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SOLUTION TO BRIOT AND BOUQUET PROBLEM ON SINGULARITIES OF DIFFERENTIAL EQUATIONS

RICARDO PÉREZ-MARCO

Abstract. We solve Briot and Bouquet problem on the existence of non-monodromic (multivalued) solutions for singularities of differential equations in the complex domain. The solution is an application of hedgehog dynamics for indifferent irrational fixed points. We present an important simplification by only using a local hedgehog for which we give a simpler and direct construction of quasi-invariant curves which does not rely on complex renormalization.

1. Introduction.

We prove the following Theorem:

Theorem 1. *Let $f(z) = e^{2\pi i\alpha}z + \mathcal{O}(z^2)$, $\alpha \in \mathbb{R} - \mathbb{Q}$ be a germ of holomorphic diffeomorphism with an indifferent irrational fixed point at* 0.

There is no orbit of f distinct from the fixed point at 0 *that converges to* 0 *by positive or negative iteration by f.*

This Theorem solves the question of C. Briot and J.-C. Bouquet on singularities of differential equations from 1856 ([7]), as well as questions of H. Dulac (1904, [10], [11]), É. Picard (1896, [28]), P. Fatou (1919, [12]), and two more recent conjectures of M. Lyubich (1986, [16]).

The Theorem is trivial when the fixed point is linearizable, so, for the rest of the article, we assume that f is not linearizable.

The main difficulty is to understand the non-linearizable dynamics. The proof relies on hedgehogs and their dynamics discovered by the author in [24]. More precisely, we have from [24] the existence of hedgehogs:

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Theorem 2 (Existence of hedgehogs)**.** *Let U be a Jordan neighborhood of 0 such that f and f^{-1} are defined and univalent on U, and continuous on $\bar{U}$.*

There exists a hedgehog K with the following properties:

- $0 \in K \subset \bar{U}$
- *K is a full, compact and connected set.*
- $K \cap \partial U \neq \emptyset$.
- $f(K) = f^{-1}(K) = K$.

Moreover, f acts continuously on the space of prime-ends of $\mathbb{C} - K$ and defines an homeomorphism of the circle of prime-ends with rotation number α.

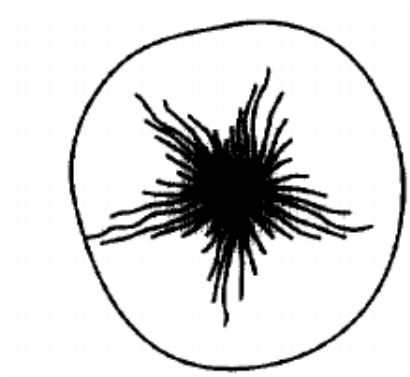

FIGURE 1. A hedgehog and its defining neighborhood.

In the proof we only need to consider local hedgehogs, i.e. a hedgehog associated to a small disk $U = \mathbb{D}_{r_0}$ with $r_0 > 0$ small enough. Let K_0 be the hedgehog associated to $\mathbb{D}_{r_0}$. The two following Theorems imply our main Theorem.

Theorem 3. *Let $(p_n/q_n)_{n\geq 0}$ be the sequence of convergents of α. We have*

$$\lim_{n\to+\infty} f^{\pm q_n}_{/K_0} = \mathrm{id}_{K_0} \ ,$$

where the convergence is uniform on K_0.

Therefore all points of the hedgehog are uniformly recurrent, and no point on the hedgehog distinct from 0 converges to 0 by positive or negative iteration by f.

Theorem 4. *Let $z_0 \in U - K_0$ such that the positive, resp. negative, orbit $(f^n(z_0))_{n\geq 0}$, resp. $(f^{-n}(z_0))_{n\geq 0}$, accumulates a point on K_0. Then this orbit accumulates all K_0,*

$$K_0 \subset \overline{(f^n(z_0))_{n\geq 0}} \ \ (\mathrm{resp.} K_0 \subset \overline{(f^{-n}(z_0))_{n\geq 0}}) \ .$$

In particular this implies that if such an orbit $(f^n(z_0))_{n\geq 0}$ (resp. $(f^{-n}(z_0))_{n\geq 0}$) accumulates $0 \in K_0$ then it cannot converge to 0. Note that if f is not linearizable then it is clear that $0 \in \partial K_0$. Indeed one can prove that the hedgehog K_0 has empty interior and $K_0 = \partial K_0$, but we don't need to use this fact. We can just prove the previous Theorem for ∂K_0.

The proof of these two Theorems are done by constructing quasi-invariant curves near the hedgehog. These are Jordan curves surrounding the hedgehog and almost invariant by high iterates of the dynamics. The quasi-invariance property is obtained for the Poincaré metric of the complement of the hedgehog in the Riemann sphere.

Therefore, it is enough to carry out the construction for local hedgehogs, and for these we have a direct and simpler construction of quasi-invariant curves, that does not rely on complex renormalization techniques. Classical one real dimensional estimates for smooth circle diffeomorphism combined with an hyperbolic version of Denjoy-Yoccoz Lemma in order to control the complex orbits for analytic circle diffeomorphisms, are enough. This gives an important simplification for local hedgehogs of the proof of the main Theorem that was announced in [21].

2. Historical introduction on Briot and Bouquet problem.

In 1856 C. Briot and J.-C. Bouquet published a foundational article [7] on the local solutions of differential equations in the complex domain. They are particularly interested in how a local solution determines uniquely the holomorphic function through analytic continuation. They consider a first order differential equation of a differential equation of the form

$$\frac{dy}{dx} = f(x, y) \ ,$$

where f is a meromorphic function of the two complex variables $(x, y) \in \mathbb{C}^2$ in a neighborhood of a point (x_0, y_0). A. Cauchy proved his fundamental Theorem on existence and uniqueness of local solutions[1]: If f is finite and holomorphic in a neighborhood of (x_0, y_0) then there exists a unique holomorphic local solution $y(x)$ satisfying the initial conditions

$$y(x_0) = y_0 \ .$$

In their terminology, Briot and Bouquet talk about "solutions monogènes et monodromes", "monogène" or monogenic meaning $\mathbb{C}$-differentiable, i.e. holomorphic, and "monodrome" or monodromic meaning univalued, since they also consider multivalued solutions with non-trivial monodromy at $x_0 \in \mathbb{C}$.

[1]What is called today in Calculus books Cauchy-Lipschitz Theorem.

Briot and Bouquet start their article by giving a simple proof of Cauchy Theorem by the majorant series method. Then they consider the situation where f is infinite or has a singularity at (x_0, y_0). They observe that even in Cauchy's situation, we may get to such a point by a global analytic continuation of any solution. We assume for now on that $(x_0, y_0) = (0, 0)$. Writing down f as the quotient of two holomorphic germs

$$f(x, y) = \frac{A(x, y)}{B(x, y)} \ ,$$

they study the situation when $A(0, 0) = B(0, 0) = 0$ (they call these singularities "of the form $\frac{0}{0}$"). This is done in Chapter III, starting in section 75 of [7]. After a simple change of variables, the equation reduces to

$$x\frac{dy}{dx} = ay + bx + \mathcal{O}(2) \ ,$$

and a discussion starts considering the different cases for different values of the coefficients $a, b \in \mathbb{C}$. They prove the remarkable Theorem that if a is not a positive integer, then there always exists a holomorphic solution $y(x)$ in a neighborhood of 0 vanishing at 0 (Theorem XXVIII in section 80 of [7]). They show that this holomorphic solution is the only monodromic one and in their proof of uniqueness (in section 81) the equation is put in the form

$$x\frac{dy}{dx} = y(a + \mathcal{O}(2)) \ .$$

In this last form the holomorphic solution corresponds to $y = 0$.

After that they proceed to show that when the real part of a is positive there are infinitely many non-monodromic solutions (section 82 in [7]), i.e. holomorphic solutions $y(x)$ that are multivalued around $0 \in \mathbb{C}$.

They make the claim in section 85 in [7] that when the real part of a is negative there are no other solutions, not even non-monodromic, other than the holomorphic solution found.

The proof of this statement contains a gap. Starting with the new form of the differential equation

$$x\frac{dy}{dx} = y(a + \mathcal{O}_y(1)) + xy\varphi(x, y) \ ,$$

they transform it into

$$\frac{dy}{y} + (A + By + \ldots)\, dy = a\,\frac{dt}{t} + \psi(x, y)\, dt \ ,$$

where $A + By + \ldots$ is a holomorphic function of y near 0 and ψ is holomorphic near $(0,0)$. Assuming by contradiction the existence of another solution, integration of the equation over a path from x_1 to x, $y_1 = y(x_1)$, gives

$$\log\left(\frac{y}{y_1}\right) + (A(y-y_1) + \ldots) = \log\left(\frac{x}{x_1}\right)^a + \int_{x_1}^{x} \psi(x, y(x))\, dx \ .$$

They pretend that this is of the form

$$\log\left(\frac{y(x)}{y_1}\right) = \log\left(\frac{x}{x_1}\right)^a + \epsilon,$$

where ϵ is a small quantity, vanishing for $x = x_1$, and very small when $x \to 0$, to get a contradiction using that for $\Re a < 0$, $\Re \log(x/x_1)^a \to +\infty$ when $x \to 0$ but $\Re \log(y/y_1) \to -\infty$ if $y(x) \to 0$.

Unfortunately ϵ is not small because since $y(x)$ is not monodromic, the integral

$$\int_{x_1}^{x} \psi(x, y(x))\, dx$$

is not monodromic either, and if the path of integration spirals around 0 it can get arbitrarily large.

É. Picard observes ([28] Vol. II p.314 and p.317, 1893, see also Vol. III p.27 and 29, 1896) that with some implicit assumptions (that are not in [7]) the argument is correct if we approach $x = 0$ along a path of finite length where the argument of $y(x)$ stays bounded or with a tangent at 0, trying (not very convincingly) to rebate L. Fuchs that pointed out the error in [13]. H. Poincaré does not mention the error in his article [29] where he states Briot and Bouquet result without any restriction, and in his Thesis [30] where he studies the case where the real part of a is positive (and carefully avoids discussing further the other problematic case).

Picard, in his first edition of his "Traité d'Analyse" ([28], Vol. III, page 30, 1896), casts no doubt about the correction of Briot and Bouquet statement:

"Il resterait à démontrer que ces deux intégrales sont, en dehors de toute hypothèse, les seules qui passent par l'origine ou qui s'en rapprochent indéfiniment. Je dois avouer que je ne possède pas une démonstration rigoureuse de cette proposition, qui ne paraît cependant pas douteuse."[2]

[2] *"It remains to prove that these two solutions are, without any assumption, the only ones passing through the origin or accumulating it. I have to admit that I don't have a proof of this fact but it doesn't seem doubtful."*

He refers to the two Briot-Bouquet holomorphic solutions $y(x)$ and $x(y)$. His belief is probably reinforced by the saddle picture for real solutions that clearly only exhibit two real solutions in $\mathbb{R}^2$ passing through the singularity.

A major progress came with the Thesis of H. Dulac published in 1904 in the Journal of the École Polytechnique [10]. He proves the existence of an infinite number of distinct non-monodromic solutions when a is a negative rational number, thus proving than Briot and Bouquet original claim is always false in the rational situation. From the introduction of [10] we can read

"... on sait depuis bien longtemps, qu'il n'existe que deux courbes intégrales réelles passant par l'origine. En est-il de même dans le champ complexe ? C'est une question qui restait en suspens et que les géomètres penchaient à trancher par l'affirmative (Picard, Traité d'Analyse, II (sic)[3]*, p. 30). Or je prouve, au contraire, tout au moins dans le cas où α est rationnel, qu'il existe une infinité d'intégrales $y(x)$ s'annulant avec x (x tendant vers zéro suivant une loi convenable) ..."* [4]

After Dulac's result Picard changed the quoted text in later editions of his Traité d'Analyse ([28], Vol. III, 3rd edition, p.30, 1928) into:

"On a longtemps présumé que ces intégrales sont, en dehors de toute hypothèse, les seules qui passent par l'origine ou qui s'en rapprochent indéfiniment. Dans un excellent travail sur les points singuliers des équations différentielles, M. Dulac a démontré que la question était très complexe. Prenons, par exemple, l'équation

$$x\frac{dy}{dx} + y(\nu + \ldots) = 0 \ ,$$

où ν est positif, équation à laquelle peut toujours se ramener le cas où λ est négatif. M. Dulac examine particulièrement le cas où ν est un nombre rationnel p/q, et montre qu'il y a alors, en général, une infinité d'intégrales pour lesquelles x et y tendent vers 0." [5]

[3]Volume III is the correct reference.

[4]"... from long time ago we know that there are only two real solutions passing through the origin. Is it the same in the complex? This is a question that remained open and that the geometers were inclined to decide in the affirmative (Picard, Traité d'Analyse, II (sic), p. 30). But, on the contrary, I prove, at least in the case when α is rational, that there are infinitely many solutions $y(x)$ vanishing with x (x converging to 0 under a suitable law) ..."

[5]"For a long time it was believed that, without any further condition, these are the only solutions passing through or accumulating the origin.

In an excellent work on the singular points of differential equations, M. Dulac has proved that the question is very complex. Take for instance the equation

$$x\frac{dy}{dx} + y(\nu + \ldots) = 0 \ ,$$

Dulac insisted in his Thesis that he had no answer for the irrational case ([10] p.4):

"1. ν est irrationnel. On a un col. $H(x,y)$ existe formellement, mais est divergent, au moins dans certains cas. S'il y a des intégrales pour lesquelles x et y tendent simultanément vers 0, et si l'on désigne par ω et θ les arguments de x et y, quels que soient m et n, $|x^m y^n \omega|$ et $|x^m y^n \theta|$ croissent indéfiniment. Je ne puis me prononcer sur l'existence de ces intégrales."[6]

The expression $yx^\nu H(x,y)$ is a formal first integral of the solutions and he discuss its convergence in p.20. It is well known to Dulac that convergence of H solves the problem.

Then 30 years later he recalls that the problem remains unsolved ([11] p.31):

"Dans le cas 2 (ν irrationnel, $h(x,y)$ divergent), on ne sait s'il existe des solutions nulles autres que $x=0$, $y=0$. Ce sont là deux questions qu'il y aurait grand intérêt à élucider."[7]

Many results obtained by these distinguished geometers of the XIXth century where rediscovered in modern times, sometimes with a different point of view or language. The original problem of singularities of differential equations of the form $\frac{0}{0}$ (according to Briot and Bouquet terminology) is equivalent to study solutions of the holomorphic vector field $X=(B,A)$ near $(0,0)\in\mathbb{C}^2$,

$$\begin{cases} \dot{x} &= B(x,y) \\ \dot{y} &= A(x,y) \end{cases}$$

The local geometry corresponds also to the study the holomorphic foliations on $\mathbb{C}^2$ near the singular point $(0,0)$ defined by the differential form

$$A(x,y)dx - B(x,y)dy = 0 \ .$$

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where ν is positive, equation that we can always reduce the case where λ is negative.

M. Dulac examines specially the case where ν is a rational number p/q, and proves that in general there are an infinite number of solutions for which x and y converge to 0."

[6] "1.ν is irrational. We have a saddle. $H(x,y)$ exists formally, but is divergent, at least in certain cases. If there are solutions x and y which tend simultaneously to 0, and if we note ω and θ the arguments of x and y, then for all m and n, $|x^m y^n \omega|$ and $|x^m y^n \theta|$ must grow indefinitely. I cannot decide on the existence of such solutions."

[7] "In case 2 (ν irrational, $h(x,y)$ divergent), we don't know if there are null solutions other than $x=0$, $y=0$."

The situation of Briot and Bouquet problem corresponds to an irreducible singularity with a non-degenerate linear part,

$$(\alpha y + \mathcal{O}(2))dx + (x + \mathcal{O}(2))dy = 0 \ ,$$

where $-\alpha = a$ is Briot and Bouquet coefficient.

When $\alpha \in \mathbb{C} - \mathbb{R}_+$, and α is neither a negative integer nor the inverse of a negative integer, we are in the Poincaré domain and the singularity is equivalent to the linear one. When α is a negative integer or its inverse, then we can conjugate the singularity to a finite Poincaré-Dulac normal form (see [2] section 24). We assume α real and positive $\alpha > 0$, which defines, in modern terminology, a singularity in the Siegel domain. The singularity is formally linearizable, but the convergence of the linearization presents problems of Small Divisors. Precisely in this situation Dulac already proved in his Thesis the existence of non-linearizable singularities in section 12. This is a notable achievement that anticipates in several decades the non-linearization results for indifferent fixed points. The existence of Briot and Bouquet holomorphic solution proves the existence of two leaves of the holomorphic foliation crossing transversally at $(0, 0)$. This means that the singularity can be put into the form

$$\alpha y(1 + \mathcal{O}(2))dx + x(1 + \mathcal{O}(2))dy = 0 \ .$$

Again $y = 0$ corresponds to the Briot and Bouquet holomorphic solution. It is now easy to make the link with the original Briot and Bouquet $\frac{0}{0}$ singulatities of differential equations. Each solution $y(x)$, distinct from the only monodromic solution $y(x) = 0$, with initial data (x_0, y_0) close to $(0, 0)$, has a graph over the x-axes that corresponds to the leaf of the foliation passing through the point (x_0, y_0). The multivaluedness or non-monodromic character of the solution can be seen in the intersection of that leaf with a transversal $\{x = x_0\}$. The y-coordinates of these points of intersection give the different values taken by the non-monodromic solution that are obtained by following a path in the leaf that projects onto the x-axes into a path circling around $x = 0$.

The topology of the foliation is understood through a holonomy construction (see [17], and for the rational case see [9]): Taking a transversal $\{x = x_0\}$ and lifting the circle $C(0, |x_0|) \subset \{y = 0\}$ in nearby leaves, the return map following this lift in the negative orientation, defines a germ of holomorphic diffeomorphism in one complex variable with a fixed point at $(x_0, 0) \subset \{x = x_0\}$. Taking a local chart in this complex line, we have a local holomorphic diffeomorphism $f \in \mathrm{Diff}(\mathbb{C}, 0)$, $f(0) = 0$, and linearizing the equations we can compute its linear part at 0,

$$f(z) = e^{2\pi i\alpha} z + \mathcal{O}(z^2) \ .$$

(to see this, note that yx^α is a first integral of the linearized differential form, thus is invariant of the solutions in the first order) Thus we get a germ of holomorphic

diffeomorphism with an indifferent irrational fixed point. It is obvious from the classical point of view that the local dynamics near 0 of this return map contains the information about the non-monodromic solutions starting at $x = x_0$. Thus we transform our original problem into a problem of holomorphic dynamics. Note that we can also reconstruct all the foliation and a neighborhood of $(0,0) \in \mathbb{C}^2$ minus the leave $\{y = 0\}$ by continuing the continuing the complex leaves from the transversal. J.-F. Mattei and P. Moussu proved in [17] that two singularities in the Siegel domain with conjugated holonomies are indeed conjugated in $\mathbb{C}^2$ by "pushing" the conjugacy along these leaves and using Riemann removability Theorem in $\mathbb{C}^2$. J.-Ch. Yoccoz and the author proved in [27] that the set of dynamical conjugacy classes of holonomies is in bijection with the set of conjugacy classes of singularities in the Siegel domain. The rational case was previously treated by J. Martinet and J.-P. Ramis ([18], [19]) by identifying the conjugacy invariants. This establish a full dictionary of the two problems. In particular, an interesting corollary is that Brjuno diophantine condition is optimal for analytic linearization of the singularity.

For our problem, the existence of non-monodromic solutions vanishing with x when $x \to 0$ following an appropriate path is equivalent to finding a leave that accumulates the singularity $(0,0)$ but distinct from the Briot and Bouquet leaves $\{x = 0\}$ and $\{y = 0\}$ and a path γ on this leave converging to $(0,0)$. This path γ projects properly in the $\{y = 0\}$ plane into a spiral around $(0,0)$ and converging to $(0,0)$. The path γ is homothopic in the leave to a path above $C(0, |x_0|)$ such that the iterates of the return map converve to $(x_0, 0)$. Since $\pi_1(\mathbb{C}^*) \approx \mathbb{Z}$, this gives an orbit of the return map that has a positive or negative orbit converging to the indifferent fixed point. Conversely, if we have such an orbit of the return map, we can push homothopicaly the path in the leave close to $\{x = 0\}$ to make it converging to $(0,0)$ (just using continuity of the foliation).

Proposition 5. *When $\alpha \in \mathbb{R}_+ - \mathbb{Q}$, Briot and Bouquet non-existence of non-monodromic solutions vanishing at 0 is equivalent to the existence of an orbit distinct from 0 that converges to 0 by iteration by the return map f or f^{-1}.*

Since linearizable dynamics don't have this property, we see that C-L. Siegel linearization theorem ([31], 1942) shows that Briot and Bouquet statement is true when $\alpha \in \mathbb{R}_+ - \mathbb{Q}$ satisfies the arithmetic linearization condition that was improved later by A.D. Brjuno ([8]) to the so called Brjuno's condition

$$\sum_{n=0}^{+\infty} \frac{\log q_{n+1}}{q_n} < +\infty \ .$$

The sequence of $(p_n/q_n)_{n\geq 0}$ are the convergents of α. The positive answer to Briot and Bouquet question in the linearizable case, that corresponds to $H(x,y)$ being convergent in Dulac's notation, was already well known to Dulac in [10].

Indeed the non-existence of non-monodromic for singularities of differential equations were well understood in the linearizable case, since H. Poincaré [29] because linearization is equivalent to the existence of a first integral of the system of the form (see [10] section [])

$$I(x,y) = yx^{\alpha}H(x,y) \ .$$

Note also that to have non-monodromic solutions $y(x)$ that accumulate into (but not converge to) 0 when $x \to 0$ is a simpler problem that is equivalent for the monodromy dynamics to have an orbit that accumulates 0 by positive or negative iteration. This was solved in general in [24] by the discovery of hedgehogs and the result that almost all points in the hedgehog for the harmonic measure have a dense orbit in the hedgehog.

What remains to be elucidated for the Briot and Bouquet problem is the non-linearizable case, and more precisely the following problem:

Problem 6. *Let $\alpha \in \mathbb{R} - \mathbb{Q}$ and $f(z) = e^{2\pi i\alpha}z + \mathcal{O}(z^2)$ be a germ of holomorphic diffeomorphism with an indifferent fixed point at 0. Does there exists $z_0 \neq 0$ such that*

$$\lim_{n\to+\infty} f^n(z_0) = 0 \ .$$

P. Fatou was confronted to this problem in his pioneer study of the dynamics of rational functions ([12], 1919) without knowing the relation to Briot and Bouquet problem. About fixed points ("points doubles" in Fatou's terminology) of holomorphic germs, which are indifferent, irrational and non-linearizable, Fatou writes [12] p.220-221:

"Il reste à étudier les points doubles dont le multiplicateur est de la forme $e^{i\alpha}$, α étant un nombre réel inconmensurable avec π. Nous ne savons que fort peu de choses sur ces points doubles, dont l'étude du point de vue qui nous occupe paraît très difficile. (...) Existe-t-il alors des domaines dont les conséquents tendent vers le point double ? Nous ne pouvons actuellement ni en donner d'exemple, ni prouver que la chose soit impossible ..." [8]

[8] *"It remains to study fixed points with a multiplier of the form $e^{i\alpha}$, α being a real number incommensurable with π. We know little about these fixed points, and their study from our point of view appears very hard. (...) Are there any domain such that the positive iterates converge to the fixed point? We cannot give examples nor rule out this possibility."*

Fatou's question is related to the question of the non-existence of wandering components of the Fatou set for rational functions. This was only proved in 1985 by D. Sullivan [33]. Note that we do indeed have domains (that are not Fatou components) converging by iteration to rational indifferent fixed point as the local analysis of the rational case shows.

The non-existence of domains converging to an indifferent irrational fixed point was also conjectured by M. Lyubich in [16] p.73 (Conjecture 1.2), apparently unaware of Fatou's question. Lyubich also conjectured (Conjecture 1.5 (a) [16] p.77) that for any indifferent irrational non-linearizable fixed point there is a critical orbit that converges to the fixed point.

The author proved in [24] the Moussu-Dulac Criterium : f is not linearizable if and only if f has an orbit accumulating the fixed point 0. We may think that this could give support to the existence of a converging orbit. The discovery of hedgehogs gave new tools for the understanding of the non-linearizable dynamics. Indeed, hedgehogs are the central tool in the final solution of all this problems:

Theorem 7. *There is no orbit converging by positive or negative iteration to an indifferent irrational fixed point of an holomorphic map and distinct from the fixed point.*

Therefore, the Briot and Bouquet problem has a positive solution in the irrational case. The questions of Dulac, Picard, Fatou are solved. Lyubich's Conjecture 1.2 in [16] has a positive answer, but conjecture 1.5 (a) in [16] is false: For a generic rational function, there is no critical point converging to an indifferent irrational non-linearizable periodic orbits. There may be pre-periodic critical points to this orbit, but this is clearly non-generic. We may formulate a proper conjecture that has better chances to hold true:

Conjecture 8. *Let f be a rational function of degree 2 or more, with an indifferent irrational non-linearizable fixed point z_0. There exists a critical point c_0 of f, such that*

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{j=0}^{n-1} \delta_{f^j(c_0)} \to \delta_{z_0} \ .$$

Theorem 7 was announced in [21] and a complete proof was given in the unpublished manuscript [25]. The proof given here concentrates on this particular Theorem and the solution of Briot and Bouquet problem, and not the many other properties of general hedgehog's dynamics. The proof follows the same lines as in [25], but we have incorporated several new ideas that greatly improve and simplify the technical part of construction of quasi-invariant curves that are fundamental in the study of

the hedgehog dynamics. It was recently noticed in [26] an hyperbolic interpretation of Denjoy-Yoccoz Lemma that controls orbits of an analytic circle diffeomorphism g in a complex neighborhood of the circle. Then, when we control the non-linearity $||D \log Dg||_{C^0}$ of g, we can construct directly the quasi-invariant curves without complex renormalization. The second observation if that in the proof of Theorem 7 we can work with local hedgehogs (small hedgehogs). Then the associated circle diffeomorphism has a small non-linearity and the construction of quasi-invariant curves is easier.

3. Analytic circle diffeomorphisms.

3.1. **Notations.** We denote by $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ the abstract circle, and $\mathbb{S}^1 = E(\mathbb{T})$ its embedding in the complex plane $\mathbb{C}$ given by the exponential mapping $E(x) = e^{2\pi i x}$.

We study analytic diffeomorphisms of the circle, but we prefer to work at the level of the universal covering, the real line, with its standard embedding $\mathbb{R} \subset \mathbb{C}$. We denote by $D^\omega(\mathbb{T})$ the space of non decreasing analytic diffeomorphisms g of the real line such that, for any $x \in \mathbb{R}$, $g(x+1) = g(x) + 1$, which is the commutation to the generator of the deck transformations $T(x) = x + 1$. An element of the space $D^\omega(\mathbb{T})$ has a well defined rotation number $\rho(g) \in \mathbb{R}$. The order preserving diffeomorphism g is conjugated to the rigid translation $T_{\rho(g)} : x \mapsto x + \rho(g)$, by an orientation preserving homeomorphism $h : \mathbb{R} \to \mathbb{R}$, such that $h(x+1) = h(x) + 1$.

For $\Delta > 0$, we note $B_\Delta = \{z \in \mathbb{C}; |\Im z| < \Delta\}$, and $A_\Delta = E(B_\Delta)$. The subspace $D^\omega(\mathbb{T}, \Delta) \subset D^\omega(\mathbb{T})$ is composed by the elements of $D^\omega(\mathbb{T})$ which extend analytically to a holomorphic diffeomorphism, denoted again by g, such that g and g^{-1} are defined on a neighborhood of $\bar{B}_\Delta$.

3.2. **Real estimates.** We refer to [36] for the results on this section. We assume that the orientation preserving circle diffeomorphism g is C^3 and that the rotation number $\alpha = \rho(g)$ is irrational. We consider the convergents $(p_n/q_n)_{n\geq 0}$ of α obtained by the continued fraction algorithm (see [14] for notations and basic properties of continued fractions).

For $n \geq 0$, we define the map $g_n(x) = g^{q_n}(x) - p_n$ and the intervals $I_n(x) = [x, g_n(x)]$, $J_n(x) = I_n(x) \cup I_n(g_n^{-1}(x)) = [g_n^{-1}(x), g_n(x)]$. Let $m_n(x) = g^{q_n}(x) - x - p_n = \pm|I_n(x)|$, $M_n = \sup_{\mathbb{R}} |m_n(x)|$, and $m_n = \min_{\mathbb{R}} |m_n(x)|$. Topological linearization is equivalent to $\lim_{n\to+\infty} M_n = 0$. This is always true for analytic diffeomorphisms by Denjoy's Theorem, that holds for C^1 diffeomorphisms such that $\log Dg$ has bounded variation.

Since g is topologically linearizable, combinatorics of the irrational translation (or the continued fration algorithm) shows:

Lemma 9. *Let $x \in \mathbb{R}$, $0 \leq j < q_{n+1}$ and $k \in \mathbb{Z}$ the intervals $g^j \circ T^k(I_n(x))$ have disjoint interiors, and the intervals $g^j \circ T^k(J_n(x))$ cover $\mathbb{R}$ at most twice.*

We have the following estimates on the Schwarzian derivatives of the iterates of f, for $0 \leq j \leq q_{n+1}$,

$$\left|Sg^j(x)\right| \leq \frac{M_n e^{2V} S}{|I_n(x)|^2} \ ,$$

with $S = ||Sg||_{C^0(\mathbb{R})}$ and $V = \mathrm{Var} \log Dg$.

This implies a control of the non-linearity of the iterates (Corollary 3.18 in [36]):

Proposition 10. *For* $0 \leq j \leq 2q_{n+1}$, $c = \sqrt{2S}e^V$,

$$||D \log Dg^j||_{C^0(\mathbb{R})} \leq c \, \frac{M_n^{1/2}}{m_n} \ .$$

These give estimates on g_n. More precisely we have (Corollary 3.20 in [36]):

Proposition 11. *For some constant* $C > 0$, *we have*

$$|| \log Dg_n||_{C^0(\mathbb{R})} \leq C M_n^{1/2} \ .$$

Corollary 12. *For any* $\epsilon > 0$, *there exists* $n_0 \geq 1$ *such that for* $n \geq n_0$, *we have*

$$||Dg_n - 1||_{C^0(\mathbb{R})} \leq \epsilon \ .$$

Proof. Take $n_0 \geq 1$ large enough so that for $n \geq n_0$, $CM_n^{1/2} < \min(\frac{2}{3}\epsilon, \frac{1}{2})$, then use Proposition 11 and $|e^w - 1| \leq \frac{3}{2}|w|$ for $|w| < 1/2$. □

Corollary 13. *For any* $\epsilon > 0$, *there exists* $n_0 \geq 1$ *such that for* $n \geq n_0$, *for any* $x \in \mathbb{R}$ *and* $y \in I_n(x)$ *we have*

$$1 - \epsilon \leq \frac{m_n(y)}{m_n(x)} \leq 1 + \epsilon \ .$$

Proof. We have $Dm_n(x) = Dg_n(x) - 1$, and

$$|m_n(y) - m_n(x)| \leq ||Dm_n||_{C^0(\mathbb{R})}|y - x| \leq ||Dg_n - 1||_{C^0(\mathbb{R})}|m_n(x)| \ .$$

We conclude using Lemma 12. □

4. Hyperbolic Denjoy-Yoccoz Lemma.

With these real estimates for the iterates, and, more precisely, a control on the non-linearity, we can use them to control orbits in a complex neighborhood. We give here a version of Denjoy-Yoccoz lemma (Proposition 4.4 in [36]) that is convenient for our purposes.

Given $\Delta > 0$, we consider $g \in D^\omega(\mathbb{T}, \Delta)$ such that $\inf_{B_\Delta} \Re Dg > 0$ so that $\log Dg$ is a well defined univalued holomorphic function in B_Δ. Given $g \in D^\omega(\mathbb{T})$ we get always this for a $\Delta > 0$ small enough (as in [36]), but here we don't need to make the assumption that for a given g, Δ is small enough.

We do assume that we have a small non-linearity in B_Δ, more precisely,

$$\tau = ||D \log Dg||_{C^0(B_\Delta)} < 1/9 \ .$$

Lemma 14. *Let $n_0 \geq 1$ large enough such that for all $n \geq n_0$, $M_n < \Delta/2$.*

For $x_0 \in \mathbb{R}$, let $0 < y_0 \leq 1$ and

$$z_0 = x_0 + i m_n(x_0) y_0 \ .$$

Then for $0 \leq j \leq q_{n+1}$, $y_j \in \mathbb{C}$, $\Re y_j > 0$, is well defined by

$$z_j = g^j(z_0) = g^j(x_0) + i m_n(g^j(x_0)) y_j \ ,$$

and we have

$$|y_j - y_0| \leq \frac{3}{4} y_0 \ .$$

Proof. For $0 < t \leq 1$ we define more generally

$$z_{0,t} = x_0 + i m_n(x_0) t y_0 \ ,$$

and we prove that $y_{j,t} \in \mathbb{C}$, $\Re y_{j,t} > 0$, is well defined by

$$z_{j,t} = g^j(z_{0,t}) = g^j(x_0) + i m_n(g^j(x_0)) y_{j,t} \ ,$$

and that we have

$$|y_{j,t} - y_{0,t}| \leq \frac{3}{4} y_{0,t} \ .$$

Note that this last inequality implies $\Re y_{j,t} \leq \frac{7}{4} y_{0,t}$. The lemma corresponds to the case $t = 1$.

We prove this result by induction on $0 \leq j < q_{n+1}$ starting from $j = 0$ for which the result is obvious. Assuming it has been proved up to $0 \leq j - 1 < q_{n+1}$, then we have

$$0 < \Im z_{j-1,t} \leq M_n \Re y_{j-1,t} \leq M_n \frac{7}{4} t y_0 < \frac{7}{8} \Delta < \Delta \ ,$$

so $z_{j-1,t} \in B_\Delta$ and we can iterate once more and $z_{j,t} = g(z_{j-1,t})$ is well defined. We need to prove the estimate for $y_{j,t}$. By the chain rule we have

$$\log Dg^j(z_{0,t}) = \sum_{l=0}^{j-1} \log Dg(z_{l,t}) \ .$$

Therefore, we have

$$\begin{aligned}
\left|\log Dg^j(z_{0,t}) - \log Dg^j(x_0)\right| &\leq \sum_{l=0}^{j-1} |\log Dg(z_{l,t}) - \log Dg(x_l)| \\
&\leq \tau \sum_{l=0}^{j-1} |z_{l,t} - x_l| \\
&\leq \tau \sum_{l=0}^{j-1} |m_n(x_l)||y_{l,t}| \\
&\leq \frac{7}{4}\tau t y_0 \sum_{l=0}^{j-1} |m_n(x_l)| \\
&\leq \frac{7}{4}\tau \sum_{l=0}^{j-1} |m_n(x_l)| \ .
\end{aligned}$$

Considering the j-iterate of g on the interval $I_n(x_0)$, we obtain a point $\zeta \in]x_0, g^{q_n}(x_0) - p_n[$ such that,

$$Dg^j(\zeta) = \frac{m_n(x_j)}{m_n(x_0)} \ ,$$

and

$$\left|\log Dg^j(\zeta) - \log Dg^j(x_0)\right| \leq \tau |m_n(x_0)| \leq \tau \sum_{l=0}^{j-1} |m_n(x_l)| \ .$$

Adding the two previous inequalities, we have

$$\left|\log Dg^j(z_{0,t}) - \log \frac{m_n(x_j)}{m_n(x_0)}\right| \leq \frac{11}{4}\tau \sum_{l=0}^{j-1} |m_n(x_l)| \ .$$

The intervals $I_n(x_l)$, $0 \leq l < q_{n+1}$, being disjoint modulo 1, we have

$$\sum_{l=0}^{q_{n+1}-1} |m_n(x_l)| < 1 \ .$$

So we obtain

$$\left|\log Dg^j(z_{0,T}) - \log \frac{m_n(x_j)}{m_n(x_0)}\right| \leq \frac{11}{4}\,\tau\ ,$$

and taking the exponential (using $|e^w - 1| \leq 3/2|w|$, for $|w| < 1/2$, since $\tau < 1/9$ and $\frac{11}{4}\,\tau < \frac{1}{2}$), we have

$$\left|Dg^j(z_{0,t}) - \frac{m_n(x_j)}{m_n(x_0)}\right| \leq \frac{33}{8}\,\tau \frac{m_n(x_j)}{m_n(x_0)}.$$

Now, integrating along the vertical segment $[x_0, z_{0,t}]$ we get

$$\left|g^j(z_{0,t}) - g^j(x_0) - iy_0 m_n(x_j)\right| \leq \frac{33}{8}\,\tau y_{0,t}|m_n(x_j)|\ ,$$

which, using $\tau < 1/9$, finally gives

$$|y_{j,t} - y_{0,t}| \leq \frac{11}{24}\,y_{0,t} < \frac{3}{4}\,y_{0,t}\ .$$

□

4.1. **Flow interpolation in $\mathbb{R}$.** Since g is analytic, from Denjoy's Theorem we know that $g_{/\mathbb{R}}$ is topologically linearizable, i.e. there exists an non-decreasing homeomorphism $h : \mathbb{R} \to \mathbb{R}$, such that for $x \in \mathbb{R}$, $h(x+1) = h(x) + 1$, and

$$h^{-1} \circ g \circ h = T_\alpha\ ,$$

where $T_\alpha : \mathbb{R} \to \mathbb{R}$, $x \mapsto x + \alpha$.

We can embed g into a topological flow on the real line $(\varphi_t)_{t\in\mathbb{R}}$ defined for $t \in \mathbb{R}$ by $\varphi_t = h \circ T_{t\alpha} \circ h^{-1}$. When g is analytically linearizable the diffeomorphisms of this flow are analytic circle diffeomorphisms, but in general, when g is not analytically linearizable the maps φ_t are only homeomorphism of the real line, although for $t \in \mathbb{Z} + \alpha^{-1}\mathbb{Z}$, φ_t is analytic since φ_t is an iterate of g composed by an integer translation. This can happen that for other values of t, where φ_t can be an analytic diffeomorphism from the analytic centralizer of g since $\varphi_t \circ g = g \circ \varphi_t$. We refer to [22] for more information on this fact and examples of uncountable analytic centralizers for non-analytically linearizable dynamics. Now $(\varphi_t)_{t\in[0,1]}$ is an isotopy from the identity to g. The flow $(\varphi_t)_{t\in\mathbb{R}}$ is a one parameter subgroup of homeomorphisms of the real line commuting to the translation by 1.

4.2. **Flow interpolation in $\mathbb{C}$.** There are different complex extensions of the flow $(\varphi_t)_{t\in\mathbb{R}}$ suitable for our purposes. For each $n \geq 0$, we can extend this topological flow to a topological flow $\mathcal{F}_n$ in $\mathbb{C}$ by defining, for $z_0 = x_0 + i\,|m_n(x_0)|y_0 \in \mathbb{C}$, with $x_0, y_0 \in \mathbb{R}$,

$$\varphi_t^{(n)}(z_0) = z_0(t) = \varphi_t(x_0) + i\,|m_n(\varphi_t(x_0))|y_0\ .$$

We denote $\Phi_{z_0}^{(n)}$ the flow line passing through z_0,

$$\Phi_{z_0}^{(n)} = (\varphi_t^{(n)}(z_0))_{t\in\mathbb{R}}.$$

4.3. **Hyperbolic Denjoy-Yoccoz Lemma.** We are now ready to give a geometric version of Denjoy-Yoccoz Lemma. We denote by d_P the Poincaré distance in the upper half plane.

Lemma 15 (Hyperbolic Denjoy-Yoccoz Lemma)**.** *Let $\Delta > 0$ and $g \in D^\omega(\mathbb{T}, \Delta)$ such that*

$$||D \ \log \ Dg||_{C^0(B_\Delta)} < 1/9 \ .$$

Let $n_0 \geq 1$ large enough such that for all $n \geq n_0$, $M_n < \Delta/2$.

Let $z_0 = x_0 + i|m_n(x_0)|y_0$, with $0 < y_0 < 1$, so $z_0 \in B_\Delta$. Then for $0 \leq j \leq q_{n+1}$ we have that the $(g^j(z_0))$ piece of orbit follows at bounded distance the flow $\mathcal{F}_n$ for the Poincaré metric of the upper half plane. More precisely we have

$$d_P(g^j(z_0), \varphi_j^{(n)}(z_0)) \leq C_0 \ ,$$

for some constant $C_0 > 0$ (we can take $C_0 = 3$).

Proof. We just use Lemma 14 reminding that the Poincaré metric in the upper half plane is given by $|ds| = \frac{|d\xi|}{\Im\xi}$ and

$$\begin{aligned} d_P(z_j, \varphi_j^{(n)}(z_0)) &\leq \int_{[z_j, \varphi_j^{(n)}(z_0)]} \frac{|d\xi|}{\Im\xi} \\ &\leq |m_n(x_j)| \, . \, |y_j - y_0| \, \frac{1}{\inf_{\xi\in[z_j,\varphi_j^{(n)}(z_0)]} \Im\xi} \\ &\leq |m_n(x_j)| \, . \, |y_j - y_0| \, \frac{4}{|m_n(x_j)| \, y_0} \\ &\leq 4 \, \frac{|y_j - y_0|}{y_0} \leq 3 = C_0 \end{aligned}$$

where in the second inequality we used that $\Re y_j \geq \frac{1}{4} y_0$ which follow from $|y_j - y_0| \leq \frac{3}{4} y_0$ that we also used in the last inequality. □

5. Quasi-invariant curves for local hedgehogs.

Now we construct quasi-invariant curves for g under the previous assumptions: $g \in D^\omega(\mathbb{T}, \Delta)$ and

$$\tau = ||D \log Dg||_{C^0(B_\Delta)} < 1/9 \ .$$

Theorem 16 (Quasi-invariant curves)**.** *Let g be an analytic circle diffeomorphism with irrational rotation number α. Let $(p_n/q_n)_{n\geq 0}$ be the sequence of convergents of α given by the continued fraction algorithm.*

Given $C_0 > 0$ there is $n_0 \geq 0$ large enough such that there is a sequence of Jordan curves $(\gamma_n)_{n\geq n_0}$ for g which are homotopic to $\mathbb{S}^1$ and exterior to $\overline{\mathbb{D}}$ such that all the iterates g^j, $0 \leq j \leq q_n$, are defined in a neighborhood of the closure of the annulus U_n bounded by $\mathbb{S}^1$ and γ_n, and we have

$$\mathcal{D}_P(g^j(\gamma_n), \gamma_n) \leq C_0 \ ,$$

where $\mathcal{D}_P$ denotes the Hausdorff distance between compact sets associated to d_P, the Poincaré distance in $\mathbb{C} - \overline{\mathbb{D}}$. We also have for any $z \in \gamma_n$, $d_P(g^{q_n}(z), z) \leq C_0$, that is,

$$||g^{q_n} - \mathrm{id}||_{C^0_P(\gamma_n)} \leq C_0 \ .$$

We choose the flow lines $\gamma_{n+1} = \Phi^{(n)}_{z_0}$, with $y_0 > 1/2$ and $n \geq n_0$ for $n_0 \geq 1$ large enough, for the quasi-invariant curves of the Theorem. These flow lines are graphs over $\mathbb{R}$. Given an interval $I \subset \mathbb{R}$, we label $\tilde{I}^{(n)}$ the piece of $\Phi^{(n)}_{z_0}$ over I.

Lemma 17. *There is $n_0 \geq 1$ such that for $n \geq n_0$ and for any $x \in \mathbb{R}$, the piece $\tilde{I}^{(n)}_n(x)$ has bounded Poincaré diameter.*

Proof. Let $z = x + i\,|m_n(x)|y_0$ be the current point in $\tilde{I}^{(n)}_n(x)$. We have

$$dz = (1 \pm i\,(Dg_n(x) - 1)y_0)\ dx \ .$$

For any $\epsilon_0 > 0$, choosing $n_0 \geq 1$ large enough, for $n \geq n_0$, according to Lemma 12 we have

$$\left|\frac{dz}{dx} - 1\right| \leq \epsilon_0 \ .$$

Therefore, we have

$$l_P(\tilde{I}^{(n)}_n(x_0)) = \int_{\tilde{I}^{(n)}_n(x_0)} \frac{1}{|m_n(x)|\,y_0}\,|dz| \leq \int_{I_n(x_0)} \frac{1}{|m_n(x)|\,y_0}\,(1+\epsilon_0)\,dx \ .$$

Now using Lemma 13 with $\epsilon = \epsilon_0$ and increasing n_0 if necessary, we have

$$l_P(\tilde{I}^{(n)}_n(x)) \leq \int_{I_n(x_0)} \frac{1}{|m_n(x_0)|\,y_0}\,\frac{1+\epsilon_0}{1-\epsilon_0}\,dx \leq \frac{1}{y_0}\frac{1+\epsilon_0}{1-\epsilon_0} \leq 2\,\frac{1+\epsilon_0}{1-\epsilon_0} \leq C \ .$$

□

We assume $n \geq n_0$ from now on in this section and the next one.

Lemma 18. *For $0 \leq j < q_{n+1}$ and any $x \in \mathbb{R}$, the pieces $(g^j \circ T^k(\tilde{J}_n^{(n)}(x)))_{0 \leq j \leq q_{n+1}, k \in \mathbb{Z}}$ have bounded Poincaré diameter and cover $\Phi_{z_0}^{(n)}$.*

Proof. From Lemma 17 any $\tilde{I}_n^{(n)}(x)$ has bounded Poincaré diameter, thus also any $\tilde{J}_n^{(n)}(x) = \tilde{I}_n^{(n)}(x) \cup \tilde{I}_n^{(n)}(g_n^{-1}(x))$. Moreover, we have $g^j \circ T^k(J_n(x)) = J_n(g^j \circ T^k(x))$, and all $\tilde{J}_n^{(n)}(g^j \circ T^k(x))$ have also bounded Poincaré diameter. From Lemma 9 these pieces cover $\Phi_{z_0}^{(n)}$. □

Corollary 19. *For some $C_0 > 0$, the flow orbit $(\varphi_{j,k}^{(n)}(z_0))_{0 \leq j < q_{n+1}, k \in \mathbb{Z}}$ is C_0-dense in $\Phi_{z_0}^{(n)}$ for the Poincaré metric.*

We prove the first property stated in Theorem 16:

Proposition 20. *Let $\gamma_n = \Phi_{z_0}^{(n-1)}$ for some z_0 from the previous lemma, then we have, for $0 \leq j \leq q_n$,*

$$\mathcal{D}_P(g^j(\gamma_n), \gamma_n) \leq 2C_0$$

Proof. We prove this Proposition for $n+1$ instead of n (the proposition is stated to match n in Theorem 16). It follows from the hyperbolic Denjoy-Yoccoz Lemma that the orbit $(g^j \circ T^k(z_0))_{0 \leq j < q_{n+1}, k \in \mathbb{Z}}$ is C_0-close to flow orbit $(\varphi_{j,k}^{(n)}(z_0))_{0 \leq j < q_{n+1}, k \in \mathbb{Z}}$, and from Corollary 19 we have that a $2C_0$-neighborhood of $g^j(\gamma_{n+1})$ contains γ_{n+1}. Conversely, since we can chooose any $z_0 \in \gamma_{n+1}$, we also have that $g^j(\gamma_{n+1})$ is in a C_0-neighborhood of γ_{n+1}. □

We prove the second property of Theorem 16. We observe that $g^{q_{n+1}}(z_0) \in \tilde{J}_n^{(n)}(x_0)$, that $z_0 \in \tilde{J}_n^{(n)}(x_0)$, and that $\tilde{J}_n^{(n)}(x_0)$ has a bounded Poincaré diameter by Lemma 18. Thus we get (taking a larger $C_0 > 0$ if necessary):

Proposition 21. *For any $z_0 \in \Phi^{(n)}$, we have*

$$d_P(z_0, g^{q_{n+1}}(z_0)) \leq C_0 \ .$$

6. Osculating orbit.

We prove the existence of an osculating orbit.

Theorem 22 (Oscullating orbit)**.** *With the above hypothesis, for $n \geq n_o$ there exists a quasi-invariant curves $\gamma_n = \Phi_{z_0}^{(n-1)}$ such that the orbit $(g^j(z_0))_{0\leq j\leq q_n}$ is such that the union of Poincaré balls*

$$U_n = \bigcup_{0\leq j<q_n, k\in\mathbb{Z}} B_P(g^j(z_0)+k, C_0) \ ,$$

separates $\mathbb{R}$ from $\{\Im z > H\}$ with $H > 0$ large enough, and any orbit $(g^j(w_0))_{j\in\mathbb{Z}}$ with $\Im w_0 > H$ with an iterate between γ_n and $\mathbb{R}$ has, for any $0 \leq j \leq q_n$, an iterate in

$$\bigcup_{k\in\mathbb{Z}} B_P(g^j(z_0)+k, C_0) \ .$$

From Lemma 18 we get the property that the hyperbolic balls $B_P(\varphi_{t+k}^{(n)}(z_0), C_0)$ cover $\Phi_{z_0}^{(n)}$.

Lemma 23. *We have that*

$$U_n = \bigcup_{0\leq j<q_{n+1}, k\in\mathbb{Z}} B_P(\varphi_{t+k}^{(n)}(z_0), C_0)$$

is a neighborhood of the flow line $\Phi_{z_0}^{(n)}$

Proof. We prove Theorem 22. In the following argument C_0 will denote several universal constants. Enlarging the constant C_0, and using Lemma 15 we can replace the points $\varphi_{t+k}^{(n)}(z_0)$ by the points $g^j(z_0)+k$ in the orbit of z_0 in Lemma 23. Also, any orbit that jumps over γ_n (by positive or negative iteration) as in Theorem 22 has to visit a C_0-neighborhood of γ_n , and will be C_0-close to a point $z_1 \in \gamma_n$ and then will be C_0-close to the q_n-orbit of z_1 modulo 1. Finally we can replace z_1 by z_0 using that each point of the q_n-orbit of z_1 is C_0-close to a point in the q_n-orbit of z_0 modulo 1 (enlarge C_0 if need be). □

7. Proof of the main Theorem.

We prove Theorems 3 and 4 that imply the main Theorem. We prove first the following preliminary Lemma that will allow us to work only with local hedgehogs.

Lemma 24. *Let $g_n \in D^\omega(\mathbb{T}, \Delta_n)$ with $\rho(g_n) = \alpha$ and $\Delta_n \to +\infty$. Then $g_n \to R_\alpha$ uniformly on compact sets of $\mathbb{C}^*$ and*

$$\lim_{n\to+\infty} ||D\log Dg_n||_{C^0(\mathbb{R})} = 0 \ .$$

Proof. Let $\tilde{g}_n$ be the associated circle diffeomorphism. The sequence $(\tilde{g}_n)$ is a normal family in $\mathbb{C}^*$ (bounded inside $\mathbb{D}$, and outside is the reflection across the unit circle), and any accumulation point is not constant since the unit circle is in the image of all

g_n. Then by Hurwitz theorem any limit is an automorphism of $\mathbb{C}^*$, that extends to 0 by Riemann's theorem, and so gives an automorphism of the plane leaving the unit circle invariant. The rotation number on the circle depends continuously on $\tilde{g}_n$ and is constant equal to α, therefore the only possible limit of the sequence (g_n) is R_α. Since $D \log DR_\alpha = 0$ we get the last statement. □

We consider now the hedgehog K_0 given by Theorem 2 for the domain $U = \mathbb{D}_{r_0}$, and we use the relation between hedgehogs and analytic circle diffeomorphisms presented in [24] to construct a circle diffeomorphism g_0.

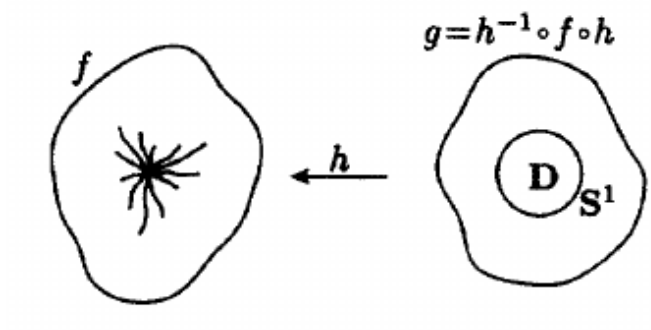

FIGURE 2. Relation between hedgehogs and circle maps.

We consider a conformal representation $h_0 : \mathbb{C} - \overline{\mathbb{D}} \to \mathbb{C} - K_0$ ($\mathbb{D}$ is the unit disk), and we conjugate the dynamics to a univalent map g_0 in an annulus V having the circle $\mathbb{S}^1 = \partial\mathbb{D}$ as the inner boundary,

$$g_0 = h_0^{-1} \circ f \circ h_0 : V \to \mathbb{C} \ .$$

The topology of K_0 is complex ([4], [5], [22]) and in particular K_0 is never locally connected, and h_0 does not extend to a continuous correspondence between $\mathbb{S}^1$ and ∂K_0. Nevertheless, f extends continuously to Caratheodory's prime-end compactification of $\mathbb{C} - K_0$. This shows that g_0 extends continuously to $\mathbb{S}^1$ and its Schwarz reflection defines an analytic map of the circle defined on $V \cup \mathbb{S}^1 \cup \bar{V}$, where $\bar{V}$ is the reflected annulus of V. Then it is not difficult to see that g_0 is an analytic circle diffeomorphism. We can also prove that g_0 has rotation number α. This is harder to prove in general (for an aribtrary hedgehog), but it is not difficult to show that we can pick K_0 so that the rotation number of g_0 is α (see [24] Lemma III.3.3) that is enough for our purposes. We choose such a K_0. Therefore, the dynamics in a complex neighborhood of K_0 corresponds to the dynamics of an analytic circle diffeomorphism with rotation number α.

There is no risk of confusion and we denote also g_0 the lift to $\mathbb{R}$.

Theorem 25. *Let $\epsilon_0 > 0$ and $\Delta > 0$ be given*

For $r_0 > 0$ small enough we have $g_0 \in D^\omega(\mathbb{T}, \Delta)$ and

$$||D \log Dg_0||_{C^0(\mathbb{R})} < \epsilon_0 \ .$$

Proof. When $r_0 \to 0$, we have $K_0 \to \{0\}$ and the annulus where g_0 and g_0^{-1} are defined has a modulus $M_0 \to +\infty$. Therefore, by Grötsch extremal problem, for $r_0 > 0$ small enough we have $g_0 \in D^\omega(\mathbb{T}, \Delta)$. From Lemma 24

$$\lim_{r_0 \to 0} ||D \log Dg_0||_{C^0(\mathbb{R})} = 0 \ ,$$

and the result follows. □

Let $\epsilon_0 = 1/9$ and $\Delta > 0$ be as in Section 5 and Section 6. We fix now $r_0 > 0$ small enough such that $g_0 \in D^\omega(\mathbb{T}, \Delta)$, $\rho(g_0) = \alpha$, and

$$||D \log Dg_0||_{C^0(\mathbb{R})} < \epsilon_0 \ ,$$

so that the hypothesis of Theorem 16 are fulfilled for g_0. Now we can apply Theorem 22 and find a sequence $(\gamma_n)_{n \geq n_0}$ of quasi-invariant curves for g_0. We transport them by h_0 to get a sequence of Jordan curves $(\eta_n)_{n \geq n_0}$

$$\eta_n = h_0(\gamma_n) \ .$$

We have

$$||g_0^{q_n} - \mathrm{id}||_{C^0_P(\gamma_n)} \leq C_0 \ ,$$

therefore, for the Poincaré metric of the exterior of the hedgehog,

$$||f^{q_n} - \mathrm{id}||_{C^0_P(\eta_n)} \leq C_0 \ ,$$

and, since $\eta_n \to K_0$, for the euclidean metric, we have

$$||f^{q_n} - \mathrm{id}||_{C^0(\eta_n)} = \epsilon_n \to 0 \ .$$

Thus, if Ω_n is the Jordan domain bounded by η_n, by the maximum principle we have

$$||f^{q_n} - \mathrm{id}||_{C^0(\Omega_n)} = \epsilon_n \to 0 \ .$$

Since Ω_n is a neighborhood of K_0, $K_0 \subset \bar{\Omega}_n$, we have

$$||f^{q_n} - \mathrm{id}||_{C^0(K_0)} = \epsilon_n \to 0 \ .$$

This proves Theorem 3 for the positive iterates (same proof for the negative ones, or just apply the result to f^{-1}).

We prove Theorem 4 for K_0, or more precisely for ∂K_0 that was noted before that is enough for proving the Main Theorem (the hedgehog K_0 has empty interior and $K_0 = \partial K_0$, but we don't need to use this fact). For the proof of Theorem 4 we

transport by h_0 the Poincaré C_0-dense orbit $(g_0^j(z_0))_{0\leq j\leq q_n}$ given by Theorem 22. Let $\zeta_0 = h_0(z_0)$ and $\mathcal{O}_n = (f^j(\zeta_0))_{0\leq j\leq q_n}$ be this orbit. Since $\eta_n \to \partial K_0$, we have, for $\epsilon_n \to 0$

$$\mathcal{D}(\mathcal{O}_n, \partial K_0) \leq \mathcal{D}(\mathcal{O}_n, \eta_n) + \mathcal{D}(\eta_n, \partial K_0) \leq \epsilon_n \ ,$$

where $\mathcal{D}$ denotes the Hausdorff distance for the euclidean metric.

Then any orbit starting outside of η_n with an iterate inside η_n must visit ϵ_n-close for the euclidean metric any point of ∂K_0 that is strictly larger than $\{0\}$.

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