

We have

$$\begin{aligned} J_m^{(1)} &= - \int_{t_1}^T dt \int \frac{dx}{(1+x^2)^{j/2}} \gamma_m(t, x) \frac{\partial^2 U_m}{\partial t \partial x}(t, x) \\ &= - \int_{t_1}^T dt \int \frac{dx}{(1+x^2)^{j/2}} \int d\tau \gamma(\tau) \rho_m(t-\tau, x-\tilde{b}(\tau)) \frac{\partial^2 U_m}{\partial t \partial x}(t, x) \\ &= - \int d\tau \int ds \int dy \mathbf{1}_{\{t_1 < \tau+s < T\}} \gamma(\tau) \rho_m(s, y) \frac{\partial^2 U_m}{\partial t \partial x}(\tau+s, \tilde{b}(\tau)+y) \frac{1}{(1+(y+\tilde{b}(\tau))^2)^{j/2}} \\ &= - \int_{t_1}^{T+\frac{1}{m}} \gamma(\tau) \eta_m(\tau) d\tau, \end{aligned}$$

where

$$\begin{aligned} \eta_m(\tau) &= \int_{t_1-\tau}^{T-\tau} ds \int \frac{dy}{(1+(y+\tilde{b}(\tau))^2)^{j/2}} \rho_m(s, y) \frac{\partial^2 U_m}{\partial t \partial x}(\tau+s, \tilde{b}(\tau)+y) \\ &= \int_{t_1-\tau}^{T-\tau} ds \int \frac{\rho_m(s, y) dy}{(1+(y+\tilde{b}(\tau))^2)^{j/2}} \int \int ds' dy' \rho_m(s', y') \frac{\partial^2 U}{\partial t \partial x}(\tau+s-s', \tilde{b}(\tau)+y-y'). \end{aligned}$$

Note that $\frac{\partial^2 U}{\partial t \partial x} = 0$ on the open set $S = \{(t, x) \mid t > 0, x < \tilde{b}(t)\}$ (which is the interior set of the stopping region), so that

$$\eta_m(\tau) = \int_{t_1-\tau}^{T-\tau} ds \int \frac{\rho_m(s, y) dy}{(1+(y+\tilde{b}(\tau))^2)^{j/2}} \int \int ds' dy' \rho_m(s', y') \bar{W}(\tau, s-s', y-y'),$$

where

$$\bar{W}(\tau, \theta, z) = \frac{\partial^2 U}{\partial t \partial x}(\tau+\theta, \tilde{b}(\tau)+z) \mathbf{1}_{\{\tilde{b}(\tau)+z \geq \tilde{b}(\tau+\theta)\}}.$$

Since $\frac{\partial^2 U}{\partial t \partial x}(\tau, \tilde{b}(\tau)) \geq 0$, for $\tau > 0$, we have

$$\eta_m(\tau) \geq \int_{t_1-\tau}^{T-\tau} ds \int \int \int \frac{\rho_m(s, y) dy \rho_m(s', y') ds' dy'}{(1+(y+\tilde{b}(\tau))^2)^{j/2}} D(\tau, s-s', y-y'),$$

where

$$D(\tau, \theta, z) = \left(\frac{\partial^2 U}{\partial t \partial x}(\tau+\theta, \tilde{b}(\tau)+z) - \frac{\partial^2 U}{\partial t \partial x}(\tau, \tilde{b}(\tau)) \right) \mathbf{1}_{\{\tilde{b}(\tau)+z \geq \tilde{b}(\tau+\theta)\}}.$$

Hence (since $\gamma \geq 0$)

$$J_m^{(1)} \leq - \int_{t_1}^{T+\frac{1}{m}} \gamma(\tau) \varepsilon_m(\tau) d\tau,$$

with

$$\varepsilon_m(\tau) = \int_{t_1-\tau}^{T-\tau} ds \int \int \int \frac{\rho_m(s, y) dy \rho_m(s', y') ds' dy'}{(1+(y+\tilde{b}(\tau))^2)^{j/2}} D(\tau, s-s', y-y').$$

We have

$$|\varepsilon_m(\tau)| \leq \int_{t_1-\tau}^{T-\tau} D_m(\tau) ds = (T - t_1)D_m(\tau),$$

where

$$D_m(\tau) = \sup_{|\theta| \leq 1/m, |z| \leq 2/m} \left(|D(\tau, \theta, z)| \mathbf{1}_{\{\tilde{b}(\tau) + z \geq \tilde{b}(\tau + \theta)\}} \right).$$

Due to the continuity properties of $\frac{\partial^2 U}{\partial t \partial x}$, as $m \rightarrow \infty$, the function D_m converges to 0, uniformly on the interval $[t_1, T + 1]$. Therefore, we have

$$\limsup_{m \rightarrow \infty} J_m^{(1)} \leq 0. \quad (7)$$

We now examine $J_m^{(2)}$. We have, using the boundedness of γ ,

$$\begin{aligned} |J_m^{(2)}| &\leq \int_{t_1}^T dt \int \frac{dx}{(1+x^2)^{j/2}} |\gamma_m(t, x)| \left| \frac{\partial^2 u_m}{\partial t \partial x}(t, x) \right| \\ &\leq C \int_{t_1}^T dt \int \frac{dx}{(1+x^2)^{j/2}} \int d\tau \rho_m(t - \tau, x - \tilde{b}(\tau)) \left| \frac{\partial^2 u_m}{\partial t \partial x}(t, x) \right|. \end{aligned}$$

Note that

$$\left\| \frac{\partial^2 u_m}{\partial t \partial x}(t, \cdot) \right\|_{\infty} \leq \left\| \frac{\partial^2 u_{\varphi}}{\partial t \partial x}(t, \cdot) \right\|_{\infty} \leq \frac{C}{t}.$$

Hence

$$|J_m^{(2)}| \leq C \int_{t_1}^T \frac{dt}{t} \int dx \int d\tau \rho_m(t - \tau, x - \tilde{b}(\tau)) = C \ln \frac{T}{t_1}. \quad (8)$$

It follows from (6), (7) and (8) that

$$\limsup_{m \rightarrow \infty} \int_{t_1}^T \left| \frac{\partial W_m}{\partial t}(t, \cdot) \right|_j^2 dt \leq C \left(1 + t_1^{2\varepsilon} + \ln \frac{T}{t_1} \right),$$

which proves the lemma. \diamond

3.3 Proof of Theorem 3.1

For the proof of Theorem 3.1, we will work on the equation satisfied by $\partial \tilde{U} / \partial t$. Let

$$V = \frac{\partial \tilde{U}}{\partial t}.$$

We have

$$-\frac{\partial V}{\partial t} + (A - r)V = \frac{\partial h}{\partial t},$$

where

$$h(t, x) = (A - r)\varphi(x) \mathbf{1}_{\{x \leq \tilde{b}(t)\}}, \quad t > 0, \quad x \in \mathbb{R}.$$

The following lemma will clarify the computation of the derivative $\partial h / \partial t$ in the sense of distributions.

Lemma 3.5. Define the function I on $(0, +\infty) \times \mathbb{R}$ by

$$I(t, x) = \mathbf{1}_{\{x \leq \tilde{b}(t)\}}, \quad t > 0, \quad x \in \mathbb{R}.$$

The distribution $\partial I / \partial t$ applied to a compactly supported C^∞ function ρ on $(0, +\infty) \times \mathbb{R}$ is given by

$$\left\langle \frac{\partial I}{\partial t}, \rho \right\rangle = \int \tilde{b}'(t) \rho(t, \tilde{b}(t)) dt.$$

This can be written (less precisely): $\frac{\partial I}{\partial t}(t, \cdot) = \tilde{b}'(t) \delta_{\tilde{b}(t)}$.

Proof: We have

$$\begin{aligned} \left\langle \frac{\partial I}{\partial t}, \rho \right\rangle &= - \left\langle I, \frac{\partial \rho}{\partial t} \right\rangle \\ &= - \int dt \int dx I(t, x) \frac{\partial \rho}{\partial t}(t, x) \end{aligned}$$

Let J be the range of \tilde{b} . We have $J = (\tilde{b}(\infty), \tilde{b}(0))$. Note that, if $x \leq \tilde{b}(\infty)$, $I(t, x) = 1$ for all $t > 0$ and if $x \geq \tilde{b}(0)$ $I(t, x) = 0$ for all $t > 0$, so that, in both cases, $\int I(t, x) \frac{\partial \rho}{\partial t}(t, x) dt = 0$. Therefore

$$\begin{aligned} \left\langle \frac{\partial I}{\partial t}, \rho \right\rangle &= - \int_J dx \int dt \mathbf{1}_{\{x \leq \tilde{b}(t)\}} \frac{\partial \rho}{\partial t}(t, x) \\ &= - \int_J dx \int dt \mathbf{1}_{\{t \leq \tilde{b}^{-1}(x)\}} \frac{\partial \rho}{\partial t}(t, x) \\ &= - \int_J dx \rho(\tilde{b}^{-1}(x), x) \\ &= \int \tilde{b}'(t) \rho(t, \tilde{b}(t)) dt. \end{aligned}$$

Here, we have used the fact that \tilde{b} is strictly decreasing (which is proved in [8]), but we can also approximate \tilde{b} by the strictly decreasing functions $\tilde{b}_\varepsilon(t) = -\varepsilon t + \tilde{b}(t)$ to derive the formula. In fact, we only need \tilde{b} to be C^1 : indeed, we can replace $\tilde{b}(t)$ by $\tilde{b}_\mu(t) = -\mu t + \tilde{b}(t)$ and choose μ so that \tilde{b}_μ is strictly increasing in a neighborhood of the time projection of the support of ρ . \diamond

We now proceed with the proof of Theorem 3.1. As in the proof of Lemma 3.4, we introduce a regularizing sequence ρ_m , and set

$$V_m = V * \rho_m \quad \text{and} \quad \chi_m = \frac{\partial h}{\partial t} * \rho_m,$$

so that

$$-\frac{\partial V_m}{\partial t} + (A - r)V_m = \chi_m$$

Note that the functions V_m, χ_m are C^∞ with bounded derivatives. For any fixed $t > 0$, multiply by $\partial V_m / \partial t$ and integrate with respect to ν_j to get

$$-\left| \frac{\partial V_m}{\partial t}(t, \cdot) \right|_j^2 - a_j \left(V_m(t, \cdot), \frac{\partial V_m}{\partial t}(t, \cdot) \right) = \int \chi_m(t, x) \frac{\partial V_m}{\partial t}(t, x) \nu_j(dx).$$

We have

$$a_j \left(V_m(t, \cdot), \frac{\partial V_m}{\partial t}(t, \cdot) \right) = \frac{1}{2} \frac{d}{dt} (\tilde{a}_j(V_m(t, \cdot), V_m(t, \cdot))) + \bar{a}_j \left(V_m(t, \cdot), \frac{\partial V_m}{\partial t}(t, \cdot) \right).$$

By integrating with respect to time, we get, if $0 < t_1 < T$,

$$\begin{aligned} & - \int_{t_1}^T \left| \frac{\partial V_m}{\partial t}(t, \cdot) \right|_j^2 dt + \frac{1}{2} [\tilde{a}_j(V_m(t_1, \cdot), V_m(t_1, \cdot)) - \tilde{a}_j(V_m(T, \cdot), V_m(T, \cdot))] = \\ & \int_{t_1}^T \bar{a}_j \left(V_m(t, \cdot), \frac{\partial V_m}{\partial t}(t, \cdot) \right) dt + \int_{t_1}^T \left(\chi_m(t, \cdot), \frac{\partial V_m}{\partial t}(t, \cdot) \right)_j dt. \end{aligned}$$

Hence

$$\begin{aligned} \int_{t_1}^T \left| \frac{\partial V_m}{\partial t}(t, \cdot) \right|_j^2 dt & \leq C \|V_m(t_1, \cdot)\|_j^2 - \int_{t_1}^T \bar{a}_j \left(V_m(t, \cdot), \frac{\partial V_m}{\partial t}(t, \cdot) \right) dt \\ & \quad - \int_{t_1}^T \left(\chi_m(t, \cdot), \frac{\partial V_m}{\partial t}(t, \cdot) \right)_j dt \\ & \leq C \left(\|V_m(t_1, \cdot)\|_j^2 + \int_{t_1}^T \|V_m(t, \cdot)\|_j \left| \frac{\partial V_m}{\partial t}(t, \cdot) \right|_j dt \right) + J_m(t_1, T), \end{aligned}$$

with

$$J_m(t_1, T) = - \int_{t_1}^T \left(\chi_m(t, \cdot), \frac{\partial V_m}{\partial t}(t, \cdot) \right)_j dt.$$

Using the inequality

$$2 \|V_m(t, \cdot)\|_j \left| \frac{\partial V_m}{\partial t}(t, \cdot) \right|_j \leq \varepsilon \left| \frac{\partial V_m}{\partial t}(t, \cdot) \right|_j^2 + \frac{1}{\varepsilon} \|V_m(t, \cdot)\|_j^2,$$

we derive

$$\frac{1}{2} \int_{t_1}^T \left| \frac{\partial V_m}{\partial t}(t, \cdot) \right|_j^2 dt \leq C \left(\|V_m(t_1, \cdot)\|_j^2 + \int_{t_1}^T \|V_m(t, \cdot)\|_j^2 dt \right) + J_m(t_1, T). \quad (9)$$

We now study $J_m(t_1, T)$. Note that, for any fixed $t > 0$,

$$\left(\chi_m(t, \cdot), \frac{\partial V_m}{\partial t}(t, \cdot) \right)_j = \int \nu_j(dx) \frac{\partial V_m}{\partial t}(t, x) \frac{\partial h}{\partial t} * \rho_m(t, x)$$

We have

$$\frac{\partial h}{\partial t}(t, \cdot) = (A - r)\varphi \frac{\partial I}{\partial t}(t, \cdot),$$

so that, using Lemma 3.5, and the notation $\gamma(t) = -(A - r)\varphi(\tilde{b}(t))$

$$\frac{\partial h}{\partial t} * \rho_m(t, x) = - \int d\tau \rho_m(t - \tau, x - \tilde{b}(\tau)) \tilde{b}'(\tau) \gamma(\tau),$$

Recall that $\gamma(\tau) \geq 0$. Hence

$$\begin{aligned}
\left(\chi_m(t, \cdot), \frac{\partial V_m}{\partial t}(t, \cdot) \right)_j &= - \int \nu_j(dx) \frac{\partial V_m}{\partial t}(t, x) \int d\tau \rho_m(t - \tau, x - \tilde{b}(\tau)) \tilde{b}'(\tau) \gamma(\tau) \\
&= - \int d\tau \int \frac{dx}{(1+x^2)^{j/2}} \frac{\partial V_m}{\partial t}(t, x) \rho_m(t - \tau, x - \tilde{b}(\tau)) \tilde{b}'(\tau) \gamma(\tau) \\
&= - \int d\tau \int \frac{dy}{(1+(y+\tilde{b}(\tau))^2)^{j/2}} \frac{\partial V_m}{\partial t}(t, y + \tilde{b}(\tau)) \rho_m(t - \tau, y) \tilde{b}'(\tau) \gamma(\tau).
\end{aligned}$$

Going back to $J_m(t_1, T)$, we have

$$J_m(t_1, T) = - \int_{t_1}^T dt \int d\tau \int dy \frac{\partial V_m}{\partial t}(t, y + \tilde{b}(\tau)) \rho_m(t - \tau, y) \bar{\gamma}_j(\tau, y),$$

with

$$\bar{\gamma}_j(\tau, y) = - \frac{1}{(1+(y+\tilde{b}(\tau))^2)^{j/2}} \tilde{b}'(\tau) \gamma(\tau).$$

Note that $\bar{\gamma}_j(\tau, y) \geq 0$. We have

$$\begin{aligned}
J_m(t_1, T) &= - \int d\tau \int dt \int dy \mathbf{1}_{\{t_1 < t < T\}} \frac{\partial V_m}{\partial t}(t, y + \tilde{b}(\tau)) \rho_m(t - \tau, y) \bar{\gamma}_j(\tau, y) \\
&= - \int d\tau \int ds \int dy \mathbf{1}_{\{t_1 < \tau + s < T\}} \frac{\partial V_m}{\partial t}(\tau + s, y + \tilde{b}(\tau)) \rho_m(s, y) \bar{\gamma}_j(\tau, y).
\end{aligned}$$

Observe that

$$\frac{d}{d\tau} \left(V_m(\tau + s, y + \tilde{b}(\tau)) \right) = \frac{\partial V_m}{\partial t}(\tau + s, y + \tilde{b}(\tau)) + \frac{\partial V_m}{\partial x}(\tau + s, y + \tilde{b}(\tau)) \tilde{b}'(\tau),$$

so that

$$\frac{\partial V_m}{\partial t}(\tau + s, y + \tilde{b}(\tau)) = \frac{d}{d\tau} \left(V_m(\tau + s, y + \tilde{b}(\tau)) \right) - \frac{\partial V_m}{\partial x}(\tau + s, y + \tilde{b}(\tau)) \tilde{b}'(\tau).$$

Hence

$$J_m(t_1, T) = \hat{J}_m(t_1, T) + \bar{J}_m(t_1, T),$$

with

$$\hat{J}_m(t_1, T) = - \int d\tau \int ds \int dy \mathbf{1}_{\{t_1 < \tau + s < T\}} \frac{d}{d\tau} \left(V_m(\tau + s, y + \tilde{b}(\tau)) \right) \rho_m(s, y) \bar{\gamma}_j(\tau, y)$$

and

$$\bar{J}_m(t_1, T) = + \int d\tau \int ds \int dy \mathbf{1}_{\{t_1 < \tau + s < T\}} \frac{\partial V_m}{\partial x}(\tau + s, y + \tilde{b}(\tau)) \tilde{b}'(\tau) \rho_m(s, y) \bar{\gamma}_j(\tau, y).$$

We have, using integration by parts,

$$\begin{aligned}
\hat{J}_m(t_1, T) &= - \int ds \int dy \rho_m(s, y) \left(\int_{t_1-s}^{T-s} \frac{d}{d\tau} \left(V_m(\tau + s, y + \tilde{b}(\tau)) \right) \bar{\gamma}_j(\tau, y) d\tau \right) \\
&= - \int ds \int dy \rho_m(s, y) V_m(T, y + \tilde{b}(T-s)) \bar{\gamma}_j(T-s, y) \\
&\quad + \int ds \int dy \rho_m(s, y) V_m(t_1, y + \tilde{b}(t_1-s)) \bar{\gamma}_j(t_1-s, y) \\
&\quad + \int ds \int dy \rho_m(s, y) \int_{t_1-s}^{T-s} V_m(s + \tau, y + \tilde{b}(\tau)) \frac{\partial \bar{\gamma}_j}{\partial \tau}(\tau, y) d\tau.
\end{aligned}$$

Note that, due to the continuity of $V(= \partial\tilde{U}/\partial t)$ on $(0, \infty) \times \mathbb{R}$, the sequence V_m converges uniformly to V on compact sets. We also have the continuity of $\tilde{\gamma}_j$ and $\partial\tilde{\gamma}_j/\partial\tau$ (due to the fact that \tilde{b} is C^2). We easily deduce thereof that

$$\lim_{m \rightarrow \infty} \hat{J}_m(t_1, T) = -V(T, \tilde{b}(T))\tilde{\gamma}_j(T, 0) + V(t_1, \tilde{b}(t_1))\tilde{\gamma}_j(t_1, 0) + \int_{t_1}^T V(\tau, \tilde{b}(\tau)) \frac{\partial\tilde{\gamma}_j}{\partial\tau}(\tau, 0) d\tau, \quad (10)$$

and the convergence is uniform with respect to t_1 , as long as t_1 remains in a compact set of the form $[\xi, T]$, where $0 < \xi < T$. For $\bar{J}_m(t_1, T)$, we have

$$\bar{J}_m(t_1, T) = \bar{J}_m^{(1)}(t_1, T) + \bar{J}_m^{(2)}(t_1, T),$$

with

$$\bar{J}_m^{(1)}(t_1, T) = + \int d\tau \int ds \int dy \mathbf{1}_{\{t_1 < \tau + s < T\}} \frac{\partial^2 U_m}{\partial t \partial x}(\tau + s, y + \tilde{b}(\tau)) \tilde{b}'(\tau) \rho_m(s, y) \tilde{\gamma}_j(\tau, y)$$

and

$$\bar{J}_m^{(2)}(t_1, T) = - \int d\tau \int ds \int dy \mathbf{1}_{\{t_1 < \tau + s < T\}} \frac{\partial^2 u_m}{\partial t \partial x}(\tau + s, y + \tilde{b}(\tau)) \tilde{b}'(\tau) \rho_m(s, y) \tilde{\gamma}_j(\tau, y).$$

We deal with $\bar{J}_m^{(1)}(t_1, T)$ in the same way as for the proof of (7). Using the fact that $\tilde{b}'(\tau)\tilde{\gamma}_j(\tau, y) \leq 0$, we have $\bar{J}_m^{(1)}(t_1, T) \leq \tilde{J}_m^{(1)}(t_1, T)$, with

$$\tilde{J}_m^{(1)}(t_1, T) = \int d\tau \int ds \int dy \mathbf{1}_{\{t_1 < \tau + s < T\}} \int \int ds' dy' \rho_m(s', y') D(\tau, s - s', y - y') \tilde{b}'(\tau) \rho_m(s, y) \tilde{\gamma}_j(\tau, y),$$

where

$$D(\tau, \theta, z) = \left(\frac{\partial^2 U}{\partial t \partial x}(\tau + \theta, \tilde{b}(\tau) + z) - \frac{\partial^2 U}{\partial t \partial x}(\tau, \tilde{b}(\tau)) \right) \mathbf{1}_{\{\tilde{b}(\tau) + z \geq \tilde{b}(\tau + \theta)\}}.$$

Due to the continuity properties of $\frac{\partial^2 U}{\partial t \partial x}$, we have

$$\lim_{m \rightarrow \infty} \tilde{J}_m^{(1)}(t_1, T) = 0,$$

and the convergence is uniform with respect to t_1 , as long as t_1 remains in $[\xi, T]$. On the other hand, due to the continuity of $\frac{\partial^2 u}{\partial t \partial x}$, we have

$$\lim_{m \rightarrow \infty} \bar{J}_m^{(2)}(t_1, T) = - \int_{t_1}^T d\tau \frac{\partial^2 u}{\partial t \partial x}(\tau, \tilde{b}(\tau)) \tilde{b}'(\tau) \tilde{\gamma}_j(\tau, 0),$$

uniformly with respect to $t_1 \in [\xi, T]$. At this stage, we can state that $J_m(t_1, T) \leq \bar{J}_m(t_1, T)$, with $\tilde{J}_m(t_1, T) = \hat{J}_m(t_1, T) + \tilde{J}_m^{(1)}(t_1, T) + \bar{J}_m^{(2)}(t_1, T)$, and

$$\lim_{m \rightarrow \infty} \sup_{t_1 \in [\xi, T]} |\tilde{J}_m(t_1, T) - \tilde{J}(t_1, T)| = 0,$$

where

$$\begin{aligned} \tilde{J}(t_1, T) &= -V(T, \tilde{b}(T))\tilde{\gamma}_j(T, 0) + V(t_1, \tilde{b}(t_1))\tilde{\gamma}_j(t_1, 0) + \int_{t_1}^T V(\tau, \tilde{b}(\tau)) \frac{\partial\tilde{\gamma}_j}{\partial\tau}(\tau, 0) d\tau \\ &\quad - \int_{t_1}^T d\tau \frac{\partial^2 u}{\partial t \partial x}(\tau, \tilde{b}(\tau)) \tilde{b}'(\tau) \tilde{\gamma}_j(\tau, 0). \end{aligned}$$

Since $\partial U/\partial t$ vanishes along the exercise boundary, we have $V(t, \tilde{b}(t)) = -\frac{\partial u_\varphi}{\partial t}(t, \tilde{b}(t))$, so that

$$\begin{aligned} & -V(T, \tilde{b}(T))\bar{\gamma}_j(T, 0) + V(t_1, \tilde{b}(t_1))\bar{\gamma}_j(t_1, 0) + \int_{t_1}^T V(\tau, \tilde{b}(\tau))\frac{\partial \bar{\gamma}_j}{\partial \tau}(\tau, 0)d\tau = \\ & \quad \frac{\partial u_\varphi}{\partial t}(T, \tilde{b}(T))\bar{\gamma}_j(T, 0) - \frac{\partial u_\varphi}{\partial t}(t_1, \tilde{b}(t_1))\bar{\gamma}_j(t_1, 0) - \int_{t_1}^T \frac{\partial u_\varphi}{\partial t}(\tau, \tilde{b}(\tau))\frac{\partial \bar{\gamma}_j}{\partial \tau}(\tau, 0)d\tau \\ & = \int_{t_1}^T \frac{d}{d\tau} \left(\frac{\partial u_\varphi}{\partial t}(\tau, \tilde{b}(\tau)) \right) \bar{\gamma}_j(\tau, 0)d\tau, \end{aligned}$$

so that

$$\begin{aligned} \tilde{J}(t_1, T) &= \int_{t_1}^T \left[\frac{d}{d\tau} \left(\frac{\partial u_\varphi}{\partial t}(\tau, \tilde{b}(\tau)) \right) - \frac{\partial^2 u_\varphi}{\partial t \partial x}(\tau, \tilde{b}(\tau))\tilde{b}'(\tau) \right] \bar{\gamma}_j(\tau, 0)d\tau \\ &= \int_{t_1}^T \frac{\partial^2 u_\varphi}{\partial t^2}(\tau, \tilde{b}(\tau))\bar{\gamma}_j(\tau, 0)d\tau. \end{aligned}$$

We now go back to (9) and integrate with respect to t_1 to derive

$$\begin{aligned} \frac{1}{2} \int_\xi^T dt_1 \int_{t_1}^T \left| \frac{\partial^2 \tilde{U}_m}{\partial t^2}(t, \cdot) \right|_j^2 dt &\leq C \left(\int_\xi^T \|V_m(t_1, \cdot)\|_j^2 dt_1 + \int_\xi^T \left(\int_{t_1}^T \|V_m(t, \cdot)\|_j^2 dt \right) dt_1 \right) \\ &\quad + \int_\xi^T dt_1 \tilde{J}_m(t_1, T). \end{aligned}$$

Hence

$$\frac{1}{2} \int_\xi^T (t - \xi) \left| \frac{\partial^2 \tilde{U}_m}{\partial t^2}(t, \cdot) \right|_j^2 dt \leq C \int_\xi^T \|V_m(t, \cdot)\|_j^2 dt + \int_\xi^T dt_1 \tilde{J}_m(t_1, T).$$

Note that

$$\begin{aligned} \lim_{m \rightarrow \infty} \int_\xi^T \|V_m(t, \cdot)\|_j^2 dt &= \int_\xi^T \|V(t, \cdot)\|_j^2 dt \\ &\leq C \left(1 + \ln \frac{T}{\xi} \right), \end{aligned}$$

where the last inequality follows from Lemma 3.4. Moreover,

$$\begin{aligned} \lim_{m \rightarrow \infty} \int_\xi^T dt_1 \tilde{J}_m(t_1, T) &= \int_\xi^T dt_1 \tilde{J}(t_1, T) \\ &= \int_\xi^T (t - \xi) \frac{\partial^2 u_\varphi}{\partial t^2}(t, \tilde{b}(t))\bar{\gamma}_j(t, 0)dt. \end{aligned}$$

Hence

$$\frac{1}{2} \int_\xi^T (t - \xi) \left| \frac{\partial^2 \tilde{U}}{\partial t^2}(t, \cdot) \right|_j^2 dt \leq C \left(1 + \ln \frac{T}{\xi} \right) + \int_\xi^T (t - \xi) \frac{\partial^2 u_\varphi}{\partial t^2}(t, \tilde{b}(t))\bar{\gamma}_j(t, 0)dt.$$

Theorem 3.1 now follows from the following lemma, which relies on the asymptotic behavior of the exercise boundary near maturity (see [1], [7]).

Lemma 3.6. *We have*

$$\int_{\xi}^T (t - \xi) \left| \frac{\partial^2 u_{\varphi}}{\partial t^2}(t, \tilde{b}(t)) \right| |\tilde{b}'(t)| dt \leq C \left(1 + |\ln \xi|^{\beta}\right), \quad \text{with } \beta = \begin{cases} 3/2, & \text{if } d \leq r, \\ 1, & \text{if } d > r. \end{cases}$$

Proof: We have

$$\left\| \frac{\partial^2 u_{\varphi}}{\partial t^2}(t, \cdot) \right\|_{\infty} \leq \frac{C}{t} \left\| \frac{\partial u_{\varphi}}{\partial t}(t, \cdot) \right\|_{\infty} \leq \frac{C}{t^{3/2}}$$

Hence

$$\begin{aligned} \int_{\xi}^T (t - \xi) \left| \frac{\partial^2 u_{\varphi}}{\partial t^2}(t, \tilde{b}(t)) \right| |\tilde{b}'(t)| dt &\leq C \int_{\xi}^T \frac{t - \xi}{t^{3/2}} |\tilde{b}'(t)| dt \\ &\leq C \int_{\xi}^T \frac{1}{\sqrt{t}} |\tilde{b}'(t)| dt. \end{aligned}$$

Now, since $\tilde{b}'(t) \leq 0$, we have

$$\begin{aligned} \int_{\xi}^T \frac{1}{\sqrt{t}} |\tilde{b}'(t)| dt &= - \int_{\xi}^T \frac{1}{\sqrt{t}} \tilde{b}'(t) dt \\ &= - \left(\frac{\tilde{b}(T) - \tilde{b}(0)}{\sqrt{T}} - \frac{\tilde{b}(\xi) - \tilde{b}(0)}{\sqrt{\xi}} \right) - \frac{1}{2} \int_{\xi}^T \frac{1}{t^{3/2}} (\tilde{b}(t) - \tilde{b}(0)) dt \\ &\leq \frac{\tilde{b}(0) - \tilde{b}(T)}{\sqrt{T}} + \frac{1}{2} \int_{\xi}^T \frac{1}{t^{3/2}} (\tilde{b}(0) - \tilde{b}(t)) dt \end{aligned}$$

If $d \leq r$, we have $\tilde{b}(0) - \tilde{b}(t) \leq C\sqrt{t|\ln t|}$ for t close to 0, so that

$$\int_{\xi}^T \frac{1}{t^{3/2}} (\tilde{b}(0) - \tilde{b}(t)) dt \leq C \int_{\xi}^T \frac{1}{t} \sqrt{|\ln t|} dt \leq C(1 + |\ln \xi|^{3/2}).$$

If $d > r$, we have $\tilde{b}(0) - \tilde{b}(t) \leq C\sqrt{t}$ for t close to 0, so that

$$\int_{\xi}^T \frac{1}{t^{3/2}} (\tilde{b}(0) - \tilde{b}(t)) dt \leq C \int_{\xi}^T \frac{1}{t} dt = C \ln(T/\xi).$$

◇

4 Upper bound for $P_0^{(n)} - P_0$

In order to derive an upper bound for $P_0^{(n)} - P_0$, we relate this quantity to u using (1) as follows:

$$\begin{aligned} P_0^{(n)} - P_0 &= \sup_{\tau \in \mathcal{T}_{0,T}^{(n)}} \mathbb{E} \left(e^{-r\tau} g(\mu_0 \tau + B_{\tau}^{(n)}) \right) - u(0, 0) \\ &\leq \sup_{\tau \in \mathcal{T}_{0,T}^{(n)}} \mathbb{E} \left(u(\tau, B_{\tau}^{(n)}) - u(0, 0) \right) \\ &= \sup_{\tau \in \mathcal{T}_{0,T}^{(n)}} \mathbb{E} \left(\sum_{j=1}^{\tau/h} \mathcal{D}u((j-1)h, B_{(j-1)h}^{(n)}) \right). \end{aligned}$$

We observe that $\mathcal{D}u \leq \tilde{\mathcal{D}}u$, and recall from [6] (Lemma 4.1) that $\sup_{0 \leq j \leq n-1} \mathbb{E} \left| \mathcal{D}u(jh, B_{jh}^{(n)}) \right| \leq Ch$, so that

$$\begin{aligned} P_0^{(n)} - P_0 &\leq \mathbb{E} \left(\sum_{j=1}^{n-2} \left| \tilde{\mathcal{D}}u(jh, B_{jh}^{(n)}) \right| \right) + O(h) \\ &\leq C_{k,X} h \sqrt{2} \int_h^{T-h} \frac{ds}{\sqrt{s}} \int \frac{dy}{1+|y|^k} \left| \frac{\partial^3 u}{\partial t \partial x^2}(s, y) \right| + O(h). \end{aligned} \quad (11)$$

Here, we have a regularity problem, since u is not C^3 . This problem can be fixed as follows. By convolution, one can approximate u by a sequence u_m which is smooth, uniformly bounded and satisfies $\delta u_m \leq 0$, and $\mathcal{D}u_m \leq \tilde{\mathcal{D}}u_m$. We need the following variant of Lemma 3.1.

Lemma 4.1. *If ρ is a Radon measure on $(0, T) \times \mathbb{R}$ and q a nonnegative integrable function on $(0, T) \times \mathbb{R}$, with $q(t, x) = 0$ for $t \notin (0, a)$, where a satisfies $0 < a < h$, we have*

$$\int_h^{T-h} \frac{ds}{\sqrt{s}} \int \frac{dy}{(1+|y|^2)^{k/2}} |\rho * q(s, y)| \leq 2^{k/2} \int_{h-a}^{T-h} \int_{\mathbb{R}} \frac{|\rho(dt, dz)|}{\sqrt{t}(1+z^2)^{k/2}} \int_0^a ds \int_{-\infty}^{\infty} q(s, x)(1+x^2)^{k/2} dx.$$

Proof: We have

$$\begin{aligned} \int_h^{T-h} \frac{ds}{\sqrt{s}} \int \frac{dy}{(1+|y|^2)^{k/2}} |\rho * q(s, y)| &\leq \\ &\int_h^{T-h} \frac{ds}{\sqrt{s}} \int \frac{dy}{(1+|y|^2)^{k/2}} \int \int |\rho(dt, dz)| q(s-t, y-z) \\ &= \int \int |\rho(dt, dz)| \int_h^{T-h} \frac{ds}{\sqrt{s}} \int \frac{dy}{(1+|y|^2)^{k/2}} q(s-t, y-z) \\ &\leq \int_{h-a}^{T-h} \int |\rho(dt, dz)| \int_0^a \frac{d\theta}{\sqrt{t+\theta}} \int \frac{dy}{(1+|y|^2)^{k/2}} q(\theta, y-z) \\ &\leq \int_{h-a}^{T-h} \int \frac{|\rho(dt, dz)|}{\sqrt{t}} \int_0^a d\theta \int \frac{dy}{(1+|y|^2)^{k/2}} q(\theta, y-z) \\ &= \int_{h-a}^{T-h} \int \frac{|\rho(dt, dz)|}{\sqrt{t}(1+z^2)^{k/2}} \int_0^a d\theta \int \frac{(1+z^2)^{k/2} dx}{(1+|x+z|^2)^{k/2}} q(\theta, x) \\ &\leq 2^{k/2} \int_{h-a}^{T-h} \int \frac{|\rho(dt, dz)|}{\sqrt{t}(1+z^2)^{k/2}} \int_0^a d\theta \int dx (1+x^2)^{k/2} q(\theta, x), \end{aligned}$$

where the last inequality follows from $\frac{1+z^2}{1+(x+z)^2} \leq 2(1+x^2)$. \diamond

Using Lemma 4.1 and the fact that $\partial^3 u / (\partial t \partial x^2)$ is a Radon measure, we derive the correct version of (11), namely

$$P_0^{(n)} - P_0 \leq C_{k,X} h \sqrt{2} \int_h^{T-h} \frac{1}{\sqrt{s}} \int \frac{1}{1+|y|^k} \left| \frac{\partial^3 u}{\partial t \partial x^2}(ds, dy) \right| + O(h). \quad (12)$$

If we introduce the function $\tilde{u} := u - \bar{u}$, we have, using the fact that $\delta\bar{u} = 0$,

$$\begin{aligned}
\int_h^{T-h} \frac{1}{\sqrt{s}} \int \frac{1}{1+|y|^k} \left| \frac{\partial^3 u}{\partial t \partial x^2}(ds, dy) \right| &\leq \int_h^{T-h} \frac{1}{\sqrt{s}} \int \frac{1}{1+|y|^k} \left| \frac{\partial^3 \tilde{u}}{\partial t \partial x^2}(ds, dy) \right| \\
&\quad + 2 \int_h^{T-h} \frac{ds}{\sqrt{s}} \int \frac{dy}{1+|y|^k} \left| \frac{\partial^2 \bar{u}}{\partial t^2}(s, y) \right| \\
&\leq \int_h^{T-h} \frac{1}{\sqrt{s}} \int \frac{1}{1+|y|^k} \left| \frac{\partial^3 \tilde{u}}{\partial t \partial x^2}(ds, dy) \right| \\
&\quad + 2C_T \int_h^{T-h} \frac{ds}{\sqrt{s}(T-s)} \\
&\leq \int_h^{T-h} \frac{1}{\sqrt{s}} \int \frac{1}{1+|y|^k} \left| \frac{\partial^3 \tilde{u}}{\partial t \partial x^2}(ds, dy) \right| + C_T |\ln h|,
\end{aligned}$$

where we have used Proposition 3.1.

We now need to estimate $\int_h^{T-h} \frac{1}{\sqrt{s}} \int \frac{1}{1+|y|^k} \left| \frac{\partial^3 \tilde{u}}{\partial t \partial x^2}(ds, dy) \right|$. Recall from Remark 2.1 that

$$\frac{\partial \tilde{u}}{\partial t}(t, x) + \frac{1}{2} \frac{\partial^2 \tilde{u}}{\partial x^2}(t, x) = \zeta(t, x) \mathbb{1}_{\{x \leq \hat{b}(t)\}},$$

where $\zeta(t, x) = e^{-rt}(A-r)\varphi(\ln(S_0) + \mu t + \sigma x)$ and $\hat{b}(t) = (\tilde{b}(T-t) - \mu t - \ln(S_0)) / \sigma$. By differentiating with respect to t , we derive the following expression

$$\frac{\partial^2 \tilde{u}}{\partial t^2}(t, x) + \frac{1}{2} \frac{\partial^3 \tilde{u}}{\partial t \partial x^2}(t, x) = \frac{\partial \zeta}{\partial t}(t, x) \mathbb{1}_{\{x \leq \hat{b}(t)\}} + \zeta(t, \hat{b}(t)) \hat{b}'(t) \delta_{\hat{b}(t)}, \quad (13)$$

where we have used Lemma 3.5. Note that

$$\sup_{x \leq \hat{b}(t)} \left| \frac{\partial \zeta}{\partial t}(t, x) \right| < \infty \quad \text{and} \quad \sup_{0 < t < T} \left| \zeta(t, \hat{b}(t)) \right| < \infty.$$

Moreover, $|\hat{b}'(t)| \leq |\tilde{b}'(T-t)| + |\mu/\sigma|$, so that

$$\begin{aligned}
\int_0^T \frac{dt}{\sqrt{t}} |\hat{b}'(t)| &\leq \int_0^{T/2} \frac{dt}{\sqrt{t}} |\tilde{b}'(T-t)| + \int_{T/2}^T \frac{dt}{\sqrt{t}} |\tilde{b}'(T-t)| + 2|\mu/\sigma|\sqrt{T} \\
&\leq \sup_{0 \leq t \leq T/2} |\tilde{b}'(T-t)| \int_0^{T/2} \frac{dt}{\sqrt{t}} + \sqrt{\frac{2}{T}} \int_{T/2}^T |\tilde{b}'(T-t)| dt + 2|\mu/\sigma|\sqrt{T} \\
&= \sup_{0 \leq t \leq T/2} |\tilde{b}'(T-t)| \int_0^{T/2} \frac{dt}{\sqrt{t}} + \sqrt{\frac{2}{T}} (\tilde{b}(0) - \tilde{b}(T/2)) + 2|\mu/\sigma|\sqrt{T} < \infty.
\end{aligned}$$

Hence we have

$$\int_h^{T-h} \frac{1}{\sqrt{s}} \int \frac{1}{1+|y|^k} \left| \frac{\partial^3 \tilde{u}}{\partial t \partial x^2}(ds, dy) \right| \leq C_T + 2 \int_h^{T-h} \frac{ds}{\sqrt{s}} \int \frac{1}{1+|y|^k} \left| \frac{\partial^2 \tilde{u}}{\partial t^2}(s, y) \right|.$$

Now, using the Cauchy-Schwarz inequality and Theorem 3.1, we have

$$\begin{aligned}
\int_h^{T-h} \frac{ds}{\sqrt{s}} \int \frac{dy}{1+|y|^k} \left| \frac{\partial^2 \tilde{u}}{\partial t^2}(s, y) \right| &\leq C \left(\int_h^{T-h} \frac{ds}{s(T-s-\frac{h}{2})} \right)^{1/2} \left(\int_h^{T-h} ds (T-s-\frac{h}{2}) \left| \frac{\partial^2 \tilde{u}}{\partial t^2}(s, \cdot) \right|_k^2 \right)^{1/2} \\
&\leq C \sqrt{|\ln h|} \left(\int_h^{T-\frac{h}{2}} ds (T-s-\frac{h}{2}) \left| \frac{\partial^2 \tilde{u}}{\partial t^2}(s, \cdot) \right|_k^2 \right)^{1/2} \\
&= C \sqrt{|\ln h|} \left(\int_{h/2}^{T-h} dt (t-\frac{h}{2}) \left| \frac{\partial^2 \tilde{u}}{\partial t^2}(T-t, \cdot) \right|_k^2 \right)^{1/2} \\
&\leq C \sqrt{|\ln h|^{1+\beta}},
\end{aligned}$$

with $\beta = 1$ if $d > r$ and $\beta = 3/2$ if $d \leq r$. The last inequality follows from Theorem 3.1 and Lemma 3.4, and the connection between the derivatives of the functions \tilde{U} and \tilde{u} (see Remark 2.1; we also use the classical bounds $\|\partial U/\partial t(t, \cdot)\|_\infty + \|\partial^2 U/\partial x^2(t, \cdot)\|_\infty \leq C/\sqrt{t}$). We conclude that

$$P_0^{(n)} - P_0 \leq C \frac{(\ln n)^\alpha}{n},$$

with $\alpha = 1$ if $d > r$ and $\alpha = 5/4$ if $d \leq r$.

5 Lower bound for $P_0^{(n)} - P_0$

For the derivation of the lower bound, we use the stopping time introduced in [6] (see the proof of Theorem 5.6). Namely

$$\tau = \tau_1 \mathbb{1}_{\{\tau_1 < T-h\}} + T \mathbb{1}_{\{\tau_1 = T-h\}},$$

where

$$\tau_1 = \inf \left\{ t \in [0, T-h] \mid t/h \in \mathbb{N} \text{ and } d(B_t^{(n)}, I_{t+h}) \leq \sqrt{h} \|X\|_\infty + |\mu_0| h \right\}.$$

Here, $I_t = \{x \in \mathbb{R} \mid u(t, x) = g(t, x + \mu_0 t)\}$. Note that, if $t < T$, $I_t = (-\infty, \hat{b}(t)]$.

The modification of τ_1 into τ is motivated by the unboundedness of $\partial u/\partial t$ near T . We have, due to the definition of τ_1 ,

$$\begin{aligned}
P^{(n)} - P &\geq \mathbb{E} \left(e^{-r\tau} g(\mu_0 \tau + B_\tau^{(n)}) - u(0, 0) \right) \\
&= \mathbb{E} \left(e^{-r\tau} g(\mu_0 \tau + B_\tau^{(n)}) - u(\tau, B_\tau^{(n)}) + u(\tau, B_\tau^{(n)}) - u(0, 0) \right).
\end{aligned}$$

We have

$$\begin{aligned}
\mathbb{E} \left(u(\tau, B_\tau^{(n)}) - u(0, 0) \right) &= \mathbb{E} \left(\sum_{j=1}^{\tau/h} \mathcal{D}u((j-1)h, B_{(j-1)h}^{(n)}) \right) \\
&= \mathbb{E} \left(\sum_{j=1}^{(\tau/h) \wedge (n-2)} \mathcal{D}u(jh - h, B_{jh-h}^{(n)}) \right) \\
&\quad + \mathbb{E} \left(\mathbf{1}_{\{\tau=T\}} \sum_{j=n-1}^n \mathcal{D}u(jh - h, B_{jh-h}^{(n)}) \right) \\
&= \mathbb{E} \left(\sum_{j=1}^{(\tau/h) \wedge (n-2)} \mathcal{D}u(jh - h, B_{jh-h}^{(n)}) \right) + O(h),
\end{aligned}$$

where the last equality follows from Lemma 4.1 of [6]. Now, if $j < (\tau/h) \wedge (n-2)$, we have $d(B_{jh}^{(n)}, I_{jh+h}) > \sqrt{h} \|X\|_\infty + |\mu_0|h$, so that

$$B_{jh}^{(n)} > \hat{b}(jh + h) + \sqrt{h} \|X\|_\infty + |\mu_0|h.$$

We then have, for $s \in [jh, jh + h]$ and $z \in [0, \sqrt{h}]$

$$\begin{aligned}
B_{jh}^{(n)} + zX &> \hat{b}(jh + h) + zX + \sqrt{h} \|X\|_\infty + |\mu_0|h \\
&\geq \hat{b}(jh + h) + |\mu_0|h \\
&= \hat{b}(s) + \hat{b}(jh + h) + \mu_0(jh + h) - (\hat{b}(s) + \mu_0s) - \mu_0(jh + h - s) + |\mu_0|h \\
&\geq \hat{b}(s),
\end{aligned}$$

the last inequality coming from the fact that $t \mapsto \hat{b}(t) + \mu_0t$ is increasing. We can now assert that, for $j < (\tau/h) \wedge (n-2)$, $\mathcal{D}u(B_{jh}^{(n)}, jh) = \tilde{\mathcal{D}}u(B_{jh}^{(n)}, jh)$, so that

$$\begin{aligned}
\mathbb{E} \left(u(\tau, B_\tau^{(n)}) - u(0, 0) \right) &\leq \mathbb{E} \left(\sum_{j=1}^{n-2} \left| \tilde{\mathcal{D}}u(jh, B_{jh}^{(n)}) \right| \right) + O(h) \\
&\leq C \frac{(\ln n)^\alpha}{n},
\end{aligned}$$

with $\alpha = 1$ if $d > r$ and $\alpha = 5/4$ if $d \leq r$, as follows from the discussion in the previous section.

We now want a lower bound for

$$\mathbb{E} \left(e^{-r\tau} g(\mu_0\tau + B_\tau^{(n)}) - u(\tau, B_\tau^{(n)}) \right).$$

We have, using the equality $\{\tau \geq T - h\} = \{\tau = T\}$,

$$\begin{aligned}
u(\tau, B_\tau^{(n)}) - e^{-r\tau} g(\mu_0\tau + B_\tau^{(n)}) &= \left(u(\tau, B_\tau^{(n)}) - e^{-r\tau} g(\mu_0\tau + B_\tau^{(n)}) \right) \mathbf{1}_{\{\tau < T-h\}} \\
&= \left(u(\tau + h, B_\tau^{(n)}) - e^{-r\tau} g(\mu_0\tau + B_\tau^{(n)}) \right) \mathbf{1}_{\{\tau < T-h\}} \\
&\quad + \left(u(\tau, B_\tau^{(n)}) - u(\tau + h, B_\tau^{(n)}) \right) \mathbf{1}_{\{\tau < T-h\}}.
\end{aligned}$$

On the set $\{\tau < T - h\}$, we have $d(B_\tau^{(n)}, I_{\tau+h}) \leq \sqrt{h}\|X\|_\infty + |\mu|h$. It follows from Proposition 2.6 of [6] that

$$u(\tau + h, B_\tau^{(n)}) - e^{-r\tau}g(\mu_0\tau + B_\tau^{(n)}) \leq C \frac{(\sqrt{h}\|X\|_\infty + |\mu|h)^2}{\sqrt{T - \tau - h}}.$$

Using the estimate $\left\| \frac{\partial u}{\partial t}(t, \cdot) \right\|_\infty < C/\sqrt{T-t}$, we obtain

$$\mathbb{E} \left(u(\tau, B_\tau^{(n)}) - e^{-r\tau}g(\mu_0\tau + B_\tau^{(n)}) \right) \leq Ch\mathbb{E} \left(\frac{1}{\sqrt{T - \tau - h}} \mathbb{1}_{\{\tau \leq T-2h\}} \right).$$

The estimate $P^{(n)} - P \geq -C \frac{(\ln n)^{\bar{\alpha}}}{n}$ is now an easy consequence of Lemma 5.7 and Remark 5.8 of [6], which can be summarized in the following statement.

Lemma 5.1. *There exists a positive constant C such that*

$$\mathbb{E} \left(\frac{1}{\sqrt{T - \tau - h}} \mathbb{1}_{\{\tau \leq T-2h\}} \right) \leq C (\ln h)^\beta,$$

with

$$\beta = \begin{cases} 3/2, & \text{if } d \leq r, \\ 1, & \text{if } d > r. \end{cases}$$

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