Measurements of Second-Order Properties of Point Processes

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Abstract

Abstract-The second order statistical properties of point processes are described by the coincidence function which can be measured by a coincidence device but such measurements are long and complicated. We propose another method of measurement and we analyze its performances. The starting point is that the coincidence function can be deduced from the probability density functions of the life times (the distances between points) of the process. The idea is to transform the point process into a positive signal whose values are these distances. From an appropriate processing of this signal we deduce the coincidence function. For the validation of the method we use point processes for which the coincidence function is known. The agreement between theory and experiment is in general excellent. Finally the method is applied to measure the coincidence functions of some point processes for which no theoretical result is available.

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Index Terms

Point processes, statistical measurements, signal processing, signal representation.

I. Introduction

Point processes (PP) play an important role in many areas of physics, statistics, and engineering sciences. For example in optical communications at low levels the only available information is the set of random instants at which photons are detected. Similarly the time instants at which telephone calls arrive at a switching center is a PP.

A PP is a random distribution of points in a space. If this space is the time axis we have a time PP and its random points P_i are time instants t_i sometimes called events. All the PPs considered below are time PPs. In what follows we consider only stationary PPs characterized by the fact that their statistical properties are invariant in time.

The complete statistical description of a PP is very complicated and in most applications one is obliged to use only first- or second-order statistical properties. The first-order description of a stationary PP is contained in its density λ which is the average number of points per unit of time. The transposition to PPs of the concepts of the correlation function requires a specific analysis. The first attempt to describe second-order properties of PPs was presented by Bartlett [1], [2]. Since a PP has no correlation function, the idea was to introduce a similar function which was called the correlation density function. More recently it was indicated that this function also appears in the description of a PP by coincidence analysis [3] and this is the reason to use the term coincidence function [4] which contains all the second-order properties of a PP.

The coincidence function can be measured by a coincidence device, system widely used in Nuclear Physics. The measurements however are long and complicated and not adapted to various PPs. We propose another approach which does not start directly from the definition of the coincidence function but from its relation to the properties of the distances between successive points of the PP sometimes called life times [5], [6], [7]. Using the standard vocabulary of experimental nuclear physics, the device in order to measure the coincidence function is then no longer a coincidence circuit but a time to amplitude converter (TAC) which transforms distances between points into pulses of amplitude proportional to these distances. These amplitudes constitute a discrete-time positive signal. By an appropriate statistical signal processing of these distances it it is possible to measure the coincidence function.

Coincidence function and coincidence measurements are at the basis of some applications in statistical optics and more lately in quantum communication [4], [8], [9]. For example the structure of the coincidence function is a tool to decide whether or not an optical field is of classical or of non-classical nature [8].

In Section II we present a short review of properties of coincidence functions and we describe the principle of their measurement from the analysis of life times. We discuss theoretically the precision of the method and its limitations. In Section III we test the method by using computer experiments. There are some PPs for which the coincidence function can be calculated theoretically. By making measurements on those PPs we can validate the behavior of the method and evaluate its precision. In general the experimental measurements are in excellent agreement with theoretical results. In the last section we apply the method in the case of some PPs for which the calculation of the coincidence function is almost impossible and we present and discuss some examples.

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II. Definition and Measurement of the Coincidence Function

Let $N(t)$ be the number of points of a PP in the interval $[0, t[$, where 0 is an arbitrary origin of time. The increment $dN(\theta) = N(\theta + d\theta) - N(\theta)$ is the number of points in the interval $[\theta, \theta + d\theta[$. The second-order properties of a PP are characterized by its coincidence function $c(t)$ defined by

$$\mathsf{E}[dN(\theta)dN(\theta')] = c(\theta - \theta')d\theta d\theta'. \tag{1}$$

The name comes from the fact that this function is generated by a coincident events. Indeed a coincidence at θ and θ' is the event defined by $[dN(\theta) = 1] \cap [dN(\theta') = 1]$. For regular PPs, which are the only ones studied in this paper, the increment $dN(\theta)$ takes only the values 0 or 1 when $d\theta$ tends to 0. As a consequence the expectation value appearing in (1) is the coincidence probability. The function $b(t) = c(t)/\lambda$, where λ is the density or the PP, is called the bunching function because it describes the bunching effect appearing in the PP obtained in photodetection [4], [8], [9]. It is also sometimes called the intensity function (see p. 69 of [10]).

Note that the coincidence function is an even function of t and that it tends in general to λ^2 when t tends to infinity. This property comes from the fact that for large values of $\theta - \theta'$ the increments $dN(\theta)$ and $dN(\theta')$ become in general uncorrelated and their mean value is $\lambda d\theta$. In the case of Poisson processes these increments are independent whatever the value of $\theta - \theta'$, which implies that the coincidence function is constant and equal to λ^2.

The coincidence function satisfies a relation which will play a fundamental role in what follows. Let $x_i^{[n]}$ be the random variable (RV) equal to the distance between a point t_i of the PP and the nth point posterior to t_i, or $x_i^{[n]} = t_{i+n} - t_i$. This positive RV is sometimes called the life time of order n. Its distribution function $F_n(t)$ is the probability $P[x_i^{[n]} \leq t]$ and the PDF $f_n(t)$ of this life time is the derivative of $F_n(t)$ with respect to t. For simplicity we assume that $x_i^{[n]}$ is a continuous RV or that $F_n(t)$ has a derivative for any t. Since $x_i^{[n]} > 0$, then $f_n(t) = 0$ when $t < 0$. It can be shown (see p. 69 of [10]) that the coincidence function satisfies

$$c(t) = \lambda \sum_{n=1}^{\infty} f_n(|t|). \tag{2}$$

The principle of the proof is presented in the Appendix. This equation appears in a rather different forms in [5], [6].

This relation is the basis for the estimation of the coincidence function. For this purpose we use a TAC system which transforms the sequence of random points t_i of the PP into a sequence of values $x_i = t_{i+1} - t_i$. This positive random signal x_i describes completely the PP and any PP generates such a signal x_i.

Suppose now that we have M observations of x_i, $1 \leq i \leq M$. The problem is to deduce from these observations an estimation of the coincidence function $c(t)$. The measurement of λ appearing in (2) is obvious because the density is the inverse of the mean value of the distances x_i. For the estimation of $c(t)$ the first step is to replace the series (2) by a finite sum of S terms. As a consequence we arrive at the estimation of the truncated coincidence function $c_S(t) = \lambda \sum_{n=1}^{S} f_n(|t|)$. The choice of the appropriate value of S will be discussed later.

For a given value of S such that M/S is an integer N, and for each k satisfying $1 \leq k \leq S$, we deduce from the observation x_i a set of S signals $s_i^{[k]}$ defined by

$$s_i^{[k]} = x_{(k-1)N+i} \ , \ 1 \leq i \leq N, \tag{3}$$

and zero otherwise. From these signals we construct a set of S vectors $\mathbf{X}^{[k]}$ with N components defined by

$$X_i^{[k]} = s_i^{[k]} + s_{i+1}^{[k]} + ... + s_{i+k-1}^{[k]} \ , \ 1 \leq i \leq N. \tag{4}$$

The histogram of these components will clearly yield an estimation of the PDF $f_k(t)$. As a consequence the histogram of the components of the vector $\mathbf{X} = \mathbf{X}^{[1]} + \mathbf{X}^{[2]} + ... + \mathbf{X}^{[S]}$ yields an estimation of the truncated bunching function $b_S(t) = (1/\lambda)c_S(t)$. More precisely, consider an interval $[t, t + \Delta T[$ and let $n_k(t, \Delta T)$ be the number of samples $X_i^{[k]}$ recorded in this interval. If ΔT is sufficiently small the mean value $\bar{n}_k(t, \Delta T)$ of $n_k(t, \Delta T)$ is approximately $N f_k(t) \Delta T$. Let $n(t, \Delta T)$ the number of samples of the components of the vector $\mathbf{X}$ recorded in the same interval. From the definition of $\mathbf{X}$ and $c_S(t) = \lambda b_S(t)$ the mean value $\bar{n}(t, \Delta T)$ of $n(t, \Delta T)$ is

$$\bar{n}(\Delta T) = b_S(t)\Delta T N = b_S(t)\frac{M\Delta T}{S} \ . \tag{5}$$

The histogram of $\mathbf{X}$ yields for each experiment a value of the recorded number $n(t, \Delta T)$. If N is sufficiently large this number yields an estimation of $\bar{n}(t, \Delta T)$, and then of $b_S(t)$ or $c_S(t)$. This procedure necessarily introduces a statistical error analyzed below. Finally if we are interested only in the shape of the coincidence function, the histogram can be represented in arbitrary units and the factor $M\Delta T/S$ can be omitted.

Let us now discuss the influence of the various parameters appearing in this measurement. The two most important are the number N of samples recorded and the number S of terms used for the construction of the vector $\mathbf{X}$ or for the approximation of the series (2) by a sum of S terms. On a pure mathematical point of view, the approximation of a function like $c(t)$ defined as a series of other functions requires the discussion of the uniform convergence of the series. There are PDFs for which this uniform convergence is guaranteed, but as the method is used in the case where the PDFs $f_n(t)$ of (2) are unknown it is impossible to introduce in advance more details on this point.

The number N determines the statistical precision of the histogram and the number S determines the precision of an approximation of a series by a finite sum. These numbers are not completely independent. Indeed the total number of samples analyzed by the histogram that yields the estimation of $c_S(t)$ is $M = S \times N$. It is this number which determines the duration of the measurement or the complexity of the computer analysis. It is often necessary to introduce an upper bound of this number and in this case we must choose N and S for a given value of M. This introduces a compromise analogous to the one existing between bias and variance in many statistical measurements, as for example spectral analysis. This will be discussed more precisely in the next section.

There are two last parameters appearing in the construction of the histogram yielding the approximation of the coincidence function. The first is the range ΔR of values of t of the function $c(t)$. This range must be limited for great values of t. Indeed it is clear that $c_S(t)$ tends to 0 when $t \to \infty$ because it is a finite sum of PDFs while, as seen above, $c(t) \to \lambda^2$. The second is the number of bins of the histograms of values belonging to ΔR. Here also there is a well known compromise. Too many bins yields a better precision of the approximation of $c_S(t)$ but also decreases the number of samples belonging to each bin and therefore increases their variance. This compromise between bias and variance in estimation of the PDF of signals has been analyzed in numerous statistical signal processing papers.

Finally it must be noticed that there is a systematic error in the $k-1$ last components of the vector $\mathbf{X}^{[k]}$. In particular $X_N^{[k]} = s_N^{[k]} = x_{kN}$ which is not a sum of k terms. This kind of error also appears in all the measurements of correlation functions of random signals. There are ways to correct this error, but this does not play any role in our methods because in our experiments $N >> S$.

III. Performances of the Method

There are only a few PPs for which the coincidence function can be explicitly calculated. They will now be used to look at the agreement between theoretical results and those of computer experiments which yields an indication on the quality of the method.

A. *Poisson Processes*

The PDF of the life time of order n of a Poisson process is

$$f_n(t) = \lambda \frac{(\lambda t)^{n-1}}{(n-1)!} \exp(-\lambda t). \tag{6}$$

Inserting this expression in (2) yields $c(t) = \lambda^2$, which illustrates the fact that a Poisson process has no memory. The error when approximating (2) by a finite sum calculated for $\lambda = 1$ is then $\epsilon_S(t) = 1 - \exp(-t) \sum_{n=1}^{S} t^n/n!$.

By a simple calculation we see that we can approximate $c(t)$, which is here equal to 1, by using 10 terms with an error smaller that 10^{-2} for $t < 4$. With 15 terms this range becomes $t < 7$. Similarly by using 20 terms we can approximate $c(t)$ with an error smaller than 10^{-3} for $t < 9$. This value of S is the greatest used in the experiments described below.

Let us now present experimental results with Poisson processes. A stationary Poisson process is a renewal PP with an exponential PDF. Then the distances x_i between successive points are independent and identically distributed (IID) random variables and their common PDF is given by (6) with $n = 1$. It is easy to generate in a computer experiment such a sequence, which allows us to discuss the performance of the method in terms of the various parameters appearing in its construction. The experiments are realized with a Poisson process of density $\lambda = 1$. This implies that $b_S(t) = c_S(t)$.

In Fig. 1 we present experimental results obtained when processing $M = 2.10^6$ samples of the life times x_i. The parameters of these histograms are $S = 5$, $S = 10$, $S = 15$, $S = 20$. The widths of the bins of the histograms are 10^{-2}. This means that there are 2.10^3 values recorded in the interval [0, 20] of the figure. Three comments must be made.

First we verify experimentally the fact that the histograms are constant in an interval increasing with S. In the last histogram obtained for $S = 20$ the methods yields excellent results for $t < 12$, which means a duration twelve times greater than the mean distance between points of the Poisson process.

Secondly, as the range of possible values increases with S, the statistical precision decreases because the number of samples recorded in each bin decreases. When comparing the first and the last histograms we observe an increase of the statistical fluctuations. The only way to decrease these fluctuations is to increase the total number M of samples analyzed.

Thirdly the number of points recorded in each bin corresponds to the values deduced from (5). Applying this expression for $t = 0$ and noting that in our experiment $c_S(0) = c(0) = 1$ because $\lambda = 1$, we obtain $\bar{n}(\Delta T) = M\Delta T/S$. In this experiment $M = 2.10^6$, $\Delta T = 10^{-2}$. The values of $\bar{n}(\Delta T)$ corresponding to the four values of S used are then 4.10^4, 2.10^3, $1.33.10^3$, and 10^3, which cleanly appears on the figure.

B. Erlang Processes

Erlang processes belong to a class of PPs in which the PDF of the life time of order 1 is

$$f_1(t) = \mu(\mu t)\exp(-\mu t). \tag{7}$$

The mean value of the life time is $2/\mu$ and the density λ is then $\mu/2$. According to (6) this means that $f_1(t)$ is the PDF of the life time of order 2 of a Poisson process of density μ.

Since the PDF (7) does not define a specific PP there are various different Erlang processes. Their difference comes from the fact that, even if the PDFs of the RVs x_i are the same, the other statistical properties are different. There are of course a great variety of Erlang processes, depending on the correlation between the x_is. We shall only consider the two simplest cases.

The first case is the *renewal Erlang process*. It is the renewal PP defined by the PDF (7). It can be generated by the following procedure. Let u_i and v_i be two independent signals and suppose that they are sequences of IID RVs with the same exponential distribution. It is clear that $x_i = u_i + v_i$ is a sequence of IID RVs with the common PDF (7).

In order to calculate the coincidence or the bunching function $b(t) = c(t)/\lambda$ we use the following procedure. The Laplace transform (LT) of (7) is $F(s) = [\mu/(s+\mu)]^2$. The assumption that the PP is a renewal process yields that the LT of $f_n(t)$ is $F_n(t) = [F(t)]^n$. Then the LT of the bunching function is $B(s) = F(s)/[1 + F(s)]$ or

$$B(s) = \frac{\mu^2}{(s+\mu)^2 - \mu^2} = \frac{\mu}{2}\left[\frac{1}{s} - \frac{1}{s+2\mu}\right]. \tag{8}$$

As a result the bunching function is

$$b(t) = \frac{\mu}{2}\left[1 - \exp(-2\mu t)\right]. \tag{9}$$

The second Erlang process is defined as follows. Consider a Poisson process and let x_i be the life time of order 1, or the distance between successive points. Let $y_i = x_i + x_{i+1}$. It is clear that y_i is the life time of order 2 of the Poisson process and its PDF is given by (7). The RVs y_i however are no longer independent because, for example, x_i appears in y_i and y_{i-1}. Consider now the PP defined by the fact that y_i is the distance between successive points. It is then an Erlang process *but not a renewal process*. Changing the origin of time we can write $y^{[1]} = x_1 + x_2$ and

$$y^{[n]} = x_1 + 2(x_2 + x_3 + ... + x_n) + x_{n+1} \ , \ n > 1, \tag{10}$$

where the x_is are IID RVS with the exponential distribution $\mu \exp(-\mu t)$. We deduce that the LT of the PDF of $y^{[n]}$, the life time of order n, is

$$F_n(s) = \left(\frac{\mu}{s+\mu}\right)^2 \left(\frac{\mu}{2s+\mu}\right)^{n-1} \ , \ n \geq 1. \tag{11}$$

It is easy to calculate the LT of the bunching function defined by $B(s) = \sum_{n=1}^{\infty} F_n(s)$ and its inverse LT is

$$b(t) = \frac{\mu}{2}\left[1 - (1 - \frac{t}{\mu}) \exp(-\mu t)\right], \tag{12}$$

which must be compared to (9) obtained previously.

The experimental results for these two Erlang processes appear in Fig. 2. These figures use arbitrary units because we are only interested in the shape of the coincidence function in comparison with its theoretical value. Fig. 2.1 corresponds to the Erlang renewal process. The continuous curves are those deduced from (9) in Fig. 2.1 and (12) in Fig. 2.2 and the experimental points are obtained from an experiment with 10^6 samples and with $\mu = 1$ and $S = 10$. We see that there is excellent agreement between theory and experiment. Even if the the difference between the two Erlang processes is rather small, it appears clearly.

All the results of this section show the good performance of the method of measurement of the coincidence function introduced in this paper.

C. Poisson Process with Input Dead Time

Dead time effects appear in almost all the PPs practically used. Dead time occur when two points of a PP are so close together that they cannot both be registered. This means that some points are erased. If the value D of the dead time is very small compared to the mean distance between points, its effect can be neglected. On the other hand when the density μ of the process increases there is always a value of the density such that dead time must be taken into consideration.

The input dead time is characterized by the fact that each point t_i of the PP generates an interval $[t_i, t_i + D[$ such that all the points of the process arriving in this interval are erased. It is clear that when the density μ increases the number of points erased increases and for very large density almost all the points of the initial PP are erased. This corresponds to the classical congestion phenomenon. Note that the input dead time is sometimes called type II counter (see p. 101 of [7]).

The dead time effect transforms a given PP into another one. As in our approach a PP is defined by the life times, or the distance between successive points, the first problem is to calculate the life time after dead time in terms of the life time of the initial PP. The explicit expression of the result is almost impossible to obtain in closed form. A recursive algorithm, however, has been introduced and analyzed in [11]. This algorithm is used for the estimation of the coincidence function when the initial PP is a Poisson process. In this case the theoretical calculation of the coincidence function with input dead time is especially simple. For this it suffices to start from (1). If $\theta - \theta' < D$ the expectation is zero because one of the two points is necessarily erased. Otherwise $c(t)$ is constant and its value is λ^2. Note that λ is the density of the PP with dead time. If the density of the initial Poisson process is μ a simple calculation yields that $\lambda = \mu \exp(-\mu D)$. This expression explains the congestion phenomenon indicated above because λ tends to 0 when D tends to infinity. In the opposite case $\lambda = \mu$ if $D = 0$.

Experimental results are presented in Fig. 3. The density of the Poisson process is $\mu = 1$ and two values of the dead time are used : 0.5 and 1. The number S of terms in the sum defining $c(t)$ is 10, which ensures good precision of the results for $t < 3$. The fact that $c(t) = 0$ for $t < D$ is well verified. On the other hand the fact that $c(t)$ is constant for $t > D$ is also verified although statistical fluctuations of the estimation remain. These fluctuations are more important for $D = 1$ than for $D = 0.5$ because more points are erased in the second case than in the first. Finally, as in these experiments $M = 3.6\ 10^6$, $\Delta T = 10^{-2}$, and $S = 10$, the application of (5) with $b_S(t) = \mu \exp(-\mu D)$ yields that the mean number of points $\bar{n}(\Delta T)$ in each bin is for $t > D$ equal to 2.183 or 1.324 for the cases 1 or 2 respectively. This is very well verified in the two histograms of Fig. 3.

IV. Analysis of some Particular Point Processes

In this section we shall analyze the second-order properties of some point processes interesting in various applications but for which theoretical analysis is difficult, which justifies the experimental approach.

A. Poisson Process with Output Dead Time

This dead time, also called type I counter (see p. 101 of [7]), is characterized by the fact that each point θ_i of the Poisson process *which is not erased* generates an interval $[\theta_i, \theta_i + D[$ such that all the points t_i of the PP in this interval are erased. This means that only the points not erased contribute to the dead time. In this case when $\mu \to \infty$ the PP after dead time becomes a periodic process, or a sequence of points with equal distances D.

The calculation of the life time after output dead time in terms of the life time of the initial PP is very complicated. As previously, however, a recursive algorithm for this purpose has been introduced in [11]. We shall use this algorithm for the estimation of the coincidence function after output dead time effect when the input is a Poisson process.

Let us first present how this coincidence function can be calculated in this case. It is easy to see that the PP after dead time is a renewal process entirely defined by the PDF of its life time. This PDF is

$$f(t) = u(t-D)\mu \exp[-\mu(t-D)], \tag{13}$$

where $u(.)$ is the unit step function, D the value of the dead time and μ the density of the Poisson process. It results from this expression that the mean value of the life time after dead time is $D+1/\mu$. This quantity is the inverse of the density which then is $\lambda = \mu/[1+\mu D]$. We observe that when $\mu \to \infty$, $\lambda \to D$, which was indicated above. Furthermore when $D \to \infty$, $\lambda \to 0$, which appears with any dead time effect.

As the PP analyzed is a renewal process, its PDF $f_n(t)$ is deduced from (13) by n convolutions and the result is

$$f_n(t) = u(t-nD)\mu \frac{[\mu(t-nD)]^{n-1}}{(n-1)!} \exp[-\mu(t-nD)]. \tag{14}$$

In order to calculate the coincidence function we can use (2). There is no explicit analytic expression for $c(t)$, but its numerical calculation is possible. Indeed because of the term $u(t-nD)$ in (14) the series is always limited to a finite sum. As we must have $t-nD > 0$, the number of terms in this sum is the greatest integer smaller than t/D.

Numerical results of calculations are presented in Fig. 4 where some examples of the bunching function $b(t) = (1/\lambda)c(t)$ are presented. The value of μ is 1 and the values of D are 1/4, 2/3, 1, 3/2, in such a way that the corresponding values of the density λ are 0.8, 0.6, 0.5, 0.4. We verify that $b(t) \to \lambda$ when $t \to \infty$. In reality the values of t are limited to the interval $0 < t < 8$ and for this interval the sum (2) is truncated to 12 terms.

We see on this figure that when $D = 1/4$ the effect of the dead time on the coincidence function disappears when $t > 2/3$. For $D = 1$ this inequality becomes $t > 4$. Furthermore all these curves show a discontinuity of the derivative for the value $t = 2D$ which is the time at which the second term $f_2(t)$ appears in the sum (2).

The experiments with output dead time are much more difficult. Indeed the algorithm yielding the life time after dead time presented in [11] introduces much more complexity and calculation time. This limits the possible values of M, and thus the statistical precision of the procedure. This effect is increased by the fact that, as in any experiment with dead time, many points are erased and the number of those playing a role in the estimation tends to 0 when D increases.

Experimental results appear in Fig. 5. Estimations of the coincidence function are presented for $D = 0.5$ and $D = 1.5$. The results are in a rather good agreement with those of Fig. 4, but some statistical fluctuations remain. Their suppression would require more memory and computer time.

B. Renewal Process with Uniform Distribution

Consider a renewal PP in which the PDF $f(t)$ of the distance between successive points is a rectangular function equal to $1/2b$ when t is in the interval $[m-b, m+b]$ and zero otherwise. This kind of PDF appears in the description of jitter phenomena in which the distance between successive points is not strictly constant, which appears if $b \to 0$. The mean value of the distance is clearly m and as a result the density λ of the corresponding PP is $1/m$.

The PDFs $f_n(t)$ can be calculated by successive convolutions. The calculation is possible but rapidly becomes very complicated and no explicit expression of the convolution is available. Two points however can be noted. First the PDF $f_2(t)$ of the second order life time is a triangular function equal to $(1/4b^2)[- \mid t-2m \mid +2b]$ for $\mid t-2m \mid < 2$, and zero otherwise. In particular $f_2(2m) = 1/2b$. Secondly when $n >> 1$, $f_n(t)$ tends to have a Gaussian shape of the form $N(nm, n\sigma^2)$, where σ^2 is the variance associated with the PDF $f(t)$ and equal to $b^2/3$.

Experimental results are presented in Fig. 6. The bunching function of the PP is estimated for the values $m = 1$. The values chosen for b are 1/10, 1/3, 1/2, and 1. The number of samples used is $M = 3.10^6$, the number of terms in (2) is $S = 9$, and the width of the bins is $\Delta T = 10^{-2}$. As a consequence the coefficient $M\Delta T/S$ in (5) is $(1/3)10^4$.

These functions suggest the following comments. We note first that if $b \to 0$, the PP tends toward a periodic PP, which means a periodic distribution of points with a period equal to 1. This is easily verified experimentally but has no practical interest. The value $b = 0.1$ is still rather small and a memory of the periodic structure remains. The different PDFs $f_n(t)$ appearing in the figure do not overlap, at least for $t < 8$ used in Fig. 6.1 . We observe clearly the rectangular and triangular functions $f_1(t)$ and $f_2(t)$. For large values of t not represented in the figure the Gaussian shapes overlap in such a way that their sum becomes constant and equal to the asymptotic value 1 of $b(t)$. This, however, requires very great values of S and M which cannot be obtained in our experiments.

Fig. 6.2 corresponds to $b = 1/3$. This value has been chosen because it is the greatest value of b for which $f_1(t)$ and $f_2(t)$ do not overlap. The corresponding limit point is $t = 1+1/3$, which clearly appears on the figure. For this value of b the rectangular function $f_1(t)$ is equal to $3/2$. Applying (5) with the value calculated above yields $(1/2)10^4 = 5000$, which clearly appears in Fig. 6.2. The same can be said for $f_2(2)$ which has the same value. Finally for $t \to \infty$ the bunching function tends to 1, which yields a number of samples in each bin of $(1/3)10^4$, which appears in Figs. 7.2, 7.3, and 7.4.

Theoretical calculation of the bunching function corresponding to the values used in Figs. 6.3 and 6.4 is almost impossible. This justifies the interest of experimental results displayed on these figures.

C. Point Process with Uniform Distribution and Correlated Life Times

In this section we assume that the PDF of the life time is still rectangular, as in the previous one, but that the PP is no longer a renewal PP. This means that the distances between successive points are identically distributed (rectangular distribution) but no longer independent.

An algorithm for generating this kind of life time has been presented in [11], and its principle is outlined in the Appendix. The normalized correlation function of the first order life time is exponential, or $p^{|n|}$. If $p = 0$, we find again the case of the previous section.

Experimental results are presented in Fig. 7. The parameters of this figure are the same as in Fig. 6.2 and our objective is only to evaluate the effect of the correlation characterized by the parameter p. Fig. 7.1 corresponds to $p = 0$, the same situation as in Fig. 6.2, and we find again the same form of coincidence function. When p increases this coincidence function is strongly modified and theoretical calculations are impossible because of the complexity of the model. But the effect of a correlation is perfectly clear in Figs. 7.2, 7.3, and 7.4. In all these figures $f_1(t)$ is the same, and this appears for values of t smaller than 1.333. We have then a good example of point processes with the same distribution of life time and entirely different second order properties characterized by the coincidence functions represented in Fig. 7.

V. CONCLUSION

The second order properties of a point process are entirely contained in its coincidence function. In this paper we have shown that this function can be measured by an appropriate processing of the distance between successive points obtained for example from a time-to-amplitude converter. The theoretical foundation of this processing is an expression that yields a relation between the coincidence function and the probability density function of the life time of the PP. For its practical realization we replace a series by a sum of a finite number of terms constructed from appropriate histograms of samples of the signal. The method has been tested in the case of PPs where the coincidence function has an explicit theoretical expression as in the case of Poisson processes with or without dead time, and also of some Erlang PPs.

The experimental results are in excellent agreement with the theoretical calculations. Finally, when this method is used for the measurement of coincidence functions that cannot be obtained theoretically, the experimental results exhibit various interesting features.

VI. Appendix

The expectation (1) defining the coincidence function is equal to the probability $P\{[dN(\theta) = 1] \cap [dN(\theta') = 1]\}$. This probability can be written

$$P\{[dN(\theta) = 1] \cap [dN(\theta') = 1]\} = P[dN(\theta) = 1].P\{[dN(\theta') = 1] | [dN(\theta) = 1]\}. \tag{15}$$

We deduce from the definition of the density λ that $P[dN(\theta) = 1] = \lambda d\theta$. The PDF $f_n(\theta - \theta')$ is defined by

$$f_n(\theta - \theta')d\theta' = P\{[dN(\theta') = 1] | A_n(\theta' - \theta) \cap [dN(\theta) = 1]\}, \tag{16}$$

where $A_n(\theta' - \theta)$ is the event that there are $n - 1$ points of the PP in $[\theta + d\theta, \theta'[$. It is clear that

$$P\{[dN(\theta') = 1] | [dN(\theta) = 1]\} = \sum_{n=1}^{\infty} P\{[dN(\theta') = 1] | A_n(\theta' - \theta) \cap [dN(\theta) = 1]\} \tag{17}$$

This yields (2).

The statistical signal used in Section IV C is constructed as follows. Let u_k and v_k be two independent IID random variables. Suppose furthermore that u_k takes only the value 0 or 1 and let p be the probability that $u_k = 1$. Let $f(.)$ be the PDF of the v_k. Consider now the signal $x_k = u_k x_{k-1} + (1 - u_k)v_k$. It is clear that if the PDF of x_1 is $f(.)$, all the other x_k have the same PDF. It can be shown that this is asymptotically verified whatever the PDF of x_1. The x_ks have then the same PDF but are correlated. Their correlation function γ_k satisfies the recursion $\gamma_k = p\gamma_{k-1}$. This yields $\gamma_k = \sigma^2 p^k$, where σ^2 is the variance of the RVs v_k. For $p = 0$, $x_k = v_k$, while for $p = 1$ all the RVs x_is are equal.

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