DIRECTED POLYMERS IN HEAVY-TAIL RANDOM ENVIRONMENT AND ENTROPY-CONTROLLED LAST PASSAGE PERCOLATION

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ABSTRACT. We study the directed polymer model in dimension $1+1$ when the environment is heavy-tailed, with a decay exponent $\alpha \in (0,2)$. We give all possible scaling limits of the model in the *weak-coupling* regime, *i.e.* when the inverse temperature temperature $\beta = \beta_n$ vanishes as the size of the system n goes to infinity. When $\alpha \in (1/2, 2)$, we show that all possible transversal fluctuations $\sqrt{n} \leqslant h_n \leqslant n$ of the polymer can be achieved by tuning properly β_n, allowing to interpolate between all super-diffusive scales. Moreover, we determine the scaling limit of the model, answering a conjecture by Dey and Zygouras [18] – we actually identify five different regimes. On the other hand, when $\alpha < 1/2$, we show that there are only two regimes: the transversal fluctuations are either $\sqrt{n}$ or n. This extends the results of Auffinger and Louidor [5], and Dey and Zygouras [18], which considered only the cases where $h_n = n$, resp. $h_n = \sqrt{n}$. As a key ingredient, we introduce the *Entropy-controlled Last Passage Percolation* (E-LPP), which is a natural generalization of Hammersley's Last Passage Percolation where points can be collected by paths with the constraint to have an entropy bounded by a fixed constant – instead of a 1-Lipschitz constraint. We prove several estimates on the E-LPP in continuous and in discrete settings, which are of interest on their own.

Keywords: Directed polymer, Heavy-tail distributions, Intermediate disorder, Last Passage Percolation, Super-diffusivity.

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CONTENTS

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1. Introduction: Directed Polymers in Random Environment

1.1. **General setting.** We consider the directed polymer model: it has been introduced by Huse and Henley [24] as an effective model for an interface in the Ising model with random interactions, and is now used to describe a stretched polymer interacting with an inhomogeneous solvent. This model has gained the attention of the mathematical and physics community over the past decades – and is still widely studied –, and we refer to[14, 15, 17] for a general overview.

Let S be a nearest-neighbor simple symmetric random walk on $\mathbb{Z}^d$, $d \geqslant 1$, whose law is denoted by $\mathbf{P}$, and let $(\omega_{i,x})_{i\in\mathbb{N},\, x\in\mathbb{Z}^d}$ be a field of i.i.d. random variables – the *environment* – with law $\mathbb{P}$ (ω will denote a random variable which has the common distribution of the $\omega_{i,x}$). The *directed* random walk $(i, S_i)_{i\in\mathbb{N}_0}$ represents a polymer trajectory and interacts with its environment via a coupling parameter $\beta > 0$ (the inverse temperature). The model is defined through a Gibbs measure obtained by the following Radon-Nikodym derivative

$$\frac{\mathrm{d}\mathbf{P}^{\omega}_{n,\beta}}{\mathrm{d}\mathbf{P}}(s) \;:=\; \frac{1}{\mathbf{Z}^{\omega}_{n\,\beta}} \exp\left(\beta \sum_{i=1}^{n} \omega_{i,s_i}\right), \tag{1.1}$$

where $\mathbf{Z}^{\omega}_{n\,\beta}$ is the *partition function* of the model.

One of the main question about this model is that of the localization and super-diffusivity of paths trajectories drawn from the measure $\mathbf{P}^{\omega}_{n,\beta}$. The transversal exponent ξ describes the fluctuation of the end-point, that is $\mathbb{E}\mathbf{E}^{\omega}_{n,\beta}|S_n| \approx n^{\xi}$ as $n \to \infty$. Another quantity of interest is the fluctuation exponent χ, that describes the fluctuations of $\log \mathbf{Z}^{\omega}_{n,\beta}$, *i.e.* $|\log \mathbf{Z}^{\omega}_{n,\beta} - \mathbb{E}\log \mathbf{Z}^{\omega}_{n,\beta}| \approx n^{\chi}$ as $n \to \infty$.

This model has been widely studied in the physical and mathematical literature, in particular when $\omega_{n,x}$ have an exponential moment. In dimension $d \geqslant 3$ and if β small enough, trajectories have been shown to be diffusive ($\xi = 1/2$, $\chi = 0$) with a Brownian scaling (see [11, 16, 25]). This case is refereed to as the *weak disorder regime*. In dimensions $d = 1, 2$ for any $\beta > 0$ or $d \geqslant 3$ with β large enough, there is some evidence [13, 15] that there are some favorable narrow corridors where the trajectories tend to localize, and it is believed [20] that paths are *super-difffusive* (at least in small dimensions), *i.e.* $\xi > 1/2$. This is referred to as the *strong disorder regime*. The case of the dimension $d = 1$ as attracted much attention in recent years, in particular because the model is in the *KPZ universality class* ($\log \mathbf{Z}^{\omega}_{n,\beta}$ is seen as a discretization of the Hopf-Cole solution of the KPZ equation). It is conjectured that the transversal and fluctuation exponents are $\xi = 2/3$ and $\chi = 1/3$. Moreover, it is expected that the point-to-point partition function, when properly centered and renormalized, converges in distribution to the GUE distribution. Such scalings has been proved so far only for some special models, cf. [7, 32].

A recent and fruitful approach to proving universality results for this model has been to consider is *weak-coupling limit*, that is when the coupling parameter β is near to criticality. This means that we allow $\beta = \beta_n$ to depend on n, with $\beta_n \to 0$ as $n \to \infty$. In [1, 2] and [12], the authors let $\beta_n = \widehat{\beta} n^{-\gamma}$, $\gamma = 1/4$ for some fixed $\widehat{\beta} > 0$, and they prove that the model (one may focus on its partition function Z^{ω}_{n,β_n}) converges to a *non-trivial* (*i.e. disordered*) continuous version of the model. This is called the *intermediate disorder regime*, since it somehow interpolates between weak disorder and strong disorder behavior. More precisely, they showed that

$$\log \mathbf{Z}^{\omega}_{n,\beta_n} - n\lambda(\beta_n) \xrightarrow{(\mathrm{d})} \log \mathcal{Z}_{\sqrt{2}\widehat{\beta}}, \quad \text{as} \quad n \to \infty,$$

where $\lambda(s) := \log \mathbb{E}[e^{s\omega}]$. The process $\widehat{\beta} \mapsto \log \mathcal{Z}_{\sqrt{2}\widehat{\beta}}$ is the so called *cross-over process*, which interpolates between a Gaussian and GUE scaling as $\widehat{\beta}$ goes from 0 to ∞ (see [3]). Moreover $\mathcal{Z}_{\sqrt{2}\widehat{\beta}}$ is related to the solution of the Stochastic Heat Equation.

These results were obtained under the assumption that the environment has exponential moments, but the *universality of the limit* was conjectured to hold under the assumption of six moments, see [2]. In [18] Dey and Zygouras proved this conjecture, and suggest that this result is a part of a bigger picture (notice that when $\lambda(s)$ is not defined a different centering constant would be necessary).

1.2. **The case of a heavy-tail environment.** In the rest of the paper we will focus on the dimension $d = 1$ for simplicity — some comments about higher dimension are made in Section 2.5. We consider the case where the environment distribution ω is non-negative (for simplicity, nothing deep is hidden in that assumption) and has some heavy tail distribution: there is some $\alpha > 0$ and some slowly varying function $L(\cdot)$ such that

$$\mathbb{P}(\omega > x) = L(x)x^{-\alpha}. \tag{1.2}$$

In the case where $\beta > 0$ does not depend on n, the $\xi = 2/3, \chi = 1/3$ picture is expected to be modified, depending on the value of α. According to the heuristics – and terminology – of [10, 21], three regimes should occur, with different paths behaviors:

(a) if $\alpha > 5$, there should be a *collective* optimization and we should have $\xi = 2/3$, KPZ universality class, as in the finite exponential moment case;
(b) if $\alpha \in (2, 5)$, the optmization strategy should be *elitist*: most of the total energy collected should be via a small fraction of the points visited by the path, and we should have $\xi = \frac{\alpha+1}{2\alpha-1}$;
(c) if $\alpha \in (0, 2)$, the strategy is *individual*: the polymer targets few exceptional points, and we have $\xi = 1$. This case is treated in [5, 22].

As suggested by [18], this is part of a larger picture, when the inverse temperature β is allowed to depend on n. Setting $\beta_n = \widehat{\beta} n^{-\gamma}$ for some $\widehat{\beta} \geqslant 0$ and some $\gamma \in \mathbb{R}$ then we have three different classes of coupling. When $\gamma = 0$ we recover the standard directed polymer model, when $\gamma > 0$ we have a weak-coupling limit, while in the case $\gamma < 0$ we consider a strong-coupling limit. Let us stress that this last case has not been studied in the literature – for no apparent reason – and should also be of interest. In [34] and in [18], the authors suggest that the fluctuation exponent depends on α, γ in the following manner

$$\xi = \begin{cases} \frac{2(1-\gamma)}{3} & \text{for } \alpha \geqslant \frac{5-2\gamma}{1-\gamma},\ -\frac{1}{2} \leqslant \gamma \leqslant \frac{1}{4}, \\ \frac{1+\alpha(1-\gamma)}{2\alpha-1} & \text{for } \alpha \leqslant \frac{5-2\gamma}{1-\gamma},\ \frac{2}{\alpha} - 1 \leqslant \gamma \leqslant \frac{3}{2\alpha}. \end{cases} \tag{1.3}$$

The first part is derived in [34], based on Airy process considerations, and the second part in [18] based on a Flory argument inspired by [10]. Moreover, in the two regions of the (α, γ) plane defined by (1.3), the KPZ scaling relation $\chi = 2\xi - 1$ should hold (this has been proved in the case $\gamma = 0, \alpha > 2$ in [4]). Outside of these regions, one should have $\xi = 1/2$ (γ large) or $\xi = 1$ (γ small). This is summarized in Figure 1 below, which is the analogous of [18, Fig. 1].

This picture is far from being settled, and so far only the border cases where $\xi = 1$ or $\xi = 1/2$ have been studied: Dey and Zygouras [18] proved that $\xi = 1/2$ in the cases $\alpha > 6, \gamma = 1/4$ and $\alpha \in (1/2, 6), \gamma = 3/2\alpha$; Auffinger and Louidor [5] proved that $\xi = 1$ for $\gamma = \frac{2}{\alpha} - 1$ and $\alpha \in (0, 2)$ (their result should be easily adapted to the case $\alpha \in (0, 4)$ once we accept to have a negative γ). In both these papers, the exponent χ and the scaling limits

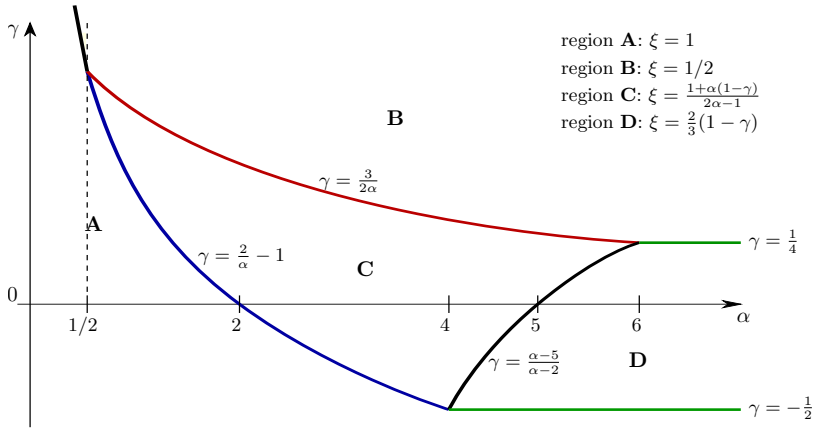

FIGURE 1. We identify four regions in the (α, γ) plane. Region **A** with $\alpha < 2$ is treated in [5] and Region **B** with $\alpha > 1/2$ in [18]. Regions **C** and **D** are still open, and the KPZ scaling relation $\chi = 2\xi - 1$ should hold in these two regions. Our main result is to establish the picture in the case $\alpha \in (0, 2)$.

are also identified. Here, we complete the picture in the case $\alpha \in (0, 2)$, going beyond the cases $\xi = 1/2$ or $\xi = 1$: we identify the correct order for the transversal fluctuations (that interpolate between $\xi = 1/2$ and $\xi = 1$), and we prove the convergence of $\log \mathbf{Z}^{\omega}_{n,\beta_n}$ in all possible intermediate disorder regimes. Let us point out that for $\alpha \in (1/2, 2)$ our result proves Conjecture 1.7 in [18].

2. MAIN RESULTS: WEAK-COUPLING LIMITS IN THE CASE $\alpha \in (0, 2)$

In this section we give a complete description of our results: from now on, we consider the case of an environment ω verifying (1.2) with $\alpha \in (0, 2)$. For the inverse temperature, we will consider arbitrary sequences $(\beta_n)_{n\geqslant 1}$, but a reference example is $\beta_n = n^{-\gamma}$ for some $\gamma \in \mathbb{R}$.

For two sequences $(a_n)_{n\geqslant 1}, (b_n)_{n\geqslant 1}$, we use the notations $a_n \sim b_n$ if $\lim_{n\to\infty} a_n/b_n = 1$, $a_n \ll b_n$ if $\lim_{n\to\infty} a_n/b_n = 0$, and $a_n \asymp b_n$ if $0 < \liminf a_n/b_n \leqslant \limsup a_n/b_n < \infty$.

2.1. First definitions and heuristics.

First of all, let us present a brief energy/entropy argument to justify what the correct transversal fluctuations of the polymer should be. Let $F(x) = \mathbb{P}(\omega \leqslant x)$ be the disorder distribution, and define the function $m(x)$ by

$$m(x) := F^{-1}\big(1 - \tfrac{1}{x}\big), \qquad \text{so we have } \ \mathbb{P}\big(\omega > m(x)\big) = \frac{1}{x}. \tag{2.1}$$

Note that the second identity characterizes $m(x)$ up to asymptotic equivalence: we have that $m(\cdot)$ is a regularly varying function with exponent $1/\alpha$.

Assuming that the transversal fluctuations are of order h_n (we necessarily have $\sqrt{n} \leqslant h_n \leqslant n$), then the amount of weight collected by a path should be of order $m(nh_n)$ (it should be dominated by the maximal value of ω in $[0, n] \times [-h_n, h_n]$). On the other hand, thanks to moderate deviations estimates for the simple random walk, the entropic cost of having fluctuations of order h_n is roughly h_n^2/n at the exponential level – at least when $h_n \gg \sqrt{n \log n}$, see (2.15) below. It therefore leads us to define h_n (seen as a function of β_n) up to asymptotic equivalence by the relation

$$\beta_n m(nh_n) \sim h_n^2/n\,. \tag{2.2}$$

In the case $\beta_n = n^{-\gamma}$ and $\alpha \in (1/2, 2)$ we recover (1.3), that is we get that $h_n = n^{\xi+o(1)}$ with $\xi = \frac{1+\alpha(1-\gamma)}{2\alpha-1}$, which is in $(1/2, 1)$ for $\gamma \in (\frac{2}{\alpha} - 1, \frac{3}{2\alpha})$. When $\alpha \in (0, 1/2)$, there is no h_n verifying (2.2) with $\sqrt{n} \ll h_n \ll n$, leading to believe that intermediate transversal fluctuations (*i.e.* $\xi \in (1/2, 1)$) cannot occur. In the following, we will therefore separate the cases $\alpha \in (1/2, 2)$ and $\alpha \in (0, 1/2)$.

2.2. **A natural candidate for the scaling limit.** Once we have identified in (2.2) the scale h_n for the transversal fluctuations, we are able to rescale both path trajectories and the field $(\omega_{i,x})$, so that we can define the rescaled "entropy" and "energy" of a path, and the corresponding continous quantities. The rescaled paths will be in the following set

$$\mathscr{D} := \big\{ s : [0, 1] \to \mathbb{R} \; ; \; s \text{ continuous and a.e. differentiable} \big\}, \tag{2.3}$$

and the (continuum) entropy of a path $s \in \mathscr{D}$ will derive from the rate function of the moderate deviation of the simple random walk (see [33] or (2.15) below), *i.e.*

$$\mathrm{Ent}(s) = \frac{1}{2} \int_0^1 \big(s'(t)\big)^2 dt \qquad \text{for } s \in \mathscr{D}. \tag{2.4}$$

As far as the continuum disorder field is concerned, we let $\mathcal{P} := \{(w_i, t_i, x_i) \colon i \geqslant 1\}$ be a Poisson Point Process on $[0, \infty) \times [0, 1] \times \mathbb{R}$ of intensity $\mu(\mathrm{d}w\mathrm{d}t\mathrm{d}x) = \frac{\alpha}{2} w^{-\alpha-1} 1_{\{w>0\}} dwdtdx$. For a quenched realization of $\mathcal{P}$, the energy of a continuous path $s \in \mathscr{D}$ is then defined by

$$\pi(s) = \pi_{\mathcal{P}}(s) := \sum_{(w,t,x)\in\mathcal{P}} w \, 1_{\{(t,x)\in s\}}, \tag{2.5}$$

where the notation $(t, x) \in s$ means that the point (t, x) is visited by the path s, that is $s_t = x$.

Then, a natural guess for the continuous scaling limit of the partition function is to consider an energy–entropy competition variational problem. For any $\beta \geqslant 0$ we let

$$\mathcal{T}_\beta := \sup_{s\in\mathscr{D}, \mathrm{Ent}(s)<+\infty} \Big\{ \beta\pi(s) - \mathrm{Ent}(s) \Big\}. \tag{2.6}$$

This variational problem was originally introduced by Dey and Zygouras [18], conjecturing that it was well defined as long as $\alpha \in (1/2, 2)$ and that it was the good candidate for the scaling limit (see [18, Conjecture 1.7]). Our first result shows that the variational problem (2.6) is indeed well defined when $\alpha \in (1/2, 2)$. In Theorem 2.4 below we prove the second part of Conjecture 1.7.

Theorem 2.1. *For $\alpha \in (1/2, 2)$ we have the scaling relation*

$$\mathcal{T}_\beta \overset{(d)}{=} \beta^{\frac{2\alpha}{2\alpha-1}} \mathcal{T}_1, \tag{2.7}$$

and $\mathcal{T}_\beta \in (0, +\infty)$ for all $\beta > 0$ a.s. Moreover, $\mathbb{E}\big[(\mathcal{T}_\beta)^\upsilon\big] < \infty$ for any $\upsilon < \alpha - 1/2$. We also have that a.s. the map $\beta \mapsto \mathcal{T}_\beta$ is continuous, and that the supremum in (2.6) *is attained by some unique continuous path s^*_β with $\mathrm{Ent}(s^*_\beta) < \infty$.*

On the other hand, for $\alpha \in (0, 1/2]$ we have $\mathcal{T}_\beta = +\infty$ for all $\beta > 0$ a.s.

Let us mention here that in [5], the authors consider the case of transversal fluctuations of order n. The natural candidate for the limit is $\widehat{\mathcal{T}}_\beta$, defined analogously to (2.6) by $\widehat{\mathcal{T}}_\beta = 0$ for $\beta = 0$, and for $\beta > 0$

$$\widehat{\mathcal{T}}_\beta = \sup_{s\in\mathrm{Lip}_1} \Big\{ \pi(s) - \frac{1}{\beta} \widehat{\mathrm{Ent}}(s) \Big\}. \tag{2.8}$$

Here the supremum is taken over the set Lip_1 of 1-Lipschitz functions, and the entropy $\widehat{\mathrm{Ent}}(s)$ derives from the rate function of the large deviations for the simple random walk, *i.e.*

$$\widehat{\mathrm{Ent}}(s) = \int_0^1 e\big(s'(t)\big)dt \quad \text{with } e(x) = \tfrac{1}{2}(1+x)\log(1+x) + \tfrac{1}{2}(1-x)\log(1-x)\,.$$

2.3. **Main results I : the case $\alpha \in (1/2, 2)$.** Our first result deals with the transversal fluctuations of the polymer: we prove that h_n defined in (2.2) indeed gives the correct order for the transversal fluctuations.

Theorem 2.2. *Assume that $\alpha \in (1/2, 2)$, that $\beta_n m(n^2) \to 0$ and $\beta_n m(n^{3/2}) \to +\infty$, and define h_n as in* (2.2)*: then $\sqrt{n} \ll h_n \ll n$. Then, there are constants c_1, c_2 and $\nu > 0$ such that for any sequences $A_n \geqslant 1$ we have for all $n \geqslant 1$*

$$\mathbb{P}\left(\mathbf{P}^{\omega}_{n,\beta_n}\big(\max_{i\leqslant n}|S_i| \geqslant A_n\, h_n\big) \geqslant n\, e^{-c_1 A_n^2 h_n^2/n}\right) \leqslant c_2\, A_n^{-\nu}\,. \tag{2.9}$$

In particular, this proves that if h_n defined in (2.2) is larger than a constant times $\sqrt{n\log n}$, then $ne^{-c_1 A h_n^2/n}$ goes to 0 as $n \to \infty$ provided that A is large enough: the transversal fluctuations are at most Ah_n, with high $\mathbb{P}$-probability. On the other hand, if h_n is much smaller than $\sqrt{n\log n}$, then this theorem does not give sharp information: we still find that the transversal fluctuations must be smaller than $A\sqrt{n\log n}$, with high $\mathbb{P}$-probability. Anyway, in the course of the demonstration of our results, it will be clear that the main contribution to the partition function comes from trajectories with transversal fluctuations of order exactly h_n.

We stress that the cases $\beta_n m(n^2) \to \beta \in (0, +\infty]$ and $\beta_n m(n^{3/2}) \to \beta \in [0, \infty)$ have already been considered by Auffinger and Louidor [5] and Dey and Zygouras [18] respectively: they find that the transversal fluctuations are of order n, resp. $\sqrt{n}$. We state their results below, see Theorem 2.3 and Theorem 2.8 respectively.

Our first series of results consist in identifying three new regimes for the transversal fluctuations ($\sqrt{n\log n} \ll h_n \ll n$, $h_n \asymp \sqrt{n\log n}$, and $\sqrt{n} \ll h_n \ll \sqrt{n\log n}$), that interpolate between the Auffinger Louidor regime ($h_n \asymp n$) and the Dey Zygouras regime ($h_n \asymp \sqrt{n}$). We now describe more precisely these five different regimes.

Regime 1: transversal fluctuations of order n. Consider the case where

$$\beta_n n^{-1} m(n^2) \to \beta \in (0, \infty]\,, \tag{R1}$$

which corresponds to having transversal fluctuations of order n. In that case, Auffinger and Louidor showed that, properly rescaled, $\log \mathbf{Z}^{\omega}_{n,\beta_n}$ converges to $\widehat{\mathcal{T}}_\beta$ defined in (2.8).

Theorem 2.3 (Regime 1,[5])**.** *Assume $\alpha \in (0, 2)$, and consider β_n such that* (R1) *holds. Then we have the following convergence*

$$\frac{1}{\beta_n m(n^2)} \log \mathbf{Z}^{\omega}_{n,\beta_n} \xrightarrow{\text{(d)}} \widehat{\mathcal{T}}_\beta\,, \qquad \text{as } n \to +\infty\,.$$

where $\widehat{\mathcal{T}}_\beta$ is defined in (2.8)*. For $\alpha \in [1/2, 2)$, we have that $\widehat{\mathcal{T}}_\beta > 0$ a.s. for all $\beta > 0$.*

Regime 2: $\sqrt{n\log n} \ll h_n \ll n$. Consider the case when

$$\beta_n n^{-1} m(n^2) \to 0 \quad \text{and} \quad \beta_n \log n^{-1} m(n^{3/2}\sqrt{\log n}) \to \infty\,, \tag{R2}$$

which corresponds to having transversal fluctuations $\sqrt{n\log n} \ll h_n \ll n$, see (2.2). We find that, properly rescaled, $\log \mathbf{Z}^{\omega}_{n,\beta_n}$ converges to $\mathcal{T}_1$ defined in (2.6) – this proves Conjecture 1.7 in [18].

Theorem 2.4 (Regime 2). *Assume that $\alpha \in (1/2, 2)$, and consider β_n such that* (R2) *holds. Defining h_n as in* (2.2)*, then $\sqrt{n \log n} \ll h_n \ll n$, and we have*

$$\frac{1}{\beta_n m(nh_n)} \Big(\log \mathbf{Z}^{\omega}_{n,\beta_n} - n\beta_n \mathbb{E}[\omega] 1_{\{\alpha \geqslant 3/2\}} \Big) \xrightarrow{(\mathrm{d})} \mathcal{T}_1 \,, \qquad \text{as } n \to +\infty \,. \tag{2.10}$$

where $\mathcal{T}_1$ is defined in (2.6).

We stress here that we need to recenter $\log \mathbf{Z}^{\omega}_{n,\beta_n}$ by $n\beta_n \mathbb{E}[\omega]$ only when necessary, that is when $n/m(nh_n)$ does not go to 0: in terms of the picture described in Figure 1, this can happen only when $\gamma \geqslant 4 - 2\alpha$, and in particular when $\alpha \geqslant 3/2$ (we stressed that fact in the statement of the theorem).

Regime 3: $h_n \asymp \sqrt{n \log n}$. Consider the case

$$\beta_n \log n^{-1} m(n^{3/2}\sqrt{\log n}) \to \beta \in (0, \infty) \,, \tag{R3}$$

which from (2.2) corresponds to transversal fluctuations $h_n \sim \beta^{1/2}\sqrt{n \log n}$, see (2.2). We find the correct scaling of $\log \mathbf{Z}^{\omega}_{n,\beta_n}$, which can be of two different natures (and go to $+\infty$ or 0), see Theorems 2.5-2.6 below.

We first need to introduce a few more notations. For a quenched continuum energy field $\mathcal{P}$ (as defined in Section 2.2), we define for a path s the number of weights w it collects:

$$N(s) := \sum_{(w,t,x) \in \mathcal{P}} 1_{\{(x,t) \in s\}} \,. \tag{2.11}$$

Then, we define a new energy-entropy variational problem: for a fixed realization of $\mathcal{P}$, define for any $k \geqslant 1$

$$\widetilde{\mathcal{T}}^{(k)}_{\beta} = \widetilde{\mathcal{T}}^{(k)}_{\beta}(\mathcal{P}) := \sup_{s \in \mathscr{D}, N(s) = k} \Big\{ \pi(s) - \mathrm{Ent}(s) - \frac{k}{2\beta} \Big\}, \quad \text{and} \quad \widetilde{\mathcal{T}}^{(\geqslant r)}_{\beta} := \sup_{k \geqslant r} \widetilde{\mathcal{T}}^{(k)}_{\beta}. \tag{2.12}$$

When $r = 0$ we denote by $\widetilde{\mathcal{T}}_{\beta}$ the quantity $\widetilde{\mathcal{T}}^{(\geqslant 0)}_{\beta}$. In Proposition 7.6, we prove that these quantities are well defined, and it is also noted that there exists $\beta_c = \beta_c(\mathcal{P}) \in (0, \infty)$ such that $\widetilde{\mathcal{T}}_{\beta} \in (0, \infty)$ if $\beta > \beta_c$ and $\widetilde{\mathcal{T}}_{\beta} = 0$ if $\beta < \beta_c$.

Theorem 2.5 (Regime 3-a). *Assume that $\alpha \in (1/2, 2)$, and consider β_n such that* (R3) *holds. Then from* (2.2) *we have $h_n \asymp \sqrt{n \log n}$, and*

$$\frac{1}{\beta_n m(nh_n)} \Big(\log \mathbf{Z}^{\omega}_{n,\beta_n} - n\beta_n \mathbb{E}[\omega] 1_{\{\alpha \geqslant 3/2\}} \Big) \xrightarrow{(\mathrm{d})} \widetilde{\mathcal{T}}_{\beta} \qquad \text{as } n \to +\infty \,. \tag{2.13}$$

(Recall that $\beta_n m(nh_n) \sim h_n^2/n \sim \beta \log n$)

If $\widetilde{\mathcal{T}}_{\beta} > 0$ ($\beta > \beta_c$) the scaling limit is therefore well identified, and $\log \mathbf{Z}^{\omega}_{n,\beta_n}$ (when recentered) grows like $\beta\widetilde{\mathcal{T}}_{\beta} \log n$ with $\beta\widetilde{\mathcal{T}}_{\beta} > 0$. On the other hand, if $\widetilde{\mathcal{T}}_{\beta} = 0$, then the above theorem gives only a trivial limit. By an extended version of Skorokhod representation theorem [27, Corollary 5.12], one can couple the discrete environment and the continuum field $\mathcal{P}$ in order to obtain an almost sure convergence in Theorem 2.5 above. Hence, it makes sense to work conditionally on $\widetilde{\mathcal{T}}^{\geqslant 1}_{\beta} < 0$ ($\beta < \beta_c$), even at the discrete level. Our next theorem says that for $\beta < \beta_c$, $\log \mathbf{Z}_{n,\beta_n}$ decays polynomially, with a random exponent $\beta\widetilde{\mathcal{T}}^{(\geqslant 1)}_{\beta} \in (-1/2, 0)$.

Theorem 2.6 (Regime 3-b, $\widetilde{\mathcal{T}}_\beta = 0$, $\beta < \beta_c$)**.** *Assume that $\alpha \in (1/2, 2)$ and that* (R3) *holds. Then, conditionally on $\{\widetilde{\mathcal{T}}_\beta^{(\geqslant 1)} < 0\}$ (i.e. $\beta < \beta_c$),*

$$\frac{1}{\beta_n m(nh_n)} \log\Big(\log \mathbf{Z}^\omega_{n,\beta_n} - n\beta_n \mathbb{E}[\omega 1_{\{\omega \leqslant 1/\beta_n\}}] 1_{\{\alpha \geqslant 1\}} \Big) \xrightarrow{(\mathrm{d})} \widetilde{\mathcal{T}}_\beta^{(\geqslant 1)}\,, \quad \text{as } n \to +\infty\,.$$

Recalling that $\beta_n m(nh_n) \sim h_n^2/n \sim \beta \log n$, we note that $\exp(\beta \widetilde{\mathcal{T}}_\beta^{(\geqslant 1)} \log n)$ goes to 0 as a (random) power $\beta \widetilde{\mathcal{T}}_\beta^{(\geqslant 1)}$ of n, with $\beta \widetilde{\mathcal{T}}_\beta^{(\geqslant 1)} \in (-1/2, 0)$.

Regime 4: $\sqrt{n} \ll h_n \ll \sqrt{n \log n}$*.* Consider the case

$$\beta_n m(n^{3/2}) \to \infty \quad \text{and} \quad \beta_n \log n^{-1} m(n^{3/2}\sqrt{\log n}) \to 0\,; \tag{R4}$$

which corresponds to having transversal fluctuations $\sqrt{n} \ll h_n \ll \sqrt{n \log n}$, see (2.2). Let us define

$$W_\beta := \widetilde{\mathcal{T}}_\beta^{(1)} + \frac{1}{2\beta} := \sup_{(w,x,t)\in\mathcal{P}} \Big\{ w - \frac{x^2}{2\beta t} \Big\}\,, \tag{2.14}$$

which is a.s. positive and finite if $\alpha \in (1/2, 2)$, see Proposition 8.4.

Theorem 2.7 (Regime 4)**.** *Assume that $\alpha \in (1/2, 2)$, and consider β_n such that* (R4) *holds. Defining h_n as in* (2.2)*, then $\sqrt{n} \ll h_n \ll \sqrt{n \log n}$, and we have*

$$\frac{1}{\beta_n m(nh_n)} \log\left(\sqrt{n} \Big(\log \mathbf{Z}^\omega_{n,\beta_n} - n\beta_n \mathbb{E}[\omega 1_{\{\omega \leqslant 1/\beta_n\}}] 1_{\{\alpha \geqslant 1\}} \Big) \right) \xrightarrow{(\mathrm{d})} W_1\,, \qquad \text{as } n \to +\infty\,.$$

Recalling that $\beta_n m(nh_n) \sim h_n^2/n \ll \log n$, we note that $\exp\big(W_1 h_n^2/n\big)$ goes to infinity (at some random rate), but slower than any power of n.

Regime 5: transversal fluctuations of order $\sqrt{n}$. Consider the case

$$\beta_n m(n^{3/2}) \to \beta \in [0, \infty)\,; \tag{R5}$$

this corresponds to having transversal fluctuations h_n of order $\sqrt{n}$. Here, we state one of the results obtained by Dey and Zygouras, [18, Theorem 1.4].

Theorem 2.8 (Regime 5, [18])**.** *Assume that $\alpha \in (1/2, 2)$, and consider β_n such that* (R5) *holds, that is $\beta_n m(n^{3/2}) \to \beta \in [0, \infty)$. Then*

$$\frac{1}{\beta_n m(n^{3/2})} \sqrt{n} \Big(\log \mathbf{Z}_{n,\beta_n} - n\beta_n \mathbb{E}\big[\omega 1_{\{\omega \leqslant m(n^{3/2})\}}\big] 1_{\alpha \geqslant 1} \Big) \xrightarrow{(\mathrm{d})} 2\mathcal{W}_\beta^{(\alpha)}\,, \qquad \text{as } n \to +\infty\,.$$

Here, $\mathcal{W}_\beta^{(\alpha)}$ is some specific α-stable random variable (defined in [18, p. 4011]*).*

Some comments about the different regimes. The intermediate regimes 2-3-4 have different behavior due to the different regimes for the *local moderate deviation*, see [33, Theorem 3]. We indeed have that for $\sqrt{n} \ll h_n \ll n$

$$\mathbf{P}(S_n = h_n) = \frac{c}{\sqrt{n}} \exp\Big(-(1 + o(1)) \frac{h_n^2}{2n} \Big)\,, \tag{2.15}$$

so that we identify three main possibilities: if $h_n \ll \sqrt{n \log n}$, then $\mathbf{P}(S_n = h_n) = n^{-1/2+o(1)}$; if $h_n \sim c\sqrt{n \log n}$ then $\mathbf{P}(S_n = h_n) = n^{-(c^2+1)/2+o(1)}$; if $h_n \gg \sqrt{n \log n}$ then $\mathbf{P}(S_n = h_n) = e^{-(1+o(1))h_n^2/n}$ which decays faster than any power of n.

This is actually reflected in the behavior of the partition function. Let us denote $\bar{\mathbf{Z}}^\omega_{n,\beta_n} = e^{-n\beta_n C_\alpha} \times \mathbf{Z}^\omega_{n,\beta_n}$ be the renormalized (when necessary) partition function. We recall that

C_α is equal either to $\mathbb{E}[\omega]1_{\{\alpha\geqslant 3/2\}}$ (Regime 2 and 3-a) or to $\mathbb{E}[\omega 1_{\{\omega\leqslant 1/\beta_n\}}]1_{\alpha\geqslant 1}$ (Regime 3-b and 4). Then we have

- In regimes 1 and 2, transversal fluctuations are $h_n \gg \sqrt{n\log n}$, and $\bar{\mathbf{Z}}_{n,\beta_n}$ grows faster than any power of n: roughly, it is of order $e^{\beta\widehat{\mathcal{T}}_\beta n}$ in Regime 1 (for $\beta<\infty$), and of order $e^{\mathcal{T}_1 h_n^2/n}$ in Regime 2.
- In Regime 3, transversal fluctuations are $h_n \asymp \sqrt{n\log n}$, and $\bar{\mathbf{Z}}_{n,\beta_n}$ goes to infinity polynomially in Regime 3-a, and it goes to 1 with a polynomial correction in Regime 3-b. This could be summarized as $\bar{\mathbf{Z}}_{n,\beta_n}\approx 1+n^{\beta\widetilde{\mathcal{T}}_\beta^{(\geqslant 1)}}$, with $\beta\widetilde{\mathcal{T}}_\beta^{(\geqslant 1)}>-1/2$: the transition between regime 3-a and 3-b occurs as $\beta\widetilde{\mathcal{T}}_\beta^{(\geqslant 1)}$ changes sign, at $\beta=\beta_c$ (note that $\beta\widetilde{\mathcal{T}}_\beta^{(\geqslant 1)}$ keeps a mark of the local limit theorem, see (2.12) and (2.15)).
- In Regimes 4, $\bar{\mathbf{Z}}_{n,\beta_n}$ goes to 1 with a correction of order $n^{-1/2}\times e^{W_1 h_n^2/n}$, with $e^{W_1 h_n^2/n}$ going to infinity slower than any power of n: this corresponds to the cost for a trajectory to visit a single site, at which the supremum in W_1 is attained. In Regime 5, $\bar{\mathbf{Z}}^\omega_{n,\beta_n}$ goes to 1 with a correction of order $n^{-1/2}$.

2.4. **Main results II : the case** $\alpha\in(0,1/2)$**.** In this case, since $n^{-1}m(n^2)/m(n^{3/2})\to\infty$, there is no sequence β_n such that $\beta_n n^{-1}m(n^2)\to 0$ and $\beta_n m(n^{3/2})\to+\infty$. First of all, we realize that Theorem 2.3 already gives a result, but a phase transition has been identified in [5, 35] when $\alpha\in(0,1/2)$.

Theorem 2.9 ([5, 35])**.** *When $\alpha\in(0,1/2)$, $\widehat{\mathcal{T}}_\beta$ defined in* (2.8) *undergoes a phase transition: there exists some $\widehat{\beta}_c=\widehat{\beta}_c(\mathcal{P})$ with $\widehat{\beta}_c\in(0,\infty)$ $\mathbb{P}$-a.s., such that $\widehat{\mathcal{T}}_\beta=0$ if $\beta\leqslant\widehat{\beta}_c$ and $\widehat{\mathcal{T}}_\beta>0$ if $\beta>\widehat{\beta}_c$.*

The fact that $\widehat{\mathcal{T}}_{\widehat{\beta}_c}=0$ was not noted in [5, 35], but simply comes from the (left) continuity of $\beta\mapsto\widehat{\mathcal{T}}_\beta$ (the proof is identical to that for $\beta\mapsto\mathcal{T}_\beta$, see Section 4.4).

In view of Theorem 2.3, the scaling limit of $\log\mathbf{Z}^\omega_{n,\beta_n}$ is identified when $\widehat{\mathcal{T}}_\beta>0$, and it is trivial when $\widehat{\mathcal{T}}_\beta=0$. Again, by an extended version of Skorokhod representation theorem [27, Corollary 5.12], we can obtain an almost sure convergence in Theorem 2.3 (an actual coupling is provided in [5]). Hence, it makes sense to work conditionally on $\widehat{\mathcal{T}}_\beta>0$ or $\widehat{\mathcal{T}}_\beta=0$, even at the discrete level. We show here that only two regimes can hold:

(1) if $\widehat{\mathcal{T}}_\beta>0$, then fluctuations are of order n, and properly rescaled, $\log\mathbf{Z}^\omega_{n,\beta_n}$ converges to $\widehat{\mathcal{T}}_\beta$ – this is Theorem 2.3;
(2) if $\widehat{\mathcal{T}}_\beta=0$, then fluctuations are of order $\sqrt{n}$, and properly rescaled, $\log\mathbf{Z}^\omega_{n,\beta_n}$ converges in distribution (conditionally on $\widehat{\mathcal{T}}_\beta=0$), as shown by the following result.

Theorem 2.10. *Assume $\alpha\in(0,1/2)$, and consider β_n so that $\beta_n n^{-1}m(n^2)\to\beta\in[0,+\infty)$. Then, on the event $\{\widehat{\mathcal{T}}_\beta=0\}$ ($\beta\leqslant\widehat{\beta}_c<\infty$), transversal fluctuations are of order $\sqrt{n}$. More precisely, for any $\varepsilon>0$, there exists some $c_0,\nu>0$ such that, for any sequence $C_n>1$ we have*

$$\mathbb{P}\Big(\mathbf{P}^\omega_{n,\beta_n}\Big(\max_{i\leqslant n}|S_i|\geqslant C_n\sqrt{n}\Big)\geqslant e^{-c_0 C_n^{2\wedge n^{1/2}}}\;\Big|\;\widehat{\mathcal{T}}_\beta=0\Big)\leqslant\varepsilon. \tag{2.16}$$

Moreover, conditionally on $\{\widehat{\mathcal{T}}_\beta=0\}$, we have that

$$\frac{\sqrt{n}}{\beta_n m(n^{3/2})}\log\mathbf{Z}^\omega_{n,\beta}\xrightarrow{(\mathrm{d})}2\mathcal{W}_0^{(\alpha)}\,,\qquad \text{as } n\to+\infty\,. \tag{2.17}$$

where $\mathcal{W}_0^{(\alpha)} := \int_{\mathbb{R}_+\times\mathbb{R}\times[0,1]} w\rho(t,x)\mathcal{P}(dw,dx,dt)$ *with* $\mathcal{P}$ *a realization of the Poisson Point Process defined in Section 2.2, and* $\rho(t,x) = (2\pi t)^{-1/2}e^{-x^2/2t}$ *is the Gaussian Heat kernel.*

Let us mention that $\mathcal{W}_0^{(\alpha)}$ is well defined and that it has an α-stable distribution, with explicit characteristic function, see Lemma 1.3 in [18]. Later on, we show that $\mathcal{W}_0^{(\alpha)}$ is finite and give some other properties, see Lemma 9.3.

We stress that Theorem 2.10 shows that when $\alpha < 1/2$, a very sharp phase transition occurs on the line $\beta_n \sim \beta n/m(n^2)$: for $\beta \leqslant \widehat{\beta}_c$, transversal fluctuations are of order $\sqrt{n}$ whereas for $\beta > \widehat{\beta}_c$ they are of order n.

2.5. **Some comments and perspectives.** We now make some comments about our results: we present some possible generalizations, and we discuss remaining open questions.

About the case $\alpha = 1/2$. In the above results, we excluded the case $\alpha = 1/2$. In that case, both $n^{-1}m(n^2)$ and $m(n^{3/2})$ are regularly varying with index 3, and there are mostly two possibilities.

(1) If $\frac{n^{-1}m(n^2)}{m(n^{3/2})} \to 0$ (which occurs for instance if $L(x) = e^{-(\log x)^b}$ for some $0 < b < 1$), then there exist sequences $(\beta_n)_{n\geqslant 1}$ such that $\beta_n n^{-1}m(n^2) \to 0$ and $\beta_n m(n^{3/2}) \to +\infty$. Then, the situation should be similar to that of Section 2.3: there should be five regimes, with transversal fluctuations h_n interpolating between $\sqrt{n}$ and n. However, we stress that for $\alpha = 1/2$ we have $\mathcal{T}_1 = +\infty$ (and also $\mathcal{W}_\beta^{(\alpha)} = +\infty$ for all $\beta > 0$), so that we do not have an analogue of Theorem 2.4 (or of Theorem 2.8). We suspect that the transversal fluctuations might not be as defined in (2.2) but could be much larger, and that a larger normalization should be needed – but this is purely conjectural.

(2) if $\frac{n^{-1}m(n^2)}{m(n^{3/2})} \to c \in (0,\infty]$ (which occurs for instance if $L(n) = (\log x)^b$ for some b), then there is no sequence $(\beta_n)_{n\geqslant 1}$ such that $\beta_n n^{-1}m(n^2) \to 0$ and $\beta_n m(n^{3/2}) \to +\infty$. Then, the situation should be similar to that of Section 2.4: there should be only two regimes, with transversal fluctuations either $\sqrt{n}$ or n. Note that the statement of Theorem 2.3 still holds, and we stress that for $\alpha = 1/2$, $\widehat{\mathcal{T}}_\beta > 0$ for all $\beta > 0$ a.s. ($\widehat{\beta}_c = 0$), see [5, Prop. 2.5]. We therefore have that

$$\text{if } \beta_n n^{-1}m(n^2) \to \beta \in (0,\infty], \qquad \frac{1}{\beta_n m(n^2)} \log \mathbf{Z}^\omega_{n,\beta_n} \to \widehat{\mathcal{T}}_\beta > 0 \ \ a.s.,$$

that is transversal fluctuations are of order n. On the other hand, in the case where $\beta_n n^{-1}m(n^2) \to 0$ (so that $\beta_n m(n^{3/2})$ also goes to 0), we can expect that the analogue of Theorem 2.10 holds (corresponding to $\beta \leqslant \widehat{\beta}_c = 0$), that is

$$\text{if } \beta_n n^{-1}m(n^2) \to 0, \qquad \frac{\sqrt{n}}{\beta_n m(n^{3/2})} \log \mathbf{Z}^\omega_{n,\beta_n} \to \mathcal{W}_0^{(\alpha)},$$

with transversal fluctuations are of order $\sqrt{n}$.

Toward the case $\alpha \in (2,5)$. When $\alpha \in (2,5)$ (more generally region **C** in Figure 1), one of the main difficulty is to find the correct centering term for $\log \mathbf{Z}^\omega_{n,\beta_n}$. Another important difficulty is that the variational problem $\mathcal{T}_\beta$ defined in (2.6) is $\mathcal{T}_\beta = +\infty$ a.s., since paths that collect many small weights bring an important contribution to $\mathcal{T}_\beta$. The main objective is therefore to prove a result of the type: there exists a function $f(\cdot)$ such that, for $\alpha \in (2,6)$

and any β_n in region **C** of Figure 1 (*i.e.* $\beta_n = n^{-\gamma}$ with $\gamma < 3/2\alpha$, $\gamma > (\alpha-5)/(\alpha-2)$, $\gamma > 2/\alpha - 1$)

$$\frac{1}{\beta_n m(nh_n)}\Big(\log \mathbf{Z}^{\omega}_{n,\beta_n} - f(\beta_n)\Big) \xrightarrow{\text{(d)}} \breve{\mathcal{T}}_1\,,$$

with h_n defined as in (2.2) (the transversal fluctuations should still be at scale h_n) and where $\breve{\mathcal{T}}_1$ is somehow a "recentered" version of the variational problem (2.6) (that is in which the contribution of the small weights has been canceled out). The difficulties are therefore serious: one needs (i) to identify the centering term $f(\beta_n)$, (ii) to make sense of the variational problem $\breve{\mathcal{T}}_1$.

Path localization. We mention that in [5], Auffinger and Louidor go beyond the convergence in distribution stated in Theorem 2.3, and show some path localization. Indeed, they prove that, under $\mathbf{P}^{\omega}_{n,\beta_n}$, path trajectories concentrate around the (unique) maximizer γ^*_{n,β_n} of the discrete analogue of the variational problem (2.8), see Theorem 2.1 in [5]; moreover this maximizer γ^*_{n,β_n} converges in distribution to the (unique) maximizer $\widehat{\gamma}^*_{\beta}$ of the variational problem (2.8). This could theoretically be applied in our setting, in that direction, we prove the existence and uniqueness of the maximizer of the continuous variational problem in Theorem 2.1. However, we omit this statement here to lighten the paper: we refer to Remark 4.5 for more details, see in particular Conjectures 4.6-4.7.

Higher dimensions, unbounded jumps and non-directed paths. Similarly to [5], our methods should work in any dimension $1+d$ (one temporal dimension, d transveral dimension). The energy-entropy balance argument would then give that the transversal fluctuations h_n are given by the relation $\beta_n m(nh_n^d) \sim h_n^2/n$ in place of (2.2): for paths with transversal scale h_n, the energy collected should be of order $\beta_n m(nh_n^d)$ while the entropy cost should remain of order h_n^2/n, at the exponential level. For $\alpha \in (0, 1+d)$, and choosing $\beta_n = n^{-\gamma}$, we should therefore find that in dimension d a similar picture to Figure 1 hold:

Case $\alpha \in (0, d/2)$

$\gamma < \frac{1+d}{\alpha} - 1$	$\gamma > \frac{1+d}{\alpha} - 1$
$\xi = 1$	$\xi = 1/2$

Case $\alpha \in (d/2, 1+d)$

$\gamma \leqslant \frac{1+d}{\alpha} - 1$	$\frac{1+d}{\alpha} - 1 < \gamma < \frac{2+d}{2\alpha}$	$\gamma \geqslant \frac{2+d}{2\alpha}$
$\xi = 1$	$\xi = \frac{1+(1-\gamma)\alpha}{2\alpha - d} \in (1/2, 1)$	$\xi = 1/2$

However, to keep things more straightforward we only work in dimension $d = 1$.

Additionally, we stress that the Entropy-controlled LPP can easily be generalized, as long as a notion of entropy can be formulated: this should enable us to deal with directed paths with unbounded jumps, or with non-directed paths. We refer here to [8], where generalizations of LPP are introduced and investigated: this opens the way to many interesting models, and in particular to the study of polymers with unbounded jumps, and more importantly to non-directed polymers in random environment.

2.6. **Organization of the rest of the paper and ideas of proof.** We now present an overview of how the rest of the paper is organized, and give some of the ideas we develop in the paper.

Part 1: Entropy-controlled Last Passage Percolation. In a first part, we introduce and develop a key tool for our analysis, the Entropy controlled LPP (E-LPP), and give some applications that are used throughout the rest of the paper.

∗ In Section 3, we define the continuous and discrete E-LPP. For a given domain $\Lambda_{t,h}$ (which is either a subset of $\mathbb{R}^2$ or of $\mathbb{Z}^2$) with fixed length t and height x, we consider m

points chosen uniformly in the domain (here the definition depends on the discrete or continuous nature of the domain). The problem is then to estimate $\mathcal{L}_m$, the maximal number of points that can be collected by paths with an entropy bounded by a fixed constant B. We define the entropy of a set Δ in (3.1), and $\mathcal{L}_m$ is thus the maximal cardinality of a set Δ with entropy smaller than B. We show that $\mathcal{L}_m$ is of order $(Bt/x^2)^{1/4}\sqrt{m}$, and we provide good estimates on the deviation probabilities in Theorem 3.1. The proof of the theorem, in Section 3.3, is not difficult: it uses a standard first moment technique on one side, and a specific strategy to collect points on the other side.

$*$ In Section 4, we provide applications of the continuous E-LPP: in particular we prove Theorem 2.1. First we prove that $\mathcal{T}_\beta = +\infty$ when $\alpha \leqslant 1/2$. Then, we prove the scaling relation (2.7), and finally we show the finiteness of the υ-th moment ($\upsilon < \alpha - 1/2$), in Section 4.3. Rougly, the idea of the proof is to decompose the variational problem (2.6) according to the value of the entropy:

$$\mathcal{T}_\beta = \sup_{B \geqslant 0} \Big\{ \beta \sup_{s \in \mathscr{D}, \mathrm{Ent}(s) = B} \pi(s) - B \Big\}. \tag{2.18}$$

Then, a simple scaling argument gives that $\sup_{s:\mathrm{Ent}(s)\leqslant B} \pi(s) \overset{(d)}{=} B^{\frac{1}{2\alpha}} \sup_{s:\mathrm{Ent}(s)\leqslant 1} \pi(s)$, and the E-LPP appears essential to show that the last supremum is finite, see in particular Lemma 4.1. Then, at a heuristic level, we get that $\mathcal{T}_\beta$ is finite because in (2.18) $B^{\frac{1}{2\alpha}} \ll B$ as $B \to \infty$ (remember that $\alpha > 1/2$). We conclude Section 4 by the proof of the contuinuity of $\beta \mapsto \mathcal{T}_\beta$ and of the existence and uniqueness of the maximizer in (2.6), relying again heavily on the E-LPP estimates of Section 3.

$*$ In Section 5, we use the estimates on the discrete E-LPP to control the discrete version of the variational problem (2.6) which we define in (5.2). More precisely, the first result of this section is Theorem 5.2 which provides estimates on the discrete variational problem, and in particular give its correct order and control its deviation probability – one of the key here is the uniformity of the estimate in the different parameters. The second result of importance is Proposition 5.3, which proves the convergence of the discrete variational problem to the continuous one. Here again, the proofs rely heavily on the estimates on the discrete E-LPP, in particular Lemma 5.4.

Part 2: Main results on the directed polymer in heavy-tail environment. In a second part, we turn to the study of the weak-coupling of the directed polymer, and prove the results stated in Sections 2.3-2.4. One of the key estimate of Section 5, Theorem 5.2, requires to work with a given height for transversal fluctuations: a recurrent idea is then to reduce to trajectories with a fixed transversal fluctuation.

$*$ In Section 6, we prove Theorem 2.2, identifying the correct transversal fluctuations. In order to make our ideas appear clearer, we first treat the case when no centering is needed (*i.e.* $\alpha < 3/2$) in Section 6.1. In Section 6.2 we adapt the proof to the case where it is needed. Let us present the ideas in the first case, where we use a rough bound $\mathbf{P}^\omega_{n,\beta_n}\big(\max_{i\leqslant n}|S_i| \geqslant A_n h_n\big) \leqslant \mathbf{Z}^\omega_{n,\beta_n}\big(\max_{i\leqslant n}|S_i| \geqslant A_n h_n\big)$, where the second term being the partition function restricted to path trajectories with $\max_{i\leqslant n}|S_i| \geqslant A_n h_n$. The key idea is to decompose this quantity into sub-parts where trajectories are controlled, and have a "fixed" transversal fluctuation

$$\mathbf{Z}^\omega_{n,\beta_n}\big(\max_{i\leqslant n}|S_i| \geqslant A_n h_n\big) = \sum_{k=\log_2 A_n+1}^{\log_2(n/h_n)} \mathbf{Z}^\omega_{n,\beta_n}\Big(\max_{i\leqslant n}|S_i| \in [2^{k-1}h_n, 2^k h_n)\Big).$$

Then, we control each term separately. Notice that forcing the random walk to reach the scale $2^{k-1}h_n$ has some entropy cost $\exp(-c2^{2k}h_n^2/n)$ so we need to understand if the partition function, when restricted to paths trajectories with $\max_{i\leqslant n}|S_i| \leqslant 2^k h_n$, compensates this cost (cf. (6.3)): we need to estimate the probability of having $\mathbf{Z}^{\omega}_{n,\beta_n}(\max_{i\leqslant n}|S_i| \leqslant 2^k h_n) \geqslant e^{c2^{2k}h_n^2/n}$. This is the purpose of Lemma 6.1, which is the central estimate of this section, and which tediously uses estimates of Section 5 – note that here the trajectories have a given transversal fluctuations bounded by $2^k h_n$.

$*$ In Section 7, we consider Regimes 2 and 3-a, proving Theorems 2.4-2.5. We regroup the two proofs since they are built on the same scheme – in both cases we have that $\log \mathbf{Z}_{n,\beta_n} \to \infty$. The proof is decomposed into three steps. In the first step (Section 7.1), we use Theorem 2.2 in order to restrict the partition function to path trajectories that have transversal fluctuations smaller than Ah_n (for some large A fixed). Then, in a second step (Section 7.2), we show that we can keep only the largest weights in the box of height Ah_n (more precisely a finite number of them), the small-weights contribution being negligible. Finally, the third step (Section 7.3) consists in proving the convergence of the large-weights partition function, and relies on the convergence of the discrete varitational problem (5.5) introduced in Section 5 – an adaptation in Regime 3-a is needed, and developed in Section 7.4.

$*$ In Section 8, we consider Regime 3-b and Regime 4, proving Theorems 2.6-2.7. Here again, we proceed in three steps, analogous to that of Section 7. In the first step (Section 8.1), we use again Theorem 2.2 to be able to restrict the partition function to trajectories with transversal fluctuations smaller than $A\sqrt{n\log n}$ (for some large A fixed). The second step (Section 8.2) consists in showing that one can restrict to large weights. Then, in the third step (Section 8.3), we observe that since our result implies that $\log \mathbf{Z}^{\omega}_{n,\beta_n}$ converges to 0, it is equivalent to studying the convergence of $\mathbf{Z}^{\omega}_{n,\beta_n} - 1$: we then reduce to showing the convergence of a finite number of terms of the polynomial chaos expansion of $\mathbf{Z}^{\omega}_{n,\beta_n} - 1$, see Lemmas 8.2-8.3. We prove this convergence in a last step: in Section 8.4, we show the convergence in Regime 3-b (Lemma 8.2), relying once again on the convergence of discrete variational problem proven in Section 5. In Section 8.5, we show the convergence in Regime 4 (Lemma 8.3), which is slightly more technical since we first need to reduce to trajectories with transversal fluctuations of order $h_n \ll \sqrt{n\log n}$.

$*$ In Section 9, we consider the case $\alpha \in (0, 1/2)$, and prove Theorem 2.10. First, in Section 9.1, we prove (2.16) *i.e.* that there cannot be intermediate transversal fluctuations between $\sqrt{n}$ and n. We use mostly the same ideas as in Section 6, decomposing the contribution to the partition function according to the scale of the path, and controlling the entropic cost vs. energy reward for each term. However in this case, we cannot use the estimates of Section 5 since they are valid only for $\alpha \in (1/2, 2)$. Luckily, some simplifications occur here, since one can bound the maximal energy collected by a path at a given scale by the sum of all weights in a box containing the path and this sum being dominated by the maximal weight in the box (this is true for $\alpha < 1$). We then turn to the convergence of the partition function in Section 9.2. The idea is similar to that of [18, Section 5], and consists in several steps: first we reduce the partition function to trajectories that stay at scale $\sqrt{n\log n}$; then we perform a polynomial chaos expansion of $\mathbf{Z}^{\omega}_{n,\beta_n} - 1$ and show that only the first term contributes; finally, we prove the convergence of the main term, see Lemma 9.2, in particular showing that the main contribution comes from trajectories that stay at scale $\sqrt{n}$.

Part 1. Entropy-controlled Last Passage Percolation and first applications

In [22], in order to show that $\widehat{\mathcal{T}}_\beta$ in (2.8) is finite, one crucial point is to show that restricting to the ℓ largest weights, the path can collect at most $\sqrt{\ell}$ of these weights. This is due to the fact that the path being Lipschitz, it can be mapped to an up/right path, and the problem is transformed into that of Hammersley's Last Passage Percolation, that we recall in Section 3.1. In our setting, since we want to treat transversal exponent strictly smaller than 1, we loose the Lipschitz property of the rescaled paths, and we need to modify the LPP problem: the idea to control the number of points that can be collected via paths which have an entropy constraint, rather than a 1-Lipschitz constraint. We call this new model *Entropy-controlled Last Passage Percolation*, or E-LPP.

3. Entropy-controlled Last-Passage Percolation (Continuous and Discrete)

The results we present here will be specific to the case of interest for us, but the techniques are very general and require very little assumption on the definition of the entropy of a path, as it will be developped in [8].

3.1. **Reminder of Hammersley's LPP.** Let us recall the original Hammersley's LPP problem of the maximal number of points that can be collected by up/right paths, also known as Ulam's problem [36] of the maximal increasing sequence.

Let $m \in \mathbb{N}$, and $(Z_i)_{1\leqslant i\leqslant m}$ be m points independently drawn uniformly on the square $[0,1]^2$, we denote the coordinates of these points $Z_i := (x_i, y_i)$ for $1 \leqslant i \leqslant m$. A sequence $(z_{i_\ell})_{1\leqslant \ell\leqslant k}$ is said to be *increasing* if $x_{i_\ell} > x_{i_{\ell-1}}$ and $y_{i_\ell} > y_{i_{\ell-1}}$ for any $1 \leqslant \ell \leqslant k$ (we set by convention $i_0 = 0$ and $z_0 = (0,0)$). Then, the question is to find the length of the longest increasing sequence among the m points, which is equivalent to finding the length of the longest increasing subsequence of a random (uniform) permutation of length m. We denote:

$$\mathcal{L}_m = \sup\big\{k : \exists\,(i_1,\dots,i_k)\ s.t.\ (Z_{i_\ell})_{1\leqslant\ell\leqslant k} \text{ is increasing}\big\}$$

Using subadditive techniques (together with a Poissonization argument), Hammersley [23] first proved that $m^{-1/2}\mathcal{L}_m$ converges a.s. and in L^1 to some constant, that was believed to be 2. It has then been proven that the constant was indeed 2, see [29, 37], and estimates related to $\mathcal{L}_m$ were improved by a series of papers, and in particular, the fluctuations of $\mathcal{L}_m$ around $2\sqrt{m}$ have been found to be or order $m^{1/6}$. More precisely Baik, Deift and Johansson [6] showed in a seminal paper that $m^{-1/6}(\mathcal{L}_m - 2\sqrt{m})$ converges in distribution to the Tracy-Widom distribution. Additionally, Johansson [26] proved that the transversal fluctuations of a path collecting the maximal number of points is of order $m^{2/3}$ – note that optimal paths are not unique, and there are in fact an exponential number of them, see [19].

3.2. **Definition of the Entropy-controlled LPP.** Operating a rotation by 45° clockwise, we may map the previous problem to that of the maximal number of points that can be collected by 1-Lipschitz paths $s : [0,1] \to \mathbb{R}$. We now introduce a new (natural) model where the Lipschitz constraint is replaced by an entropy constraint.

For $t > 0$, and a finite set $\Delta = \big\{(t_i, x_i); 1 \leqslant i \leqslant j\big\} \subset [0,t]\times\mathbb{R}$ with $|\Delta| = j \in \mathbb{N}$ and with $0 \leqslant t_1 \leqslant t_2 \leqslant \cdots \leqslant t_j \leqslant t$, we can define the entropy of Δ as

$$\mathrm{Ent}(\Delta) := \frac{1}{2}\sum_{i=1}^{j} \frac{(x_i - x_{i-1})^2}{t_i - t_{i-1}}\,. \tag{3.1}$$

where we used the convention that $(t_0, x_0) = (0, 0)$. By convention, if there exists some $1 \leqslant i \leqslant j$ such that $t_i = t_{i-1}$ then we set $\mathrm{Ent}(\Delta) = +\infty$. This corresponds to the definition (2.4) for the linear interpolation of the points of Δ: to any set Δ we can therefore canonically associated a (continuous) path $s : [0, t] \to \mathbb{R}$ with the same entropy. The set Δ is seen as a set of points a (continuous or discrete) path has to go through, and if $\Delta \subset \mathbb{N} \times \mathbb{Z}$ a standard calculation gives that $\mathbf{P}(\Delta \subset S) \leqslant e^{-\mathrm{Ent}(\Delta)}$ ($\Delta \subset S$ means that $S_{t_i} = x_i$ for all $i \leqslant |\Delta|$), where we use that for the simple random walk $\mathbf{P}(S_i = x) \leqslant e^{-x^2/2i}$ by a standard Chernoff bound argument.

Then, for any fixed $B > 0$, we will consider the maximal number of points that can be collected by paths with entropy smaller than B, among a random set Υ_m of m points, whose law is denoted $\mathbb{P}$. We now consider two types of problems, depending on how this set Υ_m is constructed:

(i) *continuous* setting: for $t, x > 0$, we consider a domain $\mathbf{\Lambda}_{t,x} := [0, t] \times [-x, x]$, and $\mathbf{\Upsilon}_m = \mathbf{\Upsilon}_m(t, x) = \{Y_1, \dots, Y_m\}$ where $(Y_i)_{1 \leqslant i \leqslant m}$ is a collection of independent r.v. chosen uniformly in $\mathbf{\Lambda}_{t,x}$;

(ii) *discrete* setting: for $n, h \in \mathbb{N}$, we consider a domain $\Lambda_{n,h} := [\![0, n]\!] \times [\![-h, h]\!]$, and $\Upsilon_m = \Upsilon_m(n, h) = \{Y_1, \dots, Y_m\}$ is a set of m distinct points taken randomly in $\Lambda_{n,h}$.

We are then able to define the Entropy-controlled LPP by

$$\mathcal{L}_m^{(B)}(t, x) = \max_{\substack{\Delta \subset \mathbf{\Upsilon}_m(t,x) \\ \mathrm{Ent}(\Delta) \leqslant B}} \big|\Delta\big|, \qquad L_m^{(B)}(n, h) = \max_{\substack{\Delta \subset \Upsilon_m(n,h) \\ \mathrm{Ent}(\Delta) \leqslant B}} \big|\Delta\big|, \tag{3.2}$$

the maximal number of points than can be included in a set Δ that has entropy smaller than B. In other words, it is the maximal number of points in $\mathbf{\Upsilon}_m$ or Υ_m that can be collected by a path of entropy smaller than B. Note that we use the different font to be able to differentiate the setting: $\mathcal{L}, \mathbf{\Lambda}, \mathbf{\Upsilon}$ for the continuous case and L, Λ, Υ for the discrete one.

We show the following result.

Theorem 3.1. *There are constants $C_0, c_0, c_0' > 0$ such that: for any $t, x, B > 0$, $n, h \geqslant 1$*

(i) continuous setting: *for all $m \geqslant 1$ and all $k \leqslant m$*

$$\mathbb{P}\Big(\mathcal{L}_m^{(B)}(t, x) \geqslant k\Big) \leqslant \Big(\frac{C_0(Bt/x^2)^{1/2}m}{k^2}\Big)^k; \tag{3.3}$$

$$\mathbb{P}\Big(\mathcal{L}_m^{(B)}(t, x) \geqslant k\Big) \geqslant 1 - \exp\Big\{-k\Big(\frac{(Bt/x^2)^{1/2}m}{k^2} \wedge \frac{m}{k} - c_0'\Big)\Big\}. \tag{3.4}$$

(ii) discrete setting: *for all $1 \leqslant m \leqslant nh$ and all $k \leqslant m$*

$$\mathbb{P}\Big(L_m^{(B)}(n, h) \geqslant k\Big) \leqslant \Big(\frac{C_0(Bn/h^2)^{1/2}m}{k^2}\Big)^k; \tag{3.5}$$

$$\mathbb{P}\Big(L_m^{(B)}(n, h) \geqslant k\Big) \geqslant 1 - \exp\Big\{-c_0k\Big(\frac{(Bn/h^2)^{1/2}m}{k^2} \wedge \frac{m}{k} - c_0'\Big)\Big\}. \tag{3.6}$$

The proof of Theorem 3.1 is not difficult but a bit technical, and we postpone it to Section 3.3. We included the lower bound in the statement of the theorem for the sake of completeness (the proof is short), but it will not be of any use in the rest of the paper.

This result shows in particular that $\mathcal{L}_m^{(B)}(t, x)$ is of order $\big((Bt/x^2)^{1/4}\sqrt{m}\big) \wedge m$, resp. $L_m^{(B)}(n, h)$ is of order $\big((Bn/h^2)^{1/4}\sqrt{m}\big) \wedge m$, as stressed by the following corollary.

Corollary 3.2. *For any $b > 0$, there is a constant $c_b > 0$ such that, for any $m \geqslant 1$, and any positive B, and any t, x, resp. n, h,*

$$\frac{1}{c_b} \leqslant \mathbb{E}\Big[\Big(\frac{\mathcal{L}_m^{(B)}(t,x)}{\big((Bt/x^2)^{1/4}\sqrt{m}\big) \wedge m}\Big)^b\Big] \leqslant c_b\,, \quad \frac{1}{c_b} \leqslant \mathbb{E}\Big[\Big(\frac{L_m^{(B)}(n,h)}{\big((Bn/h^2)^{1/4}\sqrt{m}\big) \wedge m}\Big)^b\Big] \leqslant c_b\,.$$

As an additional consequence of Theorem 3.1 (see in particular (3.7)-(3.8) below), a simple application of Borel Cantelli lemma gives, in the continuous setting, that for any fixed B, t, x we have that $\mathbb{P}$-a.s. there is some (random) m_0 such that

$$\frac{1}{2c_0'} \leqslant \frac{\mathcal{L}_m^{(B)}(t,x)}{\big((Bt/x^2)^{1/4}\sqrt{m}\big) \wedge m} \leqslant (2C_0)^{1/2} \qquad \text{for all } m \geqslant m_0\,.$$

We may also conjecture that, analogously to the standard LPP, Hammersley's method (*i.e.* Poissonization and sub-additivity) would give that there is a constant $\mathtt{C}$ such that

$$\lim_{m\to\infty} \frac{\mathcal{L}_m^{(B)}(t,x)}{(Bt/x^2)^{1/4}\sqrt{m}} = \mathtt{C} \qquad \mathbb{P}-a.s. \text{ and in } L^1\,.$$

In the discrete setting, the situation is slightly more complicated since one cannot fix B, n, h and let m go to infinity, but Theorem 3.1-(ii) gives useful *uniform* estimates to deal with the case where B, n, h go to infinity.

Proof of Corollary 3.2. We prove it in the continuous setting, the discrete one being similar.

Upper bound. From Theorem 3.1, we deduce that for any $u \geqslant (eC_0)^{1/2}$, we have

$$\mathbb{P}\Big(\mathcal{L}_m^{(B)}(t,x) \geqslant u\,(Bt/x^2)^{1/4}\sqrt{m}\Big) \leqslant \exp\Big(-u(Bt/x^2)^{1/4}\sqrt{m}\Big). \tag{3.7}$$

Applying this inequality with $u = (eC_0)^{1/2}$, and using also the *a priori* bound $\mathcal{L}_m^{(B)}(n,h) \leqslant m$, we get that for any $b > 0$

$$\begin{aligned}\mathbb{E}\Big[\Big(\frac{\mathcal{L}_m^{(B)}(t,x)}{\big((Bt/x^2)^{1/4}m^{1/2}\big) \wedge m}\Big)^b\Big] &\leqslant (eC_0)^{b/2} + \int_{(eC_0)^{b/2}}^{+\infty} \mathbb{P}\Big(\frac{\mathcal{L}_m^{(B)}(t,x)}{\big((Bt/x^2)^{1/4}m^{1/2}\big) \wedge m} > u^{1/b}\Big)du \\ &\leqslant (eC_0)^{b/2} + cst.\end{aligned}$$

Lower bound. An analogous reasoning holds for the lower bound, since for any $0 < u \leqslant (2c_0')^{-1} \wedge 1$ we have that

$$\mathbb{P}\Big(\mathcal{L}_m^{(B)} \leqslant u \times \big((Bt/x^2)^{1/4}\sqrt{m}\big) \wedge m\Big) \leqslant \exp\Big(-cu \times \big((Bt/x^2)^{1/4}\sqrt{m}\big) \wedge m\Big)\,. \tag{3.8}$$

Indeed, using (3.4), if $m \geqslant (Bt/x^2)^{1/2}$, then the probability is

$$\mathbb{P}\Big(\mathcal{L}_m^{(B)}(t,x) \leqslant u(Bt/x^2)^{1/4}\sqrt{m}\Big) \leqslant \exp\Big(-c_0u(Bt/x^2)^{1/4}\sqrt{m}\Big(\frac{1}{u^2} \wedge \frac{\sqrt{m}}{u(Bt/x^2)^{1/4}} - c_0'\Big)\Big)\,,$$

with the parenthesis inside the exponential larger than $1/u - c_0' \geqslant c_0'$. On the other hand, if $m \leqslant (Bt/x^2)^{1/2}$, then the probability is

$$\mathbb{P}\Big(\mathcal{L}_m^{(B)}(t,x) \leqslant u\,m\Big) \leqslant \exp\Big(-c_0u\,m\Big(\frac{(Bt/x^2)^{1/2}}{u^2m} \wedge \frac{1}{u} - c_0'\Big)\Big)\,,$$

with the parenthesis inside the exponential again larger than c_0'. □

Remark 3.3. On may view Theorem 3.1 as a generalization of [22, Proposition 3.3]. More precisely, we recover [22, Proposition 3.3] by considering $\Lambda_{n,n} = [\![n, n]\!]^2$ and replacing the entropy condition $\mathrm{Ent}(\Delta) \leqslant B$ with a *Lipschitz* condition, that is considering only the sets Δ whose points can be interpolated using a Lipschitz path. Let us denote $L_m^{(\mathrm{Lip})}(n)$ the LPP obtained. Now observe that if Δ satisfies the Lipschitz condition we have that $\mathrm{Ent}(\Delta) \leqslant n/2$ (recall the definition (3.1)): as a consequence it holds that $L_m^{(n/2)}(n,n) \geqslant L_m^{(\mathrm{Lip})}(n)$.

We also stress that our definition of E-LPP opens the way to many extensions: in particular as soon as one is able to properly define the entropy of a path (*i.e.* of a set Δ), one could extend the results to the case of paths with unbounded jumps or even non-directed paths: this is the object of [8], where a general notion of path-constrained LPP is developed and studied.

Let us stress here that one might want to reverse the point of view, and estimate the minimal entropy needed for a path to visit at least k points – this will be useful in Section 6. One realizes that

$$\inf_{\substack{\Delta\subset\Upsilon_m\\ |\Delta|\geqslant k}} \mathrm{Ent}(\Delta) \leqslant B \qquad \Longleftrightarrow \qquad \sup_{\substack{\Delta\subset\Upsilon_m\\ \mathrm{Ent}(\Delta)\leqslant B}} |\Delta| \geqslant k\,.$$

Hence, an easy consequence of Theorem 3.1 is that for any $k \leqslant n$ (we state it only in the discrete setting)

$$\mathbb{P}\Big(\inf_{\Delta\subset\Upsilon_m, |\Delta|\geqslant k} \mathrm{Ent}(\Delta) \leqslant B\Big) \leqslant \Big(\frac{C_0(Bn/h^2)^{1/2}m}{k^2}\Big)^k. \tag{3.9}$$

It therefore says that, with high probability, a path that collects k points in $\Upsilon_m \subset \Lambda_{n,h}$ has an entropy larger than a constant times $k^4/m^2 \times h^2/n$.

3.3. **Proof of Theorem 3.1.** We separate the upper and lower bounds, starting the proofs in the continuous setting. The discrete setting follows the same lines.

Continuous setting: upper bound. Let us consider $\mathcal{E}_k^{(t,B)}$ the set of k-uples in $[0,t]\times\mathbb{R}$ (*i.e.* up to time t) that have entropy smaller than B:

$$\mathcal{E}_k^{(t,B)} = \Big\{ \big((t_\ell, x_\ell)\big)_{1\leqslant\ell\leqslant k} \subset [0,t]\times\mathbb{R}\,;\; 0<t_1<\cdots<t_k<t\,;\; \mathrm{Ent}\big((t_\ell,x_\ell)_{1\leqslant\ell\leqslant k}\big) \leqslant B \Big\}.$$

Then, we can compute exactly the volume of $\mathcal{E}_k^{(t,B)}$.

Lemma 3.4. *We have, for any $t>0$ and $B>0$*

$$\mathrm{Vol}\big(\mathcal{E}_k^{(t,B)}\big) = C_k \times B^{k/2}t^{3k/2}, \quad \textit{with } C_k = \frac{\pi^k/\sqrt{2}}{\Gamma\big(k/2+1\big)\Gamma\big(3k/2+1\big)}\,.$$

In particular, it gives that there exists some constant C such that

$$\mathrm{Vol}\big(\mathcal{E}_k^{(t,B)}\big) \leqslant \Big(\frac{CB^{1/2}t^{3/2}}{k^2}\Big)^k.$$

Proof. The key to the computation is the induction formula below, based on the decomposition over the left-most point in $\mathcal{E}_k^{(t,B)}$ at position (u,y) (by symmetry we can assume $y\geqslant 0$): it leaves $k-1$ points with remaining time $t-u$ and entropy smaller than $B-\frac{y^2}{2u}$,

$$\mathrm{Vol}\big(\mathcal{E}_k^{(t,B)}\big) = 2\int_{u=0}^{t}\int_{y=0}^{\sqrt{2Bu}} \mathrm{Vol}\big(\mathcal{E}_{k-1}^{(t-u,B-y^2/2u)}\big)\,dydu. \tag{3.10}$$

The induction is only calculations. For $k = 1$ we have

$$\mathrm{Vol}\big(\mathcal{E}_1^{(t,B)}\big) = 2\int_{u=0}^{t}\int_{y=0}^{\sqrt{2Bu}} dudy = 2\sqrt{2B}\int_0^t u^{1/2}du = \frac{4\sqrt{2}}{3}B^{1/2}t^{3/2}\,,$$

so that we indeed have that $C_1 = \pi(\sqrt{2}\Gamma(3/2)\Gamma(5/2))^{-1}$, since $\Gamma(3/2) = \frac{\sqrt{\pi}}{2}$, $\Gamma(5/2) = \frac{3\sqrt{\pi}}{4}$.

For $k \geqslant 2$, by induction, we have

$$\mathrm{Vol}(\mathcal{E}_k^{(t,B)}) = 2C_{k-1}\int_{u=0}^{t}\int_{y=0}^{\sqrt{2Bu}} (t-u)^{3(k-1)/2}\big(B - \tfrac{y^2}{2u}\big)^{(k-1)/2}dydu.$$

Then, by a change of variable $w = y^2/(2Bu)$, we get that

$$\begin{aligned}\int_{y=0}^{\sqrt{2Bu}} \big(B - \tfrac{y^2}{2u}\big)^{(k-1)/2}dy &= B^{(k-1)/2}\int_0^1 (1-w)^{(k-1)/2}\sqrt{\frac{Bu}{2}}w^{-1/2}dw \\ &= \frac{1}{\sqrt{2}}B^{k/2}\,u^{1/2}\,\frac{\Gamma\big((k-1)/2+1\big)\Gamma(1/2)}{\Gamma(k/2+1)}\,.\end{aligned}$$

Moreover, we also have

$$\begin{aligned}\int_{u=0}^{t} u^{1/2}(t-u)^{3(k-1)/2}dx &= t^{3(k-1)/2+1/2+1}\int_0^1 v^{1/2}(1-v)^{3(k-1)/2}dv \\ &= t^{3k/2}\frac{\Gamma(3/2)\Gamma(3(k-1)/2+1)}{\Gamma(3k/2+1)}\,.\end{aligned}$$

Hence, the constant C_k verifies

$$C_k = 2C_{k-1}\times\sqrt{\pi}\frac{\Gamma\big((k-1)/2+1\big)}{\Gamma(k/2+1)}\times\frac{\sqrt{\pi}}{2}\frac{\Gamma(3(k-1)/2+1)}{\Gamma(3k/2+1)} = \frac{\pi^k/\sqrt{2}}{\Gamma\big(k/2+1\big)\Gamma\big(3k/2+1\big)}\,,$$

the last equality holding by induction, in view of the formula for C_{k-1}.

For the inequality in the second part of the lemma, we simply use properties of the Gamma function, to get that there is a constant $c > 0$ such that

$$\Gamma\big(k/2+1\big) \geqslant \big(ck\big)^{k/2} \quad \text{and} \quad \Gamma\big(3k/2+1\big) \geqslant \big(ck\big)^{3k/2}\,.$$

□

We then use Lemma 3.4 to control the probability that $\mathcal{L}_m^{(B)}(t,x)$ is larger than some k. Let us denote $\mathcal{N}_k$ the number of sets $\Delta \subset \Upsilon_m(t,x)$ with $|\Delta| = k$, that have entropy at most B. We write

$$\mathbb{P}\big(\mathcal{L}_m^{(B)}(t,x) \geqslant k\big) = \mathbb{P}(\mathcal{N}_k \geqslant 1) \leqslant \mathbb{E}[\mathcal{N}_k]\,.$$

Since all the points are exchangeable, we get

$$\mathbb{E}[\mathcal{N}_k] = \binom{m}{k}\mathbb{P}\Big(\exists\ \sigma\in\mathfrak{S}_k\ s.t.\ (Z_{\sigma(1)},\dots,Z_{\sigma(k)})\in\mathcal{E}_k^{(t,B)}\Big),$$

where $Z_1 = (t_1,x_1),\dots,Z_k = (t_k,x_k)$ are independent uniform r.v. on the domain $\mathbf{\Lambda}_{t,x}$ (with volume $2tx$). We then have that

$$\mathbb{P}\Big(\exists\ \sigma\in\mathfrak{S}_k\ s.t.\ (Z_{\sigma(1)},\dots,Z_{\sigma(k)})\in\mathcal{E}_k^{(t,B)}\Big) = k!\,\frac{\mathrm{Vol}(\mathcal{E}_k^{(t,B)})}{(2tx)^k}\,.$$

We therefore obtain, using that $\binom{m}{k} \leqslant m^k/k!$, together with Lemma 3.4

$$\mathbb{P}\big(\mathcal{L}_m^{(B)}(t,x) \geqslant k\big) \leqslant \Big(\frac{CB^{1/2}t^{1/2}m}{2xk^2}\Big)^k \tag{3.11}$$

This gives the upper bound of Theorem 3.1-(i). □

Discrete setting: upper bound. The proof follows the same idea as above. We give only the main steps, and skip some details since it is very similar. Define $E_k^{(n,B)}$ the set of k-uples in $[\![1,n]\!] \times \mathbb{Z}$ that have entropy smaller than B:

$$E_k^{(n,B)} := \Big\{ \big((t_\ell,x_\ell)\big)_{1\leqslant \ell\leqslant k} \subset [\![1,n]\!] \times \mathbb{Z}\,;\; 0<t_1<\cdots<t_k\leqslant n\,;\; \mathrm{Ent}\big((t_\ell,x_\ell)_{1\leqslant\ell\leqslant k}\big) \leqslant B \Big\}.$$

Then, we can estimate the volume (or cardinality) of $E_k^{(n,B)}$ – however not in an exact manner as in the continuous case.

Lemma 3.5. *For any $n \in \mathbb{N}$ it holds true that*

$$\mathrm{Vol}\big(E_k^{(n,B)}\big) \leqslant 2^k\, C_k \times B^{k/2} n^{3k/2}, \quad \textit{with } C_k = \frac{\pi^k/\sqrt{2}}{\Gamma\big(k/2+1\big)\Gamma\big(3k/2+1\big)}\,.$$

In particular, it gives that there exists some constant C such that

$$\mathrm{Vol}\big(E_k^{(n,B)}\big) \leqslant \Big(\frac{CB^{1/2}n^{3/2}}{k^2}\Big)^k.$$

Proof. The key to the computation is the following induction formula, similar to (3.10):

$$\mathrm{Vol}\big(E_k^{(n,B)}\big) = 2\sum_{i=1}^{n}\sum_{y=0}^{\sqrt{2Bi}} \mathrm{Vol}\big(E_{k-1}^{(n-i,B-x^2/2i)}\big). \tag{3.12}$$

The induction is again straightforward calculations: we can use the computations made in the continuous setting, together with the comparison between finite sums and Riemann integrals, *i.e.*

$$\begin{aligned} &\sum_{i=0}^{N} g(i) \leqslant \int_0^{N+1} g(z)\mathrm{d}z \qquad \text{if } g \text{ is increasing,}\\ &\sum_{i=0}^{N} g(i) \leqslant g(0) + \int_0^{N} g(z)\mathrm{d}z \qquad \text{if } g \text{ is decreasing}\,. \end{aligned} \tag{3.13}$$

Details are left to the reader. □

We use this lemma to control the probability that $L_m^{(B)}(n,h)$ is larger than some k. Again, we have $\mathbb{P}\big(L_m^{(B)}(n,h) \geqslant k\big) \leqslant \mathbb{E}[N_k]$, where N_k is the number of sets $\Delta \subset \Upsilon_m \subset \Lambda_{n,h}$ with $|\Delta| = k$, that have entropy at most B. Then,

$$\mathbb{E}[N_k] = \binom{m}{k}\mathbb{P}\Big(\exists\ \sigma\in\mathfrak{S}_k\ s.t.\ (Z_{\sigma(1)}^{(n,h)},\dots,Z_{\sigma(k)}^{(n,h)}) \in \mathcal{E}_k^{(n,B)}\Big),$$

where $(Z_1^{(n,h)},\cdots,Z_k^{(n,h)})$ are a uniform random choice of k distinct points from $\Lambda_{n,h}$ (which contains $n(2h+1)$ points) – the main difference with the continuous setting comes from the fact that the Z_i's are not independent. We therefore have that, using Lemma 3.5,

$$\mathbb{E}[N_k] = \binom{m}{k}\,\frac{\mathrm{Vol}(E_k^{(n,B)})}{\binom{2nh+n}{k}} \leqslant \frac{m^k}{(2nh)^k}\Big(\frac{CB^{1/2}}{k^2}\Big)^k.$$

We also used that $\binom{m}{k} \leqslant m^k/k!$ and that $\binom{2nh+n}{k} \geqslant (2nh+n-k)^k/k!$ with $k \leqslant n$. This concludes the proof of the upper bound in Theorem 3.1-(ii). □

Continuous setting: lower bound. For any $k \geqslant 1$, let us consider the following $4k$ sub-boxes of $\mathbf{\Lambda}_{t,x}$, for $1 \leqslant i \leqslant 4k$:

$$\mathcal{B}_i := \Big[\frac{(i-1)t}{4k}, \frac{it}{4k}\Big) \times \Big[-\frac{(tB)^{1/2}}{4k} \wedge x, \frac{(tB)^{1/2}}{4k} \wedge x\Big].$$

Then, we realize that if there are at least k boxes among the "even" boxes $\{\mathcal{B}_{2i}\}_{1\leqslant i\leqslant 2k}$ containing (at least) one point, then this set of k points has an entropy which is bounded by $k \times \frac{1}{2}\frac{(2(tB)^{1/2}/4k)^2}{t/4k} \leqslant B/2$. Hence, we get that

$$\mathbb{P}\Big(\mathcal{L}_m^{(B)}(t,x) \geqslant k\Big) \geqslant \mathbb{P}\Big(\sum_{i=1}^{2k} 1_{\{|\boldsymbol{\Upsilon}_m \cap \mathcal{B}_{2i}| \geqslant 1\}} \geqslant k\Big) = \mathbb{P}\Big(\sum_{i=1}^{2k} 1_{\{|\boldsymbol{\Upsilon}_m \cap \mathcal{B}_{2i}| = 0\}} \leqslant k\Big). \tag{3.14}$$

For the last probability, we use a union bound and the fact that the $1_{\{|\boldsymbol{\Upsilon}_m \cap \mathcal{B}_{2i}|=0\}}$ are exchangeable, to get that

$$\begin{aligned} 1 - \mathbb{P}\Big(\sum_{i=1}^{2k} 1_{\{|\boldsymbol{\Upsilon}_m \cap \mathcal{B}_{2i}|=0\}} \leqslant k\Big) &\leqslant \binom{2k}{k}\mathbb{P}\Big(\boldsymbol{\Upsilon}_m \cap \bigcup_{i=1}^{k} \mathcal{B}_i = \varnothing\Big) \\ &\leqslant 2^{2k}\Big(1 - \frac{t^{1/2}B^{1/2}}{16kx} \wedge \frac{1}{8}\Big)^m. \end{aligned} \tag{3.15}$$

In the second inequality we used that $\boldsymbol{\Upsilon}_m$ is a set of m independent random variables uniform in $\mathbf{\Lambda}_{t,x}$ (of volume $2tx$), and that $\bigcup_{i=1}^k \mathcal{B}_i$ has a volume of $\frac{t^{3/2}B^{1/2}}{8k} \wedge \frac{tx}{4}$. Then, we use that $1 - x \leqslant e^{-x}$ for any x, to get that

$$\begin{aligned} \mathbb{P}\Big(\mathcal{L}_m^{(B)}(t,x) \leqslant k\Big) &\leqslant e^{ck}e^{-c'\left(\frac{(Bt/x^2)^{1/2}}{k}\wedge 1\right)m} \\ &\leqslant \exp\left\{ck\Big(1 - c''\Big(\frac{(Bt/x^2)^{1/2}}{k} \wedge 1\Big)\frac{m}{k}\Big)\right\} \end{aligned}$$

which concludes the proof of the lower bound in Theorem 3.1-(i). □

Discrete setting: lower bound. The proof is identical to the continuous setting, the only adaptation needed being in the second inequality of (3.15). Details are left to the reader.

4. Application I: continuous heavy-tail E-LPP, proof of Theorem 2.1

As an application of Theorem 3.1, we prove Theorem 2.1. We start by proving that $\mathcal{T}_\beta = +\infty$ for $\alpha \in (0, 1/2]$. Then, for $\alpha > 1/2$, we prove the scaling relations in Theorem 2.1, which will be useful to show that $\mathbb{E}[(\mathcal{T}_\beta)^\upsilon] < \infty$. Finally, we show the continuity of $\beta \mapsto \mathcal{T}_\beta$, and we end the section by the proof of the existence and uniqueness of the maximizer.

Let $\mathcal{P} := \big\{(w_i, t_i, x_i) \colon i \geqslant 1\big\}$ be a Poisson Point Process on $[0,\infty) \times [0,1] \times \mathbb{R}$ of intensity $\mu(dwdtdx) = \frac{\alpha}{2}w^{-\alpha-1}1_{\{w>0\}}dwdtdx$, as introduced in Section 2.2.

4.1. **Case $\alpha \leqslant 1/2$.** Let us prove here that $\mathcal{T}_\beta = +\infty$ when $\alpha \in (0, 1/2]$. For any k in $\mathbb{Z}$, we define the event

$$\mathcal{G}_k := \left\{\mathcal{P} \cap [\beta^{-1}2^{2k+1}, +\infty) \times [\tfrac{1}{2}, 1] \times [2^{k-1}, 2^k] \neq \varnothing\right\},$$

On the event $\mathcal{G}_k$, we denote (w_k, t_k, x_k) a point of $\mathcal{P}$ such that $w_k \geqslant \beta^{-1}2^{2k+1}$ and $(t_k, x_k) \in [\frac{1}{2}, 1] \times [2^{k-1}, 2^k]$: considering the path going straight to (t_k, x_k) we get that

$$\mathcal{T}_\beta \geqslant \beta w_k - \frac{x_k^2}{2t_k} \geqslant 2^{2k}, \qquad \text{on the event } \mathcal{G}_k\,.$$

Then, it is just a matter of estimating $\mathbb{P}(\mathcal{G}_k)$. We stress that considering $\mathcal{M}_k$ the maximal weight in $[\frac{1}{2}, 1] \times [2^{k-1}, 2^k]$, we find that $\mathcal{M}_k$ is of order $(2^k)^{1/\alpha}$ (as a maximum of a field of independent heavy-tail random variables, or using the scaling relations below), so that we get that: if $\alpha < 1/2$, $\mathbb{P}(\mathcal{G}_k) \to 1$ as $k \to +\infty$; if $\alpha = 1/2$, there is a constant $c > 0$ such that $\mathbb{P}(\mathcal{G}_k) \geqslant c$ for all $k \in \mathbb{Z}$; if $\alpha > 1/2$, $\mathbb{P}(\mathcal{G}_k) \to 1$ as $k \to -\infty$. Note that the events $\mathcal{G}_k$ are independent, so an application of Borel-Cantelli lemma gives that for $\alpha \leqslant 1/2$, a.s. $\mathcal{G}_k$ occurs for infinitely many $k \in \mathbb{N}$: since $\mathcal{T}_\beta \geqslant 2^{2k}$ on $\mathcal{G}_k$, it implies that $\mathcal{T}_\beta = +\infty$ a.s. for $\alpha \leqslant 1/2$.

On the other hand, it also proves that when $\alpha > 1/2$, a.s. there exists some $k_0 \leqslant -1$ such that $\mathcal{G}_{k_0}$ occurs and thus $\mathcal{T}_\beta \geqslant 2^{2k_0} > 0$.

4.2. **Scaling relations.** For any $\alpha \in (0, 2)$ and $a > 0$ we consider two functions $\varphi(w, t, x) := (w, t, ax)$ and $\psi(w, t, x) := (a^{-1/\alpha}w, t, x)$ which scale space by a (hence the entropy by a^2) and weights by $a^{-1/\alpha}$ respectively. The random sets $\varphi(\mathcal{P})$ and $\psi(\mathcal{P})$ are still two Poisson Point Processes with the same law, that is $\varphi(\mathcal{P}) \overset{(\mathrm{d})}{=} \psi(\mathcal{P})$. This implies that (recall the definition (2.5))

$$\pi(as) \overset{(\mathrm{d})}{=} a^{1/\alpha}\pi(s).$$

Therefore,

$$\sup_{s\in\mathscr{D}, \mathrm{Ent}(s)<\infty} \left\{\beta\pi(s) - a^2\mathrm{Ent}(s)\right\} \overset{(\mathrm{d})}{=} \sup_{s\in\mathscr{D}, \mathrm{Ent}(s)<\infty} \left\{\beta a^{-1/\alpha}\pi(s) - \mathrm{Ent}(s)\right\}. \tag{4.1}$$

Consequently, for any $\alpha \in (0, 2)$, $a^2\mathcal{T}_{\beta/a^2} \overset{(\mathrm{d})}{=} \mathcal{T}_{\beta a^{-1/\alpha}}$. In particular, for any $\beta > 0$ it holds true that for $\alpha > 1/2$

$$\mathcal{T}_\beta \overset{(\mathrm{d})}{=} \beta^{\frac{2\alpha}{2\alpha-1}}\mathcal{T}_1\,. \tag{4.2}$$

4.3. **Finite moments of $\mathcal{T}_\beta$.** We show that for $\alpha \in (1/2, 2)$ $\mathbb{E}[(\mathcal{T}_\beta)^\upsilon] < \infty$ for any $\upsilon < \alpha - 1/2$, which readily implies that $\mathcal{T}_\beta < \infty$ a.s. For any interval $[c, d)$ with $0 \leqslant c < d$ we let

$$\mathcal{T}_\beta\big([c, d)\big) := \sup_{s\in\mathscr{D}, \mathrm{Ent}(s)\in[c,d)} \left\{\beta\pi(s) - \mathrm{Ent}(s)\right\}, \tag{4.3}$$

and we observe that $\mathcal{T}_\beta = \mathcal{T}_\beta\big([0, 1)\big) \vee \sup_{k\geqslant 0} \mathcal{T}_\beta\big([2^k, 2^{k+1})\big)$. Moreover, as in (4.1) we have

$$\mathcal{T}_\beta\big([2^k, 2^{k+1})\big) \overset{(\mathrm{d})}{=} \sup_{s:\,\mathrm{Ent}(s)\in[1,2)} \left\{2^{\frac{k}{2\alpha}}\beta\pi(s) - 2^k\,\mathrm{Ent}(s)\right\} \leqslant 2^{\frac{k}{2\alpha}}\beta \sup_{s:\mathrm{Ent}(s)\leqslant 2} \pi(s) - 2^k\,. \tag{4.4}$$

We show the following Lemma.

Lemma 4.1. *For any $a < \alpha$, we have that there is a constant $c_a > 0$ such that for any $t > 1$ we get*

$$\mathbb{P}\Big(\sup_{s\in\mathscr{D},\mathrm{Ent}(s)\leqslant 2} \pi(s) > t\Big) \leqslant c_a t^{-a}\,.$$

From this lemma and (4.4), we get that for any $t \geqslant -1$ and any k large enough so that $\beta^{-1}2^{-\frac{k}{2\alpha}}2^{-k} > 2$, we get

$$\begin{aligned}\mathbb{P}\Big(\mathcal{T}_\beta\big([2^k,2^{k+1})\big) > t\Big) &\leqslant \mathbb{P}\Big(\sup_{s\in\mathscr{D},\mathrm{Ent}(s)\leqslant 2} \pi(s) > \beta^{-1}2^{-\frac{k}{2\alpha}}(t+2^k)\Big)\\ &\leqslant c_a\beta^a 2^{k\frac{a}{2\alpha}}\big(t+2^k\big)^{-a}\,. \end{aligned} \tag{4.5}$$

Then, for any $t \geqslant 1$ and $a < \alpha$, we get by a union bound that

$$\begin{aligned}\mathbb{P}\big(\mathcal{T}_\beta > t\big) &\leqslant \sum_{k=0}^{\infty} \mathbb{P}\Big(\mathcal{T}_\beta\big([2^k,2^{k+1})\big) > t\Big)\\ &\leqslant c'_a 2^a\beta^a t^{-a}\sum_{k=0}^{\log_2 t} 2^{k\frac{a}{2\alpha}} + c'_a 2^a\beta^a \sum_{k>\log_2 t} 2^{-ak\left(1-\frac{1}{2\alpha}\right)}\\ &\leqslant c''_a\beta^a t^{-a}t^{\frac{a}{2\alpha}} + c''_a t^{-a\left(1-\frac{1}{2\alpha}\right)} \leqslant 2c''_a\beta^a t^{-a\left(1-\frac{1}{2\alpha}\right)}\,,\end{aligned}$$

where we used that $t + 2^k \geqslant t/2$ if $k \leqslant \log_2 t$, and $t + 2^k \geqslant 2^k/2$ if $k > \log_2 t$. For the second sum we also used that $1 - \frac{1}{2\alpha} > 0$ when $\alpha > 1/2$. In particular, this shows that for any $\delta > 0$, there is some constant $c_{\delta,\beta} > 0$ such that for any $t \geqslant 1$

$$\mathbb{P}\big(\mathcal{T}_\beta > t\big) \leqslant c_{\delta,\beta}t^{-(\alpha-\frac{1}{2})+\delta}\,, \tag{4.6}$$

which proves that $\mathbb{E}[(\mathcal{T}_\beta)^\upsilon] < \infty$ for any $\upsilon < \alpha - 1/2$.

Proof of Lemma 4.1. Let us recall that $\mathrm{Ent}(s) \leqslant 2$ implies that $\max|s| \leqslant 2$. Therefore we can restrict our Poisson Point Process on $\mathbb{R}_+ \times [0,1] \times [-2,2]$. In this case (cf. Section 5.2 below) we rewrite a realization of the Poisson Point Process by using its ordered statistic. We introduce $(Y_i)_{i\in\mathbb{N}}$ be an i.i.d. sequence of uniform random variables on $[0,1]\times[-2,2]$ and $(M_i)_{i\in\mathbb{N}}$ be a random sequence independent of $(Y_i)_{i\in\mathbb{N}}$ defined by $M_i = 4^{1/\alpha}(\mathcal{E}_1 + \cdots + \mathcal{E}_i)^{-1/\alpha}$ with $(\mathcal{E}_j)_{j\geqslant 1}$ an i.i.d. sequence of $\mathrm{Exp}(1)$ random variables. In such a way $\mathcal{P} \overset{(\mathrm{d})}{=} (M_i, Y_i)_{i\in\mathbb{N}}$ and $\pi(s) = \sum_{i=1}^{\infty} M_i 1_{\{Y_i\in s\}}$.

The proof is then a consequence of Theorem 3.1 (with $B = 1$), which allows to use the same ideas as in [22, Proposition 3.3] – we develop the argument used in [22] in a more robust way, which makes it easier to adapt to the discrete setting. Using the notations introduced in Section 3, for any $i \geqslant 0$, we denote $\boldsymbol{\Upsilon}_i = \{Y_1,\dots,Y_i\}$ ($\boldsymbol{\Upsilon}_0 = \varnothing$), and let $\Delta_i = \Delta_i(s) = s \cap \boldsymbol{\Upsilon}_i$ be the set of the i largest weights collected by s. The E-LPP can be written here as $\mathcal{L}_i^{(2)} := \max_{s:\mathrm{Ent}(s)\leqslant 2}|\Delta_i(s)|$ – we drop here the dependence on t, x.

Using that M_i is a non-increasing sequence, we write

$$\pi(s) = \sum_{j=0}^{\infty}\sum_{i=2^j}^{2^{j+1}-1} M_i 1_{\{Y_i\in s\}} \leqslant \sum_{j=0}^{\infty} M_{2^j}\,\mathcal{L}^{(2)}_{2^{j+1}}\,. \tag{4.7}$$

Then, we fix some $\delta > 0$ such that $1/\alpha - 1/2 > 2\delta$, and we let $C = \sum_{j=0}^{\infty} 2^{j(1/2-1/\alpha+2\delta)}$: we obtain via a union bound that

$$\mathbb{P}\Big(\sup_{\mathrm{Ent}(s)\leqslant 2} \pi(s) > t\Big) \leqslant \sum_{j=0}^{\infty} \mathbb{P}\Big(M_{2^j}\mathcal{L}^{(2)}_{2^{j+1}} > \frac{1}{C}\, t\, 2^{j(1/2-1/\alpha+2\delta)}\Big)$$
$$\leqslant \sum_{j=0}^{\infty} \Big[\mathbb{P}\Big(\mathcal{L}^{(2)}_{2^{j+1}} > C' \log t\, (2^{j+1})^{1/2+\delta)}\Big) + \mathbb{P}\Big(M_{2^j} > C''\, \frac{t}{\log t}(2^j)^{-1/\alpha+\delta}\Big)\Big]. \tag{4.8}$$

Here C' is a constant that we choose large in a moment, and C'' is a constant depending on C, C' - we also work with $t \geqslant 2$ for simplicity.

For the first probability in the sum, we obtain from Theorem 3.1-(i) that provided $C'(\log t)2^{j\delta} \geqslant 2C_0^{1/2}$

$$\mathbb{P}\Big(\mathcal{L}^{(2)}_{2^{j+1}} > C' \log t\, (2^{j+1})^{1/2+\delta)}\Big) \leqslant \Big(\frac{1}{2}\Big)^{C'(\log t)2^{j\delta}} \leqslant t^{-\log 2\, C' 2^{j\delta}}.$$

Hence, for t sufficiently large we get that

$$\sum_{j=0}^{\infty} \mathbb{P}\Big(\mathcal{L}^{(2)}_{2^{j+1}} > C' \log t\, (2^{j+1})^{1/2+\delta)}\Big) \leqslant ct^{-C'\log 2} \leqslant ct^{-a} \tag{4.9}$$

provided that we fixed C' large.

For the second probability in the sum, we recall that $M_i \overset{(\mathrm{d})}{=} 4^{1/\alpha}\mathrm{Gamma}(i)^{-1/\alpha}$, so that for any $a < \alpha$, $\mathbb{E}[(i^{1/\alpha}M_i)^a]$ is bounded by a constant independent of i. Therefore, Markov's inequality gives that

$$\mathbb{P}\Big(M_{2^j} > C''\, \frac{t}{\log t}(2^j)^{-1/\alpha+\delta}\Big) \leqslant c(\log t)^a t^{-a}(2^j)^{-a\delta},$$

so that

$$\sum_{j=0}^{\infty} \mathbb{P}\Big(M_{2^j} > C''\, \frac{t}{\log t}(2^j)^{-1/\alpha+\delta}\Big) \leqslant c(\log t)^a t^{-a}. \tag{4.10}$$

Plugging (4.9) and (4.10) into (4.8), we obtain that for any $a' < a < \alpha$ there are constants $c > 0$ such that for any $t \geqslant 2$

$$\mathbb{P}\Big(\sup_{\mathrm{Ent}(s)\leqslant 2} \pi(s) > t\Big) \leqslant 2c(\log t)^a t^{-a} \leqslant c' t^{-a'},$$

which concludes the proof. □

4.4. **Continuity of $\beta \mapsto \mathcal{T}_\beta$.** An obvious and crucial fact that we use along the way is that for any realization of $\mathcal{P}$, $\beta \mapsto \mathcal{T}_\beta$ is non-decreasing.

Left-continuity. Let us first show that $\beta \mapsto \mathcal{T}_\beta$ is left-continuous, since it is less technical. Fix $\varepsilon > 0$. For any β, there exists a path $s_\beta^{(\varepsilon)}$ with $\pi(s_\beta^{(\varepsilon)}) < \infty$ such that $\mathcal{T}_\beta \leqslant \beta\pi(s_\beta^{(\varepsilon)}) - \mathrm{Ent}(s_\beta^{(\varepsilon)}) + \varepsilon$. Using this path $s_\beta^{(\varepsilon)}$, we then simply write that for any $\delta > 0$

$$\mathcal{T}_\beta \geqslant \mathcal{T}_{\beta-\delta} \geqslant (\beta-\delta)\pi(s_\beta^{(\varepsilon)}) - \mathrm{Ent}(s_\beta^{(\varepsilon)}).$$

Letting $\delta \downarrow 0$, we get that the right hand side converges to $\beta\pi(s_\beta^{(\varepsilon)}) - \mathrm{Ent}(s_\beta^{(\varepsilon)}) \geqslant \mathcal{T}_\beta - \varepsilon$. Since ε is arbitrary, one concludes that $\lim_{\delta\uparrow 0} \mathcal{T}_{\beta-\delta} = \mathcal{T}_\beta$, that is $\beta \mapsto \mathcal{T}_\beta$ is left-continuous.

Right-continuity. It remains to prove that a.s. $\beta \mapsto \mathcal{T}_\beta$ is right-continuous. We prove a preliminary result.

Lemma 4.2. *For any $K > 0$, $\mathbb{P}$-a.s. there exists $B_0 > 0$ such that for any $0 \leqslant \beta \leqslant K$*

$$\mathcal{T}_\beta = \mathcal{T}_\beta\big([0, B_0]\big)\,, \tag{4.11}$$

where $\mathcal{T}_\beta\big([0, B_0]\big)$ is defined in (4.3).

Proof. Let us recall that $\mathcal{T}_\beta = \mathcal{T}_\beta\big([0,1)\big) \vee \sup_{k\geqslant 0} \mathcal{T}_\beta\big([2^k, 2^{k+1})\big)$. Using (4.5) with $t = -1$, for any $a < \alpha$ we have that

$$\mathbb{P}\Big(\mathcal{T}_\beta\big([2^k, 2^{k+1})\big) > -1\Big) \leqslant c_a \beta^a 2^{k\frac{a}{2\alpha}} \big(2^k - 1\big)^{-a} \leqslant c_{a,K}\, 2^{k(\frac{1}{2\alpha}-1)}.$$

Since $\frac{1}{2\alpha} - 1 < 0$, by Borel-Cantelli lemma we obtain that $\mathbb{P}$-a.s. there exists $k_0 > 0$ such that $\mathcal{T}_\beta\big([2^k, 2^{k+1})\big) \leqslant -1$ for all $k \geqslant k_0$. This concludes the proof. □

Then, since we now consider paths with entropy bounded by B_0, we can restrict the Poisson Point Process $\mathcal{P}$ to $\mathbb{R}_+ \times [0,1] \times [-\sqrt{2B_0}, \sqrt{2B_0}]$. In this case we write a realization of the Poisson Point Process by using its ordered statistic. More precisely we introduce $M_i := (8B_0)^{1/2\alpha}(\mathcal{E}_1 + \cdots + \mathcal{E}_i)^{-1/\alpha}$, where $(\mathcal{E}_i)_{i\in\mathbb{N}}$ is an i.i.d. sequence of exponential of mean 1 and $(Y_i)_{i\in\mathbb{N}}$ is a i.i.d. sequence of uniform random variables on $[0,1] \times [-\sqrt{2B_0}, \sqrt{2B_0}]$, independent of $(\mathcal{E}_i)_{i\in\mathbb{N}}$. Then, $\mathcal{P} \overset{(\mathrm{d})}{=} (M_i, Y_i)_{i\in\mathbb{N}}$ and $\pi(s) = \sum_{i=1}^\infty M_i 1_{\{Y_i \in s\}}$.

For any $\ell \in \mathbb{N}$, we let $\pi^{(\ell)} := \sum_{i=1}^\ell M_i 1_{Y_i\in s}$ be the "truncated" energy of a path: we can write for any $\beta < K$, and any $\delta > 0$ such that $\beta + \delta \leqslant K$

$$\begin{aligned}\mathcal{T}_{\beta+\delta} = \mathcal{T}_{\beta+\delta}\big([0,B_0]\big) \leqslant \sup_{s\in\mathscr{D}, \mathrm{Ent}(s)\leqslant B_0} &\big\{(\beta+\delta)\pi^{(\ell)}(s) - \mathrm{Ent}(s)\big\} \\ &+ (\beta+\delta) \sup_{s\in\mathscr{D}, \mathrm{Ent}(s)\leqslant B_0} \big|\pi(s) - \pi^{(\ell)}(s)\big|\,.\end{aligned}$$

Then, we show that

$$\max_{s\in\mathscr{D}, \mathrm{Ent}(s)\leqslant B_0} \big|\pi(s) - \pi^{(\ell)}(s)\big| \xrightarrow[\ell\to\infty]{a.s.} 0\,. \tag{4.12}$$

Hence, for any fixed ε, we can a.s. choose some ℓ_ε such that for any $\beta < K$ and any $\delta > 0$ with $\beta + \delta \leqslant K$

$$\mathcal{T}_\beta \leqslant \mathcal{T}_{\beta+\delta} \leqslant \sup_{s\in\mathscr{D}, \mathrm{Ent}(s)\leqslant B_0} \big\{(\beta+\delta)\pi^{(\ell)}(s) - \mathrm{Ent}(s)\big\} + K\varepsilon\,.$$

Then, letting $\delta \downarrow 0$, and since the supremum can now be reduced to a finite set (we consider only ℓ points), we get that for any $\beta < K$

$$\mathcal{T}_\beta \leqslant \lim_{\delta\downarrow 0} \mathcal{T}_{\beta+\delta} \leqslant \sup_{s\in\mathscr{D}, \mathrm{Ent}(s)\leqslant B_0} \big\{\beta\pi^{(\ell)}(s) - \mathrm{Ent}(s)\big\} + \varepsilon \leqslant \mathcal{T}_\beta + \varepsilon\,.$$

Since ε is arbitrary, this shows that $\lim_{\delta\downarrow 0} \mathcal{T}_{\beta+\delta} = \mathcal{T}_\beta$ a.s., that is $\beta \mapsto \mathcal{T}_\beta$ is right-continuous.

It remains to prove (4.12). For any $i \in \mathbb{N}$ we consider $\boldsymbol{\Upsilon}_i = \{Y_1, \ldots, Y_i\}$ and for any given path s we define $\Delta_i = \Delta_i(s) = s \cap \boldsymbol{\Upsilon}_i$ the set of the i largest weights collected by s. Then, let $\mathcal{L}_i^{(B_0)} = \sup_{s\in\mathscr{D}_{B_0}} |\Delta_i(s)|$, as introduced in (3.2). Realizing that $1_{\{Y_i\in s\}} = |\Delta_i(s)| - |\Delta_{i-1}(s)|$,

and integrating by parts (as done in [22]), we obtain for any $s \in \mathscr{D}_{B_0}$

$$\begin{aligned}\pi(s) - \pi^{(\ell)}(s) &= \sum_{i>\ell} M_i 1_{\{Y_i \in s\}} = \lim_{N\to\infty} \sum_{i=\ell+1}^{N} M_i\big(|\Delta_i| - |\Delta_{i-1}|\big) \\ &= \lim_{N\to\infty} \Big(\sum_{i=\ell+1}^{N-1} |\Delta_i|(M_i - M_{i+1}) + M_N|\Delta_N| - M_\ell|\Delta_\ell| \Big) \\ &\leqslant \sum_{i=\ell+1}^{\infty} \mathcal{L}_i^{(B_0)}(M_i - M_{i+1}) + \limsup_{N\to\infty} M_N \mathcal{L}_N^{(B_0)}. \end{aligned} \tag{4.13}$$

At this stage, the law of large numbers gives that $\lim_{n\to\infty} N^{1/\alpha} M_N = (8B_0)^{1/2\alpha}$ a.s., and Corollary 3.2 gives that $\limsup_{N\to\infty} N^{-1/2}\mathcal{L}_N^{(B_0)} < +\infty$ a.s. Since $\alpha < 2$, we therefore conclude that $\limsup_{N\to\infty} M_N \mathcal{L}_N^{(B_0)} = 0$ a.s.

We let $U_\ell := \sum_{i>\ell} \mathcal{L}_i^{(B_0)}(M_i - M_{i-1})$. We are going to show that there exists some ℓ_0 such that $\sum_{i>\ell_0} \mathcal{L}_i^{(B)}(M_i - M_{i-1}) < \infty$ a.s., and thus $\lim_{\ell\to\infty} U_\ell = 0$ a.s. We show that $\mathbb{E}[U_{\ell_0}^2]$ is finite for ℓ_0 large enough. For any $\varepsilon > 0$, by Cauchy-Schwarz inequality we have that

$$U_{\ell_0} \leqslant \Big(\sum_{i>\ell_0} \big(i^{-\frac{1}{2}-\varepsilon}\big)^2 \Big)^{1/2} \Big(\sum_{i>\ell_0} \big(i^{-\frac{1}{2}+\varepsilon} \mathcal{L}_i^{(B_0)}(M_i - M_{i+1})\big)^2 \Big)^{1/2}.$$

Then, we get that for ℓ_0 large enough

$$\begin{aligned}\mathbb{E}[U_{\ell_0}^2] &\leqslant C \sum_{i>\ell_0} i^{1+2\varepsilon} \mathbb{E}\big[(\mathcal{L}_i^{(B)})^2\big] \mathbb{E}\big[(M_i - M_{i-1})^2\big] \\ &\leqslant C'_{B_0} \sum_{i>\ell_0} i^{1+2\varepsilon} \times i \times i^{-2-2/\alpha} < +\infty\,. \end{aligned}$$

Here, we used Corollary 3.2 and a straightforward calculation that gives $\mathbb{E}\big[(M_i - M_{i-1})^2\big] \leqslant c i^{-2-2/\alpha}$ for i large enough (see for instance Equation (7.2) in [22]). Provided ε is small enough so that $2\varepsilon - 2/\alpha < -1$ we obtain that $\mathbb{E}[U_{\ell_0}^2] < \infty$. □

4.5. **Existence and uniqueness of the maximizer.** As a consequence of Lemma 4.2, to show that the supremum is attained and is unique in (2.6), it is enough to prove the following result.

Lemma 4.3. *For a.e. realization of $\mathcal{P}$ and for any $B > 0$ we have that*

$$s_\beta^*(B) = \operatorname*{arg\,max}_{s\in\mathscr{D}_B} \big\{\beta\pi(s) - \mathrm{Ent}(s)\big\}$$

exists, and it is unique. Here, we defined $\mathscr{D}_B := \{s \in \mathscr{D} : \mathrm{Ent}(s) \leqslant B\}$.

Proof. Our first step is to show that $\mathscr{D}_B$ is compact for the uniform norm $\|\cdot\|_\infty$. Let us observe that for any $s : [0,1] \to \mathbb{R}$, the condition $\mathrm{Ent}(s) \leqslant B$ implies that

$$|s(x) - s(y)| \leqslant \int_y^x |s'(t)|dt \leqslant (2B)^{1/2}|x-y|^{1/2}, \qquad \forall\, x, y \in [0,1],$$

so that s belongs to the Hölder Space $C^{1/2}([0,1])$. Hence, $\mathscr{D}_B$ is included in $C^{1/2}([0,1])$ which is compact for the uniform norm $\|\cdot\|_\infty$ by the Ascoli-Arzelà theorem. We therefore only need to show that $\mathscr{D}_B$ is closed for the uniform norm $\|\cdot\|_\infty$.

For this purpose we consider a convergent sequence s_n and we denote by $\mathbf{s}$ its limit. Since $\mathrm{Ent}(s_n) = \frac{1}{2}\|s_n'\|_{L^2}^2$ for all n, we have that $(s_n')_{n\in\mathbb{N}}$ belongs to the (closed) ball of

radius $(2B)^{1/2}$ of $L^2([0,1])$. By Banach–Alaoglu theorem, the sequence $(s'_n)_{n\in\mathbb{N}}$ contains a weakly convergent subsequence. This means that there exist n_k and $s^\star$ such that

$$\int_0^1 \varphi(x)s'_{n_k}(x)\mathrm{d}x \xrightarrow[k\to\infty]{} \int_0^1 \varphi(s)s^\star(x)dx, \qquad \forall \varphi \in L^2([0,1]).$$

By uniqueness of the limit (and taking $\varphi(x) = 1_{\{[0,y]\}}(x)$), this relation implies that $\mathbf{s}(y) = \int_0^y s^\star(x)dx$, that is $\mathbf{s}' = s^\star$ almost everywhere. Since the L^2 norm is weakly lower semi-continuous by the Hahn-Banach theorem – that is $\|s^\star\|_{L^2} \leqslant \liminf_{k\to\infty} \|s'_{n_k}\|_{L^2}$ – we obtain that $\mathbf{s} \in \mathscr{D}_B$, so $\mathscr{D}_B$ is closed. As a by-product of this argument we also have that the entropy function $s \mapsto \mathrm{Ent}(s)$ is lower semi-continuous on $(\mathscr{D}_B, \|\cdot\|_\infty)$.

Existence of the maximizer. Since $\mathscr{D}_B$ is compact, the existence of the maximizer comes from the fact that the function

$$t_\beta(s) := \beta\pi(s) - \mathrm{Ent}(s) \tag{4.14}$$

is upper semi-continuous, thanks to the extreme value theorem tells. Since we have already shown that $s \mapsto \mathrm{Ent}(s)$ is lower semi-continuous, we only need to prove the following.

Lemma 4.4. *For a.e. realization of $\mathcal{P}$ and for any $B > 0$ the function $s \mapsto \pi(s)$ is upper semi-continuous on $(\mathscr{D}_B, \|\cdot\|_\infty)$.*

Proof. We recall that if $s \in \mathscr{D}_B$ then $\max_{t\in[0,1]} |s(t)| \leqslant \sqrt{2B}$. Therefore, using the same notations as above, we can write a realization of the Poisson Point Process $\mathcal{P}$ by using its ordered statistic: $\mathcal{P} = (M_i, Y_i)_{i\in\mathbb{N}}$, $\pi(s) = \sum_{i=1}^\infty M_i 1_{\{Y_i\in s\}}$, and recall that for any $\ell \in \mathbb{N}$ we let $\pi^{(\ell)} := \sum_{i=1}^\ell M_i 1_{\{Y_i\in s\}}$. Thanks to (4.12), we only need to prove that for any fixed $\ell \in \mathbb{N}$ the function $s \mapsto \pi^{(\ell)}(s)$ is upper semi-continuous: then $\pi(s)$, as the uniform limit of $\pi^{(\ell)}$, is still upper semi-continuous.

For any $s \in \mathscr{D}_B$ we let $\iota_s := \boldsymbol{\Upsilon}_\ell \backslash \{s \cap \boldsymbol{\Upsilon}_\ell\}$ be the set of all points of $\boldsymbol{\Upsilon}_\ell = \{Y_1, \dots, Y_\ell\}$ that are not in s. Since there are finitely many points, we realize that there exists $\eta = \eta(s,\ell) > 0$ such that $d_H(\iota_s, \mathrm{graph}(s)) > \eta$, with d_H is the Hausdorff distance.

Given $s \in \mathscr{D}_B$, we consider a sequence $(s_n)_n$, $s_n \in \mathscr{D}_B$ such that $\lim_{n\to\infty} \|s_n - s\|_\infty = 0$. We observe that whenever $\|s_n - s\|_\infty \leqslant \eta/2$, we have that $d_H(\iota_s, \mathrm{graph}(s_n)) > \eta/2$. This means that for n large enough

$$\{s_n \cap \boldsymbol{\Upsilon}_\ell\} \subset \{s \cap \boldsymbol{\Upsilon}_\ell\},$$

which implies that $\pi^{(\ell)}(s) \geqslant \limsup_{n\to\infty} \pi^{(\ell)}(s_n)$. □

Uniquenes of the maximizer. The strategy is very similar to the one used in [5, Lemma 4.1] or [22, Lemma 4.2]. For any $s \in \mathscr{D}_B$, we let $I(s) := \{s \cap \boldsymbol{\Upsilon}_\infty\}$, where we $\boldsymbol{\Upsilon}_\infty = \{Y_i,\, i \in \mathbb{N}\}$.

Let us assume that we have two maximizers $s_1 \neq s_2$. Since $\boldsymbol{\Upsilon}_\infty$ is dense in $[0,1] \times [-\sqrt{2B}, \sqrt{2B}]$ we have that $I(s_1) \neq I(s_2)$. In particular, there exists i_0 such that $Y_{i_0} \in I(s_1)$ and $Y_{i_0} \notin I(s_2)$, and since s_1 and s_2 are two maximizers of (4.14) it means

$$\max_{s:\, Y_{i_0}\in I(s)} t_\beta(s) = \max_{s:\, Y_{i_0}\notin I(s)} t_\beta(s).$$

This implies that

$$\beta M_{i_0} = \max_{s:\, Y_{i_0}\notin I(s)} t_\beta(s) - \max_{s:\, Y_{i_0}\in I(s)} \left\{ \beta \sum_{j, j\neq i_0} M_j 1_{\{Y_j\in s\}} - \mathrm{Ent}(s) \right\}. \tag{4.15}$$

Conditioning on $(Y_j)_{j\in\mathbb{N}}$ and $(M_j)_{j\in\mathbb{N}, j\neq i_0}$ we have that the l.h.s. has a continuous distribution – the distribution of $M_{i_0}^{-\alpha}$ conditional on $(Y_j)_{j\in\mathbb{N}}$ and $(M_j)_{j\in\mathbb{N}, j\neq i_0}$ is uniform on

the interval $[M_{i_0-1}^{-\alpha}, M_{i_0+1}^{-\alpha}]$ –, while the r.h.s. is a constant – it is independent of M_{i_0}. Therefore the event (4.15) has zero probability, and by sigma sub-additivity we get that $\mathbb{P}\big(I(s_1) \neq I(s_2)\big) = 0$, which contradicts the existence of two distinct maximizers. □

Remark 4.5. Let us now explain how the fact that the maximizer of $\mathcal{T}_\beta$ is unique would help showing path concentration as in Theorem 2.1 of [5]. For any $n, h \in \mathbb{N}$ and $\beta > 0$, we consider $T_{n,h}^\beta$ defined in (5.2) below, which is the discrete analogous of $\mathcal{T}_\beta$. Then, with a similar approach as above, one should be able to show that

$$\Delta_{n,h}^\beta := \operatorname*{arg\,max}_{\Delta \subset \Lambda_{n,h}} T_{n,h_n}^\beta$$

is well defined. For any $q > 0$ we let $s_{n,qh}^\beta$ be the linear interpolation of $\Delta_{n,qh}^\beta/(n \times h)$. At this stage, the path concentration would be a consequence of the following two statements.

Conjecture 4.6. *Let $h = h_n$ and $\beta = \beta_n$ as in* (2.2)*. Then,*

$$\lim_{q\to\infty} \lim_{n\to\infty} s_{n,h_n}^{\beta_n} = s_1^*$$

Conjecture 4.7. *For any $\varepsilon > 0$ it holds that*

$$\lim_{q\to\infty} \lim_{n\to\infty} \mathbb{P}\bigg(\mathbf{P}_{n,\beta_n}^\omega \Big(\|s_{n,h_n}^{\beta_n} - S\|_\infty \geqslant \varepsilon \Big) \geqslant \varepsilon \bigg) = 0.$$

The approach to proving Conjectures 4.6 and 4.7 should be similar to that used in [5].

5. Application II: discrete heavy-tail E-LPP

5.1. **Heavy-tail E-LPP and ordered statistics.** Here, we give some results that will be central in many parts of the sequel. Let us consider a box $\Lambda_{n,h} = [\![1, n]\!] \times [\![-h, h]\!]$. For any set $\Delta \subset \Lambda_{n,h}$, we can define the (discrete) energy collected by Δ by

$$\Omega_{n,h}(\Delta) := \sum_{(i,x)\in\Delta} \omega_{i,x}\,, \tag{5.1}$$

and we are interested in the (discrete) variational problem,

$$T_{n,h}^{\beta_{n,h}} := \max_{\Delta\subset\Lambda_{n,h}} \big\{\beta_{n,h}\Omega_{n,h}(\Delta) - \mathrm{Ent}(\Delta)\big\}\,, \tag{5.2}$$

with $\beta_{n,h}$ some function of n, h (soon to be specified), and $\mathrm{Ent}(\Delta)$ is defined in (3.1).

We may rewrite the disorder in this region $(\omega_{i,x})_{(i,x)\in\Lambda_{n,h}}$ using the *ordered statistic*: we let $M_r^{(n,h)}$ be the r-th largest value of $(\omega_{i,x})_{(i,x)\in\Lambda_{n,h}}$ and $Y_r^{(n,h)} \in \Lambda_{n,h}$ its position. In such a way

$$(\omega_{i,j})_{(i,j)\in\Lambda_n} = (M_r^{(n,h)}, Y_r^{(n,h)})_{r=1}^{|\Lambda_{n,h}|}\,. \tag{5.3}$$

In the following we refer to $(M_r^{(n,h)})_{r=1}^{|\Lambda_{n,h}|}$ as the *weight* sequence. Note also that $(Y_r^{(n,h)})_{r=1}^{|\Lambda_{n,h}|}$ is simply a random permutation of the points of $\Lambda_{n,h}$. Let us state right away a lemma that will prove to be useful in the rest of the paper.

Lemma 5.1. *For any $\eta > 0$, there exists a constant c such that, for any $t > 1$ and any $\ell \leqslant nh$, we have*

$$\mathbb{P}\Big(M_\ell^{(n,h)} > tm(\tfrac{nh}{\ell}) \Big) \leqslant (ct)^{-(1-\eta)\alpha\ell}\,.$$

Proof. We simply write that by a union bound

$$\mathbb{P}\Big(M_r^{(n,h)} > tm(\tfrac{nh}{r})\Big) \leqslant \binom{nh}{r}\mathbb{P}\Big(\omega_1 > tm(\tfrac{nh}{r})\Big)^r \leqslant \Big(c\frac{nh}{r}\mathbb{P}\big(\omega_1 > tm(\tfrac{nh}{r})\big)\Big)^r .$$

Then, since $\mathbb{P}(\omega_1 > x)$ is regularly varying with exponent $-\alpha$, we get that there is a constant c_η such that for any $t \geqslant 1$

$$\mathbb{P}\big(\omega_1 > tm(\tfrac{nh}{r})\big) \leqslant c_\eta t^{-(1-\eta)\alpha}\mathbb{P}\big(\omega_1 > m(\tfrac{nh}{r})\big) = c_\eta t^{-(1-\eta)\alpha}\,\frac{nh}{r}\,,$$

where we used the definition of $m(\cdot)$ in the last identity. This concludes the proof. □

The ordered statistics allows us to redefine the energy collected by a set $\Delta \subset \Lambda_{n,h}$, and its contribution by the first ℓ weights (with $1 \leqslant \ell \leqslant |\Lambda_{n,h}|$) by

$$\Omega_{n,h}(\Delta) := \sum_{r=1}^{|\Lambda_{n,h}|} M_r^{(n,h)} 1_{\{Y_r^{(n,h)} \in \Delta\}}\,, \quad \text{and} \quad \Omega_{n,h}^{(\ell)}(\Delta) := \sum_{r=1}^{\ell} M_r^{(n,h)} 1_{\{Y_r^{(n,h)} \in \Delta\}}\,. \tag{5.4}$$

We also set $\Omega_{n,h}^{(>\ell)}(\Delta) := \Omega_{n,h}(\Delta) - \Omega_{n,h}^{(\ell)}(\Delta)$. We then define analogues of (5.2) with a restriction to the ℓ largest weights, or beyond the ℓ-th weight

$$\begin{aligned} T_{n,h}^{\beta_{n,h},(\ell)} &:= \max_{\Delta\subset\Lambda_{n,h}} \big\{\beta_{n,h}\Omega_{n,h}^{(\ell)}(\Delta) - \mathrm{Ent}(\Delta)\big\}\,, \\ T_{n,h}^{\beta_{n,h},(>\ell)} &:= \max_{\Delta\subset\Lambda_{n,h}} \big\{\beta_{n,h}\Omega_{n,h}^{(>\ell)}(\Delta) - \mathrm{Ent}(\Delta)\big\}\,. \end{aligned} \tag{5.5}$$

In this Section, we prove the following theorem.

Theorem 5.2. *The following hold true:*

- *For any $a < \alpha$, there is a constant $c_a > 0$ such that for any $1 \leqslant \ell \leqslant nh$, for any $b > 1$*

$$\mathbb{P}\Big(T_{n,h}^{\beta_{n,h},(\ell)} \geqslant b \times \big(\beta_{n,h}m(nh)\big)^{4/3}\Big(\frac{n}{h^2}\Big)^{1/3}\Big) \leqslant c_a\, b^{-3a/4}\,. \tag{5.6}$$

- *We also have that there is a constant $c > 0$ such that for any $b > 1$*

$$\mathbb{P}\Big(T_{n,h}^{\beta_{n,h},(>\ell)} \geqslant b \times \big(\beta_{n,h}m(nh/\ell)\big)^{4/3}\Big(\frac{\ell^2 n}{h^2}\Big)^{1/3}\Big) \leqslant cb^{-\alpha\ell/4} + e^{-cb^{1/4}}\,. \tag{5.7}$$

The proofs are inspired by the techniques of the previous section, but we need here to keep track of the dependence on n, h – the proof is postponed to Section 5.3. To that end, estimates obtained in Section 3 will be crucial. Note already that if $\frac{n}{h^2}\beta_{n,h}m(nh) \to \beta \in (0,\infty)$, as $n, h \to \infty$, it gives that $T_{n,h}^{\beta_{n,h},(\ell)}$ is of order $\beta^4 h^2/n$ (the lower bound can easily be found thanks to Theorem 3.1).

5.2. **Continuum limit of the discrete heavy-tail E-LPP.** In this section, we consider the limit when $n, h \to \infty$.

If we rescale $\Lambda_{n,qh}$ by $n \times h$, and we let $(\widetilde{Y}_r^{(n,h)})_{r=1}^{|\Lambda_{n,h}|}$ be the rescaled permutation, *i.e.* a random permutation of the points of the set $([0,1]\times[-q,q]) \cap (\frac{\mathbb{N}}{n} \times \frac{\mathbb{Z}}{h})$. Then for any fixed $\ell \in \mathbb{N}$,

$$\big(\widetilde{Y}_1^{(n,h)}, \dots, \widetilde{Y}_\ell^{(n,h)}\big) \overset{(\mathrm{d})}{\to} \big(Y_1, \dots, Y_\ell\big), \quad \text{as} \quad n \to \infty, \tag{5.8}$$

where $(Y_i)_{i\in\mathbb{N}}$ is an i.i.d. sequence of uniform random variables on $[0,1]\times[-q,q]$.

To find out a continuum limit for the weight sequence, we use some basic facts of the classical extreme value theory (see e.g., [31]), that is for all $\ell \in \mathbb{N}$,

$$\Big(\widetilde{M}_i^{(n,h)} := \frac{M_i^{(n,h)}}{m(nh)},\, i = 1, \dots, \ell\Big) \overset{\text{(d)}}{\to} \big(M_i,\, i = 1, \dots, \ell\big), \tag{5.9}$$

where $(M_i)_{i\in\mathbb{N}}$ is the *continuum weight sequence*. The sequence $(M_i)_{i\geqslant 1}$ can be defined as $M_i := 2^{1/\alpha}(\mathcal{E}_1 + \cdots + \mathcal{E}_i)^{-\frac{1}{\alpha}}$, where $(\mathcal{E}_i)_{i\in\mathbb{N}}$ is an i.i.d. sequence of exponential random variables of mean 1, independent of the Y_i's.

In such a way $(M_i, Y_i)_{i\in\mathbb{N}}$ is a realization of a Poisson Point Process on $[0,\infty) \times [0,1] \times [-q,q]$ of intensity $\mu(dw,dt,dx) = \frac{\alpha}{2} w^{-\alpha-1} 1_{\{w>0\}} dwdtdx$. Then, we have the following convergence in distribution for the discrete energy-entropy variational problem (5.2).

Proposition 5.3. *Suppose that $\frac{n}{h^2}\beta_{n,h} m(nh) \to \nu \in [0,\infty)$ as $n, h \to \infty$. Then, for every $\alpha \in (1/2, 2)$ and for any $q > 0$ we have the following convergence in distribution*

$$\frac{n}{h^2}\, T_{n,qh}^{\beta_{n,h}} \xrightarrow{\text{(d)}} \mathcal{T}_{\nu,q} := \sup_{s\in\mathscr{M}_q} \big\{\nu\pi(s) - \mathrm{Ent}(s)\big\}, \tag{5.10}$$

where the supremum is taken over the set $\mathscr{M}_q := \{s \in \mathscr{D}, \mathrm{Ent}(s) < \infty, \max_{t\in[0,1]} |s(t)| \leqslant q\}$. We also have the convergence

$$\frac{n}{h^2}\, T_{n,qh}^{\beta_{n,h},(\ell)} \xrightarrow{\text{(d)}} \mathcal{T}_{\nu,q}^{(\ell)}, \tag{5.11}$$

with $\mathcal{T}_{\nu,q}^{\ell}$ defined in (5.13) *below. Moreover,*

$$\mathcal{T}_{\nu,q}^{(\ell)} \overset{\ell\to\infty}{\to} \mathcal{T}_{\nu,q} \qquad \text{and} \qquad \mathcal{T}_{\nu,q} \overset{q\to\infty}{\to} \mathcal{T}_{\nu}\,, \quad a.s. \tag{5.12}$$

Proof. For any $q > 0$, we consider the Poisson Point Process restricted to $[0,1]\times[-q,q]$, and we label its elements according to the ordered statistic $(M_i, Y_i)_{i\in\mathbb{N}}$ (cf. Section 5.2). For any $\Delta \subset [0,1]\times[-q,q]$ we define $\pi^{(\ell)}(\Delta) = \sum_{i=1}^{\ell} M_i 1_{\{Y_i\in\Delta\}}$ and $\pi^{(>\ell)}(\Delta) := \pi(\Delta) - \pi^{(\ell)}(\Delta)$. In analogy with the discrete setting (cf. (5.5)), we define

$$\begin{aligned} \mathcal{T}_{\nu,q}^{(>\ell)} &= \sup_{s\in\mathscr{M}_q} \left\{\nu\pi^{(>\ell)}(s) - \mathrm{Ent}(s)\right\}, \\ \mathcal{T}_{\nu,q}^{(\ell)} &= \sup_{s\in\mathscr{M}_q} \left\{\nu\pi^{(\ell)}(s) - \mathrm{Ent}(s)\right\}. \end{aligned} \tag{5.13}$$

We now show the convergence (5.11) of the large-weights variational problem, before we prove the convergence (5.10).

Convergence of the large weights. Note that the maximum of $T_{n,qh}^{\beta_{n,h},(\ell)}$ and $\mathcal{T}_{\nu,q}^{(\ell)}$ are achieved on $\Upsilon_\ell = \Upsilon_\ell(q)$ and $\mathbf{\Upsilon}_\ell = \mathbf{\Upsilon}_\ell(q)$ respectively, that is

$$\begin{aligned} T_{n,qh}^{\beta_{n,h},(\ell)} &= \max_{\Delta\subset\Upsilon_\ell} \left\{\beta_{n,h}\Omega_{n,h}^{(\ell)}(\Delta) - \mathrm{Ent}(\Delta)\right\}, \\ \mathcal{T}_{\nu,q}^{(\ell)} &= \sup_{\Delta\subset\mathbf{\Upsilon}_\ell} \left\{\nu\pi^{(\ell)}(\Delta) - \mathrm{Ent}(\Delta)\right\}, \end{aligned} \tag{5.14}$$

where we recall, $\Upsilon_\ell(q)$ (resp. $\mathbf{\Upsilon}_\ell(q)$) is the set of the locations of the ℓ largest weights inside $\Lambda_{n,qh}$ (resp. $\mathbf{\Lambda}_{1,q}$). Since we have only a finite number of points, the proof is a consequence of (5.8) and (5.9) and the Skorokhod representation theorem.

Restriction to the large weights. To show the convergence (5.10), it is therefore enough to control the contribution of the large weights Let $\delta > 0$ such that $\frac{1}{\alpha} - \frac{1}{2} > \delta$. Using Potter's

bound (cf. [9]) we have that

$$\big(\beta_{n,h}m(nh/\ell)\big)^{\frac{4}{3}}\Big(\frac{\ell^2 n}{h^2}\Big)^{\frac{1}{3}} \leqslant c\frac{h^2}{n}\ell^{-\frac{4}{3}(\frac{1}{\alpha}-\frac{1}{2}-\delta)}.$$

Plugging it into (5.7) and taking $b = b_{\ell,\varepsilon} := \varepsilon\, \ell^{\frac{4}{3}(\frac{1}{\alpha}-\frac{1}{2}+\delta)}$, we obtain that

$$\mathbb{P}\Big(\frac{n}{h^2}T_{n,qh}^{\beta_{n,h},(>\ell)} \geqslant \varepsilon\Big) \leqslant c' b_{\ell,\varepsilon}^{-\alpha\ell/4} + e^{-c' b_{\ell,\varepsilon}^{1/4}} \overset{\ell\to\infty}{\longrightarrow} 0, \tag{5.15}$$

uniformly on n, h. Conbined with (5.11) and (5.12), this concludes the convergence (5.10).

Proof of the convergence (5.12). We show that $\mathcal{T}_{\nu,q}^{(>\ell)} \to 0$ as $\ell \to \infty$. Reasoning as in (4.7)–(4.10) we obtain that for any $a < \alpha$ and $t > 1$ it holds that

$$\mathbb{P}\Big(\sup_{\mathrm{Ent}(s)\leqslant 2} \pi^{(>\ell)}(s) > t\Big) \leqslant t^{-c2^{\ell\delta}} + ct^{-a}2^{\ell(-a/\alpha+a\delta)} \leqslant c' t^{-a}2^{-c_a\ell}.$$

This implies, reasoning as (4.4)–(4.6) that

$$\mathbb{P}\big(\mathcal{T}_{\nu,q}^{(>\ell)} > t\big) \leqslant c' t^{-a\left(1-\frac{1}{2\alpha}\right)}2^{-c'\ell}, \tag{5.16}$$

which proves that $\mathbb{E}[(\mathcal{T}_{\nu,q}^{(>\ell)})^\upsilon] \overset{\ell\to\infty}{\to} 0$ for any $\upsilon < \alpha - 1/2$. This implies that $\mathcal{T}_{\nu,q}^{(>\ell)} \to 0$ in probability by Markov's inequality, hence the a.s. convergence along a subsequence, which in turns implies the a.s. convergence thanks to the monotonicity in ℓ.

We now prove that $\mathcal{T}_{\nu,q} \to \mathcal{T}_\nu$ as $q \to \infty$. First, notice that $\mathcal{T}_\nu \geqslant \mathcal{T}_{\nu,q} \geqslant \mathcal{T}_\nu([0, q^2])$ (recall the definition (4.3)). Then, the conclusion follows from Lemma 4.2, which implies that a.s. there exists $k_0 > 0$ such that for any $k > k_0$ it holds that $\mathcal{T}_\nu = \mathcal{T}_\nu([0, k])$. Hence, a.s. $\mathcal{T}_\nu = \mathcal{T}_{\nu,q}$ for q large enough. □

5.3. **Proof of Theorem 5.2.** Let us first focus on $T_{n,h}^{\beta_{n,h},(\ell)}$. As in (4.3) in the continuous setting, we introduce, for any interval $[c, d)$,

$$T_{n,h}^{\beta_{n,h},(\ell)}\big([c, d)\big) := \max_{\Delta\subset\Lambda_{n,h},\mathrm{Ent}(\Delta)\in[c,d)} \big\{\beta_{n,h}\Omega_{n,h}^{(\ell)}(\Delta) - \mathrm{Ent}(\Delta)\big\}. \tag{5.17}$$

Then, we realize that for any $d > 0$

$$T_{n,h}^{\beta_{n,h},(\ell)} = T_{n,h}^{\beta_{n,h},(\ell)}\big([0, d)\big) \vee \sup_{k\geqslant 1} T_{n,h}^{\beta_{n,h},(\ell)}\big([2^{k-1}d, 2^k d)\big).$$

Using that

$$T_{n,h}^{\beta_{n,h},(\ell)}\big([2^{k-1}d, 2^k d)\big) \leqslant \beta_{n,h} \sup_{\Delta\,:\,\mathrm{Ent}(\Delta)\leqslant 2^k d} \Omega_{n,h}^{(\ell)}(\Delta) - 2^{k-1}d, \quad \text{for } k \geqslant 1,$$

$$T_{n,h}^{\beta_{n,h},(\ell)}\big([0, d)\big) \leqslant \beta_{n,h} \sup_{\Delta\,:\,\mathrm{Ent}(\Delta)\leqslant d} \Omega_{n,h}^{(\ell)}(\Delta),$$

with the choice $d = b\,\widehat{\beta}$ and $\widehat{\beta} := (\beta_{n,h}m(nh))^{4/3}(n/h^2)^{1/3}$, a union bound gives that

$$\begin{aligned}\mathbb{P}\Big(T_{n,h}^{\beta_{n,h},(\ell)} \geqslant b\widehat{\beta}\Big) &\leqslant \sum_{k\geqslant 0}\mathbb{P}\Big(\beta_{n,h} \sup_{\Delta\,:\,\mathrm{Ent}(\Delta)\leqslant 2^k b\widehat{\beta}} \Omega_{n,h}^{(\ell)}(\Delta) \geqslant 2^{k-1}b\widehat{\beta}\Big)\\ &\leqslant \sum_{k\geqslant 0}\mathbb{P}\Big(\sup_{\Delta\,:\,\mathrm{Ent}(\Delta)\leqslant 2^k b\widehat{\beta}} \Omega_{n,h}^{(\ell)}(\Delta) \geqslant 2^{k-1}b\, m(nh)\big(\widehat{\beta}n/h^2\big)^{1/4}\Big),\end{aligned} \tag{5.18}$$

where we use that $\widehat{\beta}$ satisfies the equation $\widehat{\beta} = \beta_{n,h}m(nh)(\widehat{\beta}n/h^2)^{1/4}$.

We then need the following lemma, analogous to Lemma 4.1.

Lemma 5.4. *For any $a < \alpha$, there exists a constant c such that for any $B \geqslant 1$, $n, h \geqslant 1$, and any $t > 1$*

$$\mathbb{P}\Big(\sup_{\Delta\,:\,\mathrm{Ent}(\Delta)\leqslant B} \Omega_{n,h}^{(\ell)}(\Delta) \geqslant t \times m(nh)\big(Bn/h^2\big)^{1/4}\Big) \leqslant ct^{-a}$$

Applying this lemma in (5.18) (with $B = 2^k b\widehat{\beta}$, $t = 2^{3k/4-1}b^{3/4}$), we get that for any $k \geqslant 0$

$$\mathbb{P}\Big(\sup_{\Delta\,:\,\mathrm{Ent}(\Delta)\leqslant 2^k b\widehat{\beta}} \Omega_{n,h}^{(\ell)}(\Delta) \geqslant 2^{k-1}bm(nh)(\widehat{\beta}n/h^2)^{1/4}\Big) \leqslant c(2^k b)^{-3a/4}\,,$$

so that summing over k in (5.18), we get Theorem 5.2.

Proof of Lemma 5.4. We mimic here the proof of Lemma 4.1, but we need to keep the dependence on the parameters n, h, B. For $i \geqslant 0$, we denote $\Upsilon_i := \{Y_1^{(n,h)}, \dots, Y_i^{(n,h)}\}$ with the $Y_j^{(n,h)}$ introduced in Section 5.1 ($\Upsilon_0 = \varnothing$), and for any Δ we let $\Delta_i := \Delta \cap \Upsilon_i$ be the restriction of Δ to the i largest weights. As in (4.7), we can write

$$\frac{1}{m(nh)(Bn/h^2)^{1/4}} \times \sup_{\Delta:\mathrm{Ent}(\Delta)\leqslant B} \Omega_{n,h}^{(\ell)}(\Delta) \leqslant \sum_{j=0}^{\log_2 \ell} \widetilde{M}_{2^j}\widetilde{L}_{2^{j+1}}\,, \tag{5.19}$$

where $\widetilde{M}_i = \widetilde{M}_i^{(n,h)} = M_i^{(n,h)}/m(nh)$ and $\widetilde{L}_i = \widetilde{L}_i^{(B)}(n,h) = L_i^{(B)}(n,h)/(Bn/h^2)^{1/4}$ are the renormalized weights and E-LPP (we drop the dependence on n, h, B for notational convenience).

As in the proof of Lemma 4.1, we fix some $\delta > 0$ such that $1/\alpha - 1/2 > 2\delta$, and as for (4.8), the probability in Lemma 5.4 is bounded by

$$\sum_{j=0}^{\log_2 \ell} \Big[\mathbb{P}\Big(\widetilde{L}_{2^{j+1}} > C'\log t\,(2^{j+1})^{1/2+\delta}\Big) + \mathbb{P}\Big(\widetilde{M}_{2^j} > C''\,\frac{t}{\log t}(2^j)^{-1/\alpha+\delta}\Big)\Big]\,. \tag{5.20}$$

For the first probability in the sum, we obtain from Theorem 3.1-(ii) that provided that $C'(\log t)2^{j\delta} \geqslant 2C_0^{1/2}$

$$\mathbb{P}\Big(\widetilde{L}_{2^{j+1}} > C'\log t\,(2^{j+1})^{1/2+\delta)}\Big) \leqslant \Big(\frac{1}{2}\Big)^{C'(\log t)2^{j\delta}} \leqslant t^{-(\log 2)C'2^{j\delta}}\,. \tag{5.21}$$

Then, the first sum in (5.20) is bounded by t^{-a} provided that C' had been fixed large enough.

For the second probability in (5.20), we use Lemma 5.1 above to get that for any $a < \alpha$

$$\mathbb{P}\Big(\widetilde{M}_{2^j} > C''\,\frac{t}{\log t}(2^j)^{-1/\alpha+\delta}\Big) \leqslant \mathbb{P}\Big(M_{2^j}^{(n,h)} > C'''\,\frac{t}{\log t}(2^j)^{\delta/2}m(nh2^{-j})\Big) \tag{5.22}$$

$$\leqslant c(\log t)^a t^{-a}(2^j)^{-a\delta}\,. \tag{5.23}$$

For the first inequality, we used Potter's bound to get that $m(nh2^{-j}) \leqslant cm(nh)(2^j)^{-1/\alpha+\delta/2}$. We conclude that the second sum in (5.20) is bounded by a constant times $(\log t)^a t^{-a}$.

All together, and possibly decreasing the value a a (by an arbitrarily small anount), this yields Lemma 5.4. □

Let us now turn to the case of $T_{n,h}^{\beta_{n,h},(>\ell)}$. We first need an analogue of Lemma 5.4.

Lemma 5.5. *There exists a constant c such that for any $B \geqslant 1$, $n, h \in \mathbb{N}$ and $0 \leqslant \ell \leqslant nh$, for any $t > 1$*

$$\mathbb{P}\Big(\sup_{\Delta\,:\,\mathrm{Ent}(\Delta)\leqslant B} \Omega_{n,h}^{(>\ell)}(\Delta) \geqslant t \times m(nh/\ell)\, \ell^{1/2} \big(Bn/h^2\big)^{1/4} \Big) \leqslant ct^{-\alpha\ell/3} + e^{-c\sqrt{t}}\,.$$

Proof. Analogously to (5.19), we get that

$$\frac{1}{m(nh/\ell)\ell^{1/2}(Bn/h^2)^{1/4}} \times \sup_{\Delta:\mathrm{Ent}(\Delta)\leqslant B} \Omega_{n,h}^{(>\ell)}(\Delta) \leqslant \sum_{j=0}^{\log_2(nh/\ell)} \frac{M_{2^j\ell}^{(n,h)}}{m(nh/\ell)} \frac{L_{2^{j+1}\ell}^{(n,h)}}{\ell^{1/2}(Bn/h^2)^{1/4}}\,, \tag{5.24}$$

Then, we get similarly to (5.21)-(5.22) that for any $\delta > 0$: (a) thanks to Theorem 3.1-(ii) we have

$$\mathbb{P}\Big(\frac{L_{2^{j+1}\ell}^{(n,h)}}{\ell^{1/2}(Bn/h^2)^{1/4}} \geqslant C'\sqrt{t}(2^{j+1})^{1/2+\delta} \Big) \leqslant \Big(\frac{1}{2}\Big)^{C'\sqrt{t}2^{j\delta}} \leqslant e^{-c\sqrt{t}\,2^{\delta j}}\,; \tag{5.25}$$

(b) thanks to Lemma 5.1 we have

$$\mathbb{P}\Big(\frac{M_{2^j\ell}^{(n,h)}}{m(nh/\ell)} \geqslant C''\sqrt{t}(2^j)^{-1/\alpha+\delta} \Big) \leqslant ct^{-\alpha\ell/3}(2^j)^{-\alpha\delta\ell/2}\,. \tag{5.26}$$

Lemma 5.5 follows from a bound analogous to (5.20). □

Then, setting $\widehat{\beta}_\ell = (\beta_{nh} m(nh/\ell))^{4/3}(\ell^2 n/h^2))^{\frac{1}{3}}$ so that $\widehat{\beta}_\ell = \beta_{n,h} m(nh/\ell)\ell^{1/2}(\widehat{\beta}_\ell n/h^2)^{1/4}$, we obtain similarly to (5.18) that

$$\begin{aligned}
\mathbb{P}\Big(T_{n,h}^{\beta_{n,h},(>\ell)} \geqslant b \times \widehat{\beta}_\ell \Big) &\leqslant \sum_{k\geqslant 0} \mathbb{P}\Big(\beta_{n,h} \sup_{\Delta\,:\,\mathrm{Ent}(\Delta)\leqslant 2^k b\widehat{\beta}_\ell} \Omega_{n,h}^{(>\ell)}(\Delta) \geqslant 2^{k-1} b\widehat{\beta}_\ell \Big) \\
&\leqslant \sum_{k\geqslant 0} \mathbb{P}\Big(\sup_{\Delta\,:\,\mathrm{Ent}(\Delta)\leqslant 2^k b\widehat{\beta}_\ell} \Omega_{n,h}^{(>\ell)}(\Delta) \geqslant 2^{k-1} b\; m(nh/\ell)(\ell^2\widehat{\beta}_\ell n/h^2)^{1/4} \Big) \\
&\leqslant \sum_{k\geqslant 0} \Big(c(2^k b)^{-\alpha\ell/4} + e^{-c2^{3k/8}b^{3/8}} \Big) \leqslant c' b^{-\alpha\ell/4} + e^{-c'b^{1/4}}\,.
\end{aligned}$$

This concludes the proof of Theorem 5.2.

Part 2. Polymers in heavy-tail random environment

In this part, we go back to the study of directed polymers in heavy tail random environment, and show our main results, with the help of the E-LPP estimates proven above.

6. Transversal fluctuations: proof of Theorem 2.2

In this section, we consider the case $\alpha \in (1/2, 2)$, and prove Theorem 2.2. First, we partition the interval $[A_n h_n, n]$ into blocks

$$B_{k,n} := [2^{k-1}h_n, 2^k h_n), \quad k = \log_2 A_n + 1, \dots, \log_2(n/h_n) + 1. \tag{6.1}$$

In such a way,

$$\mathbf{P}_{n,\beta_n}^{\omega}\big(\max_{i\leqslant n} |S_i| \geqslant A_n h_n \big) = \sum_{k=\log_2 A_n+1}^{\log_2(n/h_n)} \mathbf{P}_{n,\beta_n}^{\omega}\big(\max_{i\leqslant n} |S_i| \in B_{k,n} \big). \tag{6.2}$$

We first deal with the case where $n/m(nh_n) \overset{n\to\infty}{\to} 0$ for the sake of clarity of the exposition: in that case, $\log \mathbf{Z}^{\omega}_{n,\beta_n}$ does not need to be recentered. We treat the remaining case (in particular we have $\alpha \geqslant 3/2$) in a second step.

6.1. **Case $n/m(nh_n) \overset{n\to\infty}{\to} 0$.** We observe that the assumption $\omega \geqslant 0$ implies that the partition function $\mathbf{Z}^{\omega}_{n,\beta_n}$ is larger than one. Therefore,

$$\mathbf{P}^{\omega}_{n,\beta_n}\big(\max_{i\leqslant n}|S_i| \in B_{k,n}\big) \leqslant \mathbf{Z}^{\omega}_{n,\beta_n}\big(\max_{i\leqslant n}|S_i| \in B_{k,n}\big).$$

By using Cauchy-Schwarz inequality, we get that

$$\mathbf{Z}^{\omega}_{n,\beta_n}\big(\max_{i\leqslant n}|S_i| \in B_{k,n}\big)^2 \leqslant \mathbf{P}\big(\max_{i\leqslant n}|S_i| \geqslant 2^{k-1}h_n\big) \times \mathbf{Z}^{\omega}_{n,2\beta_n}\big(\max_{i\leqslant n}|S_i| \leqslant 2^k h_n\big)\,. \tag{6.3}$$

The first probability is bounded by $2\mathbf{P}(|S_n| \geqslant h_n) \leqslant 4\exp(-2^{2k}h_n^2/2n)$ (by Levy's inequality and a standard Chernov's bound). We are going to show the following lemma, which is the central estimate of the proof.

Lemma 6.1. *There exist some constant $q_0 > 0$ and some $\nu > 0$, such that for all $q \geqslant q_0$ we have*

$$\mathbb{P}\Big(\mathbf{Z}^{\omega}_{n,2\beta_n}\big(\max_{i\leqslant n}|S_i| \leqslant qh_n\big) \geqslant e^{\frac{1}{4}q^2\frac{h_n^2}{n}}\Big) \leqslant q^{-\nu}\Big(1 + 1 \wedge \frac{n}{m(nh_n)}\Big) \tag{6.4}$$

Therefore, if $n/m(nh_n) \overset{n\to\infty}{\to} 0$, this lemma gives that for $c_0 = 1/8$ and for k large enough (*i.e.* A_n large enough), using (6.3),

$$\mathbb{P}\Big(\mathbf{Z}^{\omega}_{n,\beta_n}\big(\max_{i\leqslant n}|S_i| \in B_{k,n}\big) \geqslant 4e^{-c_0 2^{2k}h_n^2/n}\Big) \leqslant (2^k)^{-\nu}\,.$$

Then, using that $\sum_{k>\log_2 A_n} 4e^{-c_0 2^{2k}h_n^2/n} \leqslant e^{-c_1 A_n^2 h_n^2/n}$, we get that by a union bound

$$\begin{aligned}\mathbb{P}\Big(\mathbf{P}^{\omega}_{n,\beta}\big(\max_{i\leqslant n}|S_i| \geqslant A_n h_n\big) \geqslant e^{-c_1 A_n^2 h_n^2/n}\Big)& \\ \leqslant \sum_{k=\log_2 A_n+1}^{\log_2(n/h_n)} &\mathbb{P}\Big(\mathbf{Z}^{\omega}_{n,\beta}\big(\max_{i\leqslant n}|S_i| \in B_{k,n}\big) \geqslant 4e^{-c_0 2^{2k}h_n^2/n}\Big) \\ \leqslant \sum_{k>\log_2 A_n} &2^{-\nu k} \leqslant cA_n^{-\nu}\,.\end{aligned} \tag{6.5}$$

We stress that in the case when $n/m(nh_n) \overset{n\to\infty}{\to} 0$, we do not need the additional n in front of $e^{-c_1 A_n^2 h_n^2/n}$ in (2.9).

Proof of Lemma 6.1. For simplicity, we assume in the following that qh_n is an integer. We fix $\delta > 0$ such that $(1+\delta)/\alpha < 2$ and $(1-\delta)/\alpha > 1/2$, and let

$$\mathtt{T} = \mathtt{T}_n(qh_n) = \frac{h_n^2}{n}q^{1/\alpha}(q^2h_n^2/n)^{-(1-\delta)^{3/2}/\alpha} \vee 1 \tag{6.6}$$

be a truncation level. Note that if $\alpha \leqslant (1-\delta)^{3/2}$ then we have $\mathtt{T} = 1$. We decompose the partition function into three parts: thanks to Hölder's inequality, we can write that

$$\log \mathbf{Z}^{\omega}_{n,2\beta_n}\big(\max_{i\leqslant n}|S_i| \leqslant qh_n\big) \leqslant \frac{1}{3}\log \mathbf{Z}^{(>\mathtt{T})}_{n,6\beta_n} + \frac{1}{3}\log \mathbf{Z}^{((1,\mathtt{T}])}_{n,6\beta_n} + \frac{1}{3}\log \mathbf{Z}^{(\leqslant 1)}_{n,6\beta_n}\,, \tag{6.7}$$

where the three partition functions correspond to three ranges for the weights $\beta_n\omega_{i,S_i}$:

$$\mathbf{Z}^{(>\mathtt{T})}_{n,6\beta_n} := \mathbf{E}\Big[\exp\Big(\sum_{i=1}^{n} 6\beta_n\omega_{i,S_i}1_{\{\beta_n\omega_{i,S_i}>\mathtt{T}\}}\Big)1_{\{\max_{i\leqslant n}|S_i|\leqslant qh_n\}}\Big] \tag{6.8}$$

$$\mathbf{Z}^{((1,\mathtt{T}])}_{n,6\beta_n} := \mathbf{E}\Big[\exp\Big(\sum_{i=1}^{n} 6\beta_n\omega_{i,S_i}1_{\{\beta_n\omega_{i,S_i}\in(1,\mathtt{T}]\}}\Big)1_{\{\max_{i\leqslant n}|S_i|\leqslant qh_n\}}\Big] \tag{6.9}$$

$$\mathbf{Z}^{(\leqslant 1)}_{n,6\beta_n} := \mathbf{E}\Big[\exp\Big(\sum_{i=1}^{n} 6\beta_n\omega_{i,S_i}1_{\{\beta_n\omega_{i,S_i}\leqslant 1\}}\Big)1_{\{\max_{i\leqslant n}|S_i|\leqslant qh_n\}}\Big]. \tag{6.10}$$

We now show that with high probability, these three partition functions cannot be large. Note that when $\mathtt{T}=1$, the second term is equal to 1 and we do not have to deal with it.

Term 1. For (6.8), we prove that for any $\nu < 2\alpha-1$, for q sufficiently large, for all n large enough we have

$$\mathbb{P}\Big(\log \mathbf{Z}^{(>\mathtt{T})}_{n,6\beta_n} \geqslant c_0 q^2\frac{h_n^2}{n}\Big) \leqslant q^{-\nu}. \tag{6.11}$$

We compare this truncated partition function with the partition function where we keep the first ℓ weights in the ordered statistics $(M_i^{(n,qh_n)})_{1\leqslant i\leqslant nqh_n}$. Define

$$\ell = \ell_n(qh_n) := (q^2h_n^2/n)^{1-\delta}, \quad \text{so } \mathtt{T} = \frac{h_n^2}{n}q^{1/\alpha}\times \ell^{-(1-\delta)^{1/2}/\alpha}, \tag{6.12}$$

and set

$$\mathbf{Z}^{(\ell)}_{n,6\beta_n} := \mathbf{E}\Big[\exp\Big(\sum_{i=1}^{\ell} 6\beta_n M_i^{(n,qh_n)}1_{\{Y_i^{(n,qh_n)}\in S\}}\Big)\Big]. \tag{6.13}$$

Remark that, with the definition of $\mathtt{T}$ and thanks to the relation (2.2) verified by β_n, we have that for n large enough

$$\mathbb{P}\Big(\beta_n M_\ell^{(n,qh_n)} > \mathtt{T}\Big) \leqslant \mathbb{P}\Big(M_\ell^{(n,qh_n)} \geqslant \frac{1}{2}q^{1/\alpha}\ell^{-(1-\delta)^{1/2}/\alpha}m(nh_n)\Big)$$

Then, since we have $q/\ell \leqslant 1$ (see (6.12)), we can use Potter's bound to get that for n sufficiently large

$$m\big(nqh_n/\ell\big) \leqslant (q/\ell)^{(1-\delta^2)/\alpha}m(nh_n),$$

and we obtain that provided that δ is small enough

$$\mathbb{P}\Big(\beta_n M_\ell^{(n,qh_n)} > \mathtt{T}\Big) \leqslant \mathbb{P}\Big(M_\ell^{(n,qh_n)} \geqslant c_0 q^{\delta^2/\alpha}\ell^{\delta^2/\alpha}m\big(nqh_n/\ell\big)\Big) \leqslant (cq\ell)^{-\delta^2\ell/2},$$

where we used Lemma 5.1 for the last inequality. We therefore get that, with probability larger than $1-(c\ell)^{-\delta\ell/2}$ (note that $\ell^{-\delta\ell/2}\leqslant q^{-\delta\ell/2}\leqslant q^{-4}$ for n large enough), we have that

$$\Big\{(i,x)\in[\![1,n]\!]\times[\![-qh_n,qh_n]\!];\beta_n\omega_{i,x}>\mathtt{T}\Big\} \subset \Upsilon_\ell := \big\{Y_1^{(n,qh_n)},\dots,Y_\ell^{(n,qh_n)}\big\}, \tag{6.14}$$

and hence $\mathbf{Z}^{(>T)}_{n,6\beta_n}\leqslant \mathbf{Z}^{(\ell)}_{n,6\beta_n}$.

We are therefore left to focus on the term $\mathbf{Z}^{(\ell)}_{n,6\beta_n}$: recalling the definitions (5.4) and (5.5), we get that

$$\begin{aligned}\mathbf{Z}^{(\ell)}_{n,6\beta_n} &= \sum_{\Delta\subset\Upsilon_\ell} e^{6\beta_n\Omega^{(\ell)}_{n,qh_n}(\Delta)}\mathbf{P}\big(S\cap\Upsilon_m=\Delta\big)\\ &\leqslant \sum_{\Delta\subset\Upsilon_\ell}\exp\big(6\beta_n\Omega_{n,qh_n}(\Delta)-\mathrm{Ent}(\Delta)\big) \leqslant 2^\ell\exp\Big(T^{6\beta_n,(\ell)}_{n,qh_n}\Big),\end{aligned} \tag{6.15}$$

where we used that $\mathbf{P}(\Delta \subset S) \leqslant \exp(-\mathrm{Ent}(\Delta))$ as noted before in Section 3.2. Note that we have $\ell \leqslant \frac{1}{2}c_0 q^2 h_n^2/n$ for n large enough (and $q \geqslant 1$), so we get that

$$\mathbb{P}\Big(\log \mathbf{Z}^{(\ell)}_{n,6\beta_n} \geqslant c_0 q^2 \frac{h_n^2}{n} \Big) \leqslant \mathbb{P}\Big(T^{6\beta_n,(\ell)}_{n,qh_n} \geqslant \frac{1}{2} c_0 q^2 \frac{h_n^2}{n} \Big) .$$

Then, by the definition (2.2) and thanks to Potter's bound, for any $\eta > 0$ there exists a constant c_η such that for any $q \geqslant 1$

$$\frac{\big(6\beta_n m(nqh_n)\big)^{4/3}}{(q^2h_n^2/n)^{1/3}} \leqslant c_\eta q^{(1+\eta)\frac{4}{3\alpha}-\frac{2}{3}} \frac{h_n^2}{n} = c_\eta (q^{4/3})^{(1+\eta)/\alpha-2} \times q^2 \frac{h_n^2}{n} ,$$

where we used that for any $\eta > 0$, $m(nqh_n) \leqslant c'_\eta q^{(1+\eta)/\alpha} m(nh_n)$ provided that n is large enough (Potter's bound). Therefore, provided that η is small enough so that $(1+\eta)/\alpha < 2$, an application of Theorem 5.2 gives that for q large enough (so that $b_q := \frac{c_0}{2c_\eta}(q^{4/3})^{2-(1+\eta)/\alpha}$ is large),

$$\mathbb{P}\Big(T^{6\beta_n,(\ell)}_{n,qh_n} \geqslant \frac{1}{2} c_0 q^2 \frac{h_n^2}{n} \Big) \leqslant \mathbb{P}\Big(T^{6\beta_n,(\ell)}_{n,qh_n} \geqslant b_q \times \frac{\big(6\beta_n m(nqh_n)\big)^{4/3}}{(q^2h_n^2/n)^{1/3}} \Big) \leqslant cq^{-\nu} , \tag{6.16}$$

with $\nu = 2\alpha - 1 - 2\eta$. This gives (6.11), since η is arbitrary.

Term 2. We now turn to (6.9) We consider only the case $\mathtt{T} > 1$ (and in particular we have $\alpha > (1-\delta)^{3/2}$). We show that for any $\eta > 0$, there is a constant $c_\eta > 0$ such that for q large enough and n large enough,

$$\mathbb{P}\Big(\log \mathbf{Z}^{((1,\mathtt{T}])}_{n,6\beta_n} \geqslant c_0 \big(q^2h_n^2/n\big)^{1-\eta} \Big) \leqslant \exp\big(- c_\eta (q^2h_n^2/n)^{1/3} \big) . \tag{6.17}$$

Again, we need to decompose $\mathbf{Z}^{((1,\mathtt{T}])}_{n,6\beta_n}$ according to the values of the weights. We set $\theta := (1-\delta)2/\alpha > 1$, and let

$$\ell_j := (q^2h_n^2/n)^{\theta^j(1-\delta)} = (\ell_0)^{\theta^j} , \quad \text{with } \ell_0 = \ell = (q^2h_n^2/n)^{1-\delta} \text{ as in (6.12)} \tag{6.18}$$

$$\mathtt{T}^{(j)} := \frac{h_n^2}{n} q^{1/\alpha} \times (q^2h_n^2/n)^{-\theta^j(1-\delta)^{3/2}/\alpha} = \frac{h_n^2}{n} q^{1/\alpha} \big(\ell_j\big)^{-(1-\delta)^{1/2}/\alpha} \tag{6.19}$$

for $j \in \{0, \dots, \kappa\}$ with κ the first integer such that $\theta^\kappa > \alpha/(1-\delta)^{3/2}$. We get that $\mathtt{T}^{(0)} = \mathtt{T}$, and $\mathtt{T}^{(\kappa)} < 1$. Then, thanks to Hölder inequality we may write

$$\log \mathbf{Z}^{((1,T])}_{n,6\beta_n} \leqslant \frac{1}{\kappa} \sum_{j=1}^{\kappa} \log \mathbf{Z}^{((\mathtt{T}^{(j)},\mathtt{T}^{(j-1)}])}_{n,6\kappa\beta_n} ,$$

$$\text{with} \quad \mathbf{Z}^{((\mathtt{T}^{(j)},\mathtt{T}^{(j-1)}])}_{n,6\kappa\beta_n} := \mathbf{E}\Big[\exp\Big(\sum_{i=1}^{n} 6\kappa\beta_n \omega_{i,S_i} 1_{\{\beta_n\omega_{i,S_i} \in (\mathtt{T}^{(j)},\mathtt{T}^{(j-1)}]\}} \Big) 1_{\{\max_{i\leqslant n}|S_i| \leqslant qh_n\}} \Big] .$$

To prove (6.17), it is therefore enough to prove that for any $1 \leqslant j \leqslant \kappa$, since $\ell_j \geqslant (q^2h_n^2/n)^{1-\delta}$,

$$\mathbb{P}\Big(\log \mathbf{Z}^{((\mathtt{T}^{(j)},\mathtt{T}^{(j-1)}])}_{n,6\kappa\beta_n} \geqslant 8\kappa \big(q^2h_n^2/n\big) \ell_j^{-\delta/10} \Big) \leqslant \exp\big(- c(q^2h_n^2/n)^{1/3} \big) . \tag{6.20}$$

First of all, we notice that in view of (6.18)-(6.19), with the same computation leading to (6.14), we have that with probability larger than $1 - (c\ell_j)^{-\delta\ell_j/4}$

$$\Big\{ (i,x) \in [\![1,n]\!] \times [\![-qh_n, qh_n]\!] ; \beta_n\omega_{i,x} > \mathtt{T}^{(j-1)} \Big\} \subset \Upsilon_{\ell_j} := \big\{ Y_1^{(n,qh_n)}, \dots, Y_{\ell_j}^{(n,qh_n)} \big\} . \tag{6.21}$$

On this event, and using that $\ell_j = (\ell_{j-1})^{(1-\delta)2/\alpha}$ and

$$\mathtt{T}^{(j-1)} = \frac{h_n^2}{n} q^{1/\alpha} \ell_j^{-(1-\delta)^{-1/2}/2} \leqslant \frac{h_n^2}{n} q^{1/\alpha} \ell_j^{-1/2-\delta/5}$$

(if δ is small), we have

$$\begin{aligned} \mathbf{Z}_{n,6\kappa\beta_n}^{((\mathtt{T}^{(j)},\mathtt{T}^{(j-1)}])} &\leqslant \mathbf{E}\Big[\exp\Big(6\kappa \mathtt{T}^{(j-1)} \sum_{i=1}^{\ell_j} 1_{\{Y_i^{(n,qh_n)} \in S\}} \Big) \Big] \qquad (6.22) \\ &\leqslant e^{6\kappa q^2 \frac{h_n^2}{n} \ell_j^{-\delta/10}} + \sum_{k = q^{2-\frac{1}{\alpha}} \ell_j^{1/2+\delta/10}}^{\ell_j} \sum_{\Delta \subset \Upsilon_{\ell_j}; |\Delta| = k} e^{6\kappa \frac{h_n^2}{n} q^{1/\alpha} \ell_j^{-1/2-\delta/5} k} \, \mathbf{P}\big(S \cap \Upsilon_{\ell_j} = \Delta\big) \\ &\leqslant e^{6\kappa q^2 \frac{h_n^2}{n} \ell_j^{-\delta/10}} + \sum_{k = q^{2-\frac{1}{\alpha}} \ell_j^{1/2+\delta/10}}^{\ell_j} \binom{\ell_j}{k} \exp\Big(6\kappa \frac{h_n^2}{n} q^{1/\alpha} \ell_j^{-1/2-\delta/5} k - \inf_{\Delta \subset \Upsilon_{\ell_j}, |\Delta| = k} \mathrm{Ent}(\Delta) \Big) . \end{aligned}$$

Then, we may bound $\binom{\ell_j}{k} \leqslant e^{k \log \ell_j}$. We notice from the definition of κ (and since $\theta \in (1,2)$) that there exists some $\eta > 0$ such that $\ell_j \leqslant \ell_\kappa \leqslant (q^2 h_n^2/n)^{2-\eta}$ for any $1 \leqslant j \leqslant \kappa$: it shows in particular that $\log \ell_j \leqslant \ell_j^{\delta^2} \leqslant q^2 \frac{h_n^2}{n} \ell_j^{-1/2-\delta/5}$, provided that n is sufficiently large and δ has been fixed sufficiently small. We end up with the following bound

$$\mathbf{Z}_{n,6\kappa\beta_n}^{((\mathtt{T}^{(j)},\mathtt{T}^{(j-1)}])} \leqslant e^{6\kappa q^2 \frac{h_n^2}{n} \ell_j^{-\delta/10}} + \sum_{k = q^{2-\frac{1}{\alpha}} \ell_j^{1/2+\delta/10}}^{\ell_j} \exp\Big(c q^2 \frac{h_n^2}{n} \ell_j^{-1/2-\delta/5} k - \inf_{\Delta \subset \Upsilon_{\ell_j}, |\Delta| = k} \mathrm{Ent}(\Delta) \Big).$$

Then, we may use (3.9) (with $m = \ell_j$, $h = qh_n$) to get that, for any $k \geqslant q^{2-\frac{1}{\alpha}} \ell_j^{1/2+\delta/10}$

$$\begin{aligned} \mathbb{P}\Big(\inf_{\Delta \subset \Upsilon_{\ell_j}, |\Delta| = k} \mathrm{Ent}(\Delta) \leqslant 2cq^2 \frac{h_n^2}{n} \ell_j^{-1/2-\delta/5} k \Big) &\leqslant \Big(\frac{C_0 (2c\ell_j^{-1/2-\delta/5} k)^{1/2} \ell_j}{k^2} \Big)^k \\ &\leqslant \big(c q^{\frac{3}{2\alpha}-3} \ell_j^{-\delta/4} \big)^k \leqslant \big(c\ell_j \big)^{-\delta k/4} . \qquad (6.23) \end{aligned}$$

For the last inequality, we used that $q^{\frac{3}{2\alpha}-3} \leqslant 1$, since $\alpha > 1/2$ and $q \geqslant 1$. Since we have that $q^2 \frac{h_n^2}{n} \ell_j^{-1/2-\delta/5} \geqslant 1$, we get that there is a constant $c' > 0$ such that

$$\sum_{k \geqslant q^{2-\frac{1}{\alpha}} \ell_j^{1/2+\delta/10}} e^{-cq^2 \frac{h_n^2}{n} \ell_j^{-1/2-\delta/5} k} \leqslant c' e^{-cq^2 \frac{h_n^2}{n} \ell_j^{-\delta/10}} \leqslant c'.$$

Using (6.23), we therefore obtain, via a union bound (also recalling (6.21)), that provided that n is large enough

$$\begin{aligned} \mathbb{P}\Big(\mathbf{Z}_{n,6\kappa\beta_n}^{((\mathtt{T}^{(j)},\mathtt{T}^{(j-1)}])} \geqslant e^{8\kappa q^2 \frac{h_n^2}{n} \ell_j^{-\delta/10}} \Big) &\leqslant (c\ell_j)^{-\delta \ell_j/4} + \sum_{k \geqslant q^{2-\frac{1}{\alpha}} \ell_j^{1/2+\delta/10}} \big(c\ell_j \big)^{-\delta k/4} \\ &\leqslant \big(c\ell_j \big)^{-c_\delta \ell_j^{1/2}} . \end{aligned}$$

This proves (6.20) since $\ell_j \geqslant \ell_0 = (q^2 h_n^2/n)^{1-\delta}$.

Term 3. For the last part (6.10), we prove that for arbitrary $\eta > 0$,

$$\mathbb{P}\Big(\log \mathbf{Z}^{(\leqslant 1)}_{n,6\beta_n} \geqslant c_0 q^2 \frac{h_n^2}{n} \Big) \leqslant cq^{-2} \times \begin{cases} \frac{n}{m(nh_n)} & \text{if } \alpha > 1\,, \\ \frac{n}{m(nh_n)^{(1-\eta)\alpha}} & \text{if } \alpha \leqslant 1\,. \end{cases} \tag{6.24}$$

Let us stress that in the case $\alpha \leqslant 1$ we get that for n large $m(nh_n)^{(1-\eta)\alpha} \geqslant (nh_n)^{1-2\eta}$, therefore $n/(nh_n)^{(1-\eta)\alpha}$ goes to 0 provided that η is small enough, since we are considering the case when $h_n \geqslant \sqrt{n}$. Hence, we can replace the upper bound in (6.24) by $1 \wedge (n/m(nh_n))$.

To prove (6.24), we use that $e^{6x1_{\{x\leqslant 1\}}} \leqslant 1 + e^6 x 1_{\{x\leqslant 1\}}$ for any x, and we get that

$$\begin{aligned} \mathbf{Z}^{(\leqslant 1)}_{n,6\beta_n} &\leqslant \mathbf{E}\Big[\prod_{i=1}^n \big(1 + 6e^6 \beta_n \omega_{i,s_i} 1_{\{\beta_n \omega_{i,s_i} \leqslant 1\}}\big) \Big], \\ \text{and } \ \mathbb{E}\mathbf{Z}^{(\leqslant 1)}_{n,6\beta_n} &\leqslant \mathbf{E}\Big[\prod_{i=1}^n \big(1 + 6e^6 \beta_n \mathbb{E}\big[\omega 1_{\{\omega \leqslant 1/\beta_n\}}\big]\big) \Big] \leqslant \exp\Big(6e^6 n\beta_n \mathbb{E}\big[\omega 1_{\{\omega\leqslant 1/\beta_n\}}\big] \Big)\,. \end{aligned} \tag{6.25}$$

Therefore, by Markov inequality and Jensen inequality,

$$\mathbb{P}\Big(\log \mathbf{Z}^{(\leqslant 1)}_{n,6\beta_n} \geqslant c_0 q^2 \frac{h_n^2}{n} \Big) \leqslant \frac{1}{c_0 q^2} \frac{n}{h_n^2} \log \mathbb{E}\mathbf{Z}^{(\leqslant 1)}_{n,6\beta_n} \leqslant C q^{-2} \frac{n^2 \beta_n}{h_n^2} \mathbb{E}\big[\omega 1_{\{\omega\leqslant 1/\beta_n\}}\big]\,. \tag{6.26}$$

It remains to estimate $\mathbb{E}\big[\omega 1_{\{\omega\leqslant 1/\beta_n\}}\big]$. If $\alpha > 1$ then it is bounded by $\mathbb{E}[\omega] < +\infty$: this gives the first part of (6.24), using also (2.2). If $\alpha \leqslant 1$ then for any $\delta > 0$, for n large enough we have $\beta_n \mathbb{E}\big[\omega 1_{\{\omega\leqslant 1/\beta_n\}}\big] \leqslant \beta_n^{(1-\eta)\alpha}$ for n large: by using (2.2) together with $h_n^2/n \geqslant 1$, this gives the second part of (6.24).

The conclusion of Lemma 6.1 follows by collecting the estimates (6.11)-(6.17)-(6.24) of the three terms in (6.7). □

6.2. **Remaining case ($\alpha \geqslant 3/2$).** We now consider the remaining case, *i.e.* when we do not have that $n/m(nh_n) \overset{n\to\infty}{\to} 0$. In particular, we need to have that $\alpha \geqslant 3/2$, and hence $\mathbb{E}[\omega] =: \mu < +\infty$. Then, we do not simply use that $\mathbf{Z}^\omega_{n,\beta_n} \geqslant 1$ to bound $\mathbf{P}^\omega_{n,\beta_n}\big(\max_{i\leqslant n} |S_i| \in B_{k,n}\big)$, but instead we use a re-centered partition function $\bar{\mathbf{Z}}^\omega_{n,\beta_n} = e^{-n\beta_n\mu} \mathbf{Z}^\omega_{n,\beta_n}$, so that we can write

$$\begin{aligned} \mathbf{P}^\omega_{n,\beta_n}\big(\max_{i\leqslant n} |S_i| \in B_{k,n}\big) &= \frac{1}{\bar{\mathbf{Z}}^\omega_{n,\beta_n}} \mathbf{E}\Big[\exp\Big(\sum_{i=1}^n \beta_n(\omega_{i,s_i} - \mu)\Big) 1_{\{\max_{i\leqslant n}|S_i| \in B_{k,n}\}} \Big] \\ &=: \frac{1}{\bar{\mathbf{Z}}^\omega_{n,\beta_n}} \bar{\mathbf{Z}}^\omega_{n,\beta_n}\big(\max_{i\leqslant n} |S_i| \in B_{k,n}\big)\,. \end{aligned} \tag{6.27}$$

First, we need to get a lower bound on $\bar{\mathbf{Z}}^\omega_{n,\beta_n}$.

Lemma 6.2. *For any $\delta > 0$, there is a constant $c > 0$ such that for any positive sequence $\varepsilon_n \leqslant 1$ with $\varepsilon_n \geqslant n^{-1/2}(h_n^2/n)^{\alpha-3/2+\delta}$ (this goes to 0 for δ small enough), and any $n \geqslant 1$*

$$\mathbb{P}\Big(\bar{\mathbf{Z}}^\omega_{n,\beta_n} \geqslant n^{-1}\, e^{\varepsilon_n \frac{h_n^2}{n}} \Big) \geqslant 1 - e^{-c/\varepsilon_n^{\alpha-1/2-\delta}} - e^{-c\varepsilon_n h_n^2/n}. \tag{6.28}$$

We postpone the proof of this lemma to the end of this subsection, and we now complete the proof of Theorem 2.2-(2.9). Lemma 6.2 gives that $\bar{\mathbf{Z}}^\omega_{n,\beta_n} \geqslant n^{-1}$ with overwhelming

probability: using (6.2) combined with (6.27), we get, analogously to (6.5),

$$\mathbb{P}\Big(\mathbf{P}^{\omega}_{n,\beta}\big(\max_{i\leqslant n}|S_i|\geqslant A_nh_n\big)\geqslant ne^{-c_1A_n^2h_n^2/n}\Big)$$
$$\leqslant \mathbb{P}\big(\bar{\mathbf{Z}}^{\omega}_{n,\beta_n}\leqslant n^{-1}\big)+\sum_{k=\log_2 A_n+1}^{\log_2(n/h_n)+1}\mathbb{P}\Big(\bar{\mathbf{Z}}^{\omega}_{n,\beta}\big(\max_{i\leqslant n}|S_i|\in B_{k,n}\big)\geqslant 4e^{-c_0 2^{2k}h_n^2/n}\Big)\,. \tag{6.29}$$

Then, we have a lemma which is the analogous of Lemma 6.1 for $\bar{\mathbf{Z}}^{\omega}_{n,\beta_n}$.

Lemma 6.3. *There exist some constant $q_0>0$ and some $\nu>0$, such that for all $q\geqslant q_0$ we have*

$$\mathbb{P}\Big(\bar{\mathbf{Z}}^{\omega}_{n,2\beta_n}\big(\max_{i\leqslant n}|S_i|\leqslant qh_n\big)\geqslant e^{\frac{1}{4}q^2\frac{h_n^2}{n}}\Big)\leqslant q^{-\nu}\,. \tag{6.30}$$

Proof. The proof follows the same lines as for Lemma 6.1: (6.7) still holds, with $\beta_n\omega_{i,S_i}$ replaced by $\beta_n(\omega_{i,S_i}-\mu)$ (outside of the indicator function). The bounds (6.11)-(6.17) for terms 1 and 2 still hold, since one fall back to the same estimates by using that $(\omega_{i,S_i}-\mu)\leqslant\omega_{i,S_i}$. It remains only to control only the third term: we prove that when $\mu:=\mathbb{E}[\omega]<\infty$, then for any $\delta>0$, provided that n is large enough,

$$\mathbb{P}\Big(\log\bar{\mathbf{Z}}^{(\leqslant 1)}_{n,6\beta_n}\geqslant c_0q^2\frac{h_n^2}{n}\Big)\leqslant cq^{-2}\times n^{-1/2}\Big(\frac{h_n^2}{n}\Big)^{\alpha-\frac{3}{2}+\delta}, \tag{6.31}$$

where we set analogously to (6.7)

$$\bar{\mathbf{Z}}^{(\leqslant 1)}_{n,6\beta_n}:=\mathbf{E}\Big[\exp\Big(\sum_{i=1}^{n}6\beta_n(\omega_{i,S_i}-\mu)1_{\{\beta_n\omega_{i,S_i}\leqslant 1\}}\Big)\Big]\,. \tag{6.32}$$

Then, using $h_n^2/n\leqslant n$ (if $\alpha\geqslant 3/2$, the upper bound in (6.31) is bounded by $cq^{-2}n^{\alpha-2+\delta}$ which is smaller than q^{-2} provided that δ had been fixed small enough.

To prove (6.31), we use that there is a constant c such that $e^x\leqslant 1+x+cx^2$ as soon as $|x|\leqslant 6$, so that we get similarly to (6.25) that

$$\mathbb{E}\mathbf{Z}^{(\leqslant 1)}_{n,6\beta_n}\leqslant\Big(1+\beta_n\mathbb{E}\big[(\omega-\mu)1_{\{\omega\leqslant 1/\beta_n\}}\big]+c\beta_n^2\mathbb{E}\big[(\omega-\mu)^2 1_{\{\omega\leqslant 1/\beta_n\}}\big]\Big)^n$$
$$\leqslant\exp\Big(cnL(1/\beta_n)\beta_n^{\alpha}\Big)\leqslant\exp\Big(\frac{c}{h_n}(h_n^2/n)^{\alpha+\delta}\Big)\,. \tag{6.33}$$

For the second inequality, we used that $\mathbb{E}\big[(\omega-\mu)1_{\{\omega\leqslant 1/\beta_n\}}\big]\leqslant 0$ (as soon a $1/\beta_n\geqslant\mu$), and also that $\mathbb{E}\big[(\omega-\mu)^2 1_{\{\omega\leqslant 1/\beta_n\}}\big]\leqslant cL(1/\beta_n)\beta_n^{\alpha-2}$, thanks to (1.2). The last inequality holds for any fixed δ, provided that n is large enough, and comes from using Potter's bound and the relation (2.2) to get that $L(1/\beta_n)\beta_n^{\alpha}\leqslant c'\mathbb{P}(\omega>1/\beta_n)\leqslant(nh_n)^{-1}(h_n^2/n)^{\alpha+\delta}$. Then, applying Markov and Jensen inequalities as in (6.26), we get that

$$\mathbb{P}\Big(\log\bar{\mathbf{Z}}^{(\leqslant 1)}_{n,6\beta_n}\geqslant c_0q^2\frac{h_n^2}{n}\Big)\leqslant cq^{-2}\frac{n}{h_n^3}\Big(\frac{h_n^2}{n}\Big)^{\alpha+\delta},$$

which proves (6.31). □

With Lemma 6.3 in hand, and using Cauchy-Schwarz inequality as in (6.3), we get that

$$\mathbb{P}\Big(\bar{\mathbf{Z}}^{\omega}_{n,\beta}\big(\max_{i\leqslant n}|S_i|\in B_{k,n}\big)\geqslant 2e^{-c_0 2^{2k}h_n^2/n}\Big)\leqslant(2^k)^{-\nu}\,.$$

Plugged into (6.29), this concludes the proof of Theorem 2.2-(2.9). It therefore only remains to prove Lemma 6.2.

Proof of Lemma 6.2. We need to obtain a lower bound on $\bar{\mathbf{Z}}_{n,\beta_n}$, so we use Cauchy-Schwarz inequality *backwards*: we apply Cauchy Schwarz inequality to

$$\begin{aligned}\bar{\mathbf{Z}}^{(>1)}_{n,\beta_n/2} &:= \mathbf{E}\Big[\exp\Big(\sum_{i=1}^{n}\frac{\beta_n}{2}(\omega_{i,s_i}-\mu)1_{\{\beta_n\omega_{i,s_i}>1\}}\Big)\Big]\\ &\leqslant (\bar{\mathbf{Z}}_{n,\beta_n})^{1/2}\mathbf{E}\Big[\exp\Big(\sum_{i=1}^{n}-\beta_n(\omega_{i,s_i}-\mu)1_{\{\beta_n\omega_{i,s_i}>1\}}\Big)\Big]^{1/2} =: (\bar{\mathbf{Z}}_{n,\beta_n})^{1/2}(\bar{\mathbf{Z}}^{(\leqslant 1)}_{n,-\beta_n})^{1/2},\end{aligned}$$

so that

$$\bar{\mathbf{Z}}_{n,\beta_n} \geqslant (\bar{\mathbf{Z}}^{(>1)}_{n,\beta_n/2})^2 \Big/ \bar{\mathbf{Z}}^{(\leqslant 1)}_{n,-\beta_n}\,. \tag{6.34}$$

Hence, we get that

$$\mathbb{P}\Big(\bar{\mathbf{Z}}^{\omega}_{n,\beta_n} \leqslant n^{-1}\,e^{\varepsilon_n\frac{h_n^2}{n}}\Big) \leqslant \mathbb{P}\Big(\bar{\mathbf{Z}}^{(\leqslant 1)}_{n,-\beta_n} \geqslant e^{\varepsilon_n\frac{h_n^2}{2n}}\Big) + \mathbb{P}\Big(\bar{\mathbf{Z}}^{(>1)}_{n,\beta_n/2} \leqslant n^{-1/2}\,e^{\varepsilon_n\frac{h_n^2}{4n}}\Big)\,, \tag{6.35}$$

and we deal with both terms separately.

For the first term, we use that analogously to (6.33) we have

$$\begin{aligned}\mathbb{E}\bar{\mathbf{Z}}^{(\leqslant 1)}_{n,-\beta_n} &\leqslant \Big(1-\beta_n\mathbb{E}\big[(\omega-\mu)1_{\{\omega\leqslant 1/\beta_n\}}\big] + c\beta_n^2\mathbb{E}\big[(\omega-\mu)^2 1_{\{\omega\leqslant 1/\beta_n\}}\big]\Big)^n\\ &\leqslant \Big(1+cL(1/\beta_n)\beta_n^{\alpha}\Big)^n \leqslant \exp\Big(\frac{c}{h_n}\big(h_n^2/n\big)^{\alpha+\delta/2}\Big)\,,\end{aligned} \tag{6.36}$$

Here, the difference with (6.33) is that we use for the second inequality that $-\mathbb{E}\big[(\omega-\mu)1_{\{\omega\leqslant 1/\beta_n\}}\big] = \mathbb{E}\big[(\omega-\mu)1_{\{\omega>1/\beta_n\}}\big] \leqslant cL(1/\beta_n)\beta_n^{\alpha-1}$, thanks to (1.2). Again, the second inequality holds for any fixed δ, provided that n is large enough. Using Markov's inequality, one therefore obtains that the first term in (6.35) is bounded by

$$\mathbb{P}\Big(\bar{\mathbf{Z}}^{(\leqslant 1)}_{n,-\beta_n} \geqslant e^{\varepsilon_n\frac{\varepsilon_n h_n^2}{2n}}\Big) \leqslant \exp\Big(\frac{c}{h_n}\big(h_n^2/n\big)^{\alpha+\delta} - \varepsilon_n\frac{h_n^2}{2n}\Big) \leqslant \exp\Big(-\varepsilon_n\frac{h_n^2}{4n}\Big)\,, \tag{6.37}$$

the second inequality holding provided that ε_n is larger than $n^{-1/2}\Big(\frac{h_n^2}{n}\Big)^{\alpha-\frac{3}{2}+\delta}$.

As far as the second term in (6.35) is concerned, we find a lower bound on $\mathbf{Z}^{(\geqslant 1)}_{n,\beta_n}$ by restricting to a particular set of trajectories – the idea being similar as when we proved $\mathcal{T}_{\beta>0}$ in Section 4. Consider the set

$$\mathcal{O}_n := \Big\{(i,x)\in[\![n/2,n]\!]\times[\![\varepsilon_n^{1/2}h_n, 2\varepsilon_n^{1/2}h_n]\!];\beta_n\omega_{i,x}\geqslant 2x^2/i\Big\}\,.$$

If the set $\mathcal{O}_n$ is non-empty, then pick some $(i_0,x_0)\in\mathcal{O}_n$, and consider trajectories which visit this specific site: since all other weights are non-negative ($(\omega-\mu)1_{\{\beta_n\omega>1\}}\geqslant 0$ provided $\mu<1/\beta_n$), we get that

$$\begin{aligned}\bar{\mathbf{Z}}^{(\geqslant 1)}_{n,\beta_n} &\geqslant e^{\beta_n(\omega_{i_0,x_0}-\mu)}\mathbf{P}\big(S_{i_0}=x_0\big)\\ &\geqslant \frac{c}{\sqrt{n}}\exp\Big(\beta_n\omega_{i_0,x_0}-\frac{x_0^2}{i_0}\Big) \geqslant \frac{c}{\sqrt{n}}e^{\varepsilon_n\frac{h_n^2}{n}}\,.\end{aligned} \tag{6.38}$$

We used Stone's local limit theorem [33] for the second inequality (valid provided that n is large, using also that $i_0\geqslant n/2$). For the last inequality, we used the definition of $\mathcal{O}_n$ to

bound the argument of the exponential by $x_0^2/i_0 \geqslant \varepsilon_n h_n^2/n$. Therefore, we get that

$$\mathbb{P}\Big(\bar{\mathbf{Z}}_{n,\beta_n}^{(\geqslant 1)} \leqslant \frac{c}{\sqrt{n}} e^{\varepsilon_n \frac{h_n^2}{n}}\Big) \leqslant \mathbb{P}\big(\mathcal{O}_n = \varnothing\big) = \prod_{i=n/2}^{n} \prod_{x=\varepsilon_n^{1/2} h_n}^{2\varepsilon_n^{1/2} h_n} \Big(1 - \mathbb{P}\big(\beta_n \omega > 2x^2/i\big)\Big)$$
$$\leqslant \Big(1 - \mathbb{P}\big(\omega > 4\varepsilon_n m(nh_n)\big)\Big)^{\varepsilon_n^{1/2} n h_n}.$$

For the second inequality we used that $x^2/i \geqslant \varepsilon_n h_n^2/n$ for the range considered, together with the relation (2.2) characterizing β_n. Then, we use the definition of $m(nh_n)$ together with Potter's bound to get that for any fixed $\delta > 0$, we have $\mathbb{P}\big(\omega > 4\varepsilon_n m(nh_n)\big) \geqslant c\varepsilon_n^{-\alpha+\delta}(nh_n)^{-1}$, provided that n is large enough. Therefore, we obtain that

$$\mathbb{P}\Big(\bar{\mathbf{Z}}_{n,\beta_n}^{(\geqslant 1)} \leqslant \frac{c}{\sqrt{n}} e^{\varepsilon_n \frac{h_n^2}{n}}\Big) \leqslant \exp\Big(-c\,\varepsilon_n^{\frac{1}{2}-\alpha+\delta}\Big), \tag{6.39}$$

which bounds the second term in (6.35). □

7. Regime 2 and regime 3-a

In this section we prove Theorem 2.4 and Theorem 2.5. We decompose the proof in three steps, Step 1 and Step 2 being the same for both theorems. For the third step, we give the details in regime 2, and adapt the reasoning to regime 3-a.

7.1. **Step 1: Reduction of the set of trajectories.** Recalling $\mu = \mathbb{E}[\omega]$ (which is finite for $\alpha > 1$), we define

$$\bar{\mathbf{Z}}_{n,\beta_n}^{\omega} := \mathbf{E}\Big[\exp\Big(\sum_{i=1}^{n} \beta_n\big(\omega_{i,S_i} - \mu \mathbb{1}_{\{\alpha \geqslant 3/2\}}\big)\Big)\Big] \tag{7.1}$$

We show that to prove Theorem 2.4 and Theorem 2.5 we can reduce the problem to the random walk trajectories belonging to Λ_{n,Ah_n} for some $A > 0$ (large). For any $A > 0$, we define

$$\mathcal{B}_n(A) := \Big\{(i, S_i)_{i=1}^n \colon \max_{i \leqslant n} |S_i| \leqslant Ah_n\Big\} \tag{7.2}$$

and we let

$$\bar{\mathbf{Z}}_{n,\beta_n}^{\omega}(\mathcal{B}_n(A)) := \mathbf{E}\Big[\exp\Big(\sum_{i=1}^{n} \beta_n\big(\omega_{i,s_i} - \mu \mathbb{1}_{\{\alpha \geqslant 3/2\}}\big)\Big)\mathbb{1}_{\mathcal{B}_n(A)}\Big]. \tag{7.3}$$

Relation (2.9) gives that $\mathbb{P}\Big(\mathbf{P}_{n,\beta_n}^{\omega}\big(\mathcal{B}_n(A)\big) \geqslant ne^{-c_1 A^2 h_n^2/n}\Big) \leqslant c_2 A^{-\nu_1}$, uniformly on $n \in \mathbb{N}$. This implies that

$$\mathbb{P}\Big(\Big|\log \bar{\mathbf{Z}}_{n,\beta_n}^{\omega} - \log \bar{\mathbf{Z}}_{n,\beta_n}^{\omega}(\mathcal{B}_n(A))\Big| \geqslant ne^{-c_1' A^2 h_n^2/n}\Big) \leqslant c_2 A^{-\nu_1}, \tag{7.4}$$

uniformly on $n \in \mathbb{N}$. Let us observe that in Regime 2 and regime 3-a we have that $h_n^2/n \geqslant c_\beta \log n$, therefore $ne^{-c_1' A^2 h_n^2/n}$ goes to 0 as n gets large, provided A is sufficiently large.

In such a way relation (7.4) implies

$$\lim_{n\to\infty} \frac{n}{h_n^2} \log \bar{\mathbf{Z}}_{n,\beta_n}^{\omega} = \lim_{A\to\infty} \lim_{n\to\infty} \frac{n}{h_n^2} \log \bar{\mathbf{Z}}_{n,\beta_n}^{\omega}(\mathcal{B}_n(A)). \tag{7.5}$$

7.2. **Step 2: Restriction to large weights.** In the second step of the proof we show that we can only consider the partition function $\mathbf{Z}^{\omega,(\mathtt{L})}_{n,\beta_n}$ truncated to a finite number $\mathtt{L}$ of large weights, iwth$\mathtt{L}$ independent of n. We need some intermediate truncation steps.

We start by removing the small weights. Using the notations introduced in (6.8 – 6.10) and (6.32), Hölder's inequality gives that for any $\eta \in (0,1)$

$$\Big(\bar{\mathbf{Z}}^{(>1)}_{n,(1-\eta)\beta_n}\Big)^{\frac{1}{1-\eta}}\Big(\bar{\mathbf{Z}}^{(\leqslant 1)}_{n,-(\eta^{-1}-1)\beta_n}\Big)^{-\frac{\eta}{1-\eta}} \leqslant \bar{\mathbf{Z}}^{\omega}_{n,\beta_n}(\mathcal{B}_n(A)) \leqslant \Big(\bar{\mathbf{Z}}^{(>1)}_{n,(1+\eta)\beta_n}\Big)^{\frac{1}{1+\eta}}\Big(\bar{\mathbf{Z}}^{(\leqslant 1)}_{n,(1+\eta^{-1})\beta_n}\Big)^{\frac{\eta}{1+\eta}}, \tag{7.6}$$

We observe that the condition $\beta_n\omega > 1$ implies (if $\mu < \infty$)

$$(1-2\eta)\beta_n\omega \leqslant (1-\eta)\beta_n(\omega-\mu) \qquad \text{and} \qquad (1+\eta)\beta_n(\omega-\mu) \leqslant (1+\eta)\beta_n\omega, \tag{7.7}$$

provided n is large enough. In such a way, we can safely replace $\bar{\mathbf{Z}}^{(>1)}_{n,(1-\eta)\beta_n}$ by $\mathbf{Z}^{(>1)}_{n,(1-2\eta)\beta_n}$ and $\bar{\mathbf{Z}}^{(>1)}_{n,(1+\eta)\beta_n}$ by $\mathbf{Z}^{(>1)}_{n,(1+\eta)\beta_n}$ in (7.6). The next lemma shows that the contribution given by $\log \bar{\mathbf{Z}}^{(\leqslant 1)}_{n,\rho\beta_n}$ is negligible.

Lemma 7.1. *Let $\rho \in \mathbb{R}$. Then,*

$$\frac{n}{h_n^2}\log \bar{\mathbf{Z}}^{(\leqslant 1)}_{n,\rho\beta_n} \xrightarrow{\mathbb{P}} 0, \quad \text{as } n \to \infty. \tag{7.8}$$

Proof. The case $\rho > 0$ is a consequence of the estimate in (6.25) and (6.26), while the case $\rho < 0$ is a consequence of the estimate in (6.36) and (6.37) □

We can further reduce the partition function $\mathbf{Z}^{(>1)}_{n,\nu\beta_n}$ to even (intermediate) larger weights (with $\nu > 0$).

We fix some $\delta > 0$ small, and define $\ell := (A^2h_n^2/n)^{1-\delta}$ and $\mathtt{T} = A^{1/\alpha}\frac{h_n^2}{n}\ell^{-(1-\delta)^{1/2}/\alpha}$ as in (6.12): then, Hölder's inequality gives that for any $\eta \in (0,1)$

$$\log \mathbf{Z}^{(>\mathtt{T})}_{n,\nu\beta_n} \leqslant \log \mathbf{Z}^{(>1)}_{n,\nu\beta_n} \leqslant \frac{1}{1+\eta}\log \mathbf{Z}^{(>\mathtt{T})}_{n,(1+\eta)\nu\beta_n} + \frac{\eta}{1+\eta}\log \mathbf{Z}^{((1,\mathtt{T}])}_{n,(1+\eta^{-1})\nu\beta_n}.$$

Then, (6.17) gives that for any fixed $A \geqslant 1$, and since $h_n^2/n \to \infty$, we have that for any $\rho > 0$,

$$\frac{n}{h_n^2}\log \mathbf{Z}^{((1,\mathtt{T}])}_{n,\rho\beta_n} \xrightarrow{\mathbb{P}} 0, \quad \text{as } n \to \infty. \tag{7.9}$$

Finally we show that we can only consider a finite number of large weights. We consider $\Upsilon_\ell = \{Y_1^{(n,Ah_n)}, \dots, Y_\ell^{(n,Ah_n)}\}$ with ℓ chosen above. Using (6.14), with probability larger $1-(c\ell)^{-\delta\ell/2}$ (with $\ell \to \infty$ as $n \to \infty$) we have that

$$\Xi_{\mathtt{T}} := \Big\{(i,x) \in [\![1,n]\!] \times [\![-Ah_n, Ah_n]\!]; \beta_n\omega_{i,x} > \mathtt{T}\Big\} \subset \Upsilon_\ell$$

and thus $\mathbf{Z}^{(>\mathtt{T})}_{n,\nu\beta_n} \leqslant \mathbf{Z}^{(\ell)}_{n,\nu\beta_n}$ with high probability. We let $\mathtt{L} \in \mathbb{N}$ be a fixed (large) constant. Since $|\Xi_{\mathtt{T}}| \to \infty$ as $n \to \infty$ in probability, we have that $\Upsilon_{\mathtt{L}} \subset \Xi_{\mathtt{T}}$ so that, $\mathbf{Z}^{(\mathtt{L})}_{n,\nu\beta_n} \leqslant \mathbf{Z}^{(>\mathtt{T})}_{n,\nu\beta_n}$ for large n, with high probability. By using Hölder's inequality we get,

$$\mathbf{Z}^{(\mathtt{L})}_{n,\nu\beta_n} \leqslant \mathbf{Z}^{(>\mathtt{T})}_{n,\nu\beta_n} \leqslant \Big(\mathbf{Z}^{(\mathtt{L})}_{n,\nu(1+\eta)\beta_n}\Big)^{\frac{1}{1+\eta}}\Big(\mathbf{Z}^{(\mathtt{L},\ell)}_{n,\nu(1+\eta^{-1})\beta_n}\Big)^{\frac{\eta}{1+\eta}},$$

where

$$\mathbf{Z}^{(\mathtt{L},\ell)}_{n,\beta_n} := \mathbf{E}\Big[\exp\Big(\sum_{i=\mathtt{L}+1}^{\ell}\beta_n M_i^{(n,qh_n)}1_{\{Y_i^{(n,qh_n)}\in S\}}\Big)\Big]. \tag{7.10}$$

We now show that the contribution of $\mathbf{Z}^{(\mathtt{L},\ell)}_{n,\nu(1+\eta^{-1})\beta_n}$ is negligible.

Lemma 7.2. *For any $\varepsilon \in (0,1)$ and for any $\mathtt{L} \in \mathbb{N}$ and $\rho > 0$ there exists $\delta_{\mathtt{L}}$ such that for all n*

$$\mathbb{P}\Big(\frac{n}{h_n^2} \log \mathbf{Z}^{(\mathtt{L},\ell)}_{n,\rho\beta_n} > \varepsilon \Big) \leqslant \delta_{\mathtt{L}}, \tag{7.11}$$

with $\delta_{\mathtt{L}} \to 0$ as $\mathtt{L} \to \infty$.

Proof. We let $\rho > 0$. Recalling the definition (5.4), and using that $\mathbf{P}(\Delta \subset S) \leqslant e^{\mathrm{Ent}(\Delta)}$, we have that

$$\begin{aligned} \mathbf{Z}^{(\mathtt{L},\ell)}_{n,\rho\beta_n} &\leqslant \sum_{\Delta \subset \Upsilon_\ell} e^{\rho\beta_n \Omega^{(>\mathtt{L})}_{n,qh_n}(\Delta)} \mathbf{P}\big(S \cap \Upsilon_\ell = \Delta\big) \\ &\leqslant \sum_{\Delta \subset \Upsilon_\ell} \exp\Big(\rho\beta_n \Omega^{(>\mathtt{L})}_{n,qh_n}(\Delta) - \mathrm{Ent}(\Delta) \Big) \leqslant 2^\ell \exp\Big(T^{\rho\beta_n,(>\mathtt{L})}_{n,Ah_n} \Big). \end{aligned}$$

Using that $\ell = o(h^2/n)$ and (5.15), we conclude the proof. □

Collecting the above estimates, we can conclude that

$$\lim_{n\to\infty} \frac{n}{h_n^2} \log \bar{\mathbf{Z}}^{\omega}_{n,\beta_n}(\mathcal{B}_n(A)) = \lim_{\nu\to 1} \lim_{\mathtt{L}\to\infty} \lim_{n\to\infty} \frac{n}{h_n^2} \log \mathbf{Z}^{(\mathtt{L})}_{n,\nu\beta_n}. \tag{7.12}$$

7.3. **Step 3: Regime 2. Convergence of the main term.** In remain to show the convergence of the partition function restricted to the large weights.

Proposition 7.3. *For any $\nu > 0$, and $\mathtt{L} > 0$*

$$\frac{n}{h_n^2} \log \mathbf{Z}^{(\mathtt{L})}_{n,\nu\beta_n} \overset{(d)}{\to} \begin{cases} \mathcal{T}^{(\mathtt{L})}_{\nu,A} & \text{in Regime 2,} \\ \widetilde{\mathcal{T}}^{(\mathtt{L})}_{\beta,\nu,A} & \text{in Regime 3-a,} \end{cases} \tag{7.13}$$

where $\mathcal{T}^{(\mathtt{L})}_{\beta,A}$ is defined in (5.13) *and $\widetilde{\mathcal{T}}^{(\mathtt{L})}_{\beta,\nu,A}$ in* (7.18) *below.*

Remark 7.4. One readily verifies that

∗ $\nu \mapsto \mathcal{T}^{(\mathtt{L})}_{\nu,A}$ (resp. $\nu \mapsto \widetilde{\mathcal{T}}^{(\mathtt{L})}_{\beta,\nu,A}$) is a continuous function;

∗ $\mathcal{T}^{(\mathtt{L})}_{1,A} \to \mathcal{T}_{1,A}$ (resp. $\widetilde{\mathcal{T}}^{(\mathtt{L})}_{\beta,1,A} \to \widetilde{\mathcal{T}}_{\beta,1,A}$) as $\mathtt{L} \to \infty$ (see Proposition 5.3, resp. Proposition 7.5);

∗ $\mathcal{T}_{1,A} \to \mathcal{T}_1$ (resp. $\widetilde{\mathcal{T}}_{\beta,1,A} \to \widetilde{\mathcal{T}}_\beta$) as $A \to \infty$ (see Proposition 5.3, resp. Proposition 7.5).

Therefore, the proof of Theorem 2.4 and Theorem 2.5 is a consequence of relations (7.5), (7.12) and (7.13).

Proof. We detail the proof for the Regime 2. The Regime 3-a follows similarly using the results in Section 7.4 below. To keep the notation clear we let $\nu = 1$.

Lower bound. For any $\mathtt{L} \in \mathbb{N}$ we consider a set $\Delta_{\mathtt{L}} \subset \Upsilon_{\mathtt{L}}$ which achieves the maximum of $T^{\beta_n,(\mathtt{L})}_{n,Ah_n}$, resp. of $\widetilde{T}^{\beta_n,(\mathtt{L})}_{n,Ah_n}$ defined below in (7.17) for Regime 3-a. We have

$$\mathbf{Z}^{(\mathtt{L})}_{n,\beta_n} \geqslant \exp\Big(\beta_n \Omega_{n,Ah_n}(\Delta_{\mathtt{L}}) \Big) \mathbf{P}\big(S \cap \Upsilon_{\mathtt{L}} = \Delta_{\mathtt{L}}\big).$$

Since $\mathtt{L}$ is fixed, we realize that any pair of points $(i,x),(j,y) \in \Upsilon_{\mathtt{L}}$ satisfies the condition $|i-j| \geqslant \varepsilon n$ and $|x-y| \geqslant \varepsilon h_n$ with probability at least $1 - c_\varepsilon$ with $c_\varepsilon \to 0$ as $\varepsilon \to 0$. In such a way, we can use the Stone local limit theorem [33] to get that $\mathbf{P}(S \cap \Upsilon_{\mathtt{L}} = \Delta_{\mathtt{L}}) = n^{-\frac{|\Delta_{\mathtt{L}}|}{2} + o(1)} e^{-\mathrm{Ent}(\Delta_{\mathtt{L}})}$. In the Regime 2, in which $\mathrm{Ent}(\Delta_{\mathtt{L}}) \asymp h_n^2/n \gg \log n$, this implies that

$$\mathbf{Z}^{(\mathtt{L})}_{n,\beta_n} \geqslant \exp\Big((1+o(1)) T^{\beta_n,(\mathtt{L})}_{n,Ah_n} \Big). \tag{7.14}$$

To conclude, we use Proposition 5.3-(5.11) to obtain that $T^{\beta_n,(\mathtt{L})}_{n,Ah_n}$ converges in distribution to $\mathcal{T}^{(\mathtt{L})}_{1,A}$, concluding the lower bound.

In Regime 3-a, (7.14) is replaced by

$$\mathbf{Z}^{(\mathtt{L})}_{n,\beta_n} \geqslant \exp\Big((1+o(1))\Big\{\beta_n\Omega_{n,Ah_n}(\Delta_{\mathtt{L}}) - \mathrm{Ent}(\Delta_{\mathtt{L}}) - \frac{|\Delta_{\mathtt{L}}|}{2}\log n\Big\}\Big), \tag{7.15}$$

so that $T^{\beta_n,(\mathtt{L})}_{n,Ah_n}$ is replaced by $\widetilde{T}^{\beta_n,(\mathtt{L})}_{n,Ah_n}$ defined in (7.17). Then the conclusion follows by Proposition 7.5-(7.19) below.

Upper bound. We have

$$\mathbf{Z}^{(\mathtt{L})}_{n,\beta_n} = \sum_{\Delta\subset\Upsilon_{\mathtt{L}}} e^{\beta_n\Omega^{(\mathtt{L})}_{n,qh_n}(\Delta)}\mathbf{P}\big(S\cap\Upsilon_{\mathtt{L}}=\Delta\big)$$

Using the Stone local limit theorem [33] we have that $\mathbf{P}(S\cap\Upsilon_{\mathtt{L}}=\Delta) = n^{-\frac{|\Delta|}{2}+o(1)}e^{-\mathrm{Ent}(\Delta)}$ uniformly for all $\Delta\subset\Upsilon_{\mathtt{L}}$. Since we have only a finite number of sets, we obtain that

$$\mathbf{Z}^{(\mathtt{L})}_{n,\beta_n} \leqslant 2^{\mathtt{L}}\exp\Big((1+o(1))T^{\beta_n,(\mathtt{L})}_{n,Ah_n}\Big), \tag{7.16}$$

which concludes the proof of the upper bound, again thanks to the convergence proven in Proposition 5.3-(5.11). In Regime 3-a, using the Stone local limit theorem, we can safely replace $T^{\beta_n,(\mathtt{L})}_{n,Ah_n}$ by $\widetilde{T}^{\beta_n,(\mathtt{L})}_{n,Ah_n}$ defined below in (7.17), and also conclude thanks to Proposition 7.5-(7.19). □

7.4. **Step 3: Regime 3.a. Complements for the convergence of the main term.** We end here the proof of Theorem 2.5 by stating the results needed to complete Step 3 above in the case of regime 3.a. In analogy with (5.2), and in view of the local limit theorem (2.15), we define

$$\begin{aligned}
\widetilde{T}^{\beta_{n,h}}_{n,h} &:= \max_{\Delta\subset\Lambda_{n,h}}\big\{\beta_{n,h}\Omega_{n,h}(\Delta) - \mathrm{Ent}(\Delta) - \frac{|\Delta|}{2}\log n\big\}\,,\\
\widetilde{T}^{\beta_{n,h},(\ell)}_{n,h} &:= \max_{\Delta\subset\Lambda_{n,h}}\big\{\beta_{n,h}\Omega^{(\ell)}_{n,h}(\Delta) - \mathrm{Ent}(\Delta) - \frac{|\Delta|}{2}\log n\big\}
\end{aligned} \tag{7.17}$$

In the next result we state the convergence of $\frac{n}{h^2}\widetilde{T}^{\beta_{n,h}}_{n,h}$ and $\frac{n}{h^2}\widetilde{T}^{\beta_{n,h},(\ell)}_{n,h}$, analogously to Proposition 5.3.

Proposition 7.5. *Suppose that $\frac{n}{h^2}\beta_{n,h}m(nh)\to\nu\in(0,\infty)$ as $n,h\to\infty$ and $h\sim\beta^{1/2}\sqrt{\log n}$, with $\beta>0$. Then, for every $\alpha\in(1/2,2)$ and for any $q>0$, $\ell\in\mathbb{N}$ we have the following convergence in distribution*

$$\frac{n}{h^2}\,\widetilde{T}^{\beta_{n,h}}_{n,qh} \xrightarrow{(\mathrm{d})} \widetilde{\mathcal{T}}_{\beta,\nu,q} := \sup_{s\in\mathscr{M}_q}\Big\{\nu\pi(s) - \mathrm{Ent}(s) - \frac{N(s)}{2\beta}\Big\}\,, \tag{7.18}$$

where the supremum is taken over the set $\mathscr{M}_q := \{s\in\mathscr{D}\colon \mathrm{Ent}(s)<\infty, \max_{t\in[0,1]}|s(t)|\leqslant q\}$. We also have the convergence

$$\frac{n}{h^2}\,\widetilde{T}^{\beta_{n,h},(\ell)}_{n,qh} \xrightarrow{(\mathrm{d})} \widetilde{\mathcal{T}}^{(\ell)}_{\beta,\nu,q} := \sup_{s\in\mathscr{M}_q}\Big\{\nu\pi^{(\ell)}(s) - \mathrm{Ent}(s) - \frac{N(s)}{2\beta}\Big\}\,. \tag{7.19}$$

Moreover, we have

$$\widetilde{\mathcal{T}}^{(\ell)}_{\beta,\nu,q} \xrightarrow{(\mathrm{d})} \widetilde{\mathcal{T}}_{\beta,\nu,q} \ \text{ as } \ell\to\infty, \qquad \text{and} \qquad \widetilde{\mathcal{T}}_{\beta,\nu,q} \xrightarrow{(\mathrm{d})} \widetilde{\mathcal{T}}_{\beta,\nu} \ \text{ as } q\to\infty\,.$$

The proof is identical to the proof of Proposition 5.3 – using also that $\frac{n}{h_n^2}\log n \to \frac{1}{\beta}$ in regime 3 –, for this reason it is omitted.

To conclude, let us show that $\widetilde{\mathcal{T}}_\beta^{(\geqslant r)}$ defined in (2.12) is well defined.

Proposition 7.6. *For any $r \geqslant 0$ the quantities $\widetilde{\mathcal{T}}_\beta^{(\geqslant r)}$ are well defined and for any $\beta > 0$*

$$-\frac{1}{2\beta} < \widetilde{\mathcal{T}}_\beta^{(\geqslant 1)} \leqslant \widetilde{\mathcal{T}}_\beta < \infty. \tag{7.20}$$

Moreover $\widetilde{\mathcal{T}}_\beta \geqslant 0$, and we have $\widetilde{\mathcal{T}}_\beta > 0$ if and only if $\widetilde{\mathcal{T}}_\beta^{(\geqslant 1)} > 0$. Finally, there is a critical value

$$\beta_c = \inf\{\beta \colon \widetilde{\mathcal{T}}_\beta > 0\} \in (0, \infty).$$

Proof. Since $\widetilde{\mathcal{T}}_\beta^{(0)} = 0$, we obtain that $\widetilde{\mathcal{T}}_\beta \in [0, \infty)$. As a by-product we also have that $\widetilde{\mathcal{T}}_\beta > 0$ if and only if $\widetilde{\mathcal{T}}_\beta^{(\geqslant 1)} > 0$; and in that case $\widetilde{\mathcal{T}}_\beta = \widetilde{\mathcal{T}}_\beta^{(\geqslant 1)}$. Additionally, we have

$$W_\beta - \frac{1}{2\beta} \leqslant \widetilde{\mathcal{T}}_\beta^{(\geqslant 1)} \leqslant \widetilde{\mathcal{T}}_\beta \leqslant \Big(\mathcal{T}_1 - \frac{1}{2\beta}\Big) \vee 0,$$

with W_β and $\mathcal{T}_1$ defined in (2.14) and (2.6) respectively. Proposition 8.4 and Theorem 2.1 ensure that for $\beta > 0$, $W_\beta \in (0, \infty)$ and $\widetilde{\mathcal{T}}_1 < \infty$, showing(7.20).

It remains to show that $\beta_c \in (0, \infty)$, by observing that $\beta \mapsto \beta W_\beta$ and $\beta \mapsto (\beta\mathcal{T}_1 - 1/2) \vee 0$ are monotone functions which converge to 0 as $\beta \to 0$. □

8. Regime 3-b and regime 4

In this section we prove Theorem 2.6 and Theorem 2.7. We decompose the proof in three steps (analogously to what is done in Section 7), Step 1 and Step 2 being the same for both regimes 3-b and 2. For the third step, we separate regime 3-b and regime 4, which have different behaviors. Note that in both regimes there is a constant $c_\beta > 0$ such that $h_n \leqslant c\sqrt{n\log n}$ (in regime 4, we have $h_n \ll \sqrt{n\log n}$).

Let us define here, analogously to (7.1), the recntered partition function

$$\bar{\mathbf{Z}}_{n,\beta_n}^\omega := \mathbf{E}\Big[\exp\Big(\sum_{i=1}^n \beta_n\big(\omega_{i,s_i} - \mathbb{E}[\omega 1_{\omega\leqslant 1/\beta_n}]1_{\{\alpha\geqslant 1\}}\big)\Big)\Big]. \tag{8.1}$$

Then, roughly speaking, we show that $\log \bar{\mathbf{Z}}_{n,\beta_n}^\omega$ is of order $n^{-1/2}\exp(Xh_n^2/n)$, with $X = \widetilde{\mathcal{T}}_\beta^{(\geqslant 1)} + \frac{1}{2\beta}$ in the regime 3-b (where $h_n^2/n \sim \beta\log n$), and with $X = W_1$ in regime 4. In all cases, we will have $\log \bar{\mathbf{Z}}_{n,\beta_n}^\omega = o(1)$ (recall that in regime 3-b, $\widetilde{\mathcal{T}}_\beta^{(\geqslant 1)} < 0$).

8.1. Step 1. Reduction of the set of trajectories.

We proceed as for Step 1 in Section 7: for any $A > 0$ (fixed large in a moment), we define

$$\mathcal{A}_n := \Big\{(i, S_i) \,:\, \max_{i\leqslant n}|S_i| \leqslant A\sqrt{n\log n}\Big\}. \tag{8.2}$$

Then, we let $\bar{\mathbf{Z}}_{n,\beta_n}^\omega(\mathcal{A}_n)$ be the (normalized) partition function restricted to trajectories in $\mathcal{A}_n$. Relation (2.9) gives that, analogously to (7.4)

$$\mathbb{P}\Big(\Big|\log \bar{\mathbf{Z}}_{n,\beta_n}^\omega - \log \bar{\mathbf{Z}}_{n,\beta_n}^\omega(\mathcal{A}_n)\Big| \geqslant ne^{-c_1A^2\log n}\Big) \leqslant c_2A^{-\nu_1}. \tag{8.3}$$

Hence, we fix A large enough so that $e^{-c_0 A^2 \log n} \leqslant n^{-3}$. This shows that with high probability $\log \bar{\mathbf{Z}}^{\omega}_{n,\beta_n} = \log \bar{\mathbf{Z}}^{\omega}_{n,\beta_n}(\mathcal{A}_n) + O(n^{-2})$. In such a way, in the following we can safely focus only on the partition function with trajectories restricted to $\mathcal{A}_n$.

8.2. **Step 2. Restriction to large weights.** We now fix $\eta \in (0,1)$, small. The same Hölder inequalities as in (7.6) hold for $\mathbf{Z}^{\omega}_{n,\beta_n}(\mathcal{A}_n)$, so that we can write, with similar notations as in (6.8)-(6.10) (the restriction to trajectories in $\mathcal{A}_n$ does not appear in the notations)

$$\log \bar{\mathbf{Z}}^{\omega}_{n,\beta_n}(\mathcal{A}_n) \begin{cases} \leqslant \dfrac{1}{1+\eta} \log \mathbf{Z}^{(>1)}_{n,(1+\eta)\beta_n} + \dfrac{\eta}{1+\eta} \log \bar{\mathbf{Z}}^{(\leqslant 1)}_{n,(1+\eta^{-1})\beta_n}\,, \\ \geqslant \dfrac{1}{1-\eta} \log \mathbf{Z}^{(>1)}_{n,(1-2\eta)\beta_n} - \dfrac{\eta}{1-\eta} \log \bar{\mathbf{Z}}^{(\leqslant 1)}_{n,-(\eta^{-1}-1)\beta_n}\,. \end{cases} \tag{8.4}$$

We used also (7.7) to be able to bound below $\bar{\mathbf{Z}}^{(>1)}_{n,(1-\eta)\beta_n}$ by $\mathbf{Z}^{(>1)}_{n,(1-2\eta)\beta_n}$ (using that $\beta_n \mathbb{E}[\omega 1_{\{\omega \leqslant 1/\beta_n\}}] \ll 1$ when $\alpha \geqslant 1$). Then, we need to get a more precise statement than Lemma 7.1 to deal with $\bar{\mathbf{Z}}^{(\leqslant 1)}_{n,\rho\beta_n}$.

Lemma 8.1. *For any $\rho \in \mathbb{R}$,*

$$\Big(\frac{h_n^2}{n}\Big)^{-3\alpha} \sqrt{n} \log \bar{\mathbf{Z}}^{(\leqslant 1)}_{n,\rho\beta_n} \xrightarrow{\mathbb{P}} 0\,, \qquad \text{as } n \to \infty\,.$$

Proof. We will simply control the first moment of $\bar{\mathbf{Z}}^{(\leqslant 1)}_{n,\rho\beta_n} - 1$. The idea is similar to that used to obtain (6.24) and (6.31). We divide the proof into two cases: when $\alpha < 1$ so that there is no renormalization necessary in (8.1), and when $\alpha \in [1,2)$.

Let us start with the case $\alpha < 1$: using that $|\rho|\beta_n \omega_{i,S_i} \leqslant |\rho|$ on the event $\{\beta_n \omega_{i,S_i} \leqslant 1\}$, we get that there exists a constant c_ρ such that

$$e^{\sum_{i=1}^n \rho \beta_n \omega_{i,S_i} 1_{\{\beta_n \omega_{i,S_i} \leqslant 1\}}} \leqslant \prod_{i=1}^n \big(1 + c_\rho \beta_n \omega_{i,S_i} 1_{\{\beta_n \omega_{i,S_i} \leqslant 1\}}\big)\,. \tag{8.5}$$

By independence, and since $\mathbb{P}(\omega > t)$ is regularly varying, we get that for n sufficiently large

$$\begin{aligned} \mathbb{E}[\beta_n \omega_{i,x} 1_{\{\beta_n \omega_{i,x} \leqslant 1\}}] &\leqslant \int_0^{1/\beta_n} \beta_n \mathbb{P}(\omega > t) dt \leqslant c\, L(1/\beta_n) \beta_n^{\alpha} \\ &\leqslant c\mathbb{P}\big(\omega > 1/\beta_n\big) \leqslant \frac{c'}{n h_n} \Big(\frac{h_n^2}{n}\Big)^{2\alpha}. \end{aligned} \tag{8.6}$$

For the last inequality we used Potter's bound, and the definition of β_n, *i.e.* the fact that $\beta_n \sim \frac{h_n^2}{n} m(n h_n)$. Therefore, in view of (8.5) and using that $h_n \geqslant \sqrt{n}$, we get that for n sufficiently large (how large depends on ρ)

$$\mathbb{E}\big[\bar{\mathbf{Z}}^{(\leqslant 1)}_{n,\rho\beta_n} - 1\big] \leqslant \Big(1 + c'_\rho \frac{(h_n^2/n)^{2\alpha}}{n^{3/2}}\Big)^n - 1 \leqslant 2c'_\rho n^{-1/2} \Big(\frac{h_n^2}{n}\Big)^{2\alpha}. \tag{8.7}$$

This concludes the proof in the case $\alpha < 1$ by using Markov's inequality, since $h_n^2/n \to +\infty$.

In the case $\alpha \in [1,2)$, we use the expansion $e^x \leqslant 1 + x + c_\rho x^2$ for all $|x| \leqslant 2|\rho|$, to get analogously to (8.5), and setting $\mu_n := \mathbb{E}[\omega 1_{\{\omega \leqslant 1/\beta_n\}}] \ll 1/\beta_n$,

$$\begin{aligned} \mathbb{E}\Big[\bar{\mathbf{Z}}^{(\leqslant 1)}_{n,\rho\beta_n}\Big] &\leqslant \Big(1 + \rho\beta_n \mathbb{E}\big[(\omega - \mu_n) 1_{\{\omega \leqslant 1/\beta_n\}}\big] + c_\rho \beta_n^2 \mathbb{E}\big[(\omega - \mu_n)^2 1_{\{\omega \leqslant 1/\beta_n\}}\big]\Big)^n \\ &\leqslant \exp\Big(c\, n \mathbb{P}(\omega > 1/\beta_n)\Big) \leqslant 1 + c n^{-1/2} \Big(\frac{h_n^2}{n}\Big)^{2\alpha}, \end{aligned}$$

obtaining the same upper bound as in (8.7). To obtain the above inequality, we used that

$$\mathbb{E}[(\omega-\mu_n)1_{\{\omega\leqslant 1/\beta_n\}}] = \mu_n\mathbb{P}(\omega>1/\beta_n)\leqslant \beta_n^{-1}\mathbb{P}(\omega>1/\beta_n)\,,$$
$$\mathbb{E}[(\omega-\mu_n)^2 1_{\{\omega\leqslant 1/\beta_n\}}]\leqslant \mathbb{E}[\omega^2 1_{\{\omega\leqslant 1/\beta_n\}}]\leqslant cL(1/\beta_n)\beta_n^{\alpha-2}\,,$$

where the last inequality follows similarly to (8.6). One concludes that (8.7) also holds when $\alpha\geqslant 1$, and the lemma follows by Markov's inequality. □

Therefore, in view of (8.4) and Lemma 8.1, we have that for both regimes 3-b and 4

$$\lim_{n\to\infty}\frac{n}{h_n^2}\log\Big(\sqrt{n}\log\bar{\mathbf{Z}}^{\omega}_{n,\beta_n}(\mathcal{A}_n)\Big)=\lim_{\nu\to 1}\lim_{n\to\infty}\frac{n}{h_n^2}\log\Big(\sqrt{n}\log\mathbf{Z}^{(>1)}_{n,\nu\beta_n}\Big)\,. \tag{8.8}$$

Note that in the case of regime 3-b, $h_n^2/n\sim\beta\log n$, so the limit is that of

$$\frac{1}{\beta\log n}\log\Big(\log\mathbf{Z}^{(>1)}_{n,\nu\beta_n}\Big)+\frac{1}{2\beta}\,.$$

For simplicity of notations, we will consider only the case $\nu=1$ in the following.

8.3. **Step 3. Reduction of the main term.** In both regimes 3-b and 4, we show that $\log\mathbf{Z}^{(>1)}_{n,\beta_n}$ goes to 0, and we identify at which rate: to do so, it is equivalent to identify the rate at which $\mathbf{Z}^{(>1)}_{n,\beta_n}-1$ goes to 0. The behavior for regimes 3-b and 4 are different, since the main contribution to $\mathbf{Z}^{(>1)}_{n,\beta_n}-1$ may come from several large weights in regime 3-b, whereas it comes from a single large weight in regime 4, as it will be reflected in the proof.

Let us define $\ell=\ell(\omega)$ the number of $(i,x)\in\Lambda_{n,A_n}=[\![1,n]\!]\times[\![-A_n,A_n]\!]$ (with the notation $A_n=A\sqrt{n\log n}$ for simplicity) such that $\beta_n\omega_{i,x}\geqslant 1$, and let us denote

$$\Big\{(i,x)\in\Lambda_{n,A_n};\beta_n\omega_{i,x}\geqslant 1\Big\}=\Upsilon_\ell:=\big\{Y_1^{(n,A_n)},\dots,Y_\ell^{(n,A_n)}\big\}\,, \tag{8.9}$$

with $Y_i^{(n,A_n)}$ the ordered statistic, as in Section 5. We have that

$$\mathbb{E}[\ell]=\sum_{(i,x)\in\Lambda_{n,An}}\mathbb{P}(\beta_n\omega_{i,x}\geqslant 1)\leqslant 2An^{3/2}\sqrt{\log n}\Big(\frac{h_n^2}{n}\Big)^{2\alpha}\frac{1}{nh_n}\,, \tag{8.10}$$

where we used that $\mathbb{P}(\omega\geqslant 1/\beta_n)\leqslant(h_n^2/n)^{2\alpha}(nh_n)^{-1}$ for n large enough, thanks to (2.2) and Potter's bound. Since $h_n^2/n\leqslant c\log n$, $h_n\gg\sqrt{n}$, (8.10) implies that $\ell\leqslant(\log n)^{3\alpha}$ with probability going to 1 (we also used that $\frac{1}{2}+2\alpha<3\alpha$).

Hence, decomposing $\mathbf{Z}^{(>1)}_{n,\beta_n}$ according to the number of sites in Υ_ℓ visited, we can write for any fixed $k_0>0$,

$$\sum_{k=1}^{k_0}\mathbf{U}_k\leqslant\mathbf{Z}^{(>1)}_{n,\beta_n}-1=\sum_{k=1}^{\ell}\mathbf{U}_k\,,\quad\text{with }\mathbf{U}_k=\sum_{\Delta\subset\Upsilon_\ell,|\Delta|=k}e^{\beta_n\Omega_{n,A_n}(\Delta)}\mathbf{P}\big(S\cap\Upsilon_\ell=\Delta\big)\,. \tag{8.11}$$

In regime 3-b, the main contribution comes from one of the $\mathbf{U}_k$'s for some $k\geqslant 1$, whereas in regime 4 only the term $\mathbf{U}_1$ will contribute.

Let us now show that, with high probability, we can replace the upper bound in (8.11) by considering only a finite number of terms. For this purpose, notice that $\ell\leqslant(\log n)^{3\alpha}$ and $\min\{|i-j|,(i,x)\neq(j,y)\in\Upsilon_\ell\}\geqslant n/(\log n)^{10\alpha}$ with probability going to 1. Then, we can use the Stone local limit theorem [33] to have that for any $\Delta\subset\Upsilon_\ell$

$$\mathbf{P}\big(S\cap\Upsilon_\ell=\Delta\big)\leqslant cn^{-(\frac{1}{2}-\eta)|\Delta|}e^{-\mathrm{Ent}(\Delta)}\,,$$

where $\eta > 0$ is independent of Δ and can be chosen arbitrary small (by changing the value of the constant c).

As a consequence, using that $\binom{\ell}{k} \leqslant \ell^k$ and $\ell \leqslant (\log n)^{3\alpha}$, we have for any $1 \leqslant k_1 \leqslant \ell$

$$\sum_{k=k_1}^{\ell} \mathbf{U}_k = \sum_{k=k_1}^{\ell} \sum_{\Delta \subset \Upsilon_\ell, |\Delta|=k} e^{\beta_n \Omega_{n,A_n}(\Delta)} \mathbf{P}\big(S \cap \Upsilon_\ell = \Delta\big) \tag{8.12}$$

$$\leqslant e^{T^{\beta_n}_{n,A_n}} \sum_{k=k_1}^{\ell} \ell^k \, n^{-k(\frac{1}{2}-\eta)} \leqslant c\, e^{T^{\beta_n}_{n,A_n}} \, n^{-k_1(\frac{1}{2}-\eta')}.$$

Recalling Proposition 5.3 (and the fact that $h_n^2/n \leqslant c \log n$) we have that $T^{\beta_n}_{n,A_n} \leqslant C \log n$ with probability going to 1 as $C \to \infty$. Therefore, we obtain that (8.12) is $O(n^{-2})$ with probability close to 1, provided that k_1 is sufficiently large – this will turn out to be negligible, see Lemma 8.2. Hence, we have shown that with probability close to 1, we can keep a finite number of terms in (8.11).

This can actually be improved in regime 4, where we can keep only one term: indeed, since in that case $h_n^2/n = o(\log n)$, we get that for any fixed $\gamma > 0$, $T^{\beta_n}_{n,A_n} \leqslant \gamma \log n$ with probability going to one. Hence, we get that in regime 4, we can take $k_1 = 2$ in (8.12) and obtain that $\sum_{k=2}^{\ell} \mathbf{U}_k = O(n^{-3/4})$ with probability close to 1, which will turn out to be negligible, see Lemma 8.3.

It remains to show the following lemmas, proving the convergence of the main term in regimes 3-b and 4.

Lemma 8.2. *In regime 3* (R3) *(recall $h_n^2/n \sim \beta \log n$), for any $K > 0$ we have that*

$$\frac{n}{h_n^2} \log\Big(\sum_{k=1}^{K} \mathbf{U}_k\Big) \xrightarrow{(\mathrm{d})} \sup_{1 \leqslant k \leqslant K} \widetilde{\mathcal{T}}^{(k)}_{\beta,A}, \tag{8.13}$$

where $\widetilde{\mathcal{T}}^{(k)}_{\beta,A} := \sup_{s \in \mathscr{M}_A, N(s)=k} \big\{\pi(s) - \mathrm{Ent}(s) - \frac{k}{2\beta}\big\}$, with $\mathscr{M}_A$ defined below (7.18).

Note that we have $\sup_{k \geqslant 1} \widetilde{\mathcal{T}}^{(k)}_{\beta,A} < 0$ in regime 3-b: this lemma proves that $\sum_{k=1}^{K} \mathbf{U}_k$ goes to 0 in probability, and hence $\mathbf{Z}^{(>1)}_{n,\beta_n} - 1$ also goes to 0 in probability. This is needed to replace the study of $\log \mathbf{Z}^{(>1)}_{n,\beta_n}$ by that of $\mathbf{Z}^{(>1)}_{n,\beta_n} - 1$, and it is actually the only place where the definition of regime 3-b is used.

Lemma 8.3. *In regime 4* (R4)*, we have that*

$$\frac{n}{h_n^2} \log\Big(\sqrt{n}\, \mathbf{U}_1\Big) \xrightarrow{(\mathrm{d})} W_1, \tag{8.14}$$

with W_1 defined in (2.14).

Here also, this proves that $\mathbf{U}_1 \to 0$ in probability, and hence so does $\mathbf{Z}^{(>1)}_{n,\beta_n} - 1$.

8.4. **Regime 3-b: convergence of the main term.** In this section, we prove Lemma 8.2.

Reduction to finitely many weights. First of all, we fix some $\mathtt{L}$ large and show that the main contribution comes from the $\mathtt{L}$ largest weights. We define

$$\mathbf{U}^{(\mathtt{L})}_k := \sum_{\Delta \subset \Upsilon_{\mathtt{L}}, |\Delta|=k} e^{\beta_n \Omega_{n,A_n}(\Delta)} \mathbf{P}\big(S \cap \Upsilon_\ell = \Delta\big), \tag{8.15}$$

where $\Upsilon_{\mathtt{L}} = \{Y_1^{n,A_n}, \dots, Y_{\mathtt{L}}^{n,A_n}\}$ is the set of $\mathtt{L}$ largest weights in Λ_{n,A_n} (note that $\Upsilon_{\mathtt{L}} \subset \Upsilon_\ell$ for n large enough). Then we have that $\mathbf{U}_k \geqslant \mathbf{U}_k^{(\mathtt{L})}$, and

$$\begin{aligned}\sum_{k=1}^K \mathbf{U}_k &\leqslant \sum_{k=1}^K \sum_{\Delta\subset\Upsilon_{\mathtt{L}},|\Delta|=k} \sum_{\Delta'\subset\Upsilon_\ell\setminus\Upsilon_{\mathtt{L}},|\Delta'|\leqslant K} e^{\beta_n\Omega_{n,A_n}(\Delta)+\beta_n\Omega_{n,A_n}(\Delta')}\mathbf{P}\big(S\cap\Upsilon_\ell = \Delta\cup\Delta'\big)\\ &\leqslant \sum_{k=1}^K \sum_{\Delta\subset\Upsilon_{\mathtt{L}},|\Delta|=k} e^{\beta_n\Omega_{n,A_n}(\Delta)}\mathbf{P}\big(S\cap\Upsilon_{\mathtt{L}} = \Delta\big)\times\exp\Big(K\beta_n M_{\mathtt{L}}^{(n,A_n)}\Big)\\ &= \exp\Big(K\beta_n M_{\mathtt{L}}^{(n,A_n)}\Big)\sum_{k=1}^K \mathbf{U}_k^{(\mathtt{L})}.\end{aligned}$$

In the second inequality, we simply bounded $\Omega_{n,A_n}(\Delta')$ by $KM_{\mathtt{L}}^{(n,A_n)}$ uniformly for $\Delta'\subset\Upsilon_\ell\setminus\Upsilon_{\mathtt{L}}$, with $|\Delta'|\leqslant K$. Then, since $\beta_n\sim c_\beta(\log n)/m(nh_n)\sim c_{\beta,A}(\log n)/m(nA_n)$ as $n\to\infty$, we get that $K\beta_n M_{\mathtt{L}}^{(n,A_n)}\leqslant 2c_{\beta,A}KM_{\mathtt{L}}^{(n,A_n)}/m(nA_n)\times\log n$. For any fixed $\varepsilon>0$, we can fix $\mathtt{L}$ large enough so that for large n we have $M_{\mathtt{L}}^{(n,A_n)}/m(nA_n)\leqslant\varepsilon/(2Kc_{\beta,A})$ with probability larger than $1-\varepsilon$. We conclude that there exists some $\varepsilon_{\mathtt{L}}$ with $\varepsilon_{\mathtt{L}}to0$ as $\mathtt{L}\to\infty$ such that

$$0\leqslant\sum_{k=1}^K(\mathbf{U}_k-\mathbf{U}_k^{(\mathtt{L})})\leqslant n^{\varepsilon_{\mathtt{L}}}\sum_{k=1}^K\mathbf{U}_k^{(\mathtt{L})}.$$

Since $h_n^2/n\sim\beta\log n$, this proves that

$$\lim_{n\to\infty}\frac{n}{h_n^2}\log\Big(\sum_{k=1}^K\mathbf{U}_k\Big)=\lim_{\mathtt{L}\to\infty}\lim_{n\to\infty}\frac{n}{h_n^2}\log\Big(\sum_{k=1}^K\mathbf{U}_k^{(\mathtt{L})}\Big). \tag{8.16}$$

Convergence of the remaining term. We finally prove that

$$\frac{n}{h_n^2}\log\Big(\sum_{k=1}^K\mathbf{U}_k^{(\mathtt{L})}\Big)\xrightarrow{(\mathrm{d})}\max_{1\leqslant k\leqslant K}\widetilde{\mathcal{T}}_{\beta,A}^{(k,\mathtt{L})} \tag{8.17}$$

where $\widetilde{\mathcal{T}}_{\beta,A}^{(k,\mathtt{L})}$ is the restriction of $\widetilde{\mathcal{T}}_{\beta,A}^{(k)}$ to the $\mathtt{L}$ largest weights in $[0,1]\times[-A,A]$, that is

$$\widetilde{\mathcal{T}}_{\beta,A}^{(k,\mathtt{L})}:=\sup_{s\in\mathscr{M}_A,N(s)=k}\Big\{\pi^{(\mathtt{L})}(s)-\mathrm{Ent}(s)-\frac{k}{2\beta}\Big\}$$

In analogy with Proposition 7.5, one shows that $\widetilde{\mathcal{T}}_{\beta,A}^{(k,\mathtt{L})}\to\widetilde{\mathcal{T}}_{\beta,A}^{(k)}$ as $\mathtt{L}\to\infty$, which completes the proof.

The proof of (8.17) comes from the rewriting

$$\begin{aligned}\sum_{k=1}^K\mathbf{U}_k^{(\mathtt{L})}&=\sum_{\Delta\subset\Upsilon_{\mathtt{L}},|\Delta|\leqslant K}e^{\beta_n\Omega_{n,A_n}(\Delta)}\mathbf{P}\big(S\cap\Upsilon_{\mathtt{L}}=\Delta\big)\\ &=\sum_{\Delta\subset\Upsilon_{\mathtt{L}},|\Delta|\leqslant K}\exp\Big(\beta_n\Omega_{n,A_n}(\Delta)-\mathrm{Ent}(\Delta)-\frac{|\Delta|}{2}\log n+o(K)\Big),\end{aligned}$$

where for the last inequality we used Stone local limit theorem [33] (using that any two points in $\Upsilon_{\mathtt{L}}$ have abscissa differing by at least εn with probability going to 1 as $\varepsilon\to0$)

to get that $\mathbf{P}\big(S \cap \Upsilon_{\mathtt{L}} = \Delta\big) = n^{-\frac{|\Delta|}{2}+o(1)} e^{-\mathrm{Ent}(\Delta)}$ uniformly for $\Delta \subset \Upsilon_{\mathtt{L}}$. Since there are finitely many terms in the sum, we get that analogously to (7.14)-(7.16),

$$\sum_{k=1}^{K} \mathbf{U}_k^{(\mathtt{L})} = e^{o(\log n)} \times \exp\Big(\max_{\Delta\subset\Upsilon_{\mathtt{L}},|\Delta|\leqslant K} \Big\{ \beta_n \Omega_{n,A_n}(\Delta) - \mathrm{Ent}(\Delta) - \frac{|\Delta|}{2}\log n \Big\}\Big).$$

At this stage we write

$$\max_{\Delta\subset\Upsilon_{\mathtt{L}},|\Delta|\leqslant K} \Big\{ \beta_n \Omega_{n,A_n}(\Delta) - \mathrm{Ent}(\Delta) - \frac{|\Delta|}{2}\log n \Big\} = \max_{1\leqslant k\leqslant K} \widetilde{T}_{n,h}^{\beta_{n,h},(k,\mathtt{L})},$$

$$\text{where} \qquad \widetilde{T}_{n,h}^{\beta_{n,h},(k,\mathtt{L})} := \max_{\Delta\subset\Upsilon_{\mathtt{L}},|\Delta|=k} \Big\{ \beta_n \Omega_{n,A_n}(\Delta) - \mathrm{Ent}(\Delta) - \frac{k}{2}\log n \Big\}$$

To complete the proof of (8.17) we only have to show that

$$\frac{n}{h_n^2}\log\Big(\sum_{k=1}^{K} \mathbf{U}_k^{(\mathtt{L})}\Big) = o(1) + \frac{n}{h_n^2}\max_{1\leqslant k\leqslant K} \widetilde{T}_{n,h}^{\beta_{n,h},(k,\mathtt{L})} \xrightarrow{\text{(d)}} \max_{1\leqslant k\leqslant K} \widetilde{\mathcal{T}}_{\beta,A}^{(k,\mathtt{L})}. \tag{8.18}$$

In analogy with (7.17) and Proposition 7.5, we have that for any fixed k,

$$\frac{n}{h_n^2}\widetilde{T}_{n,h}^{\beta_{n,h},(k,\mathtt{L})} \xrightarrow{\text{(d)}} \widetilde{\mathcal{T}}_{\beta,A}^{(k,\mathtt{L})}.$$

As for the convergence of (5.11), since we have only a finite number of points, the proof is a consequence of (5.8) and (5.9) and the Skorokhod representation theorem – we use also that $\frac{n}{h_n^2}\log n \to \frac{1}{\beta}$. Since the maximum is taken over a finite number of terms, this shows (8.18) and concludes the proof.

8.5. **Regime 4: convergence of the main term.** First of all, we show briefly that W_β is well defined, before we turn to the proof of Lemma 8.3. One of the difficulties here is that the reduction to trajectories operated in Section 8.1 (to trajectories with $\max_{i\leqslant n}|S_i| \leqslant A\sqrt{n\log n}$) is not adapted here, since the transversal fluctuations are of order $h_n \ll \sqrt{n\log n}$. Therefore, we have to further reduce the set of trajectories in $\mathbf{U}_1$.

Well-posedness and properties of W_β. We prove the following proposition.

Proposition 8.4. *Assume that $\alpha \in (1/2, 1)$. Then for every $\beta > 0$, $W_\beta \in (0,\infty)$ almost surely.*

Proof. Recalling the definition (2.14) of W_β. We fix a region $\mathcal{D}_\varepsilon := [\frac{1}{2}, 1]\times[-\varepsilon,\varepsilon]$, for $\varepsilon > 0$. In such a way we have that

$$W_\beta \geqslant \sup_{(w,t.x)\in\mathcal{P};(t,x)\in\mathcal{D}_\varepsilon} \big\{\, w \,\big\} - \frac{\varepsilon^2}{\beta}. \tag{8.19}$$

We observe that

$$\max_{(w,t,x)\in\mathcal{P};(t,x)\in\mathcal{D}_\varepsilon} \big\{\, w \,\big\} \overset{\text{(d)}}{=} (2\varepsilon)^{1/\alpha}\mathrm{Exp}(1)^{-1/\alpha}.$$

Therefore, since $\frac{1}{\alpha} < 2$, the r.h.s. of (8.19) is a.s. positive provided ε is sufficiently small.

For an upper bound, we simply observe that $W_\beta \leqslant \mathcal{T}_\beta < \infty$ a.s. □

Proof of Lemma 8.3. We denote $p(i,x) := \mathbf{P}(S_i = x)$ for the random walk kernel. For $A > 0$ fixed and $\delta > 0$, we split $\sqrt{n}\,\mathbf{U}_1$ into three parts:

$$\begin{aligned}\sqrt{n}\,\mathbf{U}_1 &:= \sum_{(i,x)\in\Upsilon_\ell} e^{\beta_n\omega_{i,x}}\sqrt{n}p(i,x)\\ &= \Big(\sum_{(i,x)\in\Upsilon_\ell,|x|>Ah_n} + \sum_{(i,x)\in\Upsilon_\ell,i<\delta n,|x|\leqslant Ah_n} + \sum_{(i,x)\in\Upsilon_\ell,i\geqslant\delta n,|x|\leqslant Ah_n}\Big) e^{\beta_n\omega_{i,x}}\sqrt{n}p(i,x)\,. \quad (8.20)\end{aligned}$$

The main term is the last one, and we now give three lemmas to control the three terms.

Lemma 8.5. *There exist constants c and $\nu > 0$ such that for all n sufficiently large, for any $A > 1$*

$$\mathbb{P}\Big(\sum_{(i,x)\in\Upsilon_\ell,|x|>Ah_n} e^{\beta_n\omega_{i,x}}\sqrt{n}p(i,x) > A\Big(\frac{h_n^2}{n}\Big)^{3\alpha}\Big) \leqslant cA^{-\nu}\,. \quad (8.21)$$

Lemma 8.6. *There exist some $c,\nu > 0$ such that, for any $A > 1$ and $0 < \delta < A^{-1}$, we get that for n sufficiently large,*

$$\mathbb{P}\Big(\frac{n}{h_n^2}\log\Big(\sum_{(i,x)\in\Upsilon_\ell,i<\delta n,|x|\leqslant Ah_n} e^{\beta_n\omega_{i,x}}\sqrt{n}p(i,x)\Big) \geqslant (\delta A)^{\frac{1}{4\alpha}}\Big) \leqslant c(\delta A)^{1/2}\,. \quad (8.22)$$

And finally, for last term, we have the convergence.

Lemma 8.7. *We have that*

$$\frac{n}{h_n^2}\log\Big(\sum_{(i,x)\in\Upsilon_\ell,i\geqslant\delta n,|x|\leqslant Ah_n} e^{\beta_n\omega_{i,x}}\sqrt{n}p(i,x)\Big) \xrightarrow{(\mathrm{d})} W_1(\delta,A) := \max_{\substack{(w,t,x)\in\mathcal{P}\\ t>\delta,|x|\leqslant A}}\Big\{w - \frac{x^2}{2t}\Big\}\,.$$

Now, let us observe that taking the limit $\delta \downarrow 0$, and $A \uparrow \infty$, we readily obtain that $W_1(\delta,A) \to W_1$ (by monotonicity). Therefore, combining Lemmas 8.5-8.6-8.7, we conclude the proof of Lemma 8.3. □

Proof of Lemma 8.5. Let us consider the event

$$\mathcal{G}(n,A) := \Big\{\beta_n\omega_{i,x} \leqslant \frac{x^2}{8i} \text{ for any } |x| > Ah_n,\ 1 \leqslant i \leqslant n\Big\}. \quad (8.23)$$

Using this event to split the probability (and Markov's inequality), we have that, recalling the definition (8.9) of Υ_ℓ

$$\begin{aligned}&\mathbb{P}\Big(\sum_{(i,x)\in\Upsilon_\ell,|x|>Ah_n} e^{\beta_n\omega_{i,x}}\sqrt{n}p(i,x) > A\Big(\frac{h_n^2}{n}\Big)^{3\alpha}\Big)\\ &\leqslant \frac{1}{A}\Big(\frac{h_n^2}{n}\Big)^{-3\alpha}\mathbb{E}\Big[\sum_{i=1}^n\sum_{|x|>Ah_n} e^{x^2/8i}\sqrt{n}p(i,x)1_{\{\beta_n\omega_{i,x}\geqslant 1\}}\Big] + \mathbb{P}\Big(\mathcal{G}(n,A)^c\Big)\,. \quad (8.24)\end{aligned}$$

Using again that $\mathbb{P}(\omega \geqslant 1/\beta_n) \leqslant (h_n^2/n)^{2\alpha}(nh_n)^{-1}$ and that $p(i,x) \leqslant e^{-x^2/4i}$ uniformly in the range considered (provided that n is large enough), we get that the first term is bounded by

$$\frac{1}{A}\Big(\frac{h_n^2}{n}\Big)^{-\alpha}\frac{\sqrt{n}}{nh_n}\sum_{i=1}^n\sum_{|x|>Ah_n} e^{-x^2/8i} \leqslant \Big(\frac{h_n^2}{n}\Big)^{-\alpha}.$$

In the last inequality, we used that the sum over x is bounded by a constant independent of i, and also that $\sqrt{n}/h_n \to 0$. The first term in (8.24) therefore goes to 0 as $n \to \infty$, and we are left to control $\mathbb{P}(\mathcal{G}(n,A)^c)$. A union bound gives

$$\mathbb{P}\big(\mathcal{G}(n,A)^c\big) \leqslant \sum_{i=1}^{n} \sum_{x=Ah_n}^{+\infty} \mathbb{P}\Big(\beta_n \omega_{i,x} \geqslant \frac{x^2}{8i}\Big) \leqslant n \sum_{k=0}^{+\infty} \sum_{x=2^k Ah_n}^{2^{k+1}Ah_n} \mathbb{P}\Big(\beta_n \omega \geqslant 2^{2k} A^2 \frac{h_n^2}{8n}\Big)$$
$$\leqslant 2Anh_n \sum_{k=0}^{\infty} 2^k \mathbb{P}\Big(\omega \geqslant \frac{1}{10} 2^{2k} A^2 m(nh_n)\Big)\,, \tag{8.25}$$

where we used the definition (2.2) of h_n for the last inequality, with n large enough. Then, using the definition of $m(nh_n)$ and Potter's bound, we obtain that for any $\eta > 0$ (chosen such that $1 - 2\alpha + 2\eta < 0$) there is a constant $c > 0$ such that for n large enough

$$\mathbb{P}\big(\mathcal{G}(n,A)^c\big) \leqslant cAnh_n \sum_{k \geqslant 1} 2^k (2^{2k}A^2)^{-\alpha+\eta} \frac{1}{nh_n} \leqslant c' A^{1-2\alpha+2\eta}\,,$$

where the sum over k is finite because $1 - 2\alpha + 2\eta < 0$. This concludes the proof of Lemma 8.5. □

Proof of Lemma 8.6. Decomposing over the event

$$\mathcal{M}_n(\delta, A) = \Big\{ \max_{i < \delta n, |x| \leqslant Ah_n} \beta_n \omega_{i,x} \leqslant \frac{1}{2} (\delta A)^{\frac{1}{4\alpha}} \frac{h_n^2}{n} \Big\}\,,$$

and using Markov's inequality, we get that (similarly to (8.24))

$$\mathbb{P}\Big(\sum_{(i,x) \in \Upsilon_\ell, i < \delta n, |x| \leqslant Ah_n} e^{\beta_n \omega_{i,x}} \sqrt{n} p(i,x) \geqslant \exp\Big((\delta A)^{\frac{1}{4\alpha}} \frac{h_n^2}{n} \Big) \Big) \tag{8.26}$$
$$\leqslant e^{-\frac{1}{2}(\delta A)^{\frac{1}{4\alpha}} \frac{h_n^2}{n}} \mathbb{E}\Big[\sum_{i=1}^{\delta n} \sum_{|x| \leqslant Ah_n} \sqrt{n} p(i,x) 1_{\{\beta_n \omega_{i,x} \geqslant 1\}} \Big] + \mathbb{P}\big(\mathcal{M}_n(\delta,A)^c\big)\,.$$

We use again that $\mathbb{P}(\omega \geqslant 1/\beta_n) \leqslant (h_n^2/n)^{2\alpha} (nh_n)^{-1}$, and the fact that $\sum_x p(i,x) = 1$ for any $i \in \mathbb{N}$, to get that the first term is bounded by

$$e^{-\frac{1}{2}(\delta A)^{\frac{1}{4\alpha}} \frac{h_n^2}{n}} \Big(\frac{h_n^2}{n}\Big)^{2\alpha} \frac{n\sqrt{n}}{nh_n} \to 0 \quad \text{as } n \to \infty\,.$$

For the remaining term, using that $\beta_n^{-1} h_n^2/n \sim m(nh_n)$, we have by a union bound that for n large enough

$$\mathbb{P}\big(\mathcal{M}_n(\delta,A)^c\big) \leqslant \delta Anh_n \mathbb{P}\Big(\omega > \frac{1}{4}(\delta A)^{\frac{1}{4\alpha}} m(nh_n)\Big) \leqslant c\delta Anh_n \times \big((\delta A)^{\frac{1}{4\alpha}}\big)^{-2\alpha} \frac{1}{nh_n}\,,$$

where we used Potter's bound (with $(\delta A)^{\frac{1}{4\alpha}}$ small) and the definition of $m(nh_n)$ for the last inequality (for n large). This concludes the proof of Lemma 8.6. □

Proof of Lemma 8.7. The Stone local limit theorem [33] (see (2.15)) gives that, for fixed $A > 0, \delta > 0$, there exists $c > 0$ such that uniformly for $\delta n \leqslant i \leqslant n$, $|x| \leqslant Ah_n$,

$$\frac{1}{c}\, e^{-x^2/2i} \leqslant \sqrt{i}\, p(i,x) \leqslant c\, e^{-x^2/2i}\,. \tag{8.27}$$

Since $\sqrt{n/i} \geqslant 1$ for all $i \leqslant n$, we get the lower bound

$$\sum_{i=\delta n}^{n} \sum_{|x|\leqslant Ah_n} e^{\beta_n \omega_{i,x}} \sqrt{n} p(i,x) 1_{\{\beta_n \omega_{i,x} \geqslant 1\}} \geqslant c \exp\Big(\beta_n W_n(\delta, A)\Big), \tag{8.28}$$

where $W_n(\delta, A)$ is a discrete analogue of $W_1(\delta, A)$, that is

$$W_n(\delta, A) := \max_{\substack{|x|\leqslant Ah_n,\, i=\delta n,\dots,n \\ \beta_n \omega_{i,x} \geqslant 1}} \Big\{\omega_{i,x} - \frac{x^2}{2\beta_n i}\Big\}. \tag{8.29}$$

On the other hand, we get that $\sqrt{n/i} \leqslant \delta^{-1/2}$ for $i \geqslant \delta n$, so that from (8.27) we get

$$\sum_{i=\delta n}^{n} \sum_{|x|\leqslant Ah_n} e^{\beta_n \omega_{i,x}} \sqrt{n} p(i,x) 1_{\{\beta_n \omega_{i,x} \geqslant 1\}} \leqslant \frac{c}{\sqrt{\delta}} e^{\beta_n W_n(\delta, A)} \sum_{i=1}^{n} \sum_{|x|\leqslant Ah_n} 1_{\{\beta_n \omega_{i,x} \geqslant 1\}}. \tag{8.30}$$

Now, we have that $\mathbb{P}(\omega > 1/\beta_n) \leqslant (h_n^2/n)^{2\alpha}(nh_n)^{-1}$ as already noticed, so that

$$\mathbb{E}\Big[\sum_{i=1}^{n} \sum_{|x|\leqslant Ah_n} 1_{\{\beta_n \omega_{i,x} \geqslant 1\}}\Big] \leqslant A \left(\frac{h_n^2}{n}\right)^{2\alpha}. \tag{8.31}$$

Overall, combining (8.28) with (8.30)-(8.31), we get that with probability going to 1 as $n \to \infty$,

$$\left|\log\Big(\sum_{(i,x)\in\Upsilon_\ell, i\geqslant\delta n, |x|\leqslant Ah_n} e^{\beta_n \omega_{i,x}} \sqrt{n} p(i,x)\Big) - \beta_n W_n(\delta, A)\right| \leqslant (2\alpha+1) \log \frac{h_n^2}{n}.$$

To conclude the proof of Lemma 8.6, it therefore remains to show that

$$\frac{n}{h_n^2} \times \beta_n W_n(\delta, A) \xrightarrow[n\to\infty]{\text{(d)}} W_1(\delta, A), \tag{8.32}$$

where $W_1(\delta, A)$ is defined in Lemma 8.6.

We fix $\varepsilon > 0$ and we consider $\widetilde{W}_n(\varepsilon, \delta, A)$ the truncated version of $W_n(\delta, A)$ in which we replace the condition $\{\beta_n \omega_{i,x} \geqslant 1\}$ by $\{\beta_n \omega_{i,x} > \varepsilon \frac{h_n^2}{n}\}$, that is

$$\widetilde{W}_n(\varepsilon, \delta, A) := \max_{\substack{|x|\leqslant Ah_n,\, i=\delta n,\dots,n \\ \beta_n \omega_{i,x} > \varepsilon \frac{h_n^2}{n}}} \Big\{\omega_{i,x} - \frac{x^2}{2\beta_n i}\Big\}. \tag{8.33}$$

In such a way, and since $\varepsilon h_n^2/n \geqslant 1$ for large n, we have

$$\frac{n}{h_n^2}\beta_n \widetilde{W}_n(\varepsilon, \delta, A) \leqslant \frac{n}{h_n^2}\beta_n W_n(\delta, A) \leqslant \frac{n}{h_n^2}\beta_n \widetilde{W}_n(\varepsilon, \delta, A) + \varepsilon.$$

To prove (8.32) we need to show that

$$\frac{n}{h_n^2} \times \beta_n \widetilde{W}_n(\varepsilon, \delta, A) \xrightarrow[n\to\infty]{\text{(d)}} \widetilde{W}_1(\varepsilon, \delta, A) := \max_{\substack{(w,t,x)\in\mathcal{P} \\ t>\delta, |x|\leqslant A, w>\varepsilon}} \Big\{w - \frac{x^2}{2t}\Big\}, \tag{8.34}$$

and then let $\varepsilon \downarrow 0$ – notice that we have $\widetilde{W}_1(\varepsilon, \delta, A) \leqslant W_1(\delta, A) \leqslant \widetilde{W}_1(\varepsilon, \delta, A) + \varepsilon$ so that $\widetilde{W}_1(\varepsilon, \delta, A) \to W_1(\delta, A)$ as $\varepsilon \downarrow 0$.

We observe that a.s. there are only finitely many $\omega_{i,x}$ in $[\![1,n]\!] \times [\![-Ah_n, Ah_n]\!]$ that are larger than $\varepsilon m(nh_n) \sim \beta_n^{-1}\varepsilon h_n^2/n$. This is a consequence of Markov's inequality and Borel-Cantelli Lemma. Indeed, for any $K \in \mathbb{N}$ we have

$$\mathbb{P}\Big(\Big| \big\{(i,x) \in [\![1,n]\!] \times [\![-Ah_n, Ah_n]\!] \colon \omega_{i,x} \geqslant \varepsilon m(nh_n)\big\}\Big| > 2^K \Big) \\ \leqslant 2^{-K}(2Anh_n)\mathbb{P}\Big(\omega \geqslant \varepsilon m(nh_n)\Big) \leqslant C_\varepsilon 2^{-K}.$$

Therefore, the convergence (8.34) is a straightforward consequence of the Skorokhod representational theorem. □

9. Case $\alpha \in (0, 1/2)$

In the first part of this section we prove (2.16). In the second part, we prove the convergence (2.17).

9.1. Transversal fluctuations: proof of (2.16).

Paths cannot be at an intermediate scale. We start by showing that there exists $c_0, c, \nu > 0$ such that for any sequences $C_n > 1$ and $\delta_n \in (0,1)$ (which may go to ∞, resp. 0, as $n \to \infty$) and for any $n \geqslant 1$

$$\mathbb{P}\Big(\mathbf{P}^\omega_{n,\beta_n}\big(\max_{i\leqslant n}|S_i| \in [C_n\sqrt{n}, \delta_n n)\big) \leqslant e^{-c_0 C_n^2} + e^{-c_0 n^{1/2}}\Big) \geqslant 1 - c\delta_n^\nu + n^{-\frac{1-2\alpha}{4}+\varepsilon}. \tag{9.1}$$

To prove it, we use a decomposition into blocks, as we did in Section 6. Here, we have to partition the interval $[C_n\sqrt{n}, \delta_n n)$ into $[C_n\sqrt{n}, n^{3/4}) \cup [n^{3/4}, \delta_n n)$ (one of these intervals might be empty), obtaining

$$\mathbf{P}^\omega_{n,\beta_n}\Big(\max_{i\leqslant n}|S_i| \in [C_n\sqrt{n}, \delta n)\Big) \\ = \mathbf{P}^\omega_{n,\beta_n}\Big(\max_{i\leqslant n}|S_i| \in [C_n\sqrt{n}, n^{3/4})\Big) + \mathbf{P}^\omega_{n,\beta_n}\Big(\max_{i\leqslant n}|S_i| \in (n^{3/4}, \delta_n n)\Big). \tag{9.2}$$

For the first term, we partition the interval $[C_n\sqrt{n}, n^{3/4})$ into smaller blocks $D_{k,n} := [2^k\sqrt{n}, 2^{k+1}\sqrt{n})$, with $k = \log_2 C_n, \dots, \log_2 n^{1/4} - 1$. Let us define

$$\Sigma(n,h) = \sum_{i=1}^n \sum_{x\in[\![-h,h]\!]} \omega_{i,x} \tag{9.3}$$

the sum of all weights in $[\![1,n]\!] \times [\![-h,h]\!]$. Then, we write similarly to (6.2) (we also use that $\mathbf{Z}^\omega_{n,\beta_n} \geqslant 1$, which is harmless here since no recentering term is needed)

$$\begin{aligned}\mathbf{P}^\omega_{n,\beta_n}\Big(\max_{i\leqslant n}|S_i| \in [C_n\sqrt{n}, n^{3/4})\Big) &\leqslant \sum_{k=\log_2 C_n}^{\log_2 n^{1/4}} \mathbf{Z}^\omega_{n,\beta_n}\big(\max_{i\leqslant n}|S_i| \in D_{k,n}\big) \\ &\leqslant \sum_{k=\log_2 C_n}^{\log_2 n^{1/4}-1} e^{\beta_n \Sigma(n, 2^{k+1}\sqrt{n})}\mathbf{P}\big(\max_{i\leqslant n}|S_i| \in D_{k,n}\big) \\ &\leqslant \sum_{k=\log_2 C_n}^{\log_2 n^{1/4}} \exp\Big(\beta_n\Sigma(n, 2^{k+1}\sqrt{n}) - c2^{2k}\Big)\end{aligned}$$

where for the last inequality we used a standard estimate for the deviation probability of a random walk $\mathbf{P}\big(\max_{i\leqslant n}|S_i|\geqslant 2^k\sqrt{n}\big)\leqslant e^{-c2^{2k}}$, see for example [28, Prop. 2.1.2-(b)]. Therefore, on the event

$$\Big\{\forall\, k=\log_2 C_n,\dots,\log_2 n^{1/4},\ \beta_n\Sigma(n,2^{k+1}\sqrt{n})\leqslant \frac{c}{2}2^{2k}\Big\} \tag{9.4}$$

we have that

$$\mathbf{P}^{\omega}_{n,\beta}\Big(\max_{i\leqslant n}|S_i|\in[C_n\sqrt{n},n^{3/4})\Big)\leqslant \sum_{k=\log_2 C_n}^{\log_2 n^{1/4}} e^{-\frac{c}{2}2^{2k}}\leqslant c'e^{-\frac{c}{2}C_n^2}. \tag{9.5}$$

For the second term in (9.2), we partition the interval $(n^{3/4},\delta_n n)$ into blocks $E_{n,k}:=[2^{-k-1}n,2^{-k}n)$, $k=\log_2(1/\delta_n),\dots,\log_2 n^{1/4}-1$. Exactly as above we use the large deviation estimate $\mathbf{P}\big(\max_{i\leqslant n}|S_i|\geqslant 2^{-k+1}n\big)\leqslant e^{-c2^{-2k}n}$ (see e.g [28, Prop. 2.1.2-(b)]), and we obtain that on the event

$$\Big\{\forall\, k=\log_2(1/\delta_n),\dots,\log_2 n^{1/4},\ \beta_n\Sigma(n,2^{-kn})\leqslant \frac{c}{2}2^{-2k}n\Big\} \tag{9.6}$$

we have

$$\mathbf{P}^{\omega}_{n,\beta}\Big(\max_{i\leqslant n}|S_i|\in(n^{3/4},\delta_n n)\Big)\leqslant \sum_{k=\log_2(1/\delta_n)}^{\log_2 n^{1/4}} e^{-\frac{c}{2}2^{-2k}n}\leqslant c'e^{-\frac{c}{2}n^{1/2}}. \tag{9.7}$$

It now only remains to show that the complementary events of (9.4) and (9.6) have small probability. We start with (9.6). Using that $\beta_n\leqslant 2\beta n/m(n^2)$ for n large, we get by a union bound that

$$\mathbb{P}\Big(\exists\, k\geqslant\log_2 1/\delta_n\,,\ \beta_n\Sigma(n,2^{-k}n)>\frac{c}{2}2^{-2k}n\Big)\leqslant \sum_{k\geqslant\log_2 1/\delta_n}\mathbb{P}\Big(\Sigma(n,2^{-k}n)>c_\beta 2^{-2k}m(n^2)\Big) \tag{9.8}$$

Then, by Potter's bound we have that $m(2^{-k+1}n^2)\leqslant 2^{-2k}m(n^2)$ since $\alpha<1/2$ (recall $m(\cdot)$ (2.1) is regularly varying with exponent $1/\alpha$). As a consequence, the last probability in (9.8) is in the so-called one-jump large deviation domain (see [30, Thm. 1.1], we are using $\alpha<1$ here), that is

$$\mathbb{P}\Big(\Sigma(n,2^{-k}n)>c_\beta 2^{-2k}m(n^2)\Big)\sim 2^{-k+1}n^2\mathbb{P}\big(\omega>c_\beta 2^{-2k}m(n^2)\big)\,.$$

Therefore, using again Potter's bound, we get that for arbitrary η there is some constant c such that

$$\mathbb{P}\Big(\Sigma(n,2^{-k}n)>c_\beta 2^{-2k}m(n^2)\Big)\leqslant c(2^{2k})^{\alpha+\eta}n^{-2}$$

where we also used that $\mathbb{P}(\omega>m(n^2))=n^{-2}$. Therefore, taking η small enough so that $2\alpha-1+2\eta<0$, we obtain that (9.8) is bounded by a constant times

$$\sum_{k\geqslant\log_2 1/\delta_n}2^{k(2\alpha-1+2\eta)}\leqslant c\delta_n^{1-2\alpha+2\eta}\,.$$

Similarly, for (9.4), we have by a union bound that

$$\mathbb{P}\Big(\exists\, k\in\{\log_2 C_n,\dots,\log_2 n^{1/4}\},\ \beta_n\Sigma(n,2^{k+1}\sqrt{n})>\frac{c}{2}2^{2k}\Big)$$
$$\leqslant\sum_{k=\log_2 C_n}^{\log_2 n^{1/4}}\mathbb{P}\Big(\Sigma(n,2^{k+1}\sqrt{n})>c_\beta 2^{2k}n^{-1}m(n^2)\Big)\,. \tag{9.9}$$

Then again, we notice that $m(2^{k+2}n^{3/2}) \leqslant 2^{2k}n^{-1}m(n^2)$ (using Potter's bound, as $\alpha < 1/2$). Hence, the last probability in (9.9) is in the one-jump large deviation domain (see [30, Thm. 1.1]), that is

$$\mathbb{P}\Big(\Sigma(n, 2^{k+1}\sqrt{n}) > c2^{2k}n^{-1}m(n^2)\Big) \leqslant c2^k n^{3/2}\mathbb{P}\big(\omega > c_\beta 2^{2k}n^{-1}m(n^2)\big)$$

Then, we also get that for any $\eta > 0$ we have that there is a constant $c > 0$ such that

$$\mathbb{P}\big(\omega > c_\beta 2^{2k}n^{-1}m(n^2)\big) \leqslant c(2^{2k}n^{-1})^{-\alpha-\eta}\,,$$

so that provided that $1 - 2\alpha - 2\eta > 0$, (9.9) is bounded by a constant times

$$\sum_{k=\log_2 C_n}^{\log_2 n^{1/4}} 2^{k(1-2\alpha-2\eta)}n^{\alpha-\frac{1}{2}+\eta} \leqslant cn^{-\frac{1}{4}(1-2\alpha-2\eta)}\,.$$

Paths cannot be at scale n conditionnaly on $\widehat{\mathcal{T}}_\beta = 0$. We have shown in (9.1) that paths cannot be on an intermediate scale: it remains to prove that on the event $\widehat{\mathcal{T}}_\beta = 0$, paths cannot be at scale n. For this purpose we use [5, Theorem 2.1] and [35, Theorem 1.8], which ensure that for any δ and $\varepsilon > 0$ there exists $\nu > 0$ such that

$$\mathbb{P}\Big(\mathbf{P}^\omega_{n,\beta_n}\big(\max_{i\leqslant n}|S_i| \in (\delta n, n]\big) \leqslant e^{-n\nu} \;\Big|\; \widehat{\mathcal{T}}_\beta = 0\Big) \geqslant 1 - \varepsilon. \tag{9.10}$$

Therefore, we get that for any $\varepsilon > 0$ and $\delta > 0$, combining (9.1) with (9.10), for any sequence $C_n > 1$, provided that n is large enough we have

$$\mathbb{P}\Big(\mathbf{P}^\omega_{n,\beta_n}\big(\max_{i\leqslant n}|S_i| \geqslant C_n\sqrt{n}\big) \geqslant e^{-c_0C_n^2} + e^{-c_0n^{1/2}} + e^{-n\nu} \;\Big|\; \widehat{\mathcal{T}}_\beta = 0\Big) \leqslant c\delta^\nu + 2\varepsilon\,, \tag{9.11}$$

which concludes the proof of (2.16).

9.2. **Convergence in distribution conditionally on $\widehat{\mathcal{T}}_\beta = 0$, proof of (2.17).** In the following, we will consider the case where $\beta_n n^{-1}m(n^2) \to \beta$ with $\beta < \infty$. In the case $\beta = +\infty$, we would indeed have that $\widehat{\mathcal{T}}_\beta > 0$. The proof follows the same idea as that of [18, Thm. 1.4] (and similar steps as above), but with many adaptations (and simplifications) in our case. We will also focus on the case $\beta > 0$, in which $\frac{\sqrt{n}}{\beta_n m(n^{3/2})}$ goes to infinity as a regularly varying function with exponent $\frac{2}{\alpha} - \frac{1}{2} - \frac{3}{2\alpha} = \frac{1-\alpha}{2\alpha} > 0$ (If $\beta = 0$, it goes to infinity faster).

Step 1. Reduction of the set of trajectories. Equation (2.16) (with $C_n = A\sqrt{\log n}$) gives that, with $\mathbb{P}$ probability larger than $1-\varepsilon$ (conditionally on $\widehat{\mathcal{T}}_\beta = 0$), we have $\mathbf{P}^\omega_{n,\beta_n}\big(\max_{i\leqslant n}|S_i| \leqslant A\sqrt{n\log n}\big) \geqslant 1 - e^{-c_0A\log n}$ provided that n is large enough. We therefore get

$$\mathbb{P}\Big(\big|\log \mathbf{Z}^\omega_{n,\beta_n} - \log \mathbf{Z}^\omega_{n,\beta_n}(\mathcal{A}_n)\big| \leqslant n^{-c_0A} \;\Big|\; \widehat{\mathcal{T}}_\beta = 0\Big) \geqslant 1 - \varepsilon\,, \tag{9.12}$$

where $\mathcal{A}_n$ is defined in (8.2). Note that, provided A has been fixed large enough, we have that $\frac{\sqrt{n}}{\beta_n m(n^{3/2})}n^{-c_0A} \to 0$ as $n \to \infty$: we conclude that, for any $\varepsilon > 0$

$$\mathbb{P}\left(\frac{\sqrt{n}}{\beta_n m(n^{3/2})}\big|\log \mathbf{Z}^\omega_{n,\beta_n} - \log \mathbf{Z}^\omega_{n,\beta_n}(\mathcal{A}_n)\big| > \varepsilon \;\middle|\; \widehat{\mathcal{T}}_\beta = 0\right) \leqslant \varepsilon\,, \tag{9.13}$$

provided that n is large enough. We will therefore focus on $\log \mathbf{Z}^\omega_{n,\beta_n}(\mathcal{A}_n)$.

As in Section 8, we use the notation $A_n = A\sqrt{n\log n} = C_n\sqrt{n}$ and $\Lambda_{n,A_n} = [\![1,n]\!] \times [\![-A_n, A_n]\!]$.

Step 2. Truncation of the weights. We let $k_n := m(n^{3/2}\log n)$ be a sequence of truncation levels, and $\widetilde{\omega}_x := \omega_x 1_{\{\omega_x \leqslant k_n\}}$ be the truncated environment. Then, we have that

$$\mathbb{P}\Big(\mathbf{Z}^{\omega}_{n,\beta_n}(\mathcal{A}_n) \neq \mathbf{Z}^{\widetilde{\omega}}_{n,\beta_n}(\mathcal{A}_n)\Big) = \mathbb{P}\big(\max_{(i,x)\in\Lambda_{n,A_n}} \omega_{i,x} > m(n^{3/2}\log n)\big) \leqslant \frac{2A}{\sqrt{\log n}} \overset{n\to\infty}{\to} 0\,, \tag{9.14}$$

where we used a union bound for the last inequality, together with the definition of $m(\cdot)$ (2.1). Henceforth we can safely replace $\mathbf{Z}^{\omega}_{n,\beta_n}(\mathcal{A}_n)$ with the truncated partition function $\mathbf{Z}^{\widetilde{\omega}}_{n,\beta_n}(\mathcal{A}_n)$.

Step 3. Expansion of the partition function. We write again $p(i,x) = \mathbf{P}(S_i = x)$ for the random walk kernel, and let $\lambda_n(t) = \log \mathbb{E}[e^{t\widetilde{\omega}_x}]$. Then, expanding

$$\exp\Big(\sum_{i=1}^{n}\big(\beta_n\omega_{i,S_i} - \lambda_n(\beta_n)\big)\Big) = \prod_{(i,x)\in\Lambda_{n,A_n}} \big(1 + e^{\beta_n\widetilde{\omega}_{i,x}-\lambda_n(\beta_n)} - 1\big)^{1_{\{S_i=x\}}},$$

we obtain

$$e^{-n\lambda_n(\beta_n)}\mathbf{Z}^{\widetilde{\omega}}_{n,\beta_n}(\mathcal{A}_n) = 1 + \sum_{(i,x)\in\Lambda_{n,A_n}} \big(e^{\beta_n\widetilde{\omega}_{i,x}-\lambda_n(\beta_n)} - 1\big)p(i,x) + \mathbf{R}_n, \tag{9.15}$$

with

$$\mathbf{R}_n := \sum_{k=2}^{\infty} \sum_{\substack{1\leqslant i_1<\cdots<i_k\leqslant n\\ |x_i|\leqslant A_n,\, i=1,\dots,k}} \prod_{j=1}^{k} \big(e^{\beta_n\widetilde{\omega}_{j,x_j}-\lambda_n(\beta_n)} - 1\big)p_n(i_j - i_{j-1}, x_j - x_{j-1})\,. \tag{9.16}$$

Lemma 9.1. *We have that for n large*

$$\mathbb{P}\left(\frac{\sqrt{n}}{\beta_n m(n^{3/2})}\mathbf{R}_n \geqslant n^{-1/4}\right) \leqslant \frac{(\log n)^{4/\alpha}}{\sqrt{n}} \to 0\,.$$

In particular, $\mathbf{R}_n \to 0$ in probability.

Proof. Note that $\mathbb{E}[\mathbf{R}_n] = 0$, so it will be enough to control the second moment of $\mathbf{R}_n$. Since the $\widetilde{\omega}_{i,x}$ are independent and that $\mathbb{E}[e^{\beta_n\widetilde{\omega}_{i,x}-\lambda_n(\beta_n)} - 1] = 0$, we have

$$\begin{aligned}\mathbb{E}[\mathbf{R}_n^2] &= \sum_{k=2}^{\infty} \sum_{\substack{1\leqslant i_1<\cdots<i_k\leqslant n\\ |x_i|\leqslant A_n,\, i=1,\dots,k}} \big(e^{\lambda_n(2\beta_n)-\lambda_n(\beta_n)} - 1\big)^k \prod_{j=1}^{k} p_n(i_j - i_{j-1}, x_j - x_{j-1})^2\\ &\leqslant \sum_{k=2}^{\infty} \big(e^{\lambda_n(2\beta_n)} - 1\big)^k \Big(\sum_{i=1}^{n}\sum_{x\in\mathbb{Z}} p(i,x)^2\Big)^k. \end{aligned} \tag{9.17}$$

First, we have that

$$\sum_{i=1}^{n}\sum_{x\in\mathbb{Z}} p(i,x)^2 = \mathbf{E}^{\otimes 2}\Big[\sum_{i=1}^{n} 1_{\{S_n = S'_n\}}\Big] \leqslant c\sqrt{n}\,,$$

where S and S' are two independent simple random walks. Then, since $\beta_n\widetilde{\omega} \leqslant \beta_n k_n \to 0$, we can write $e^{2\beta_n\widetilde{\omega}} \leqslant 1 + 3\beta_n\widetilde{\omega}$ for n large, so that

$$\begin{aligned} e^{\lambda_n(2\beta_n)} - 1 &\leqslant 3\beta_n\mathbb{E}[\widetilde{\omega}] = 3\beta_n\int_0^{k_n}\mathbb{P}(\omega > u)\mathrm{d}u\\ &\leqslant c\beta_n L(k_n)k_n^{1-\alpha} \leqslant \frac{c\beta_n k_n}{n^{3/2}\log n}\,. \end{aligned} \tag{9.18}$$

To estimate the integral we used the tail behavior of $\mathbb{P}(\omega > u)$ (1.2) (see [9, Theorem 1.5.8]), while for the last inequality, we used that $k_n = m(n^{3/2}\log n)$ and the definition (2.1) of $m(\cdot)$, so that $L(k_n)k_n^{-\alpha} \sim n^{-3/2}(\log n)^{-1}$. We therefore get that for n large enough

$$\mathbb{E}[\mathbf{R}_n^2] \leqslant \sum_{k\geqslant 2}\Big(\frac{\beta_n k_n}{n}\Big)^k \leqslant 2\Big(\frac{\beta_n k_n}{n}\Big)^2.$$

To conclude, by Potter's bounds we get that $k_n \leqslant m(n^{3/2})(\log n)^{2/\alpha}$ for n large, so that

$$\mathbb{E}[\mathbf{R}_n^2] \leqslant \Big(\frac{\beta_n m(n^{3/2})}{\sqrt{n}}\Big)^2 \times \frac{(\log n)^{\frac{4}{\alpha}}}{n}, \tag{9.19}$$

and the conclusion of the lemma follows by using Markov's inequality. □

Going back to (9.15), we get that

$$\begin{aligned}\mathbf{Z}^{\widetilde{\omega}}_{n,\beta_n}(\mathcal{A}_n) &= e^{(n-1)\lambda_n(\beta_n)}\Big(e^{\lambda_n(\beta_n)} + \sum_{(i,x)\in\Lambda_{n,A_n}}\big(e^{\beta_n\widetilde{\omega}_{i,x}} - e^{\lambda_n(\beta_n)}\big)p(i,x) + e^{\lambda_n(\beta_n)}\mathbf{R}_n\Big)\\ &= e^{(n-1)\lambda_n(\beta_n)}\Big(1 + \mathbf{V}_n + \mathbf{W}_n + e^{\lambda_n(\beta_n)}\mathbf{R}_n\Big), \end{aligned}\tag{9.20}$$

with

$$\mathbf{V}_n := \sum_{(i,x)\in\Lambda_{n,A_n}}\big(e^{\beta_n\widetilde{\omega}_{i,x}} - 1\big)p(i,x) \quad \text{and} \quad \mathbf{W}_n := (e^{\lambda_n(\beta_n)} - 1)\big(1 - \sum_{(i,x)\in\Lambda_{n,A_n}} p(i,x)\big).$$

We show below that $\lim_{n\to\infty}\mathbf{W}_n = 0$ and that $\mathbf{V}_n$ converges in probability to 0, so that using also Lemma 9.1, we get

$$\frac{\sqrt{n}}{\beta_n m(n^{3/2})}\log\mathbf{Z}^{\widetilde{\omega}}_{n,\beta_n}(\mathcal{A}_n) = \frac{\sqrt{n}}{\beta_n m(n^{3/2})}\mathbf{V}_n + \frac{\sqrt{n}}{\beta_n m(n^{3/2})}\Big((n-1)\lambda_n(\beta_n) + \mathbf{W_n}\Big) + o(1).\tag{9.21}$$

Before we prove the convergence of the first term (see Lemma 9.2), we show that the second term goes to 0 – note that this implies that $\lim_{n\to\infty}\mathbf{W}_n = 0$ since $\beta_n n^{-1/2}m(n^{3/2}) \to 0$. We write that

$$\big|(n-1)\lambda_n(\beta_n) + \mathbf{W_n}\big| \leqslant (n-1)\big|e^{\lambda_n(\beta_n)} - 1 - \lambda_n(\beta_n)\big| + \Big|n - \sum_{(i,x)\in\Lambda_{n,A_n}} p(i,x)\Big|. \tag{9.22}$$

For the second term, using standard large deviation probabilities for the simple random walk (e.g. [28, Prop. 2.1.2-(b)]), we get that there is a constant $c > 0$ such that

$$n - \sum_{(i,x)\in\Lambda_{n,A_n}} p(i,x) = \sum_{i=1}^n \mathbf{P}(S_i > A\sqrt{n\log n}) \leqslant ne^{-cA^2\log n}. \tag{9.23}$$

For the first term, since we have $\lambda_n(\beta_n) \to 0$, we get that for n large enough

$$\big|e^{\lambda_n(\beta_n)} - 1 - \lambda_n(\beta_n)\big| \leqslant \lambda_n(\beta_n)^2 \leqslant \Big(\frac{\beta_n m(n^{3/2})}{n^{3/2}}(\log n)^{2/\alpha}\Big)^2, \tag{9.24}$$

where for the second inequality we used (9.18) (note that $\lambda_n(\beta_n) \leqslant e^{\lambda_n(\beta_n)} - 1$), together with the fact that $k_n \leqslant m(n^{3/2})(\log n)^{2/\alpha}$.

Hence plugging (9.23) and (9.24) into (9.22), we get that provided that A is large enough,

$$\frac{\sqrt{n}}{\beta_n m(n^{3/2})}\big|(n-1)\lambda_n(\beta_n) + \mathbf{W_n}\big| \leqslant \frac{\beta_n m(n^{3/2})}{n^{3/2}}(\log n)^{4/\alpha} + o(1) \to 0 \quad \text{as } n\to\infty.$$

so that the second term in (9.21) goes to 0 as $n \to \infty$, proving also that $\mathbf{W}_n \to 0$ (recall also $\beta_n n^{-1/2} m(n^{3/2}) \to 0$).

Step 4. Convergence of the main term. We conclude the proof by showing the convergence in distribution of the first term in (9.21) – which proves also that $\mathbf{V}_n$ goes to 0 in probability, since $\beta_n n^{-1/2} m(n^{3/2}) \to 0$.

Lemma 9.2. *We have the following convergence in distribution,*

$$\frac{\sqrt{n}}{\beta_n m(n^{3/2})} \mathbf{V}_n := \frac{\sqrt{n}}{\beta_n m(n^{3/2})} \sum_{(i,x)\in\Lambda_{n,A_n}} \big(e^{\beta_n \widetilde{\omega}_{i,x}} - 1\big) p(i,x) \xrightarrow[n\to\infty]{(d)} \mathcal{W}_0^{(\alpha)} ,$$

with $\mathcal{W}_0^{\alpha}$ defined in Theorem 2.10.

Proof. First of all, since $\beta_n \widetilde{\omega}_{i,x} \leqslant \beta_n k_n \to 0$ as $n \to \infty$ (and using that $0 \leqslant e^x - 1 - x \leqslant x^2$ for x small), we have that for n large

$$0 \leqslant \mathbf{V}_n - \beta_n \sum_{(i,x)\in\Lambda_{n,A_n}} \widetilde{\omega}_{i,x} p(i,x) \leqslant \sum_{(i,x)\in\Lambda_{n,A_n}} \big(\beta_n \widetilde{\omega}_{i,x}\big)^2 p(i,x) . \tag{9.25}$$

Then, we can estimate the expectation of the upper bound, using that similarly to (9.18) we have $\mathbb{E}[(\widetilde{\omega})^2] \leqslant cL(k_n)k_n^{2-\alpha} \sim ck_n^2/(n^{3/2}\log n)$. Using also that $k_n \leqslant m(n^{3/2})(\log n)^{2/\alpha}$ for n large, we obtain that

$$\begin{aligned}\frac{\sqrt{n}}{\beta_n m(n^{3/2})} \mathbb{E}\Big[\sum_{(i,x)\in\bar{\Lambda}_n} \big(\beta_n \widetilde{\omega}_{i,x}\big)^2 p(i,x)\Big] &\leqslant c \frac{k_n}{m(n^{3/2})} \beta_n k_n \, n^{-1} \sum_{i=1}^{n} \sum_{x\in\mathbb{Z}} p(i,x) \\ &\leqslant c(\log n)^{2/\alpha} \beta_n k_n \overset{n\to\infty}{\to} 0 .\end{aligned}$$

The proof of the lemma is therefore reduced to showing the convergence in distribution of the following term

$$\begin{aligned}&\frac{\sqrt{n}}{m(n^{3/2})} \sum_{(i,x)\in\bar{\Lambda}_n} \widetilde{\omega}_{i,x} p(i,x) \\ &\qquad = \sum_{i=1}^{n} \sum_{|x|\leqslant K\sqrt{n}} \frac{\widetilde{\omega}_{i,x}}{m(n^{3/2})} \sqrt{n} p(i,x) + \sum_{i=1}^{n} \sum_{K\sqrt{n}<|x|\leqslant A_n} \frac{\widetilde{\omega}_{i,x}}{m(n^{3/2})} \sqrt{n} p(i,x),\end{aligned} \tag{9.26}$$

where we fixed some level $K > 0$ (we take the limit $K \to \infty$ in the end).

First term in (9.26). Let us first show that the first term converges in distribution to

$$\mathcal{W}_{0,K}^{(\alpha)} := 2 \int_{\mathbb{R}_+} \int_0^1 \int_{-K}^{K} w\rho(t,x) \mathcal{P}(\mathrm{d}w\mathrm{d}t\mathrm{d}x), \tag{9.27}$$

where $\rho(t,x) := (2\pi t)^{-1/2} e^{-x^2/2t}$ is the Gaussian kernel and $\mathcal{P}(w,t,x)$ a Poisson Point Process on $\mathbb{R}_+ \times [0,1] \times \mathbb{R}$ of intensity $\mu(\mathrm{d}w\mathrm{d}t\mathrm{d}x) = \frac{\alpha}{2} w^{-\alpha-1} 1_{\{w>0\}} \mathrm{d}w\mathrm{d}t\mathrm{d}x$. Let us stress already that since $\mathcal{W}_0^{(\alpha)} < \infty$ a.s. (see [18, Lemma 1.3] or Lemma 9.3), one readily gets that $\mathcal{W}_{0,K}^{(\alpha)} \to \mathcal{W}_0^{(\alpha)}$ as $K \to \infty$ (by monotonicity).

We observe that the random field

$$\{(m(n^{3/2})^{-1}\omega_{i,x}, n^{-1}i,\, n^{-1/2}x) \colon i + x \text{ even}\,, |x| \leqslant K\sqrt{n}\} \tag{9.28}$$

converges to the restriction of $\mathcal{P}$ to $\mathbb{R}_+ \times (0,1) \times (-K,K)$. For the proof of this fact we refer to [18] (see relation (38) and below). Then, the method is similar to [18, p. 4036]

(we use similar notations). We let $\mathscr{P}_\delta$ be a partition $\mathbb{R}_+ \times (0,1) \times \mathbb{R}$ into disjoint rectangles of size $\delta > 0$. For any $\varpi \in \mathfrak{P}_\delta$ we denote by $(w_\varpi, x_\varpi, t_\varpi)$ its center. In such a way we have

$$\begin{aligned}
&\sum_{i=1}^{n} \sum_{|x| \leqslant K\sqrt{n}} \frac{\widetilde{\omega}_{i,x}}{m(n^{3/2})} \sqrt{n} p_n(i,x) \\
&\qquad = \sum_{\varpi \in \mathscr{P}_\delta} \sum_{i=1}^{n} \sum_{|x| \leqslant K\sqrt{n}} \frac{\widetilde{\omega}_{i,x}}{m(n^{3/2})} \sqrt{n} p\big(n\frac{i}{n}, \sqrt{n}\frac{x}{\sqrt{n}}\big) 1_{\big\{\big(\frac{\omega_{i,x}}{m(n^{3/2})}, \frac{i}{n}, \frac{x}{\sqrt{n}}\big) \in \varpi\big\}} \\
&\qquad = \sum_{\varpi \in \mathscr{P}_\delta} (1 + o_\delta(1)) w_\varpi 2\rho(t_\varpi, x_\varpi) \sum_{i=1}^{n} \sum_{|x| \leqslant K\sqrt{n}} 1_{\big\{\big(\frac{\omega_{i,x}}{m(n^{3/2})}, \frac{i}{n}, \frac{x}{\sqrt{n}}\big) \in \varpi\big\}}.
\end{aligned}$$

For the last identity, we used the local limit theorem for the convergence of $\sqrt{n}p(\cdot)$ to $2\rho(\cdot)$, and used the notation $o_\delta(1)$ to denote the error which is negligible as $\delta \to 0$. Moreover, because of the convergence of the field (9.28) toward $\mathcal{P}$, we have

$$\sum_{i=1}^{n} \sum_{|x| \leqslant K\sqrt{n}} 1_{\big\{\big(\frac{\omega_{i,x}}{m(n^{3/2})}, \frac{i}{n}, \frac{x}{\sqrt{n}}\big) \in \varpi\big\}} \to \mathcal{P}(\varpi), \quad \text{as} \quad n \to 0.$$

Taking the limit $n \to \infty$ followed by the limit $\delta \to 0$, we conclude that the first term in (9.26) converges in distribution to $\mathcal{W}_{0,K}$.

Second term in (9.26). To conclude the proof, it remains to show that the second term in (9.26) goes to 0 in probability as $K \to \infty$, uniformly in n: for any K (large), we have for n sufficiently large

$$\mathbb{P}\Big(\sum_{i=1}^{n} \sum_{K\sqrt{n} < |x| \leqslant A_n} \frac{\widetilde{\omega}_{i,x}}{m(n^{3/2})} \sqrt{n} p(i,x) \geqslant K^{-1} \Big) \leqslant c e^{-\alpha K}. \tag{9.29}$$

To prove (9.29), we decompose the sum into smaller parts with $|x| \in (2^{k-1}K\sqrt{n}, 2^k K\sqrt{n}]$ for $k = 1, 2 \ldots$. By a union bound, we have

$$\begin{aligned}
&\mathbb{P}\Big(\sum_{i=1}^{n} \sum_{|x| > K\sqrt{n}} \frac{\widetilde{\omega}_{i,x}}{m(n^{3/2})} \sqrt{n} p(i,x) \geqslant K^{-1} \Big) \\
&\qquad \leqslant \sum_{k=1}^{\infty} \mathbb{P}\Big(\sum_{i=1}^{n} \sum_{|x| = 2^{k-1}K\sqrt{n}}^{2^k K\sqrt{n}} \frac{\omega_{i,x}}{m(n^{3/2})} \sqrt{n} p(i,x) \geqslant K^{-1} 2^{-k} \Big) \\
&\qquad \leqslant \sum_{k \geqslant 1} \mathbb{P}\Big(\sum_{i=1}^{n} \sum_{|x| \leqslant 2^k K\sqrt{n}} \omega_{i,x} \geqslant e^{c'(2^k K)^2} m\big(n^{3/2}\big) \Big)
\end{aligned} \tag{9.30}$$

In the last inequality, we usee that there is a constant c such that for any k, uniformly in $i \in \{1, \ldots, n\}$ and $|x| \geqslant 2^{k-1}K\sqrt{n}$, we have $\sqrt{n}p(i,x) \leqslant e^{-c(2^k K)^2} \leqslant 2^{-k} K^{-1} e^{-c'(2^k K)^2}$ (since $K 2^k \geqslant 1$).

Now, we use that $m(2^{k+1}Kn^{3/2}) \geqslant (2^k K)^{-2/\alpha} m(n^{3/2})$ by Potter's bound, and also that for all k $e^{c'(2^k K)^2} (2^k K)^{-2/\alpha} \geqslant e^{2^k K}$ if K is large, to realize that the last probability in (9.30) is in the one-jump large deviation domain (see [30, Thm. 1.1], we use here that $\alpha < 1$):

there is a constant $c > 0$ such that for all $k \geqslant 1$

$$\mathbb{P}\Big(\sum_{i=1}^{n}\sum_{|x|\leqslant 2^k K\sqrt{n}} \omega_{i,x} \geqslant e^{2^k K} m\big(2^{k+1}Kn^{3/2}\big)\Big) \leqslant c2^k K n^{3/2}\mathbb{P}\Big(\omega \geqslant e^{2^k K} m\big(2^k K n^{3/2}\big)\Big) \leqslant c e^{-\frac{\alpha}{2}2^k K}.$$

The second inequality comes from Potter's bound, provided that $e^{2^k K}$ is large enough, and also used the definition (2.1) of $m(\cdot)$. Plugged in (9.30), we get that

$$\mathbb{P}\Big(\sum_{i=1}^{n}\sum_{|x|>K\sqrt{n}} \frac{\widetilde{\omega}_{i,x}}{m(n^{3/2})}\sqrt{n}p(i,x) \geqslant \varepsilon\Big) \leqslant c\sum_{k\geqslant 1} e^{-\frac{\alpha}{2}2^k K} \leqslant ce^{-\alpha K},$$

which is (9.29). □

We conclude the section by showing that the quantities $\mathcal{W}_{0,K}^{(\alpha)}$ and $\mathcal{W}_0^{(\alpha)}$ are well defined (you can also see [18, Lemma 1.3], here we slightly improve the result) and that the truncated quantity $\mathcal{W}_{0,K}^{(\alpha)}$ converges to $\mathcal{W}_0^{(\alpha)}$ in distribution.

Lemma 9.3. *For any $p \in \mathbb{N}$, we have that a.s.*

$$\int_{\mathbb{R}_+}\int_{(0,1)}\int_{\mathbb{R}} w^p \rho(t,x)\mathcal{P}(dw,dt,dx) < \infty. \tag{9.31}$$

Moreover, we have that $\mathcal{W}_{0,K}^{(\alpha)} \overset{(d)}{\to} \mathcal{W}_0^{(\alpha)}$ as $K \to \infty$, where $\mathcal{W}_{0,K}^{(\alpha)}$ is defined in (9.27).

Proof. The convergence of $\mathcal{W}_{0,K}^{(\alpha)}$ is a consequence of (9.31). To prove (9.31) we consider

$$\mathcal{D}_1 := \big\{(w,t,x)\colon |w| > 1 + x^{2/\alpha}\big\}, \quad \mathcal{D}_2 := \big\{(w,t,x)\colon 1 < |w| < 1 + x^{2/\alpha}\big\},$$
$$\mathcal{D}_3 := \big\{(w,t,x)\colon |w| \leqslant 1\big\}.$$

We have that

$$\mathcal{P}(\mathcal{D}_1) = \int_{\mathcal{D}_1} \frac{\alpha}{2} w^{-\alpha-1} 1_{\{w>0\}} dwdtdx = 2\int_{\mathbb{R}} \Big(1 + x^{2/\alpha}\Big)^{-\alpha} \mathrm{d}x < \infty.$$

This means that there are only a finite number of points of $\mathcal{P}$ in $\mathcal{D}_1$.

On $\mathcal{D}_2$ we have that $w^p\rho(t,x) \leqslant 2^p x^{2p/\alpha}\rho(t,x)$, therefore

$$\mathbb{E}^{\mathcal{P}}\Big[\int_{\mathcal{D}_2} w^p\rho(t,x)\mathcal{P}(dw,dt,dx)\Big] \leqslant 2^p \int_{(0,1)}\int_{\mathbb{R}} x^{2p/\alpha}\rho(t,x)dtdx < \infty.$$

In the same way one checks that $\mathbb{E}^{\mathcal{P}}\big[\int_{\mathcal{D}_3} w^p\rho(x,t)\mathcal{P}(dw,dt,dx)\big] < \infty$ and this concludes the proof. □

REFERENCES

[1] T. Alberts, K. Khanin, and J. Quastel. The continuum directed random polymer. *J. Stat. Phys.*, 154:305–326, 2014.

[2] T. Alberts, K. Khanin, and J. Quastel. The intermediate disorder regime for directed polymers in dimension 1 + 1. *Ann. Probab.*, 42:1212–1256, 2014.

[3] G. Amir, I. Corwin, and J. Quastel. Probability distribution of the free energy of the continuum directed random polymer in 1 + 1 dimensions. *Commun. Pure Appl. Math.*, 64:466–537, 2011.

[4] A. Auffinger and M. Damron. The scaling relation $\chi = 2\xi - 1$ for directed polymers in a random environment. *ALEA, Lat. Am. J. Probab. Math. Stat.*, 10(2):857–880, 2013.

[5] A. Auffinger and O. Louidor. Directed polymers in random environment with heavy tails. *Comm. on Pure and Applied Math.*, 64:183–204, 2011.

[6] J. Baik, P. Deift, and K. Johansson. On the ditribution of the longest increasing subsequence of random permutations. *J. Amer. Math. Soc.*, 12:1119–1178, 1999.
[7] M. Balázs, J. Quastel, and T. Seppäläinen. Fluctuation exponent of the kpz/stochastic burgers equation. *J. Amer. Math. Soc.*, 24(3):683–708, 2011.
[8] Q. Berger and N. Torri. Directed and non-directed path constrained last-passage percolation. *preprint*, 2018.
[9] N. H. Bingham, C. M. Goldie, and J. L. Teugels. *Regular Variation*, volume 27 of *Encyclopedia of Mathematics and its applications*. Cambridge University Press, 1987.
[10] G. Biroli, J.P. Bouchaud, and M. Potters. Extreme value problems in random matrix theory and other disordered systems. *J. Stat. Mech.*, 7:P07019, 15 pp., 2007.
[11] E. Bolthausen. A note on diffusion of directed polymer in a random environment. *Commun. Math. Phys.*, 123:529–534, 1989.
[12] F. Caravenna, R. Sun, and N. Zygouras. Polynomial chaos and scaling limits of disordered systems. *J. Eur. Math. Soc. (JEMS)*, 19:1–65, 2017.
[13] P. Carmona and Y. Hu. Strong disorder implies strong localization for directed polymers in a random environment. *ALEA*, 2:217–229, 2006.
[14] F. Comets. *Directed Polymers in Random Environments*, volume 2175 of *Ecole d'Eté de probabilités de Saint-Flour*. Springer International Publishing, 2016.
[15] F. Comets, T. Shiga, and N. Yoshida. Probabilistic analysis of directed polymers in a random environment: a review. *Stochastic analysis on large scale interacting systems*, 39, Math. Soc. Japan, Tokyo:115–142, 2004.
[16] F. Comets and N. Yoshida. Directed polymers in a random environment are diffusive at weak disorder. *Ann. Probab.*, 34(5):1746–1770, 2006.
[17] F. den Hollander. *Random polymers*, volume 1974 of *Ecole d'Eté de probabilités de Saint-Flour*. Springer-Verlag Berlin Heidelberg, 2007.
[18] P. S. Dey and N. Zygouras. High temperature limits for $(1+1)$-dimensional directed polymer with heavy-tailed disorder. *Ann. Probab.*, 44(6):4006–4048, 2016.
[19] H. Duminil-Copin, H. Kesten, F. Nazarov, Y. Peres, and V. Sidoravicius. On the number of maximal paths in directed last-passage percolation. *arXiv:1801.05777 [math.PR]*, 2018.
[20] D. S. Fisher and D. A. Huse. Directed paths in random potential. *Phys. Rev. B*, 43:10728–10742, 1991.
[21] T. Geudré, P. Le Doussal, J.-P. Bouchaud, and A. Rosso. Ground-state statistics of directed polymers with heavy-tailed disorder. *Phys. Rev. E*, 91, 2016.
[22] B. Hambly and J. B. Martin. Heavy tails in last-passage percolation. *Probability Theory and Related Fields*, 137:227–275, 2007.
[23] J. M. Hammersley. A few seedlings of research. In University of California Press, editor, *Proc. Sixth Berkeley Symp. Math. Statist. and Probab.*, volume 1, pages 345–394, 1972.
[24] D. A. Huse and C. L. Henley. Pinning and roughening of domain walls in ising systems due to random impurities. *Phys. Rev. Lett.*, 54:2708–2711, 1985.
[25] J.Z. Imbrie and T. Spencer. Diffusion of directed polymer in a random environment. *J. Stat. Phys.*, 52:608–626, 1988.
[26] K. Johansson. Transversal fluctuations for increasing subsequences on the plane. *Probab. Theory Rel. Fields*, 116(4):445–456, 1999.
[27] O. Kallenberg. *Foundations of Modern Probability*. Springer Probability and its Applications. Springer-Verlag New York, 2 edition, 2002.
[28] G. F. Lawler and V. Limic. *Random Walk: A Modern Introduction*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2010.
[29] B. F. Logan and L. A. Shepp. A variational problem for random young tableaux. *Advances in Math.*, 26:206–222, 1977.
[30] A. V. Nagaev. Large deviations of sums of independent random variables. *Ann. Probab.*, 7(5):745–789, 1979.
[31] R. Resnick. *Extreme values, regular variation and point process*. Series in Operations Research and Financial Engineering. New York, Springer, 1987.
[32] T. Seppäläinen. Scaling for a one-dimensional directed polymer with boundary conditions. *Ann. Probab.*, 40(1):19–73, 2012.
[33] C. Stone. On local and ratio limit theorems. *Proc. Fifth Berkeley Symp. on Math. Statist. and Prob. (Univ. of Calif. Press, 1967)*, 2:217–224, 1967.

[34] K. Khanin T. Alberts and J. Quastel. Intermediate disorder regime for directed polymers in dimension 1+1. *Phys. Rev. Lett.*, 105, 2010.
[35] N. Torri. Pinning model with heavy tailed disorder. *Stochastic Processes and their Applications*, 126:542–571, 2016.
[36] S. M. Ulam. Monte Carlo calculations in problems of mathematical physics. In *Modern mathematics for the engineer: Second series*, pages 261–281. McGraw-Hill, New York, 1961.
[37] A. M. Vershik and S. V. Kerov. Asymptotics of the plancherel measure of the symmetric group and the limiting form of young tables. *Soviet Math. Dokl.*, 18:527–531, 1977.

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