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# Pure spinors and a construction of the $E_*$ - Lie algebras

Marcus J. Slupinski and Robert J. Stanton

*This paper is dedicated to our friend and collaborator Gestur Ólafsson.*

**ABSTRACT.** Let  $(V, g)$  be a  $2n$ -dimensional hyperbolic space and  $C(V, g)$  its Clifford algebra.  $C(V, g)$  has a  $\mathbb{Z}$ -grading,  $C^k$ , and an algebra isomorphism  $C(V, g) \cong \text{End}(S)$ ,  $S$  the space of spinors. É. Cartan defined operators  $L_k : \text{End}(S) \rightarrow C^k$  which are involved in the definition of pure spinors. We shall give a more refined study of the operator  $L_2$ , in fact, obtain explicit formulae for it in terms of spinor inner products and combinatorics, as well as the matrix of it in a basis of pure spinors. Using this information we give a construction of the exceptional Lie algebras  $e_6, e_7, e_8$  completely within the theory of Clifford algebras and spinors.

## 1. Introduction

Constructions of exceptional Lie algebras over quite general fields have been given by many people and from various perspectives. While the list is too long to give, error free, we must mention Freudenthal and Tits. The perspective of this paper is that of spinors. When the base field  $k$  is  $\mathbb{R}$ , the classification by É. Cartan of irreducible Riemannian symmetric spaces of the noncompact type already provides an example, the case of  $e_8$ , for which the pair  $(\mathfrak{so}(16, \mathbb{R}), S_{\pm})$  occurs. More recently, J. F. Adams [Ad] gave a construction of compact exceptional Lie groups using compact spin groups and the relationship of some to Jordan algebras. Moroianu-Semmelman [MoSe] gave a construction of exceptional Lie algebras of compact type by refining Kostant's [K] invariant 4-tensor characterization of certain holonomy representations and coupled with the compact spinor material from [Ad]. Our point of view is to present natural properties of Clifford algebras and their spinors for a hyperbolic space over a very general field  $k$ , and then to derive the existence of the  $E_*$ -series using these properties and the combinatorics of pure spinors, thus a construction intrinsic to spinor algebra.

The paper is essentially self-contained and written with Lie theorists in mind, such as a master like Gestur, hence includes some standard material on spinors known to experts. We begin with a Clifford algebra  $C(V, g)$  and its basic isomorphism with  $\text{End}(S)$ ,  $S$  the spinors. Then we relate properties of  $C(V, g)$  and  $S$ , including fundamental material about the spinor norm. The key tool in the paper is the operator introduced by Cartan,  $L_2 : S \times S \rightarrow C_2$ . After we give a new description of  $L_2$ , we obtain an explicit formula for  $L_2$  in terms of Clifford elements. The formula is intrinsic to Clifford theory as the operator is completely specified by various spinor norms and combinatorics. Then using the basis of pure spinors we compute the matrix of  $L_2$  and express its entries in terms of

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spinor norms and combinatorics. This treatment is completely general for a hyperbolic space  $(V, g)$  and a field  $k$  of characteristic not 2 or 3.

In the last section we specialise the formula for the matrix of  $L_2$  to three specific dimensions and show that various entries of the matrix vanish for combinatorial reasons yielding a Jacobi identity for the various Lie algebras in the  $E_*$ -series.

There are several potential future directions. The choice of a hyperbolic  $g$  was made to avoid field extensions of  $k$  - indeed there are metrics of other signature that could be considered. Also, the combinatorics that arise in the computations mirror properties of the Weyl group quotient that parametrises Schubert cells in the flag variety of projectivised pure spinors. We did not consider whether other topological properties of the cells are responsible for the various combinatorial identities. Finally, since the spinor algebra is a universal linear construction, we expect the spinor algebra constructions, in particular  $L_2$ , to transfer to vector bundles.

## 2. Background on Spinors

Let  $V$  be a  $2n$ -dimensional vector space over a field  $k$  of characteristic not 2 or 3. We shall assume that  $V$  has a nondegenerate symmetric bilinear form  $g$  of Witt index  $n$  (i.e. a hyperbolic form). The hyperbolic case allows us to give a rather complete presentation of the results without any base extension of  $k$ . This was highlighted by Chevalley and today seems even more relevant. A good reference for much of the basic material of this section is [Ch].

### 2.1. Clifford algebra.

Let  $C = C(V, g)$  be the Clifford algebra of  $V$  with respect to  $g$ . Then  $C$  has the usual  $\mathbb{Z}_2$  grading  $C = C_+ \oplus C_-$  inherited from the tensor algebra of  $V$ . As  $g$  is hyperbolic,  $C$  is isomorphic to the algebra of  $2^n \times 2^n$  matrices over  $k$ . We can choose a  $2^n$ -dimensional  $k$ -vector space  $S$ , up to equivalence, called the space of spinors, and obtain an algebra isomorphism

$$C \cong \text{End}(S).$$

Hence  $C$  has a natural trace that we denote  $\text{Tr} : C(V, g) \rightarrow k$ .

The vector space  $V$  is naturally included in  $C_-$  so, from now on, we consider  $V$  as a subset of  $C$ . By the universal property of  $\Lambda^*(V)$ , the exterior algebra of  $V$ , one can extend the inclusion of  $V$  into  $C_-$  to an  $O(V, g)$ -equivariant linear (but not algebra) isomorphism  $Q : \Lambda^*(V) \rightarrow C$  by defining

$$Q(v_1 \wedge \cdots \wedge v_k) = \frac{1}{k!} \sum_{\sigma} (-1)^{\sigma} v_{\sigma(1)} \cdots v_{\sigma(k)},$$

at least if  $k$  is of characteristic 0. If  $v_1, \dots, v_k$  are orthogonal this formula implies that

$$Q(v_1 \wedge \cdots \wedge v_k) = v_1 \cdots v_k$$

and one uses this property to characterize  $Q$  when  $k$  is of positive characteristic (see [Ch]).

We set  $C^k = Q(\Lambda^k(V))$ . Hence if  $\{v_1, \dots, v_{2n}\}$  is a basis of  $V$  and  $i_1 < i_2 < \cdots < i_k$  then the collection  $\{Q(v_{i_1} \wedge \cdots \wedge v_{i_k})\}$  is a basis of  $C^k$ . The collection of subspaces  $C^k$  then give  $C$  the structure of a  $\mathbb{Z}$  graded vector space.

$C$  is also a filtered algebra where  $D^k$  is generated by products of at most  $k$  elements of  $V$ . We have then an isomorphism of the associated  $\mathbb{Z}$  graded space determined by the filtration onto the  $\mathbb{Z}$  graded  $Q(\Lambda^*(V))$ .

The following commutator relations are well known:

$$[C^1, C^1] \subseteq C^2, \quad [C^2, C^m] \subseteq C^m, \quad [C^2, C^{2n}] = 0.$$

Consequently,  $C^2$ ,  $C^1 \oplus C^2$  and  $C^2 \oplus C^{2n}$  are Lie algebras.

The composition with  $Q$  of any  $\mathfrak{o}(V, g)$ -equivariant isomorphism

$$\mathfrak{o}(V, g) \cong \Lambda^2(V)$$

defines a Lie algebra isomorphism  $\mathfrak{o}(V, g) \cong C^2$ .

Similarly one shows easily that the orthogonal Lie algebra of a vector space of dimension  $2n+1$  of maximal Witt index is isomorphic to  $C^1 \oplus C^2$ .

The canonical anti-automorphism of order 2 of  $C(V, g)$ , namely the one extending  $v \mapsto v$  for  $v \in V$ , is inherited from the tensor algebra. It will be denoted  $x \mapsto x^T$ . Using the canonical anti-automorphism  $T$  and the trace  $Tr$  one can give  $C$  a norm, namely  $\|c\|^2 = Tr(c^T c)$ .

PROPOSITION 2.1. *Let  $g_\Lambda$  be the natural extension of  $g$  to  $\Lambda^*(V)$ . Then*

$$2^n g_\Lambda(\alpha, \beta) = Tr(Q(\alpha)^T Q(\beta)),$$

*i.e.  $Q$  is a multiple of an isometry.*

REMARK 2.2. *By an orthonormal basis of  $V$  we mean a basis  $\{e_1, \dots, e_{2n}\}$  which satisfies*

$$g(e_i, e_j) = \pm \delta_{ij}.$$

*Orthonormal bases exist because  $(V, g)$  is isometric to an orthogonal sum of hyperbolic planes. Later we will use ordered orthonormal bases.*

For the  $\mathbb{Z}$  grading  $C = \bigoplus C^k$  and with respect to an orthonormal basis there is a formula for the projection  $\pi_k : C \rightarrow C^k$ :

$$\pi_k(c) = \frac{1}{2^n} \sum_{i_1 < \dots < i_k} g(e_{i_1}, e_{i_1}) \dots g(e_{i_k}, e_{i_k}) Tr(e_{i_k} \dots e_{i_1} c) e_{i_1} \dots e_{i_k},$$

as follows easily from the fact that  $\{Q(v_{i_1} \wedge \dots \wedge v_{i_k})\}$  is a basis of  $C^k$  and  $Q(e_{i_1} \wedge \dots \wedge e_{i_k}) = e_{i_1} \dots e_{i_k}$ .

There is a natural  $\mathbb{Z}_2$  grading of  $S$ ,  $S = S_1 \oplus S_2$ , into a direct sum of two  $2^{n-1}$  dimensional subspaces compatible with the  $\mathbb{Z}_2$  graded action of  $C$ , i.e.,

$$C_+ \cdot S_1 \subseteq S_1, \quad C_+ \cdot S_2 \subseteq S_2, \quad C_- \cdot S_1 \subseteq S_2 \text{ and } C_- \cdot S_2 \subseteq S_1.$$

Elements of  $S_1$  or  $S_2$  are called half-spinors. An element of  $C$  that implements such a grading of  $S$ , i.e. which is the identity on one half-spinor space and minus the identity on the other, will be called a grading element. They are usually denoted by  $\varepsilon$  and are elements of  $C^{2n}$ . Note that if  $\{e_1, \dots, e_{2n}\}$  is an orthonormal basis of  $V$  then

$$\varepsilon = e_1 \dots e_{2n}$$

satisfies  $\varepsilon^2 = 1$  and is a grading element. The graded decomposition of  $S$  corresponding to this particular  $\varepsilon$  will be denoted  $S = S_+ \oplus S_-$ .

PROPOSITION 2.3. *If  $\varepsilon$  is a grading element, then for all  $1 \leq k \leq 2n$  and all  $c \in C$ ,*

$$\pi_{2n-k}(\varepsilon c) = \varepsilon \pi_k(c).$$

*Proof.* This is essentially III.4.3 in [Ch].

QED

## 2.2. Spinor norms.

Recall that  $V$  is of dimension  $2n$  and that  $g$  is hyperbolic.

DEFINITION 2.4. *A spinor norm is a bilinear map  $B : S \times S \rightarrow k$  such that*

$$B(v \cdot \phi, \psi) = B(\phi, v \cdot \psi) \quad \forall v \in V, \forall \phi, \psi \in S.$$

*Hence, given a non-zero spinor norm  $B$ , the canonical anti-automorphism of  $C$ ,  $x \mapsto x^T$ , is, via the isomorphism  $C \cong \text{End}(S)$ , the transpose relative to  $B$ .*

PROPOSITION 2.5. (*É. Cartan*) *The space of spinor norms is one dimensional.*

The symmetry type of a spinor norm, symmetric or alternating, as well as the type of its restriction to the half-spinor spaces is summarized in the next result.

PROPOSITION 2.6. *Let  $B$  be a spinor norm.*

- (i) *If  $n \equiv 0 \pmod{4}$  then  $B$  is symmetric and even, i.e.,  $B(S_1, S_2) = 0$ .*
- (ii) *If  $n \equiv 1 \pmod{4}$  then  $B$  is symmetric and odd, i.e.,  $B(S_1, S_1) = B(S_2, S_2) = 0$ .*
- (iii) *If  $n \equiv 2 \pmod{4}$  then  $B$  is antisymmetric and even.*
- (iv) *If  $n \equiv 3 \pmod{4}$  then  $B$  is antisymmetric and odd.*

We shall fix a non-degenerate spinor norm,  $B$ . It is natural to have the relationship of the  $\mathbb{Z}$  grading of  $C$  to  $B$ . If  $\phi, \psi \in S$ ,  $1 \leq k \leq 2n$  and  $v_1, \dots, v_k \in V$  are orthogonal, it is clear that

$$B(v_1 \cdots v_k \cdot \phi, \psi) = (-1)^{\frac{1}{2}k(k-1)} B(\phi, v_1 \cdots v_k \cdot \psi).$$

Hence we have

COROLLARY 2.7. *If  $k \equiv 2$  or  $3 \pmod{4}$ , then*

$$B(c \cdot \phi, \psi) + B(\phi, c \cdot \psi) = 0 \quad \forall c \in C^k,$$

*i.e., a spinor norm is invariant under the action of  $C^k$  iff  $k \equiv 2$  or  $k \equiv 3 \pmod{4}$ .*

Since  $B$  can have either symmetry type, symmetric or alternating, we denote by  $\text{aut}(S, B)$  the endomorphisms of  $S$  that leave invariant the spinor norm.

COROLLARY 2.8. *Identifying  $C(V, g)$  with  $\text{End}(S)$ , we have*

$$\bigoplus_{k \equiv 2 \text{ or } 3 \pmod{4}} C^k = \text{aut}(S, B).$$

*Proof.* By Corollary 2.7 the LHS is included in the RHS. The result follows from a dimension count for the corresponding symmetry type of  $B$ :

$$\sum_{k \equiv 2 \text{ or } 3 \pmod{4}} \binom{2n}{k} = 2^{n-1}(2^n - 1) = \dim \mathfrak{so}(S, B) \quad (n \equiv 0, 1 \pmod{4})$$

and

$$\sum_{k \equiv 2 \text{ or } 3 \pmod{4}} \binom{2n}{k} = 2^{n-1}(2^n + 1) = \dim \mathfrak{sp}(S, B) \quad (n \equiv 2, 3 \pmod{4}).$$

QED

COROLLARY 2.9. *If  $\varepsilon$  is any grading element and we set*

$$\text{aut}_{\pm}(S, B) = \{c \in \text{aut}(S, B) : c\varepsilon = \pm \varepsilon c\}$$

*then*

$$\bigoplus_{k \equiv 2 \pmod{4}} C^k = \text{aut}_{+}(S, B), \quad \bigoplus_{k \equiv 3 \pmod{4}} C^k = \text{aut}_{-}(S, B).$$

*Proof.* This is immediate from the previous Corollary.

QED

From Prop. 2.6 we see that the spinor norm  $B$  is even if and only if  $n$  is even. In this case  $B$  restricts to nondegenerate forms  $B_1$  and  $B_2$  on the half-spinor spaces  $S_1$  and  $S_2$  respectively. Thus for  $n$  even and grading element  $\varepsilon$ , the Lie algebra  $\text{aut}_{+}(S, B)$  is a direct product

$$\text{aut}_{+}(S, B) \cong \text{aut}(S_1, B_1) \oplus \text{aut}(S_2, B_2).$$

To realise this decomposition of  $\text{aut}_{+}(S, B)$  in the Clifford algebra we use the grading element.

COROLLARY 2.10. *Let  $n$  be even. Let  $\varepsilon \in C(V, g)$  be a grading element and  $S = S_1 \oplus S_2$  the associated grading (the  $\pm 1$  eigenspaces of  $\varepsilon$ ). Set  $\varepsilon_{\pm} = \frac{(1 \pm \varepsilon)}{2}$ .*

(i) *Then*

$$\bigoplus_{k \equiv 2 \pmod{4}} C^k = \left( \bigoplus_{k \equiv 2 \pmod{4}, k \leq n} C^k \varepsilon_+ \right) \oplus \left( \bigoplus_{k \equiv 2 \pmod{4}, k \leq n} C^k \varepsilon_- \right).$$

(ii) *With respect to the decomposition  $S = S_1 \oplus S_2$ , the two summands of (i) are:*

$$\bigoplus_{k \equiv 2 \pmod{4}, k \leq n} C^k \varepsilon_+ = \left( \begin{array}{c|c} \mathbf{aut}(S_1, B_1) & 0 \\ \hline 0 & 0 \end{array} \right)$$

and

$$\bigoplus_{k \equiv 2 \pmod{4}, k \leq n} C^k \varepsilon_- = \left( \begin{array}{c|c} 0 & 0 \\ \hline 0 & \mathbf{aut}(S_2, B_2) \end{array} \right)$$

*Proof.* Part (i) basically follows from the fact that

$$1 = \frac{1}{2}(1 + \varepsilon) + \frac{1}{2}(1 - \varepsilon)$$

decomposes the identity of  $C$  as a sum of two orthogonal idempotents. Part (ii) is straightforward. QED

REMARK 2.11. *If  $k < n$ , both  $C^k \varepsilon_+$  and  $C^k \varepsilon_-$  are isomorphic to  $\Lambda^k(V)$  as absolutely irreducible  $\mathfrak{so}(V, g)$ -representations. If  $k = n$ , É. Cartan showed that  $C^n \varepsilon_+$  and  $C^n \varepsilon_-$  are absolutely irreducible, non-isomorphic representations of the same dimension. The proposition therefore gives an explicit reduction of  $\mathbf{aut}(S_1, B_1)$  and  $\mathbf{aut}(S_2, B_2)$  into their  $\mathfrak{so}(V, g)$ -irreducible components.*

### 2.3. Tensor Product $S \otimes S$ .

As usual, a choice of  $B$  on  $S$  gives a  $C^2$ -equivariant isomorphism  $\tau : S \otimes S \rightarrow \mathbf{End}(S)$ :

$$\tau(\phi \otimes \psi)(\xi) = B(\phi, \xi)\psi.$$

Since

$$B(\tau(\phi \otimes \psi)(\xi), \eta) = (-1)^{\frac{1}{2}n(n-1)} B(\xi, \tau(\psi \otimes \phi)(\eta)),$$

it follows that

$$\begin{cases} \tau(\phi \otimes \psi - \psi \otimes \phi) \in \mathbf{aut}(S, B) & \text{if } n \equiv 0, 1 \pmod{4}, \\ \tau(\phi \otimes \psi + \psi \otimes \phi) \in \mathbf{aut}(S, B) & \text{if } n \equiv 2, 3 \pmod{4}. \end{cases}$$

Identifying  $\mathbf{End}(S)$  with the Clifford algebra  $C$  and using the preceding Remark one can reduce symmetric and antisymmetric spinors as  $\mathfrak{so}(V, g)$ -representations.

PROPOSITION 2.12.

(i) *If  $n \equiv 0, 1 \pmod{4}$  then  $\tau$  induces  $\mathfrak{so}(V, g)$ -equivariant isomorphisms:*

$$\Lambda^2(S) \cong \bigoplus_{k \equiv 2 \text{ or } 3 \pmod{4}} C^k, \quad S^2(S) \cong \bigoplus_{k \equiv 0 \text{ or } 1 \pmod{4}} C^k.$$

(ii) *If  $n \equiv 2, 3 \pmod{4}$  then  $\tau$  induces  $\mathfrak{so}(V, g)$ -equivariant isomorphisms:*

$$\Lambda^2(S) \cong \bigoplus_{k \equiv 0 \text{ or } 1 \pmod{4}} C^k, \quad S^2(S) \cong \bigoplus_{k \equiv 2 \text{ or } 3 \pmod{4}} C^k.$$

### 3. Cartan's operator $L_2$

The operators  $L_2$  and  $L_{2n}$  to be defined in this section are among the operators  $L_k$  that appear in Cartan and Chevalley, where mostly they are used to characterise pure spinors. The interesting properties of  $L_2$  to be described herein appear to be new.

Composing  $\tau : S \otimes S \rightarrow \text{End}(S)$  with the projection  $\pi_2 : C \rightarrow C^2$  we can define a  $C^2$ -equivariant map  $L_2 : S \times S \rightarrow C^2$  :

$$(1) \quad L_2(\phi, \psi) = \frac{1}{2^n} \sum_{i < j} g(e_i, e_i)g(e_j, e_j)B(\phi, e_j e_i \cdot \psi)e_i e_j,$$

where  $\{e_1, \dots, e_{2n}\}$  is any orthonormal basis of  $V$ .

PROPOSITION 3.1.

- (i) If  $n \equiv 0 \pmod{4}$  then  $L_2$  is antisymmetric and even, i.e.  $L_2(S_1, S_2) = 0$ .
- (ii) If  $n \equiv 1 \pmod{4}$  then  $L_2$  is antisymmetric and odd, i.e.  $L_2(S_1, S_1) = L_2(S_2, S_2) = 0$ .
- (iii) If  $n \equiv 2 \pmod{4}$  then  $L_2$  is symmetric and even.
- (iv) If  $n \equiv 3 \pmod{4}$  then  $L_2$  is symmetric and odd.

$L_2$  has an interesting formulation in terms of orbit maps.

DEFINITION 3.2. For  $\phi \in S$  we define  $\phi : V \rightarrow S$  and  $\phi^* : S \rightarrow V$  by

$$(2) \quad \phi(v) = v \cdot \phi, \quad \phi^*(\psi) = \sum_i g(e_i, e_i)B(\phi, e_i \cdot \psi)e_i$$

where  $\{e_1, \dots, e_{2n}\}$  is any orthonormal basis of  $V$ .

The maps  $\phi$  and  $\phi^*$  are adjoints for the respective norms, i.e.,

$$(3) \quad B(\phi(v), \psi) = g(v, \phi^*(\psi)).$$

Thus we have

$$\begin{cases} g(\phi^* \circ \psi(v_1), v_2) = B(\phi(v_2), \psi(v_1)), \\ g(\psi^* \circ \phi(v_2), v_1) = B(\psi(v_1), \phi(v_2)), \end{cases}$$

hence

$$g(\phi^* \circ \psi(v_1), v_2) = (-1)^{\frac{1}{2}n(n-1)} g(\psi^* \circ \phi(v_2), v_1).$$

Similarly, equation (3) implies

$$B(\psi \circ \phi^*(\xi), \eta) = g(\phi^*(\xi), \psi^*(\eta)) = B(\phi \circ \psi^*(\eta), \xi),$$

hence

$$B(\psi \circ \phi^*(\xi), \eta) = (-1)^{\frac{1}{2}n(n-1)} B(\xi, \phi \circ \psi^*(\eta)).$$

The next Proposition follows immediately from the above.

PROPOSITION 3.3. Let  $\phi, \psi \in S$  be spinors. Then

$$(-1)^{\frac{1}{2}n(n-1)} \psi^* \circ \phi - \phi^* \circ \psi, \quad (-1)^{\frac{1}{2}n(n-1)} \psi^* \circ \phi + \phi^* \circ \psi$$

are respectively antisymmetric and symmetric endomorphisms of  $(V, g)$ , while

$$(-1)^{\frac{1}{2}n(n-1)} \psi \circ \phi^* - \phi \circ \psi^*, \quad (-1)^{\frac{1}{2}n(n-1)} \psi \circ \phi^* + \phi \circ \psi^*$$

are respectively antisymmetric and symmetric endomorphisms of  $(S, B)$ .

Consequently the first expression of Proposition 3.3 defines a bilinear form on  $S$  with values in  $\mathfrak{so}(V, g)$ , and even/odd depending on  $n \bmod 4$ . By Proposition 3.1, the bilinear form  $L_2$  also takes values in  $\mathfrak{so}(V, g)$  and can be seen to have the same parity properties. Analogously, the third expression also defines a bilinear form on  $S$  with the same parity properties as  $L_2$  but taking values in  $\mathfrak{aut}(S, B)$ . Taken together, they suggest a type of curvature operator. The exact relationship between them will be described next.

In fact it will be convenient to ‘renormalise’  $L_2$  as follows:

DEFINITION 3.4. Define  $\tilde{L}_2 : S \times S \rightarrow C^2$  by

$$\tilde{L}_2 = 2^{n-1}L_2$$

PROPOSITION 3.5.

(i) For all  $\phi, \psi \in S$ ,

$$2\tilde{L}_2(\phi, \psi) = (-1)^{\frac{1}{2}n(n-1)}\psi^* \circ \phi - \phi^* \circ \psi$$

in the sense that for all  $v \in V$ ,

$$(4) \quad 2[\tilde{L}_2(\phi, \psi), v] = (-1)^{\frac{1}{2}n(n-1)}\psi^* \circ \phi(v) - \phi^* \circ \psi(v).$$

(ii) For all  $\phi, \psi \in S$  and all  $v \in V$ ,

$$(5) \quad 2B(\phi, \psi)v = (-1)^{\frac{1}{2}n(n-1)}\psi^* \circ \phi(v) + \phi^* \circ \psi(v).$$

*Proof.* Let  $\{e_1, \dots, e_{2n}\}$  be an orthonormal basis of  $V$ . For any  $e_k$  in this basis, by equation (1) we have

$$[\tilde{L}_2(\phi, \psi), e_k] = \frac{1}{2} \sum_{i < j} g(e_i, e_i)g(e_j, e_j)B(\phi, e_j e_i \cdot \psi)[e_i e_j, e_k],$$

and using

$$[e_i e_j, e_k] = \begin{cases} 0 & \text{if } i \neq k, j \neq k, \\ -2g(e_k, e_k)e_j & \text{if } i = k, \\ 2g(e_k, e_k)e_i & \text{if } j = k, \end{cases}$$

this simplifies to

$$[\tilde{L}_2(\phi, \psi), e_k] = \sum_{i \neq k} g(e_i, e_i)g(e_k, e_k)^2 B(\phi, e_k e_i \cdot \psi)e_i,$$

which, since  $g(e_k, e_k)^2 = 1$ , reduces to

$$(6) \quad [\tilde{L}_2(\phi, \psi), e_k] = \sum_{i \neq k} g(e_i, e_i)B(\phi, e_k e_i \cdot \psi)e_i.$$

To calculate the RHS of equation (4) acting on  $e_k$  we have

$$\begin{aligned} (-1)^{\frac{1}{2}n(n-1)}\psi^* \circ \phi(e_k) &= (-1)^{\frac{1}{2}n(n-1)} \sum_i g(e_i, e_i)B(\psi, e_i e_k \cdot \phi)e_i \\ &= \sum_i g(e_i, e_i)B(e_i e_k \cdot \phi, \psi)e_i \\ (7) \quad &= \sum_i g(e_i, e_i)B(\phi, e_k e_i \cdot \psi)e_i, \end{aligned}$$

and

$$(8) \quad \phi^* \circ \psi(e_k) = \sum_i g(e_i, e_i)B(\phi, e_i e_k \cdot \psi)e_i.$$

Since  $e_k$  is an arbitrary basis element, both parts of the proposition follow from equations (6), (7) and (8).

QED

In the same way there is a  $C^2$ -equivariant map  $L_{2n} : S \times S \rightarrow C^{2n}$  obtained by composing  $\tau : S \otimes S \rightarrow \text{End}(S) = C$  and  $\pi_{2n} : C \rightarrow C^{2n}$ . Explicitly,

$$(9) \quad L_{2n}(\phi, \psi) = \frac{1}{2^n} g(e_1, e_1) \dots g(e_{2n}, e_{2n}) B(\phi, e_{2n} \dots e_1 \cdot \psi) e_1 \dots e_{2n},$$

where  $\{e_1, \dots, e_{2n}\}$  is any orthonormal basis of  $V$ . The symmetry properties of  $L_{2n}$  follow readily from the preceding.

PROPOSITION 3.6.

- (i) If  $n \equiv 0 \pmod{4}$  then  $L_{2n}$  is symmetric and even.
- (ii) If  $n \equiv 1 \pmod{4}$  then  $L_{2n}$  is antisymmetric and odd.
- (iii) If  $n \equiv 2 \pmod{4}$  then  $L_{2n}$  is symmetric and even.
- (iv) If  $n \equiv 3 \pmod{4}$  then  $L_{2n}$  is antisymmetric and odd.

### 3.1. Graded spinor norms.

Spinor norms are invariant under  $C^2$  but not under  $C^1$ , as follows from Corollary 2.7. In order to get something invariant under the action of the Lie algebra  $C^1 \oplus C^2$  one can use a grading element.

PROPOSITION 3.7. *Let  $B$  be a spinor norm and  $\varepsilon \in C$  be a grading element. Define the associated graded spinor norm  $B_\varepsilon : S \times S \rightarrow k$  by*

$$B_\varepsilon(\phi, \psi) = B(\varepsilon \cdot \phi, \psi) \forall \phi, \psi \in S.$$

Then

$$B_\varepsilon(v \cdot \phi, \psi) = -B_\varepsilon(\phi, v \cdot \psi) \forall v \in V, \forall \phi, \psi \in S,$$

and graded spinor norms are characterised by this property.

If  $\phi, \psi \in S$ ,  $1 \leq k \leq 2n$  and  $v_1, \dots, v_k \in V$  are orthogonal, it is clear that

$$B_\varepsilon(v_1 \dots v_k \cdot \phi, \psi) = (-1)^{\frac{1}{2}k(k+1)} B_\varepsilon(\phi, v_1 \dots v_k \cdot \psi)$$

and hence we have

COROLLARY 3.8. *A graded spinor norm is invariant for the action of the Lie algebra  $C^1 \oplus C^2$ .*

PROPOSITION 3.9. *Let  $B_\varepsilon$  be a graded spinor norm.*

- (i) If  $n \equiv 0 \pmod{4}$  then  $B_\varepsilon$  is symmetric and even.
- (ii) If  $n \equiv 1 \pmod{4}$  then  $B_\varepsilon$  is antisymmetric and odd.
- (iii) If  $n \equiv 2 \pmod{4}$  then  $B_\varepsilon$  is antisymmetric and even.
- (iv) If  $n \equiv 3 \pmod{4}$  then  $B_\varepsilon$  is symmetric and odd.

COROLLARY 3.10. *If  $\phi, \psi \in S$ ,  $1 \leq k \leq 2n$  and  $v_1, \dots, v_k \in V$  are orthogonal, then*

$$B_\varepsilon(\phi, v_1 \dots v_k \cdot \psi) = (-1)^{\frac{1}{2}n(n+1) + \frac{1}{2}k(k+1)} B_\varepsilon(\psi, v_1 \dots v_k \cdot \phi).$$

Using  $B_\varepsilon$  we can define a  $C^1 \oplus C^2$ -equivariant map  $\tau_\varepsilon : S \times S \rightarrow \text{End}(S)$  by

$$\tau_\varepsilon(\phi, \psi)(\xi) = B_\varepsilon(\phi, \xi)\psi.$$

One can now repeat the preceding but using the graded versions.

Using  $B_\varepsilon$  we can define a  $C^1 \oplus C^2$ -equivariant map  $L_\varepsilon : S \times S \rightarrow C^1 \oplus C^2$  by composing  $\tau_\varepsilon : S \otimes S \rightarrow \text{End}(S) = C$  and  $\pi_1 \oplus \pi_2 : C \rightarrow C^1 \oplus C^2$ . (To avoid excessive notation we suppress the subscripts 1, 2.) Explicitly,

$$L_\varepsilon(\phi, \psi) = \frac{1}{2^n} \left( \sum_i g(e_i, e_i) B_\varepsilon(\phi, e_i \cdot \psi) e_i + \sum_{i < j} g(e_i, e_i) g(e_j, e_j) B_\varepsilon(\phi, e_j e_i \cdot \psi) e_i e_j \right),$$

where  $\{e_1, \dots, e_{2n}\}$  is any orthonormal basis of  $V$ .

PROPOSITION 3.11.

- (i) If  $n \equiv 0 \pmod{4}$  then  $L_\varepsilon$  is antisymmetric,  $\pi_2 \circ L_\varepsilon$  is even and  $\pi_1 \circ L_\varepsilon$  is odd.
- (ii) If  $n \equiv 1 \pmod{4}$  then  $L_\varepsilon$  is symmetric,  $\pi_2 \circ L_\varepsilon$  is odd and  $\pi_1 \circ L_\varepsilon$  is even.
- (iii) If  $n \equiv 2 \pmod{4}$  then  $L_\varepsilon$  is symmetric,  $\pi_2 \circ L_\varepsilon$  is even and  $\pi_1 \circ L_\varepsilon$  is odd.
- (iv) If  $n \equiv 3 \pmod{4}$  then  $L_\varepsilon$  is antisymmetric,  $\pi_2 \circ L_\varepsilon$  is odd and  $\pi_1 \circ L_\varepsilon$  is even.

#### 4. Maximal isotropic subspaces and polarisations

A maximal isotropic subspace of  $V$  is an  $n$ -dimensional subspace  $\mathcal{I}$  of  $V$  such that the restriction of  $g$  to  $\mathcal{I}$  vanishes. By Witt's theorem the group  $O(V, g)$  acts transitively on the collection of maximal isotropic subspaces. The stabiliser of  $\mathcal{I}$ ,  $S(\mathcal{I})$ , is a maximal parabolic subgroup of  $O(V, g)$  (see e.g. [Wo]). The natural map from  $S(\mathcal{I})$  to  $GL(\mathcal{I})$  is surjective, giving rise to the exact sequence of groups

$$1 \rightarrow A \rightarrow S(\mathcal{I}) \rightarrow GL(\mathcal{I}) \rightarrow 1.$$

A description of  $A$  can be obtained as follows.

If  $a \in A$ , then  $(a - Id_V)(V) \subseteq \mathcal{I}$ , hence there is a unique  $\alpha : V \rightarrow V$  such that

$$(10) \quad (i) \quad a = Id_V + \alpha$$

$$(11) \quad (ii) \quad \alpha|_{\mathcal{I}} = 0, \alpha(V) \subseteq \mathcal{I} \text{ and } g(\alpha(v), w) + g(v, \alpha(w)) = 0.$$

This shows that  $A$  is abelian and  $a \mapsto \alpha$  identifies it with a subgroup of the vector space  $Hom(V/\mathcal{I}, \mathcal{I})$ .

Since  $\mathcal{I}$  is maximal isotropic the metric defines a duality pairing  $V/\mathcal{I} \otimes \mathcal{I} \rightarrow k$ , and hence there is an  $S(\mathcal{I})$ -equivariant isomorphism

$$Hom(V/\mathcal{I}, \mathcal{I}) \cong \mathcal{I} \otimes \mathcal{I}.$$

One checks that the composition of maps  $A \hookrightarrow Hom(V/\mathcal{I}, \mathcal{I}) \cong \mathcal{I} \otimes \mathcal{I}$  is an  $S(\mathcal{I})$ -equivariant isomorphism of  $A$  with  $\Lambda^2(\mathcal{I})$ , the space of antisymmetric two tensors on  $\mathcal{I}$ . Notice that  $A$  acts trivially on  $A$ ,  $\mathcal{I}$ , and  $V/\mathcal{I}$  so that this factors to a  $S(\mathcal{I})/A \cong GL(\mathcal{I})$ -equivariant isomorphism

$$(12) \quad A \cong \Lambda^2(\mathcal{I}),$$

as a module for  $GL(\mathcal{I})$ . Finally, under this isomorphism the cone of decomposable elements in  $\Lambda^2(\mathcal{I})$  is the image of the set  $T$  of elements  $a$  of  $A$  with the property that there exist  $v, w \in \mathcal{I}$  such that

$$a(x) = x + g(x, v)w - g(x, w)v \quad \forall x \in V.$$

Alternatively,  $T$  is the set of elements of  $A$  which are the identity on some  $2n - 2$  dimensional subspace of  $V$  containing  $\mathcal{I}$ . The subspace determines the group element essentially uniquely and then restriction of the above isomorphism to  $T$  corresponds to the Plücker embedding of its orthogonal in  $\Lambda^2(\mathcal{I})$ .

We can realise  $S(\mathcal{I})$  as a group of affine transformations of an affine space of which  $A$  is the group of translations. Consider the exact sequence of vector spaces:

$$0 \rightarrow \mathcal{I} \rightarrow V \xrightarrow{p} V/\mathcal{I} \rightarrow 0,$$

and let

$$\begin{aligned} \mathcal{A} &= \{s \in Hom(V/\mathcal{I}, V) : p \circ s = Id_{V/\mathcal{I}} \text{ and } \text{Im } s \text{ is maximal isotropic}\} \\ &= \{s \in Hom(V/\mathcal{I}, V) : g(s \circ p(v), i) = g(v, i) \forall v \in V, i \in \mathcal{I} \text{ and } \text{Im } s \text{ is maximal isotropic}\} \end{aligned}$$

be the space of isotropic splittings. (One can also identify  $\mathcal{A}$  with the space of maximal isotropic complements of  $\mathcal{I}$  by  $s \in \mathcal{A} \mapsto \text{Im } s$ ).

This is not a linear subspace of  $\text{Hom}(V/\mathcal{I}, V)$  but it is stable under the natural action of  $S(\mathcal{I})$ :

$$f \mapsto g \circ f \circ g^{-1}$$

and then the group  $A$  acts on  $\mathcal{A}$  as follows: if  $s \in \mathcal{A}$  and if we write  $a \in A$  as  $a = \text{Id}_V + \alpha$ , then  $a$  maps  $s$  to  $s' = s + \alpha$ . To check that  $s' \in \mathcal{A}$  first note that  $p \circ s' = p \circ s = \text{Id}_{V/\mathcal{I}}$  and so  $s'$  is a splitting. Further,

$$(13) \quad g(s' \circ p(v), s' \circ p(w)) = g(s \circ p(v), \alpha \circ p(w)) + g(\alpha \circ p(v), s \circ p(w))$$

$$(14) \quad = g(v, \alpha \circ p(w)) + g(\alpha \circ p(v), w)$$

$$(15) \quad = g(v, \alpha(w)) + g(\alpha(v), w) = 0,$$

and hence  $s' \in \mathcal{A}$ . It is clear that  $s \mapsto s'$  defines a free group action of  $A$  on  $\mathcal{A}$ .

To see that  $A$  acts transitively, take  $s, s' \in \mathcal{A}$ . Then  $\alpha = s' - s \in \text{Hom}(V/\mathcal{I}, \mathcal{I})$  and

$$(16) \quad 0 = g(s' \circ p(v), s' \circ p(w)) = g(s \circ p(v), \alpha \circ p(w)) + g(\alpha \circ p(v), s \circ p(w))$$

$$(17) \quad = g(v, \alpha \circ p(w)) + g(\alpha \circ p(v), w),$$

and the difference  $s' - s$  of the two isotropic splittings is in  $A$ .

With respect to this affine structure the group  $S(\mathcal{I})$  acts by affine transformations on  $\mathcal{A}$ . The tangent space at any point of  $\mathcal{A}$  is canonically isomorphic to  $A$  and hence carries a cone structure induced by  $T \subset A$  and this is clearly preserved by the affine action of  $S(\mathcal{I})$ . From the theory of 3-graded Lie algebras one can see that  $S(\mathcal{I})$  is exactly the group of affine transformations of  $\mathcal{A}$  which preserve this cone structure.

REMARK 4.1. *Of course for vector spaces  $V$  defined over  $\mathbb{R}$  or  $\mathbb{C}$  some of the above is standard (e.g. [Wo]).*

Let  $\mathcal{I}$  be a fixed maximal isotropic subspace of  $V$ . An  $\mathcal{I}$ -polarisation of  $(V, g)$  is a decomposition  $V = \mathcal{I} \oplus \mathcal{E}$  such that  $\mathcal{E}$  is a maximal isotropic subspace of  $V$ , i.e. an element of  $\mathcal{A}$  above. Choose a basis  $i_1, \dots, i_n$  of  $\mathcal{I}$  and let  $e_1, \dots, e_n$  be the basis of  $\mathcal{E}$  satisfying

$$g(e_a, i_b) = \frac{1}{2}\delta_{ab}.$$

The Clifford algebra relations  $xy + yx = 2g(x, y)\text{Id}$  imply that for all  $1 \leq a, b \leq n$ ,

$$(18) \quad \begin{aligned} i_a i_b + i_b i_a &= 0 \\ e_a e_b + e_b e_a &= 0 \\ i_a e_b + e_b i_a &= \delta_{ab}. \end{aligned}$$

In terms of this basis the grading operator has the following expression:

PROPOSITION 4.2.

$$\varepsilon = (i_1 - e_1)(i_1 + e_1) \dots (i_n - e_n)(i_n + e_n), \text{ in particular } \varepsilon \in C^{2n}.$$

For a maximal isotropic subspace  $\mathcal{I}$ , an  $\mathcal{I}$ -polarisation  $V = \mathcal{I} \oplus \mathcal{E}$ , and bases  $\{i_a\}, \{e_a\}$  of  $\mathcal{I}$  and  $\mathcal{E}$  as above, since  $g(i_a \pm e_a, i_b \pm e_b) = \pm\delta_{ab}$  and  $g(i_a + e_a, i_b - e_b) = 0$ , the set  $\{i_a \pm e_a : 1 \leq a \leq n\}$  is an orthonormal basis of  $V$  as used in §1. For  $1 \leq a \leq n$ , set

$$E_a = e_a + i_a, E_{\bar{a}} = e_a - i_a.$$

Then  $\{E_1, E_{\bar{1}}, \dots, E_n, E_{\bar{n}}\}$  is an orthonormal basis with  $g(E_a, E_a) = 1$  and  $g(E_{\bar{a}}, E_{\bar{a}}) = -1$ . We order this basis by

$$1 < \bar{1} < 2 < \bar{2} < \dots < n < \bar{n}.$$

The use of  $e_i$  in two different ways in a basis, as in §1 and here in §5, hopefully does not lead to confusion.

### 5. Pure spinors

É. Cartan found a beautiful relationship between the maximal isotropic subspaces of  $V$  and a distinguished subset of spinors in  $S$ . More precisely, he showed that to each maximal isotropic subspace  $\mathcal{I}$  there is a unique (up to scalar multiplication) nonzero element  $\mathbf{v}_{\mathcal{I}} \in S$  such that

$$i_a \cdot \mathbf{v}_{\mathcal{I}} = 0 \quad \forall 1 \leq a \leq n.$$

In the language of É. Cartan  $\mathbf{v}_{\mathcal{I}}$  is called a pure spinor, and in the language of physics a vacuum.

Take an  $\mathcal{I}$ -polarisation of  $(V, g)$ , so that  $V = \mathcal{I} \oplus \mathcal{E}$ , and with bases as in §4. Then the grading operator,  $\varepsilon \in C$ , defined there determines the spaces of half-spinors

$$\varepsilon \cdot \psi_{\pm} = \pm \psi_{\pm} \quad \forall \psi_{\pm} \in S_{\pm}.$$

Repeated use of (18) shows that  $\mathbf{v}_{\mathcal{I}}$  is in the half-spinor space  $S_+$  associated to  $\varepsilon$ .

A basis for  $S$  is obtained by applying successive “creation operators”  $e_a$  to the “vacuum”  $\mathbf{v}_{\mathcal{I}}$ , so that  $S_+$  is then the space of “even particle states”,  $S_-$  the space of “odd particle states”:

$$\begin{aligned} S_+ &= \langle \mathbf{v}_{\mathcal{I}}, e_{i_1} e_{i_2} \dots e_{i_k} \cdot \mathbf{v}_{\mathcal{I}} : 1 \leq i_1 < i_2 < \dots < i_k \leq n, k \text{ is even} \rangle \\ S_- &= \langle e_{i_1} e_{i_2} \dots e_{i_k} \cdot \mathbf{v}_{\mathcal{I}} : 1 \leq i_1 < i_2 < \dots < i_k \leq n, k \text{ is odd} \rangle. \end{aligned}$$

(19)

Notice that  $e_{i_1} e_{i_2} \dots e_{i_k} \cdot \mathbf{v}_{\mathcal{I}}$  is a pure spinor with

$$\text{Ann}_V(e_{i_1} e_{i_2} \dots e_{i_k} \cdot \mathbf{v}_{\mathcal{I}}) = \langle e_p, i_q : p \in \{i_1, \dots, i_k\}, q \notin \{i_1, \dots, i_k\} \rangle,$$

a maximal isotropic subspace.

REMARK 5.1. *The basis of pure spinors is an effective computational tool in spinor algebra. Essentially, its use converts computations to combinatorial statements about the parameters for pure spinors.*

An easy example of this is the matrix of the spinor norm  $B$ . It is helpful to compare this to Proposition 2.6.

PROPOSITION 5.2. *If  $K = \{k_1, \dots, k_p\}$  and  $J = \{j_1, \dots, j_q\}$  are ordered subsets of  $\{1, 2, \dots, n\}$  we set  $e_K = e_{k_1} \dots e_{k_p}$ ,  $e_J = e_{j_1} \dots e_{j_q}$  and  $e_{\emptyset} = 1$ . Then*

$$(20) \quad (a) \quad B(e_K \cdot \mathbf{v}_{\mathcal{I}}, e_J \cdot \mathbf{v}_{\mathcal{I}}) \neq 0 \quad \Rightarrow \quad K \cap J = \emptyset \text{ and } K \cup J = \{1, 2, \dots, n\} \text{ i.e. } J = K^c,$$

$$(21) \quad (b) \quad B(\mathbf{v}_{\mathcal{I}}, e_1 e_2 e_3 \dots e_n \cdot \mathbf{v}_{\mathcal{I}}) \neq 0.$$

For another easy example, take an  $\mathcal{I}$ -polarisation  $V = \mathcal{I} \oplus \mathcal{E}$  and define the following element of the Lie algebra  $C^2$ :

$$(22) \quad H = \frac{1}{2} \sum_1^n (e_a i_a - i_a e_a).$$

This is independent of the choice of bases  $\{i_a\}, \{e_a\}$  above. It is an element of  $C^2$  since the set  $\{e_a e_b, i_a i_b : 1 \leq a < b \leq n\} \cup \{e_a i_b - i_b e_a : 1 \leq a, b \leq n\}$  is a basis of  $C^2$ .

PROPOSITION 5.3. *For  $1 \leq a \leq n$  and  $1 \leq i_1 < \dots < i_k \leq n$ ,*

$$(a) \quad [H, e_a] = e_a; \quad (b) \quad [H, i_a] = -i_a; \quad (c) \quad H e_{i_1} e_{i_2} \dots e_{i_k} \cdot \mathbf{v}_{\mathcal{I}} = \left(k - \frac{n}{2}\right) e_{i_1} e_{i_2} \dots e_{i_k} \cdot \mathbf{v}_{\mathcal{I}}.$$

$H$  is a useful substitute for what is called the number operator in physics,  $N = \sum_{a=1}^{a=n} e_a i_a$ , which is not in  $C^2$ .

## 6. Computations of $\tilde{L}_2$

### 6.1. The operator $\tilde{L}_2$ .

Recall the definition of the operator  $L_2$ :

$$L_2(\phi, \psi) = \frac{1}{2^n} \sum_{i < j} g(e_i, e_i) g(e_j, e_j) B(\phi, e_j e_i \cdot \psi) e_i e_j.$$

Using the basis  $\{E_a, E_{\bar{b}}\}$  from §4 we obtain an alternative expression for  $\tilde{L}_2 = 2^{n-1} L_2$  which is more convenient for the computation of the matrix of  $\tilde{L}_2$ .

**PROPOSITION 6.1.** *In terms of the basis of  $C^2$ :  $\{e_a e_b, i_a i_b : 1 \leq a < b \leq n\} \cup \{e_a i_b - i_b e_a : 1 \leq a, b \leq n\}$  we have*

$$(23) \quad \begin{aligned} \tilde{L}_2(\psi_1, \psi_2) &= \sum_{a \neq b} B(e_a e_b \cdot \psi_1, \psi_2) i_a i_b + \sum_{a \neq b} B(i_a i_b \cdot \psi_1, \psi_2) e_a e_b + \sum_{a \neq b} B(e_a i_b \cdot \psi_1, \psi_2) (i_a e_b - e_b i_a) \\ &+ \frac{1}{2} \sum_a B((e_a i_a - i_a e_a) \cdot \psi_1, \psi_2) (i_a e_a - e_a i_a). \end{aligned}$$

*Proof.* In a basis  $\{E_a, E_{\bar{b}}\}$

$$\tilde{L}_2(\psi_1, \psi_2) = \frac{1}{2} \sum_{i < j} g(E_i, E_i) g(E_j, E_j) B(\psi_1, E_j E_i \cdot \psi_2) E_i E_j$$

and using the ordered basis this sum can be split into two subsums:

$$(24) \quad \frac{1}{2} \sum_i g(E_i, E_i) g(E_{\bar{i}}, E_{\bar{i}}) B(\psi_1, E_{\bar{i}} E_i \cdot \psi_2) E_i E_{\bar{i}}$$

and

$$(25) \quad \begin{aligned} &\frac{1}{2} \sum_{i < j} \left( g(E_i, E_i) g(E_j, E_j) B(\psi_1, E_j E_i \cdot \psi_2) E_i E_j \right. \\ &\quad + g(E_i, E_i) g(E_{\bar{j}}, E_{\bar{j}}) B(\psi_1, E_{\bar{j}} E_i \cdot \psi_2) E_i E_{\bar{j}} \\ &\quad + g(E_{\bar{i}}, E_{\bar{i}}) g(E_j, E_j) B(\psi_1, E_j E_{\bar{i}} \cdot \psi_2) E_{\bar{i}} E_j \\ &\quad \left. + g(E_{\bar{i}}, E_{\bar{i}}) g(E_{\bar{j}}, E_{\bar{j}}) B(\psi_1, E_{\bar{j}} E_{\bar{i}} \cdot \psi_2) E_{\bar{i}} E_{\bar{j}} \right). \end{aligned}$$

Since

$$E_a E_{\bar{a}} = -e_a i_a + i_a e_a, \quad E_{\bar{a}} E_a = -i_a e_a + e_a i_a$$

the sum (24) reduces to

$$(26) \quad -\frac{1}{2} \sum_a B(\psi_1, (e_a i_a - i_a e_a) \cdot \psi_2) (i_a e_a - e_a i_a) = \frac{1}{2} \sum_a B((e_a i_a - i_a e_a) \cdot \psi_1, \psi_2) (i_a e_a - e_a i_a)$$

and this is the last term in (23).

To simplify (25) we first observe that

$$(27) \quad \begin{aligned} E_a E_b &= e_{ab} + e_a i_b + i_a e_b + i_a i_b, & E_a E_{\bar{b}} &= e_a e_b - e_a i_b + i_a e_b - i_a i_b, \\ E_{\bar{a}} E_b &= e_{ab} + e_a i_b - i_a e_b - i_a i_b, & E_{\bar{a}} E_{\bar{b}} &= e_a e_b - e_a i_b - i_a e_b + i_a i_b, \end{aligned}$$

Hence for fixed  $a < b$ , the coefficient of  $e_{ab}$  in (25) is

$$\frac{1}{2} (B(\psi_1, E_b E_a \cdot \psi_2) - B(\psi_1, E_{\bar{b}} E_a \cdot \psi_2) - B(\psi_1, E_b E_{\bar{a}} \cdot \psi_2) + B(\psi_1, E_{\bar{b}} E_{\bar{a}} \cdot \psi_2))$$

which can be written

$$\frac{1}{2} B(\psi_1, (E_b - E_{\bar{b}})(E_a - E_{\bar{a}}) \cdot \psi_2),$$

that is

$$2B(\psi_1, i_b i_a \cdot \psi_2) = 2B(i_a i_b \cdot \psi_1, \psi_2).$$

Summing over all  $a < b$  we get

$$2 \sum_{1 \leq a < b \leq n} B(i_a i_b \cdot \psi_1, \psi_2) e_{ab}$$

which is the second term in (23). Similarly, looking at the coefficients of  $i_a i_b$  and  $i_a e_b - e_b i_a$  in (25) we get the first and third terms of (23). QED

## 6.2. The matrix of $\tilde{L}_2$ .

Next we use the basis of pure spinors to simplify the expression. Let  $\mathbf{v}_{\mathcal{I}}$  be a pure spinor defined by the maximal isotropic subspace  $\mathcal{I}$ . Recall from §5 the basis of pure spinors and take  $\psi_1 = e_I \cdot \mathbf{v}_{\mathcal{I}}$  and  $\psi_2 = e_J \cdot \mathbf{v}_{\mathcal{I}}$ . Most terms in the formula for  $\tilde{L}_2(e_I \cdot \mathbf{v}_{\mathcal{I}}, e_J \cdot \mathbf{v}_{\mathcal{I}})$  vanish by Proposition 5.2:

- (a)  $B(e_a e_b e_I \cdot \mathbf{v}_{\mathcal{I}}, e_J \cdot \mathbf{v}_{\mathcal{I}}) \neq 0 \iff I \cap J = \emptyset$  and  $I^c \cap J^c = \{a, b\}$ .
- (b)  $B(i_a i_b e_I \cdot \mathbf{v}_{\mathcal{I}}, e_J \cdot \mathbf{v}_{\mathcal{I}}) \neq 0 \iff I \cap J = \{a, b\}$  and  $I^c \cap J^c = \emptyset$ .
- (c)  $B(e_a i_b e_I \cdot \mathbf{v}_{\mathcal{I}}, e_J \cdot \mathbf{v}_{\mathcal{I}}) \neq 0$  and  $a \neq b \iff I \cap J = \{b\}$  and  $I^c \cap J^c = \{a\}$ .
- (d)  $B((e_a i_a - i_a e_a) e_I \cdot \mathbf{v}_{\mathcal{I}}, e_J \cdot \mathbf{v}_{\mathcal{I}}) \neq 0 \iff I \cap J = \emptyset$  and  $I^c \cap J^c = \emptyset$ .

REMARK 6.2. Note that  $(I \cap J) \cup (I^c \cap J^c) = (I \Delta J)^c$  where  $I \Delta J$  denotes the symmetric difference of the sets  $I$  and  $J$ .

From this we can calculate the matrix of  $\tilde{L}_2(e_I \cdot \mathbf{v}_{\mathcal{I}}, e_J \cdot \mathbf{v}_{\mathcal{I}})$  in the basis of particle states  $\{e_K \cdot \mathbf{v}_{\mathcal{I}}\}$ .

PROPOSITION 6.3.  $\tilde{L}_2(e_I \cdot \mathbf{v}_{\mathcal{I}}, e_J \cdot \mathbf{v}_{\mathcal{I}}) \neq 0$  iff  $I$  and  $J$  satisfy one of (a), (b), (c), (d) above. In those cases in terms of the basis  $\{e_K \cdot \mathbf{v}_{\mathcal{I}}\}$  we have

(a) If  $I \cap J = \emptyset$  and  $I^c \cap J^c = \{a, b\}$  then

$$\tilde{L}_2(e_I \cdot \mathbf{v}_{\mathcal{I}}, e_J \cdot \mathbf{v}_{\mathcal{I}}) e_K \cdot \mathbf{v}_{\mathcal{I}} = 2B(e_a e_b e_I \cdot \mathbf{v}_{\mathcal{I}}, e_J \cdot \mathbf{v}_{\mathcal{I}}) i_a i_b e_K \cdot \mathbf{v}_{\mathcal{I}}.$$

(b) If  $I \cap J = \{a, b\}$  and  $I^c \cap J^c = \emptyset$  then

$$\tilde{L}_2(e_I \cdot \mathbf{v}_{\mathcal{I}}, e_J \cdot \mathbf{v}_{\mathcal{I}}) e_K \cdot \mathbf{v}_{\mathcal{I}} = 2B(i_a i_b e_I \cdot \mathbf{v}_{\mathcal{I}}, e_J \cdot \mathbf{v}_{\mathcal{I}}) e_a e_b e_K \cdot \mathbf{v}_{\mathcal{I}}.$$

(c) If  $I \cap J = \{b\}$  and  $I^c \cap J^c = \{a\}$  then

$$\tilde{L}_2(e_I \cdot \mathbf{v}_{\mathcal{I}}, e_J \cdot \mathbf{v}_{\mathcal{I}}) e_K \cdot \mathbf{v}_{\mathcal{I}} = 2B(e_a i_b e_I \cdot \mathbf{v}_{\mathcal{I}}, e_J \cdot \mathbf{v}_{\mathcal{I}}) i_a e_b e_K \cdot \mathbf{v}_{\mathcal{I}}.$$

(d) If  $I \cap J = \emptyset$  and  $I^c \cap J^c = \emptyset$  then

$$\tilde{L}_2(e_I \cdot \mathbf{v}_{\mathcal{I}}, e_J \cdot \mathbf{v}_{\mathcal{I}}) e_K \cdot \mathbf{v}_{\mathcal{I}} = \frac{1}{2} B(e_I \cdot \mathbf{v}_{\mathcal{I}}, e_J \cdot \mathbf{v}_{\mathcal{I}}) (n - 2|I \cap K| - 2|I^c \cap K^c|) e_K \cdot \mathbf{v}_{\mathcal{I}}.$$

REMARK 6.4. Given two pure spinors  $\psi, \psi'$  such that  $\tilde{L}_2(\psi, \psi') \neq 0$ , the intersection properties of the associated maximal isotropic subspaces  $\text{Ann}_V(\psi), \text{Ann}_V(\psi')$  determine  $\tilde{L}_2(\psi, \psi')$  up to a constant:

- if  $B(\psi, \psi') = 0$  then  $\dim(\text{Ann}_V(\psi) \cap \text{Ann}_V(\psi')) = 2$  and  $\tilde{L}_2(\psi, \psi')$  is proportional to  $Q(\omega)$  for any nonzero  $\omega \in \Lambda^2(\text{Ann}_V(\psi) \cap \text{Ann}_V(\psi'))$  (see [EC]).
- if  $B(\psi, \psi') \neq 0$  then  $\dim(\text{Ann}_V(\psi) \cap \text{Ann}_V(\psi')) = 0$  and  $\tilde{L}_2(\psi, \psi')$  is proportional to the operator  $H$  associated to the polarisation  $V = \text{Ann}_V(\psi) \oplus \text{Ann}_V(\psi')$  (see (22)).

This is a weaker but ‘geometric’ version of Proposition 6.3. For example, if  $\psi = e_I \cdot \mathbf{v}_{\mathcal{I}}$  and  $\psi' = e_J \cdot \mathbf{v}_{\mathcal{I}}$  are as in Proposition 6.3 (a), we have

$$B(\psi, \psi') = 0, \quad \text{Ann}_V(\psi) \cap \text{Ann}_V(\psi') = \langle i_a, i_b \rangle$$

and this result implies that  $\tilde{L}_2(\psi, \psi')$  is proportional to  $Q(i_a \wedge i_b) = i_a i_b$ , whereas more importantly Proposition 6.3 also gives the constant of proportionality.

There is still some simplification possible in the parameters  $I, J, K$ . Looking at the above more closely we see that  $\tilde{L}_2(e_I \cdot \mathbf{v}_I, e_J \cdot \mathbf{v}_I) e_K \cdot \mathbf{v}_I$  is ‘symmetric’ in  $I, J, K$  in the following sense.

COROLLARY 6.5.

(i) If either  $I \cap J \cap K \neq \emptyset$  or  $I^c \cap J^c \cap K^c \neq \emptyset$  then

$$\tilde{L}_2(e_I \cdot \mathbf{v}_I, e_J \cdot \mathbf{v}_I) e_K \cdot \mathbf{v}_I = 0.$$

(ii) If  $I \cap J \cap K = \emptyset$  and  $I^c \cap J^c \cap K^c = \emptyset$  then

$$\tilde{L}_2(e_I \cdot \mathbf{v}_I, e_J \cdot \mathbf{v}_I) e_K \cdot \mathbf{v}_I \text{ is proportional to } e_{(I \cap J) \cup (J \cap K) \cup (K \cap I)} \cdot \mathbf{v}_I.$$

Now three subsets  $I, J, K$  of  $\{1, 2, \dots, n\}$  satisfying the conditions

$$(28) \quad I \cap J \cap K = I^c \cap J^c \cap K^c = \emptyset$$

define a partition of  $\{1, 2, \dots, n\}$  into six disjoint subsets:

$$\{1, 2, \dots, n\} = (I \cap J) \cup (J \cap K) \cup (K \cap I) \cup (I^c \cap J^c) \cup (J^c \cap K^c) \cup (K^c \cap I^c)$$

and in terms of this partition

$$(29) \quad \begin{aligned} I &= (I \cap J) \cup (K \cap I) \cup (J^c \cap K^c) \\ J &= (J \cap K) \cup (I \cap J) \cup (K^c \cap I^c) \\ K &= (K \cap I) \cup (J \cap K) \cup (I^c \cap J^c). \end{aligned}$$

The simplest example of three subsets satisfying the conditions (28) is given by three pairwise disjoint subsets of  $\{1, 2, \dots, n\}$  whose union is  $\{1, 2, \dots, n\}$ . In fact this is the general case.

PROPOSITION 6.6. *Let  $I, J, K$  be three oriented subsets of  $\{1, 2, \dots, n\}$ . Suppose that  $I \cap J \cap K = \emptyset$  and  $I^c \cap J^c \cap K^c = \emptyset$ . Then there is a polarisation  $V = \mathcal{T}' \oplus \mathcal{E}'$  and oriented subsets  $I', J', K'$  of  $\{1, 2, \dots, n\}$  such that*

$$(i) \quad e_I \cdot \mathbf{v}_I = e'_{I'} \cdot \mathbf{v}_{\mathcal{T}'}, \quad e_J \cdot \mathbf{v}_I = e'_{J'} \cdot \mathbf{v}_{\mathcal{T}'}, \quad \text{and} \quad e_K \cdot \mathbf{v}_I = e'_{K'} \cdot \mathbf{v}_{\mathcal{T}'}$$

$$(ii) \quad I' \cap J' = \emptyset, \quad K' = I^c \cap J^c, \quad \text{and} \quad I' \cup J' \cup K' = \{1, 2, \dots, n\}.$$

(iii) *If  $I, J, K$  are of the same parity then  $I', J', K'$  are of the same parity, i.e.  $S_+$  or  $S_-$ .*

*Proof.* Set

$$\mathcal{T}' = \text{Vect} \langle e_a, i_b : a \in (I^c \cap J^c) \cup (J^c \cap K^c) \cup (K^c \cap I^c), b \in (I \cap J) \cup (J \cap K) \cup (K \cap I) \rangle,$$

$$\mathcal{E}' = \text{Vect} \langle e_a, i_b : a \in (I \cap J) \cup (J \cap K) \cup (K \cap I), b \in (I^c \cap J^c) \cup (J^c \cap K^c) \cup (K^c \cap I^c) \rangle.$$

Then it is clear that  $V = \mathcal{T}' \oplus \mathcal{E}'$  is a polarisation, and that

$$\mathbf{v}_{\mathcal{T}'} := e_{(I \cap J) \cup (J \cap K) \cup (K \cap I)} \cdot \mathbf{v}_I$$

is a pure spinor defined by  $\mathcal{T}'$ . It is equally clear that if

$$I' = (J^c \cap K^c) \cup (J \cap K),$$

$$J' = (K^c \cap I^c) \cup (K \cap I),$$

$$K' = (I^c \cap J^c) \cup (I \cap J),$$

then  $I' \cap J' = J' \cap K' = K' \cap I' = \emptyset$  and  $I' \cup J' \cup K' = \{1, 2, \dots, n\}$ .

Define

$$e'_a = \begin{cases} e_a & \text{if } a \in (I^c \cap J^c) \cup (J^c \cap K^c) \cup (K^c \cap I^c), \\ i_a & \text{if } a \in (I \cap J) \cup (J \cap K) \cup (K \cap I) \end{cases}$$

and

$$i'_a = \begin{cases} e_a & \text{if } a \in (I \cap J) \cup (J \cap K) \cup (K \cap I), \\ i_a & \text{if } a \in (I^c \cap J^c) \cup (J^c \cap K^c) \cup (K^c \cap I^c). \end{cases}$$

Then  $\{e'_a, i'_a : 1 \leq a \leq n\}$  satisfy the Clifford relations

$$(30) \quad \begin{aligned} i'_a i'_b + i'_b i'_a &= 0 \\ e'_a e'_b + e'_b e'_a &= 0 \\ i'_a e'_b + e'_b i'_a &= \delta_{ab} \end{aligned}$$

and up to signs,

$$e'_{I'} \cdot \mathbf{v}_{I'} = e^{J^c \cap K^c} i_{J \cap K} e^{(I \cap J) \cup (J \cap K) \cup (K \cap I)} \cdot \mathbf{v}_{I'} = e^{J^c \cap K^c} e^{(I \cap J) \cup (K \cap I)} \cdot \mathbf{v}_{I'}.$$

Since  $(J^c \cap K^c) \cup (I \cap J) \cup (K \cap I) = I$  this means (up to signs)

$$e'_{I'} \cdot \mathbf{v}_{I'} = e_I \cdot \mathbf{v}_{I'}.$$

Similarly  $e'_{J'} \cdot \mathbf{v}_{I'} = e_J \cdot \mathbf{v}_{I'}$  and  $e'_{K'} \cdot \mathbf{v}_{I'} = e_K \cdot \mathbf{v}_{I'}$ . This proves (i). Parts (ii) and (iii) follow immediately. QED

**PROPOSITION 6.7.** *Let  $I, J, K$  be three oriented subsets of  $\{1, 2, \dots, n\}$  that are pairwise disjoint,  $I \cap J = J \cap K = K \cap I = \emptyset$ , and  $I^c \cap J^c \cap K^c = \emptyset$ . Then*

(i)  $|I| + |J| + |K| = n$ .

(ii) If  $|K|$  is not equal to 0 or 2, then  $\tilde{L}_2(e_I \cdot \mathbf{v}_{I'}, e_J \cdot \mathbf{v}_{I'}) e_K \cdot \mathbf{v}_{I'} = 0$ .

(iii) If  $|K| = 0$  then

$$\tilde{L}_2(e_I \cdot \mathbf{v}_{I'}, e_J \cdot \mathbf{v}_{I'}) e_K \cdot \mathbf{v}_{I'} = B(e_I \cdot \mathbf{v}_{I'}, e_J \cdot \mathbf{v}_{I'}) (|I| - \frac{n}{2}) \mathbf{v}_{I'}.$$

(iv) If  $|K| = 2$  and  $K = \{\vec{ba}\}$  then

$$\tilde{L}_2(e_I \cdot \mathbf{v}_{I'}, e_J \cdot \mathbf{v}_{I'}) e_K \cdot \mathbf{v}_{I'} = 2B(e_{ab} e_I \cdot \mathbf{v}_{I'}, e_J \cdot \mathbf{v}_{I'}) \mathbf{v}_{I'}.$$

*Proof.* Part (i) is clear since as we observed before  $I, J, K$  define a partition of  $\{1, \dots, n\}$ . Parts (ii), (iii) and (iv) follow from Proposition 6.3 since  $I \cap J = \emptyset$  and  $K = I^c \cap J^c$ . QED

In §8 we will need an expression for  $L_{2n}$ . Recall from (9)

$$L_{2n}(\psi_1, \psi_2) = \frac{1}{2^n} g(e_1, e_1) \dots g(e_{2n}, e_{2n}) B(\psi_1, e_{2n} \dots e_1 \cdot \psi_2) e_1 \dots e_{2n}.$$

A simplification of this in terms of pure spinors is rather straightforward.

**PROPOSITION 6.8.**

$$L_{2n}(\psi_1, \psi_2) = \frac{1}{2^n} B(\psi_1, \varepsilon \cdot \psi_2) \varepsilon.$$

### 6.3. Résumé.

Let

- $(V, g)$  be a  $2n$ -dimensional vector space with a hyperbolic metric  $g$ ;
- $S$  be a space of spinors (i.e., we identify  $C(V, g)$  with  $\text{End}(S)$  for some  $2^n$ -dimensional vector space  $S$ ;
- $B : S \times S \rightarrow k$  be a spinor norm (Cartan form).
- $V = \mathcal{I} \oplus \mathcal{E}$  be a polarisation of  $V$ ;
- $\mathbf{v}_{\mathcal{I}}$  be a pure spinor associated to  $\mathcal{I}$  (i.e.,  $v \cdot \mathbf{v}_{\mathcal{I}} = 0$  for all  $v \in \mathcal{I}$ );

- $i_1, \dots, i_n$  and  $e_1, \dots, e_n$  be bases of  $\mathcal{I}$  and  $\mathcal{E}$  respectively such that

$$(31) \quad \begin{aligned} i_a i_b + i_b i_a &= 0 \\ e_a e_b + e_b e_a &= 0 \\ i_a e_b + e_b i_a &= \delta_{ab}; \end{aligned}$$

- $\tilde{L}_2 : S \times S \rightarrow C^2(V, g)$  be the normalised projection operator:

$$(32) \quad \begin{aligned} \tilde{L}_2(\psi_1, \psi_2) &= \sum_{a \neq b} B(e_a e_b \psi_1, \psi_2) i_a i_b + \sum_{a \neq b} B(i_a i_b \psi_1, \psi_2) e_a e_b + \sum_{a \neq b} B(e_a i_b \psi_1, \psi_2) (i_a e_b - e_b i_a) \\ &+ \frac{1}{2} \sum_a B((e_a i_a - i_a e_a) \cdot \psi_1, \psi_2) (i_a e_a - e_a i_a). \end{aligned}$$

Then for all oriented subsets  $I, J, K$  of  $\{1, \dots, n\}$  we have shown that:

- $\tilde{L}_2(e_I \cdot \mathbf{v}_I, e_J \cdot \mathbf{v}_I) e_K \cdot \mathbf{v}_I = 0$  unless  $I \cap J \cap K = \emptyset$  and  $I^c \cap J^c \cap K^c = \emptyset$ .
- If  $I \cap J \cap K = \emptyset$  and  $I^c \cap J^c \cap K^c = \emptyset$  there is a polarisation  $V = \mathcal{I}' \oplus \mathcal{E}'$  and oriented subsets  $I', J', K'$  of  $\{1, \dots, n\}$  such that

$$(33) \quad \begin{aligned} (i) \quad e_I \cdot \mathbf{v}_I &= e_{I'} \cdot \mathbf{v}_{I'}, \quad e_J \cdot \mathbf{v}_I = e_{J'} \cdot \mathbf{v}_{I'}, \quad e_K \cdot \mathbf{v}_I = e_{K'} \cdot \mathbf{v}_{I'}. \\ (ii) \quad I' \cap J' &= \emptyset \text{ and } K' = I'^c \cap J'^c. \end{aligned}$$

PROPOSITION 6.9.

Let  $I, J, K$  be three oriented subsets of  $\{1, 2, \dots, n\}$  satisfying  $I \cap J = \emptyset$  and  $K = I^c \cap J^c$ . Then

- (i)  $|I| + |J| + |K| = n$ .
- (ii) If  $|K|$  is not equal to 0 or 2, then  $\tilde{L}_2(e_I \cdot \mathbf{v}_I, e_J \cdot \mathbf{v}_I) e_K \cdot \mathbf{v}_I = 0$ .
- (iii) If  $|K| = 0$  then

$$\tilde{L}_2(e_I \cdot \mathbf{v}_I, e_J \cdot \mathbf{v}_I) e_K \cdot \mathbf{v}_I = B(e_I \cdot \mathbf{v}_I, e_J \cdot \mathbf{v}_I) (|I| - \frac{n}{2}) \mathbf{v}_I.$$

- (iv) If  $|K| = 2$  and  $K = \{\vec{ba}\}$  then

$$\tilde{L}_2(e_I \cdot \mathbf{v}_I, e_J \cdot \mathbf{v}_I) e_K \cdot \mathbf{v}_I = 2B(e_{ab} e_I \cdot \mathbf{v}_I, e_J \cdot \mathbf{v}_I) \mathbf{v}_I.$$

- In all cases  $\tilde{L}_2(e_I \cdot \mathbf{v}_I, e_J \cdot \mathbf{v}_I) e_K \cdot \mathbf{v}_I$  is proportional to  $e_{(I \cap J)(J \cap K)(K \cap I)} \cdot \mathbf{v}_I$ .

## 7. Potential Lie algebra structures

As Cartan's operator  $\tilde{L}_2$  (or  $\tilde{L}_2 + L_{2n}$ ) maps from  $S_i \times S_i$  (or  $S \times S$ ) to  $C^2$  (or  $C^2 \oplus C^{2n}$ ) it provides a natural candidate for a type of "curvature" operator on  $S_i$  (or  $S$ ). Cartan's calculation of curvature operators for symmetric spaces then motivates possible Lie triple system structures.

If  $n \equiv 0 \pmod{4}$ , we can now define ( $i = 1, 2$ ) a unique antisymmetric map

$$(C^2 \oplus S_i) \times (C^2 \oplus S_i) \rightarrow C^2 \oplus S_i$$

such that

$$(34) \quad [A, B] = AB - BA \quad \text{if } A, B \in C^2;$$

$$(35) \quad [A, \psi] = A \cdot \psi \quad \text{if } A \in C^2, \psi \in S_i;$$

$$(36) \quad [\phi, \psi] = \tilde{L}_2(\phi, \psi) \quad \text{if } \phi, \psi \in S_i.$$

Similarly, If  $n \equiv 1 \pmod{4}$ , we can define a unique antisymmetric map

$$(C^2 \oplus C^{2n} \oplus S) \times (C^2 \oplus C^{2n} \oplus S) \rightarrow C^2 \oplus C^{2n} \oplus S$$

such that

$$(37) \quad [A, B] = AB - BA \quad \text{if } A, B \in C^2 \oplus C^{2n}$$

$$(38) \quad [A, \psi] = A \cdot \psi \quad \text{if } A \in C^2 \oplus C^{2n}, \psi \in S$$

$$(39) \quad [\phi, \psi] = \tilde{L}_2(\phi, \psi) + L_{2n}(\phi, \psi) \quad \text{if } \phi, \psi \in S.$$

The question is: do these brackets define Lie algebra structures on  $C^2 \oplus S_i$  ( $n = 0 \pmod{4}$ ) and  $C^2 \oplus C^{2n} \oplus S$  ( $n = 1 \pmod{4}$ ) respectively? Since  $C^2, C^2 \oplus C^{2n}$  are Lie algebras, since  $S_i, S$  are representations and since  $\tilde{L}_2 : S_i \times S_i \rightarrow C^2$  and  $\tilde{L}_2 + L_{2n} : S \times S \rightarrow C^2 \oplus C^{2n}$  are equivariant maps, this will be the case if and only if the following Jacobi identities are satisfied:

$$\tilde{L}_2(\psi_1, \psi_2) \cdot \psi_3 + \tilde{L}_2(\psi_2, \psi_3) \cdot \psi_1 + \tilde{L}_2(\psi_3, \psi_1) \cdot \psi_2 = 0 \quad (n = 0 \pmod{4})$$

and respectively if for some  $a, b \in k^*$

$$(a\tilde{L}_2 + bL_{2n})(\psi_1, \psi_2) \cdot \psi_3 + (a\tilde{L}_2 + bL_{2n})(\psi_2, \psi_3) \cdot \psi_1 + (a\tilde{L}_2 + bL_{2n})(\psi_3, \psi_1) \cdot \psi_2 = 0 \quad (n = 1 \pmod{4}).$$

Note that if  $n = 8$ ,  $C^2 \oplus S_i$  is of dimension 248 and that if  $n = 5$ ,  $C^2 \oplus C^{2n} \oplus S$  is of dimension 78.

## 8. Spinor constructions of exceptional Lie algebras

### 8.1. Construction of split $e_8$ .

Let  $(V, g)$  be a sixteen-dimensional vector space with a nondegenerate hyperbolic symmetric bilinear form  $g$ . Choose a 256-dimensional space of spinors  $S = S_1 \oplus S_2$  and an isomorphism  $C(V, g) \cong \text{End}(S)$ . Since  $n = 8 = 0 \pmod{4}$ ,  $B : S \times S \rightarrow k$  is even symmetric and  $\tilde{L}_2 : S \times S \rightarrow C^2(V, g)$  is even antisymmetric (Proposition 2.6 and Proposition 3.1).

On the 248-dimensional vector space

$$E = C^2(V, g) \oplus S_1$$

following the procedure above we define  $[\ , \ ] : E \otimes E \rightarrow E$  to be the unique antisymmetric bilinear map such that:

$$(40) \quad [A, B] = AB - BA \quad \text{if } A, B \in C^2;$$

$$(41) \quad [A, \psi] = A \cdot \psi \quad \text{if } A \in C^2, \psi \in S_1;$$

$$(42) \quad [\phi, \psi] = \tilde{L}_2(\phi, \psi) \quad \text{if } \phi, \psi \in S_1.$$

where  $\tilde{L}_2$  is given by (23). The bracket  $[\ , \ ]$  defines a Lie algebra structure on  $E$  iff

$$(43) \quad \tilde{L}_2(\psi_1, \psi_2) \cdot \psi_3 + \tilde{L}_2(\psi_2, \psi_3) \cdot \psi_1 + \tilde{L}_2(\psi_3, \psi_1) \cdot \psi_2 = 0 \quad \forall \psi_1, \psi_2, \psi_3 \in S_1.$$

Choose a polarisation  $V = \mathcal{I} \oplus \mathcal{E}$  of  $V$  such that a pure spinor  $\mathbf{v}_{\mathcal{I}}$  corresponding to  $\mathcal{I}$  is in  $S_1$  and bases  $i_1, \dots, i_n$  and  $e_1, \dots, e_n$  of respectively  $\mathcal{I}$  and  $\mathcal{E}$  such that

$$(44) \quad \begin{aligned} i_a i_b + i_b i_a &= 0 \\ e_a e_b + e_b e_a &= 0 \\ i_a e_b + e_b i_a &= \delta_{ab}. \end{aligned}$$

Then

$$\{e_I \cdot \mathbf{v}_{\mathcal{I}} : |I| \text{ is even or } \emptyset\}$$

is a basis of  $S_1$  and to prove the Jacobi identity (43) it is sufficient to prove that

$$(45) \quad \tilde{L}_2(e_I \cdot \mathbf{v}_{\mathcal{I}}, e_J \cdot \mathbf{v}_{\mathcal{I}}) \cdot e_K \cdot \mathbf{v}_{\mathcal{I}} + \tilde{L}_2(e_J \cdot \mathbf{v}_{\mathcal{I}}, e_K \cdot \mathbf{v}_{\mathcal{I}}) \cdot e_I \cdot \mathbf{v}_{\mathcal{I}} + \tilde{L}_2(e_K \cdot \mathbf{v}_{\mathcal{I}}, e_I \cdot \mathbf{v}_{\mathcal{I}}) \cdot e_J \cdot \mathbf{v}_{\mathcal{I}} = 0$$

for all even subsets  $I, J, K$  of  $\{1, 2, 3, 4, 5, 6, 7, 8\}$ .

By the résumé above, all terms in this equation vanish unless  $I \cap J \cap K = \emptyset$  and  $I^c \cap J^c \cap K^c = \emptyset$ . So in fact we need only prove (45) for even subsets of  $\{1, 2, 3, 4, 5, 6, 7, 8\}$  satisfying these two

conditions. But then, again by the résumé, by changing the polarisation if necessary, we can always assume that

$$I \cap J = \emptyset, \quad K = I^c \cap J^c$$

and then all terms in (45) vanish unless one of the sets  $I, J, K$  has 0 or 2 elements. So in the end, to prove that the bracket  $[ \ , \ ]$  defines a Lie algebra structure on  $E$  it remains to prove only that

$$(46) \quad \tilde{L}_2(e_I \cdot \mathbf{v}_I, e_J \cdot \mathbf{v}_I) \cdot e_K \cdot \mathbf{v}_I + \tilde{L}_2(e_J \cdot \mathbf{v}_I, e_K \cdot \mathbf{v}_I) \cdot e_I \cdot \mathbf{v}_I + \tilde{L}_2(e_K \cdot \mathbf{v}_I, e_I \cdot \mathbf{v}_I) \cdot e_J \cdot \mathbf{v}_I = 0$$

for those oriented subsets  $I, J, K$  of  $\{1, 2, 3, 4, 5, 6, 7, 8\}$  such that

- $|I|, |J|$  and  $|K|$  are even;
- $|I| + |J| + |K| = 8$ ;
- $I \cap J = J \cap K = K \cap I = \emptyset$ ;
- One of  $|I|, |J|$  or  $|K|$  is equal to 0 or 2.

Up to permutations of  $I, J$  and  $K$  the only possibilities are

- (i)  $|I| = 0, |J| = 0$  and  $|K| = 8$ ;
- (ii)  $|I| = 0, |J| = 2$  and  $|K| = 6$ ;
- (iii)  $|I| = 0, |J| = 4$  and  $|K| = 4$ ;
- (iv)  $|I| = 2, |J| = 2$  and  $|K| = 4$ .

Since  $|K| \neq 0$  and  $|K| \neq 2$  in all four cases we have

$$\tilde{L}_2(e_I \cdot \mathbf{v}_I, e_J \cdot \mathbf{v}_I) \cdot e_K \cdot \mathbf{v}_I = 0$$

and hence proving (46) reduces to proving that

$$(47) \quad \tilde{L}_2(e_J \cdot \mathbf{v}_I, e_K \cdot \mathbf{v}_I) \cdot e_I \cdot \mathbf{v}_I + \tilde{L}_2(e_K \cdot \mathbf{v}_I, e_I \cdot \mathbf{v}_I) \cdot e_J \cdot \mathbf{v}_I = 0$$

for all subsets  $I, J, K$  of  $\{1, 2, 3, 4, 5, 6, 7, 8\}$  satisfying one of (i), (ii), (iii) or (iv).

**Case (i):** We have  $I = J = \emptyset$  and  $K = \{1, 2, 3, 4, 5, 6, 7, 8\}$ . By Proposition 6.9 (iii) this means that

$$\tilde{L}_2(e_J \cdot \mathbf{v}_I, e_K \cdot \mathbf{v}_I) \cdot e_I \cdot \mathbf{v}_I = -4B(\mathbf{v}_I, e_K \cdot \mathbf{v}_I)\mathbf{v}_I$$

and

$$\tilde{L}_2(e_K \cdot \mathbf{v}_I, \mathbf{v}_I) \cdot e_J \cdot \mathbf{v}_I = 4B(e_K \cdot \mathbf{v}_I, \mathbf{v}_I)\mathbf{v}_I.$$

Hence

$$\tilde{L}_2(e_J \cdot \mathbf{v}_I, e_K \cdot \mathbf{v}_I) \cdot e_I \cdot \mathbf{v}_I + \tilde{L}_2(e_K \cdot \mathbf{v}_I, e_I \cdot \mathbf{v}_I) \cdot e_J \cdot \mathbf{v}_I = 0$$

since  $B$  is symmetric.

**Case (ii):** We have  $I = \emptyset$  and without loss of generality we can suppose  $J = \{1, 2\}$  and  $K = \{3, 4, 5, 6, 7, 8\}$ . By Proposition 6.9(iii) this means

$$\tilde{L}_2(e_J \cdot \mathbf{v}_I, e_K \cdot \mathbf{v}_I) \cdot e_I \cdot \mathbf{v}_I = -2B(e_{12}\mathbf{v}_I, e_{345678} \cdot \mathbf{v}_I)\mathbf{v}_I$$

and by Proposition 6.9(iv) that

$$\tilde{L}_2(e_K \cdot \mathbf{v}_I, e_I \cdot \mathbf{v}_I) \cdot e_J \cdot \mathbf{v}_I = 2B(e_{21} \cdot e_{345678} \cdot \mathbf{v}_I, \mathbf{v}_I)\mathbf{v}_I.$$

Hence from  $B(e_{12} \cdot \mathbf{v}_I, e_{345678} \cdot \mathbf{v}_I) = B(e_2 \cdot \mathbf{v}_I, e_1 e_{345678} \cdot \mathbf{v}_I) = B(\mathbf{v}_I, e_2 e_1 e_{345678} \cdot \mathbf{v}_I)$  and the fact that  $B$  is symmetric it follows that

$$\tilde{L}_2(e_J \cdot \mathbf{v}_I, e_K \cdot \mathbf{v}_I) \cdot e_I \cdot \mathbf{v}_I + \tilde{L}_2(e_K \cdot \mathbf{v}_I, e_I \cdot \mathbf{v}_I) \cdot e_J \cdot \mathbf{v}_I = 0$$

**Case (iii):** We have  $I = \emptyset$  and without loss of generality we can suppose  $J = \{1, 2, 3, 4\}$  and  $K = \{5, 6, 7, 8\}$ . By Proposition 6.9 (iii) this means ( $4 = \frac{8}{2}!$ ) that

$$\tilde{L}_2(e_J \cdot \mathbf{v}_I, e_K \cdot \mathbf{v}_I) \cdot e_I \cdot \mathbf{v}_I = 0$$

and by Proposition 6.9 (ii) that

$$\tilde{L}_2(e_K \cdot \mathbf{v}_I, \mathbf{v}_I) \cdot e_J \cdot \mathbf{v}_I = 0.$$

It follows immediately that

$$\tilde{L}_2(e_J \cdot \mathbf{v}_I, e_K \cdot \mathbf{v}_I) \cdot e_I \cdot \mathbf{v}_I + \tilde{L}_2(e_K \cdot \mathbf{v}_I, e_I \cdot \mathbf{v}_I) \cdot e_J \cdot \mathbf{v}_I = 0$$

**Case (iv):** We can suppose without loss of generality that  $I = \{1, 2\}$ ,  $J = \{3, 4\}$  and  $K = \{5, 6, 7, 8\}$ . By Proposition 6.9(iv) this means that

$$\tilde{L}_2(e_J \cdot \mathbf{v}_I, e_K \cdot \mathbf{v}_I) \cdot e_I \cdot \mathbf{v}_I = -2B(e_{12}e_{34} \cdot \mathbf{v}_I, e_{5678} \cdot \mathbf{v}_I)\mathbf{v}_I$$

and that

$$\tilde{L}_2(e_K \cdot \mathbf{v}_I, e_I \cdot \mathbf{v}_I) \cdot e_J \cdot \mathbf{v}_I = 2B(e_{43}e_{5678} \cdot \mathbf{v}_I, e_{12} \cdot \mathbf{v}_I)\mathbf{v}_I.$$

Hence from  $B(e_{12}e_{34} \cdot \mathbf{v}_I, e_{5678} \cdot \mathbf{v}_I) = B(\mathbf{v}_I, e_4e_3e_2e_1e_{5678} \cdot \mathbf{v}_I)$  and the fact that  $B$  is symmetric it follows that

$$\tilde{L}_2(e_J \cdot \mathbf{v}_I, e_K \cdot \mathbf{v}_I) \cdot e_I \cdot \mathbf{v}_I + \tilde{L}_2(e_K \cdot \mathbf{v}_I, e_I \cdot \mathbf{v}_I) \cdot e_J \cdot \mathbf{v}_I = 0.$$

PROPOSITION 8.1. *The Lie algebra constructed above is simple.*

*Proof.* The Lie bracket we have just defined on the 248-dimensional vector space

$$E = C^2(V, g) \oplus S_1$$

has the following properties:

- (i)  $C^2(V, g)$  is a simple Lie subalgebra;
- (ii) the bracket of  $C^2(V, g)$  with  $S_1$  defines a nontrivial irreducible representation of  $C^2(V, g)$  on  $S_1$ ;
- (iii)  $\dim(C^2(V, g)) > \dim(S_1)$ ;
- (iv)  $[S_1, S_1] = C^2(V, g)$ .

We denote by  $\pi_{C^2} : E \rightarrow C^2(V, g)$  and  $\pi_{S_1} : E \rightarrow S_1$  the projections defined by the direct sum decomposition  $E = C^2(V, g) \oplus S_1$ . Clearly these are  $C^2(V, g)$ -equivariant maps whose respective kernels are  $S_1$  and  $C^2(V, g)$ .

If  $\mathcal{I}$  is a nonzero ideal in  $E$  then  $\pi_{C^2}(\mathcal{I})$  cannot be equal to  $\{0\}$  - if so then  $\mathcal{I} \subseteq S_1$  which would imply  $\mathcal{I} = S_1$  (cf. (ii)) and this is impossible since  $S_1$  is not an ideal (cf. (iv)). From the equivariance of  $\pi_{C^2}$  and the irreducibility of  $C^2(V, g)$  it then follows that

$$(48) \quad \pi_{C^2}(\mathcal{I}) = C^2(V, g)$$

and similarly, since  $C^2(V, g)$  is not an ideal (cf. (ii)), we have

$$(49) \quad \pi_{S_1}(\mathcal{I}) = S_1.$$

The rank theorem for  $\pi_{C^2} : \mathcal{I} \rightarrow C^2(V, g)$  and (48) imply

$$\dim(\mathcal{I}) \geq \dim(C^2(V, g))$$

and by (iii) this means

$$\dim(\mathcal{I} \cap C^2(V, g)) > 0$$

which by (i) implies

$$(50) \quad \mathcal{I} \cap C^2(V, g) = C^2(V, g).$$

It now follows from (49), (50) and the rank theorem for  $\pi_{S_1} : \mathcal{I} \rightarrow S_1$  that

$$\dim(\mathcal{I}) = \dim(\mathcal{I} \cap C^2(V, g)) + \dim(\pi_{S_1}(\mathcal{I})) = \dim(C^2(V, g)) + \dim(S_1) = \dim(E)$$

and hence

$$\mathcal{I} = E.$$

QED

### 8.2. Construction of split $e_7$ .

Let  $(V, g)$  be a twelve-dimensional vector space with a nondegenerate hyperbolic symmetric bilinear form  $g$ . Choose a 64-dimensional space of spinors  $S = S_1 \oplus S_2$  and an isomorphism  $C(V, g) \cong \text{End}(S)$ . Since  $n = 6 = 2 \pmod{4}$ ,  $B : S \times S \rightarrow k$  is even antisymmetric and  $\tilde{L}_2 : S \times S \rightarrow C^2(V, g)$  is even symmetric (Proposition 2.6 and Proposition 3.1). Hence  $(S_1, B, C^2(V, g), L)$  is a symplectic representation of the Lie algebra  $C^2(V, g)$  possessing a natural equivariant symmetric bilinear form  $\tilde{L}_2$  with values in  $C^2(V, g)$ .

Following [Fa] or [GSSR] we can define a Lie bracket on the 133-dimensional space

$$E = C^2(V, g) \oplus \mathfrak{sl}(2, k) \oplus S_1 \otimes k^2$$

if  $\tilde{L}_2$  (or a multiple of  $\tilde{L}_2$ ) satisfies the equation

$$(51) \quad \begin{aligned} \tilde{L}_2(\psi_1, \psi_2) \cdot \psi_3 - \tilde{L}_2(\psi_1, \psi_3) \cdot \psi_2 &= -B(\psi_1, \psi_2)\psi_3 + B(\psi_1, \psi_3)\psi_2 + 2B(\psi_2, \psi_3)\psi_1 \\ \forall \psi_1, \psi_2, \psi_3 \in S_1. \end{aligned}$$

Choose a polarisation  $V = \mathcal{I} \oplus \mathcal{E}$  of  $V$  such that a pure spinor  $\mathbf{v}_{\mathcal{I}}$  corresponding to  $\mathcal{I}$  is in  $S_1$  and bases  $i_1, \dots, i_n$  and  $e_1, \dots, e_n$  of respectively  $\mathcal{I}$  and  $\mathcal{E}$  such that

$$(52) \quad \begin{aligned} i_a i_b + i_b i_a &= 0 \\ e_a e_b + e_b e_a &= 0 \\ i_a e_b + e_b i_a &= \delta_{ab}. \end{aligned}$$

Then

$$S_1 = \text{Vect}\{e_I \cdot \mathbf{v}_{\mathcal{I}} : |I| \text{ is even}\}$$

and to prove the identity (51) it is sufficient to prove that

$$(53) \quad \begin{aligned} \tilde{L}_2(e_I \cdot \mathbf{v}_{\mathcal{I}}, e_J \cdot \mathbf{v}_{\mathcal{I}}) \cdot e_K \cdot \mathbf{v}_{\mathcal{I}} - \tilde{L}_2(e_I \cdot \mathbf{v}_{\mathcal{I}}, e_K \cdot \mathbf{v}_{\mathcal{I}}) \cdot e_J \cdot \mathbf{v}_{\mathcal{I}} = \\ -B(e_I \cdot \mathbf{v}_{\mathcal{I}}, e_J \cdot \mathbf{v}_{\mathcal{I}})e_K \cdot \mathbf{v}_{\mathcal{I}} + B(e_I \cdot \mathbf{v}_{\mathcal{I}}, e_K \cdot \mathbf{v}_{\mathcal{I}})e_J \cdot \mathbf{v}_{\mathcal{I}} + 2B(e_J \cdot \mathbf{v}_{\mathcal{I}}, e_K \cdot \mathbf{v}_{\mathcal{I}})e_I \cdot \mathbf{v}_{\mathcal{I}} \end{aligned}$$

for all even oriented subsets  $I, J, K$  of  $\{1, 2, 3, 4, 5, 6\}$ .

By the résumé above, the two terms on the LHS of this equation vanish unless  $I \cap J \cap K = \emptyset$  and  $I^c \cap J^c \cap K^c = \emptyset$  and by Proposition 5.2 the three terms on the RHS vanish if  $I \cap J \cap K \neq \emptyset$ . Hence we need only prove (53) for even subsets of  $\{1, 2, 3, 4, 5, 6\}$  satisfying  $I \cap J \cap K = \emptyset$  and  $I^c \cap J^c \cap K^c = \emptyset$ . But then, again by the résumé, by changing the polarisation if necessary, we can always assume that

$$I \cap J = \emptyset, \quad K = I^c \cap J^c$$

and then all terms on the LHS of (53) vanish unless one of the sets  $I, J, K$  has 0 or 2 elements. So in fact to prove (53) for all even oriented subsets  $I, J, K$  of  $\{1, 2, 3, 4, 5, 6\}$  it is sufficient to prove that

$$(54) \quad \begin{aligned} \tilde{L}_2(e_I \cdot \mathbf{v}_{\mathcal{I}}, e_J \cdot \mathbf{v}_{\mathcal{I}}) \cdot e_K \cdot \mathbf{v}_{\mathcal{I}} - \tilde{L}_2(e_I \cdot \mathbf{v}_{\mathcal{I}}, e_K \cdot \mathbf{v}_{\mathcal{I}}) \cdot e_J \cdot \mathbf{v}_{\mathcal{I}} = \\ -B(e_I \cdot \mathbf{v}_{\mathcal{I}}, e_J \cdot \mathbf{v}_{\mathcal{I}})e_K \cdot \mathbf{v}_{\mathcal{I}} + B(e_I \cdot \mathbf{v}_{\mathcal{I}}, e_K \cdot \mathbf{v}_{\mathcal{I}})e_J \cdot \mathbf{v}_{\mathcal{I}} + 2B(e_J \cdot \mathbf{v}_{\mathcal{I}}, e_K \cdot \mathbf{v}_{\mathcal{I}})e_I \cdot \mathbf{v}_{\mathcal{I}} \end{aligned}$$

for all oriented subsets  $I, J, K$  of  $\{1, 2, 3, 4, 5, 6\}$  such that

- $|I|, |J|$  and  $|K|$  are even;
- $|I| + |J| + |K| = 6$ ;
- $I \cap J = J \cap K = K \cap I = \emptyset$ ;
- One of  $|I|, |J|$  or  $|K|$  is equal to 0 or 2.

Up to permutations of  $I, J, K$  the only possibilities are

- (i)  $|I| = 0, |J| = 0$  and  $|K| = 6$ ;
- (ii)  $|I| = 0, |J| = 2$  and  $|K| = 4$ ;
- (iii)  $|I| = 2, |J| = 2$  and  $|K| = 2$ .

**Case (i):** We have  $I = J = \emptyset$  and  $K = \{1, 2, 3, 4, 5, 6\}$ . By Proposition 6.9(ii) this means that

$$\tilde{L}_2(e_I \cdot \mathbf{v}_I, e_J \cdot \mathbf{v}_I) \cdot e_K \cdot \mathbf{v}_I = 0$$

and by Proposition 6.9(iii) that

$$\tilde{L}_2(e_I \cdot \mathbf{v}_I, e_K \cdot \mathbf{v}_I) \cdot e_J \cdot \mathbf{v}_I = -3B(\mathbf{v}_I, e_{123456} \cdot \mathbf{v}_I)\mathbf{v}_I.$$

Hence the LHS of (53) is

$$3B(\mathbf{v}_I, e_{123456} \cdot \mathbf{v}_I)\mathbf{v}_I$$

which is equal to the RHS of (53) since by Proposition 5.2,

$$-B(e_I \cdot \mathbf{v}_I, e_J \cdot \mathbf{v}_I)e_K \cdot \mathbf{v}_I + B(e_I \cdot \mathbf{v}_I, e_K \cdot \mathbf{v}_I)e_J \cdot \mathbf{v}_I + 2B(e_J \cdot \mathbf{v}_I, e_K \cdot \mathbf{v}_I)e_I \cdot \mathbf{v}_I$$

reduces to

$$0 + B(\mathbf{v}_I, e_{123456} \cdot \mathbf{v}_I)\mathbf{v}_I + 2B(\mathbf{v}_I, e_{123456} \cdot \mathbf{v}_I)e_I \cdot \mathbf{v}_I = 3B(\mathbf{v}_I, e_{123456} \cdot \mathbf{v}_I)\mathbf{v}_I.$$

**Case (ii):** We have  $I = \emptyset$  and without loss of generality we can suppose  $J = \{1, 2\}$  and  $K = \{3, 4, 5, 6\}$ . By Proposition 6.9(ii) this means that

$$\tilde{L}_2(e_I \cdot \mathbf{v}_I, e_J \cdot \mathbf{v}_I) \cdot e_K \cdot \mathbf{v}_I = 0$$

and by Proposition 6.9(iv) that

$$\tilde{L}_2(e_I \cdot \mathbf{v}_I, e_K \cdot \mathbf{v}_I) \cdot e_J \cdot \mathbf{v}_I = 2B(e_{21}\mathbf{v}_I, e_{3456} \cdot \mathbf{v}_I)\mathbf{v}_I.$$

Hence the LHS of (53) is

$$-2B(e_{21}\mathbf{v}_I, e_{3456} \cdot \mathbf{v}_I)\mathbf{v}_I$$

which is equal to the RHS of (53) since by Proposition 5.2,

$$-B(e_I \cdot \mathbf{v}_I, e_J \cdot \mathbf{v}_I)e_K \cdot \mathbf{v}_I + B(e_I \cdot \mathbf{v}_I, e_K \cdot \mathbf{v}_I)e_J \cdot \mathbf{v}_I + 2B(e_J \cdot \mathbf{v}_I, e_K \cdot \mathbf{v}_I)e_I \cdot \mathbf{v}_I$$

reduces to

$$0 + 0 + 2B(e_{12} \cdot \mathbf{v}_I, e_{3456} \cdot \mathbf{v}_I)\mathbf{v}_I = -2B(e_{21}\mathbf{v}_I, e_{3456} \cdot \mathbf{v}_I)\mathbf{v}_I.$$

**Case (iii):** Without loss of generality we can suppose  $I = \{1, 2\}$ ,  $J = \{3, 4\}$  and  $K = \{5, 6\}$ . By Proposition 6.9(iv) this means that

$$\tilde{L}_2(e_I \cdot \mathbf{v}_I, e_J \cdot \mathbf{v}_I) \cdot e_K \cdot \mathbf{v}_I = 2B(e_{65}e_{12} \cdot \mathbf{v}_I, e_{34} \cdot \mathbf{v}_I)\mathbf{v}_I$$

and that

$$\tilde{L}_2(e_I \cdot \mathbf{v}_I, e_K \cdot \mathbf{v}_I) \cdot e_J \cdot \mathbf{v}_I = 2B(e_{43}e_{12} \cdot \mathbf{v}_I, e_{56} \cdot \mathbf{v}_I)\mathbf{v}_I.$$

Hence the LHS of (53) is

$$2B(e_{65}e_{12} \cdot \mathbf{v}_I, e_{34} \cdot \mathbf{v}_I)\mathbf{v}_I - 2B(e_{43}e_{12} \cdot \mathbf{v}_I, e_{56} \cdot \mathbf{v}_I)\mathbf{v}_I = 0.$$

The RHS of (53) also vanishes since each term of

$$-B(e_I \cdot \mathbf{v}_I, e_J \cdot \mathbf{v}_I)e_K \cdot \mathbf{v}_I + B(e_I \cdot \mathbf{v}_I, e_K \cdot \mathbf{v}_I)e_J \cdot \mathbf{v}_I + 2B(e_J \cdot \mathbf{v}_I, e_K \cdot \mathbf{v}_I)e_I \cdot \mathbf{v}_I$$

vanishes by Proposition 5.2.

**PROPOSITION 8.2.** *The Lie algebra constructed above is simple.*

*Proof.* The Lie bracket we have just defined on the 133-dimensional space

$$E = C^2(V, g) \oplus sl(2, k) \oplus S_1 \otimes k^2$$

has the properties:

- (i)  $C^2(V, g)$  and  $sl(2, k)$  are commuting simple Lie subalgebras;
- (ii) the bracket of  $C^2(V, g) \oplus sl(2, k)$  with  $S_1 \otimes k^2$  defines a faithful, irreducible representation of  $C^2(V, g) \oplus sl(2, k)$  on  $S_1 \otimes k^2$ ;
- (iii)  $[S_1 \otimes k^2, S_1 \otimes k^2] = C^2(V, g) \oplus sl(2, k)$ .

Pick any standard semisimple  $h$  in  $sl(2, k)$ . Then  $\text{ad}(h) : E \rightarrow E$  is diagonalisable with eigenvalues  $\{0, \pm 1, \pm 2\}$  and:

- $C^2(V, g) \oplus sl(2, k) = E_+$  is the sum of the eigenspaces corresponding to even eigenvalues;
- $S_1 \otimes k^2 = E_-$  is the sum of the eigenspaces corresponding to odd eigenvalues.

Let  $\mathcal{I}$  be a nonzero ideal in  $E$ . Then  $[h, \mathcal{I}] \subseteq \mathcal{I}$  and hence

$$\mathcal{I} = \mathcal{I} \cap E_+ \oplus \mathcal{I} \cap E_-.$$

If  $\mathcal{I} \cap E_- = \{0\}$  then  $\mathcal{I} = \mathcal{I} \cap E_+ \neq \{0\}$  which is impossible since no nontrivial ideal of  $E$  can be contained in  $E_+$  ( $E_+$  acts faithfully on  $E_-$  by (ii)). Hence  $\mathcal{I} \cap E_- \neq \{0\}$ .

If  $\mathcal{I} \cap E_- \neq \{0\}$  then in fact  $\mathcal{I} \cap E_- = E_-$  since  $\mathcal{I} \cap E_-$  is stable under  $E_+$  and  $E_-$  is an irreducible representation of  $E_+$  (cf (ii)). However if  $\mathcal{I}$  contains  $E_-$  it contains  $E_+$  by (iii) and hence  $\mathcal{I} = E$ .

QED

### 8.3. Construction of split $e_6$ .

Let  $(V, g)$  be a ten-dimensional vector space with a nondegenerate hyperbolic symmetric bilinear form  $g$ . Choose a 32-dimensional space of spinors  $S = S_1 \oplus S_2$  and an isomorphism  $C(V, g) \cong \text{End}(S)$ . Since  $n = 5 = 1 \pmod{4}$ ,  $B : S \times S \rightarrow k$  is odd symmetric (Proposition 2.6),  $\tilde{L}_2 : S \times S \rightarrow C^2(V, g)$  is odd antisymmetric (Proposition 3.1) and  $L_{10} : S \times S \rightarrow C^{10}(V, g)$  is odd antisymmetric (Proposition 3.6).

Consider the 78-dimensional vector space

$$E = C^{10}(V, g) \oplus C^2(V, g) \oplus S.$$

Following the procedure in §7 define  $[\ , \ ] : E \otimes E \rightarrow E$  to be the unique antisymmetric bilinear map such that:

$$(55) \quad [A, B] = AB - BA \quad \text{if } A, B \in C^2 \oplus C^{10}$$

$$(56) \quad [A, \psi] = A \cdot \psi \quad \text{if } A \in C^2 \oplus C^{10}, \psi \in S$$

$$(57) \quad [\psi_1, \psi_2] = 2\tilde{L}_2(\psi_1, \psi_2) + 96L_{10}(\psi_1, \psi_2) \quad \text{if } \psi_1, \psi_2 \in S.$$

Since  $C^2, C^2 \oplus C^{10}$  are Lie algebras, since  $S$  is a representation and since  $\tilde{L}_2 : S \times S \rightarrow C^2$  and  $2\tilde{L}_2 + 96L_{10} : S \times S \rightarrow C^2 \oplus C^{10}$  are equivariant maps, this map defines a Lie bracket on  $E$  if and only if the following Jacobi identities are satisfied:

$$(58) \quad (2\tilde{L}_2 + 96L_{10})(\psi_1, \psi_2) \cdot \psi_3 + (2\tilde{L}_2 + 96L_{10})(\psi_2, \psi_3) \cdot \psi_1 + (2\tilde{L}_2 + 96L_{10})(\psi_3, \psi_1) \cdot \psi_2 = 0 \\ \forall \psi_1, \psi_2, \psi_3 \in S.$$

Recall that if  $\varepsilon \in C^{10}(V, g)$  is a grading operator then

$$L_{10}(\psi_1, \psi_2) = \frac{1}{2^5} B(\psi_1, \varepsilon \cdot \psi_2) \varepsilon$$

so that this is equivalent to

$$(59) \quad 2\tilde{L}_2(\psi_1, \psi_2) \cdot \psi_3 + 2\tilde{L}_2(\psi_2, \psi_3) \cdot \psi_1 + 2\tilde{L}_2(\psi_3, \psi_1) \cdot \psi_2 = \\ -3B(\psi_1, \varepsilon \cdot \psi_2) \varepsilon \cdot \psi_3 - 3B(\psi_2, \varepsilon \cdot \psi_3) \varepsilon \cdot \psi_1 - 3B(\psi_3, \varepsilon \cdot \psi_1) \varepsilon \cdot \psi_2 \\ \forall \psi_1, \psi_2, \psi_3 \in S.$$

A purist would say that in this example we should use the graded spinor norm described in §3.1. However, the calculations in §6 were all done with the usual spinor norm. For the sake of ease of verifying the result we shall not use the graded norm but just the usual spinor norm.

Choose a polarisation  $V = \mathcal{I} \oplus \mathcal{E}$  of  $V$  such that a pure spinor  $\mathbf{v}_{\mathcal{I}}$  corresponding to  $\mathcal{I}$  is in  $S_1$  and bases  $i_1, \dots, i_n$  and  $e_1, \dots, e_n$  of respectively  $\mathcal{I}$  and  $\mathcal{E}$  such that

$$(60) \quad \begin{aligned} i_a i_b + i_b i_a &= 0 \\ e_a e_b + e_b e_a &= 0 \\ i_a e_b + e_b i_a &= \delta_{ab}. \end{aligned}$$

Then

$$S_1 = \text{Vect}\{e_I \cdot \mathbf{v}_{\mathcal{I}} : |I| \text{ is even}\}, \quad S_2 = \text{Vect}\{e_I \cdot \mathbf{v}_{\mathcal{I}} : |I| \text{ is odd}\}$$

and to prove the identity (59) it is sufficient to prove that

$$(61) \quad 2\tilde{L}_2(e_I \cdot \mathbf{v}_{\mathcal{I}}, e_J \cdot \mathbf{v}_{\mathcal{I}}) \cdot e_K \cdot \mathbf{v}_{\mathcal{I}} + 2\tilde{L}_2(e_J \cdot \mathbf{v}_{\mathcal{I}}, e_K \cdot \mathbf{v}_{\mathcal{I}}) \cdot e_I \cdot \mathbf{v}_{\mathcal{I}} + 2\tilde{L}_2(e_K \cdot \mathbf{v}_{\mathcal{I}}, e_I \cdot \mathbf{v}_{\mathcal{I}}) \cdot e_J \cdot \mathbf{v}_{\mathcal{I}} =$$

$$-3B(e_I \cdot \mathbf{v}_{\mathcal{I}}, \varepsilon \cdot e_J \cdot \mathbf{v}_{\mathcal{I}}) \varepsilon \cdot e_K \cdot \mathbf{v}_{\mathcal{I}} - 3B(e_J \cdot \mathbf{v}_{\mathcal{I}}, \varepsilon \cdot e_K \cdot \mathbf{v}_{\mathcal{I}}) \varepsilon \cdot e_I \cdot \mathbf{v}_{\mathcal{I}} - 3B(e_K \cdot \mathbf{v}_{\mathcal{I}}, \varepsilon \cdot e_I \cdot \mathbf{v}_{\mathcal{I}}) \varepsilon \cdot e_J \cdot \mathbf{v}_{\mathcal{I}}$$

for all oriented subsets  $I, J, K$  of  $\{1, 2, 3, 4, 5\}$ .

If  $|I|, |J|, |K|$  are of the same parity, all terms in this expression vanish ( $\tilde{L}_2$  and  $B$  are odd) and the identity is true. If  $|I|, |J|, |K|$  are not of the same parity two of them must be even and one odd, and without loss of generality we can suppose  $|I|, |J|$  are even and  $|K|$  is odd. Since  $\tilde{L}_2$  and  $B$  are odd the identity (61) then reduces to

$$(62) \quad \begin{aligned} 2\tilde{L}_2(e_J \cdot \mathbf{v}_{\mathcal{I}}, e_K \cdot \mathbf{v}_{\mathcal{I}}) \cdot e_I \cdot \mathbf{v}_{\mathcal{I}} + 2\tilde{L}_2(e_K \cdot \mathbf{v}_{\mathcal{I}}, e_I \cdot \mathbf{v}_{\mathcal{I}}) \cdot e_J \cdot \mathbf{v}_{\mathcal{I}} = \\ 3B(e_J \cdot \mathbf{v}_{\mathcal{I}}, \varepsilon \cdot e_K \cdot \mathbf{v}_{\mathcal{I}}) \varepsilon \cdot e_I \cdot \mathbf{v}_{\mathcal{I}} - 3B(e_K \cdot \mathbf{v}_{\mathcal{I}}, \varepsilon \cdot e_I \cdot \mathbf{v}_{\mathcal{I}}) \varepsilon \cdot e_J \cdot \mathbf{v}_{\mathcal{I}}. \end{aligned}$$

By the résumé above, the two terms on the LHS of this equation vanish unless  $I \cap J \cap K = \emptyset$  and  $I^c \cap J^c \cap K^c = \emptyset$  and by Proposition 5.2 the three terms on the RHS vanish if  $I \cap J \cap K \neq \emptyset$ .

If  $I \cap J \cap K = \emptyset$  and  $I^c \cap J^c \cap K^c = \emptyset$  then by changing the polarisation if necessary, we can always assume that

$$I \cap J = \emptyset, \quad K = I^c \cap J^c,$$

and then all terms on the LHS of (62) vanish unless one of the sets  $I, J, K$  has 0 or 2 elements. Up to permutations of  $I, J, K$  the only possibilities are

- (i)  $|I| = 0, |J| = 0$  and  $|K| = 5$ ;
- (ii)  $|I| = 0, |J| = 2$  and  $|K| = 3$ ;
- (iii)  $|I| = 0, |J| = 4$  and  $|K| = 1$ ;
- (iv)  $|I| = 2, |J| = 2$  and  $|K| = 1$ .

**Case (i):** We have  $I = J = \emptyset$  and  $K = \{1, 2, 3, 4, 5\}$ . By Proposition 6.9(iii) this means that

$$2\tilde{L}_2(e_J \cdot \mathbf{v}_{\mathcal{I}}, e_K \cdot \mathbf{v}_{\mathcal{I}}) \cdot e_I \cdot \mathbf{v}_{\mathcal{I}} = 2B(\mathbf{v}_{\mathcal{I}}, e_{12345} \cdot \mathbf{v}_{\mathcal{I}}) \left(0 - \frac{5}{2}\right) \mathbf{v}_{\mathcal{I}}$$

and that

$$2\tilde{L}_2(e_K \cdot \mathbf{v}_{\mathcal{I}}, e_I \cdot \mathbf{v}_{\mathcal{I}}) \cdot e_J \cdot \mathbf{v}_{\mathcal{I}} = 2B(e_{12345} \cdot \mathbf{v}_{\mathcal{I}}, \mathbf{v}_{\mathcal{I}}) \left(5 - \frac{5}{2}\right) \mathbf{v}_{\mathcal{I}}$$

Hence the LHS of (62) vanishes since  $B$  is symmetric as does the RHS for the same reason.

**Case (ii):** We have  $I = \emptyset$  and without loss of generality we can suppose  $J = \{1, 2\}$  and  $K = \{3, 4, 5\}$ . By Proposition 6.9(iii) this means that

$$2\tilde{L}_2(e_J \cdot \mathbf{v}_{\mathcal{I}}, e_K \cdot \mathbf{v}_{\mathcal{I}}) \cdot e_I \cdot \mathbf{v}_{\mathcal{I}} = 2B(e_{12} \mathbf{v}_{\mathcal{I}}, e_{345} \cdot \mathbf{v}_{\mathcal{I}}) \left(2 - \frac{5}{2}\right) \mathbf{v}_{\mathcal{I}},$$

and by Proposition 6.9(iv) that

$$2\tilde{L}_2(e_K \cdot \mathbf{v}_{\mathcal{I}}, e_I \cdot \mathbf{v}_{\mathcal{I}}) \cdot e_J \cdot \mathbf{v}_{\mathcal{I}} = 4B(e_{21} \cdot e_{345} \cdot \mathbf{v}_{\mathcal{I}}, \mathbf{v}_{\mathcal{I}}) \mathbf{v}_{\mathcal{I}}.$$

Hence the LHS of (62) reduces to

$$3B(e_{21} \cdot e_{345} \cdot \mathbf{v}_{\mathcal{I}}, \mathbf{v}_{\mathcal{I}}) \mathbf{v}_{\mathcal{I}}.$$

The RHS of (62) is

$$3B(e_J \cdot \mathbf{v}_I, \varepsilon \cdot e_K \cdot \mathbf{v}_I) \varepsilon \cdot e_I \cdot \mathbf{v}_I - 3B(e_K \cdot \mathbf{v}_I, \varepsilon \cdot e_I \cdot \mathbf{v}_I) \varepsilon \cdot e_J \cdot \mathbf{v}_I$$

which by Proposition 5.2 also reduces to

$$3B(e_{12} \cdot \mathbf{v}_I, e_{345} \cdot \mathbf{v}_I) \mathbf{v}_I - 0 = 3B(e_{21} \cdot e_{345} \cdot \mathbf{v}_I, \mathbf{v}_I) \mathbf{v}_I.$$

**Case (iii):** We have  $I = \emptyset$  and without loss of generality we can suppose  $J = \{1, 2, 3, 4\}$  and  $K = \{5\}$ . By Proposition 6.9(iii) this means that

$$2\tilde{L}_2(e_J \cdot \mathbf{v}_I, e_K \cdot \mathbf{v}_I) \cdot e_I \cdot \mathbf{v}_I = 2B(e_{1234} \cdot \mathbf{v}_I, e_5 \cdot \mathbf{v}_I) \left(4 - \frac{5}{2}\right) \mathbf{v}_I$$

and by Proposition 6.9(ii) that

$$2\tilde{L}_2(e_K \cdot \mathbf{v}_I, e_I \cdot \mathbf{v}_I) \cdot e_J \cdot \mathbf{v}_I = 0$$

Hence the LHS of (62) reduces to

$$3B(e_{12345} \cdot \mathbf{v}_I, \mathbf{v}_I) \mathbf{v}_I.$$

The RHS of (62) is

$$3B(e_J \cdot \mathbf{v}_I, \varepsilon \cdot e_K \cdot \mathbf{v}_I) \varepsilon \cdot e_I \cdot \mathbf{v}_I - 3B(e_K \cdot \mathbf{v}_I, \varepsilon \cdot e_I \cdot \mathbf{v}_I) \varepsilon \cdot e_J \cdot \mathbf{v}_I$$

which by Proposition 5.2 also reduces to

$$3B(e_{1234} \mathbf{v}_I, e_5 \cdot \mathbf{v}_I) \mathbf{v}_I - 0 = 3B(e_{12345} \cdot \mathbf{v}_I, \mathbf{v}_I) \mathbf{v}_I.$$

**Case (iv):** We can suppose without loss of generality that  $I = \{1, 2\}$ ,  $J = \{3, 4\}$  and  $K = \{5\}$ . By Proposition 6.9(iv) this means

$$2\tilde{L}_2(e_J \cdot \mathbf{v}_I, e_K \cdot \mathbf{v}_I) \cdot e_I \cdot \mathbf{v}_I = 4B(e_{21} \cdot e_{34} \cdot \mathbf{v}_I, e_5 \cdot \mathbf{v}_I) \mathbf{v}_I$$

and

$$2\tilde{L}_2(e_K \cdot \mathbf{v}_I, e_I \cdot \mathbf{v}_I) \cdot e_J \cdot \mathbf{v}_I = 4B(e_{43} \cdot e_5 \cdot \mathbf{v}_I, e_{12} \cdot \mathbf{v}_I) \mathbf{v}_I.$$

Since

$$B(e_{21} \cdot e_{34} \cdot \mathbf{v}_I, e_5 \cdot \mathbf{v}_I) \mathbf{v}_I = B(e_{34} \cdot e_{21} \cdot \mathbf{v}_I, e_5 \cdot \mathbf{v}_I) \mathbf{v}_I = B(e_{21} \cdot \mathbf{v}_I, e_{43} \cdot e_5 \cdot \mathbf{v}_I) \mathbf{v}_I$$

and  $B$  is symmetric, the LHS of (62) vanishes as do all terms on the RHS by Proposition 5.2.

**PROPOSITION 8.3.** *The Lie algebra constructed above is simple.*

*Proof.* The Lie bracket we have just defined on the 78-dimensional space

$$E = C^2(V, g) \oplus C^{10}(V, g) \oplus S$$

has the properties:

- (i)  $C^2(V, g)$  is a simple Lie subalgebra and  $C^{10}(V, g)$  is its one-dimensional commutant in  $E$ ;
- (ii) there exists  $\varepsilon \in C^{10}(V, g)$  such that;  $\text{ad}(\varepsilon) : E \rightarrow E$  is diagonalisable with eigenvalues  $\{0, \pm 1\}$  and  $S = E_{-1} \oplus E_1$  is the decomposition of  $S$  into faithful, non-isomorphic irreducible  $C^2(V, g)$ -modules.
- (iii)  $[E_{-1}, E_1] = C^2(V, g) \oplus C^{10}(V, g)$ .

Let  $\mathcal{I}$  be a nonzero ideal in  $E$ . Then  $[\varepsilon, \mathcal{I}] \subseteq \mathcal{I}$  and hence

$$\mathcal{I} = \mathcal{I} \cap E_0 \oplus \mathcal{I} \cap E_{-1} \oplus \mathcal{I} \cap E_1.$$

If  $\mathcal{I} \cap E_{-1} = \mathcal{I} \cap E_1 = \{0\}$  then  $\mathcal{I} = \mathcal{I} \cap E_0 \neq \{0\}$  which is impossible since no nontrivial ideal of  $E$  can be contained in  $E_0$  by (ii) ( $E_0$  acts faithfully on  $S$  by (ii)). Hence either  $\mathcal{I} \cap E_{-1} \neq \{0\}$  or  $\mathcal{I} \cap E_1 \neq \{0\}$ .

If  $\mathcal{I} \cap E_{-1} \neq \{0\}$  then in fact  $\mathcal{I} \cap E_{-1} = E_{-1}$  since  $\mathcal{I} \cap E_{-1}$  is stable under  $E_0$  and  $E_{-1}$  is an irreducible representation of  $E_0$  (cf (ii)). However if  $\mathcal{I}$  contains  $E_{-1}$  it contains  $E_0$  by (iii) and hence  $\mathcal{I} = E$ . Similarly, if  $\mathcal{I} \cap E_1 \neq \{0\}$  then  $\mathcal{I} = E$  and the proposition is proved. QED

### References

- [Ad] J. F. Adams, *Lectures on exceptional Lie groups* Edited by Zafer Mahmud and Mamoru Mimura. With a foreword by J. Peter May. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, (1996) xiv+122 pp.
- [EC] É. Cartan, *Leçons sur la théorie des spineurs* Vol. I, Actualités Sci. Indust. No. 643, Hermann, Paris, (1938); Vol. II, Actualités Sci. Indust. No. 701, (1938).
- [Ch] C. Chevalley, *The Algebraic Theory of Spinors and Clifford Algebras* Collected Works, v.2 Springer Verlag Berlin, Heidelberg (1997).
- [Fa] J.R. Faulkner, *A construction of Lie algebras from a class of ternary algebras* Trans. A.M.S. **155**, 397-408 (1971).
- [K] B. Kostant, *On invariant skew-tensors* Proc. Natl. Acad. Sci. USA **42**, 148-151 (1956).
- [MoSe] A. Moroianu and U. Semmelmann, *Invariant four-forms and symmetric pairs* Ann. Global. Anal. Geom. **43**, 107-121 (2013).
- [Se] G. B. Seligman, *Rational methods in Lie algebras* Lecture Notes in Pure and Applied Mathematics, v.17 Marcel Dekker, Inc., New York-Basel, viii+346 p. (1976).
- [GSSR] M. J. Slupinski and R. J. Stanton, *The geometry of special symplectic representations* Journal of Algebra **428**, 149-189, 15 April 2015.
- [Wo] J. A. Wolf, *Unitary representations of maximal parabolic subgroups of the classical groups* Mem. Amer. Math. Soc. 8 (1976), no. 180, iii+193 pp.

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