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Learning in non atomic anonymous games with applications to first order mean field games

Saeed Hadikhanloo*

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Abstract

We introduce a model of non atomic anonymous games with the player dependent action sets. We propose several learning procedures based on the well-known fictitious play and online mirror descent and prove their convergence to equilibrium under the classical monotonicity condition. Typical examples are first-order mean field games.

1 Introduction

Our goal in this article is to propose learning procedures for first order mean field games with monotone costs. Mean field games (MFGs) were introduced simultaneously by Lasry and Lions [14][15][15][13] and Huang, Caines and Malhamé [11][12]. In short, these games are differential games with an infinite number of players. The equilibrium configuration is usually described by a PDE system (of a Hamilton-Jacobi coupled with a Fokker-Planck equations); but we can give a weaker description of an equilibrium in terms of distribution over the set of trajectories.

Let us describe this weaker formulation. Let $\Gamma = \mathcal{AC}([0, T]; \mathbb{R}^d)$ be the set of all absolutely continuous paths in \mathbb{R}^d , endowed with the uniform norm. For a given $m_0 \in \mathcal{P}(\mathbb{R}^d)$, called the initial measure, we denote

$$\mathcal{P}_{m_0}(\Gamma) = \{\eta \in \mathcal{P}(\Gamma) ; e_0\#\eta = m_0\},$$

the set of all measures over set of trajectories Γ such that the induced measure $e_0\#\eta$ at instant $t = 0$ equals to our given initial measure m_0 . We call a distribution, $\eta^* \in \mathcal{P}_{m_0}(\Gamma)$ an equilibrium distribution if for η^* -almost every γ , we have

$$\gamma \in \operatorname{argmin}_{z \in \Gamma, z(0) = \gamma(0)} \left\{ \int_0^T \left(L(z(t), \dot{z}(t)) + f(z(t), e_t\#\eta^*) \right) dt + g(z(T), e_T\#\eta^*) \right\}$$

This formulation is known in the literature; for example, Cardliaguet and Hadikhanloo [8] worked on the potential first order mean field games and proved the convergence of fictitious play to the equilibrium distribution. They also showed that under strong assumptions on the data, one can obtain the first order MFG solution from an equilibrium distribution.

In this article our goal is to give algorithms for finding such distribution equilibria. We mostly work with two algorithms similar to learning procedures in games, called fictitious play and online

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mirror descent. For example, fictitious play in this framework reads as follows. Let an arbitrary $\eta^1 \in \mathcal{P}_{m_0}(\Gamma)$ and construct recursively $\eta^k, \bar{\eta}^k \in \mathcal{P}_{m_0}(\Gamma)$ with the following rule. Let η^{k+1} be such that for η^{k+1} -almost every γ , we have

$$\gamma \in \operatorname{argmin}_{z \in \Gamma, z(0) = \gamma(0)} \left\{ \int_0^T \left(L(z(t), \dot{z}(t)) + f(z(t), e_t \# \bar{\eta}^k) \right) dt + g(z(T), e_T \# \bar{\eta}^k) \right\},$$

and then $\bar{\eta}^{k+1} = \frac{k}{k+1} \bar{\eta}^k + \frac{1}{k+1} \eta^{k+1}$. We prove that such procedure converge to an equilibrium distribution in the case of monotonicity of f and g .

As explained above, our main purpose is to work in the framework of first-order MFGs; however, the approach can be used for a larger class of games and hence we work under a more general model, that is the framework of non atomic anonymous games. This framework is closely related to the model used by Mas-Colell [16], Blanchet and Carlier [3][4][5]. Contrasting to their approach, we work with a model that the action sets are not identical for all players and they depend on the types of players. This is the case in first-order MFGs; the players choose the paths with fixed (player dependent) initial positions as their actions.

We provided sufficient conditions proving the existence of an equilibrium. Moreover, we proved the uniqueness of the equilibrium under an adapted monotonicity notion. The strict monotonicity yields the uniqueness of the Nash equilibrium in several games (Lasry and Lions [14],[15], Hofbauer and Sandholm [10], Blanchet and Carlier [4]). In non atomic anonymous games with (not necessarily strict) monotone costs, equilibrium uniqueness is a direct consequence of monotonicity and an additional assumption, called the unique minimiser condition.

After setting a proper framework, the next question we deal with, is to propose algorithms similar to learning procedures in games, ensuring convergence to equilibria. There are several learning procedures in static games with finitely many players and/or a finite number of actions per player (see for example the monograph [9]). Here we extend two of the most known of them to non atomic anonymous games: fictitious play and online mirror descent.

Fictitious play was introduced by Brown [6]. The procedure is as follows. Let a fixed game be played for many rounds. At every round, the players set their belief about the actions of other players, equal to the average of empirical frequency of actions played in previous rounds. Then they choose their best action against to this belief. Convergence toward an equilibrium has been proved for different classes of finite games, for example potential games (Monderer, Shapley [19]), zero sum games (Robinson [21]) and 2×2 games (Miyasawa [18]). Cardaliaguet, Hadikhannoo [8] proved the convergence of a similar procedure (still called fictitious play) in first and second order potential MFGs. We should note that our approach in this article, covers a different class of first-order MFGs, that are the ones with monotone costs.

The second procedure we consider is the online mirror descent (OMD). The method was first introduced by Nemirovski, Yudin [20], as a generalization of standard gradient descent. The form of the algorithm is closely related to the notion of no-regret procedures in online optimization. A good explanatory introduction can be found in Shalev Shwartz[24]. Roughly speaking, the procedure deals with two variables, a primal one and a dual one. They are revised at every round; the dual is revised by using the sub-gradient of the objective function and the primal is obtained by a quasi projection via a strongly convex penalty function on the convex domain. Mertikopolous [17] proved the convergence of OMD to equilibria in the class of games with convex action sets and concave costs. Here we examine the convergence properties of OMD in monotone anonymous games.

In the proof of convergence of both procedures to the equilibrium, we define a value $\phi_n \in \mathbb{R}, n \in \mathbb{N}$ measuring how much the actual profile of action at step n is far from being an equilibrium; in fictitious play the quantity ϕ_n is calculated by using the best response function and in OMD by the Fenchel coupling. We then prove that indeed $\lim_{n \rightarrow \infty} \phi_n = 0$; this gives our desired convergence toward the equilibrium.

Here is how the paper is organized: in section 3 a general model of anonymous game is proposed. The notion of Nash equilibrium is reviewed and the existence is proved under general continuity conditions. Then we define monotonicity in terms of the cost function, and its consequence on the uniqueness of the Nash equilibrium. Section 4 is devoted to the definition of fictitious play and its convergence under Lipschitz conditions. Section 5 deals with the online mirror descent algorithm and its convergence. Section 6 shows that the first order MFG can be considered as an example of anonymous games and shows that the previous results can be applied under suitable conditions. For sake of completeness, we provide in the Appendix some disintegration theorems which are used in the proofs.

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2 Preliminaries

For a measure space X let $\mathcal{P}(X)$ denotes the set of probability measures on X .

Definition 2.1. A correspondence $A : I \rightarrow V, A(i) = A_i$ is called continuous if:

- it is upper semi continuous i.e. the graph $\{(i, a) \in I \times V \mid a \in A_i\}$ is closed in $I \times V$,
- it is lower semi continuous i.e. for every open set $U \subseteq V$ the set $\{i \in I \mid A_i \cap U \neq \emptyset\}$ is open in I .

3 Non atomic anonymous games

3.1 Model

Let us introduce our general model of anonymous game G . Let I be the set of players and $\lambda \in \mathcal{P}(I)$ a prior non atomic probability measure on I modelling the repartition of players on I . Let V be a measure space. For every player $i \in I$, let $A_i \subset V$ be the action set of i . Define the set of admissible profiles of actions

$$\mathcal{A} = \{\Psi : I \rightarrow V \text{ measurable} \mid \Psi(i) \in A_i \text{ for } \lambda\text{-almost every } i \in I\}.$$

We identify the action profiles up to λ -zero measure subsets of I , i.e. $\Psi_1 = \Psi_2$ iff $\Psi_1(i) = \Psi_2(i)$ for λ -almost every $i \in I$. The induced measure of a typical profile $\Psi \in \mathcal{A}$ on the set of actions, that captures the portion of players who have chosen a given subset of actions, is denoted by $\Psi\#\lambda \in \mathcal{P}(V)$. More precisely, $\Psi\#\lambda$ is the push-forward of measure of λ by application Ψ , that is for

every measurable set $B \subseteq V$ we have $\Psi \# \lambda(B) = \lambda(\Psi^{-1}(B))$. Since the set consisting of measures $\Psi \# \lambda$ for all admissible profiles Ψ , may be different from $\mathcal{P}(V)$, it is sufficient to work with:

$$\mathcal{P}_G(V) = \{ \eta \in \mathcal{P}(V) \mid \exists \Psi \in \mathcal{A} : \eta = \Psi \# \lambda \}.$$

For every $i \in I$ let $c_i : \mathcal{A} \rightarrow \mathbb{R}$ be the cost paid by player i . We call the game anonymous, if for every player $i \in I$, there exists $J_i : A_i \times \mathcal{P}_G(V) \rightarrow \mathbb{R}$ such that $c_i(\Psi) = J_i(\Psi(i), \Psi \# \lambda)$. In other words, $J_i(a, \eta)$ captures the cost endured by a typical player $i \in I$, whose action is $a \in A_i$ while facing the distribution of actions $\eta \in \mathcal{P}(V)$ chosen by other players. We consider here anonymous games where the players have identical cost function, i.e. there is $J : V \times \mathcal{P}_G(V) \rightarrow \mathbb{R}$ such that for every $i \in I$ we have $J_i = J$. We use the following notation for referring to such game:

$$G = (I, \lambda, V, (A_i)_{i \in I}, J).$$

Example 3.1 (Population Game [10]). *Set $I = [0, 1]$ be the set of players and λ the Lebesgue measure as the distribution of players on I . Let $N \in \mathbb{N}$ represents the number of populations in the game i.e. there is a partition of players $I_1, I_2, \dots, I_N \subseteq I$ where for every $1 \leq p \leq N, I_p \subseteq I$ represents the set of players belonging to population p . For every player $i \in I$ suppose the set of actions A_i is finite and depends only on the population where the player i comes from, i.e. for every population p there is S_p such that for all $i \in I_p$ we have $A_i = S_p$. Set $V = \cup_p S_p$. For every population p the cost function has the form $J_p : S_p \times \Delta(V) \rightarrow \mathbb{R}$ where $J_p(a, (m_j)_{1 \leq j \leq |V|})$ is the cost payed by a typical player in population p whose action is $a \in S_p$ while facing $(m_j)_{1 \leq j \leq |V|}$ where for every $1 \leq j \leq |V|$, $m_j \geq 0$ is the portion of players who have chosen action $j \in V$. The form of the cost function illustrates the fact that the population games are anonymous.*

Example 3.2. *In section 5, we show that the First order MFG is an anonymous game with suitable actions sets and cost function.*

3.2 Nash equilibria

Inspired from the notion of Nash equilibrium in non atomic games (see Schmeidler [23], Mas-Colell [16]), we omit the effect of λ -zero measure subsets of players in the definition of equilibria:

Definition 3.1. *A profile $\tilde{\Psi} \in \mathcal{A}$ is called a Nash equilibrium if*

$$\tilde{\Psi}(i) \in \arg \min_{a \in A_i} J(a, \tilde{\Psi} \# \lambda) \quad \text{for } \lambda\text{-almost every } i \in I.$$

The corresponding distribution $\tilde{\eta} = \tilde{\Psi} \# \lambda$ is called a Nash (or equilibrium) distribution.

One can note that the definition of Nash equilibrium highly depends on the prior distribution of players λ . The following theorem gives a sufficient condition under which the game possesses at least one equilibrium. Let I be a topological and V be a metric space (with $\mathcal{B}(I), \mathcal{B}(V)$ as their σ -fields). Suppose the A_i 's are uniformly bounded for λ -almost every $i \in I$, i.e. there exist $M > 0, v \in V$ such that:

$$\text{for } \lambda\text{-almost every } i \in I \text{ and every } a \in A_i : \quad d_V(v, a) < M. \quad (1)$$

This condition gives us $\mathcal{P}_G(V) \subseteq \mathcal{P}_1(V)$ where:

$$\mathcal{P}_1(V) = \{ \eta \in \mathcal{P}(V) \mid \exists v \in V : \int_V d_V(v, a) d\eta(a) < +\infty \}$$

endowed with the metric:

$$d_1(\eta_1, \eta_2) = \sup_{h: V \rightarrow \mathbb{R}, 1\text{-Lipschitz}} \int_V h(a) d(\eta_1 - \eta_2)(a).$$

For technical reasons we work with closure convex hull of $\mathcal{P}_G(V)$ i.e. $\overline{\text{cov}(\mathcal{P}_G(V))}$.

Definition 3.2. *We say $G = (I, \lambda, V, (A_i)_{i \in I}, J)$ satisfies the unique minimiser condition, if for every $\eta \in \overline{\text{cov}(\mathcal{P}_G(V))}$, there exists $I_\eta \subseteq I$ with $\lambda(I \setminus I_\eta) = 0$, such that for all $i \in I_\eta$ there is exactly one $a \in A_i$ minimizing $J(\cdot, \eta)$ in A_i .*

Informally, the definition says facing to every distribution of actions, (almost) every player has a unique best response.

For more detailed theorems about set valued maps, see [2].

Assumption 3.1. *Here are the assumptions we consider for the non atomic anonymous games:*

1. *the correspondence $A : I \rightarrow V$, $A(i) = A_i$ is continuous and compact valued,*
2. *there is an extension $J : V \times \overline{\text{cov}(\mathcal{P}_G(V))} \rightarrow \mathbb{R}$ which is lower semi-continuous,*
3. *the function $\text{Min} : I \times \mathcal{P}_G(V) \rightarrow \mathbb{R}$, $\text{Min}(i, \eta) := \min_{a \in A_i} J(a, \eta)$ is continuous,*
4. *$\overline{\text{cov}(\mathcal{P}_G(V))}$ is compact,*
5. *G satisfies the unique minimiser condition.*

Theorem 3.1. *Let $G = (I, \lambda, V, (A_i)_{i \in I}, J)$ be an anonymous game. Suppose the assumptions (3.1) hold. Then G will admit at least a Nash equilibrium.*

Assumptions (3.1)(1-4) provide enough continuity and compactness conditions we need for the fixed point theorem. The assumption (3.1)(5) allows us to prove the existence of pure Nash equilibrium; it is crucial as well for the uniqueness of equilibrium and convergence results in learning procedures that we will propose. So we add it here as an assumption for being coherent in the entire chapter. Before we start the proof let us provide some lemmas which will be used here and in the rest of paper:

Lemma 3.1. *Define the best response correspondence as follows*

$$BR : I \times \overline{\text{cov}(\mathcal{P}_G(V))} \rightarrow V, \quad BR(i, \eta) = \arg \min_{a \in A_i} J(a, \eta).$$

If the assumptions (3.1) hold, then for every $\eta \in \overline{\text{cov}(\mathcal{P}_G(V))}$ the correspondence $BR(\cdot, \eta) : I \rightarrow V$, that is almost everywhere singleton, is almost everywhere continuous and hence measurable.

Proof. Fix $\eta \in \overline{\text{cov}(\mathcal{P}_G(V))}$. According to the unique minimiser condition there exists $I_\eta \subseteq I$ with $\lambda(I \setminus I_\eta) = 0$, such that $BR(i, \eta)$ is singleton for every $i \in I_\eta$. We will show the continuity of the restricted best response function $BR(\cdot, \eta) : I_\eta \rightarrow V$ which completes our proof. Consider $i, i_n \in I_\eta$ such that $i_n \rightarrow i$. Set $a_n = BR(i_n, \eta)$. The set $\{a_n\}_{n \in \mathbb{N}}$ is pre-compact since $A : I \rightarrow V$ is a compact valued correspondence and hence $A(\{i_n\}_{n \in \mathbb{N}} \cup \{i\}) = \cup_n A_{i_n} \cup A_i$ is compact. Suppose $\tilde{a} \in V$ is an accumulation point of $\{a_n\}_{n \in \mathbb{N}}$. So there is a sub-sequence $\{a_{n_k}\}_{k \in \mathbb{N}}$ such that $\lim_{k \rightarrow \infty} a_{n_k} = \tilde{a}$.

We have $\tilde{a} \in A_i$ since the correspondence $A : I \rightarrow V$ is upper semi continuous and $a_n \in A_{i_n}$. By definition $J(a_n, \eta) = \text{Min}(i_n, \eta)$ which gives:

$$J(\tilde{a}, \eta) \leq \liminf_{n_k} J(a_{n_k}, \eta) = \liminf_{n_k} \text{Min}(i_{n_k}, \eta) = \text{Min}(i, \eta),$$

since the Min function is continuous. It yields $\tilde{a} = BR(i, \eta)$. So every accumulation point of $\{a_n\}_{n \in \mathbb{N}}$ should be $BR(i, \eta)$ which shows $a_n \rightarrow BR(i, \eta)$. \square

Lemma 3.2. *Define the best response distribution function $\Theta : \overline{\text{cov}(\mathcal{P}_G(V))} \rightarrow \mathcal{P}_G(V)$ as follows:*

$$\Theta(\eta) = BR(\cdot, \eta) \# \lambda, \quad \text{for every } \eta \in \overline{\text{cov}(\mathcal{P}_G(V))}.$$

If the assumptions (3.1) hold then Θ is continuous.

Proof. Let $\eta_m \rightarrow \eta$. If $J = I_\eta \cap_{n \in \mathbb{N}} I_{\eta_n}$ then we have $\lambda(I \setminus J) = 0$. One can show as for Lemma 3.1 that for every $i \in J$:

$$BR(i, \eta_m) \rightarrow BR(i, \eta).$$

Since the A_i 's are uniformly bounded for λ -almost every $i \in J$, the dominated Lebesgue convergence theorem implies $\int_I d_V(BR(i, \eta_m), BR(i, \eta)) d\lambda(i) \rightarrow 0$. Thus $\Theta(\eta_m) \xrightarrow{d_1} \Theta(\eta)$ since:

$$\begin{aligned} d_1(\Theta(\eta_m), \Theta(\eta)) &= \sup_{f: V \rightarrow \mathbb{R}, 1\text{-Lipschitz}} \int_V f(v) d(\Theta(\eta_m) - \Theta(\eta))(v) = \\ & \sup_{f: V \rightarrow \mathbb{R}, 1\text{-Lipschitz}} \int_I (f(BR(i, \eta_m)) - f(BR(i, \eta))) d\lambda(i) \leq \int_I d_V(BR(i, \eta_m), BR(i, \eta)) d\lambda(i) \rightarrow 0. \end{aligned}$$

\square

Proof of Theorem 3.1. Consider the best response distribution function Θ defined in Lemma 3.2. We have by definition

$$\Theta(\overline{\text{cov}(\mathcal{P}_G(V))}) \subset \mathcal{P}_G(V) \subset \overline{\text{cov}(\mathcal{P}_G(V))},$$

which implies that the image of Θ is pre-compact. Since Θ is continuous (Lemma 3.2) and $\overline{\text{cov}(\mathcal{P}_G(V))}$ is convex, by the Schauder's fixed point theorem, there is $\tilde{\eta} \in \overline{\text{cov}(\mathcal{P}_G(V))}$ such that $\Theta(\tilde{\eta}) = \tilde{\eta}$. Since $\Theta(\tilde{\eta}) = BR(\cdot, \tilde{\eta}) \# \lambda \in \mathcal{P}_G(V)$ so if we set $\tilde{\Psi}(\cdot) = BR(\cdot, \tilde{\eta}) \in \mathcal{A}$ then

$$\tilde{\Psi} \# \lambda = \tilde{\eta}, \quad \tilde{\Psi}(i) \in \arg \min_{a \in A_i} J(a, \tilde{\eta}) \quad \text{for } \lambda\text{-almost every } i \in I.$$

This means $\tilde{\Psi}$ is the desired Nash equilibrium. \square

3.3 Anonymous games with monotone cost

Here we give a definition of monotonicity and its additional consequences on the structure of the game and its equilibria.

Definition 3.3. *The anonymous game $G = (I, \lambda, V, (A_i)_{i \in I}, J)$ has a monotone cost J if for any $\eta, \eta' \in \overline{\text{cov}(\mathcal{P}_G(V))}$:*

$$\int_V |J(a, \eta)| \, d\eta'(a) < +\infty,$$

and

$$\int_V (J(a, \eta) - J(a, \eta')) \, d(\eta - \eta')(a) \geq 0.$$

We call J a strict monotone cost function if the later inequality holds strictly for $\eta \neq \eta'$.

This condition is usually interpreted as the aversion of players for choosing actions that are chosen by many of players i.e. congestion avoiding effect.

Remark 3.1. *If J is monotone and if $\tilde{\Psi} \in \mathcal{A}$ is a Nash equilibrium, then for every $\Psi \in \mathcal{A}$ we have:*

$$\text{if } \tilde{\eta} = \tilde{\Psi} \# \lambda, \eta = \Psi \# \lambda : \int_V J(a, \eta) \, d(\eta - \tilde{\eta})(a) \geq \int_V J(a, \tilde{\eta}) \, d(\eta - \tilde{\eta})(a) \geq 0.$$

Proof. Since J is monotone we have $\int_V (J(a, \eta) - J(a, \tilde{\eta})) \, d(\eta - \tilde{\eta})(a) \geq 0$ and so:

$$\int_V J(a, \eta) \, d(\eta - \tilde{\eta})(a) \geq \int_V J(a, \tilde{\eta}) \, d(\eta - \tilde{\eta})(a).$$

On the other hand

$$\int_V J(a, \tilde{\eta}) \, d(\eta - \tilde{\eta})(a) = \int_I \left(J(\Psi(i), \tilde{\eta}) - J(\tilde{\Psi}(i), \tilde{\eta}) \right) \, d\lambda(i)$$

by the definition of push-forward measures. Since $\tilde{\Psi}$ is an equilibrium, for λ -almost every $i \in I$, we have $J(\Psi(i), \tilde{\eta}) - J(\tilde{\Psi}(i), \tilde{\eta}) \geq 0$, which gives our result. \square

The strict monotonicity yields the uniqueness of the Nash equilibrium in different frameworks, e.g. Haufbauer, Sandholm [10], Blanchet, Carlier [4], Lasry, Lions [14]. In the following we show that in non atomic anonymous games, the monotonicity and unique minimiser conditions are sufficient for the uniqueness of the equilibrium.

Theorem 3.2. *Consider a game $G = (I, \lambda, V, (A_i)_{i \in I}, J)$. Then the game G admits at most one Nash equilibrium if J is monotone and G satisfies the unique minimiser condition.*

Proof. Let $\Psi_1, \Psi_2 \in \mathcal{A}$ be two Nash equilibria. We will show that $\Psi_1(i) = \Psi_2(i)$ for λ -almost every $i \in I$. Set $\eta_i = \Psi_i \# \lambda$ for $i = 1, 2$. Since Ψ_1 is an equilibrium, we have:

$$\int_I (J(\Psi_1(i), \eta_1) - J(\Psi_2(i), \eta_1)) \, d\lambda(i) \leq 0,$$

since $J(\Psi_1(i), \eta_1) \leq J(\Psi_2(i), \eta_1)$ for λ -almost every $i \in I$. On the other hand:

$$\int_I (J(\Psi_1(i), \eta_1) - J(\Psi_2(i), \eta_1)) \, d\lambda(i) = \int_V J(a, \eta_1) \, d(\eta_1 - \eta_2)(a),$$

from the definition since $\Psi_i \# \lambda = \eta_i$ for $i = 1, 2$. So

$$\int_V J(a, \eta_1) \, d(\eta_1 - \eta_2)(a) \leq 0 \quad \text{and (similarly)} \quad \int_V J(a, \eta_2) \, d(\eta_2 - \eta_1)(a) \leq 0.$$

Summing up the last inequalities gives us:

$$\int_V (J(a, \eta_1) - J(a, \eta_2)) \, d(\eta_1 - \eta_2)(a) \leq 0.$$

Hence by monotonicity of J we should have the equality in the previous inequalities. So for λ -almost every $i \in I$, one has $J(\Psi_1(i), \eta_1) = J(\Psi_2(i), \eta_1)$. Since $\Psi_1(i) \in A_i$ is the unique minimiser of $J(\cdot, \eta_1)$ on A_i so $\Psi_1(i) = \Psi_2(i)$ for λ -almost every $i \in I$. \square

Remark 3.2. *One can similarly show that if J is strictly monotone and not necessarily satisfies the unique minimizer condition, then there exists at most one Nash equilibrium distribution.*

4 Fictitious play in anonymous games

Here we introduce a learning procedure similar to the fictitious play and prove its convergence to the unique Nash equilibrium under monotonicity condition.

Let $G = (I, \lambda, V, (A_i)_{i \in I}, J)$. For technical reasons, we suppose that assumptions (3.1) hold throughout this section. Suppose G is being played repeatedly on discrete rounds $n = 1, 2, \dots$. At every round, the players set their belief equals to the average of the action distribution observed in the previous rounds and then react their best to such belief. At the end of the round players revise their beliefs by a new observation. More formally, consider $\Psi_1 \in \mathcal{A}$, $\bar{\eta}_1 = \eta_1 = \Psi_1 \# \lambda \in \mathcal{P}_G(V)$ an arbitrary initial belief. Construct recursively $(\Psi_n, \eta_n, \bar{\eta}_n) \in \mathcal{A} \times \mathcal{P}_G(V) \times \text{cov}(\mathcal{P}_G(V))$ for $n = 1, 2, \dots$ as follows:

$$\begin{aligned} (i) \quad \Psi_{n+1}(i) &= BR(i, \bar{\eta}_n), \quad \text{for } \lambda\text{-almost every } i \in I, \\ (ii) \quad \eta_{n+1} &= \Psi_{n+1} \# \lambda, \\ (iii) \quad \bar{\eta}_{n+1} &= \frac{1}{n+1} \sum_{k=1}^{n+1} \eta_k = \frac{n}{n+1} \bar{\eta}_n + \frac{1}{n+1} \eta_{n+1}. \end{aligned} \tag{2}$$

One should notice that by assumption (3.1)(5) and Lemma 3.1 the expressions in (2)(i, ii) are well defined. We will show now that this procedure converges to the Nash Equilibrium when G is monotone.

Theorem 4.1. *Consider a non atomic anonymous game $G = (I, \lambda, V, (A_i)_{i \in I}, J)$ satisfying assumptions 3.1. Suppose the cost function J is monotone and there exists $C > 0$ such that for all $a, b \in V$, $\eta, \eta' \in \text{cov}(\mathcal{P}_G(V))$:*

$$\begin{aligned} |J(a, \eta) - J(a, \eta') - J(b, \eta) + J(b, \eta')| &\leq C \, d_V(a, b) \, d_1(\eta, \eta'), \\ |J(a, \eta) - J(a, \eta')| &\leq C \, d_1(\eta, \eta'). \end{aligned} \tag{3}$$

Construct $(\Psi_n, \eta_n, \bar{\eta}_n) \in \mathcal{A} \times \mathcal{P}_G(V) \times \text{cov}(\mathcal{P}_G(V))$ for $n \in \mathbb{N}$ by applying the fictitious play procedure proposed in (2). Then:

$$\eta_n, \bar{\eta}_n \xrightarrow{d_1} \tilde{\eta}$$

where $\tilde{\eta} \in \mathcal{P}_G(V)$ is the unique Nash equilibrium distribution.

Inspired from [10], the proof requires several steps. The key idea is to use the quantities $\phi_n \in \mathbb{R}$ defined by

$$\phi_n = \int_V J(a, \bar{\eta}_n) \, d(\bar{\eta}_n - \eta_{n+1})(a), \quad \text{for every } n \in \mathbb{N}.$$

Since the best response distribution of $\bar{\eta}_n$ is η_{n+1} , the quantity ϕ_n describes how much $\bar{\eta}_n$ is far from being an equilibrium. By using monotonicity and the regularity conditions, one gets

$$\forall n \in \mathbb{N}: \quad \phi_{n+1} - \phi_n \leq -\frac{1}{n+1}\phi_n + \frac{\epsilon_n}{n},$$

for suitable $\{\epsilon_n\}_{n \in \mathbb{N}}$ such that $\lim_{n \rightarrow \infty} \epsilon_n = 0$. We show the later inequality is sufficient to prove $\lim_{n \rightarrow \infty} \phi_n = 0$ and then we conclude that the accumulation points of $\bar{\eta}_n, \eta_n$ is the equilibrium distribution $\tilde{\eta}$. As one will see, the unique minimiser assumption plays a key role in Lemma 4.2 and hence in our main result.

Lemma 4.1. *Consider a sequence of real numbers $\{\phi_n\}_{n \in \mathbb{N}}$ such that $\liminf_n \phi_n \geq 0$. If there exists a real sequence $\{\epsilon_n\}_{n \in \mathbb{N}}$ such that $\lim_{n \rightarrow \infty} \epsilon_n = 0$ and :*

$$\forall n \in \mathbb{N}: \quad \phi_{n+1} - \phi_n \leq -\frac{1}{n+1}\phi_n + \frac{\epsilon_n}{n},$$

then $\lim_{n \rightarrow \infty} \phi_n = 0$.

Proof. Let $b_n = n\phi_n$ for every $n \in \mathbb{N}$. We have:

$$\forall n \in \mathbb{N}: \quad \frac{b_{n+1}}{n+1} - \frac{b_n}{n} \leq -\frac{b_n}{n(n+1)} + \frac{\epsilon_n}{n},$$

which implies $b_{n+1} \leq b_n + (n+1)\epsilon_n/n \leq b_n + 2|\epsilon_n|$. Then we get $b_n \leq b_1 + 2\sum_{i=1}^{n-1} |\epsilon_i|$ for $n \in \mathbb{N}$ and so:

$$0 \leq \liminf_n \phi_n \leq \limsup_n \phi_n \leq \limsup_n \frac{b_1 + 2\sum_{i=1}^{n-1} |\epsilon_i|}{n} = 0.$$

which proves $\lim_{n \rightarrow \infty} \phi_n = 0$. □

Lemma 4.2. *Let $(\eta_n)_{n \in \mathbb{N}}$ be defined by (2). Then*

$$d_1(\bar{\eta}_n, \bar{\eta}_{n+1}) = O(1/n), \quad \lim_{n \rightarrow \infty} d_1(\eta_n, \eta_{n+1}) = 0.$$

Proof. Let $M > 0, v \in V$ be chosen from (1). For every 1-Lipschitz continuous map $h : V \rightarrow \mathbb{R}$ we have:

$$\begin{aligned} \left| \int_V h(a) d(\bar{\eta}_{n+1} - \bar{\eta}_n) \right| &= \frac{1}{n+1} \left| \int_V h(a) d(\eta_{n+1} - \bar{\eta}_n)(a) \right| \\ &= \frac{1}{n+1} \left| \int_V (h(a) - h(v)) d(\eta_{n+1} - \bar{\eta}_n)(a) \right| \\ &\leq \frac{1}{n+1} \left(\int_V d_V(a, v) d\eta_{n+1}(a) + \frac{1}{n} \sum_{k=1}^n \int_V d_V(a, v) d\eta_k(a) \right). \end{aligned} \tag{4}$$

By the definition we have:

$$\int_V d_V(a, v) d\eta_k(a) = \int_V d_V(\Psi_k(i), v) d\lambda(i) \leq M, \quad \text{for every } k \in \mathbb{N}.$$

So we can write

$$\left| \int_V h(a) d(\bar{\eta}_{n+1} - \bar{\eta}_n) \right| \leq \frac{2M}{n+1},$$

or $d_1(\bar{\eta}_n, \bar{\eta}_{n+1}) \leq \frac{2M}{n+1}$ since h is an arbitrary 1-Lipschitz continuous function.

For the second part of the lemma, let us consider the best response distribution function Θ defined in Lemma 3.2. Since Θ is continuous (Lemma 3.2) and $\text{cov}(\mathcal{P}_G(V))$ is compact, there exists a non decreasing continuity modulus

$$\omega : \mathbb{R}^+ \rightarrow \mathbb{R}^+, \quad \lim_{x \rightarrow 0^+} \omega(x) = 0$$

such that:

$$\forall \eta_1, \eta_2 \in \overline{\text{cov}(\mathcal{P}_G(V))} : \quad d_1(\Theta(\eta_1), \Theta(\eta_2)) \leq \omega(d_1(\eta_1, \eta_2)).$$

Since for all $n \in \mathbb{N}$ we have $\bar{\eta}_n \in \overline{\text{cov}(\mathcal{P}_G(V))}$ and $\Theta(\bar{\eta}_n) = \eta_{n+1}$ we have

$$0 \leq d_1(\eta_{n+1}, \eta_{n+2}) = d_1(\Theta(\bar{\eta}_n), \Theta(\bar{\eta}_{n+1})) \leq \omega(d_1(\bar{\eta}_n, \bar{\eta}_{n+1})).$$

It gives our desired result since $d_1(\bar{\eta}_n, \bar{\eta}_{n+1}) = O(1/n)$. □

The proof of previous lemma relies heavily on the unique minimizer assumption. Instead without it, one cannot conclude that η_n, η_{n+1} are close even if $\bar{\eta}_n, \bar{\eta}_{n+1}$ are so. Even for $\bar{\eta}_n = \bar{\eta}_{n+1}$, one might have very different best responses η_n and η_{n+1} .

Proof of Theorem 4.1. Let $\{\phi_n\}_{n \in \mathbb{N}}$ be defined by:

$$\phi_n = \int_V J(a, \bar{\eta}_n) d(\bar{\eta}_n - \eta_{n+1})(a), \quad \text{for every } n \in \mathbb{N}.$$

We have $\phi_n \geq 0$ for all $n \in \mathbb{N}$. Indeed, rewriting the definition of ϕ_n , we have:

$$\phi_n = \int_I \frac{1}{n} \sum_{k=1}^n (J(\Psi_k(i), \bar{\eta}_n) - J(BR(i, \bar{\eta}_n), \bar{\eta}_n)) d\lambda(i),$$

and the positiveness comes from the definition of the best response. We now prove that exists $C > 0$ such that:

$$\phi_{n+1} - \phi_n \leq -\frac{1}{n+1} \phi_n + C \frac{d_1(\eta_n, \eta_{n+1}) + 1/n}{n}, \quad \text{for every } n \in \mathbb{N}. \quad (5)$$

Let us rewrite $\phi_{n+1} - \phi_n = A + B$, where:

$$\begin{aligned} A &= \int_V J(a, \bar{\eta}_{n+1}) d\bar{\eta}_{n+1}(a) - \int_V J(a, \bar{\eta}_n) d\bar{\eta}_n(a), \\ B &= \int_V J(a, \bar{\eta}_n) d\eta_{n+1}(a) - \int_V J(a, \bar{\eta}_{n+1}) d\eta_{n+2}(a). \end{aligned}$$

We have:

$$\begin{aligned} B &\leq \int_V J(a, \bar{\eta}_n) d\eta_{n+2}(a) - \int_V J(a, \bar{\eta}_{n+1}) d\eta_{n+2}(a) \\ &= \int_V (J(a, \bar{\eta}_n) - J(a, \bar{\eta}_{n+1})) d\eta_{n+2}(a) \\ &\leq \int_V (J(a, \bar{\eta}_n) - J(a, \bar{\eta}_{n+1})) d\eta_{n+1}(a) + \frac{C}{n} d_1(\eta_{n+1}, \eta_{n+2}), \end{aligned}$$

since by (3) and Lemma 4.2 there exists C such that the function $J(\cdot, \bar{\eta}_n) - J(\cdot, \bar{\eta}_{n+1}) : V \rightarrow \mathbb{R}$ is a C/n -Lipschitz continuous function. Let us rewrite the expression A as follows:

$$\begin{aligned} A &= \int_V J(a, \bar{\eta}_{n+1}) \, d(\bar{\eta}_n + \frac{1}{n+1}(\eta_{n+1} - \bar{\eta}_n))(a) - \int_V J(a, \bar{\eta}_n) \, d\bar{\eta}_n(a) \\ &= \int_V (J(a, \bar{\eta}_{n+1}) - J(a, \bar{\eta}_n)) \, d\bar{\eta}_n(a) + \frac{1}{n+1} \int_V J(a, \bar{\eta}_{n+1}) \, d(\eta_{n+1} - \bar{\eta}_n)(a) \\ &\leq \int_V (J(a, \bar{\eta}_{n+1}) - J(a, \bar{\eta}_n)) \, d\bar{\eta}_n(a) + \frac{1}{n+1} \int_V J(a, \bar{\eta}_n) \, d(\eta_{n+1} - \bar{\eta}_n)(a) + \frac{C}{n^2} \end{aligned}$$

since by (3) and Lemma 4.2 we have $|J(a, \bar{\eta}_n) - J(a, \bar{\eta}_{n+1})| \leq C \, d_1(\bar{\eta}_{n+1}, \bar{\eta}_n) = O(1/n)$. So

$$A \leq \int_V (J(a, \bar{\eta}_{n+1}) - J(a, \bar{\eta}_n)) \, d\bar{\eta}_n(a) - \frac{\phi_n}{n+1} + \frac{C}{n^2}.$$

Then if we set $\epsilon_n = C(d_1(\eta_{n+1}, \eta_{n+2}) + 1/n)$, by using the above inequalities for A, B , we have :

$$\begin{aligned} A + B &\leq \int_V (J(a, \bar{\eta}_{n+1}) - J(a, \bar{\eta}_n)) \, d(\bar{\eta}_n - \eta_{n+1})(a) - \frac{\phi_n}{n+1} + \frac{\epsilon_n}{n} \\ &= -(n+1) \int_V (J(a, \bar{\eta}_{n+1}) - J(a, \bar{\eta}_n)) \, d(\bar{\eta}_{n+1} - \bar{\eta}_n)(a) - \frac{\phi_n}{n+1} + \frac{\epsilon_n}{n} \\ &\leq -\frac{\phi_n}{n+1} + \frac{\epsilon_n}{n}, \end{aligned} \tag{6}$$

and the last inequality comes from the monotonicity assumption. By Lemmas 4.1 and 4.2, the inequality (5) implies $\phi_n \rightarrow 0$. Let $(\eta, \bar{\eta}) \in \overline{\mathcal{P}_G(V)} \times \overline{\text{cov}(\mathcal{P}_G(V))}$ be an accumulation point of the set $\{(\eta_{n+1}, \bar{\eta}_n)\}_{n \in \mathbb{N}}$. We have $\eta = \Theta(\bar{\eta})$ due to the continuity of best response distribution function Θ (Lemma 3.2) and the fact that $\eta_{n+1} = \Theta(\bar{\eta}_n)$.

Take an arbitrary $\theta \in \mathcal{P}_G(V)$. Since J is lower semi-continuous we have (see [1] section 5.1.1):

$$\begin{aligned} \int_V J(a, \bar{\eta}) \, d(\bar{\eta} - \theta)(a) &\leq \liminf \int_V J(a, \bar{\eta}) \, d(\bar{\eta}_n - \theta)(a) = \liminf \int_V J(a, \bar{\eta}_n) \, d(\bar{\eta}_n - \theta)(a) \\ &= \liminf \int_V J(a, \bar{\eta}_n) \, d(\eta_{n+1} - \theta)(a) + \phi_n \leq \liminf \phi_n = 0 \end{aligned}$$

since $\eta_{n+1} = \Theta(\bar{\eta}_n)$ and $\int_V J(a, \bar{\eta}_n) \, d(\eta_{n+1} - \theta)(a) \leq 0$ for every $\theta \in \mathcal{P}_G(V)$. So:

$$\forall \theta \in \mathcal{P}_G(V) : \int_V J(a, \bar{\eta}) \, d(\bar{\eta} - \theta)(a) \leq 0. \tag{7}$$

We rewrite the above inequality as follows: since $\bar{\eta} \in \overline{\text{cov}(\mathcal{P}_G(V))}$ by Corollary 8.1 we can disintegrate it with respect to $(A_i)_{i \in I}$ i.e. there are $\{\bar{\eta}^i\}_{i \in I} \subseteq \mathcal{P}(V)$ such that for λ -almost every $i \in I$ we have $\text{supp}(\bar{\eta}^i) \subset A_i$ and for every integrable function $h : V \rightarrow \mathbb{R}$:

$$\int_V h(a) \, d\bar{\eta}(a) = \int_I \int_{A_i} h(a) \, d(\bar{\eta}^i)(a) \, d\lambda(i).$$

Specially for $h = J(\cdot, \eta)$ we have:

$$\int_V J(a, \bar{\eta}) \, d\bar{\eta}(a) = \int_I \int_{A_i} J(a, \bar{\eta}) \, d(\bar{\eta}^i)(a) \, d\lambda(i),$$

and for all $\Psi \in \mathcal{A}$:

$$\int_V J(a, \bar{\eta}) \, d(\Psi \# \lambda)(a) = \int_I J(\Psi(i), \bar{\eta}) \, d\lambda(i) = \int_I \int_{A_i} J(\Psi(i), \bar{\eta}) \, d(\bar{\eta}^i)(a) \, d\lambda(i).$$

Combining the previous equalities with (7), gives us:

$$\forall \Psi \in \mathcal{A}: \quad \int_I \int_{A_i} (J(a, \bar{\eta}) - J(\Psi(i), \bar{\eta})) \, d(\bar{\eta}^i)(a) \, d\lambda(i) = \int_V J(a, \bar{\eta}) \, d(\bar{\eta} - \Psi \# \lambda)(a) \leq 0.$$

In particular if $\Psi = BR(\cdot, \bar{\eta})$ we have:

$$\int_I \int_{A_i} (J(a, \bar{\eta}) - J(BR(i, \bar{\eta}), \bar{\eta})) \, d(\bar{\eta}^i)(a) \, d\lambda(i) \leq 0,$$

which gives the equality by definition of best response action. So by unique minimizer we have $\bar{\eta}^i = \delta_{BR(i, \bar{\eta})}$ for λ -almost every $i \in I$. It means $\bar{\eta} = BR(\cdot, \bar{\eta}) \# \lambda$ or $\bar{\eta} = \Theta(\bar{\eta})$. Hence $\bar{\eta} = \eta$ and they are both equal to $\bar{\eta} \in \mathcal{P}_G(V)$, the unique fixed point of Θ , or equivalently, the unique equilibrium distribution. \square

5 Online mirror descent

Here we investigate the convergence to a Nash equilibrium by applying Online Mirror Descent (OMD) in anonymous games. The form of OMD algorithm is closely related to the online optimization and no regret algorithms. The reader can find a good explanatory note in [24]. The goal of the algorithm is to act optimally in online manner by "minimizing" a function that itself changes at each step. In the game frameworks, the cost function changes due to change of the actions chosen by adversaries in each round. As one can notice in the following, we need the structure of vector space for the action sets.

5.1 Preliminaries

Before we propose the main OMD, let us review some definitions and lemmas.

Definition 5.1. *Let $(W, \|\cdot\|_W)$ be a normed vector space. For $K > 0$ we say that $h : W \rightarrow \mathbb{R}$ is a K -strongly convex function if*

$$\forall a_1, a_2 \in W, \forall \lambda \in [0, 1]: \quad h(\lambda a_1 + (1 - \lambda)a_2) \leq \lambda h(a_1) + (1 - \lambda)h(a_2) - K\lambda(1 - \lambda)\|a_1 - a_2\|_W^2.$$

Definition 5.2. *The Fenchel conjugate of a function $h : W \rightarrow \mathbb{R}$ on a set $A \subseteq W$ is defined by:*

$$h_A^* : W^* \rightarrow \mathbb{R} \cup \{+\infty\}: \quad h_A^*(y) = \sup_{a \in A} \langle y, a \rangle - h(a), \quad \text{for all } y \in W^*$$

and the related maximiser correspondence by:

$$Q_A : W^* \rightarrow A: \quad Q_A(y) = \arg \max_{a \in A} \langle y, a \rangle - h(a), \quad \text{for all } y \in W^*.$$

Remark 5.1. *The corresponding Q_A is not empty if A is weakly closed and h is weakly lower semi-continuous and coercive, i.e. $\lim_{\|a\|_W \rightarrow \infty} h(a)/\|a\|_W = +\infty$.*

If W be a Hilbert space (so $W^* = W$) and $h(a) = \frac{1}{2}\|a\|_W^2$ then the correspondence Q_A will be the classical projection on A :

$$Q_A(y) = \arg \max_{a \in A} \langle y, a \rangle_W - \frac{1}{2}\|a\|_W^2 = \arg \max_{a \in A} -\|y - a\|_W^2 = \pi_A(y).$$

Lemma 5.1. *Let $h : W \rightarrow \mathbb{R}$ be a K -strongly convex function and A a convex subset of W . For any $y_1, y_2 \in W^*$ let $a_i \in Q_A(y_i)$, $i = 1, 2$. Then we have:*

$$2K\|a_1 - a_2\|_W^2 \leq \langle y_1 - y_2, a_1 - a_2 \rangle.$$

It implies $\|a_1 - a_2\|_W \leq \frac{1}{2K}\|y_1 - y_2\|_{W^}$. In particular if $y_1 = y_2$ then $a_1 = a_2$ i.e. the correspondence $Q_A(y)$ is either empty or single valued for every $y \in W^*$.*

Proof. Since A is convex, for every $\epsilon \in (0, 1]$ we have $(1 - \epsilon)a_1 + \epsilon a_2 \in A$. By definition:

$$\langle y_1, a_1 \rangle - h(a_1) \geq \langle y_1, (1 - \epsilon)a_1 + \epsilon a_2 \rangle - h((1 - \epsilon)a_1 + \epsilon a_2),$$

and K -strongly convex condition for h gives:

$$h((1 - \epsilon)a_1 + \epsilon a_2) \leq (1 - \epsilon)h(a_1) + \epsilon h(a_2) - K\epsilon(1 - \epsilon)\|a_1 - a_2\|^2.$$

So by combining the above inequalities:

$$\langle y_1, a_1 \rangle - h(a_1) \geq \langle y_1, (1 - \epsilon)a_1 + \epsilon a_2 \rangle - (1 - \epsilon)h(a_1) - \epsilon h(a_2) + K\epsilon(1 - \epsilon)\|a_1 - a_2\|^2,$$

which gives:

$$\epsilon \langle y_1, a_1 - a_2 \rangle \geq \epsilon h(a_1) - \epsilon h(a_2) + K\epsilon(1 - \epsilon)\|a_1 - a_2\|^2.$$

After dividing the both sides by ϵ and then tending $\epsilon \rightarrow 0^+$ we will get:

$$\langle y_1, a_1 - a_2 \rangle \geq h(a_1) - h(a_2) + K\|a_1 - a_2\|^2.$$

By exchanging the role of (a_1, y_1) and (a_2, y_2) we have:

$$\langle y_2, a_2 - a_1 \rangle \geq h(a_2) - h(a_1) + K\|a_2 - a_1\|^2.$$

It yields the desired result if we sum up the two last inequalities. □

Definition 5.3. *Let $F : W \rightarrow \mathbb{R}$ be a convex function. We say that $v \in W^*$ is a sub-gradient of F at $a \in W$ if:*

$$\forall b \in W : F(b) - F(a) \geq \langle v, b - a \rangle,$$

and set $\partial F(a) \subseteq W^$ the set of all sub-gradients at a .*

One can notice that if $F : W \rightarrow \mathbb{R}$ is differentiable (in sense of Fréchet) at $a \in W$, then $\partial F(a) = \{DF(a)\}$.

5.2 OMD algorithm and convergence result

Consider an anonymous game $G = (I, \lambda, V, (A_i)_{i \in I}, J)$. Suppose that the following conditions hold:

- there is a normed vector space $(W, \|\cdot\|_W)$ such that

$$\bigcup_{i \in I} A_i \subseteq W \subseteq V,$$

and let $h : W \rightarrow \mathbb{R}$ be a K -strongly convex function for a real $K > 0$.

- for every $i \in I$ the action sets A_i are weakly closed in W and h is weakly lower semi-continuous and coercive (and hence Q_{A_i} is single valued by Remark 5.1),
- for every $(a, \eta) \in W \times \mathcal{P}_G(V)$ the function $J(\cdot, \eta) : W \rightarrow \mathbb{R}$ is convex and exists a subgradient $y(a, \eta) \in \partial_a J(\cdot, \eta) \subseteq W^*$,

Let $\{\beta_n\}_{n \in \mathbb{N}}$ be a sequence of real positive numbers. Set an arbitrary initial measurable functions $\Psi_0 \in \mathcal{A}$, $\eta_0 = \Psi_0 \sharp \lambda$, $\Phi_0 : I \rightarrow W^*$. The following procedure (8) is called the *Online Mirror Descent (OMD)* on anonymous game G :

$$\begin{aligned} (i) \quad \Phi_{n+1}(i) &= \Phi_n(i) - \beta_n y(\Psi_n(i), \eta_n), \quad \text{for every } i \in I \\ (ii) \quad \Psi_{n+1}(i) &= Q_{A_i}(\Phi_{n+1}(i)), \quad \text{for every } i \in I \\ (iii) \quad \eta_{n+1} &= \Psi_{n+1} \sharp \lambda. \end{aligned} \tag{8}$$

Theorem 5.1. *Suppose one applies the OMD algorithm proposed in (8) for $\beta_n = \frac{1}{n}$. Suppose the following conditions hold:*

1. the game G satisfies assumptions (3.1),
2. for every $i \in I$ the action sets A_i are convex and exists $M > 0$ such that for λ -almost every $i \in I$ we have $\|a\|_W \leq M$ for all $a \in A_i$ and we have $R(M) := \sup_{\|a\| \leq M} |h(a)| < +\infty$,
3. the map $\Phi_0 : I \rightarrow W^*$ is bounded,
4. the cost function J is monotone,
5. there exists $\delta > 0$ such that for λ -almost every $i \in I$ and all $a \in A_i, \eta \in \mathcal{P}_G(V)$,

$$\|y(a, \eta)\|_{W^*} \leq \delta. \tag{9}$$

Then $\eta_n = \Psi_n \sharp \lambda$ converges to $\tilde{\eta} = \tilde{\Psi} \sharp \lambda$ where $\tilde{\eta} \in \mathcal{P}_G(V)$ is the unique Nash equilibrium distribution.

Remark 5.2. *For every $y, z \in W^*$ and any $A \subseteq W$ we have :*

$$\forall a \in Q_A(y) : \quad h_A^*(y) - h_A^*(z) \leq \langle y - z, a \rangle.$$

This is obvious since $h_A^(y) - \langle y, a \rangle + h(a) = 0 \leq h_A^*(z) - \langle z, a \rangle + h(a)$.*

Proof of Theorem 5.1. Let $\tilde{\Psi} \in \mathcal{A}$ be a Nash equilibrium profile. Define the real sequence $\{\phi_n\}_{n \in \mathbb{N}}$ as follows:

$$\forall n \in \mathbb{N} : \quad \phi_n = \int_I \left(h(\tilde{\Psi}(i)) + h_{A_i}^*(\Phi_n(i)) - \langle \Phi_n(i), \tilde{\Psi}(i) \rangle \right) d\lambda(i).$$

By definition of Fenchel conjugate we know $\phi_n \geq 0$. For making the rest of argument well-defined, we first show that ϕ_n is indeed finite. We have

$$\int_I \left(h(\tilde{\Psi}(i)) + h_{A_i}^*(\Phi_n(i)) - \langle \Phi_n(i), \tilde{\Psi}(i) \rangle \right) d\lambda(i) = \int_I \left(h(\tilde{\Psi}(i)) - h(\Psi_n(i)) - \langle \Phi_n(i), \tilde{\Psi}(i) - \Psi_n(i) \rangle \right) d\lambda(i)$$

since $\Psi_n(i) = Q_{A_i}(\Phi_n(i))$ for λ -almost every $i \in I$. Moreover,

$$|h(\tilde{\Psi}(i)) - h(\Psi_n(i)) - \langle \Phi_n(i), \tilde{\Psi}(i) - \Psi_n(i) \rangle| \leq 2R(M) + 2\|\Phi_n(i)\|_{W^*} M,$$

since $\|\tilde{\Psi}(i)\|_W, \|\Psi_n(i)\|_W \leq M$ for λ -almost every $i \in I$. By (8)(i) we have:

$$\forall n \in \mathbb{N} : \quad \|\Phi_n\|_\infty \leq \delta \left(1 + \frac{1}{2} + \dots + \frac{1}{n-1} \right) + \|\Phi_0\|_\infty$$

which yields $|\phi_n| < \infty$. Let us compute the difference $\phi_{n+1} - \phi_n$:

$$\phi_{n+1} - \phi_n = \int_I \left(h_{A_i}^*(\Phi_{n+1}(i)) - h_{A_i}^*(\Phi_n(i)) - \langle \Phi_{n+1}(i) - \Phi_n(i), \tilde{\Psi}(i) \rangle \right) d\lambda(i)$$

So from Remark 5.2:

$$\begin{aligned} \phi_{n+1} - \phi_n &\leq \int_I \langle \Phi_{n+1}(i) - \Phi_n(i), \Psi_{n+1}(i) - \tilde{\Psi}(i) \rangle d\lambda(i) \\ &= -\beta_n \int_I \langle y(\Psi_n(i), \eta_n), \Psi_{n+1}(i) - \tilde{\Psi}(i) \rangle d\lambda(i) \\ &= -\beta_n \int_I \left(\langle y(\Psi_n(i), \eta_n), \Psi_n(i) - \tilde{\Psi}(i) \rangle + \langle y(\Psi_n(i), \eta_n), \Psi_{n+1}(i) - \Psi_n(i) \rangle \right) d\lambda(i) \\ &\leq -\beta_n \alpha_n + C\beta_n^2 \end{aligned}$$

where $\alpha_n = \int_I \langle y(\Psi_n(i), \eta_n), \Psi_n(i) - \tilde{\Psi}(i) \rangle d\lambda(i)$ and since by condition (9) we have:

$$\begin{aligned} |\langle y(\Psi_n(i), \eta_n), \Psi_{n+1}(i) - \Psi_n(i) \rangle| &\leq \delta \|\Psi_{n+1}(i) - \Psi_n(i)\|_W \\ &\leq \frac{\delta}{2K} \|\Phi_{n+1}(i) - \Phi_n(i)\|_{W^*} = \beta_n \frac{\delta}{2K} \|y(\Psi_n(i), \eta_n)\|_{W^*} \leq \beta_n \frac{\delta^2}{2K}. \end{aligned}$$

By definition of the sub-gradient we have:

$$\forall b \in W : \quad \langle y(a, \eta_n), a - b \rangle \geq J(a, \eta_n) - J(b, \eta_n).$$

So:

$$\begin{aligned} \alpha_n &= \int_I \langle y(\Psi_n(i), \eta_n), \Psi_n(i) - \tilde{\Psi}(i) \rangle d\lambda(i) \geq \int_I \left(J(\Psi_n(i), \eta_n) - J(\tilde{\Psi}(i), \eta_n) \right) d\lambda(i) = \\ &\int_X J(a, \eta_n) d(\eta_n - \tilde{\eta})(a) \geq \int_X J(a, \tilde{\eta}) d(\eta_n - \tilde{\eta})(a) = \psi_n \geq 0, \end{aligned}$$

by Remark 3.1. Since $\beta_n = \frac{1}{n}$ we have:

$$\sum_{n=1}^N \frac{\psi_n}{n} \leq \sum_{n=1}^N \frac{\alpha_n}{n} = \sum_{n=1}^N \beta_n \alpha_n \leq \sum_{n=1}^N \left(\phi_n - \phi_{n+1} + \frac{C}{n^2} \right) = \phi_1 - \phi_{N+1} + \sum_{n=1}^N \frac{C}{n^2} < +\infty \quad (10)$$

so $\sum_{n=1}^{\infty} \frac{\psi_n}{n} < +\infty$. We show then that $|\psi_{n+1} - \psi_n| = O(1/n)$. We remind from Lemma ?? that this yields $\lim_{n \rightarrow \infty} \psi_n = 0$. We have

$$\psi_{n+1} - \psi_n = \int_X J(a, \tilde{\eta}) \, d(\eta_{n+1} - \eta_n)(a) = \int_X (J(\Psi_{n+1}(i), \tilde{\eta}) - J(\Psi_n(i), \tilde{\eta})) \, d\lambda(i),$$

and from the definition of sub-gradient:

$$\langle y(\Psi_n(i), \tilde{\eta}), \Psi_{n+1}(i) - \Psi_n(i) \rangle \leq J(\Psi_{n+1}(i), \tilde{\eta}) - J(\Psi_n(i), \tilde{\eta}) \leq \langle y(\Psi_{n+1}(i), \tilde{\eta}), \Psi_{n+1}(i) - \Psi_n(i) \rangle$$

so $|J(\Psi_{n+1}(i), \tilde{\eta}) - J(\Psi_n(i), \tilde{\eta})| = O(1/n)$ which gives $|\psi_{n+1} - \psi_n| = O(1/n)$.

Since $\mathcal{P}_G(V)$ is pre-compact, there exist a sequence $\{n_i\}_{i \in \mathbb{N}} \subseteq \mathbb{N}$ and $\eta' \in \overline{\mathcal{P}_G(V)}$ such that $\lim_{i \rightarrow \infty} \eta_{n_i} = \eta'$. Since $J(\cdot, \tilde{\eta}) : V \rightarrow \mathbb{R}$ is lower semi-continuous, we have:

$$\int_V J(a, \tilde{\eta}) \, d(\eta' - \tilde{\eta})(a) \leq \liminf_i \int_V J(a, \tilde{\eta}) \, d(\eta_{n_i} - \tilde{\eta}) = \liminf_i \psi_{n_i} = 0,$$

which yields $\eta' = \tilde{\eta}$ due to the Corollary 8.1 and the definition of Nash equilibrium distribution. So every accumulation point of set $\{\eta_n\}_{n \in \mathbb{N}} \subseteq \mathcal{P}_G(V)$ is $\tilde{\eta}$ which gives $\lim_{n \rightarrow \infty} \eta_n = \tilde{\eta}$ since $\mathcal{P}_G(V)$ is pre-compact. \square

6 Application to first order MFG

6.1 Model

Let us show the first-order mean field games are special case of non atomic anonymous games proposed in section 3. Set $I = \mathbb{R}^d$ with the usual topology, as the set of players and $m_0 \in \mathcal{P}(I)$ a given non atomic Borel probability measure on \mathbb{R}^d . Let $V = \mathcal{C}^0([0, T], \mathbb{R}^d)$ endowed with the supremum norm $\|\gamma\|_{\infty} = \sup_{t \in [0, T]} \|\gamma(t)\|$. For each player $i \in \mathbb{R}^d$ let $A_i = S_{i, M} \subseteq \mathcal{C}^0([0, T], \mathbb{R}^d)$ where:

$$\forall x \in \mathbb{R}^d, M > 0 : \quad S_{x, M} := \{ \gamma \in \mathcal{AC}([0, T], \mathbb{R}^d) \mid \gamma(0) = x, \int_0^T \|\dot{\gamma}(t)\|^2 \, dt \leq M \}, \quad (11)$$

where $\mathcal{AC}([0, T], \mathbb{R}^d)$ denotes the set absolutely continuous function from $[0, T]$ to \mathbb{R}^d . We will explain later how to choose $M > 0$ properly.

Let $H^1([0, T], \mathbb{R}^d)$ defined as

$$H^1([0, T], \mathbb{R}^d) = \left\{ \gamma \in \mathcal{AC}([0, T], \mathbb{R}^d) \mid \int_0^T \|\dot{\gamma}(t)\|^2 \, dt < +\infty \right\}.$$

We denote $\mathcal{P}_1(\mathcal{C}^0([0, T], \mathbb{R}^d))$ be the set of Borel probability measures with finite first moment on $\mathcal{C}^0([0, T], \mathbb{R}^d)$. Set for every $t \in [0, T]$ the evaluation function $e_t : \mathcal{C}^0([0, T], \mathbb{R}^d) \rightarrow \mathbb{R}^d$ as $e_t(\gamma) = \gamma(t)$. The MFG cost function $J : \mathcal{C}^0([0, T], \mathbb{R}^d) \times \mathcal{P}_1(\mathcal{C}^0([0, T], \mathbb{R}^d)) \rightarrow \mathbb{R}$ is defined as follows:

$$J(\gamma, \eta) = \begin{cases} \int_0^T (L(\gamma(t), \dot{\gamma}(t)) + f(\gamma(t), e_t \# \eta)) \, dt + g(\gamma(T), e_T \# \eta), & \text{if } \gamma \in H^1([0, T], \mathbb{R}^d) \\ +\infty & \text{otherwise,} \end{cases}$$

We call the anonymous game $G = (\mathbb{R}^d, m_0, \mathcal{C}^0([0, T], \mathbb{R}^d), (S_{i,M})_{i \in \mathbb{R}^d}, J)$ a *first-order mean field game*.

Remark 6.1. For every admissible profile of actions $\Psi : I \rightarrow \mathcal{C}^0([0, T], \mathbb{R}^d)$ that $\Psi(i) \in S_{i,M}$, for $\eta = \Psi \# m_0$ we have:

$$d_1(e_t \# \eta, e_s \# \eta) \leq \int_{\Gamma} \|\gamma(t) - \gamma(s)\| d\eta(\gamma) \leq \sqrt{|t-s|} \int_{\Gamma} \sqrt{\int_t^s \|\dot{\gamma}(r)\|^2 dr} d\eta(\gamma) \leq \sqrt{M|t-s|},$$

due to definition of M in (11). That means for every $\eta \in \mathcal{P}_G(V)$ the map $t \rightarrow e_t \# \eta$ is $\frac{1}{2}$ -Holder continuous.

Suppose that the following conditions hold for the data:

Assumption 6.1. Let

1. m_0 has a compact support,
2. for every $x \in \mathbb{R}^d$ the map $L(x, \cdot) : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is twice differentiable and there exists $C > 0$ such that for all $(x, v) \in \mathbb{R}^d \times \mathbb{R}^d$ we have:

$$\frac{1}{C} I_d \leq D_{vv} L(x, \cdot) \leq C I_d, \quad \|L_x(x, v)\| \leq C,$$

3. the functions $f, g : \mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d) \rightarrow \mathbb{R}$ are continuous and for every $m \in \mathcal{P}_1(\mathbb{R}^d)$ the maps $f(\cdot, m), g(\cdot, m) : \mathbb{R}^d \rightarrow \mathbb{R}$ are $\mathcal{C}^1(\mathbb{R}^d; \mathbb{R})$,
4. suppose that there exist $C > 0$ such that:

$$\forall x \in \mathbb{R}^d, m \in \mathcal{P}_1(\mathbb{R}^d) : \quad \|f_x(x, m)\|, \|g_x(x, m)\| \leq C.$$

Remark 6.2 ([7], Theorem 7.2.4). If conditions 6.1(2,3,4) hold, then there is at least one minimizer of variational problem

$$\min_{\gamma \in \mathcal{AC}([0, T], \mathbb{R}^d), \gamma(0)=x} \int_0^T (L(\gamma(t), \dot{\gamma}(t)) + f(\gamma(t), e_t \# \eta)) dt + g(\gamma(T), e_T \# \eta). \quad (12)$$

The minimizer $\gamma : [0, T] \rightarrow \mathbb{R}^d$ belongs to $\mathcal{C}^1([0, T], \mathbb{R}^d)$, $L_v(\gamma(t), \dot{\gamma}(t))$ is absolutely continuous and

$$\frac{d}{dt} L_v(\gamma(t), \dot{\gamma}(t)) = L_x(\gamma(t), \dot{\gamma}(t)) + f_x(\gamma(t), e_t \# \eta), \quad \text{for almost every } t \in [0, T], \quad (13)$$

with $\dot{\gamma}(T) = -g_x(\gamma(T), e_T \# \eta)$. In addition there is $M > 0$ such that $\|\dot{\gamma}\|_{\infty} \leq \sqrt{M/T}$ for every solution of (13). This is the way we set M in (11) as a function of constants of data in 6.1(2,3,4).

The following remark asserts that the definition of action sets in (11) and conditions in 6.1(2,3) imply the assumptions (3.1) for first order mean field game.

Remark 6.3. If $K \subseteq \mathbb{R}^d$ be compact such that $\text{supp}(m_0) \subseteq K$, then

1. for $x \in K$ we have $\|\gamma\|_{\infty} \leq \max_{y \in K} \|y\| + MT$, for $\gamma \in S_{x,M}$, which gives the condition (1),

2. the correspondence $A : I \rightarrow V$, $A(i) = S_{i,M}$ is continuous and by Arzela-Ascoli $S_{i,M}$ is compact for all $i \in I$,
3. the convexity of $L(x, \cdot)$ implies that for every $\eta \in \mathcal{P}(V)$ the function $J(\cdot, \eta) : \mathcal{C}^0([0, T], \mathbb{R}^d) \rightarrow \mathbb{R}$ is lower semi-continuous,
4. the function $\text{Min} : I \times \mathcal{P}_G(V) \rightarrow \mathbb{R}$, $\text{Min}(i, \eta) := \min_{a \in A_i} J(a, \eta)$ is continuous,
5. by Arzela-Ascoli the following set:

$$S = S_{K,M} = \left\{ \gamma \in \mathcal{C}^0([0, T], \mathbb{R}^d) \mid \gamma(0) \in K, \int_0^T \|\dot{\gamma}(t)\|^2 dt \leq M \right\}$$

is precompact in V . For every $\eta \in \text{cov}(\mathcal{P}_G(V))$ we have $\text{supp}(\eta) \subseteq S$, so $\text{cov}(\mathcal{P}_G(V))$ is tight and hence it is pre-compact in $(\mathcal{P}_1(V), d_1)$,

6. the minimiser of problem (12) is unique as is explained in section ???. Hence the unique minimiser condition holds.

Corollary 6.1. *The first-order MFGs defined above, satisfies the assumptions (3.1) and hence by Theorem 3.1 has at least a Nash equilibrium $\tilde{\Psi} \in \mathcal{A}$. If we set $\tilde{\eta} = \tilde{\Psi} \# m_0$ and $e_t \# \tilde{\eta} = \tilde{m}_t$ for all $t \in [0, T]$, then for m_0 -almost every $i \in \mathbb{R}^d$:*

$$\tilde{\Psi}(i) = \underset{\gamma \in H^1([0, T], \mathbb{R}^d), \gamma(0)=i}{\text{argmin}} \int_0^T (L(\gamma(t), \dot{\gamma}(t)) + f(\gamma(t), \tilde{m}_t)) dt + g(\gamma(T), \tilde{m}_T).$$

The measure $\tilde{\eta}$ is an equilibrium distribution in sense of (??). Under stronger assumptions, by following section ??? we can construct the first order MFG system solution (u, m) from the equilibrium distribution $\tilde{\eta}$ as in (??).

We prove that the uniqueness of equilibrium is a consequence of the monotonicity of f, g and the unique minimizer condition. This is the counterpart for the uniqueness result in [14].

Lemma 6.1. *If $f, g : \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \rightarrow \mathbb{R}$ are monotone, then the MFG cost function will be so.*

Proof. Let $\eta_1, \eta_2 \in \mathcal{P}(V)$. If we define $m_{i,t} = e_t \# \eta_i$ for $i = 1, 2$ and $t \in [0, T]$, we then have:

$$\int_V (J(\gamma, \eta_1) - J(\gamma, \eta_2)) d(\eta_1 - \eta_2)(\gamma) = \int_V \left(\int_0^T (f(\gamma(t), m_{1,t}) - f(\gamma(t), m_{2,t})) dt + g(\gamma(T), m_{1,T}) - g(\gamma(T), m_{2,T}) \right) d(\eta_1 - \eta_2)(\gamma) = A + B$$

where

$$A = \int_0^T \left(\int_{\mathbb{R}^d} (f(x, m_{1,t}) - f(x, m_{2,t})) d(m_{1,t} - m_{2,t})(x) \right) dt \geq 0$$

$$B = \int_{\mathbb{R}^d} (g(x, m_{1,T}) - g(x, m_{2,T})) d(m_{1,T} - m_{2,T})(x) \geq 0,$$

since the couplings f, g are monotone. □

Corollary 6.2. *The monotone first order MFG satisfying assumptions 6.1 possesses a unique equilibrium.*

6.2 Fictitious play in monotone first order MFG

The fictitious play in first-order MFG takes such form: for initial profile of actions

$$\Psi_1 \in \mathcal{A}, \bar{\eta}_1 = \eta_1 = \Psi_1 \# \lambda \in \mathcal{P}(V)$$

the players play as follows at round $n = 1, 2, \dots$:

$$\begin{aligned} (i) \quad \Psi_{n+1}(i) &= \arg \max_{\gamma \in H^1, \gamma(0)=i} \int_0^T (L(\gamma(t), \dot{\gamma}(t)) + f(\gamma(t), e_t \# \bar{\eta}_n)) dt + g(\gamma(T), e_T \# \bar{\eta}_n), \\ (ii) \quad \eta_{n+1} &= \Psi_{n+1} \# \lambda, \\ (iii) \quad \bar{\eta}_{n+1} &= \frac{1}{n+1} \sum_{i=1}^{n+1} \eta_i. \end{aligned} \tag{14}$$

where (i) holds for m_0 -almost every $i \in \mathbb{R}^d$. Here we apply the convergence result in fictitious play (Section 3) for monotone first-order MFG. We suppose the assumptions 6.1 (and hence (3.1)) conditions hold.

Lemma 6.2. *If $f, g : m \rightarrow f(\cdot, m), g(\cdot, m)$ are Lipschitz from $\mathcal{P}(\mathbb{R}^d)$ to $\mathcal{C}^1(\mathbb{R}^d)$ then there is a constant $C > 0$ such that:*

$$\begin{aligned} |J(\gamma, \eta) - J(\gamma, \eta') - J(\gamma', \eta) + J(\gamma', \eta')| &\leq C \|\gamma - \gamma'\|_\infty d_1(\eta, \eta') \\ |J(\gamma, \eta) - J(\gamma, \eta')| &\leq C d_1(\eta, \eta') \end{aligned}$$

for every $\gamma, \gamma' \in H^1([0, T], \mathbb{R}^d)$ and $\eta, \eta' \in \mathcal{P}(V)$.

Proof. Since $f : m \rightarrow f(\cdot, m)$ is Lipschitz from $\mathcal{P}(\mathbb{R}^d)$ to $\mathcal{C}^1(\mathbb{R}^d)$ there is $C > 0$ such that:

$$\|f(\cdot, m) - f(\cdot, m')\|_{\mathcal{C}^1} \leq C d_1(m, m'), \quad \|g(\cdot, m) - g(\cdot, m')\|_{\mathcal{C}^1} \leq C d_1(m, m')$$

which means that for every $x, x' \in \mathbb{R}^d$ we have

$$\begin{aligned} |f(x, m) - f(x, m') - f(x', m) + f(x', m')| &\leq C \|x - x'\| d_1(m, m'), \\ |f(x, m) - f(x, m')| &\leq C d_1(m, m'). \end{aligned}$$

Similar inequalities hold with respect to g . We have:

$$\begin{aligned} &|J(\gamma, \eta) - J(\gamma, \eta') - J(\gamma', \eta) + J(\gamma', \eta')| \\ &\leq \int_0^T |f(\gamma(t), e_t \# \eta) - f(\gamma(t), e_t \# \eta') + f(\gamma'(t), e_t \# \eta) - f(\gamma'(t), e_t \# \eta')| dt \\ &\quad + |g(\gamma(T), e_T \# \eta) - g(\gamma(T), e_T \# \eta') - g(\gamma'(T), e_T \# \eta) + g(\gamma'(T), e_T \# \eta')| \\ &\leq C \int_0^T \|\gamma(t) - \gamma'(t)\| d_1(e_t \# \eta, e_t \# \eta') dt + \|\gamma(T) - \gamma'(T)\| d_1(e_T \# \eta, e_T \# \eta') \\ &\leq C \int_0^T \|\gamma - \gamma'\|_\infty d_1(\eta, \eta') dt + \|\gamma - \gamma'\|_\infty d_1(\eta, \eta') = (CT + 1) \|\gamma - \gamma'\|_\infty d_1(\eta, \eta'), \end{aligned}$$

and

$$\begin{aligned} |J(\gamma, \eta) - J(\gamma, \eta')| &\leq \int_0^T |f(\gamma(t), e_t \# \eta) - f(\gamma(t), e_t \# \eta')| dt + |g(\gamma(T), e_T \# \eta) - g(\gamma(T), e_T \# \eta')| \\ &\leq C \int_0^T d_1(e_t \# \eta, e_t \# \eta') dt + d_1(e_T \# \eta, e_T \# \eta') \leq (CT + 1) d_1(\eta, \eta'). \end{aligned}$$

□

Corollary 6.3. *If $f, g : m \rightarrow f(\cdot, m), g(\cdot, m)$ are Lipschitz, then by Lemma 6.2, the convergence result of fictitious play (Theorem 4.1) holds for the first-order monotone MFG.*

6.3 Online mirror descent in monotone first order MFG

Here we use the convergence result proved in section 4 for the first-order MFG with a monotone convex cost function J . Let us suppose that the couplings f, g are monotone and $L(\cdot, \cdot), f(\cdot, m), g(\cdot, m)$ are convex for every $m \in \mathcal{P}(\mathbb{R}^d)$. It easily yields that J is monotone (by Lemma 6.1) and for every $\eta \in \mathcal{P}(V)$, the function $J(\cdot, \eta) : H^1([0, T], \mathbb{R}^d) \rightarrow \mathbb{R}$ is convex.

Remark 6.4. *We propose an example of data L, f, g such that they are convex in x, v inputs. Before we start the precise definition, let us point out that we can relax the condition 6.1(4) and replace it with the following assumption. Suppose that there exists $M > 0$ in (11) such that for all solution γ of Euler-Lagrange equation (13) with $\eta \in \mathcal{P}_G(V)$, we have $\|\dot{\gamma}\|_\infty \leq \sqrt{M/T}$. This assumption with the conditions (6.1)(1,2,3) give the existence of equilibrium as in Corollary 6.1.*

For the example, set

$$L(x, v) = \frac{1}{2} \|v\|^2, \quad f(x, m) = \alpha \langle x, \mathbb{E}_m z \rangle, \quad g(x, m) = \beta \langle x, \mathbb{E}_m z \rangle,$$

for some $\alpha, \beta > 0$ where $\mathbb{E}_m z = \int_{\mathbb{R}^d} z \, dm(z)$. Set $\alpha, \beta, R > 0$ with

$$\alpha T^2 R + \beta T R + \sup_{x \in \text{supp}(m_0)} \|x\| \leq R.$$

and the constant $M > 0$ in (11) with $M = T(\alpha T R + \beta R)^2$. For every $\eta \in \mathcal{P}_G(V)$ we have

$$\text{for } \eta\text{-almost every } \gamma: \|\gamma(t)\| \leq \sup_{x \in \text{supp}(m_0)} \|x\| + \sqrt{MT} \leq \sup_{x \in \text{supp}(m_0)} \|x\| + \alpha T^2 R + \beta T R \leq R.$$

Hence for every $\eta \in \mathcal{P}_G(V)$:

$$\sup_{t \in [0, T], x \in \text{supp}(e_t \# \eta)} \|x\| \leq R. \quad (15)$$

The Euler Lagrange equation (13) in this example read as

$$\frac{d}{dt} \dot{\gamma}(t) = \alpha \mathbb{E}_{e_t \# \eta} z, \quad \text{for almost every } t \in [0, T], \quad (16)$$

and $\dot{\gamma}(T) = \beta \mathbb{E}_{e_T \# \eta} z$, $\gamma(0) \in \text{supp}(m_0)$. That yields

$$\sup_{t \in [0, T]} \|\dot{\gamma}(t)\| \leq \alpha T R + \beta R, \quad \sup_{t \in [0, T]} \|\gamma(t)\| \leq R, \quad (17)$$

for all $\eta \in \mathcal{P}_G(V)$ since (15) holds. That means for every $\eta \in \mathcal{P}_G(V)$ the optimal trajectories γ satisfies (17) and hence $\|\dot{\gamma}\|_\infty \leq \sqrt{M/T}$.

Let us set $W = H^1([0, T], \mathbb{R}^d)$ endowed with inner product:

$$\forall \gamma_1, \gamma_2 \in W : \langle \gamma_1, \gamma_2 \rangle_W = \langle \gamma_1(0), \gamma_2(0) \rangle_{\mathbb{R}^d} + \int_0^T \langle \dot{\gamma}_1(t), \dot{\gamma}_2(t) \rangle_{\mathbb{R}^d} dt.$$

We clearly have

$$\bigcup_{i \in I} A_i \subseteq W \subseteq V,$$

and A_i are uniformly bounded in W for m_0 -almost every $i \in I$. For integrable functions $F, D \in L^2([0, T], \mathbb{R})$ and $G \in \mathbb{R}$ we define $y = [[F, D, G]] \in W^*$ by:

$$\langle y, \gamma \rangle = \int_0^T (F(t) \cdot \gamma(t) + D(t) \cdot \dot{\gamma}(t)) dt + G \cdot \gamma(T), \quad \text{for every } \gamma \in W$$

After a few computation we have:

$$\langle y, \gamma \rangle = \int_0^T \left(\int_t^T F(s) ds + D(t) + G \right) \cdot \dot{\gamma}(t) dt + \left(\int_0^T F(s) ds + G \right) \cdot \gamma_0.$$

We can find $\gamma_y \in W$ as a representation of $y \in W^*$ i.e. for all $\gamma \in H^1([0, T], \mathbb{R}^d)$ we have $\langle y, \gamma \rangle = \langle \gamma_y, \gamma \rangle_W$. The representation γ_y corresponding to y should solve

$$\gamma_y(0) = \int_0^T F(s) ds + G, \quad \frac{d}{dt}(\gamma_y)(t) = \int_t^T F(s) ds + D(t) + G. \quad (18)$$

or

$$\gamma_y(t) = \int_0^T F(s) \min(t, s) ds + \int_0^t D(s) ds + (t+1)G + \int_0^T F(s) ds. \quad (19)$$

By assumptions 6.1(2,3) and using dominated Lebesgue convergence theorem, we can conclude that the function $J(\cdot, \eta) : W \rightarrow \mathbb{R}$ is differentiable for every $\eta \in \mathcal{P}(V)$. So the sub-differential set is singleton $\partial J(\cdot, \eta)(\gamma) = \{D_\gamma J(\gamma, \eta)\} \subseteq W^*$ and the derivative is calculated by:

$$\begin{aligned} \forall z \in W : \quad \langle D_\gamma J(\gamma, \eta), z \rangle &= \lim_{\epsilon \rightarrow 0} \frac{J(\gamma + \epsilon z, \eta) - J(\gamma, \eta)}{\epsilon} \\ &= \int_0^T (L_x(\gamma_t, \dot{\gamma}(t)) \cdot z_t + L_v(\gamma_t, \dot{\gamma}(t)) \cdot \dot{z}_t + f_x(\gamma(t), e_t \# \eta) \cdot z_t) dt + g_x(\gamma(T), e_T \# \eta) \cdot z_T \end{aligned}$$

or according to our representation:

$$D_\gamma J(\gamma, \eta) = [[L_x(\gamma(\cdot), \dot{\gamma}(\cdot)) + f_x(\gamma(\cdot), e_{(\cdot)} \# \eta), L_v(\gamma(\cdot), \dot{\gamma}(\cdot)), g_x(\gamma(T), e_T \# \eta)]].$$

So by the computation in (19) the gradient $\nabla_\gamma J(\gamma, \eta) \in W$ is obtained as follows:

$$\begin{aligned} \nabla_\gamma J(\gamma, \eta)(t) &= \int_0^T (L_x(\gamma_s, \dot{\gamma}_s) + f_x(\gamma_s, e_s \# \eta)) \min(t, s) ds + \int_0^t L_v(\gamma_s, \dot{\gamma}_s) ds \\ &\quad + (t+1)g_x(\gamma(T), e_T \# \eta) + \int_0^T (L_x(\gamma_s, \dot{\gamma}_s) + f_x(\gamma_s, e_s \# \eta)) ds. \end{aligned} \quad (20)$$

Theorem 6.1. *Suppose a first-order MFG satisfies the assumptions 6.1. If the cost function J is monotone and convex w.r.t. first argument, then the online mirror descent algorithm proposed in (8) for $h : W \rightarrow \mathbb{R}$, $h(\gamma) = \frac{1}{2} \|\gamma\|_{H^1}^2$ and $\beta_n = \frac{1}{n}$ ($n \in \mathbb{N}$), converges to the unique first-order mean field game equilibrium.*

Proof. The function $h : W \rightarrow \mathbb{R}$, $h(\gamma) = \frac{1}{2}\|\gamma\|_{H^1}^2$ is $\frac{1}{2}$ -strongly convex function and lower semi-continuous for the weak topology, so the mirror projection Q_{A_i} will have singleton values.

The game satisfies the assumptions (3.1). Since the assumptions 6.1 hold, there is $C' > 0$ such that:

$$\forall \gamma \in H^1, \eta \in \mathcal{P}_G(V) : \|D_\gamma J(\gamma, \eta)\|_{W^*} \leq C'(\|\dot{\gamma}\|_{L^2} + 1).$$

So all of the conditions in Theorem 5.1 are satisfied and the desired convergence result holds. \square

Remark 6.5. *Since the space $H^1([0, T], \mathbb{R}^d)$ is Hilbert, we identify it by its dual space. Hence by choice $h(\gamma) = \frac{1}{2}\|\gamma\|_{H^1}^2$ we have:*

$$Q_{A_i}(\gamma) = \pi_{A_i}(\gamma) = \frac{\min(\|\dot{\gamma}\|_{L^2}, \sqrt{M})}{\|\dot{\gamma}\|_{L^2}}(\gamma - \gamma_0) + i.$$

by the choice of A_i . Then, the OMD algorithm have such form

$$\begin{aligned} (i) \quad \Phi_{n+1}(i) &= \Phi_n(i) - \frac{1}{n}\nabla J(\Psi_n(i), \eta_n), & \text{for every } i \in I \\ (ii) \quad \Psi_{n+1}(i) &= \frac{\min(\|\dot{\Phi}_{n+1}(i)\|_{L^2}, \sqrt{M})}{\|\dot{\Phi}_{n+1}(i)\|_{L^2}}(\Phi_{n+1}(i) - \Phi_{n+1}(i)_0) + i, & \text{for every } i \in I \\ (iii) \quad \eta_{n+1} &= \Psi_{n+1}\#\lambda. \end{aligned} \quad (21)$$

or in explicit way it takes the following form: let $\hat{\gamma}_{0,x} = 0$ for every $x \in \mathbb{R}^d$ and:

$$\begin{aligned} \hat{\gamma}_{n+1,x}(t) &= \hat{\gamma}_{n,x}(t) - \frac{1}{n} \int_0^T (L_x(\gamma_{n,x}(s), \dot{\gamma}_{n,x}(s)) + f_x(\gamma_{n,x}(s), e_s\#\eta_n)) \min(t, s) \, ds \\ &\quad - \frac{1}{n} \int_0^t L_v(\gamma_{n,x}(s), \dot{\gamma}_{n,x}(s)) \, ds - \frac{t}{n} g_x(\gamma_{n,x}(T), e_T\#\eta_n), \\ \gamma_{n+1,x} &= c_{n+1}\hat{\gamma}_{n+1,x} + x, \quad c_{n+1} = \frac{\min(\|\dot{\hat{\gamma}}_{n+1,x}\|_{L^2}, \sqrt{M})}{\|\dot{\hat{\gamma}}_{n+1,x}\|_{L^2}}, \\ \eta_{n+1} &= \gamma_{n+1}\#\lambda. \end{aligned} \quad (22)$$

7 Appendix

Here we extend the disintegration Theorem 5.3.1 in [1], and demonstrate its modification that is used in the precedent proofs. Suppose I a Polish space and V a metric space. Let $A : I \rightarrow V$ be a correspondence with $A(i) = A_i$. For a Borel probability measure $\lambda \in \mathcal{P}(I)$ we say $\eta \in \mathcal{P}(V)$ *disintegrates* with respect to $(A_i)_{i \in I}$ if there are $\{\eta^i\}_{i \in I} \subset \mathcal{P}(V)$ such that for λ -almost every $i \in I$ we have $\text{supp}(\eta^i) \subseteq A_i$, and for every bounded measurable $f : V \rightarrow \mathbb{R}$:

$$\int_V f(a) \, d\eta(a) = \int_I \int_V f(a) \, d\eta^i(a) \, d\lambda(i).$$

Theorem 7.1. *Suppose $A : I \rightarrow V$ be upper semi continuous. Let $\{\eta_n\}_{n \in \mathbb{N}} \subseteq \mathcal{P}_1(V)$ with $\eta_n \rightarrow \eta$ in weak sense. If for every $n \in \mathbb{N}$, η_n disintegrates with respect to $(A_i)_{i \in I}$ then the same holds true for η .*

Proof. For every $n \in \mathbb{N}$, define $m_n \in \mathcal{P}(I \times V)$ as follows: for every bounded measurable $f : I \times V \rightarrow \mathbb{R}$ let:

$$\int_{I \times V} f(i, a) dm_n(i, a) = \int_I \int_V f(i, a) d\eta_n^i(a) d\lambda(i).$$

Obviously $\pi_I \# m_n = \lambda, \pi_V \# m_n = \eta_n$ where π_I, π_V are respectively projections of $I \times V$ on I, V . Since $\{\eta_n\}$ are tight and I is a Polish space, for every $\epsilon > 0$, there is a compact set $I_\epsilon \subseteq I, K_\epsilon \subseteq V$ such that $\lambda(I \setminus I_\epsilon), \eta_n(V \setminus K_\epsilon) < \epsilon$ for all $n \in \mathbb{N}$. In addition

$$\begin{aligned} m_n(I_\epsilon \times K_\epsilon) &\geq 1 - m_n(I \times V \setminus K_\epsilon) - m_n(I \setminus I_\epsilon \times V) \\ &= 1 - \eta_n(V \setminus K_\epsilon) - \lambda(I \setminus I_\epsilon) \geq 1 - 2\epsilon, \end{aligned}$$

which means the set $\{m_n\}_{n \in \mathbb{N}}$ is tight too. Hence there exists $m \in \mathcal{P}(I \times V)$ and a subsequence $\{m_{n_k}\}_{k \in \mathbb{N}}$ such that $m_{n_k} \rightarrow m$. We directly have $\eta_{n_k} = \pi_V \# m_{n_k} \rightarrow \pi_V \# m$ which means $\pi_V \# m = \eta$. On the other hand, due to the disintegration theorem (see [1] Theorem 5.3.1) there are $m^i \in \mathcal{P}(V)$ for every $i \in I$, such that for every bounded measurable $f : I \times V \rightarrow \mathbb{R}$:

$$\int_{I \times V} f(i, a) dm(i, a) = \int_I \int_V f(i, a) dm^i(a) d\lambda(i).$$

So since the second marginal of m is η , we can write: for every bounded measurable $f : I \times V \rightarrow \mathbb{R}$:

$$\int_V f(a) d\eta(a) = \int_I \int_V f(a) dm^i(a) d\lambda(i).$$

So what is left is to show that for λ -almost every $i \in I$ we have $\text{supp}(m^i) \subseteq A_i$. Set $f : I \times V \rightarrow \mathbb{R}$ as $f(i, a) = 1_{a \in A_i}$. We know the function f is upper semi continuous since the correspondence $A : I \rightarrow V, A(i) = A_i$ is upper semi continuous. For every $n \in \mathbb{N}$ we have:

$$\int_{I \times V} f(i, a) dm_n(i, a) = \int_I \int_V f(i, a) d\eta_n^i(a) d\lambda(i) = 1.$$

Hence

$$1 = \limsup_k \int_{I \times V} f(i, a) dm_{n_k}(i, a) \leq \int_{I \times V} f(i, a) dm(i, a) \leq 1,$$

so $\int_{I \times V} f(i, a) dm(i, a) = 1$ which is equivalent to say for λ -almost every $i \in I$ we have $\text{supp}(m^i) \subseteq A_i$. \square

Corollary 7.1. *Every element $\eta \in \overline{\text{cov}(\mathcal{P}_G(V))}$ disintegrates with respect to $(A_i)_{i \in I}, \lambda \in \mathcal{P}(I)$.*

Proof. Let $\mathcal{S} \subset \mathcal{P}(V)$ be the set of all measures which disintegrates with respect to $(A_i)_{i \in I}$. Clearly \mathcal{S} is convex and due to Theorem 8.1 it is closed. Also, we have $\mathcal{P}_G(V) \subseteq \mathcal{S}$ since for all $\Psi \in \mathcal{A}$ we have for every bounded measurable $f : I \times V \rightarrow \mathbb{R}$:

$$\int_V f(a) d(\Psi \# \lambda)(a) = \int_I \int_V f(a) d\delta_{\Psi(i)}(a) d\lambda(i),$$

hence it gives $\overline{\text{cov}(\mathcal{P}_G(V))} \subseteq \mathcal{S}$. \square

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