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Quadratic Zonotopes
An extension of Zonotopes to Quadratic Arithmetics

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Abstract. Affine forms are a common way to represent convex sets of $\mathbb{R}$ using a base of error terms $\epsilon \in [-1, 1]^m$. Quadratic forms are an extension of affine forms enabling the use of quadratic error terms $\epsilon_i \epsilon_j$.
In static analysis, the zonotope domain, a relational abstract domain based on affine forms has been used in a wide set of settings, e.g. set-based simulation for hybrid systems, or floating point analysis, providing relational abstraction of functions with a cost linear in the number of errors terms.
In this paper, we propose a quadratic version of zonotopes. We also present a new algorithm based on semi-definite programming to project a quadratic zonotope, and therefore quadratic forms, to intervals. All presented material has been implemented and applied on representative examples.

Keywords: affine form, quadratic form, affine vectors, quadratic vectors, zonotopes, static analysis

1 Affine arithmetics and Static Analysis

Context. Affine arithmetics was introduced in the 90s by Comba and Stolfi [CS93] as an alternative to interval arithmetics, allowing to avoid some pessimistic computation like the cancellation:

$$x - x = [a, b] -_{\mathcal{I}} [a, b] = [a - b, b - a] \neq [0, 0]$$

It relies on a representation of convex subsets of $\mathbb{R}$ keeping dependencies between variables: e.g. $x \in [-1, 1]$ will represented as $0 + 1 * \epsilon_1$ while another variable $y \in [-1, 1]$ will be represented by another ϵ term: $y = 0 + 1 * \epsilon_2$. Therefore $x - x$ will be precisely computed as $\epsilon_1 - \epsilon_1 = 0$ while $x - y$ will result in $\epsilon_1 - \epsilon_2$, i.e. denoting the interval $[-2, 2]$.

In static analysis, affine forms lifted to abstract environments, as vectors of affine forms, are a friendly alternative to costly relational domains. They provide cheap and scalable relational abstractions: their complexity is linear in the number of error terms – the ϵ_i – while most relational abstract domains have a complexity at least cubic. Since their geometric concretization characterizes a zonotope, i.e. a symmetric convex polytope, they are commonly known as zonotopic abstract domains.

However since zonotopes are not fitted with lattice structure, their use in pure abstract interpretation using a Kleene iteration schema is not common. The definition of an abstract domain based on affine forms requires the definition of an upper bound and lower bound operators since no least upper bound and greatest lower bound exist in general. Choices vary from the computation of a precise minimal upper bound to a coarser upper bound that tries to maintain relationship among variables and error terms. For example, the choices of [GGP09] try to compute such bounds while preserving as much as possible the error terms of the operands, providing a precise way to approximate a functional.

Related works. Zonotopes are mainly used in static analysis to support the formal verification of critical systems performing floating point computation, e.g. aircraft controllers. One can mention a first line of works in which zonotopes are used to precisely over-approximate set of values: 1. hybrid system simulation, for example set-based simulation [BMC12]; 2. or floating point error propagation [Gou13]. In those cases, a join operator is not necessarily needed nor a partial order check.

A second line of work tries to rely on this representation to perform classical abstract interpretation. Zonotopes are then fitted with a computable partial order and a join e.g. [GGP12,GPV12]. The approach of [GGP09] is available in the open-source library APRON [JM09].

Back in the applied mathematics community, variants of affine arithmetics have been studied in [MT06] among which the quadratic extension of affine forms allowing to express terms in $\epsilon_i \epsilon_j$.

Contributions. In the paper, we ambition at using zonotopes based on this quadratic arithmetics. We propose an abstraction based on an extension of zonotopic abstract domains to quadratic arithmetic. Our approach fully handles floating point computations and performs the necessary rounding to obtain a sound result. Furthermore, while keeping the complexity reasonable, i.e. quadratic instead of linear in the error terms, quadratic forms are best suited to represent non linear computations such as multiplication. Interestingly, the geometric concretization a set of quadratic forms characterizes a non convex, non symmetric subset of $\mathbb{R}^n$, while still being fitted with an algebraic structure.

Paper structure. A first section presents quadratic forms as introduced in [MT06]. Then Sec. 3 presents our extension of zonotopes to quadratic arithmetics. Sec. 4 motivates our floating point implementation. Sec. 5 proposes a more precise way to project quadratic zonotopes to intervals using semi-definite programming (SDP) solvers. Finally Sec. 6 addresses our implementation and the evaluation of the approach with respect to existing domains (intervals, affine zonotopes variants).

2 Formal Preliminaries: Quadratic forms

We formally introduce here some definitions from [MT06] defining quadratic forms. We refer the interested reader to this publication for a wider comparison in a global optimization setting.

Quadratic forms. A (not so) recent extension of affine arithmetics is quadratic arithmetics [MT06]. It is a comparable representation of values fitted with similar arithmetics operators but quadratic forms also considers products of two errors terms, i.e. in $\epsilon_i\epsilon_j$. A quadratic form is also parametrized by additional error terms used to encode non linear errors: $\epsilon_\pm \in [-1,1], \epsilon_+ \in [0,1]$ and $\epsilon_- \in [-1,0]$. Let us define the set $\mathbf{C}^m \triangleq [-1,1]^m \times [-1,1] \times [0,1] \times [-1,0]$. A quadratic form on m noise symbols is a function q from $\mathbf{C}^m$ to $\mathbb{R}$ defined for all $t = (\epsilon, \epsilon_\pm, \epsilon_+, \epsilon_-) \in \mathbf{C}^m$by $q(t) = c + b^\intercal\epsilon + \epsilon^\intercal A\epsilon + c_\pm\epsilon_\pm + c_-\epsilon_- + c_+\epsilon_+$. A quadratic form is thus characterized by a 6-tuple $(c, (b)_m, (A)_{m^2}, c_\pm, c_+, c_-) \in \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^{m\times m} \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+$. To simplify, we will use the terminology quadratic form for both the function defined on $\mathbf{C}^m$ and the 6-tuple. We denote by $\mathcal{Q}^m$ the set of quadratic forms.

Geometric interpretation. Let $q \in \mathcal{Q}^m$. Since q is continuous, the image of $\mathbf{C}^m$ by q is a closed bounded interval. In our context, the image of $\mathbf{C}^m$ by q defines its geometric interpretation.

Definition 1 (Concretization of quadratic forms). *The concretization map of a quadratic form $\gamma_\mathcal{Q} : \mathcal{Q}^m \to \wp(\mathbb{R})$ is defined by:*

$$\gamma_\mathcal{Q}(q) = \{x \in \mathbb{R} \,|\, \exists\, t \in \mathbf{C}^m \ \text{ s.t. } \ x = q(t)\}$$

Remark 1. We can have $\gamma_\mathcal{Q}(q) = \gamma_\mathcal{Q}(q')$ with $q \neq q'$ e.g. $q = \epsilon_1^2$ and $q' = \epsilon_2^2$.

The concretization of q consists in computing the infimum and the supremum of q over $\mathbf{C}^m$ i.e. the values:

$$\mathbf{b}^q \triangleq \inf\{q(x) \mid x \in \mathbf{C}^m\} \qquad \text{and} \qquad \mathbf{B}^q \triangleq \sup\{q(x) \mid x \in \mathbf{C}^m\} \ . \tag{1}$$

To compute $\mathbf{b}^q$ and $\mathbf{B}^q$ is reduced to solve a non-convex quadratic problem which is NP-hard [Vav90]. The approach described in [MT06] uses simple inequalities to give a safe over-approximation of $\gamma_\mathcal{Q}(q)$. The interval provided by this approach is $[\mathbf{b}^q_{MT}, \mathbf{B}^q_{MT}]$ defined as follows:

$$\begin{cases} \mathbf{b}^q_{MT} \triangleq c - \sum_{i=1}^m |b_i| - \sum_{\substack{i,j=1,\ldots,m \\ j\neq i}} |A_{ij}| + \sum_{i=1}^m [A_{ii}]^- - c_- - c_\pm \\ \mathbf{B}^q_{MT} \triangleq c + \sum_{i=1}^m |b_i| + \sum_{\substack{i,j=1,\ldots,m \\ j\neq i}} |A_{ij}| + \sum_{i=1}^m [A_{ii}]^+ + c_+ + c_\pm \end{cases} \tag{2}$$

where for all $x \in \mathbb{R}$, $[x]^+ = x$ if $x > 0$ and 0 otherwise and $[x]^- = x$ if $x < 0$ and 0 otherwise.

In practice, we use $\gamma^{MT}_\mathcal{Q}(q) \triangleq [\mathbf{b}^q_{MT}, \mathbf{B}^q_{MT}]$ instead of $\gamma_\mathcal{Q}(q)$. In Sec. 5, we will present a tighter safe over-approximation of $\gamma_\mathcal{Q}(q)$ using SDP.

We will need a "reverse" map to the concretization map $\gamma_\mathcal{Q}$: a map which associates to an interval a quadratic form. We call this map the *abstraction map*. Note that the abstraction map produces a fresh noise symbol.

First, we introduce some notations for intervals. Let $\mathcal{I}$ be the set of closed bounded real intervals i.e. $\{[a,b] \mid a,b \in \mathbb{R}, a \leqslant b\}$ and $\overline{\mathcal{I}}$ its unbounded extension, i.e. $a \in \mathbb{R} \cup \{-\infty\}, b \in \mathbb{R} \cup \{+\infty\}$. $\forall [a,b] \in \mathcal{I}$, we define two functions $\lg([a,b]) = (b-a)/2$ and $\mathrm{mid}([a,b]) = (b+a)/2$. Let $\sqcup_{\mathcal{I}}$ be the classical join of $\mathcal{I}$ that is $[a,b] \sqcup_{\mathcal{I}} [c,d] \triangleq [\min(a,c), \max(b,d)]$. Let $\sqcap_{\overline{\mathcal{I}}}$ be the classical meet of intervals.

Definition 2 (Abstraction). *The abstraction map $\alpha_{\mathcal{Q}} : \mathcal{I} \to \mathcal{Q}^1$ is defined by:*

$$\alpha_{\mathcal{Q}}([a_1,a_2]) = (c,(b)_1,(0)_1,0,0,0) \text{ where } c = \mathrm{mid}\left([a_1,a_2]\right) \text{ and } b = \lg\left([a_1,a_2]\right) \quad .$$

Property 1 (Concretization of abstraction). $\gamma_{\mathcal{Q}}\left(\alpha_{\mathcal{Q}}\left([a_1,a_2]\right)\right) = [a_1,a_2]$.

Arithmetic operators. Quadratic forms are fitted with arithmetic operators which complexity is quadratic in the number of error terms. We give here the definitions of the arithmetics operators:

Definition 3 (Arithmetics operator in $\mathcal{Q}$). *Addition, negation, multiplication by scalar are defined by:*

$$\begin{aligned}
&(c,(b)_m,(A)_{m^2},c_\pm,c_+,c_-) +_{\mathcal{Q}} (c',(b')_m,(A')_{m^2},c'_\pm,c'_+,c'_-) = \\
&\qquad (c+c',(b+b')_m,(A+A')_{m^2},c_\pm+c'_\pm,c_++c'_+,c_-+c'_-) \\
&-_{\mathcal{Q}}(c,(b)_m,(A)_{m^2},c_\pm,c_+,c_-) = (-c,(-b)_m,(-A)_{m^2},c_\pm,c_-,c_+) \\
&\lambda *_{\mathcal{Q}} (c,(b)_m,(A)_{m^2},c_\pm,c_+,c_-) = (\lambda c,\lambda(b)_m,\lambda(A)_{m^2},|\lambda|c_\pm,|\lambda|c_+,|\lambda|c_-)
\end{aligned}$$

The multiplication is more complex since it introduces additional errors.

$$\begin{aligned}
&(c,(b)_m,(A)_{m^2},c_\pm,c_+,c_-) \times_{\mathcal{Q}} (c',(b')_m,(A')_{m^2},c'_\pm,c'_+,c'_-) = \\
&\begin{cases} (cc',c'(b)_m,+c(b')_m,c'(A)_{m^2}+c(A')_{m^2}+(b)_m(b')^\intercal_m,c''_\pm,c''_+,c''_- \text{ with} \\ c''_x = c''_{x_1}+c''_{x_2}+c''_{x_3}+c''_{x_4}, \forall x \in \{+,-,\pm\} \end{cases}
\end{aligned}$$

Each c''_{x_i} accounts for multiplicative errors with more than quadratic degree, obtained in the following four sub terms: (1) $\epsilon^\intercal A\epsilon \times \epsilon^\intercal A'\epsilon$ (2) $b^\intercal\epsilon \times \epsilon^\intercal A'\epsilon$ and $b'^\intercal\epsilon \times \epsilon^\intercal A\epsilon$ (3) multiplication of a matrix element in A, A' times an error term in $\pm,+,-$ (4) multiplication between error terms or with constant c, c'. Their precise definition can be found in [MT06, §3].

3 Quadratic Zonotopes: a zonotopic extension of quadratic forms to environments

Quadratic vectors are the lift to environments of quadratic forms. They provide a p-dimensional environment in which each dimension/variable is associated to a quadratic form. As for the affine sets used in zonotopic domains [GP09], the different variables share (some) error terms, this characterizes a set of relationships between variables, when varying the values of ϵ within $[-1,1]^m$. The geometric interpretation of quadratic vectors are non convex non symmetric subsets of $\mathbb{R}^p$. In the current paper, we call them Quadratic Zonotopes to preserve the analogy with affine sets and zonotopes.

Example 1 (quadratic vector). Let us consider the following quadratic vector q:

$$q = \begin{cases} x = -1 + \epsilon_1 - \epsilon_2 - \epsilon_{1,1} \\ y = 1 + 2\epsilon_2 + \epsilon_{1,2} \end{cases}$$

Fig. 1 represents its associated geometric interpretation, a quadratic zonotope.

We denote by $\mathcal{Z}^p_{\mathcal{Q}^m}$ such quadratic vectors of dimension p: $(q^p) \in \mathcal{Z}^p_{\mathcal{Q}^m} = \left(c^p, (b)^p_m, (A)^p_{m^2}, c^p_\pm, c^p_+, c^p_-\right) \in \mathbb{R}^p \times \mathbb{R}^{p\times m} \times \mathbb{R}^{p\times m\times m} \times \mathbb{R}^p_+ \times \mathbb{R}^p_+ \times \mathbb{R}^p_+$.

The Zonotope domain is then a parametric relational abstract domain, parametrized by the vector of m error terms. In practice, its definition mimics a non relational domain based on an abstraction $\mathcal{Z}^p_{\mathcal{Q}^m}$ of $\wp(\mathbb{R}^p)$. Operators are (i) assignment of a variable of the zonotope to a new value defined by an arithmetic expression, using the semantics evaluation of expressions in $\mathcal{Q}$ and the substitution in the quadratic vector; (ii) guard evaluation, i.e. constraint over a zonotope, using the classical combination of forward and backward evaluations of expressions [Min04, §2.4.4].

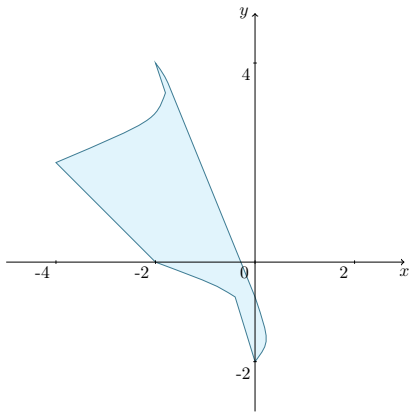


Fig. 1: Zonotopic concretization of the quadratic vector $q \in \mathcal{Z}^p_{\mathcal{Q}^m}$ of Ex. 1: $\gamma_{\mathcal{Z}_\mathcal{Q}}(q)$

Geometric interpretation and box projection. One can consider the geometric interpretation as the concretization of a quadratic vector to a quadratic zonotope.

From now on, for all $n \in \mathbb{N}$, $[n]$ denotes the set of integers $\{1, \ldots, n\}$.

Definition 4 (Concretization in $\mathcal{Z}^p_{\mathcal{Q}^m}$). *The concretization map $\gamma_{\mathcal{Z}_\mathcal{Q}} : \mathcal{Z}^p_{\mathcal{Q}^m} \mapsto \wp(\mathbb{R}^p)$ is defined for all $q = (q_1, \ldots, q_p) \in \mathcal{Z}^p_{\mathcal{Q}^m}$ by:*

$$\gamma_{\mathcal{Z}_\mathcal{Q}}(q) = \{x \in \mathbb{R}^p \,|\, \exists\, t \in \mathbf{C}^m \text{ s.t. } \forall\, k \in [p],\ x_k = q_k(t)\} \quad .$$

Remark 2. Characterizing explicitly such subset of $\mathbb{R}^p$ as a set of constraint is not easy. A classical (affine) zonotope is the image of a polyhedron (hypercube) by an affine map, hence it is a polyhedron and can be represented by a conjunction of affine inequalities. In the quadratic vectoring, such representation as conjunction of quadratic or at most polynomial inequalities is not proven to exist. This makes the concretization of a quadratic set difficult to compute precisely.

To ease the later interpretation of computed values, we rely on a naive projection to boxes: each quadratic form of the quadratic vector is concretized as an interval using $\gamma_\mathcal{Q}$.

Preorder structure. We can fit quadratic vectors with a preorder relying on the geometric inclusion provided by the map $\gamma_{\mathcal{Z}_\mathcal{Q}}$.

Definition 5 (Preorder in $\mathcal{Z}^p_{\mathcal{Q}^m}$). *The preorder $\sqsubseteq_{\mathcal{Z}_\mathcal{Q}}$ over $\mathcal{Z}^p_{\mathcal{Q}^m}$ is defined by:*

$$x \sqsubseteq_{\mathcal{Z}_\mathcal{Q}} y \iff \gamma_{\mathcal{Z}_\mathcal{Q}}(x) \subseteq \gamma_{\mathcal{Z}_\mathcal{Q}}(y) \ .$$

Remark 3. Since $\gamma_{\mathcal{Z}_\mathcal{Q}}$ is not computable, $x \sqsubseteq_{\mathcal{Z}_\mathcal{Q}} y$ is not decidable. Note also that, from Remark 1, the binary relation $\sqsubseteq_{\mathcal{Z}_\mathcal{Q}}$ cannot be antisymmetric and thus cannot be an order.

Remark 4. The least upper bound of $Z \subseteq \mathcal{Z}^p_{\mathcal{Q}^m}$ i.e. an element z' s.t.
$\left(\forall z \in Z, z \sqsubseteq_{\mathcal{Z}_\mathcal{Q}} z' \wedge \forall z'' \in \mathcal{Z}^p_{\mathcal{Q}^m}, \forall z \in Z, z \sqsubseteq_{\mathcal{Z}_\mathcal{Q}} z''\right) \implies z' \sqsubseteq_{\mathcal{Z}_\mathcal{Q}} z''$ does not necessarily exists.

Related work [GP09,GGP09,GGP10,GGP12,GPV12] addressed this issue by providing various flavors of join operator computing a safe upper bound or a minimal upper bound. Classical Kleene iteration scheme was adapted to fit this loose framework without least upper bound computation. Note that, in general, the aforementioned zonotopic domains do not rely on the geometric interpretation as the concretization to $\wp(\mathbb{R})$.

We detail here the join operator considered in this paper. It is the lift of the operator proposed in [GP09] to quadratic vectors. The motivation of this operator is to provide an upper bound while minimizing the set of error terms lost in the computation.

First we introduce a useful function argmin: it cancels values of opposite sign but provides the argument with the minimal absolute value when provided with two values of the same sign:

Definition 6 (Argmin). *We define for all $a \in \mathbb{R}$, $\mathrm{sgn}(a) = 1$ if $a \geqslant 0$ and -1 otherwise. The* argmin *function,* $\mathrm{argmin} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is defined as: $\forall a, b \in \mathbb{R}, \mathrm{argmin}(a, b) = \mathrm{sgn}(a) \min(|a|, |b|)$ if $ab \geqslant 0$ and 0 otherwise.*

We also need the projection map which selects a specific coordinate of a quadratic vector.

Definition 7 (Projection). *The projection map $\pi_k : \mathcal{Z}^p_{\mathcal{Q}^m} \to \mathcal{Q}^m$ is defined by: $\forall\, q = (q_1, \ldots, q_p) \in \mathcal{Z}^p_{\mathcal{Q}^m},\ \forall\, k \in [p],\ \pi_k(q) = q_k$.*

When a quadratic form q is defined before a new noise symbol creation, we have to extend q to take into account this fresh noise symbol.

Definition 8 (Extension). *Let $i, j \in \mathbb{N}$. The extension map $\mathrm{ext}_{i,j} : \mathcal{Q}^m \to \mathcal{Q}^{i+j+m}$ is defined by: $\forall\, q = (c, (b)_m, (A)^2_m, c_\pm, c_+, c_-) \in \mathcal{Q}^m, \mathrm{ext}_{i,j}(q) = (c, (b')_{i+j+m}, (A')_{(i+j+m)^2}, c_\pm, c_+, c_-) \in \mathcal{Q}^n$ where $b'_k = b_{k-i}$ if $i+1 \leqslant k \leqslant m+i$ and 0 otherwise and $A'_{k,l} = A_{k-i,l-i}$ if $i+1 \leqslant k, l \leqslant m+i$ and 0 otherwise.*

Property 2 (Extension properties). Let $i, j \in \mathbb{N}$.

1. Let $t = (\epsilon, \epsilon_\pm, \epsilon_+, \epsilon_-) \in \mathbf{C}^m$ and $t' = (\epsilon', \epsilon_\pm, \epsilon_+, \epsilon_-) \in \mathbf{C}^{m+i+j}$ s. t. $\forall\, i+1 \leqslant k \leqslant m+i$, $\epsilon'_k = \epsilon_{k-i}$. Then $q(\epsilon, \epsilon_\pm, \epsilon_+, \epsilon_-) = \mathrm{ext}_{i,j}(q)(\epsilon', \epsilon_\pm, \epsilon_+, \epsilon_-)$.
2. For all $q \in \mathcal{Q}^m$, $\gamma_{\mathcal{Q}}(q) = \gamma_{\mathcal{Q}}(\mathrm{ext}_{i,j}(q))$.

Now, we can give a formal definition of the upper bound of two quadratic vectors.

Definition 9 ($\sqcup_{\mathcal{Z}_\mathcal{Q}}$: Upper bound computation in $\mathcal{Z}^p_{\mathcal{Q}^m}$). *The upper bound* $\sqcup_{\mathcal{Z}_\mathcal{Q}} : \mathcal{Z}^p_{\mathcal{Q}^m} \times \mathcal{Z}^p_{\mathcal{Q}^m} \to \mathcal{Z}^p_{\mathcal{Q}^{m+p}}$ *is defined, for all* $q = (c, b, A, c_\pm, c_+, c_-)\,, q' = \left(c', b', A', c'_\pm, c'_+, c'_-\right) \in \mathcal{Z}^p_{\mathcal{Q}^m}$ *by:*

$$q \sqcup_{\mathcal{Z}_\mathcal{Q}} q' = \left(\mathrm{ext}_{0,p}(q''_k)\right)_{k\in[p]} + q^e \in \mathcal{Z}^p_{\mathcal{Q}^{m+p}}$$

where $q'' = (c'', (b'')^p_m, (A'')^p_{m^2}, c''^p_\pm, c''^p_+, c''^p_-) \in \mathcal{Z}^p_{\mathcal{Q}^m}$ *with, for all* $k \in [p]$*:*

- $(c'')_k = \mathrm{mid}(\gamma_\mathcal{Q}(\pi_k(q)) \cup \gamma_\mathcal{Q}(\pi_k(q')))$;
- $\forall\, t \in \{\pm, +, -\}, c''_{t,k} = \mathrm{argmin}(c_{t,k}, c'_{t,k})$;
- $\forall\, i \in [m],\ (b'')_{k,i} = \mathrm{argmin}(b_{k,i}, b'_{k,i})$;
- $\forall\, i, j \in [m],\ (A'')_{k,i,j} = \mathrm{argmin}(A_{k,i,j}, A'_{k,i,j})$;

and $\forall\, k \in [p]$, $q^e_k = \mathrm{ext}_{(m+k-1),(p-k)}\left(\alpha_\mathcal{Q}\left(C_k \sqcup_\mathcal{I} C'_k\right)\right)$ *with* $C_k = \gamma_\mathcal{Q}(\pi_k(q) - \pi_k(q''))$ *and* $C'_k = \gamma_\mathcal{Q}(\pi_k(q') - \pi_k(q''))$.

Let us denote the Minkowski sum and the Cartesian product of sets by respectively $D_1 \oplus D_2 = \{d_1 + d_2 \mid d_1 \in D_1,\ d_2 \in D_2\}$ and $\prod_i^n D_i = \{(d_1, \ldots, d_n) \mid \forall\, i \in [n],\ d_i \in D_i\}$. We have the nice characterization of the concretization of the upper bound given by Lemma 1.

Lemma 1. *By construction of* q'' *and* q^e *previously defined:*

$$\gamma_{\mathcal{Z}_\mathcal{Q}}\left(\left(\mathrm{ext}_{0,p}(q''_k)\right)_{k\in[p]} + q^e\right) = \gamma_{\mathcal{Z}_\mathcal{Q}}(q'') \oplus \prod_{k=1}^{p} \gamma_{\mathcal{Q}^{m+p}}(q^e_k)$$

Proof. See Appendix.

Now, we state at Theorem 1 that the $\sqcup_{\mathcal{Z}_\mathcal{Q}}$ operator computes an upper bound of its operands with respect to the preorder $\sqsubseteq_{\mathcal{Z}_\mathcal{Q}}$.

Theorem 1 (Soundness of the upper bound operator). *For all* $q, q' \in \mathcal{Z}^p_{\mathcal{Q}^m}$, $q \sqsubseteq_{\mathcal{Z}_\mathcal{Q}} q \sqcup_{\mathcal{Z}_\mathcal{Q}} q'$ *and* $q' \sqsubseteq_{\mathcal{Z}_\mathcal{Q}} q \sqcup_{\mathcal{Z}_\mathcal{Q}} q'$.

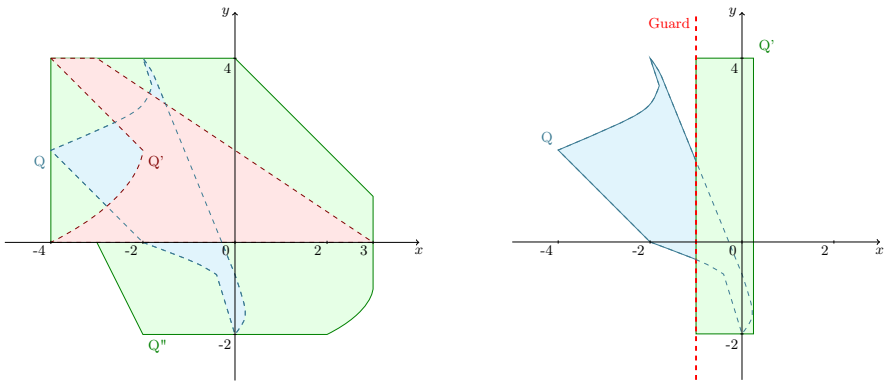
Proof. See Appendix.

Example 2. Let Q and Q' be two quadratic vectors:

$$Q = \begin{cases} x = -1 + \epsilon_1 - \epsilon_2 - \epsilon_{1,1} \\ y = 1 + 2\epsilon_2 + \epsilon_{1,2} \end{cases} \qquad Q' = \begin{cases} x = -2\epsilon_2 - \epsilon_{1,1} + \epsilon_+ \\ y = 1 + \epsilon_1 + \epsilon_2 + \epsilon_{1,2} \end{cases}$$

The resulted quadratic vector $Q'' = Q \sqcup_{\mathcal{Z}_\mathcal{Q}} Q'$ is

$$Q'' = \begin{cases} x = -\epsilon_2 - \epsilon_{1,1} + 2\epsilon_3 \\ y = 1 + \epsilon_2 + \epsilon_{1,2} + \epsilon_4 \end{cases}$$



(a) Upper bound computation (b) Guard evaluation

Fig. 2: Zonotopic concretization of operations on Quadratic Zonotopes

Transfer functions. The two operators `guard` and `assign` over the expressions *RelExpr* and *Expr* are defined like in a non relational abstract domain, as described in [Min04, §2.4.4]. Each operator relies on the forward semantics of numerical expressions, computed within arithmetics operators in $\mathcal{Q}$:

Definition 10 (Semantics of expressions). *Let $\mathcal{V}$ be a finite set of variables. We denote by $[\![\cdot]\!]_{\mathcal{Q}}(\mathcal{V} \to \mathcal{Q}) \to \mathcal{Q}$ the semantics evaluation of an expression in an environment mapping variables to quadratic forms.*

$$\begin{aligned} [\![v]\!]_{\mathcal{Q}}(Env) &= \pi_k(Env) \text{ where } k \in [p] \text{ is the index of } v \in \mathcal{V} \text{ in } Env \\ [\![e_1 \text{ bop } e_2]\!]_{\mathcal{Q}}(Env) &= [\![e_1]\!]_{\mathcal{Q}}(Env) \text{ bop}_{\mathcal{Q}} [\![e_2]\!]_{\mathcal{Q}}(Env) \\ [\![\text{uop } e]\!]_{\mathcal{Q}}(Env) &= \text{uop}_{\mathcal{Q}}[\![e]\!]_{\mathcal{Q}}(Env) \end{aligned}$$

Guards, i.e. tests, are enforced through the classical combination of forward and backward operators. Backward operators are the usual fallback operators, e.g. $[\![x+y]\!]^{\leftarrow} = (x \sqcap_{\mathcal{Q}} ([\![x+y]\!] -_{\mathcal{Q}} y), y \sqcap_{\mathcal{Q}} ([\![x+y]\!] -_{\mathcal{Q}} x))$ where $\sqcap_{\mathcal{Q}}$ denotes the meet of quadratic forms. As for upper bound computation, no best lower bound exists and such meet operator in $\mathcal{Q}$ has to compute a safe but imprecise upper bound of maximal lower bounds.

The meet over $\mathcal{Q}^m$ works as follows: it projects each argument to intervals using $\gamma_{\mathcal{Q}}$, performs the meet computation and reinject with a fresh noise symbolthe resulting closed bounded interval to $\mathcal{Q}$ thank to $\alpha_{\mathcal{Q}}$. Whereas, the meet over $\mathcal{Z}^p_{\mathcal{Q}^m}$ is the lift to the meet over $\mathcal{Q}^m$ to quadratic vectors. Formally:

Definition 11 ($\sqcap_{\mathcal{Q}}, \sqcap_{\mathcal{Z}_{\mathcal{Q}}}$: Approximations of maximal lower bounds). *The meet $\sqcap_{\mathcal{Q}} : \mathcal{Q}^m \times \mathcal{Q}^m \to \mathcal{Q}^1$ is defined by:*

$$\forall x, y \in \mathcal{Q}^m, x \sqcap_{\mathcal{Q}} y \triangleq \alpha_{\mathcal{Q}} \left(\gamma_{\mathcal{Q}}(x) \sqcap_{\mathcal{I}} \gamma_{\mathcal{Q}}(y)\right).$$

The meet $\sqcap_{\mathcal{Z}_{\mathcal{Q}}} : \mathcal{Z}^p_{\mathcal{Q}^m} \times \mathcal{Z}^p_{\mathcal{Q}^m} \to \mathcal{Z}^p_{\mathcal{Q}^p}$ is defined, for all $x, y \in \mathcal{Z}^p_{\mathcal{Q}^m}$ by $z = x \sqcap_{\mathcal{Z}_{\mathcal{Q}}} y \in \mathcal{Z}^p_{\mathcal{Q}^p}$ where:

$$\forall i \in [p], z_i = \pi_i(x) \sqcap_{\mathcal{Q}} \pi_i(y) \text{ when } \pi_i(x) \neq \pi_i(y), \pi_i(x) \text{ otherwise.}$$

Example 3. Let Q be the following quadratic vector. The meet with the constraint $x + 1 \geqslant 0$ produces the resulting quadratic vector Q':

$$Q = \begin{cases} x = -1 + \epsilon_1 - \epsilon_2 - \epsilon_{1,1} \\ y = 1 + 2\epsilon_2 + \epsilon_{1,2} \end{cases} \quad Q' = \begin{cases} x = -\frac{3}{8} + \frac{5}{8}\epsilon_3 \\ y = 1 + 2\epsilon_2 + \epsilon_{1,2} \end{cases}$$

Proof. $Guard(Q, x + 1 \geqslant 0) = Q \sqcap_{\mathcal{Z}_\mathcal{Q}} (\alpha_\mathcal{Q}(\gamma_\mathcal{Q}(x +_\mathcal{Q} 1) \sqcap_\mathcal{I} [0, +\infty[) -_\mathcal{Q} 1)$. We use the more precise concretization over-approximation map $\gamma_\mathcal{Q}^{SDP}$ that will be introduced in Sec. 5: $\gamma_\mathcal{Q}^{SDP}(\epsilon_1 - \epsilon_2 - \epsilon_{1,1}) = [-3, 1.25]$. We focus on x since the meet is performed component-wise and $\alpha_\mathcal{Q}(\gamma_\mathcal{Q}(\epsilon_1 - \epsilon_2 - \epsilon_{1,1}) \sqcap_\overline{\mathcal{I}} [0, +\infty[) -_\mathcal{Q} 1 = \alpha_\mathcal{Q}([-3, 1.25] \sqcap_\overline{\mathcal{I}} [0, +\infty[) -_\mathcal{Q} 1 = \alpha_\mathcal{Q}([0, 1.25]) -_\mathcal{Q} 1 = (5/8 + 5/8\epsilon_3) -_\mathcal{Q} 1 = -3/8 + 5/8\epsilon_3$ where ϵ_3 is a fresh error term introduced by $\alpha_\mathcal{Q}$.

4 Floating point computations

All the operators presentation above assumed a real semantics. As usual when analyzing programs, the domain has to be adapted to deal with floating point arithmetics.

We recall that our use of quadratic zonotopes is to precisely over-approximate reachable values as set of reals. We relied on the approach proposed by Stolfi and De Figueiro [SDF97], creating a new error term for each operation. Other approaches such as generalized intervals [Han75] are typically used in Fluctuat [Gou13]. Their definition in the quadratic setting is given in [MT06]. However, according to [SDF97] the approach with error terms instead of interval arithmetics is more precise but can generate an important number of error terms.

In this specific case of quadratic forms, the term in $\epsilon_\pm$ is used to accumulate floating point errors: the number of error terms does not evolve due to floating point computation. The extension to zonotopes is direct since numerical operations are evaluated at form level.

We illustrate the extension to quadratic form of [SDF97]:

Addition. According to Knuth [Knu97, §4.2.2] algorithm, the exact computation of $u + v$ is $u + v + e$ where $e = (u - ((u + v) - v)) + (v - ((u + v) - u))$ with all operations performed in floating point arithmetics. Let $e^+(u, v)$ be such additive error e.

We consider the addition of two quadratic forms $x = (x_0, (x_i), (x_{ij}), x_\pm, x_+, x_-)$ and $y = (y_0, (y_i), (y_{ij}), y_\pm, y_+, y_-)$. The addition of x and y is modified to considered these generated errors:

$$\begin{aligned} &(x_0, (x_i), (x_{ij}), x_\pm, x_+, x_-) +_\mathcal{Q} (y_0, (y_i), (y_{ij}), y_\pm, y_+, y_-) = \\ &(x_0 + y_0, (x_i + y_i), (x_{ij} + y_{ij}), x_\pm + y_\pm + rounded_err, x_+ + y_+, x_- + y_-) \end{aligned}$$

where

- $err = \sum_{i,j=1}^{n} e^+(x_{ij}, y_{ij}) + \sum_{i=0}^{n} e^+(x_i, y_i) + e^+(x_\pm, y_\pm) + e^+(x_+, y_+) + e^+(x_-, y_-)$.
- rup denotes the rounding up;
- $rounded_err = max(|rup(err)|, |-rup(-err)|)$

External multiplication. Similarly, the algorithm of Dekker and Veltkamp characterizes the multiplicative error obtained when computing $u \times v$. It relies on a constant C depending on the precision used. For single precision floats, $C = 2^{27} + 1$. We denote by $e^{\times}(u, v)$ such multiplicative error and refer the interested reader to Dekker's paper [Dek71].

The operator $*_{\mathcal{Q}}$ is modified to account such multiplicative errors:

$$\lambda *_{\mathcal{Q}} (x_0, (x_i), (x_{ij}), x_{\pm}, x_+, x_-) = (\lambda x_0, \lambda(x_i), \lambda(x_{ij}), |\lambda| x_{\pm} + r_err, |\lambda| x_+, |\lambda| x_-)$$

where

- $err = \sum_{i=1}^{n} e^{\times}(\lambda, x_i) + \sum_{i,j=1}^{n} e^{\times}(\lambda, x_{ij}) + e^{\times}(\lambda, x_{\pm}) + e^{\times}(\lambda, x_-) + e^{\times}(\lambda, x_+)$.
- $r_err = max(|rup(err)|, |-rup(-err)|)$.

All other operators behave similarly: each operation computing an addition or a product generates an additive and a multiplicative error, respectively, accumulated in the $x_{\pm}$ term.

5 Improving concretization using SDP

In this part, we propose a method based on semi-definite programming to compute an over-approximation of the interval concretization of a quadratic form. This method provides tighter bounds than $\mathbf{b}^q_{MT}$ and $\mathbf{B}^q_{MT}$ defined at Equation (2).

Let consider a quadratic form $q = (c^q, (b^q)_m, (A^q)_m, c^q_{\pm}, c^q_+, c^q_-) \in \mathcal{Q}^m$. Recall that $\mathbf{C}^m = [-1, 1]^m \times [-1, 1] \times [0, 1] \times [-1, 0]$, we remind that the concretization of q is the interval defined $[\mathbf{b}^q, \mathbf{B}^q]$ where $\mathbf{b}^q = \inf\{q(x) \mid x \in \mathbf{C}^m\}$ and $\mathbf{B}^q = \sup\{q(x) \mid x \in \mathbf{C}^m\}$.

In general, a standard quadratic form r from $\mathbb{R}^{m+3}$ to $\mathbb{R}$ is defined by $x \mapsto r(x) = x^{\intercal} A^r x + b^{r\intercal} x + c^r$ with a $(m+3) \times (m+3)$ symmetric matrix A^r, a vector of $\mathbb{R}^{m+3}$, b^r and a scalar c^r. We can cast q into a standard quadratic form r_q, leading to $r_q(x) = q(x)$ for all $x \in \mathbf{C}^m$. Indeed, it suffices to take the following data :

$$A^{r_q} = \begin{pmatrix} \tilde{A} & 0_{m\times 3} \\ 0_{3\times(m+3)} & \end{pmatrix} \text{ with } \tilde{A} = \frac{A^q + A^{q\intercal}}{2},\ b^{r_q\intercal} = \left(b^{q\intercal}, c^q_{\pm}, c^q_+, c^q_-\right) \text{ and } c^{r_q} = c^q$$

Let us denote by tr, the trace function which associates to a matrix the sum of its diagonal elements and let $x \in \mathbb{R}^{m+3}$. A simple calculus yields to:

$$r_q(x) = \operatorname{tr}(\mathbf{M}^{r_q} X) \text{ where } \mathbf{M}^{r_q} = \begin{pmatrix} A^{r_q} & \frac{1}{2} b^{r_q} \\ \frac{1}{2} b^{r_q\intercal} & c^{r_q} \end{pmatrix} \text{ and } X = \begin{pmatrix} x \\ 1 \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix}^{\intercal}.$$

To only deal with matrices, we have to translate the constraints on the vector x into constraints on the matrix X. Let us introduce the set $\mathcal{C}^m$ of $(m+4) \times (m+4)$ symmetric matrices Y such that:

$$\forall\, i,j \in [m+3],\ i<j,\ Y_{i,j} \in [-1,1] \quad (3a)$$
$$\forall\, i \in [m+1],\ Y_{i,(m+4)} \in [-1,1] \quad (3b)$$
$$\forall\, i \in [m+3],\ Y_{i,i} \in [0,1] \quad (3c)$$
$$Y_{(m+2),(m+3)} \in [-1,0] \quad (3d)$$
$$Y_{(m+2),(m+4)} \in [0,1] \quad (3e)$$
$$Y_{(m+3),(m+4)} \in [-1,0] \quad (3f)$$
$$Y_{(m+4),(m+4)} = 1 \quad (3g)$$

Note by symmetry of Y, for all $i,j \in [m+3],\ i<j,\ Y_{j,i} \in [-1,1]$; for all $i \in [m+1],\ Y_{(m+4),i} \in [-1,1]$; $Y_{(m+4),(m+3)} \in [-1,0]$ and $Y_{(m+4),(m+2)} \in [0,1]$.

$$Y = [-1,1] \cap \begin{array}{c} \begin{pmatrix} + & & & & \\ & \ddots & & \top & \\ & & \ddots & & \\ & \top & & \begin{array}{c|c|c} + & - & + \\ \hline - & + & - \\ \hline + & - & \{1\} \end{array} \end{pmatrix} \begin{array}{l} \\ \\ m+2 \\ m+3 \\ m+4 \end{array} \\ \begin{array}{ccc} m\times 2 & m\times 3 & m\times 4 \end{array} \end{array}$$

We denote by $\mathbb{S}_n^+$ the set of semi-definite positive matrices of size $n \times n$ i.e. the $n \times n$ symmetric matrices M such that for all $y \in \mathbb{R}^n$, $y^\intercal M y \geqslant 0$. We recall that the rank of a matrix is the number of linearly independent rows (or columns). We denote by $\mathrm{rk}(M)$, the rank of the matrix M.

Lemma 2 (Constraint translation). *The following statement holds:*

$$\left\{X \in \mathbb{S}_{m+4}^+ \mid \mathrm{rk}(X)=1,\ X \in \mathcal{C}^m\right\} = \left\{X \in \mathbb{S}_{m+4}^+ \mid \exists\, x \in \mathbf{C}^m \ \text{ s.t. } X = \begin{pmatrix} x \\ 1 \end{pmatrix}\begin{pmatrix} x \\ 1 \end{pmatrix}^\intercal\right\} .$$

Proof. See Appendix.

Lemma 2 allows to conclude that optimizing r_q over $\mathbf{C}^m$ and optimizing $X \mapsto \mathrm{tr}(\mathbf{M}^{r_q} X)$ over $\{X \in \mathbb{S}_{m+4}^+ \mid \mathrm{rk}(x)=1,\ X \in \mathcal{C}^m\ \}$ is the same. However, the rank one constraint on X leads to a non-convex problem which makes it difficult to solve. A natural and a commonly used relaxation is to remove the rank constraint to get a linear problem over semi-definite positive matrices. This discussion is formulated as Proposition 1.

Proposition 1. *The interval bounds of the concretization of q can be computed from the two following non-convex semi-definite programs:*

$$\mathbf{b}^q = \inf \quad tr(\mathbf{M}^{r_q} X) \quad \text{s.t.} \begin{cases} X \in \mathcal{C}^m \\ X \in \mathbb{S}_{m+4}^+ \\ \mathrm{rk}(X) = 1 \end{cases} \quad \textit{and} \quad \mathbf{B}^q = \sup \quad tr(\mathbf{M}^{r_q} X) \quad \text{s.t.} \begin{cases} X \in \mathcal{C}^m \\ X \in \mathbb{S}_{m+4}^+ \\ \mathrm{rk}(X) = 1 \end{cases}$$

By removing the rank constraint:

$$\mathbf{b}^q_{SDP} = \inf \quad tr(\mathbf{M}^{r_q} X) \leqslant \mathbf{b}^q \quad \text{s.t.} \begin{cases} X \in \mathcal{C}^m \\ X \in \mathbb{S}_{m+4}^+ \end{cases} \quad \textit{and} \quad \mathbf{B}^q_{SDP} = \sup \quad tr(\mathbf{M}^{r_q} X) \geqslant \mathbf{B}^q \quad \text{s.t.} \begin{cases} X \in \mathcal{C}^m \\ X \in \mathbb{S}_{m+4}^+ \end{cases}$$

Finally, the interval bounds of the concretization are safely approximated by using $\mathbf{b}^q_{SDP}$ and $\mathbf{B}^q_{SDP}$ and we write $\gamma^{SDP}_{\mathcal{Q}}(q) \triangleq [\mathbf{b}^q_{SDP}, \mathbf{B}^q_{SDP}]$. Moreover, those bounds improve the ones provides by [MT06].

Theorem 2 (Bounds improvements). *Let $q \in \mathcal{Q}^m$. The following inequalities hold:*

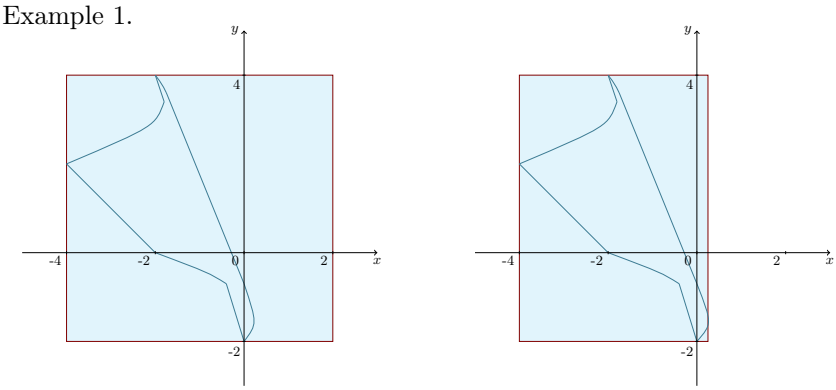
$$\gamma_{\mathcal{Q}}(q) \subseteq \gamma^{SDP}_{\mathcal{Q}}(q) \subseteq \gamma^{MT}_{\mathcal{Q}}(q) \text{ i.e. } \mathbf{b}^q_{MT} \leqslant \mathbf{b}^q_{SDP} \leqslant \mathbf{b}^q \ \wedge \ \mathbf{B}^q \leqslant \mathbf{B}^q_{SDP} \leqslant \mathbf{B}^q_{MT} \ .$$

Proof. See Appendix.

In term of complexity, SDP problems can be solved in polynomial time to an arbitrary prescribed precision by the ellipsoid method [GLS88]. More precisely, let $\alpha > 0$ be a given rational, suppose that the input data of a semi-definite program are rational and suppose that an integer N is known, such that the feasible set lies inside the ball of the radius N around zero. Then a feasible solution – the value of which is at most at a distance α from the optimal value – can be found in a time that is polynomial in the number of bits of the input data and in $-log(\alpha)$. This latter feasible solution can be found in polynomial time by interior point methods [NN94] if a strictly feasible solution is available. The advantage of interior methods is that they are very efficient in practice. We refer the reader to [RP96] for more information.

Corollary 1. *The reals $\mathbf{b}^q_{SDP}$ and $\mathbf{B}^q_{SDP}$ can be computed in polynomial time.*

The Figure 3 illustrates such concretization on the quadratic zonotopes defined in Example 1.



(a) using [MT06] (b) relying on SDP programming

Fig. 3: Projection to intervals

In terms of related works, the use of semidefinite programming to compute interval concretisation of nonlinear operation for affine forms already appeared in [Gho11, Prop. 5.1.2]. This approach appears to be the dual version of the semidefinite programs that we presented in this paper.

6 Experimentation

All presented materials has been implemented in an open-source tool written in OCaml[1]. This tool is used for teaching purpose and only consider simple imperative programs without function calls. It implements interval analysis, affine and quadratic zonotopes and provides a binding to APRON to evaluate the more complex T1P domain [GGP09], an affine zonotope domain with constraints. The reduced concretization is not integrated through the use of the CSDP or Mosek SDP solvers. Due to the increase cost in terms of computation time, it can be globally or locally activated when calling the analyzer.

The quadratic zonotope domain has been evaluated on examples provided in APRON T1P source code, or Fluctuat distribution, as well as simple iterative schemes. We present here the results obtained on an arctan function, the example of [CS93] and the Householder function analyzed in [GGP09].

Let us first consider the arctan function defined in Figure 4 and the analysis results in Table 1.

```
if (x > 1.) {
 y = 1.5708 − 1/x∗(1−C1/x^2+C2/x^4+C3/x^6+
        C4/x^8+C5/x^10+C6/x^12+C7/x^14+C8/x^16)
}
if (x < 1.) {
 y = −1.5708 − 1/x∗(1−C1/x^2+C2/x^4+C3/x^6+
        C4/x^8+C5/x^10+C6/x^12+C7/x^14+C8/x^16)
}
else {
 y = x∗(1−C1∗x^2+C2∗x^4+C3∗x^6+
        C4∗x^8+C5∗x^10+C6∗x^12+C7∗x^14+C8∗x^16)
}
```

with the constants defined as:

C1	0.0028662257
C2	−0.0161657367
C3	0.0429096138
C4	−0.0752896400
C5	0.1065626393
C6	−0.1420889944
C7	0.1999355085
C8	−0.3333314528

Fig. 4: Arctan program

	$x \in [-1, 1]$	$x \in [-10, 10]$
Domain	Bounds	Bounds
Interval	[-1.919149, 1.919149]	[-1.919149, 1.919149]
Affine Zonotopes	[-1.919149, 1.919149]	[-2.364846, 2.364846]
Quadratic Zonotopes	[-1.002866, 1.002866]	[-1.597501, 1.591769]
Affine Constrainted Zonotopes (Apron)	[-1.349407, 1.349407]	[-1.477535, 1.477535]

Table 1: Arctan program analysis results

[1] Tool and experiments available at `https://cavale.enseeiht.fr/QuadZonotopes/`

In [CS93] the authors considered the function $\sqrt{(}x^2+x-1/2)/\sqrt{(}x^2+1/2)$ and the precision obtained using affine arithmetics while evaluating the function on a partition of the input range as sub-intervals. Figure 5a illustrates the obtained results with our different abstractions for a division from 1 to 14, higher partition divisions (eg. 500) converge in terms of precision. Quadratic zonotopes shows here to be a good alternative to interval or affine zonotopes abstractions.

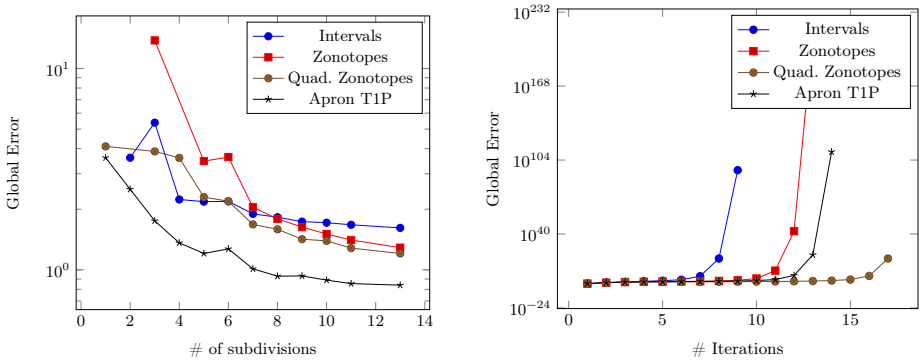
About the Householder function, it converges towards $1/\sqrt{A}$:

$$x_0 = 2^{-4}$$
$$x_{n+1} = x_n(1 + \tfrac{1}{2}(1 - Ax_n^2) + \tfrac{3}{8}(1 - Ax_n^2)^2)$$

We analyzed it using loop unrolling with $A \in [16, 20]$ and compared the global errors obtained at the $i-th$ iterate: the difference between the max and min values. Figure 5b presents the precision obtained with different analyses. Quadratic zonotopes provides here better bounds than affine or interval analysis and shows to scale better than all other analyses.

For the sake of comparison we also compared with the zonotope analysis provided in Apron as the T1P (Taylor 1 plus) abstract domain [GGP09]. We recall that this domain is not just based on affine arithmetics but also embed linear relationships between error terms in the zonotope. It shows better performance than affine zonotopes and a similar extension of our quadratic zonotopes should be considered.

Finally a disappointing result is the cost of the optimized concretization. This algorithm can potentially be used when converting a quadratic form to intervals and impact the computation of meet and join operators. However since meet operators are widely used in backward semantics [Min04], the global cost impacts widely the timing results. Moreover, in most cases the precision is not widely impacted.



(a) Stolfi [CS93] example evaluated on partitioned input range

(b) Householder precision wrt. number of unrolling.

Generally speaking, T1P abstract domain gives in most of the case better bounds in presence of several guards in the program. It provides a way to keep the noise symbols when computing "meets" in such a way that the relationships between the variables are maintained. On the other hand, the quadratic zonotope abstract domain is very efficient when working with polynomial program computing non-linear operations such as products. It is also more effective than the affine set abstract domain in term of precision but our implementation does not rely on the constrained mechanisms developed in T1P.

All experiments are developed in the Appendix 7.

7 Conclusion

Zonotopic abstractions are the current more promising analyses when it comes to the formal verification of floating point computations such as the ones found in aircraft controllers. The presented analysis seems an interesting alternative to affine zonotopes, increasing precision while keeping the complexity quadratic in the number of error terms. Quadratic zonotopes seems more suited than linear abstractions when analyzing non linear functions such as multiplications. Among the zoology of abstract domains, they belong to the small set of algebraic domains with non convex and non symmetric concretization. This may be later of great impact, e.g. when considering properties involving positivity of products of negative error terms.

Perspectives. On the theoretical side, it would be interesting to compare the abstraction generated by quadratic form with respect to the classical zonotopes, generated by affine forms. While graphically speaking quadratic zonotopes seem strictly included in their affine counterpart, the existence of a Galois connection between the two abstraction is non trivial to exhibit, if ever it exists.

On the application side, our comparison in the benchmarks with affine zonotopes was a little bit biased (against us) since we considered a naive meet operator like the one we provided. The work of [GGP09], represented as Apron T1P in the benchmark evaluation, proposes to enrich zonotopes with linear constraints over error terms to encode intersection. This extension of the domain seems a feasible approach in our setting. It would then allow a stronger comparison between affine and quadratic zonotopes.

Last, both affine and quadratic arithmetics can be seen, respectively, as a first and second order Taylor polynomial abstraction. It would be interesting to evaluate how this approach can be extended and how it combines with other methods aiming at regaining precision such as branch-and-bound algorithms.

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Appendix

Proofs of Join Soundness

Proof (Proof of Property 2).

1. Let $i, j \in \mathbb{N}$. Let $\epsilon \in [-1,1]^m$ and $\epsilon' \in [-1,1]^{m+i+j}$ such that for all $i+1 \leqslant k \leqslant m+i \epsilon'_k = \epsilon_{k-i}$. Let $(\epsilon_\pm, \epsilon_+, \epsilon_-) \in [-1,1] \times [0,1] \times [-1,0]$. By definition of A', b' and ϵ', we have:
$$\begin{aligned}
&\mathrm{ext}_{i,j}(q)(\epsilon', \epsilon_\pm, \epsilon_+, \epsilon_-) \\
&= \sum_{l=i+1}^{m+i} \sum_{n=i+1}^{m+i} \epsilon'_l A'_{l,n} \epsilon'_n + \sum_{n=i+1}^{m+i} b'_n \epsilon'_n + c_\pm \epsilon_\pm + c_+ \epsilon_+ + c_- \epsilon_- \\
&= \sum_{l=i+1}^{m+i} \sum_{n=i+1}^{m+i} \epsilon_{l-i} A_{l-i,n-i} \epsilon_{n-i} + \sum_{n=i+1}^{m+i} b_{n-i} \epsilon_{n-i} + c_\pm \epsilon_\pm + c_+ \epsilon_+ + c_- \epsilon_- \\
&= \sum_{l=1}^{m} \sum_{n=1}^{m} \epsilon_l A_{l,n} \epsilon_n + \sum_{n=1}^{m} b_n \epsilon_n + c_\pm \epsilon_\pm + c_+ \epsilon_+ + c_- \epsilon_- = q(\epsilon, \epsilon_\pm, \epsilon_+, \epsilon_-)
\end{aligned}$$
2. From the first point:
   - $\forall\, t \in \mathbf{C}^m, \exists\, t' \in \mathbf{C}^{m+p}$ s. t. $q(t) = \mathrm{ext}_{i,j}(q)(t')$ hence $\gamma_{\mathcal{Q}}(q) \subseteq \gamma_{\mathcal{Q}}(\mathrm{ext}_{i,j}(q))$.
   - $\forall\, t' \in \mathbf{C}^{m+p}, \exists\, t \in \mathbf{C}^m$ s. t. $q(t) = \mathrm{ext}_{i,j}(q)(t')$ hence $\gamma_{\mathcal{Q}}(q) \supseteq \gamma_{\mathcal{Q}}(\mathrm{ext}_{i,j}(q))$.

Proof (Proof of Lemma 1). By definition, $\alpha_{\mathcal{Q}}$ creates a fresh noise symbol when it is called, then q_k^e do not share noise symbols with anyone else and a q_k^e only depends on the $m+k$-th noise symbol:

$$\begin{aligned}
&\forall\, (\epsilon_\pm, \epsilon_+, \epsilon_-), (\epsilon'_\pm, \epsilon'_+, \epsilon'_-) \in [-1,1] \times [0,1] \times [-1,0],\ \forall\, k \in [p], \\
&\forall\, u, v \in [-1,1]^{m+p} \text{ s. t. } u_{m+k} = v_{m+k},\ q_k^e(u, \epsilon_\pm, \epsilon_+, \epsilon_-) = q_k^e(v, \epsilon'_\pm, \epsilon'_+, \epsilon'_-)
\end{aligned} \tag{4}$$

Let $t = (\epsilon, \epsilon_\pm, \epsilon_+, \epsilon_-) \in \mathbf{C}^{m+p}$. Define $\epsilon'' \in [-1,1]^m$ such that $\epsilon''_k = \epsilon_k$ for all $k \in [m]$ and $t'' = (\epsilon'', \epsilon_\pm, \epsilon_+, \epsilon_-) \in \mathbf{C}^m$. From Property 2 and since $q'' \in \mathcal{Z}^p_{\mathcal{Q}^m}$, for all $k \in [p]$, $q''_k(t'') = \mathrm{ext}_{0,p}(q''_k)(t)$. Now define for all $k \in [p]$, $\epsilon^k \in [-1,1]^{m+p}$ such that $\epsilon^k_i = \epsilon_{m+k}$ if $i = m+k$ and 0 otherwise and $t^k = (\epsilon^k, \epsilon'_\pm, \epsilon'_+, \epsilon'_-)$. From Equation (4), for all $k \in \{1, \ldots, p\}$, $q_k^e(t) = q_k^e(t^k)$. Hence, for all $k \in [p]$, $(\mathrm{ext}_{0,p}(q''_k) + q_k^e)(t) = \mathrm{ext}_{0,p}(q''_k)(t) + q_k^e(t) = q''_k(t'') + q_k^e(t^k)$. Finally, for all $t \in \mathbf{C}^{m+p}$, there exists $t'' \in \mathbf{C}^{m+p}$ and $t^1, \ldots, t^p \in \mathbf{C}^{m+p}$ such that for all $k \in [p]$, $\mathrm{ext}_{0,p}(q''_k)(t) + q_k^e(t) = q''_k(t'') + q_k^e(t^k)$ and thus $\gamma_{\mathcal{Z}_{\mathcal{Q}}}((\mathrm{ext}_{0,p}(q''_k))_{k \in [p]} + q^e) \subseteq \gamma_{\mathcal{Z}_{\mathcal{Q}}}(q'') \oplus \prod_{k=1}^p \gamma_{\mathcal{Q}^{m+p}}(q_k^{err})$.

Now let us take $t'' = (\epsilon'', \epsilon_\pm, \epsilon_+, \epsilon_-) \in \mathbf{C}^m$ and $t^1, \ldots, t^p \in \mathbf{C}^{m+p}$. Let us define $t = (\epsilon, \epsilon_\pm, \epsilon_+, \epsilon_-) \in \mathbf{C}^{m+p}$ by for all $i \in [m+p]$, $\epsilon_i = \epsilon''_i$ if $i \in [m]$ and $\epsilon_i = \epsilon^i$ if $m+1 \leqslant i \leqslant m+p$, then from Property 2, $\mathrm{ext}_{0,p}(q'')(t) = q''(t'')$ and from Equation (4), $q^e(t) = (q_k^e(t^k))_{1 \leqslant k \leqslant p}$. Finally, for all $t'' \in \mathbf{C}^m$ and $t^1, \ldots, t^p \in \mathbf{C}^{m+p}$, there exists $t \in \mathbf{C}^{m+p}$ such that $q''(t'')+) = (q_k(t^k))_{k \in [p]} = (\mathrm{ext}_{0,p}(q'') + q^e)(t)$ and thus $\gamma_{\mathcal{Z}_{\mathcal{Q}}}(\mathrm{ext}_{0,p}(q'') + q^e) \supseteq \gamma_{\mathcal{Z}_{\mathcal{Q}}}(q'') \oplus \prod_{k=1}^p \gamma_{\mathcal{Q}^{m+p}}(q_k^e)$.

Proof (Proof of Theorem 1). We only prove that $q \sqsubseteq_{\mathcal{Z}_{\mathcal{Q}}} q \sqcup_{\mathcal{Z}_{\mathcal{Q}}} q'$ i.e. $\gamma_{\mathcal{Z}_{\mathcal{Q}}}(q) \subseteq \gamma_{\mathcal{Z}_{\mathcal{Q}}}(\mathrm{ext}_{0,p}(q'') + q^e)$. By Lemma 1, it is the same to prove that $\gamma_{\mathcal{Z}_{\mathcal{Q}}}(q) \subseteq \gamma_{\mathcal{Z}_{\mathcal{Q}}}(q'') \oplus \prod_{k=1}^p \gamma_{\mathcal{Q}^{m+p}}(q_k^e)$. This is equivalent to show that for all $t \in \mathbf{C}^m$, there

exists $t'' \in \mathbf{C}^m$ such that for all $k \in [p]$, $q_k(t) - q''_k(t'') \in \gamma_{\mathcal{Q}}(q^e_k) = \left[\mathbf{b}^{q^e_k}, \mathbf{B}^{q^e_k}\right]$. Let for all $k \in [p]$, $r_k = q_k - q''_k \in \mathcal{Q}^m$ and $r'_k = q'_k - q''_k \in \mathcal{Q}^m$. From Property 1 and the second point of Property 2, we have $\left[\mathbf{b}^{q^e_k}, \mathbf{B}^{q^e_k}\right] = \gamma_{\mathcal{Q}}(r_k) \sqcup_{\mathcal{I}} \gamma_{\mathcal{Q}}(r'_k)$ and from the definition of $\sqcup_{\mathcal{I}}$, we have $\mathbf{b}^{q^e_k} = \min\left(\mathbf{b}^{r^k}, \mathbf{b}^{r'_k}\right)$ and $\mathbf{B}^{q^e_k} = \max\left(\mathbf{B}^{r^k}, \mathbf{B}^{r'_k}\right)$. Finally, it suffices to show that that for all $t \in \mathbf{C}^m$, there exists $t'' \in \mathbf{C}^m$ such that for all $k \in [p]$, $q_k(t) - q''_k(t'') \geqslant \min\left(\mathbf{b}^{r^k}, \mathbf{b}^{r'_k}\right)$ and $q_k(t) - q''_k(t'') \leqslant \max\left(\mathbf{B}^{r^k}, \mathbf{B}^{r'_k}\right)$. Let $t \in \mathbf{C}^m$ and let us take $t'' = t$, we have for all $k \in [p]$, $\min(\mathbf{b}^{r_k}, \mathbf{b}^{r'_k}) \leqslant \mathbf{b}^{r_k} \leqslant q_k(t'') - q''_k(t'') = q_k(t) - q''_k(t'') \leqslant \mathbf{B}^{r_k} \leqslant \max\left(\mathbf{B}^{r^k}, \mathbf{B}^{r'_k}\right)$.

Proofs of bounds improvements

We remind a classical result on semidefinite positive matrices: $(X \in \mathbb{S}^+_n \wedge \mathrm{rk}(X) = 1)$ if and only if $\exists\, x \in \mathbb{R}^n$ s.t. $X = xx^\intercal$.

Proof (Proof of Lemma 2). Recall that $\mathbf{C}^m = [-1,1]^m \times [-1,1] \times [0,1] \times [-1,0]$. Let $X \in \mathbb{S}^+_{m+4}$ such that $\mathrm{rk}(X) = 1$ and $X \in \mathcal{C}^m$. Since $X \in \mathbb{S}^+_{m+4}$ and $\mathrm{rk}(X) = 1$, there exists $u \in \mathbb{R}^{m+3}$ and $v \in \mathbb{R}$ such that $X = xx^\intercal$ with $x = (u\ v)$ and thus for all $i, j \in [m+4]$, $X_{i,j} = x_i x_j$. Now since $X \in \mathcal{C}^m$, from Constraint (3g), $x_{m+4}x_{m+4} = v^2 = 1$ and then $v \in \{-1, 1\}$. Using the fact that $X = (-x)(-x)^\intercal$, we can choose $v = 1$. Now from Constraint (3c), for all $i \in [m+3]$, $X_{i,m+4} = x_i x_{m+4} = x_i = u_i \in [-1, 1]$. Finally, from Constraint (3e), we get $X_{m+2,m+4} = x_{m+2}x_{m+4} = x_{m+2} = u_{m+2} \in [0, 1]$ and from Constraint (3f), $X_{m+3,m+4} = x_{m+3}x_{m+4} = x_{m+3} = u_{m+3} \in [-1, 0]$. We conclude that $u \in \mathbf{C}^{m+3}$.

Now let us take X of the form $X = xx^\intercal$ such that $x = (u\ v)$ with $u \in \mathbf{C}^{m+3}$ and $x_{m+4} = v = 1$. Since for all $i, j \in [m+4]$, $X_{i,j} = x_i x_j$ and $u \in \mathbf{C}^{m+3}$, we have readily Constraints (3).

Proof (Proof of Theorem 2). We only prove the proposition for the upper bounds, the proof for the lower bounds can be done similarly. Let $X \succeq 0$ be in $\mathcal{C}^m$. We can write X as a the following block matrix (the notations around the matrix indicates the sizes of the blocks):

$$\begin{array}{c} \begin{array}{ccc} \overbrace{\quad\quad}^{m} & \overbrace{\quad\quad}^{3} & \overbrace{\quad\quad}^{1} \end{array} \\ \left(\begin{array}{ccc} X^A & X^0 & X^b \\ X^{0\intercal} & X^{00} & X^c \\ X^{b\intercal} & X^{c\intercal} & 1 \end{array}\right) \begin{array}{l} \}m \\ \}3 \\ \}1 \end{array} \end{array}$$

We now rely on the fact that tr is linear and satisfies for all square matrices M, $\mathrm{tr}(M^\intercal) = \mathrm{tr}(M)$ and for all matrices M, N such that MN and NM are square matrices, $\mathrm{tr}(MN) = \mathrm{tr}(NM)$. Considering this and the symmetry of X, we have after simplifications: $\mathrm{tr}\left(\mathbf{M}^{r_q} X\right) = \mathrm{tr}\left(A^q X^A\right) + b^{q\intercal} X^b + X^{c\intercal}(c_\pm, c_+, c_-) + c^q$. Constraints (3a) and (3c) yield to $\mathrm{tr}\left(A^q X^A\right) \leqslant \sum_{i=1}^m \sum_{\substack{j=1,\ldots,m \\ j \neq i}} |(A^q)_{i,j}| + \sum_{i=1}^m [(A^q)_{i,i}]^+$. Constraint (3b) implies that $b^{q\intercal} X^{b_q} \leqslant \sum_{j=1}^m |b^q{}_j|$ and Constraints (3e) and (3f) imply that $X^{c\intercal}(c_\pm, c_+, c_-) \leqslant c_\pm + c_+$. By summation, we conclude that $\mathrm{tr}\left(\mathbf{M}^{r_q} X\right) \leqslant \mathbf{B}^q_{MT}$ for all $X \succeq 0$ in $\mathcal{C}^m$ and then $\mathbf{B}^q_{SDP} \leqslant \mathbf{B}^q_{MT}$.

Experiments

The following table summarizes the numerical results obtained when comparing interval arithmetics, zonotopic domains with affine arithmetics and the quadratic extension. The T1P domain is an extension of the affine zonotopes with additional linear constraints over error terms.

Numerical values have been truncated to ease their display. However, even in presence of similar values, the highlighted ones were more precise before the truncation.

	Intervals		Affine Z.		Quad. Z.		T1P	
	val	*ms*	*val*	*ms*	*val*	*ms*	*val*	*ms*
atan	$[-1.91, 1.91]$	7	$[-1.91, 1.91]$	11	$[-1.00, 1.00]$	3	$[-1.34, 1.34]$	13
atan10	$[-1.91, 1.91]$	0	$[-2.36, 2.36]$	6	$[-1.59, 1.59]$	8	$[-1.47, 1.47]$	8
stolfi1	$[0., \infty]$	0	$[-\infty, +\infty]$	0	$[-0.85, 3.25]$	5	$[0., 3.60]$	6
stolfi2	$[0., 3.60]$	7	$[-\infty, +\infty]$	3	$[-\infty, +\infty]$	6	$[-0., 2.51]$	5
stolfi3	$[0., 5.38]$	6	$[-2.98, 10.81]$	3	$[-0.62, 3.24]$	4	$[-0., 1.75]$	6
stolfi4	$[0., 2.23]$	11	$[-\infty, +\infty]$	8	$[-0.33, 3.26]$	0	$[0.35, 1.71]$	8
stolfi5	$[0., 2.18]$	0	$[-0.42, 3.03]$	4	$[0.08, 2.38]$	4	$[0.34, 1.55]$	7
stolfi6	$[0., 2.18]$	3	$[-0.71, 2.91]$	9	$[0.11, 2.30]$	11	$[0.38, 1.65]$	8
stolfi7	$[0., 1.89]$	6	$[0.19, 2.23]$	3	$[0.29, 1.97]$	6	$[0.45, 1.46]$	8
stolfi8	$[-0., 1.82]$	4	$[0.20, 2.]$	0	$[0.29, 1.88]$	3	$[0.47, 1.40]$	3
stolfi9	$[0., 1.73]$	10	$[0.26, 1.89]$	6	$[0.33, 1.75]$	10	$[0.47, 1.40]$	7
stolfi10	$[0., 1.71]$	11	$[0.32, 1.82]$	0	$[0.36, 1.75]$	5	$[0.49, 1.38]$	0
stolfi11	$[0., 1.67]$	3	$[0.34, 1.74]$	3	$[0.39, 1.67]$	4	$[0.51, 1.36]$	15
stolfi13	$[0., 1.61]$	10	$[0.38, 1.67]$	0	$[0.41, 1.61]$	8	$[0.51, 1.35]$	6
stolfi30	$[0.35, 1.43]$	10	$[0.48, 1.44]$	13	$[0.48, 1.43]$	16	$[0.53, 1.31]$	11
stolfi40	$[0.40, 1.40]$	11	$[0.49, 1.40]$	15	$[0.50, 1.40]$	14	$[0.53, 1.31]$	15
stolfi50	$[0.43, 1.38]$	12	$[0.50, 1.38]$	24	$[0.50, 1.38]$	26	$[0.53, 1.30]$	18
stolfi55	$[0.44, 1.37]$	14	$[0.51, 1.37]$	28	$[0.51, 1.37]$	27	$[0.53, 1.30]$	19
stolfi100	$[0.48, 1.34]$	27	$[0.52, 1.34]$	69	$[0.52, 1.34]$	52	$[0.54, 1.30]$	30
stolfi200	$[0.51, 1.32]$	40	$[0.53, 1.32]$	272	$[0.53, 1.32]$	209	$[0.54, 1.30]$	48
stolfi300	$[0.52, 1.31]$	50	$[0.53, 1.31]$	710	$[0.53, 1.31]$	439	$[0.54, 1.30]$	90
stolfi400	$[0.52, 1.31]$	67	$[0.53, 1.31]$	1359	$[0.53, 1.31]$	842	$[0.54, 1.30]$	111
Ln1px	$[-0.00, 0.46]$	4	$[-0.00, 0.40]$	8	$[-0.00, 0.40]$	5	$[-5.14e-05, 0.40]$	3
householder_orig	$[0.11, 0.11]$	5	$[0.11, 0.11]$	7	$[0.11, 0.11]$	0	$[0.11, 0.11]$	2
householder_orig	$[0.17, 0.18]$	0	$[0.17, 0.18]$	8	$[0.17, 0.18]$	0	$[0.17, 0.18]$	7
householder_orig	$[0.21, 0.24]$	4	$[0.21, 0.24]$	7	$[0.21, 0.24]$	0	$[0.21, 0.24]$	9
householder_orig	$[0.17, 0.29]$	6	$[0.22, 0.25]$	6	$[0.22, 0.24]$	11	$[0.22, 0.25]$	3
householder_orig	$[0.03, 0.42]$	0	$[0.22, 0.25]$	10	$[0.22, 0.24]$	3	$[0.22, 0.25]$	3
householder_orig	$[-0.90, 1.66]$	7	$[0.22, 0.25]$	12	$[0.22, 0.24]$	5	$[0.22, 0.25]$	0
householder_orig	$[-1117.82, 1899.48]$	11	$[0.22, 0.25]$	26	$[0.22, 0.24]$	7	$[0.22, 0.25]$	3
householder_orig	$[-2.18e^{+18}, 3.70e^{+18}]$	3	$[0.22, 0.25]$	16	$[0.22, 0.24]$	3	$[0.22, 0.25]$	14

	Intervals		Affine Z.		Quad. Z.		T1P	
	val	*ms*	*val*	*ms*	*val*	*ms*	*val*	*ms*
householder_orig	$[-6.19e^{+94}, 1.05e^{+95}]$	4	$[0.22, 0.25]$	32	$[0.22, 0.25]$	12	$[0.22, 0.25]$	11
householder_orig	$[-\infty, \infty]$	4	$[0.22, 0.25]$	39	$[0.22, 0.25]$	7	$[0.22, 0.25]$	4
householder_orig	$[-\infty, \infty]$	8	$[0.22, 0.25]$	34	$[0.22, 0.25]$	6	$[0.22, 0.25]$	7
householder_orig	$[-\infty, \infty]$	7	$[0.22, 0.25]$	37	$[0.22, 0.25]$	4	$[0.22, 0.25]$	8
householder_orig	$[-\infty, \infty]$	11	$[0.22, 0.25]$	20	$[0.22, 0.25]$	7	$[0.22, 0.25]$	3
householder_orig	$[-\infty, \infty]$	0	$[0.22, 0.25]$	22	$[0.22, 0.25]$	6	$[0.22, 0.25]$	7
householder_orig	$[-\infty, \infty]$	3	$[0.22, 0.25]$	11	$[0.22, 0.25]$	7	$[0.22, 0.25]$	10
householder_orig	$[-\infty, \infty]$	4	$[0.22, 0.25]$	21	$[0.23, 0.26]$	3	$[0.22, 0.25]$	9
householder_orig	$[-\infty, \infty]$	7	$[0.22, 0.25]$	21	$[0.24, 0.27]$	6	$[0.22, 0.25]$	11
householder	$[0.11, 0.11]$	7	$[0.11, 0.11]$	2	$[0.11, 0.11]$	4	$[0.11, 0.11]$	7
householder	$[0.17, 0.18]$	5	$[0.17, 0.18]$	10	$[0.17, 0.18]$	10	$[0.17, 0.18]$	3
householder	$[0.21, 0.24]$	4	$[0.21, 0.24]$	7	$[0.21, 0.24]$	5	$[0.21, 0.24]$	3
householder	$[0.17, 0.29]$	0	$[0.21, 0.25]$	6	$[0.22, 0.25]$	11	$[0.21, 0.25]$	5
householder	$[0.03, 0.42]$	4	$[0.20, 0.26]$	12	$[0.21, 0.25]$	20	$[0.21, 0.26]$	11
householder	$[-0.90, 1.66]$	6	$[0.19, 0.27]$	4	$[0.21, 0.25]$	16	$[0.20, 0.26]$	12
householder	$[-1117.82, 1899.48]$	0	$[0.17, 0.29]$	11	$[0.21, 0.25]$	13	$[0.19, 0.27]$	4
householder	$[-2.18e^{+18}, 3.70e^{+18}]$	3	$[0.12, 0.34]$	10	$[0.21, 0.25]$	9	$[0.17, 0.29]$	7
householder	$[-6.19e^{+94}, 1.05e^{+95}]$	3	$[-0.16, 0.64]$	6	$[0.21, 0.26]$	16	$[0.14, 0.33]$	10
householder	$[-\infty, \infty]$	3	$[-14.08, 14.55]$	8	$[0.20, 0.26]$	15	$[0.00, 0.47]$	9
householder	$[-\infty, \infty]$	0	$[-98003775.36, 98003775.83]$	11	$[0.20, 0.27]$	22	$[-1.45, 2.24]$	9
householder	$[-\infty, \infty]$	7	$[-1.35e^{+42}, 1.35e^{+42}]$	11	$[0.18, 0.30]$	16	$[-5335.40, 8226.73]$	10
householder	$[-\infty, \infty]$	4	$[-6.88e^{+212}, 6.88e^{+212}]$	9	$[0.15, 0.35]$	11	$[-3.66e^{+21}, 5.65e^{+21}]$	5
householder	$[-\infty, \infty]$	3	$[-\infty, +\infty]$	13	$[0.04, 0.52]$	11	$[-5.61e^{+110}, 8.65e^{+110}]$	7
householder	$[-\infty, \infty]$	5	$[-\infty, +\infty]$	7	$[-1.41, 2.27]$	11	$[-\infty, \infty]$	6
householder	$[-\infty, \infty]$	5	$[-\infty, +\infty]$	10	$[-2570.28, 2582.02]$	11	$[-\infty, \infty]$	16
householder	$[-\infty, \infty]$	0	$[-\infty, +\infty]$	10	$[-2.85e^{+18}, 2.85e^{+18}]$	7	$[-\infty, \infty]$	6
householder	$[-\infty, \infty]$	5	$[-\infty, +\infty]$	17	$[-4.62e^{+93}, 4.62e^{+93}]$	18	$[-\infty, \infty]$	11
controller	$[-0.21, 0.21]$	7	$[-0.15, 0.15]$	53	$[-0.15, 0.15]$	21	$[-0.15, 0.15]$	10
controller	$[-0.42, 0.42]$	12	$[-0.19, 0.19]$	191	$[-0.19, 0.19]$	20	$[-0.19, 0.19]$	18
controller	$[-0.65, 0.65]$	20	$[-0.20, 0.20]$	518	$[-0.20, 0.20]$	49	$[-0.20, 0.20]$	21
controller	$[-1.25, 1.25]$	30	$[-0.23, 0.23]$	1865	$[-0.23, 0.23]$	74	$[-0.23, 0.23]$	35
SinCos	$[0.86, 1.18]$	8	$[0.99, 1.01]$	10	$[0.99, 1.00]$	7	$[0.99, 1.00]$	2