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Sharp exponential inequalities in survey sampling: conditional Poisson sampling schemes

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Abstract

This paper is devoted to establishing exponential bounds for the probabilities of deviation of a sample sum from its expectation, when the variables involved in the summation are obtained by sampling in a finite population according to a rejective scheme, generalizing sampling without replacement, and by using an appropriate normalization. In contrast to Poisson sampling, classical deviation inequalities in the i.i.d. setting do not straightforwardly apply to sample sums related to rejective schemes, due to the inherent dependence structure of the sampled points. We show here how to overcome this difficulty, by combining the formulation of rejective sampling as Poisson sampling conditioned upon the sample size with the Escher transformation. In particular, the Bennett/Bernstein type bounds established highlight the effect of the asymptotic variance σ_N^2 of the (properly standardized) sample weighted sum and are shown to be much more accurate than those based on the negative association property shared by the terms involved in the summation. Beyond its interest in itself, such a result for rejective sampling is crucial, insofar as it can be extended to many other sampling schemes, namely those that can be accurately approximated by rejective plans in the sense of the total variation distance.

AMS 2015 subject classification: 60E15, 6205.

Keywords and phrases: Exponential inequality ; Poisson survey

scheme; rejective sampling; survey sampling; Escher transformation;
coupling

1 Introduction

Whereas many upper bounds for the probability that a sum of independent real-valued (integrable) random variables exceeds its expectation by a specified threshold value $t \in \mathbb{R}$ are documented in the literature (see *e.g.* [10] and the references therein), very few results are available when the random variables involved in the summation are sampled from a finite population according to a given survey scheme and next appropriately normalized (using the related survey weights as originally proposed in [20] for approximating a total). The sole situation where results in the independent setting straightforwardly carry over to survey samples (without replacement) corresponds to the case where the variables are sampled independently with possibly unequal weights, *i.e.* Poisson sampling. For more complex sampling plans, the dependence structure between the sampled variables makes the study of the fluctuations of the resulting weighted sum approximating the total (referred to as the *Horvitz-Thompson total estimate*) very challenging. The case of basic *sampling without replacement* (SWOR in abbreviated form) has been first considered in [19], and refined in [25] and [2]. In contrast, the asymptotic behavior of the Horvitz-Thompson estimator as N tends to infinity is well-documented in the literature. Following in the footsteps of the seminal contribution [18], a variety of limit results (*e.g.* consistency, asymptotic normality) have been established for Poisson sampling and next extended to rejective sampling viewed as conditional Poisson sampling given the sample size and to sampling schemes that are closed to the latter in a *coupling* sense in [24] and [4]. Although the nature of the results established in this paper are nonasymptotic, these arguments (conditioning upon the sampling size and coupling) are involved in their proofs.

It is indeed the major purpose of this article to extend tail bounds proved for SWOR to the case of rejective sampling, a fixed size sampling scheme generalizing it. The approach we develop is thus based on viewing rejective sampling as conditional Poisson sampling given the sample size and writing then the deviation probability as a ratio of two quantities: the joint probability that a Poisson sampling-based total estimate exceeds the threshold t and the size of the cardinality of the Poisson sample equals the (deterministic) size n of the rejective plan considered in the numerator and the probability that the Poisson sample size is equal to n in the denominator.

Whereas a sharp lower bound for the denominator can be straightforwardly derived from a local Berry-Esseen bound proved in [13] for sums of independent, possibly non identically distributed, Bernoulli variables, an accurate upper bound for the numerator can be established by means of an appropriate exponential change of measure (*i.e.* Escher transformation), following in the footsteps of the method proposed in [27], a refinement of the classical argument of Bahadur-Rao's theorem in order to improve exponential bounds in the independent setting. The tail bounds (of Bennett/Bernstein type) established by means of this method are shown to be sharp in the sense that they explicitly involve the 'small' asymptotic variance of the Horvitz-Thompson total estimate based on rejective sampling, in contrast to those proved by using the *negative association* property of the sampling scheme.

The article is organized as follows. A few key concepts pertaining to survey theory are recalled in section 2, as well as specific properties of Poisson and rejective sampling schemes. For comparison purpose, preliminary tail bounds in the (conditional) Poisson case are stated in section 3. The main results of the paper, sharper exponential bounds for conditional Poisson sampling namely, are proved in section 4, while section 5 explains how they can be extended to other sampling schemes, sufficiently close to rejective sampling in the sense of the total variation norm. A few remarks are finally collected in section 6 and some technical details are deferred to the Appendix section.

2 Background and Preliminaries

As a first go, we start with briefly recalling basic notions in survey theory, together with key properties of (conditional) Poisson sampling schemes. Here and throughout, the indicator function of any event \mathcal{E} is denoted by $\mathbb{I}\{\mathcal{E}\}$, the power set of any set E by $\mathcal{P}(E)$, the variance of any square integrable r.v. Y by $Var(Y)$, the cardinality of any finite set E by $\#E$ and the Dirac mass at any point a by δ_a . For any real number x , we set $x^+ = \max\{x, 0\}$, $x^- = \max\{-x, 0\}$, $\lceil x \rceil = \inf\{k \in \mathbb{Z} : x \leq k\}$ and $\lfloor x \rfloor = \sup\{k \in \mathbb{Z} : k \leq x\}$.

2.1 Sampling schemes and Horvitz-Thompson estimation

Consider a finite population of $N \geq 1$ distinct units, $\mathcal{I}_N = \{1, \dots, N\}$ say, a survey sample of (possibly random) size $n \leq N$ is any subset $s = \{i_1, \dots, i_{n(s)}\} \in \mathcal{P}(\mathcal{I}_N)$ of size $n(s) = n$. A sampling design without

replacement is defined as a probability distribution R_N on the set of all possible samples $s \in \mathcal{P}(\mathcal{I}_N)$. For all $i \in \mathcal{I}_N$, the probability that the unit i belongs to a random sample S defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and drawn from distribution R_N is denoted by $\pi_i = \mathbb{P}\{i \in S\} = R_N(\{i\})$. The π_i 's are referred to as *first order inclusion probabilities*. The *second order inclusion probability* related to any pair $(i, j) \in \mathcal{I}_N^2$ is denoted by $\pi_{i,j} = \mathbb{P}\{(i, j) \in S^2\} = R_N(\{i, j\})$ (observe that $\pi_{i,i} = \pi_i$). Here and throughout, we denote by $\mathbb{E}[\cdot]$ the \mathbb{P} -expectation and by $Var(Z)$ the conditional variance of any \mathbb{P} -square integrable r.v. $Z : \Omega \rightarrow \mathbb{R}$.

The random vector $\epsilon_N = (\epsilon_1, \dots, \epsilon_N)$ defined on $(\Omega, \mathcal{F}, \mathcal{P})$, where $\epsilon_i = \mathbb{I}\{i \in S\}$ fully characterizes the random sample $S \in \mathcal{P}(\mathcal{I}_N)$. In particular, the sample size $n(S)$ is given by $n = \sum_{i=1}^N \epsilon_i$, its expectation and variance by $\mathbb{E}[n(S)] = \sum_{i=1}^N \pi_i$ and $Var(n(S)) = \sum_{1 \leq i, j \leq N} \{\pi_{i,j} - \pi_i \pi_j\}$ respectively. The 1-dimensional marginal distributions of the random vector ϵ_N are the Bernoulli distributions $Ber(\pi_i) = \pi_i \delta_1 + (1 - \pi_i) \delta_0$, $1 \leq i \leq N$ and its covariance matrix is $\Gamma_N = (\pi_{i,j} - \pi_i \pi_j)_{1 \leq i, j \leq N}$.

We place ourselves here in the *fixed-population* or *design-based* sampling framework, meaning that we suppose that a fixed (unknown) real value x_i is assigned to each unit $i \in \mathcal{I}_N$. As originally proposed in the seminal contribution [20], the Horvitz-Thompson estimate of the population total $S_N = \sum_{i=1}^N x_i$ is given by

$$\widehat{S}_{\pi_N}^{\epsilon_N} = \sum_{i=1}^N \frac{\epsilon_i}{\pi_i} x_i = \sum_{i \in S} \frac{1}{\pi_i} x_i, \quad (1)$$

with $0/0 = 0$ by convention. Throughout the article, we assume that the π_i 's are all strictly positive. Hence, the conditional expectation of (1) is $\mathbb{E}[\widehat{S}_{\pi_N}^{\epsilon_N}] = S_N$ and, in the case where the size of the random sample is deterministic, its variance is given by

$$Var(\widehat{S}_{\pi_N}^{\epsilon_N}) = \sum_{i < j} \left(\frac{x_i}{\pi_i} - \frac{x_j}{\pi_j} \right)^2 \times (\pi_i \pi_j - \pi_{i,j}). \quad (2)$$

The goal of this paper is to establish accurate bounds for tail probabilities

$$\mathbb{P}\{\widehat{S}_{\pi_N}^{\epsilon_N} - S_N > t\}, \quad (3)$$

where $t \in \mathbb{R}$, when the sampling scheme ϵ_N is *rejective*, a very popular sampling plan that generalizes *random sampling without replacement* and can be expressed as a conditional Poisson scheme, as recalled in the following subsection for clarity. One may refer to [14] for instance for an excellent account of survey theory, including many more examples of sampling designs.

2.2 Poisson and conditional Poisson sampling

Undoubtedly, one of the simplest sampling plan is the *Poisson survey scheme* (without replacement), a generalization of *Bernoulli sampling* originally proposed in [17] for the case of unequal weights: the ϵ_i 's are independent and the sampling distribution P_N is thus entirely determined by the first order inclusion probabilities $\mathbf{p}_N = (p_1, \dots, p_N) \in]0, 1[^N$:

$$\forall s \in \mathcal{P}(\mathcal{I}_N), \quad P_N(s) = \prod_{i \in s} p_i \prod_{i \notin s} (1 - p_i). \quad (4)$$

Observe in addition that the behavior of the quantity (1) can be investigated by means of results established for sums of independent random variables. However, the major drawback of this sampling plan lies in the random nature of the corresponding sample size, impacting significantly the variability of (1). The variance of the Poisson sample size is given by $d_N = \sum_{i=1}^N p_i(1-p_i)$, while the variance of (1) is in this case:

$$\text{Var} \left(\widehat{S}_{\pi_N}^{\epsilon_N} \right) = \sum_{i=1}^N \frac{1-p_i}{p_i} x_i^2.$$

For this reason, *rejective sampling*, a sampling design R_N of fixed size $n \leq N$, is often preferred in practice. It generalizes the *simple random sampling without replacement* (where all samples with cardinality n are equally likely to be chosen, with probability $(N-n)!/n!$, all the corresponding first and second order probabilities being thus equal to n/N and $n(n-1)/(N(N-1))$ respectively). Denoting by $\pi_N^R = (\pi_1^R, \dots, \pi_N^R)$ its first order inclusion probabilities and by $\mathcal{S}_n = \{s \in \mathcal{P}(\mathcal{I}_N) : \#s = n\}$ the subset of all possible samples of size n , it is defined by:

$$\forall s \in \mathcal{S}_n, \quad R_N(s) = C \prod_{i \in s} p_i^R \prod_{i \notin s} (1 - p_i^R), \quad (5)$$

where $C = 1 / \sum_{s \in \mathcal{S}_n} \prod_{i \in s} p_i^R \prod_{i \notin s} (1 - p_i^R)$ and the vector of parameters $\mathbf{p}_N^R = (p_1^R, \dots, p_N^R) \in]0, 1[^N$ yields first order inclusion probabilities equal to the π_i^R 's and is such that $\sum_{i=1}^N p_i^R = n$. Under this latter additional condition, such a vector \mathbf{p}_N^R exists and is unique (see [15]) and the related representation (5) is then said to be *canonical*. Notice incidentally that any vector $\mathbf{p}'_N \in]0, 1[^N$ such that $p_i^R/(1-p_i^R) = cp'_i/(1-p'_i)$ for all $i \in \{1, \dots, n\}$ for some constant $c > 0$ can be used to write a representation of R_N of the same type as (5). Comparing (5) and (4) reveals that rejective R_N sampling

of fixed size n can be viewed as Poisson sampling given that the sample size is equal to n . It is for this reason that rejective sampling is usually referred to as *conditional Poisson sampling*. For simplicity's sake, the superscript R is omitted in the sequel. One must pay attention not to get the π_i 's and the p_i 's mixed up (except in the SWOR case, where these quantities are all equal to n/N): the latter are the first order inclusion probabilities of P_N , whereas the former are those of its conditional version R_N . However they can be related by means of the results stated in [18] (see Theorem 5.1 therein, as well as Lemma 6 in section 4 and [8]): $\forall i \in \{1, \dots, N\}$,

$$\pi_i(1 - p_i) = p_i(1 - \pi_i) \times (1 - (\tilde{\pi} - \pi_i)/d_N^* + o(1/d_N^*)), \quad (6)$$

$$p_i(1 - \pi_i) = \pi_i(1 - p_i) \times (1 - (\tilde{p} - p_i)/d_N + o(1/d_N)), \quad (7)$$

where $d_N^* = \sum_{i=1}^N \pi_i(1 - \pi_i)$, $d_N = \sum_{i=1}^N p_i(1 - p_i)$, $\tilde{\pi} = (1/d_N^*) \sum_{i=1}^N \pi_i^2(1 - \pi_i)$ and $\tilde{p} = (1/d_N) \sum_{i=1}^N (p_i)^2(1 - p_i)$.

Since the major advantage of conditional Poisson sampling lies in its reduced variance property (compared to Poisson sampling in particular, see the discussion in section 4), focus is next on exponential inequalities involving a variance term, of Bennett/Bernstein type namely.

3 Preliminary Results

As a first go, we establish tail bounds for the Horvitz-Thompson estimator in the case where the variables are sampled according to a Poisson scheme. We next show how to exploit the *negative association* property satisfied by rejective sampling in order to extend the latter to conditional Poisson sampling. Of course, this approach do not account for the reduced variance property of Horvitz-Thompson estimates based on rejective sampling, it is the purpose of the next section to improve these first exponential bounds.

3.1 Tails bounds for Poisson sampling

As previously observed, bounding the tail probability (3) is easy in the Poisson situation insofar as the variables summed up in (1) are independent though possibly non identically distributed (since the inclusion probabilities are not assumed to be all equal). The following theorem thus directly follows from well-known results related to tail bounds for sums of independent random variables.

Theorem 1. (POISSON SAMPLING) *Assume that the survey scheme ϵ_N defines a Poisson sampling plan with first order inclusion probabilities $p_i > 0$, with $1 \leq i \leq N$. Then, we almost-surely have: $\forall t > 0, \forall N \geq 1$,*

$$\mathbb{P} \left\{ \widehat{S}_{\mathbf{p}_N}^{\epsilon_N} - S_N > t \right\} \leq \exp \left(- \frac{\sum_{i=1}^N \frac{1-p_i}{p_i} x_i^2}{\left(\max_{1 \leq i \leq N} \frac{x_i}{p_i} \right)^2} H \left(\frac{\max_{1 \leq i \leq N} \frac{|x_i|}{p_i} t}{\sum_{i=1}^N \frac{1-p_i}{p_i} x_i^2} \right) \right) \quad (8)$$

$$\leq \exp \left(\frac{-t^2}{\frac{2}{3} \max_{1 \leq i \leq N} \frac{|x_i|}{p_i} + 2 \sum_{i=1}^N \frac{1-p_i}{p_i} x_i^2} \right), \quad (9)$$

where $H(x) = (1+x) \log(1+x) - x$ for $x \geq 0$.

Bounds (8) and (9) straightforwardly result from Bennett inequality [3] and Bernstein exponential inequality [5] respectively, when applied to the independent random variables $(\epsilon_i/p_i)x_i$, $1 \leq i \leq N$. By applying these results to the variables $-(\epsilon_i/p_i)x_i$'s, the same bounds naturally hold for the deviation probability $\mathbb{P}\{\widehat{S}_{\mathbf{p}_N}^{\epsilon_N} - S_N < -t\}$ (and, incidentally, for $\mathbb{P}\{|\widehat{S}_{\mathbf{p}_N}^{\epsilon_N} - S_N| > t\}$ up to a factor 2). Details, as well as extensions to other deviation inequalities (see *e.g.* [16]), are left to the reader.

3.2 Exponential inequalities for sums of negatively associated random variables

For clarity, we first recall the definition of *negatively associated random variables*, see [22].

Definition 1. *Let Z_1, \dots, Z_n be random variables defined on the same probability space, valued in a measurable space (E, \mathcal{E}) . They are said to be negatively associated iff for any pair of disjoint subsets A_1 and A_2 of the index set $\{1, \dots, n\}$*

$$\text{Cov}(f((Z_i)_{i \in A_1}), g((Z_j)_{j \in A_2})) \leq 0, \quad (10)$$

for any real valued measurable functions $f : E^{\#A_1} \rightarrow \mathbb{R}$ and $g : E^{\#A_2} \rightarrow \mathbb{R}$ that are both increasing in each variable.

The following result provides tail bounds for sums of negatively associated random variables, which extends the usual Bennett/Bernstein inequalities in the i.i.d. setting, see [3] and [5].

Theorem 2. Let Z_1, \dots, Z_N be square integrable negatively associated real valued random variables such that $|Z_i| \leq c$ a.s. and $\mathbb{E}[Z_i] = 0$ for $1 \leq i \leq N$. Let a_1, \dots, a_N be non negative constants and set $\sigma^2 = \frac{1}{N} \sum_{i=1}^N a_i^2 \text{Var}(Z_i)$. Then, for all $t > 0$, we have: $\forall N \geq 1$,

$$\mathbb{P} \left\{ \sum_{i=1}^N a_i Z_i \geq t \right\} \leq \exp \left(-\frac{N\sigma^2}{c^2} H \left(\frac{ct}{N\sigma^2} \right) \right) \quad (11)$$

$$\leq \exp \left(-\frac{t^2}{2N\sigma^2 + \frac{2ct}{3}} \right). \quad (12)$$

Before detailing the proof, observe that the same bounds hold true for the tail probability $\mathbb{P} \left\{ \sum_{i=1}^N a_i Z_i \leq -t \right\}$ (and for $\mathbb{P} \left\{ \left| \sum_{i=1}^N a_i Z_i \right| \geq t \right\}$ as well, up to a multiplicative factor 2). Refer also to Theorem 4 in [21] for a similar result in a more restrictive setting (*i.e.* for tail bounds related to sums of *negatively related* r.v.'s) and to [26] as well.

PROOF.

The proof starts off with the usual Chernoff method: for all $\lambda > 0$,

$$\mathbb{P} \left\{ \sum_{i=1}^N a_i Z_i \geq t \right\} \leq \exp \left(-t\lambda + \log \mathbb{E} \left[e^{t \sum_{i=1}^N a_i Z_i} \right] \right). \quad (13)$$

Next, observe that, for all $t > 0$, we have

$$\begin{aligned} \mathbb{E} \left[\exp \left(t \sum_{i=1}^n a_i Z_i \right) \right] &= \mathbb{E} \left[\exp(t a_n Z_n) \exp \left(t \sum_{i=1}^{n-1} a_i Z_i \right) \right] \\ &\leq \mathbb{E} [\exp(t a_n Z_n)] \mathbb{E} \left[\exp \left(t \sum_{i=1}^{n-1} a_i Z_i \right) \right] \\ &\leq \prod_{i=1}^n \mathbb{E} [\exp(t a_i Z_i)], \end{aligned} \quad (14)$$

using the property (10) combined with a descending recurrence on i . The proof is finished by plugging (14) into (13) and optimizing finally the resulting bound w.r.t. $\lambda > 0$, just like in the proof of the classic Bennett/Bernstein inequalities, see [3] and [5]. \square

The first assertion of the theorem stated below reveals that any rejective scheme ϵ_N^* forms a collection of negatively related r.v.'s, the second one appearing then as a direct consequence of Theorem 2. We underline that

many sampling schemes (e.g. Rao-Sampford sampling, Pareto sampling, Srinivasan sampling) of fixed size are actually described by random vectors ϵ_N with negatively associated components, see [11] or [23], so that exponential bounds similar to that stated below can be proved for such sampling plans.

Theorem 3. *Let $N \geq 1$ and $\epsilon_N^* = (\epsilon_1^*, \dots, \epsilon_N^*)$ be the vector of indicator variables related to a rejective plan on \mathcal{I}_N with first order inclusion probabilities $(\pi_1, \dots, \pi_N) \in]0, 1]^N$. Then, the following assertions hold true.*

(i) *The binary random variables $\epsilon_1^*, \dots, \epsilon_N^*$ are negatively related.*

(ii) *For any $t \geq 0$ and $N \geq 1$, we have:*

$$\begin{aligned} \mathbb{P} \left\{ \widehat{S}_{\pi}^{\epsilon_N^*} - S_N \geq t \right\} &\leq 2 \exp \left(- \frac{\sum_{i=1}^N \frac{1-\pi_i}{\pi_i} x_i^2}{\left(\max_{1 \leq i \leq N} \frac{x_i}{\pi_i} \right)^2} H \left(\frac{\max_{1 \leq i \leq N} \frac{|x_i|}{\pi_i} t/2}{\sum_{i=1}^N \frac{1-\pi_i}{\pi_i} x_i^2} \right) \right) \\ &\leq 2 \exp \left(\frac{-t^2/4}{\frac{2}{3} \max_{1 \leq i \leq N} \frac{|x_i|}{\pi_i} t + 2 \sum_{i=1}^N \frac{1-\pi_i}{\pi_i} x_i^2} \right). \end{aligned}$$

PROOF.

Considering the usual representation of the distribution of $(\epsilon_1, \dots, \epsilon_N)$ as the conditional distribution of a sample of independent Bernoulli variables $(\epsilon_1^*, \dots, \epsilon_N^*)$ conditioned upon the event $\sum_{i=1}^N \epsilon_i^* = n$ (see subsection 2.2), Assertion (i) is a straightforward consequence from Theorem 2.8 in [22] (see also [1]). Assertion (i) shows in particular that Theorem 2 can be applied to the random variables $\{(\epsilon_i^*/\pi_i - 1)x_i^+ : 1 \leq i \leq N\}$ and to the random variables $\{(\epsilon_i^*/\pi_i - 1)x_i^- : 1 \leq i \leq N\}$ as well. Using the union bound, we obtain that

$$\begin{aligned} \mathbb{P} \left\{ \widehat{S}_{\pi}^{\epsilon_N^*} - S_N \geq t \right\} &\leq \mathbb{P} \left\{ \sum_{i=1}^N \left(\frac{\epsilon_i^*}{\pi_i} - 1 \right) x_i^+ \geq t/2 \right\} \\ &\quad + \mathbb{P} \left\{ \sum_{i=1}^N \left(\frac{\epsilon_i^*}{\pi_i} - 1 \right) x_i^- \leq -t/2 \right\}, \end{aligned}$$

and a direct application of Theorem 2 to each of the terms involved in this bound straightforwardly proves Assertion (ii). \square

The negative association property permits to handle the dependence of the terms involved in the summation. However, it may lead to rather loose probability bounds. Indeed, except the factor 2, the bounds of Assertion (ii) exactly correspond to those stated in Theorem 1, as if the ϵ_i^* 's were independent, whereas the asymptotic variance σ_N^2 of $\widehat{S}_{\boldsymbol{\pi}}^{\epsilon^*}$ can be much smaller than $\sum_{i=1}^N (1 - \pi_i)x_i^2/\pi_i$. It is the goal of the subsequent analysis to improve these preliminary results and establish exponential bounds involving the asymptotic variance σ_N^2 .

Remark 1. (SWOR) *We point out that in the specific case of sampling without replacement, i.e. when $\pi_i = n/N$ for all $i \in \{1, \dots, N\}$, the inequality stated in Assertion (ii) is quite comparable (except the factor 2) to that which can be derived from the Chernoff bound given in [19], see Proposition 2 in [2].*

4 Main Results - Exponential Inequalities for Rejective Sampling

The main results of the paper are stated and discussed in the present section. More accurate deviation probabilities related to the total estimate (1) based on a rejective sampling scheme $\boldsymbol{\epsilon}_N^*$ of (fixed) sample size $n \leq N$ with first order inclusion probabilities $\boldsymbol{\pi}_N = (\pi_1, \dots, \pi_N)$ and canonical representation $\mathbf{p}_N = (p_1, \dots, p_N)$ are now investigated. Consider $\boldsymbol{\epsilon}_N$ a Poisson scheme with \mathbf{p}_N as vector of first order inclusion probabilities. As previously recalled, the distribution of $\boldsymbol{\epsilon}_N^*$ is equal to the conditional distribution of $\boldsymbol{\epsilon}_N$ given $\sum_{i=1}^N \epsilon_i = n$:

$$(\epsilon_1^*, \epsilon_2^*, \dots, \epsilon_N^*) \stackrel{d}{=} (\epsilon_1, \dots, \epsilon_N) \mid \sum_{i=1}^N \epsilon_i = n. \quad (15)$$

Hence, we almost-surely have: $\forall t > 0, \forall N \geq 1$,

$$\mathbb{P} \left\{ \widehat{S}_{\boldsymbol{\pi}_N}^{\boldsymbol{\epsilon}_N^*} - S_N > t \right\} = \mathbb{P} \left\{ \sum_{i=1}^N \frac{\epsilon_i}{\pi_i} x_i - S_N > t \mid \sum_{i=1}^N \epsilon_i = n \right\}. \quad (16)$$

As a first go, we shall prove tail bounds for the quantity

$$\widehat{S}_{\mathbf{p}_N}^{\boldsymbol{\epsilon}_N^*} \stackrel{def}{=} \sum_{i=1}^N \frac{\epsilon_i^*}{p_i} x_i. \quad (17)$$

Observe that this corresponds to the HT estimate of the total $\sum_{i=1}^N \frac{p_i}{\pi_i} x_i$. Refinements of relationships (6) and (7) between the p_i 's and the π_i 's shall next allow us to obtain an upper bound for (16). Notice incidentally that, though slightly biased (see Assertion (i) of Theorem 5), the statistic (17) is commonly used as an estimator of S_N , insofar as the parameters p_i 's are readily available from the canonical representation of ϵ_N^* , whereas the computation of the π_i 's is much more complicated. One may refer to [12] for practical algorithms dedicated to this task. Hence, Theorem 4 is of practical interest to build non asymptotic confidence intervals for the total S_N .

Asymptotic variance. Recall that $d_N = \sum_{i=1}^N p_i(1 - p_i)$ is the variance $Var(\sum_{i=1}^N \epsilon_i)$ of the size of the Poisson plan ϵ_N and set

$$\theta_N = \frac{\sum_{i=1}^N x_i(1 - p_i)}{d_N}.$$

As explained in [6], the quantity θ_N is the coefficient of the linear regression relating $\sum_{i=1}^N \frac{\epsilon_i}{p_i} x_i - S_N$ to the sample size $\sum_{i=1}^N \epsilon_i$. We may thus write

$$\sum_{i=1}^N \frac{\epsilon_i}{p_i} x_i - S_N = \theta_N \times \sum_{i=1}^N \epsilon_i + r_N,$$

where the residual r_N is orthogonal to $\sum_{i=1}^N \epsilon_i$. Hence, we have the following decomposition

$$Var \left(\sum_{i=1}^N \frac{\epsilon_i}{p_i} x_i \right) = \sigma_N^2 + \theta_N^2 d_N, \quad (18)$$

where

$$\sigma_N^2 = Var \left(\sum_{i=1}^N (\epsilon_i - p_i) \left(\frac{x_i}{p_i} - \theta_N \right) \right) \quad (19)$$

is the asymptotic variance of the statistic $\widehat{S}_{\mathbf{p}_N}^{\epsilon_N^*}$, see [18]. In other words, the variance reduction resulting from the use of a rejective sampling plan instead of a Poisson plan is equal to $\theta_N^2 d_N$, and can be very large in practice. A sharp Bernstein type probability inequality for $\widehat{S}_{\mathbf{p}_N}^{\epsilon_N^*}$ should thus involve σ_N^2 rather than the Poisson variance $Var(\sum_{i=1}^N (\epsilon_i/p_i)x_i)$. Using the fact that

$\sum_{i=1}^N (\epsilon_i - p_i) = 0$ on the event $\{\sum_{i=1}^N \epsilon_i = n\}$, we may now write:

$$\begin{aligned} \mathbb{P} \left\{ \widehat{S}_{\mathbf{p}_N}^{\epsilon_N^*} - S_N > t \right\} &= \mathbb{P} \left\{ \sum_{i=1}^N \frac{\epsilon_i}{p_i} x_i - S_N > t \mid \sum_{i=1}^N \epsilon_i = n \right\} \\ &= \frac{\mathbb{P} \left\{ \sum_{i=1}^N (\epsilon_i - p_i) \frac{x_i}{p_i} > t, \sum_{i=1}^N \epsilon_i = n \right\}}{\mathbb{P} \left\{ \sum_{i=1}^N \epsilon_i = n \right\}} \\ &= \frac{\mathbb{P} \left\{ \sum_{i=1}^N (\epsilon_i - p_i) \left(\frac{x_i}{p_i} - \theta_N \right) > t, \sum_{i=1}^N \epsilon_i = n \right\}}{\mathbb{P} \left\{ \sum_{i=1}^N \epsilon_i = n \right\}}. \end{aligned} \quad (20)$$

Based on the observation that the random variables $\sum_{i=1}^N (\epsilon_i - p_i)(x_i/p_i - \theta_N)$ and $\sum_{i=1}^N (\epsilon_i - p_i)$ are uncorrelated, Eq. (20) thus permits to establish directly the CLT $\sigma_N^{-1}(\widehat{S}_{\mathbf{p}_N}^{\epsilon_N^*} - S_N) \Rightarrow \mathcal{N}(0, 1)$, provided that $d_N \rightarrow +\infty$, as $N \rightarrow +\infty$, simplifying asymptotically the ratio, see [18]. Hence, the asymptotic variance of $\widehat{S}_{\mathbf{p}_N}^{\epsilon_N^*} - S_N$ is the variance σ_N^2 of the quantity $\sum_{i=1}^N (\epsilon_i - p_i)(x_i/p_i - \theta_N)$, which is less than that of the Poisson HT estimate (18), since it eliminates the variability due to the sample size. We also point out that Lemma 6 proved in the Appendix section straightforwardly shows that the "variance term" $\sum_{i=1}^N x_i^2(1 - \pi_i)/\pi_i$ involved in the bound stated in Theorem 2 is always larger than $(1 + 6/d_N)^{-1} \sum_{i=1}^N x_i^2(1 - p_i)/p_i$.

The desired result here is non asymptotic and accurate exponential bounds are required for both the numerator and the denominator of (20). It is proved in [18] (see Lemma 3.1 therein) that, as $N \rightarrow +\infty$:

$$\mathbb{P} \left\{ \sum_{i=1}^N \epsilon_i = n \right\} = (2\pi d_N)^{-1/2} (1 + o(1)). \quad (21)$$

As shall be seen in the proof of the theorem stated below, the approximation (21) can be refined by using a local Berry-Essen bound or the results in [13] and we thus essentially need to establish an exponential bound for the numerator with a constant of order $d_N^{-1/2}$, sharp enough so as to simplify the resulting ratio bound and cancel off the denominator. We shall prove that this can be achieved by using a similar argument as that considered in [7] for establishing an accurate exponential bound for i.i.d. 1-lattice random vectors, based on a device introduced in [27] for refining Hoeffding's inequality.

Theorem 4. Let $N \geq 1$. Suppose that ϵ_N^* is a rejective scheme of size $n \leq N$ with canonical parameter $\mathbf{p}_N = (p_1, \dots, p_N) \in]0, 1[^N$. Then, there exist universal constants C and D such that we have for all $t > 0$ and for all $N \geq 1$,

$$\begin{aligned} \mathbb{P} \left\{ \widehat{S}_{\mathbf{p}_N}^{\epsilon_N^*} - S_N > t \right\} &\leq C \exp \left(- \frac{\sigma_N^2}{\left(\max_{1 \leq j \leq N} \frac{|x_j|}{p_j} \right)^2} H \left(\frac{t \max_{1 \leq j \leq N} \frac{|x_j|}{p_j}}{\sigma_N^2} \right) \right) \\ &\leq C \exp \left(- \frac{t^2}{2 \left(\sigma_N^2 + \frac{1}{3} t \max_{1 \leq j \leq N} \frac{|x_j|}{p_j} \right)} \right), \end{aligned}$$

as soon as $\min\{d_N, d_N^*\} \geq 1$ and $d_N \geq D$.

An overestimated value of the constant C can be deduced by a careful examination of the proof given below. Before we detail it, we point out that the exponential bound in Theorem 4 involves the asymptotic variance of (17), in contrast to bounds obtained by exploiting the *negative association* property of the ϵ_i^* 's.

Remark 2. (SWOR (BIS)) *We underline that, in the particular case of sampling without replacement (i.e. when $p_i = \pi_i = n/N$ for $1 \leq i \leq N$), the Bernstein type exponential inequality stated above provides a control of the tail similar to that obtained in [2], see Theorem 2 therein, with $k = n$. In this specific situation, we have $d_N = n(1 - n/N)$ and $\theta_N = S_N/n$, so that the formula (19) then becomes*

$$\sigma_N^2 = \left(1 - \frac{n}{N}\right) \frac{N^2}{n} \left\{ \frac{1}{N} \sum_{i=1}^N x_i^2 - \left(\frac{1}{N} \sum_{i=1}^N x_i \right)^2 \right\}.$$

The control induced by Theorem 4 is actually slightly better than that given by Theorem 2 in [2], insofar as the factor $(1 - n/N)$ is involved in the variance term, rather than $(1 - (n-1)/N)$, that is crucial when considering situations where n gets close to N (see the discussion preceded by Proposition 2 in [2]).

PROOF. We first introduce additional notations. Set $Z_i = (\epsilon_i - p_i)(x_i/p_i - \theta_N)$ and $m_i = \epsilon_i - p_i$ for $1 \leq i \leq N$ and, for convenience, consider the standardized variables given by

$$\mathcal{Z}_N = n^{1/2} \frac{1}{N} \sum_{1 \leq i \leq N} Z_i \text{ and } \mathcal{M}_N = d_N^{-1/2} \sum_{1 \leq i \leq N} m_i.$$

As previously announced, the proof is based on Eq. (20). The lemma below first provides a sharp lower bound for the denominator, $\mathbb{P}^* \{\mathcal{M}_N = 0\}$ with the notations above. As shown in the proof given in the Appendix section, it can be obtained by applying the local Berry-Esseen bound established in [13] for sums of independent (and possibly non identically) Bernoulli random variables.

Lemma 1. *Suppose that Theorem 4's assumptions are fulfilled. Then, there exist universal constants C_1 and D such that: $\forall N \geq 1$,*

$$\mathbb{P}\{\mathcal{M}_N = 0\} \geq C_1 \frac{1}{\sqrt{d_N}}, \quad (22)$$

provided that $d_N \geq D$.

The second lemma gives an accurate upper bound for the numerator. Its proof can be found in the Appendix section.

Lemma 2. *Suppose that Theorem 4's assumptions are fulfilled. Then, we have for all $x \geq 0$, and for all $N \geq 1$ such that $\min\{d_N, d_N^*\} \geq 1$:*

$$\begin{aligned} \mathbb{P}\{\mathcal{Z}_N \geq x, \mathcal{M}_N = 0\} &\leq C \frac{1}{\sqrt{d_N}} \times \\ &\exp\left(-\frac{\text{Var}\left(\sum_{i=1}^N Z_i\right)}{\left(\max_{1 \leq j \leq N} \frac{|x_j|}{p_j}\right)^2} h\left(\frac{N}{\sqrt{n}} \frac{x \max_{1 \leq j \leq N} \frac{|x_j|}{p_j}}{\text{Var}\left(\sum_{i=1}^N Z_i\right)}\right)\right) \\ &\leq C_2 \frac{1}{\sqrt{d_N}} \exp\left(-\frac{N^2 x^2 / n}{2\left(\text{Var}\left(\sum_{i=1}^N Z_i\right) + \frac{1}{3} \frac{N}{\sqrt{n}} x \max_{1 \leq j \leq N} \frac{|x_j|}{p_j}\right)}\right), \end{aligned}$$

where $C_2 < +\infty$ is a universal constant.

The bound stated in Theorem 4 now directly results from Eq. (20) combined with Lemmas 1 and 2, with $x = t \frac{\sqrt{n}}{N}$. \square

Even if the computation of the biased statistic (17) is much more tractable from a practical perspective, we now come back to the study of the HT total estimate (1). The first part of the result stated below provides an estimation of the bias that replacement of (1) by (17) induces, whereas its second part finally gives a tail bound for (1).

Theorem 5. *Suppose that the assumptions of Theorem 4 are fulfilled and set $M_N = (6/d_N) \sum_{i=1}^N |x_i|/\pi_i$. The following assertions hold true.*

(i) For all $N \geq 1$, we almost-surely have:

$$\left| \widehat{S}_{\pi_N}^{\epsilon_N^*} - \widehat{S}_{\mathbf{p}_N}^{\epsilon_N^*} \right| \leq M_N.$$

(ii) There exist universal constants C and D such that, for all $t > M_N$ and for all $N \geq 1$, we have:

$$\begin{aligned} & \mathbb{P} \left\{ \widehat{S}_{\pi_N}^{\epsilon_N^*} - S_N > t \right\} \leq \\ & C \exp \left(- \frac{\sigma_N^2}{\left(\max_{1 \leq j \leq N} \frac{|x_j|}{p_j} \right)^2} H \left(\frac{N}{\sqrt{n}} \frac{(t - M_N) \max_{1 \leq j \leq N} \frac{|x_j|}{p_j}}{\sigma_N^2} \right) \right) \\ & \leq C \exp \left(- \frac{N^2 (t - M_N)^2 / n}{2 \left(\sigma_N^2 + \frac{1}{3} \frac{N}{\sqrt{n}} (t - M_N) \max_{1 \leq j \leq N} \frac{|x_j|}{p_j} \right)} \right), \end{aligned}$$

as soon as $\min\{d_N, d_N^*\} \geq 1$ and $d_N \geq D$.

The proof is given in the Appendix section. We point out that, for nearly uniform weights, *i.e.* when $c_1 n/N \leq \pi_i \leq c_2 n/N$ for all $i \in \{1, \dots, N\}$ with $0 < c_1 \leq c_2 < +\infty$, if there exists $K < +\infty$ such that $\max_{1 \leq i \leq N} |x_i| \leq K$ for all $N \geq 1$, then the bias term M_N is of order $o(N)$, provided that $\sqrt{N}/n \rightarrow 0$ as $N \rightarrow +\infty$.

5 Extensions to more general sampling schemes

We finally explain how the results established in the previous section for rejective sampling may permit to control tail probabilities for more general sampling plans. A similar argument is used in [4] to derive CLT's for HT estimators based on complex sampling schemes that can be approximated by more simple sampling plans, see also [6]. Let \widetilde{R}_N and R_N be two sampling plans on the population \mathcal{I}_N and consider the *total variation metric*

$$\|\widetilde{R}_N - R_N\|_1 \stackrel{def}{=} \sum_{s \in \mathcal{P}(\mathcal{I}_N)} \left| \widetilde{R}_N(s) - R_N(s) \right|,$$

as well as the *Kullback-Leibler divergence*

$$D_{KL}(R_N \| \widetilde{R}_N) \stackrel{def}{=} \sum_{s \in \mathcal{P}(\mathcal{I}_N)} R_N(s) \log \left(\frac{R_N(s)}{\widetilde{R}_N(s)} \right).$$

Equipped with these notations, we can state the following result.

Lemma 3. *Let ϵ_N and $\tilde{\epsilon}_N$ be two schemes defined on the same probability space and drawn from plans R_N and \tilde{R}_N respectively and let $\mathbf{p}_N \in]0, 1]^N$. Then, we have: $\forall N \geq 1, \forall t \in \mathbb{R}$,*

$$\begin{aligned} \left| \mathbb{P} \left\{ \widehat{S}_{\mathbf{p}_N}^{\epsilon_N} - S_N > t \right\} - \mathbb{P} \left\{ \widetilde{S}_{\mathbf{p}_N}^{\tilde{\epsilon}_N} - S_N > t \right\} \right| &\leq \| \tilde{R}_N - R_N \|_1 \\ &\leq \sqrt{2D_{KL}(R_N || \tilde{R}_N)}. \end{aligned}$$

PROOF. The first bound immediately results from the following elementary observation:

$$\begin{aligned} \mathbb{P} \left\{ \widehat{S}_{\mathbf{p}_N}^{\epsilon_N} - S_N > t \right\} - \mathbb{P} \left\{ \widetilde{S}_{\mathbf{p}_N}^{\tilde{\epsilon}_N} - S_N > t \right\} = \\ \sum_{s \in \mathcal{P}(\mathcal{I}_N)} \mathbb{I} \left\{ \sum_{i \in s} x_i / p_i - S_N > t \right\} \times \left(R_N(s) - \tilde{R}_N(s) \right), \end{aligned}$$

while the second bound is the classical Pinsker inequality. \square

In practice, R_N is typically the rejective sampling plan investigated in the previous subsection (or eventually the Poisson sampling scheme) and \tilde{R}_N a sampling plan from which the Kullback-Leibler divergence to R_N asymptotically vanishes, *e.g.* the rate at which $D_{KL}(R_N || \tilde{R}_N)$ decays to zero has been investigated in [4] when \tilde{R}_N corresponds to Rao-Sampford, successive sampling or Pareto sampling under appropriate regular conditions (see also [9]). Lemma 3 combined with Theorem 4 or Theorem 5 permits then to obtain upper bounds for the tail probabilities $\mathbb{P} \left\{ \widehat{S}_{\mathbf{p}_N}^{\epsilon_N} - S_N > t \right\}$.

6 Conclusion

In this article, we proved Bernstein-type tail bounds to quantify the deviation between a total and its Horvitz-Thompson estimator when based on conditional Poisson sampling, extending (and even slightly improving) results proved in the case of basic sampling without replacement. The original proof technique used to establish these inequalities relies on expressing the deviation probabilities related to a conditional Poisson scheme as conditional probabilities related to a Poisson plan. This permits to recover tight exponential bounds, involving the asymptotic variance of the Horvitz-Thompson estimator. Beyond the fact that rejective sampling is of prime importance in the practice of survey sampling, extension of these tail bounds to sampling schemes that can be accurately approximated by rejective sampling in the total variation sense is also discussed.

Appendix - Technical Proofs

Proof of Lemma 1

For clarity, we first recall the following result.

Theorem 6. ([13], Theorem 1.3) *Let $(Y_{j,n})_{1 \leq j \leq n}$ be a triangular array of independent Bernoulli random variables with means $q_{1,n}, \dots, q_{n,n}$ in $(0, 1)$ respectively. Denote by $\sigma_n^2 = \sum_{i=1}^n q_{i,n}(1 - q_{i,n})$ the variance of the sum $\Sigma_n = \sum_{i=1}^n Y_{i,n}$ and by $\nu_n = \sum_{i=1}^n q_{i,n}$ its mean. Considering the cumulative distribution function (cdf) $F_n(x) = \mathbb{P}\{\sigma_n^{-1}(\Sigma_n - \nu_n) \leq x\}$, we have: $\forall n \geq 1$,*

$$\sup_{k \in \mathbb{Z}} \left| F_n(x_{n,k}) - \Phi(x_{n,k}) - \frac{1 - x_{n,k}^2}{6\sigma_n} \phi(x_{n,k}) \left\{ 1 - \frac{2 \sum_{i=1}^n q_{i,n}^2(1 - q_{i,n})}{\sigma_n^2} \right\} \right| \leq \frac{C}{\sigma_n^2},$$

where $x_{n,k} = \sigma_n^{-1}(k - \nu_n + 1/2)$ for any $k \in \mathbb{Z}$, $\Phi(x) = (2\pi)^{-1/2} \int_{-\infty}^x \exp(-z^2/2) dz$ is the cdf of the standard normal distribution $\mathcal{N}(0, 1)$, $\phi(x) = \Phi'(x)$ and $C < +\infty$ is a universal constant.

Observe first that we can write:

$$\begin{aligned} \mathbb{P}\{\mathcal{M}_N = 0\} &= \mathbb{P}\left\{ \sum_{i=1}^N (\epsilon_i - p_i) \in]-1/2, 1/2[\right\} \\ &= \mathbb{P}\left\{ d_N^{-1/2} \sum_{i=1}^N m_i \leq \frac{1}{2} d_N^{-1/2} \right\} - \mathbb{P}\left\{ d_N^{-1/2} \sum_{i=1}^N m_i \leq -\frac{1}{2} d_N^{-1/2} \right\}. \end{aligned}$$

Applying Theorem 6 to bound the first term of this decomposition (with $k = \nu_n$ and $x_{n,k} = 1/(2\sqrt{d_N})$) directly yields that

$$\begin{aligned} \mathbb{P}\left\{ \frac{\sum_{i=1}^N m_i}{\sqrt{d_N}} \leq \frac{1}{2\sqrt{d_N}} \right\} &\geq \Phi\left(\frac{1}{2\sqrt{d_N}}\right) \\ &\quad + \frac{1 - \frac{1}{4d_N}}{6\sqrt{d_N}} \phi\left(\frac{1}{2\sqrt{d_N}}\right) \left\{ 1 - \frac{2 \sum_{i=1}^n p_i^2(1 - p_i)}{d_N} \right\} - \frac{C}{d_N}. \end{aligned}$$

For the second term, its application with $k = \nu_n - 1$ entails that:

$$\begin{aligned} -\mathbb{P}\left\{ \frac{1}{2\sqrt{d_N}} \sum_{i=1}^N m_i \geq -\frac{1}{2\sqrt{d_N}} \right\} &\leq -\Phi\left(-\frac{1}{2\sqrt{d_N}}\right) \\ &\quad - \frac{1 - \frac{1}{4d_N}}{6\sqrt{d_N}} \phi\left(-\frac{1}{2\sqrt{d_N}}\right) \left\{ 1 - \frac{2 \sum_{i=1}^n p_i^2(1 - p_i)}{d_N} \right\} - \frac{C}{d_N}. \end{aligned}$$

If $d_N \geq 1$, it follows that

$$\begin{aligned} \mathbb{P}\{\mathcal{M}_N = 0\} &\geq \Phi\left(\frac{1}{2\sqrt{d_N}}\right) - \Phi\left(-\frac{1}{2\sqrt{d_N}}\right) - \frac{2C}{d_N} \\ &= 2 \int_0^{\frac{1}{2\sqrt{d_N}}} \phi(t) dt - \frac{2C}{d_N} \geq \left(\phi(1/2) - \frac{2C}{\sqrt{d_N}}\right) \frac{1}{\sqrt{d_N}}. \end{aligned}$$

We thus obtain the desired result for $d_N \geq D$, where $D > 0$ is any constant strictly larger than $4C^2\phi^2(1/2)$.

6.1 Proof of Lemma 2

Observe that

$$\text{Var}\left(\sum_{i=1}^N Z_i\right) = \sum_{i=1}^N \text{Var}(Z_i) = \text{Var}\left(\sum_{i=1}^N \epsilon_i^* \frac{x_i}{p_i}\right) = \text{Var}\left(\widehat{S}_{\mathbf{p}_N}^{\epsilon_N^*}\right). \quad (23)$$

Let $\psi_N(u) = \log \mathbb{E}^*[\exp(\langle u, (\mathcal{Z}_N, \mathcal{M}_N) \rangle)]$, $u = (u_1, u_2) \in \mathbb{R}^+ \times \mathbb{R}$, be the log-Laplace of the 1-lattice random vector $(\mathcal{Z}_N, \mathcal{M}_N)$, where $\langle \cdot, \cdot \rangle$ is the usual scalar product on \mathbb{R}^2 . Denote by $\psi_N^{(1)}(u)$ and $\psi_N^{(2)}(u)$ its gradient and its Hessian matrix respectively. Consider now the conditional probability measure $\mathbb{P}_{u,N}^*$ given \mathcal{D}_N defined by the Esscher transform

$$d\mathbb{P}_{u,N} = \exp(\langle u, (\mathcal{Z}_N, \mathcal{M}_N) \rangle - \psi_N(u)) d\mathbb{P}. \quad (24)$$

The $\mathbb{P}_{u,N}$ -expectation is denoted by $\mathbb{E}_{u,N}[\cdot]$, the covariance matrix of a $\mathbb{P}_{u,N}$ -square integrable random vector Y under $\mathbb{P}_{u,N}$ by $\text{Var}_{u^*,N}(Y)$. With $x = t\sqrt{n}/N$, by exponential change of probability measure, we can rewrite the numerator of (20) as

$$\begin{aligned} \mathbb{P}\{\mathcal{Z}_N \geq x, \mathcal{M}_N = 0\} &= \mathbb{E}_{u,N} \left[e^{\psi_N(u) - \langle u, (\mathcal{Z}_N, \mathcal{M}_N) \rangle} \mathbb{I}\{\mathcal{Z}_N \geq x, \mathcal{M}_N = 0\} \right] \\ &= H(u) \mathbb{E}_{u,N} \left[e^{-\langle u, (\mathcal{Z}_N - x, \mathcal{M}_N) \rangle} \mathbb{I}\{\mathcal{Z}_N \geq x, \mathcal{M}_N = 0\} \right], \end{aligned}$$

where we set $H(u) = \exp(-\langle u, (x, 0) \rangle + \psi_N(u))$. Now, as ψ_N is convex, the point defined by

$$u^* = (u_1^*, 0) = \arg \sup_{u \in \mathbb{R}_+ \times \{0\}} \{\langle u, (x, 0) \rangle - \psi_N(u)\}$$

is such that $\psi_N^{(1)}(u^*) = (x, 0)$. Since $\mathbb{E}[\exp(\langle u, (\mathcal{Z}_N, \mathcal{M}_N) \rangle)] = \exp(\psi_N(u))$, by differentiating one gets

$$\mathbb{E}[e^{\langle u^*, (\mathcal{S}_N, \mathcal{M}_N) \rangle} (\mathcal{S}_N, \mathcal{M}_N)] = \psi_N^{(1)}(u^*) e^{\psi_N(u^*)} = (x, 0) e^{\psi_N(u^*)},$$

so that $\mathbb{E}_{u^*,N}[(\mathcal{Z}_N, \mathcal{M}_N)] = (x, 0)$ and $\text{Var}_{u^*,N}[(\mathcal{Z}_N, \mathcal{M}_N)] = \psi_N^{(2)}(u^*)$. Choosing $u = u^*$, integration by parts combined with straightforward changes of variables yields

$$\mathbb{E}_{u^*,N}[e^{-\langle u^*, (\mathcal{Z}_N - x, \mathcal{M}_N) \rangle}] \mathbb{I}\{\mathcal{Z}_N \geq x, \mathcal{M}_N = 0\} \leq \mathbb{P}_{u^*,N}^* \{\mathcal{M}_N = 0\}.$$

Hence, we have the bound:

$$\mathbb{P}\{\mathcal{Z}_N \geq x, \mathcal{M}_N = 0\} \leq H(u^*) \times \mathbb{P}_{u^*,N} \{\mathcal{M}_N = 0\}. \quad (25)$$

We shall bound each factor involved in (29) separately. We start with bounding $H(u^*)$, which essentially boils down to bounding $\mathbb{E}[e^{\langle u^*, (\mathcal{Z}_N, \mathcal{M}_N) \rangle}]$.

Lemma 4. *Under Theorem 4's assumptions, we have:*

$$H(u^*) \leq \exp \left(- \frac{\text{Var} \left(\sum_{i=1}^N Z_i \right)}{\left(\max_{1 \leq j \leq N} |x_j|/p_j \right)^2} h \left(\frac{N}{\sqrt{n}} \frac{x \max_{1 \leq j \leq N} |x_j|/p_j}{\text{Var} \left(\sum_{i=1}^N Z_i \right)} \right) \right) \quad (26)$$

$$\leq \exp \left(- \frac{N^2 x^2 / n}{2 \left(\text{Var} \left(\sum_{i=1}^N Z_i \right) + \frac{1}{3} \frac{N}{\sqrt{n}} x \max_{1 \leq j \leq N} |x_j|/p_j \right)} \right), \quad (27)$$

where $h(x) = (1+x) \log(1+x) - x$ for $x \geq 0$.

PROOF. Using the standard argument leading to the Bennett-Bernstein bound, observe that: $\forall i \in \{1, \dots, N\}, \forall u_1 > 0$,

$$\mathbb{E}[e^{u_1 Z_i}] \leq \exp \left(\frac{\exp \left(u_1 \max_{1 \leq j \leq N} \frac{|x_j|}{p_j} \right) - 1 - u_1 \max_{1 \leq j \leq N} \frac{|x_j|}{p_j}}{\text{Var}(Z_i) \left(\max_{1 \leq j \leq N} \frac{|x_j|}{p_j} \right)^2} \right).$$

since we \mathbb{P} -almost surely have $|Z_i| \leq \max_{1 \leq j \leq N} |x_j|/p_j$ for all $i \in \{1, \dots, N\}$. Using the independence of the Z_i 's, we obtain that: $\forall u_1 > 0$,

$$\mathbb{E}[e^{u_1 \mathcal{Z}_N}] \leq \exp \left(\text{Var} \left(\sum_{i=1}^N Z_i \right) \frac{\exp \left(\frac{\sqrt{n}}{N} u_1 \max_{1 \leq j \leq N} \frac{|x_j|}{p_j} \right) - 1 - \frac{\sqrt{n}}{N} u_1 \max_{1 \leq j \leq N} \frac{|x_j|}{p_j}}{\left(\max_{1 \leq j \leq N} \frac{|x_j|}{p_j} \right)^2} \right).$$

The resulting upper bound for $H((u_1, 0))$ being minimum for

$$u_1 = \frac{N}{\sqrt{n}} \frac{\log \left(1 + \frac{N}{\sqrt{n}} \frac{x \max_{1 \leq j \leq N} |x_j|/p_j}{\text{Var}(\sum_{i=1}^N Z_i)} \right)}{\max_{1 \leq j \leq N} |x_j|/p_j},$$

this yields

$$H(u^*) \leq \exp \left(- \frac{\text{Var} \left(\sum_{i=1}^N Z_i \right)}{\left(\max_{1 \leq j \leq N} |x_j|/p_j \right)^2} h \left(\frac{N}{\sqrt{n}} \frac{x \max_{1 \leq j \leq N} |x_j|/p_j}{\text{Var} \left(\sum_{i=1}^N Z_i \right)} \right) \right). \quad (28)$$

Using the classical inequality

$$h(x) \geq \frac{x^2}{2(1+x/3)}, \text{ for } x \geq 0,$$

we also get that

$$H(u^*) \leq \exp \left(- \frac{N^2 x^2 / n}{2 \left(\text{Var} \left(\sum_{i=1}^N Z_i \right) + \frac{1}{3} \frac{N}{\sqrt{n}} x \max_{1 \leq j \leq N} |x_j|/p_j \right)} \right). \quad (29)$$

□

We now prove the lemma stated below, which provides an upper bound for $\mathbb{P}_{u^*,N}^* \{\mathcal{M}_N = 0\}$.

Lemma 5. *Under Theorem 4's assumptions, there exists a universal constant C' such that: $\forall N \geq 1$,*

$$\mathbb{P}_{u^*,N} \{\mathcal{M}_N = 0\} \leq C' \frac{1}{\sqrt{d_N}}. \quad (30)$$

PROOF. Under the probability measure $\mathbb{P}_{u^*,N}$, the ε_i 's are still independent Bernoulli variables, with means now given by

$$\pi_i^* \stackrel{\text{def}}{=} \sum_{s \in \mathcal{P}(\mathcal{I}_N)} e^{\langle u^*, (\mathcal{Z}_N(s), \mathcal{M}_N(s)) \rangle - \psi_N(u^*)} \mathbb{I}\{i \in s\} R_N(s) > 0,$$

for $i \in \{1, \dots, N\}$. Since $\mathbb{E}_{u^*,N}[\mathcal{M}_N] = 0$, we have $\sum_{i=1}^N \pi_i^* = n$ and thus

$$d_{N,u^*} \stackrel{\text{def}}{=} \text{Var}_{u^*,N} \left(\sum_{i=1}^N \varepsilon_i \right) = \sum_{i=1}^N \pi_i^* (1 - \pi_i^*) \leq n.$$

We can thus apply the local Berry-Esseen bound established in [13] for sums of independent (and possibly non identically) Bernoulli random variables, recalled in Theorem 7.

Theorem 7. ([13], Theorem 1.2) Let $(Y_{j,n})_{1 \leq j \leq n}$ be a triangular array of independent Bernoulli random variables with means $q_{1,n}, \dots, q_{n,n}$ in $(0, 1)$ respectively. Denote by $\sigma_n^2 = \sum_{i=1}^n q_{i,n}(1 - q_{i,n})$ the variance of the sum $\Sigma_n = \sum_{i=1}^n Y_{i,n}$ and by $\nu_n = \sum_{i=1}^n q_{i,n}$ its mean. Considering the cumulative distribution function (cdf) $F_n(x) = \mathbb{P}\{\sigma_n^{-1}(\Sigma_n - \nu_n) \leq x\}$, we have: $\forall n \geq 1$,

$$\sup_x (1 + |x|^3) |F_n(x) - \Phi(x)| \leq \frac{C}{\sigma_n}, \quad (31)$$

where $\Phi(x) = (2\pi)^{-1/2} \int_{-\infty}^x \exp(-z^2/2) dz$ is the cdf of the standard normal distribution $\mathcal{N}(0, 1)$ and $C < +\infty$ is a universal constant.

Applying twice a pointwise version of the bound recalled above (for $x = 0$ and $x = 1/\sqrt{d_{N,u^*}}$), we obtain that

$$\begin{aligned} \mathbb{P}_{u^*,N} \{\mathcal{M}_N = 0\} &= \mathbb{P}_{u^*,N} \left\{ d_{N,u^*}^{-1/2} \sum_{i=1}^N m_i \leq 0 \right\} \\ &\quad - \mathbb{P}_{u^*,N} \left\{ d_{N,u^*}^{-1/2} \sum_{i=1}^N m_i \leq -d_{N,u^*}^{-1/2} \right\} \\ &\leq \frac{2C}{\sqrt{d_{N,u^*}}} + \Phi(0) - \Phi(-d_{N,u^*}^{-1/2}) \leq \left(\frac{1}{\sqrt{2\pi}} + 2C \right) \frac{1}{\sqrt{d_{N,u^*}}}, \end{aligned}$$

by means of the finite increment theorem. Finally, observe that:

$$\begin{aligned} d_{N,u^*} &= \mathbb{E}_{u^*,N} \left[\left(\sum_{i=1}^N m_i \right)^2 \right] = \mathbb{E} \left[\left(\sum_{i=1}^N m_i \right)^2 / H(u^*) \right] \\ &\geq \mathbb{E} \left[\left(\sum_{i=1}^N m_i \right)^2 \right] = d_N, \end{aligned}$$

since we proved that $H(u^*) \leq 1$. Combined with the previous bound, this yields the desired result. \square

Lemmas 4 and 5 combined with Eq. (29) leads to the bound stated in Lemma 2.

Proof of Theorem 5

We start with proving the preliminary result below.

Lemma 6. Let π_1, \dots, π_N be the first order inclusion probabilities of a rejective sampling of size n with canonical representation characterized by the Poisson weights p_1, \dots, p_N . Provided that $d_N = \sum_{i=1}^N p_i(1-p_i) \geq 1$, we have: $\forall i \in \{1, \dots, N\}$,

$$\left| \frac{1}{\pi_i} - \frac{1}{p_i} \right| \leq \frac{6}{d_N} \times \frac{1 - \pi_i}{\pi_i}.$$

PROOF. The proof follows the representation (5.14) on page 1509 of [18]. We have For all $i \in \{1, \dots, N\}$, we have:

$$\begin{aligned} \frac{\pi_i}{p_i} \frac{1 - p_i}{1 - \pi_i} &= \frac{\sum_{s \in \mathcal{P}(\mathcal{I}_N): i \in \mathcal{I}_N \setminus \{s\}} P(s) \sum_{h \in s} \frac{1 - p_h}{\sum_{j \in s} (1 - p_j) + (p_h - p_i)}}{\sum_{s \in \mathcal{P}(\mathcal{I}_N): i \in \mathcal{I}_N \setminus \{s\}} P(s)} \\ &= \frac{\sum_{s: i \in \mathcal{I}_N \setminus \{s\}} P_N(s) \sum_{h \in s} \frac{1 - p_h}{\sum_{j \in s} (1 - p_j) \left(1 + \frac{(p_h - p_i)}{\sum_{j \in s} (1 - p_j)}\right)}}{\sum_{s: i \in \mathcal{I}_N \setminus \{s\}} P_N(s)}. \end{aligned}$$

Now recall that for any $x \in]-1, 1[$, we have:

$$1 - x \leq \frac{1}{1 + x} \leq 1 - x + x^2.$$

It follows that

$$\begin{aligned} \frac{\pi_i}{p_i} \frac{1 - p_i}{1 - \pi_i} &\leq 1 - \left(\sum_{s: i \in \mathcal{I}_N \setminus \{s\}} P(s) \right)^{-1} \sum_{s: i \in \mathcal{I}_N \setminus \{s\}} P(s) \sum_{h \in s} \frac{(1 - p_h)(p_h - p_i)}{\left(\sum_{j \in s} (1 - p_j) \right)^2} \\ &\quad + \left(\sum_{s: i \in \mathcal{I}_N \setminus \{s\}} P(s) \right)^{-1} \sum_{s: i \in \mathcal{I}_N \setminus \{s\}} P(s) \sum_{h \in s} \frac{(1 - p_h)(p_h - p_i)^2}{\left(\sum_{j \in s} (1 - p_j) \right)^3} \end{aligned}$$

Following now line by line the proof on p. 1510 in [18] and noticing that $\sum_{j \in s} (1 - p_j) \geq 1/2d_N$ (see Lemma 2.2 in [18]), we have

$$\left| \sum_{h \in s} \frac{(1 - p_h)(p_h - p_i)}{\left(\sum_{j \in s} (1 - p_j) \right)^2} \right| \leq \frac{1}{\left(\sum_{j \in s} (1 - p_j) \right)} \leq \frac{2}{d_N}$$

and similarly

$$\sum_{h \in s} \frac{(1 - p_h)(p_h - p_i)^2}{\left(\sum_{j \in s} (1 - p_j) \right)^3} \leq \frac{1}{\left(\sum_{j \in s} (1 - p_j) \right)^2} \leq \frac{4}{d_N^2}.$$

This yields: $\forall i \in \{1, \dots, N\}$,

$$1 - \frac{2}{d_N} \leq \frac{\pi_i}{p_i} \frac{1 - p_i}{1 - \pi_i} \leq 1 + \frac{2}{d_N} + \frac{4}{d_N^2}$$

and

$$p_i(1 - \pi_i)\left(1 - \frac{2}{d_N}\right) \leq \pi_i(1 - p_i) \leq p_i(1 - \pi_i) \left(1 + \frac{2}{d_N} + \frac{4}{d_N^2}\right),$$

leading then to

$$-\frac{2}{d_N}(1 - \pi_i)p_i \leq \pi_i - p_i \leq p_i(1 - \pi_i) \left(\frac{2}{d_N} + \frac{4}{d_N^2}\right)$$

and finally to

$$-\frac{(1 - \pi_i)}{\pi_i} \frac{2}{d_N} \leq \frac{1}{p_i} - \frac{1}{\pi_i} \leq \frac{(1 - \pi_i)}{\pi_i} \left(\frac{2}{d_N} + \frac{4}{d_N^2}\right).$$

Since $1/d_N^2 \leq 1/d_N$ as soon as $d_N \geq 1$, the lemma is proved. \square

By virtue of lemma 6, we obtain that:

$$\left| \widehat{S}_{\pi_N}^{\epsilon_N^*} - \widehat{S}_{\mathbf{p}_N}^{\epsilon_N^*} \right| \leq \frac{6}{d_N} \sum_{i=1}^N \frac{1}{\pi_i} |x_i| = M_N$$

It follows that

$$\begin{aligned} \mathbb{P} \left\{ \widehat{S}_{\pi_N}^{\epsilon_N^*} - S_N > x \right\} &\leq \mathbb{P} \left\{ \left| \widehat{S}_{\pi_N}^{\epsilon_N^*} - \widehat{S}_{\mathbf{p}_N}^{\epsilon_N^*} \right| + \widehat{S}_{\mathbf{p}_N}^{\epsilon_N^*} - S_N > x \right\} \\ &\leq \mathbb{P} \left\{ M_N + \widehat{S}_{\mathbf{p}_N}^{\epsilon_N^*} - S_N > x \right\} \end{aligned}$$

and a direct application of Theorem 4 finally gives the desired result.

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