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Differential quasivariational inequalities in contact mechanics

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Abstract

We consider a new class of differential quasivariational inequalities, i.e. a nonlinear system that couples a differential equation with a time-dependent quasivariational inequality, both defined on abstract Banach spaces. We state and prove a general fixed principle that provides the existence and the uniqueness of the solution of the system. Then we consider a relevant particular setting for which our abstract result holds. We proceed with two examples that arise in Contact Mechanics. For each example, we describe the physical setting, the mathematical model and the assumption on the data. Then we state the variational formulation of each model, which is in the form of a differential quasivariational inequality. Finally, we apply our abstract results to provide the unique weak solvability of the corresponding contact problems.

Keywords

Differential quasivariational inequality, fixed point, contact problem, unilateral constraint, normal compliance, wear, weak solution

1. Introduction

Variational and quasivariational inequalities represent a powerful mathematical tool used in the study of various nonlinear boundary value problems with or without unilateral constraints. Their use is relevant in the study of mathematical models arising in Contact Mechanics as shown in [1–9]. The theory of variational inequalities began in the early 1960s, based on arguments of convexity and monotonicity. Basic references in the field are [7, 10–12], among others. Currently, the theory is still developing, including new results needed in the study of specific nonlinear free boundary problems. For instance, the study of a new class of quasivariational inequalities, called history-dependent variational inequalities, has been performed in [13, 14], with emphasis to the study of quasistatic contact problems. The notion of differential variational inequalities was introduced in [15]. It defines a system that couples a differential equation with a time-dependent variational inequality. Recent results in the study of differential variational inequalities have been obtained in [16, 17]. There, existence results have been provided by using arguments of semigroups of operators and fixed point.

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There are two aims in this manuscript. The first one is to provide a general existence and uniqueness result for differential quasivariational inequalities. The result is obtained under assumptions that could be easily verified in the study of quasistatic models of contact. The second aim is to illustrate how the mathematical tools we construct here can be directly applied in the study of some relevant examples of contact. This gives rise to a new approach on the analysis of the corresponding models. In this way, the results we present here represent a contribution to the development of the Mathematical Theory of Contact Mechanics.

The rest of the manuscript is structured as follows. In Section 2 we present a general fixed point principle in the study of differential quasivariational inequalities that gives rise to generic existence and uniqueness results, Theorems 2.1 and 2.2. In Section 3 we particularize Theorem 2.2 in the study of a specific class of differential quasivariational inequalities for which we obtain a new result of unique solvability, Theorem 3.1. Then, in Sections 4 and 5 we present two mathematical models of quasistatic contact for which our abstract result works. For each model, we list the assumption on the data and provide its variational formulation, which is in a form of a differential quasivariational inequality. The first model is viscoplastic and frictionless; there, the unknowns of the quasivariational inequality are the irreversible stress field and the displacement field. The second model is elastic and describes the sliding frictional contact with wear. There, the unknowns of the corresponding quasivariational inequality are the wear function and, again, the displacement field. We use our abstract tools with a specific choice of functions and operators and prove the unique weak solvability of each model.

2. A fixed point principle

Throughout this section X and V will be a normed spaces endowed with the norms $\|\cdot\|_X$ and $\|\cdot\|_V$, respectively. We denote by V^* the strong topological dual of V and by $\langle \cdot, \cdot \rangle$ the duality pairing mapping between V^* and V . Below, I denotes either a bounded interval of the form $[0, T]$ with $T > 0$, or the unbounded interval $\mathbb{R}_+ = [0, +\infty)$. We denote by $C(I; X)$ and $C(I; V)$ the space of continuous functions on I with values in X and V , respectively. Moreover, for simplicity, we use the notation $\mathcal{V} = C(I; V)$. In addition, we denote by $\mathcal{X} = C^1(I; X)$ the space of continuously differentiable functions on I with values in X . Therefore, $x \in \mathcal{X}$ if and only if $x \in C(I; X)$ and $\dot{x} \in C(I; X)$ where, here and below, \dot{x} represents the time derivative of the function x .

In the case $I = [0, T]$ the space $\mathcal{V} = C(I; V)$ will be equipped with the norm

$$\|v\|_{C([0,T];V)} = \max_{t \in [0,T]} \|v(t)\|_V.$$

It is well known that if V is a Banach space, then $C(I; V)$ is also a Banach space. Assume now that $I = \mathbb{R}_+$. It is well known that, if V is a Banach space, then $C(I; V)$ can be organized in a canonical way as a Fréchet space, i.e. a complete metric space in which the corresponding topology is induced by a countable family of seminorms. The convergence of a sequence $\{v_k\}_k$ to the element v , in the space $C(\mathbb{R}_+; V)$, can be described as follows

$$\left\{ \begin{array}{l} v_k \rightarrow v \quad \text{in } C(\mathbb{R}_+; V) \text{ as } k \rightarrow \infty \text{ if and only if} \\ \max_{r \in [0,n]} \|v_k(r) - v(r)\|_V \rightarrow 0 \text{ as } k \rightarrow \infty, \text{ for all } n \in \mathbb{N}. \end{array} \right.$$

In other words, the sequence $\{v_k\}_k$ converges to the element v in the space $C(\mathbb{R}_+; V)$ if and only if it converges to v in the space $C([0, n]; V)$ for all $n \in \mathbb{N}$. Here and below \mathbb{N} represents the set of positive integers.

The space $C^1([0, T]; X)$ will be equipped with the norm

$$\|x\|_{C^1([0,T];X)} = \max_{t \in [0,T]} \|x(t)\|_X + \max_{t \in [0,T]} \|\dot{x}(t)\|_X.$$

It is well known that if X is a Banach space then $C^1([0, T]; X)$ is a Banach space, too. Next, the convergence of a sequence $\{x_k\}_k$ to the element x , in the space $C^1(\mathbb{R}_+; X)$ can be defined as follows

$$\left\{ \begin{array}{l} x_k \rightarrow x \quad \text{in } C^1(\mathbb{R}_+; X) \text{ as } k \rightarrow \infty \text{ if and only if} \\ \max_{r \in [0,n]} \|x_k(r) - x(r)\|_X \rightarrow 0 \text{ as } k \rightarrow \infty \text{ and} \\ \max_{r \in [0,n]} \|\dot{x}_k(r) - \dot{x}(r)\|_X \rightarrow 0 \text{ as } k \rightarrow \infty, \text{ for all } n \in \mathbb{N}. \end{array} \right.$$

Let $F : I \times X \times V \rightarrow X$, $x_0 \in X$, $K \subset V$, $G : I \times X \times K \rightarrow V^*$ and $j : X \times K \times K \rightarrow \mathbb{R}$ be given. With these data, we consider the following problem.

Problem P. Find a pair of functions $x \in \mathcal{X}$ and $u \in \mathcal{V}$ such that

$$\dot{x}(t) = F(t, x(t), u(t)) \quad \forall t \in I, \quad (1)$$

$$x(0) = x_0, \quad (2)$$

$$u(t) \in K, \quad \langle G(t, x(t), u(t)), v - u(t) \rangle + j(x(t), u(t), v) - j(x(t), u(t), u(t)) \geq 0 \quad \forall v \in K, \quad t \in I. \quad (3)$$

Note that Problem **P** represents a system which couples the differential equation (1) with the quasivariational inequality (3), associated to the initial condition (2). For this reason, following the terminology in [15, 16], we refer to Problem **P** as a differential quasivariational inequality.

Our aim in what follows is to provide conditions for the solvability and the unique solvability of Problem **P**. To this end, we denote by $2^{\mathcal{X}}$ and $2^{\mathcal{V}}$ the set of parts of \mathcal{X} and \mathcal{V} , respectively, that is

$$2^{\mathcal{X}} = \{ \mathcal{A} \mid \mathcal{A} \subset \mathcal{X} \}, \quad 2^{\mathcal{V}} = \{ \mathcal{B} \mid \mathcal{B} \subset \mathcal{V} \}.$$

We say that a set $\mathcal{A} \subset \mathcal{X}$ is a singleton if it reduces to a single element. In what follows we shall consider both univalued maps and multivalued maps and, for a multivalued map $R : \mathcal{X} \rightarrow 2^{\mathcal{V}}$, we denote by $D(R)$ its domain defined by

$$D(R) = \{ x \in \mathcal{X} \mid Rx \neq \emptyset \}.$$

We recall that an element $x \in \mathcal{X}$ is called a fixed point of the univalued map $R : \mathcal{X} \rightarrow \mathcal{X}$ if $Rx = x$ and, moreover, $x \in \mathcal{X}$ is a fixed point of the multivalued map $R : \mathcal{X} \rightarrow 2^{\mathcal{X}}$ if $x \in Rx$. We adopt the same terminology and notation for a set $\mathcal{B} \subset \mathcal{V}$, and for univalued or multivalued maps S defined on \mathcal{V} .

We denote in what follows by $\mathcal{X} \times \mathcal{V}$ the Cartesian product of the spaces \mathcal{X} and \mathcal{V} . A typical element of $\mathcal{X} \times \mathcal{V}$ will be denoted by (x, u) or (x, η) . Consider also two subsets $\mathcal{P} \subset \mathcal{X} \times \mathcal{V}$ and $\mathcal{Q} \subset \mathcal{X} \times \mathcal{V}$ defined by the equivalences below

$$(x, u) \in \mathcal{P} \iff (x, u) \text{ satisfies (1) and (2)}, \quad (4)$$

$$(x, u) \in \mathcal{Q} \iff (x, u) \text{ satisfies (3)}. \quad (5)$$

It is clear that Problem **P** has at least one solution if and only if $\mathcal{P} \cap \mathcal{Q} \neq \emptyset$ and, moreover, it has a unique solution if and only if $\mathcal{P} \cap \mathcal{Q} \subset \mathcal{X} \times \mathcal{V}$ reduces to a single element, i.e. is a singleton. For this reason, our aim in what follows is to provide sufficient conditions such that $\mathcal{P} \cap \mathcal{Q} \neq \emptyset$ and, alternatively, such that $\mathcal{P} \cap \mathcal{Q}$ is a singleton. To this end, we assume in what follows that the following conditions hold

$$\text{For each } \eta \in \mathcal{V} \text{ there exists } x \in \mathcal{X} \text{ such that } (x, \eta) \in \mathcal{P}. \quad (6)$$

$$\text{For each } x \in \mathcal{X} \text{ there exists } \eta \in \mathcal{V} \text{ such that } (x, \eta) \in \mathcal{Q}. \quad (7)$$

We now define the map $R : \mathcal{V} \rightarrow 2^{\mathcal{X}}$ by

$$R\eta = \{ x \in \mathcal{X} \mid (x, \eta) \in \mathcal{P} \} \quad \text{for all } \eta \in \mathcal{V}. \quad (8)$$

In other words, for any $\eta \in \mathcal{V}$ we have the equivalence

$$x \in R\eta \iff (x, \eta) \in \mathcal{P}. \quad (9)$$

In a similar way, we define the map $S : \mathcal{X} \rightarrow 2^{\mathcal{V}}$ by

$$Sx = \{ \eta \in \mathcal{V} \mid (x, \eta) \in \mathcal{Q} \} \quad \text{for all } x \in \mathcal{X}. \quad (10)$$

We note that for each $x \in \mathcal{X}$, we have the equivalence

$$\eta \in Sx \iff (x, \eta) \in \mathcal{Q}. \quad (11)$$

Next, we consider the multivalued map $\Lambda: \mathcal{V} \rightarrow 2^{\mathcal{V}}$ defined by

$$\Lambda\eta = \bigcup_{x \in R\eta} Sx \quad \text{for all } \eta \in \mathcal{V}. \quad (12)$$

Note that this definition has sense since assumptions (6) and (7) guarantee that $D(R) = \mathcal{V}$ and $D(S) = \mathcal{X}$, respectively. Moreover, it follows from (12) that $D(\Lambda) = \mathcal{V}$. In addition, note that if R and S are univalued operators, then

$$\Lambda = SR, \quad (13)$$

where the product represents the composition of the corresponding maps. For this reason, for simplicity, we use the shorthand notation (13) even in the multivalued case.

Our first result in this section is the following.

Theorem 2.1 *Assume that (6) and (7) hold. Then, the following statements are equivalent.*

- (1) *Problem **P** has at least one solution.*
- (2) *The map Λ has a fixed point.*

Proof. Assume that Problem **P** has at least one solution or, equivalently, assume that $\mathcal{P} \cap \mathcal{Q} \neq \emptyset$. Let $(x^*, \eta^*) \in \mathcal{P} \cap \mathcal{Q}$, i.e. $(x^*, \eta^*) \in \mathcal{P}$ and $(x^*, \eta^*) \in \mathcal{Q}$. We use the definitions (9) and (11) of the maps R and S to see that $x^* \in R\eta^*$ and $\eta^* \in Sx^*$. This implies that $\eta^* \in SR\eta^*$ and, therefore, the definition (12) of Λ shows that $\eta^* \in \Lambda\eta^*$. So, we proved that

$$(x^*, \eta^*) \in \mathcal{P} \cap \mathcal{Q} \implies \eta^* \in \Lambda\eta^* \quad \text{and} \quad x^* \in R\eta^*. \quad (14)$$

Conversely, assume that $\eta^* \in \Lambda\eta^*$. Then, the definition (12) implies that there exists $x^* \in \mathcal{X}$ such that $\eta^* \in Sx^*$ and $x^* \in R\eta^*$. We now use the definitions (9) and (11) of the maps R and S , respectively, to see that $(x^*, \eta^*) \in \mathcal{P}$ and $(x^*, \eta^*) \in \mathcal{Q}$. Thus, we proved that

$$\eta^* \in \Lambda\eta^* \implies \exists x^* \in \mathcal{X} \quad \text{such that} \quad (x^*, \eta^*) \in \mathcal{P} \cap \mathcal{Q}. \quad (15)$$

Relations (14) and (15) prove that $\mathcal{P} \cap \mathcal{Q} \neq \emptyset$ if and only if Λ has a fixed point. Therefore, the statements (1) and (2) are equivalent, which concludes the proof. \square

We now reinforce conditions (6) and (7) by considering the following assumptions

$$\text{For each } \eta \in \mathcal{V} \text{ there exists a unique } x \in \mathcal{X} \text{ such that } (x, \eta) \in \mathcal{P}. \quad (16)$$

$$\text{For each } x \in \mathcal{X} \text{ there exists a unique } \eta \in \mathcal{V} \text{ such that } (x, \eta) \in \mathcal{Q}. \quad (17)$$

Note that if (16) holds then the map R is univalued and, if (17) holds, so is B . Moreover, under the above mentioned hypotheses, we have the equivalences

$$x = R\eta \iff (x, \eta) \in \mathcal{P}, \quad (18)$$

$$\eta = Sx \iff (x, \eta) \in \mathcal{Q}, \quad (19)$$

respectively. We proceed with the following result.

Theorem 2.2 *Assume that (16) and (7) hold. Then Problem **P** has a unique solution if and only if the map Λ has a unique fixed point.*

Proof. We combine (14) and (18) to see that

$$(x^*, \eta^*) \in \mathcal{P} \cap \mathcal{Q} \implies \eta^* \in \Lambda\eta^* \quad \text{and} \quad x^* = R\eta^*. \quad (20)$$

Conversely, assume that $\eta^* \in \Lambda\eta^*$ and $x^* = R\eta^*$. Then (18) implies that $(x^*, \eta^*) \in \mathcal{P}$. On the other hand, (15) implies that there exists $\tilde{x} \in \mathcal{X}$ such that $(\tilde{x}, \eta^*) \in \mathcal{P} \cap \mathcal{Q}$ and, therefore, $(\tilde{x}, \eta^*) \in \mathcal{P}$. We now use assumption (16) to see that $x^* = \tilde{x}$. So, we proved that

$$\eta^* \in \Lambda\eta^* \quad \text{and} \quad x^* = R\eta^* \implies (x^*, \eta^*) \in \mathcal{P} \cap \mathcal{Q}. \quad (21)$$

We now combine implications (20) and (21) to deduce the equivalence

$$(x^*, \eta^*) \in \mathcal{P} \cap \mathcal{Q} \iff \eta^* \in \Lambda \eta^* \quad \text{and} \quad x^* = R\eta^*. \quad (22)$$

This proves that the set $\mathcal{P} \cap \mathcal{Q}$ is a singleton if and only if the operator Λ has a unique fixed point, which concludes the proof. \square

Remark 2.3 *Note that under the assumption (16) and (7) the map R is univalued but the map S is, in general, multivalued. As a consequence, the map Λ is multivalued. Nevertheless, if (16) and (17) hold then, both R and S are univalued and equality (13) holds. In this case Theorem 2.2 states that the unique solvability of Problem **P** is equivalent with the existence of a unique fixed point of the operator $\Lambda : \mathcal{V} \rightarrow \mathcal{V}$.*

We end this section with the comment that the results above are valid in more general cases, as explained in [18]. Nevertheless, we decided to present them in the current form, which is relevant with the study of Problem **P**, as it results from the next section.

3. An existence and uniqueness result

In this section, we use Theorem 2.2 in the study of Problem **P**. To this end we consider the following assumptions

$$\left\{ \begin{array}{l} F: I \times X \times V \rightarrow X \text{ is such that} \\ \text{(a) The mapping } t \mapsto F(t, x, u): I \rightarrow X \text{ is continuous} \\ \quad \text{for all } x \in X, u \in V. \\ \text{(b) For any compact set } J \subset I \text{ there exists } L_J > 0 \\ \quad \text{such that} \\ \quad \quad \|F(t, x_1, u_1) - F(t, x_2, u_2)\|_X \\ \quad \quad \leq L_J(\|x_1 - x_2\|_X + \|u_1 - u_2\|_V) \\ \quad \quad \text{for all } x_1, x_2 \in X, u_1, u_2 \in V, t \in J. \end{array} \right. \quad (23)$$

$$x_0 \in X. \quad (24)$$

$$K \text{ is a closed convex nonempty subset of } V. \quad (25)$$

Next, we assume that the function G has a special structure, i.e.

$$\left\{ \begin{array}{l} G: I \times X \times K \rightarrow V^*, \\ G(t, x, u) = A(x, u) - f(t) \\ \text{for all } t \in I, x \in X, u \in K, \end{array} \right. \quad (26)$$

where the operator A and the function f satisfy the following conditions

$$\left\{ \begin{array}{l} A: X \times K \rightarrow V^* \text{ is such that} \\ \text{(a) There exists } L' > 0 \text{ such that} \\ \quad \|A(x_1, u) - A(x_2, u)\|_{V^*} \leq L' \|x_1 - x_2\|_X \\ \quad \text{for all } x_1, x_2 \in X, u \in K. \\ \text{(b) There exists } L'' > 0 \text{ such that} \\ \quad \|A(x, u_1) - A(x, u_2)\|_{V^*} \leq L'' \|u_1 - u_2\|_V \\ \quad \text{for all } x \in X, u_1, u_2 \in K. \\ \text{(c) There exists } m > 0 \text{ such that} \\ \quad \langle A(x, u_1) - A(x, u_2), u_1 - u_2 \rangle \geq m \|u_1 - u_2\|_V^2 \\ \quad \text{for all } x \in X, u_1, u_2 \in K. \end{array} \right. \quad (27)$$

$$f \in C(I; V^*). \quad (28)$$

Moreover, we assume that

$$\left\{ \begin{array}{l} j: X \times K \times K \rightarrow \mathbb{R} \text{ is such that} \\ \text{(a) For all } x \in X \text{ and } u \in K, j(x, u, \cdot) \text{ is convex} \\ \quad \text{and lower semicontinuous (l.s.c.) on } K. \\ \text{(b) There exist } \alpha > 0 \text{ and } \beta > 0 \text{ such that} \\ \quad j(x_1, u_1, v_2) - j(x_1, u_1, v_1) + j(x_2, u_2, v_1) - j(x_2, u_2, v_2) \\ \quad \leq \alpha \|x_1 - x_2\|_X \|v_1 - v_2\|_V + \beta \|u_1 - u_2\|_V \|v_1 - v_2\|_V \\ \quad \text{for all } x_1, x_2 \in X, u_1, u_2 \in K, v_1, v_2 \in K. \end{array} \right. \quad (29)$$

$$m > \beta. \quad (30)$$

Note that condition (27) shows that A is a Lipschitz continuous operator with respect to the first argument, uniformly with respect to the second one. It also shows that A is a Lipschitz continuous and strongly monotone operator with respect to the second argument, uniformly with respect to the first argument. Moreover, recall that m and β are the constants in (27) and (29), respectively.

Our main result in this section is the following.

Theorem 3.1 *Assume that X is a Banach space, V is a reflexive Banach space and, moreover, (23) to (30) hold. Then Problem **P** has a unique solution $(x, u) \in \mathcal{X} \times \mathcal{V}$.*

The proof of Theorem 3.1 is carried out in several steps. To provide it we need several preliminary results which will be useful to guarantee the validity of assumptions (16) and (17).

Lemma 3.2 *Let X be a Banach space and assume that*

$$\left\{ \begin{array}{l} \tilde{F}: I \times X \rightarrow X \text{ is such that} \\ \text{(a) The mapping } t \mapsto \tilde{F}(t, x): I \rightarrow X \text{ is continuous, for all } x \in X. \\ \text{(b) For any compact set } J \subset I \text{ there exists } L_J > 0 \text{ such that} \\ \quad \|\tilde{F}(t, x_1) - \tilde{F}(t, x_2)\|_X \leq L_J \|x_1 - x_2\|_X \\ \quad \text{for all } x_1, x_2 \in X, t \in J. \end{array} \right. \quad (31)$$

Then, for each $x_0 \in X$, there exists a unique function $x \in \mathcal{X}$ such that

$$\dot{x}(t) = \tilde{F}(t, x(t)) \quad \forall t \in I, \quad (32)$$

$$x(0) = x_0. \quad (33)$$

Note that Lemma 3.2 represents a version on the well-known Cauchy–Lipschitz Theorem. Its proof could be found in [18], for instance.

Lemma 3.3 *Let V be a reflexive Banach space and, besides (25), assume that*

$$\left\{ \begin{array}{l} \tilde{A}: K \rightarrow V^* \text{ is a strongly monotone Lipschitz continuous operator, i.e.} \\ \text{(a) There exists } m > 0 \text{ such that} \\ \quad \langle \tilde{A}u_1 - \tilde{A}u_2, u_1 - u_2 \rangle \geq m \|u_1 - u_2\|_V^2 \quad \text{for all } u_1, u_2 \in K. \\ \text{(b) There exists } L > 0 \text{ such that} \\ \quad \|\tilde{A}u_1 - \tilde{A}u_2\|_{V^*} \leq L \|u_1 - u_2\|_V \quad \text{for all } u_1, u_2 \in K. \end{array} \right. \quad (34)$$

$$\left\{ \begin{array}{l} \varphi: K \times K \rightarrow \mathbb{R} \text{ is such that} \\ \text{(a) For all } u \in K, \varphi(u, \cdot) \text{ is convex and lower semicontinuous} \\ \text{(l.s.c.) on } K. \\ \text{(b) There exists } \beta > 0 \text{ such that} \\ \varphi(u_1, v_2) - \varphi(u_1, v_1) + \varphi(u_2, v_1) - \varphi(u_2, v_2) \\ \leq \beta \|u_1 - u_2\|_V \|v_1 - v_2\|_V \text{ for all } u_1, u_2 \in K, v_1, v_2 \in K. \end{array} \right. \quad (35)$$

$$m > \beta. \quad (36)$$

Then, for each $f \in V^*$ there exists a unique element u such that

$$u \in K, \quad \langle \tilde{A}u, v - u \rangle + \varphi(u, v) - \varphi(u, u) \geq \langle f, v - u \rangle \quad \forall v \in K. \quad (37)$$

Lemma 3.3 represents a standard result on quasivariational inequalities which could be found in many books and surveys, see for instance [18].

We now recall with the following fixed point result, obtained in [19].

Lemma 3.4 *Let V be a Banach space and $\mathcal{S}: C(I; V) \rightarrow C(I; V)$ be an operator with the following property: for any compact set $J \subset I$ there exists $s_J > 0$ such that*

$$\|\mathcal{S}u_1(t) - \mathcal{S}u_2(t)\|_V \leq s_J \int_0^t \|u_1(s) - u_2(s)\|_V ds \quad (38)$$

for all $u_1, u_2 \in C(I; V)$, $t \in J$.

Then, \mathcal{S} has a unique fixed point, i.e. there exists a unique element $\eta^* \in C(I; V)$ such that $\mathcal{S}\eta^* = \eta^*$.

Note that here and below, when no confusion arises, we use the shorthand notation $Su(t)$ to represent the value of the function Su at the point t , i.e. $Su(t) = (Su)(t)$. An operator which satisfies condition (38) is called a history-dependent operator. This term was introduced in [13] and since it was used in many papers, see [9, 14, 18] and the references therein. There, relevant examples of such operators have been provided.

We now proceed with the following results.

Lemma 3.5 *Let X be a Banach space and assume that (23), (24) hold. Then, for each $\eta \in \mathcal{V}$, there exists a unique function $x \in \mathcal{X}$ such that*

$$\dot{x}(t) = F(t, x(t), \eta(t)) \quad \forall t \in I, \quad (39)$$

$$x(0) = x_0. \quad (40)$$

Moreover, if $x_1, x_2 \in \mathcal{X}$ represent the solution of the Cauchy problem (39), (40) for the functions $\eta_1, \eta_2 \in \mathcal{V}$, respectively, then for each compact set $J \subset I$, the following inequality holds

$$\|x_1(t) - x_2(t)\|_X \leq L_J e^{L_J t} \int_0^t \|\eta_1(s) - \eta_2(s)\|_V ds \quad \text{for all } t \in J. \quad (41)$$

Proof. Let $\eta \in \mathcal{V}$ be given and consider the function $\tilde{F}: I \times X \rightarrow X$ defined by

$$\tilde{F}(t, x) = F(t, x, \eta(t)) \quad \forall t \in I, x \in X.$$

We use assumption (23) to see that \tilde{F} satisfies condition (31). Therefore, the existence and uniqueness part of Lemma 3.5 is a direct consequence of Lemma 3.2. Assume now that $\eta_1, \eta_2 \in \mathcal{V}$, consider a compact set $J \subset I$ and let $t \in J$. We integrate equation (39) with the initial condition (40) to obtain

$$x_i(t) = \int_0^t F(s, x_i(s), \eta_i(s)) ds + x_0 \quad \forall i = 1, 2.$$

This implies that

$$x_1(t) - x_2(t) = \int_0^t (F(s, x_1(s), \eta_1(s)) - F(s, x_2(s), \eta_2(s))) ds$$

and, therefore

$$\|x_1(t) - x_2(t)\|_X \leq \int_0^t \|F(s, x_1(s), \eta_1(s)) - F(s, x_2(s), \eta_2(s))\|_X ds. \quad (42)$$

Inequality (41) follows from (42), (23)(b) and a standard Gronwall argument. \square

Lemma 3.6 *Let V be a reflexive Banach space and assume that (25), (27) to (30) hold. Then, for each $x \in \mathcal{X}$, there exists a unique function $u \in \mathcal{V}$ such that*

$$\begin{aligned} u(t) \in K, \quad \langle A(x(t), u(t)), v - u(t) \rangle + j(x(t), u(t), v) - j(x(t), u(t), u(t)) \\ \geq \langle f(t), v - u(t) \rangle \quad \forall v \in K, t \in I. \end{aligned} \quad (43)$$

Moreover, if $u_1, u_2 \in \mathcal{V}$ represent the solutions of the inequality (43) for the functions $x_1, x_2 \in \mathcal{X}$, respectively, then the following inequality holds

$$\|u_1(t) - u_2(t)\|_V \leq \frac{L' + \alpha}{m - \beta} \|x_1(t) - x_2(t)\|_X \quad \forall t \in I. \quad (44)$$

Proof. Let $x \in \mathcal{X}$ be given and let $t \in \mathbb{R}_+$ be fixed. Define the operator $\tilde{A} : K \rightarrow V^*$ and the function $\varphi : K \times K \rightarrow \mathbb{R}$ by equalities

$$\tilde{A}u = A(x(t), u), \quad \varphi(u, v) = j(x(t), u, v) \quad \forall u, v \in K.$$

Note that both the operator \tilde{A} and the function φ depend on x and t . Nevertheless, for simplicity, we do not mention explicitly this dependence. Then, using assumptions (25), (27), (28) and (29), it follows that the operator \tilde{A} satisfies condition (34), the function φ satisfies condition (35) and, moreover, the smallness condition (36) holds. Therefore, from Lemma 3.3 we deduce that there exists a unique element $u(t)$ which solves (43) at the given moment t . Now, let us show that the map $t \mapsto u(t) : I \rightarrow K$ is continuous. To this end, consider $t_1, t_2 \in I$ and, for the sake of simplicity in writing, denote $x(t_i) = x_i, u(t_i) = u_i, f(t_i) = f_i$ for $i = 1, 2$. Using (43) we obtain

$$\begin{aligned} u_1 \in K, \quad \langle A(x_1, u_1), v - u_1 \rangle + j(x_1, u_1, v) - j(x_1, u_1, u_1) \\ \geq \langle f_1, v - u_1 \rangle \quad \forall v \in K, \end{aligned} \quad (45)$$

$$\begin{aligned} u_2 \in K, \quad \langle A(x_2, u_2), v - u_2 \rangle + j(x_2, u_2, v) - j(x_2, u_2, u_2) \\ \geq \langle f_2, v - u_2 \rangle \quad \forall v \in K. \end{aligned} \quad (46)$$

Taking $v = u_2$ in (45) and $v = u_1$ in (46), and adding the resulting inequalities yield that

$$\begin{aligned} \langle A(x_1, u_1) - A(x_2, u_2), u_1 - u_2 \rangle \\ \leq j(x_1, u_1, u_2) - j(x_1, u_1, u_1) + j(x_2, u_1, u_1) - j(x_2, u_2, u_2) \\ + \langle f_1 - f_2, u_1 - u_2 \rangle. \end{aligned} \quad (47)$$

Then, writing

$$\begin{aligned} \langle A(x_1, u_1) - A(x_2, u_2), u_1 - u_2 \rangle \\ = \langle A(x_1, u_1) - A(x_1, u_2), u_1 - u_2 \rangle + \langle A(x_1, u_2) - A(x_2, u_2), u_1 - u_2 \rangle \end{aligned}$$

and using assumptions (27)(b) and (27)(c), one obtains that

$$\begin{aligned} \langle A(x_1, u_1) - A(x_2, u_2), u_1 - u_2 \rangle \\ \geq m \|u_1 - u_2\|_V^2 - L' \|x_1 - x_2\|_X \|u_1 - u_2\|_V. \end{aligned} \quad (48)$$

Next, inequalities (47), (48) and assumption (29)(b) yield

$$\begin{aligned} (m - \beta) \|u_1 - u_2\|_V & \\ \leq L' \|x_1 - x_2\|_X + \alpha \|x_1 - x_2\|_X + \|f_1 - f_2\|_{V^*}. & \end{aligned} \quad (49)$$

Inequality (49) combined with the assumption (28) implies that $t \mapsto u(t) : I \rightarrow K$ is a continuous function. This concludes the existence part of the lemma. The uniqueness part is a direct consequence of the uniqueness of the solution $u(t)$ of the quasivariational inequality (43), at each $t \in I$, guaranteed by Lemma 3.3.

Assume now that $x_1, x_2 \in \mathcal{X}$ and let $t \in I$ be given. Then using arguments similar to those used in the proof of inequality (49) we deduce that

$$(m - \beta) \|u_1(t) - u_2(t)\|_V \leq L' \|x_1(t) - x_2(t)\|_X + \alpha \|x_1(t) - x_2(t)\|_X$$

which implies (44) and concludes the proof. \square

We are now in a position to provide the proof of Theorem 3.1.

Proof. We use Theorem 2.2. To this end we use (26) to see that inequality (43) can be written in the equivalent form (3)

Therefore, using (4) and (5), we define the sets $\mathcal{P} \subset \mathcal{X} \times \mathcal{V}$ and $\mathcal{Q} \subset \mathcal{X} \times \mathcal{V}$ by equivalences

$$(x, \eta) \in \mathcal{P} \iff (39) \text{ and } (40) \text{ hold.} \quad (50)$$

$$(x, u) \in \mathcal{Q} \iff (3) \text{ holds.} \quad (51)$$

We now use Lemma 3.5 to see that condition (16) hold. Next, we use Lemma 3.6 and assumption (26) to see that condition (17) holds, too. Moreover, using Remark 2.3 it follows that the operator Λ is univalued and is defined as follows

$$\Lambda : \mathcal{V} \rightarrow \mathcal{V}, \quad \Lambda \eta = u_{x_\eta} \quad \forall \eta \in \mathcal{V}. \quad (52)$$

Here, for each $\eta \in \mathcal{V}$, $x_\eta \in \mathcal{Y}$ represents the solution of the Cauchy problem (39), (40), guaranteed by Lemma 3.5 and u_{x_η} represents the solution of the quasivariational inequality (43) for $x = x_\eta$, guaranteed by Lemma 3.6. Consider now two elements $\eta_1, \eta_2 \in \mathcal{V}$ and denote $x_{\eta_1} = x_1, x_{\eta_2} = x_2, u_{x_{\eta_1}} = u_1, u_{x_{\eta_2}} = u_2$. Then, using (52) we have

$$\Lambda \eta_1 = u_1, \quad \Lambda \eta_2 = u_2. \quad (53)$$

Let $J \subset I$ be a compact interval of time. Then, inequalities (41) and (44) show that

$$\|x_1(t) - x_2(t)\|_X \leq L_J e^{L_J t} \int_0^t \|\eta_1(s) - \eta_2(s)\|_V ds \quad \forall t \in J, \quad (54)$$

$$\|u_1(t) - u_2(t)\|_V \leq \frac{L' + \alpha}{m - \beta} \|x_1(t) - x_2(t)\|_X \quad \forall t \in J. \quad (55)$$

We now combine relations (53) to (55) to see that

$$\|\Lambda \eta_1(t) - \Lambda \eta_2(t)\|_V \leq \frac{L' + \alpha}{m - \beta} L_J e^{L_J t} \int_0^t \|\eta_1(s) - \eta_2(s)\|_V ds \quad \text{for all } t \in J$$

and, using Lemma 3.4 we deduce that the operator Λ has a unique fixed point. Theorem 3.1 is now a direct consequence of Theorem 2.2. \square

4. A viscoplastic contact problem

We start this section with additional notation and preliminaries related to the contact problems we are interested in the rest of this paper. For more details on the material presented below, see [4, 9, 18, 20]. We denote by \mathbb{S}^d ($d = 2, 3$) the space of second order symmetric tensors on \mathbb{R}^d or, equivalently, the space of symmetric matrices of order d . Typical elements in \mathbb{R}^d and \mathbb{S}^d will be denoted by $\mathbf{v} = (v_i)$ and $\boldsymbol{\tau} = (\tau_{ij})$ where, here and

below, the indices i, j, k, l run from 1 to d . Moreover, the convention summation upon the repeated indices is used. The inner product and norm on \mathbb{R}^d and \mathbb{S}^d are defined by

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= u_i v_i, & \|\mathbf{v}\| &= (\mathbf{v} \cdot \mathbf{v})^{\frac{1}{2}} & \forall \mathbf{u}, \mathbf{v} \in \mathbb{R}^d, \\ \boldsymbol{\sigma} \cdot \boldsymbol{\tau} &= \sigma_{ij} \tau_{ij}, & \|\boldsymbol{\tau}\| &= (\boldsymbol{\tau} \cdot \boldsymbol{\tau})^{\frac{1}{2}} & \forall \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{S}^d. \end{aligned}$$

Also, we use the notation $\mathbf{0}$ for the zero element of the spaces \mathbb{R}^d and \mathbb{S}^d .

Let $\Omega \subset \mathbb{R}^d$ ($d = 1, 2, 3$) be a bounded domain with Lipschitz continuous boundary Γ and let Γ_1, Γ_2 , and Γ_3 be three measurable parts of Γ such that $meas(\Gamma_1) > 0$. We use the notation $\mathbf{x} = (x_i)$ for a typical point in $\Omega \cup \Gamma$ and we denote by $\mathbf{v} = (v_i)$ the outward unit normal at Γ . Also, we use standard notation for the Lebesgue and Sobolev spaces associated to Ω and Γ and, moreover, we consider the spaces

$$V = \{ \mathbf{v} = (v_i) \in H^1(\Omega)^d : \mathbf{v} = \mathbf{0} \text{ on } \Gamma_1 \},$$

$$Q = \{ \boldsymbol{\tau} = (\tau_{ij}) \in L^2(\Omega)^{d \times d} : \tau_{ij} = \tau_{ji} \}.$$

These are real Hilbert spaces endowed with the inner products

$$(\mathbf{u}, \mathbf{v})_V = \int_{\Omega} \boldsymbol{\varepsilon}(\mathbf{u}) \cdot \boldsymbol{\varepsilon}(\mathbf{v}) \, dx, \quad (\boldsymbol{\sigma}, \boldsymbol{\tau})_Q = \int_{\Omega} \boldsymbol{\sigma} \cdot \boldsymbol{\tau} \, dx$$

and the associated norms $\|\cdot\|_V$ and $\|\cdot\|_Q$, respectively. Here $\boldsymbol{\varepsilon}$ represents the deformation operator given by

$$\boldsymbol{\varepsilon}(\mathbf{v}) = (\varepsilon_{ij}(\mathbf{v})), \quad \varepsilon_{ij}(\mathbf{v}) = \frac{1}{2} (v_{i,j} + v_{j,i}) \quad \forall \mathbf{v} \in H^1(\Omega)^d$$

where the comma indicates a partial derivative with respect the corresponding component of the spatial variable. Completeness of the space $(V, \|\cdot\|_V)$ follows from the assumption $meas(\Gamma_1) > 0$, which allows the use of Korn's inequality. As usual, we denote by V^* the strong topological dual of V and by $\langle \cdot, \cdot \rangle$ the duality pairing mapping between V^* and V .

For an element $\mathbf{v} \in V$ we still write \mathbf{v} for the trace of \mathbf{v} on the boundary and we denote by v_ν and \mathbf{v}_τ the normal and tangential components of \mathbf{v} on Γ , given by $v_\nu = \mathbf{v} \cdot \boldsymbol{\nu}$, $\mathbf{v}_\tau = \mathbf{v} - v_\nu \boldsymbol{\nu}$. Let Γ_3 be a measurable part of Γ . Then, by the Sobolev trace theorem, there exists a positive constant c_0 which depends on Ω , Γ_1 and Γ_3 such that

$$\|\mathbf{v}\|_{L^2(\Gamma_3)^d} \leq c_0 \|\mathbf{v}\|_V \quad \forall \mathbf{v} \in V. \quad (56)$$

For a regular function $\boldsymbol{\sigma} \in Q$ we use the notation σ_ν and $\boldsymbol{\sigma}_\tau$ for the normal and the tangential traces, i.e. $\sigma_\nu = (\boldsymbol{\sigma} \boldsymbol{\nu}) \cdot \boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma} \boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}$. Moreover, we recall that the divergence operator is defined by the equality $\text{Div } \boldsymbol{\sigma} = (\sigma_{ij,j})$ and, finally, the following Green's formula holds

$$\int_{\Omega} \boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}(\mathbf{v}) \, dx + \int_{\Omega} \text{Div } \boldsymbol{\sigma} \cdot \mathbf{v} \, dx = \int_{\Gamma} \boldsymbol{\sigma} \boldsymbol{\nu} \cdot \mathbf{v} \, da \quad \forall \mathbf{v} \in V. \quad (57)$$

For the contact problem we study in this section the physical setting can be resumed as follows. A viscoplastic body occupies a bounded domain $\Omega \subset \mathbb{R}^d$ with a Lipschitz continuous boundary Γ , divided into three measurable parts Γ_1, Γ_2 and Γ_3 such that $meas(\Gamma_1) > 0$. The body is subject to the action of body forces of density \mathbf{f}_0 . It is fixed on Γ_1 and time-dependent surfaces tractions of density \mathbf{f}_2 act on Γ_2 . On Γ_3 , the body is in frictionless contact with an obstacle, the so-called foundation, which is made of a hard material covered by a layer of soft material of thickness $g > 0$. The time interval of interest is I and could be either a bounded interval of the form $[0, T]$ with $T > 0$, or the unbounded interval $\mathbb{R}_+ = [0, +\infty)$. Then, the classical formulation of the contact problem is the following.

Problem \mathcal{P}_{vp} . Find a stress field $\boldsymbol{\sigma} : \Omega \times I \rightarrow \mathbb{S}^d$ and a displacement field $\mathbf{u} : \Omega \times I \rightarrow \mathbb{R}^d$ such that

$$\dot{\boldsymbol{\sigma}}(t) = \mathcal{E} \boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)) + \mathcal{G}(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{u}(t))) \quad \text{in } \Omega, \quad (58)$$

$$\text{Div } \boldsymbol{\sigma}(t) + \mathbf{f}_0(t) = \mathbf{0} \quad \text{in } \Omega, \quad (59)$$

$$\mathbf{u}(t) = \mathbf{0} \quad \text{on } \Gamma_1, \quad (60)$$

$$\boldsymbol{\sigma}(t) \boldsymbol{\nu} = \mathbf{f}_2(t) \quad \text{on } \Gamma_2, \quad (61)$$

$$\left. \begin{array}{l} u_v(t) \leq g, \quad \sigma_v(t) + p(u_v(t)) \leq 0, \\ (u_v(t) - g)(\sigma_v(t) + p(u_v(t))) = 0 \end{array} \right\} \quad \text{on } \Gamma_3, \quad (62)$$

$$\boldsymbol{\sigma}_\tau(t) = \mathbf{0} \quad \text{on } \Gamma_3, \quad (63)$$

$$\boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0, \quad \mathbf{u}(0) = \mathbf{u}_0 \quad \text{in } \Omega. \quad (64)$$

Problem \mathcal{P}_{vp} was considered in [21]. There, besides the unique solvability of the problem, based on a mixed variational formulation with Lagrange multipliers, the continuous dependence of the weak solution with respect to both the normal compliance function p and the penetration bound g was proved. Numerical simulations, which provide numerical evidence of this continuous dependence result, were also performed. Here, in this section, we study the problem by using a different method, based on our abstract result given by Theorem 3.1.

We now provide a brief description of the equations and conditions in Problem \mathcal{P}_{vp} where, in order to simplify the notation, we do not indicate explicitly the dependence of various functions on the spatial variable \mathbf{x} .

First, equation (58) represents the rate-type viscoplastic constitutive law in which we assume that elasticity tensor \mathcal{E} and the constitutive function \mathcal{G} satisfy the following conditions

$$\left\{ \begin{array}{l} \text{(a) } \mathcal{E} = (\mathcal{E}_{ijkl}) : \Omega \times \mathbb{S}^d \rightarrow \mathbb{S}^d. \\ \text{(b) } \mathcal{E}_{ijkl} = \mathcal{E}_{klij} = \mathcal{E}_{jikl} \in L^\infty(\Omega). \\ \text{(c) There exists } m_{\mathcal{E}} > 0 \text{ such that} \\ \quad \mathcal{E} \boldsymbol{\tau} \cdot \boldsymbol{\tau} \geq m_{\mathcal{E}} \|\boldsymbol{\tau}\|^2 \text{ for all } \boldsymbol{\tau} \in \mathbb{S}^d, \text{ a.e. in } \Omega. \end{array} \right. \quad (65)$$

$$\left\{ \begin{array}{l} \text{(a) } \mathcal{G} : \Omega \times \mathbb{S}^d \times \mathbb{S}^d \times \rightarrow \mathbb{S}^d. \\ \text{(b) There exists } L_{\mathcal{G}} > 0 \text{ such that} \\ \quad \|\mathcal{G}(\mathbf{x}, \boldsymbol{\sigma}_1, \boldsymbol{\varepsilon}_1) - \mathcal{G}(\mathbf{x}, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_2)\| \\ \quad \leq L_{\mathcal{G}} (\|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\| + \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|) \\ \quad \text{for all } \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \mathbf{x} \in \Omega. \\ \text{(c) The mapping } \mathbf{x} \mapsto \mathcal{G}(\mathbf{x}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}) \text{ is measurable on } \Omega, \\ \quad \text{for any } \boldsymbol{\sigma}, \boldsymbol{\varepsilon} \in \mathbb{S}^d. \\ \text{(d) The mapping } \mathbf{x} \mapsto \mathcal{G}(\mathbf{x}, \mathbf{0}, \mathbf{0}) \text{ belongs to } \mathcal{Q}. \end{array} \right. \quad (66)$$

Equation (59) is the equilibrium equation that we use since we assume that the process is quasistatic. Conditions (60) and (61) are the displacement boundary condition and traction boundary condition, respectively. We assume that the densities of body forces and surface tractions are such that

$$\mathbf{f}_0 \in C(I; L^2(\Omega)^d), \quad \mathbf{f}_2 \in C(I; L^2(\Gamma_2)^d). \quad (67)$$

Condition (62) represents a contact condition with normal compliance and unilateral constraint. It was introduced and justified in [9] and, for this reason, we do not present here in detail. We restrict ourselves to mention that this condition is derived by assuming an additive decomposition of the normal stress into two components which satisfy the Signorini condition in the form with the gap g and the normal compliance contact condition, respectively. Here p is the normal compliance function assumed to have the following properties

$$\left\{ \begin{array}{l} \text{(a) } p : \Gamma_3 \times \mathbb{R} \rightarrow \mathbb{R}_+. \\ \text{(b) There exists } L_p > 0 \text{ such that} \\ \quad |p(\mathbf{x}, r_1) - p(\mathbf{x}, r_2)| \leq L_p |r_1 - r_2| \\ \quad \text{for all } r_1, r_2 \in \mathbb{R}, \text{ a.e. } \mathbf{x} \in \Gamma_3. \\ \text{(c) } (p(\mathbf{x}, r_1) - p(\mathbf{x}, r_2))(r_1 - r_2) \geq 0 \\ \quad \text{for all } r_1, r_2 \in \mathbb{R}, \text{ a.e. } \mathbf{x} \in \Gamma_3. \\ \text{(d) The mapping } \mathbf{x} \mapsto p(\mathbf{x}, r) \text{ is measurable on } \Gamma_3, \\ \quad \text{for any } r \in \mathbb{R}. \\ \text{(e) } p(\mathbf{x}, r) = 0 \text{ for all } r \leq 0, \text{ a.e. } \mathbf{x} \in \Gamma_3. \end{array} \right. \quad (68)$$

Next, (63) is the frictionless condition and (64) represents the initial conditions in which \mathbf{u}_0 and $\boldsymbol{\sigma}_0$ denote the initial displacement and the initial stress field respectively, assumed to have the regularity

$$\mathbf{u}_0 \in V, \quad \boldsymbol{\sigma}_0 \in Q. \quad (69)$$

Consider the set of admissible displacement fields U , the irreversible stress field $\boldsymbol{\sigma}^{ir} : \Omega \times I \rightarrow \mathbb{S}^d$ and the function $\mathbf{f} : I \rightarrow V^*$ given by

$$U = \{ \mathbf{v} \in V : v_\nu \leq g \text{ a.e. on } \Gamma_3 \}, \quad (70)$$

$$\boldsymbol{\sigma}^{ir} = \boldsymbol{\sigma} - \mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}), \quad (71)$$

$$\langle \mathbf{f}(t), \mathbf{v} \rangle = \int_{\Omega} \mathbf{f}_0(t) \cdot \mathbf{v} \, dx + \int_{\Gamma_2} \mathbf{f}_2(t) \cdot \mathbf{v} \, da \quad \forall \mathbf{v} \in V. \quad (72)$$

Then using the Green formula (57), the following variational formulation of the problem could be derived.

Problem \mathcal{P}_{vp}^V . Find an irreversible stress field $\boldsymbol{\sigma}^{ir} : I \rightarrow Q$ and a displacement field $\mathbf{u} : I \rightarrow V$ such that

$$\dot{\boldsymbol{\sigma}}^{ir}(t) = \mathcal{G}(\mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}(t)) + \boldsymbol{\sigma}^{ir}(t), \boldsymbol{\varepsilon}(\mathbf{u}(t))) \quad \forall t \in I, \quad (73)$$

$$\boldsymbol{\sigma}^{ir}(0) = \boldsymbol{\sigma}_0 - \mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}_0), \quad (74)$$

$$\begin{aligned} \mathbf{u}(t) \in U, \quad & (\mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}(t)) + \boldsymbol{\sigma}^{ir}(t), \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\mathbf{u}(t)))_Q \\ & + \int_{\Gamma_3} p(u_\nu(t))(v_\nu - u_\nu(t)) \, da \geq \langle \mathbf{f}(t), \mathbf{v} - \mathbf{u}(t) \rangle_V \quad \forall \mathbf{v} \in U, t \in I. \end{aligned} \quad (75)$$

The existence of a unique solution of the Problem \mathcal{P}_{vp}^V is provided by the following result.

Theorem 4.1 Assume that (65) to (69) hold. Then, Problem \mathcal{P}_{vp}^V has a unique solution which satisfies $\boldsymbol{\sigma}^{ir} \in C^1(I; Q)$ and $\mathbf{u} \in C(I; V)$.

Proof. The proof is obtained in three steps, that we describe in what follows.

(i) *The differential quasivariational inequality.* Define the operator $A : Q \times V \rightarrow V^*$ and the functions $G : I \times Q \times V \rightarrow V^*$, $F : I \times Q \times V \rightarrow Q$, by equalities

$$\langle A(\boldsymbol{\sigma}, \mathbf{u}), \mathbf{v} \rangle = (\mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}) + \boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\mathbf{v}))_Q + \int_{\Gamma_3} p(u_\nu)v_\nu \, da, \quad (76)$$

$$G(t, \boldsymbol{\sigma}, \mathbf{u}) = A(\boldsymbol{\sigma}, \mathbf{u}) - \mathbf{f}(t), \quad (77)$$

$$F(t, \boldsymbol{\sigma}, \mathbf{u}) = \mathcal{G}(\mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}) + \boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\mathbf{u})), \quad (78)$$

for all $t \in I$, $\boldsymbol{\sigma} \in Q$, $\mathbf{u}, \mathbf{v} \in V$. Then, it is easy to see that Problem \mathcal{P}_{vp}^V is equivalent to the problem of finding an irreversible stress field $\boldsymbol{\sigma}^{ir} : I \rightarrow Q$ and a displacement field $\mathbf{u} : I \rightarrow V$ such that

$$\dot{\boldsymbol{\sigma}}^{ir}(t) = F(t, \boldsymbol{\sigma}^{ir}(t), \mathbf{u}(t)) \quad \forall t \in I, \quad (79)$$

$$\boldsymbol{\sigma}^{ir}(0) = \boldsymbol{\sigma}_0 - \mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}_0), \quad (80)$$

$$\mathbf{u}(t) \in U, \quad \langle G(t, \boldsymbol{\sigma}^{ir}(t), \mathbf{u}(t)), \mathbf{v} - \mathbf{u}(t) \rangle \geq 0 \quad \forall \mathbf{v} \in U, t \in I. \quad (81)$$

(ii) *An existence and uniqueness result.* We note that the system (79) to (81) is of the form (1) to (3) with $X = Q$, $K = U$, $\mathbf{x}_0 = \boldsymbol{\sigma}_0 - \mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}_0)$ and $j \equiv 0$, the functions G and F being defined by equalities (77) and (78), respectively. Therefore, in order to use Theorem 3.1, we check in what follows the validity of conditions (23) to (26).

First, we note that assumptions (65) and (66) on the elasticity tensor and the viscoplastic function, respectively, imply that the function (78) satisfies condition (23). Moreover, condition (69) implies (24) and (25) is clearly satisfied. In addition, using (65) and (68) and the trace inequality (56) it is easy to see that the operator A defined by (76) satisfies condition (27) with $m_A = m_{\mathcal{E}}$. On the other hand, the regularity (67) implies that the function \mathbf{f} defined by (72) has the regularity (28) and, it is obvious to see that (29), (30) and (26) hold, too. We now use Theorem 3.1 to deduce the existence of a unique solution of the differential quasivariational inequality (79) to (81), with regularity $\boldsymbol{\sigma}^{ir} \in C^1(I; Q)$, $\mathbf{u} \in C(I; V)$.

(iii) *Conclusion.* Theorem 4.1 is now a direct consequence of steps (i) and (ii) which state the equivalence between Problem \mathcal{P}_{vp}^V and the differential quasivariational inequality (79) to (81), and the unique solvability of this inequality, respectively. \square

5. An elastic contact problem with wear

The physical setting is similar to that considered in the previous section. The difference arises in the fact that now Γ_3 is assumed to be plane, the body is linearly elastic and the foundation is moving. Moreover, the contact is frictional and sliding. This implies the wear of the foundation that we model with a new surface variable, the wear function. Then, the classical formulation of the contact problem we study in this section is the following.

Problem \mathcal{P}_e . Find a stress field $\boldsymbol{\sigma} : \Omega \times I \rightarrow \mathbb{S}^d$, a displacement field $\mathbf{u} : \Omega \times I \rightarrow \mathbb{R}^d$ and a wear function $w : \Gamma_3 \times I \rightarrow \mathbb{R}$ such that

$$\boldsymbol{\sigma}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}(t)) \quad \text{in } \Omega, \quad (82)$$

$$\text{Div } \boldsymbol{\sigma}(t) + \mathbf{f}_0(t) = \mathbf{0} \quad \text{in } \Omega, \quad (83)$$

$$\mathbf{u}(t) = \mathbf{0} \quad \text{on } \Gamma_1, \quad (84)$$

$$\boldsymbol{\sigma}(t)\mathbf{v} = \mathbf{f}_2(t) \quad \text{on } \Gamma_2, \quad (85)$$

$$\left. \begin{array}{l} u_v(t) \leq g, \quad \sigma_v(t) + p(u_v(t) - w(t)) \leq 0, \\ (u_v(t) - g)(\sigma_v(t) + p(u_v(t) - w(t))) = 0 \end{array} \right\} \quad \text{on } \Gamma_3, \quad (86)$$

$$-\sigma_\tau(t) = \eta p(u_v(t) - w(t)) \mathbf{n}^*(t) \quad \text{on } \Gamma_3, \quad (87)$$

$$\dot{w}(t) = \alpha(t) p(u_v(t) - w(t)) \quad \text{on } \Gamma_3, \quad (88)$$

$$w(0) = 0 \quad \text{in } \Gamma_3, \quad (89)$$

Here, equations (82) and (83) represent the elastic constitutive law and the equilibrium equation, respectively. Moreover, (84) and (85) are the displacement and traction conditions. The boundary conditions (86) to (88) were introduced and justified in [22] and, for this reason, we do not present here in detail. We restrict ourselves to mention that (86) represents the contact condition in which the normal compliance function p satisfies assumption (68). It was derived by assuming an additive decomposition of the normal stress into two components which

satisfy the Signorini condition in the form with the gap g and the normal compliance contact condition with wear, respectively.

Condition (87) represents a sliding version of the classical Coulomb law of dry friction. Here η represents the coefficient of friction and \mathbf{n}^* is the unitary vector defined by

$$\mathbf{n}^*(t) = -\frac{\mathbf{v}^*(t)}{\|\mathbf{v}^*(t)\|}$$

where \mathbf{v}^* is the velocity of the foundation, supposed to be a non-vanishing time-dependent function in the plane of Γ_3 . Condition (87) was derived under the assumption that the velocity of the foundation is large in comparison with the tangential velocity of the elastic body. Here, we assume that the coefficient of friction and velocity of the foundation verify the following assumptions

$$\eta \in L^\infty(\Gamma_3), \quad \eta(\mathbf{x}) \geq 0 \quad \text{a.e. } \mathbf{x} \in \Gamma_3. \quad (90)$$

$$\begin{cases} \mathbf{v}^* \in C(I; \mathbb{R}^3) \text{ and there exist } v_1, v_2 > 0 \text{ such that} \\ v_1 \leq \|\mathbf{v}^*(t)\| \leq v_2 \quad \forall t \in I. \end{cases} \quad (91)$$

The differential equation (88) represents a version of Archard's law, which governs the evolution of the wear function and, again, it was derived under the assumption that the velocity of the foundation is large in comparison with the tangential velocity. Here

$$\alpha(t) = k \|\mathbf{v}^*(t)\|,$$

k being the wear coefficient, assumed to be such that

$$k \in L^\infty(\Gamma_3), \quad k(\mathbf{x}) \geq 0 \quad \text{a.e. } \mathbf{x} \in \Gamma_3. \quad (92)$$

Finally, condition (89) represents the initial condition for the wear function and shows that at the initial moment the material is new. More details concerning the modelling of the wear of contact surfaces can be found in [23–27]. The analysis of various contact models with wear can be found in [4, 28, 29].

Consider the set of admissible displacement fields given by (70). Then, using the Green formula, the following variational formulation of the problem could be derived.

Problem \mathcal{P}_e^V . Find a wear function $w : I \rightarrow L^2(\Gamma_3)$ and a displacement field $\mathbf{u} : I \rightarrow V$ such that

$$\dot{w}(t) = \alpha(t) p(u_v(t) - w(t)), \quad (93)$$

$$w(0) = 0, \quad (94)$$

$$\begin{aligned} \mathbf{u}(t) \in U, \quad (\mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}(t)), \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\mathbf{u}(t)))_Q + \int_{\Gamma_3} p(u_v(t) - w(t))(v_v - u_v(t)) da \\ + \int_{\Gamma_3} \eta p(u_v(t) - w(t)) \mathbf{n}^*(t) \cdot (\mathbf{v}_\tau - \mathbf{u}_\tau(t)) da \geq (\mathbf{f}(t), \mathbf{v} - \mathbf{u}(t))_V \quad \forall \mathbf{v} \in U, \end{aligned} \quad (95)$$

for all $t \in I$.

The existence of a unique solution of the problem \mathcal{P}_e^V is provided by the following result.

Theorem 5.1 Assume that (65), (67), (68), (90) to (92) hold. Then, there exists $e_0 > 0$ which depends only on \mathcal{E} , Ω , Γ_1 and Γ_3 such that Problem \mathcal{P}_e^V has a unique solution if $L_p \|\eta\|_{L^\infty(\Gamma_3)} < e_0$. Moreover, the solution satisfies $w \in C^1(I; L^2(\Gamma_3))$ and $\mathbf{u} \in C(I; V)$.

Proof. We use arguments similar to that used in Theorem 4.1. The steps of the proof are the following.

(i) *The differential quasivariational inequality.* Let $\mathbf{f} : I \rightarrow V^*$ be given by (72) and define the operator $A : L^2(\Gamma_3) \times V \rightarrow V^*$ and the functions $G : I \times L^2(\Gamma_3) \times V \rightarrow V^*$, $F : I \times L^2(\Gamma_3) \times V \rightarrow L^2(\Gamma_3)$, $j : L^2(\Gamma_3) \times V \times V \rightarrow \mathbb{R}$ by equalities

$$\langle A(w, \mathbf{u}), \mathbf{v} \rangle = (\mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{v}))_Q + \int_{\Gamma_3} p(u_v)v_v da, \quad (96)$$

$$G(t, w, \mathbf{u}) = A(w, \mathbf{u}) - \mathbf{f}(t), \quad (97)$$

$$F(t, w, \mathbf{u}) = \alpha(t)p(u_v(t) - w(t)), \quad (98)$$

$$j(w, \mathbf{u}, \mathbf{v}) = \int_{\Gamma_3} \eta p(u_v(t) - w(t)) \mathbf{n}^*(t) \cdot \mathbf{v}_\tau da \quad (99)$$

for all $t \in I$, $w \in L^2(\Gamma_3)$, $\mathbf{u}, \mathbf{v} \in V$. Then, it is easy to see that Problem \mathcal{P}_e^V is equivalent to the problem of finding a wear function $w : I \rightarrow L^2(\Gamma_3)$ and a displacement field $\mathbf{u} : I \rightarrow V$ such that

$$\dot{w}(t) = F(t, w(t), \mathbf{u}(t)) \quad \forall t \in I, \quad (100)$$

$$w(0) = 0, \quad (101)$$

$$\begin{aligned} \mathbf{u}(t) \in U, \quad \langle G(t, w(t), \mathbf{u}(t), \mathbf{v} - \mathbf{u}(t)) \rangle + j(w(t), \mathbf{u}(t), \mathbf{v}) \\ - j(w(t), \mathbf{u}(t), \mathbf{u}) \geq 0 \quad \forall \mathbf{v} \in U, t \in I. \end{aligned} \quad (102)$$

(ii) *An existence and uniqueness result.* It is easy to see that the system (79) to (81) is of the form (1) to (3) with $X = L^2(\Gamma_3)$, $K = U$, $x_0 = 0$, the functions G , F and j being defined by equalities (97), (98) and (99), respectively. Therefore, in order to use Theorem 3.1 we check in what follows the validity of conditions (23) to (26).

First, we note that assumptions (90) to (92) and (68) imply that the function (98) satisfies condition (23). Moreover, conditions (24) and (25) are clearly satisfied. In addition, using (65) and (68) and the trace inequality (56) it is easy to see that the operator A defined by (96) satisfies condition (27) with $m_A = m_{\mathcal{E}}$. Next, the regularity (67) implies that the function \mathbf{f} defined by (72) has the regularity given by (28) and, on the other hand it is obvious to see that the function (99) satisfies condition (29)(a). Assume now that $w_1, w_2 \in L^2(\Gamma_N)$ and $\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2 \in V$ are given. We write

$$\begin{aligned} & j(w_1, \mathbf{u}_1, \mathbf{v}_2) - j(w_1, \mathbf{u}_1, \mathbf{v}_1) + j(w_2, \mathbf{u}_2, \mathbf{v}_1) - j(w_2, \mathbf{u}_2, \mathbf{v}_2) \\ &= \int_{\Gamma_3} \eta \left(p(u_{1v}(t) - w_1(t)) - p(u_{2v}(t) - w_2(t)) \right) \mathbf{n}^*(t) \cdot (\mathbf{v}_{2\tau} - \mathbf{v}_{1\tau}) da \\ &\leq \int_{\Gamma_3} |\eta| |p(u_{1v}(t) - w_1(t)) - p(u_{2v}(t) - w_2(t))| \|\mathbf{n}^*(t)\| \|\mathbf{v}_{2\tau} - \mathbf{v}_{1\tau}\| da \leq \\ &\leq \|\eta\|_{L^\infty(\Gamma_3)} \int_{\Gamma_3} |p(u_{1v}(t) - w_1(t)) - p(u_{2v}(t) - w_2(t))| \|\mathbf{v}_1 - \mathbf{v}_2\| da, \end{aligned}$$

then we use assumption (68)(b) on the normal compliance function p combined with the trace inequality (56) to see that

$$\begin{aligned} & j(w_1, \mathbf{u}_1, \mathbf{v}_2) - j(w_1, \mathbf{u}_1, \mathbf{v}_1) + j(w_2, \mathbf{u}_2, \mathbf{v}_1) - j(w_2, \mathbf{u}_2, \mathbf{v}_2) \\ &\leq L_p c_0 \|\eta\|_{L^\infty(\Gamma_3)} \|w_1 - w_2\|_{L^2(\Gamma_3)} \|\mathbf{u}_1 - \mathbf{u}_2\|_V \\ &\quad + L_p c_0 \|\eta\|_{L^\infty(\Gamma_3)} \|\mathbf{u}_1 - \mathbf{u}_2\|_V \|\mathbf{v}_1 - \mathbf{v}_2\|_V. \end{aligned}$$

We conclude from here that the function j satisfies condition (29)(b) with $\beta = L_p c_0 \|\eta\|_{L^\infty(\Gamma_3)}$. Let

$$e_0 = \frac{m_{\mathcal{E}}}{c_0}$$

which clearly depends only on \mathcal{E} , Ω , Γ_1 and Γ_3 and assume that $L_p \|\eta\|_{L^\infty(\Gamma_3)} < e_0$. Then the smallness assumption (30) holds. Finally, note that equality (97) implies that (26) holds, too. We now use Theorem 3.1 to deduce the existence of a unique solution of the differential quasivariational inequality (100) to (102), with regularity $w \in C^1(I; L^2(\Gamma_3))$, $\mathbf{u} \in C(I; V)$.

(iii) *Conclusion.* Theorem 5.1 is now a direct consequence of steps (i) and (ii) which state the equivalence between Problem \mathcal{P}_e^V and the differential quasivariational inequality (100) to (102), and the unique solvability of this inequality, respectively. \square

We end this section with the remark that Theorem 3.1 can be used in the study of a large number of contact models which, in the variational formulation, lead to differential quasivariational inequalities of the form (1) to (3). Besides the models presented in Sections 4 and 5, additional models for which the results in this paper work could be found in [4, 8, 9], for instance. For those models, the unknown x could be an internal state variable, the bonding field, or the electric displacement field and the unknown u is the displacement field or the electric potential. The unique weak solvability of the corresponding models follows in three steps, as explained above. The main ingredient of the proof consists of checking the validity of conditions (23) to (26), with different spaces, functions and operators.

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