



HAL
open science

A Proof of the Bunkbed Conjecture for the Complete Graph at $p=1/2$

Paul de Buyer

► **To cite this version:**

Paul de Buyer. A Proof of the Bunkbed Conjecture for the Complete Graph at $p=1/2$. 2018. hal-01706589

HAL Id: hal-01706589

<https://hal.science/hal-01706589>

Preprint submitted on 12 Feb 2018

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

A PROOF OF THE BUNKBED CONJECTURE FOR THE COMPLETE GRAPH AT $p = \frac{1}{2}$

PAUL DE BUYER

ABSTRACT. The bunkbed of a graph G is the graph $G \times \{0, 1\}$. It has been conjectured that in the independent bond percolation model, the probability for $(u, 0)$ to be connected with $(v, 0)$ is greater than the probability for $(u, 0)$ to be connected with $(v, 1)$, for any vertex u, v of G . In this article, we prove this conjecture for the complete graph in the case of the independent bond percolation of parameter $p = 1/2$.

1. INTRODUCTION

Percolation theory has been widely studied over the last decades and yet, several intuitive results are very hard to prove rigorously. This is the case of the bunkbed conjecture formulated by Kasteleyn published as a remark in [13]) which investigates a notion of graph distance through percolation.

A bunkbed graph of a graph $\tilde{G} = (\tilde{V}, \tilde{E})$ is the graph $G = (V, E) = \tilde{G} \times \{0, 1\}$, to which we have added the edges that connect the vertices $(x, 0)$ to $(x, 1)$ for all vertices $x \in V$, see Figure 2.1. It is natural to distinguish vertices whether they are on the lower level, the vertices $(x, 0)$, or on the upper level, the vertices $(x, 1)$.

The bunkbed conjecture (see [5] for a more general setting) suggests that two vertices $u = (x, 0)$ and $v = (y, 0)$ on the lower level are closer than u and $v' = (y, 1)$. Closeness of two vertices has to be understood through the probability of the existence of an open path in the sense of percolation.

The percolation model is defined as follow. We open each edges of E independently with probability p and close them with probability $1 - p$ and we write \mathbb{P}_p the law associated to this percolation model. We call a configuration, an element $\omega = (\omega_e)_{e \in E} \in \{0, 1\}^E$ corresponding to the bond percolation model where $\omega_e = 0$ means that the edge e is closed and 1 means that the edge e is open. We call an open path a path of open edges and for two vertices $x, y \in V$, we write $x \leftrightarrow y$ if there exists an open path between x and y . By convention, for any configuration, a vertex is always connected to itself, i.e. $x \leftrightarrow x$. For a general introduction on percolation, see [3].

In this article we prove the bunkbed conjecture for the complete graph when the percolation parameter p is equal to $1/2$.

Theorem 1.1. *Let G be the bunkbed graph of the complete graph $\tilde{G} = K_n$. For all vertices x, y of K_n :*

$$\mathbb{P}_{\frac{1}{2}}((x, 0) \leftrightarrow (y, 0)) \geq \mathbb{P}_{\frac{1}{2}}((x, 0) \leftrightarrow (y, 1))$$

1991 *Mathematics Subject Classification.* 82B43, 60K35.

Key words and phrases. Percolation, Bunkbed graph, Combinatorics, Bunkbed Conjecture.

In the previous works of S. Linusson and M. Leander, the conjecture has been proven for the outerplanar graphs and the wheels graphs using the so-called method of minimal counter-example, see [7, 8]. However, this method might not be suitable for the complete graph due to its geometric nature. We have chosen to study the conjecture for the complete graph because we think that it should be easier to show the following proposition than the bunkbed conjecture itself: “If the bunkbed conjecture is verified for a graph \tilde{G} , then it is verified for the graph $\tilde{G} \setminus \{e\}$ where e is an edge of \tilde{G} ”.

One can note that the bunkbed conjecture is true whenever p is small enough. Indeed, in these cases, only the shortest paths can be open to connect $(x, 0)$ with $(y, 0)$ or $(y, 1)$; since the shortest path from $(x, 0)$ to $(y, 1)$ is longer than $(x, 0)$ to $(y, 0)$, the conjecture is proven. Note $d(\cdot, \cdot)$ the usual graph distance, then one can prove that for all vertices u, v and w , $d(u, v) > d(u, w) \Rightarrow \mathbb{P}_p(u \leftrightarrow v) > \mathbb{P}_p(u \leftrightarrow w)$ for sufficiently small p .

Finally, we underline some related works on bunkbed graphs. In the random walk field, an analogical problem of the reaching time of a random walk has been studied, see [1, 4, 12]. In the random directed graph field, it has been shown that it is equivalent to study random orientation of edges on the graph and percolation on the graph with Bernoulli parameter $1/2$ (as it is in our case), see [6, 8, 10].

A first approach of the problem would be to study the ratio of the probability of connection, see (1.1), and to study the derivative according to p , the Bernoulli parameter associated to the probability of opening an edge. Let $u = (x, 0)$ and $v = (y, 0)$ be two vertices on the lower level and define the vertex v' as $v' = (x, 1)$, the vertex above v , then the ratio of the probability of connection is written as:

$$(1.1) \quad \frac{\mathbb{P}_p(u \leftrightarrow v')}{\mathbb{P}_p(u \leftrightarrow v)}$$

As a result of the previous remark whenever p tends to 0, the ratio of (1.1) tends to 0, and clearly when p is equal to 1, this ratio is equal to 1. Because the events $\{u \leftrightarrow v\}$ and $\{u \leftrightarrow v'\}$ are increasing events, the derivatives can be studied using Russo’s formula. However, even using Russo’s formula, see [9, 11], it reveals itself strenuous to study it. One can notice that the derivative of $\mathbb{P}_p(u \leftrightarrow v)$ cannot be always greater than the derivative of $\mathbb{P}_p(u \leftrightarrow v')$, since they are both equal to 0 when $p = 0$ and equal to 1 when $p = 1$. However, we conjecture here that the derivative of the ratio is increasing, meaning that for all $0 < p < 1$:

$$\mathbb{P}_p[u \leftrightarrow v'] \partial_p \mathbb{P}_p[u \leftrightarrow v] \leq \mathbb{P}_p[u \leftrightarrow v] \partial_p \mathbb{P}_p[u \leftrightarrow v']$$

From now on, we note $G = (V, E)$ the bunkbed graph of the complete graph and we label the vertices of V from 1 to $2n$ such that $V = \{s_i, i \in [1; 2n] \cap \mathbb{N}\}$ and $\forall i, j \in [1; n] \cap \mathbb{N}$, $s_i \sim s_j$ and $s_i \sim s_{i+n}$ where \sim is the neighbour relation, $x \sim y \Leftrightarrow \{x, y\} \in E$. We can consider that the vertices labelled from 1 to n are the vertices of the lower level (or level 1) and the vertices labelled from $n+1$ to $2n$ are the vertices of the upper level (or level 2). In the complete graph, two vertices play the same role. Therefore, it is enough to prove the bunkbed conjecture for two vertices, here s_1 and s_n . Moreover, it is trivial to see that $\mathbb{P}(s_1 \leftrightarrow s_1) = 1 \geq \mathbb{P}(s_1 \leftrightarrow s_{n+1})$.

Our approach, based on combinatorics since $p = 1/2$, is to decompose the graph into different appropriate classes which solve the bunkbed conjecture. We count the number of ways to connect s_1 to s_n and to connect s_1 and s_{2n} . The idea of the proof is the following. We define the *main component* as the connected component

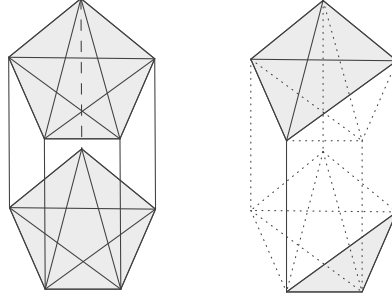


FIGURE 2.1. The bunkbed graph of K_5 and an element of $G_{3,4,2}$

that contains the vertex s_1 and all the vertices that are connected to s_1 by an open path. Then, we distinguish different classes of main component depending on the number of vertices on the lower level, on the upper level, and depending on the number of parallel vertices, notion that will be defined later. Lemma 2.1 will give the number of ways to connect the vertices of a main component. Lemmas 3.3 and 3.4 will give the number of configurations containing a main component with x vertices on the lower level, y vertices on the upper level and a key argument to properly add them together. Finally, we prove the main Theorem in section 4.

2. COVERING GRAPHS

We recall that a covering graph of a graph $G = (V, E)$ is a subgraph $G' = (V', E')$ such that G' is connected, $V' = V$ and $E' \subseteq E$. From now on, G will be used to refer to the bunkbed graph of the complete graph. We classify the subgraphs of G according to the number of vertices in the upper and lower level, as well as the number of parallel vertices. We define $G_{x,y,z}$ the set of connected subgraph of G such that $\forall G' = (V', E') \in G_{x,y,z}$:

- (1) $\exists \lambda_1, \dots, \lambda_x \in \{1, \dots, n\}, \exists \mu_1, \dots, \mu_y \in \{n+1, \dots, 2n\}$ such that $\forall i \in [1; x]$ $s_{\lambda_i} \in V'$ and $\forall j \in [1; y], s_{\mu_j} \in V', \cup_i \{s_{\lambda_i}\} \cup_j \{s_{\mu_j}\} = V$ and $\#V = x + y$
- (2) $\exists \lambda_1, \dots, \lambda_z \in \{1, \dots, n\}$ such that $\forall i, j \leq z, \lambda_i \neq \lambda_j$ and $\{s_{\lambda_i}, s_{\lambda_i+n}\} \subset V'$
- (3) $\forall x, y \in V', \{x, y\} \in E'$ iff $\{x, y\} \in E$

Graphs of $G_{x,y,z}$ can be seen as extraction of subgraph of the bunkbed graph G . Condition 1 insures that there are exactly x vertices on the lower level and exactly y vertices on the upper level. Condition number 2 insures that exactly z vertices among the y vertices on the upper level are above the x vertices of the lower level. Finally, condition 3 insures that vertices present in the extraction comes along with the corresponding edges. We will say that a graph of $G_{x,y,z}$ has x vertices of level 1 and y vertices of level 2 and z parallel vertices (e.g. figure 2.1). Moreover, one can see that two graphs G_1, G_2 of $G_{x,y,z}$ are isomorphs.

We define the function $GC : \mathbb{N}^3 \mapsto \mathbb{N}$ which gives the number $GC(x, y, z)$ of covering graphs of a graph in $G_{x,y,z}$.

Lemma 2.1. *Fix $x, x', y, y', z \in [0; n] \cap \mathbb{N}$ such that $x + y = x' + y'$ and $z \leq \min(x, x', y, y')$. If $|x - y| \geq |x' - y'|$ then $GC(x, y, z) \geq GC(x', y', z)$*

Remark 2.2. The function GC is symmetric in its first two coordinates, meaning that for all x, y, z , we have $GC(x, y, z) = GC(y, x, z)$.

Proof. To prove this lemma, by iteration, it is sufficient to prove the inequality $GC(x+1, y, z) \geq GC(x, y+1, z)$ for $x > y \geq z$. For this matter, we need to give an upper bound and a lower bound of $GC(x, y, z)$. As an upper bound we use a trivial one: we bound the number of covering graphs by the number of possible graph knowing that one vertical edge need to be open. Since in a complete graph with n vertices, there are $n(n-1)/2$ edges, therefore we have:

$$GC(x, y, z) \leq 2^{\frac{x(x-1)}{2}} (2^z - 1) 2^{\frac{y(y-1)}{2}}$$

As a lower bound, we consider the only case where we connect by at least one vertical edge a covering graph K_x and a covering graph of K_y :

$$GC(x, y, z) \geq GC(x, 0, 0) \times (2^z - 1) \times GC(0, y, 0)$$

Therefore, one has:

$$\begin{aligned} \frac{GC(x+1, y, z)}{GC(x, y+1, z)} &\geq \frac{GC(x+1, 0, 0) \times (2^z - 1) \times GC(0, y, 0)}{2^{\frac{x(x-1)}{2}} (2^z - 1) 2^{\frac{y(y+1)}{2}}} \\ &= GC(x+1, 0, 0) \times 2^{-\frac{x(x-1)}{2}} \times GC(0, y, 0) \times 2^{-\frac{y(y-1)}{2}} \times 2^{-y} \end{aligned}$$

By [2], the number of covering graphs or number of connected labelled graph with n vertices is given by the following approximation:

$$GC(n, 0, 0) = 2^{\frac{n(n-1)}{2}} (1 - 2n2^{-n} + o(2^{-n}))$$

We slightly modify this approximation in the case $n \geq 7$:

$$GC(n, 0, 0) \geq 2^{\frac{n(n-1)}{2}} (1 - 3n2^{-n})$$

Then, we split the problems into different cases. First case, whenever $x \geq 10$ and $y \geq 7$. Since $x \geq y+1$, one has:

$$\begin{aligned} \frac{GC(x+1, y, z)}{GC(x, y+1, z)} &\geq 2^{x-y} (1 - 3x2^{-x}) (1 - 3y2^{-y}) \\ &\geq 2 \times (1 - 3 \times 10 \times 2^{-10}) \times (1 - 3 \times 7 \times 2^{-7}) \geq 1 \end{aligned}$$

Second case, when $x \geq 10$ and $y < 7$, then $x \geq y+4$:

$$\begin{aligned} \frac{GC(x+1, y, z)}{GC(x, y+1, z)} &\geq 2^{x-y-1} (1 - 3x2^{-x}) \\ &\geq 2^3 \times (1 - 3 \times 10 \times 2^{-10}) \geq 1 \end{aligned}$$

Finally, when $10 > x > y$, we have computed the result by computer which ends the proof. \square

3. RESULT ON THE SIZE OF THE CLASSES

We define G^1 the set of connected subgraphs of G containing the vertices s_1 and s_n , and G^2 the set of connected subgraphs of G containing the vertices s_1 and s_{2n} . Moreover, we define the set of graph $G^1_{x,y,z} = G^1 \cap G_{x,y,z}$ and the set $G^2_{x,y,z} = G^2 \cap G_{x,y,z}$. We define the functions $q_1 : \mathbb{N}^3 \mapsto \mathbb{N}$ and $q_2 : \mathbb{N}^3 \mapsto \mathbb{N}$

such that $q_1(x, y, z) = \#G_{x,y,z}^1$ and $q_2(x, y, z) = \#G_{x,y,z}^2$. Finally, we define the function $q : \mathbb{N}^3 \mapsto \mathbb{N}$ such that

$$(3.1) \quad q(x, y, z) = \begin{cases} q_1(x, y, z) - q_2(x, y, z) + q_1(y, x, z) - q_2(y, x, z) & \text{if } x \neq y \\ q_1(x, x, z) - q_2(x, x, z) & \text{if } x = y \end{cases}$$

Before giving the main result on the function q , we give two preliminary results on the exact value of the functions q_1 and q_2 .

Lemma 3.1. *For all $x, y \geq z \geq 1$ such that $x + y - z \leq n$*

$$q_1(x, y, z) = \frac{(n-2)!x(x-1)}{(x-z)!z!(n-x-y+z)!(y-z)!}$$

Proof. First, if $z = 0$ and $y > 0$, then a graph cannot be connected. Moreover, s_0 and s_n have to be in the set of vertices of the graph of $G_{x,y,z}^1$, x has to be greater than 2 otherwise $G_{x,y,z}^1$ is an empty set. Then, we have to choose the $x-2$ vertices of level 1 among the $n-2$ vertices left, distribute z vertices of level 2 on top of the x vertices previously chosen, and choose $y-z$ vertices among the $n-x$ vertices left. Therefore, we can write:

$$q_1(x, y, z) = \binom{n-2}{x-2} \times \binom{x}{z} \times \binom{n-x}{y-z} \times \mathbb{1}_{x \geq 2}$$

Then, one can note that for all $k \in \mathbb{N}$, the following equality holds:

$$(3.2) \quad \frac{1}{(x-k)!} \mathbb{1}_{x \geq k} = \frac{1}{x!} \prod_{i=0}^k (x-i)$$

So we can write:

$$\begin{aligned} q_1(x, y, z) &= \frac{(n-2)!}{(n-x)!(x-2)!} \times \frac{x!}{(x-z)!z!} \times \frac{(n-x)!}{(n-x-y+z)!(y-z)!} \mathbb{1}_{x \geq 2} \\ &= \frac{(n-2)!x(x-1)}{(x-z)!z!(n-x-y+z)!(y-z)!} \end{aligned}$$

□

Lemma 3.2. *For all $x, y \geq z \geq 1$ such that $x + y - z \leq n$:*

$$q_2(x, y, z) = \frac{(n-2)!(xy-z)}{(x-z)!z!(n-x-y+z)!(y-z)!}$$

Proof. First, we notice that for any graph in $G_{x,y,z}^2$, s_0 and s_{2n} belong to the set of vertices. Then, to count the number of graphs G in $G_{x,y,z}^2$, we distinguish 4 different cases: either s_n and s_{n+1} belong to G ; either s_n belongs to G but not s_{n+1} ; either s_n does not belong to G but s_{n+1} does; either s_n and s_{n+1} don't belong to G . We can write:

$$\begin{aligned}
 q_2(x, y, z) &= \binom{n-2}{x-2} \times \binom{x-2}{z-2} \times \binom{n-x}{y-z} \times \mathbb{1}_{x \geq 2, y \geq 2, z \geq 2} \\
 &+ \binom{n-2}{x-2} \times \binom{x-2}{z-1} \times \binom{n-x}{y-z} \times \mathbb{1}_{x > \max(1, z), y < n} \\
 &+ \binom{n-2}{x-1} \times \binom{x-1}{z-1} \times \binom{n-x-1}{y-z-1} \times \mathbb{1}_{x < n, y > \max(1, z)} \\
 &+ \binom{n-2}{x-1} \times \binom{x-1}{z} \times \binom{n-x-1}{y-z-1} \times \mathbb{1}_{\max(1, z) < x < n, \max(1, z) < y < n}
 \end{aligned}$$

Therefore, using (3.2), we have:

$$\begin{aligned}
 q_2(x, y, z) &= \frac{(n-2)!}{(x-z)!z!(n-x-y+z)!(y-z)!} \times z(z-1) \\
 &+ \frac{(n-2)!}{(x-z)!z!(n-x-y+z)!(y-z)!} \times z(x-z) \\
 &+ \frac{(n-2)!}{(x-z)!z!(n-x-y+z)!(y-z)!} \times z(y-z) \\
 &+ \frac{(n-2)!}{(x-z)!z!(n-x-y+z)!(y-z)!} \times (x-z)(y-z) \\
 &= \frac{(n-2)!}{(x-z)!z!(n-x-y+z)!(y-z)!} (xy-z)
 \end{aligned}$$

Using lemmas 3.1 and 3.2, we have that:

$$q_1(x, y, z) - q_2(x, y, z) = \frac{(n-2)!(x^2 - x - xy + z)}{(x-z)!z!(n-x-y+z)!(y-z)!}$$

Recall the definition of the function q in (3.1), we have for all $x \geq z \geq 1$:

$$q(x, x, z) = \frac{(n-2)!(z-x)}{(x-z)!(x-z)!z!(n-2x+z)!}$$

And for all $x, y \geq z \geq 1$:

$$(3.3) \quad q(x, y, z) = \frac{(n-2)!(x^2 - 2xy + y^2 - x - y + 2z)}{(x-z)!(y-z)!z!(n-x-y+z)!}$$

Note that:

$$(3.4) \quad q(x, y, z) \leq 0 \Leftrightarrow x \in \left[y + \frac{1 - \sqrt{8y - 8z + 1}}{2}; y + \frac{1 + \sqrt{8y - 8z + 1}}{2} \right]$$

□

Lemma 3.3. *For all $k \geq z$, the following inequality holds:*

$$\sum_{i=0}^{k-z} q(k+i, k-i, z) = 0$$

Proof. To prove the theorem, it is actually easier to prove that:

$$\sum_{i=1}^{k-z} q(k+i, k-i, z) = -q(k, k, z)$$

Using as arguments of q the triplet $(k+i, k-i, z)$, some factors of (3.3) become independent of i . Indeed, we get the following equality:

$$q(k+i, k-i, z) = \frac{4i^2 - 2k + 2z}{(k+i-z)!(k-i-z)!} \times \frac{(n-2)!}{z!(n-2k+z)!}$$

Therefore, proving the Lemma is equivalent to prove that:

$$\sum_{i=1}^{k-z} \frac{4i^2 - 2k + 2z}{(k+i-z)!(k-i-z)!} = \frac{k-z}{(k-z)!(k-z)!}$$

Then, it is enough to see that:

$$\begin{aligned} \frac{k-z}{(k-z)!(k-z)!} &= \frac{4-2k+2z}{(k+1-z)!(k-1-z)!} \\ &+ \frac{3(k+2-z)}{(k+2-z)!(k-2-z)!} \mathbb{1}_{k-z \geq 2} \end{aligned}$$

and that for all $k-z > i \geq 2$:

$$\begin{aligned} \frac{(2i-1)(k+i-z)}{(k+i-z)!(k-i-z)!} &= \frac{4i^2 - 2k + 2z}{(k+i-z)!(k-i-z)!} \\ &+ \frac{(2i+1)(k+i+1-z)}{(k+i+1-z)!(k-i-1-z)!} \end{aligned}$$

Whenever $i = k-z$, then:

$$\frac{(2i-1)(k+i-z)}{(k+i-z)!(k-i-z)!} = \frac{4(k-z)^2 - 2k + 2z}{(2k-2z)!}$$

which ends the proof. \square

Lemma 3.4. *For all $k \geq z$, the following inequality holds:*

$$\sum_{i=0}^{k-z} q(k+i+1, k-i, z) = 0$$

Proof. The proof goes in the same way as lemma 3.3. Indeed, it is enough to prove that:

$$\sum_{i=1}^{k-z} \frac{2i^2 + 2i - k + z}{(k+i+1-z)!(k-i-z)!} = \frac{k-z}{(k+1-z)!(k-z)!}$$

Then, we have:

$$\begin{aligned} \frac{k-z}{(k+1-z)!(k-z)!} &= \frac{4-k+z}{(k+2-z)!(k-1-z)!} \\ &+ \frac{2(k-z+3)}{(k+3-z)!(k-2-z)!} \mathbb{1}_{k-z \geq 2} \end{aligned}$$

Then, for all $k-z > i \geq 2$, the following equality holds:

$$\begin{aligned} \frac{i(k+i+1-z)}{(k+i-z)!(k-i-z)!} &= \frac{2i^2 + 2i - k - z}{(k+i-z)!(k-i-z)!} \\ &+ \frac{(i+1)(k+i+2-z)}{(k+i+1-z)!(k-i-1-z)!} \end{aligned}$$

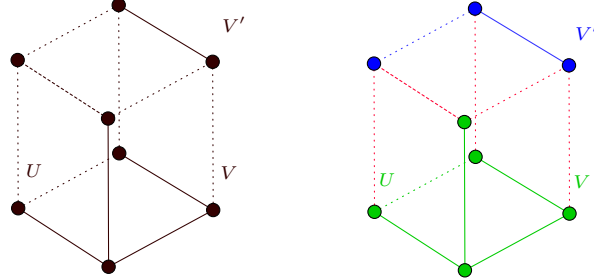


FIGURE 4.1. Decomposition of a configuration

Whenever $i = k - z$,

$$\frac{i(k+i+1-z)}{(k+i+1-z)!(k-i-1-z)!} = \frac{2(k-z)^2 + 2(k-z) - k + z}{(2k-2z+1)!}$$

which ends the proof. \square

4. PROOF OF THE MAIN THEOREM

Recall that $G = (V, E)$ is the bunkbed graph associated with the complete graph K_n . We recall here the idea of the proof of the main theorem. We split the configuration depending on the number of vertices of the main component and then split again depending on the number of vertices on the lower/upper level. Figure 4.1 gives a decomposition of a configuration and we give some explanation about the figure. For simplicity we have not drawn a complete graph. Then, edges drawn in solid lines are open edges and edges drawn in dotted lines are closed edges. Green vertices and green edges correspond to the main component. Red edges are the adjacent edges of the main component that need to be closed, if not, the main component would expand. Blue vertices and blue edges correspond to the “outside” of the main component. In this sense, we define $O(x, y, z)$ the number of “outside” edges when the main component has x vertices in the lower level, y vertices in the upper level and z parallel vertices. A simple computation gives that:

$$O(x, y, z) = n(n - x - y) + \frac{1}{2}(x^2 - x + y^2 - y) + z$$

As well as the following relations:

$$\begin{aligned} O(x, y, z) &= O(y, x, z) \\ O(x+1, y, z) - O(x, y+1, z) &= x - y \end{aligned}$$

The second relation can be understood in the spirit of Lemma 2.1 through the following statement: $x + y = x' + y'$ and $|x - y| \geq |x' - y'|$ implies $O(x, y, z) \geq O(x', y', z)$. All the quantities developed before allows us to express the probability of connection between two vertices. First, note that since the Bernoulli parameter of the percolation p is equal to $1/2$, therefore every configuration has the same probability, i.e. for any configuration ω , $\mathbb{P}_{1/2}(\omega) = 2^{-\#E}$. Moreover,

knowing that the main component of s_0 , noted $MC(s_0)$, is a subgraph of a graph in $G_{x,y,z}^1$, there are $GC(x, y, z)$ way for it to be connected, and $O(x, y, z)$ outside edges which won't affect the connectivity of s_0 and s_n . Furthermore, there are $q_1(x, y, z)$ to choose the subgraph corresponding to the main component. Thus, we have the following equalities:

$$\begin{aligned}
 & \mathbb{P}_{1/2}(s_0 \leftrightarrow s_n) \\
 = & \sum_{x,y,z} \sum_{MC(s_0) \text{ subgraph of } \tilde{G} \in G_{x,y,e}^1} 2^{O(x,y,z)} \mathbb{P}(\omega) \\
 = & 2^{-\#E} \sum_{x,y,z} 2^{O(x,y,z)} GC(x, y, z) q_1(x, y, z) \\
 = & 2^{-\#E} \sum_{k \geq 0} \sum_{z \geq 0} \sum_{i \in \mathbb{Z}} 2^{O(k+i, k-i, z)} GC(k+i, k-i, z) q_1(k+i, k-i, z) \\
 & + 2^{-\#E} \sum_{k \geq 0} \sum_{z \geq 0} \sum_{i \in \mathbb{Z}} 2^{O(k+i+1, k-i, z)} GC(k+i+1, k-i, z) q_1(k+i+1, k-i, z)
 \end{aligned}$$

The third equality is obtained by an operation of renumbering. In the same way, we have that:

$$\begin{aligned}
 & \mathbb{P}_{1/2}(s_0 \leftrightarrow s_{2n}) \\
 = & 2^{-\#E} \sum_{k \geq 0} \sum_{z \geq 0} \sum_{i \in \mathbb{Z}} 2^{O(k+i, k-i, z)} GC(k+i, k-i, z) q_2(k+i, k-i, z) \\
 & + 2^{-\#E} \sum_{k \geq 0} \sum_{z \geq 0} \sum_{i \in \mathbb{Z}} 2^{O(k+i+1, k-i, z)} GC(k+i+1, k-i, z) q_2(k+i+1, k-i, z)
 \end{aligned}$$

For all k and z , because of the symmetry of the function GC and the function O , one has:

$$\begin{aligned}
 & \sum_{i \in \mathbb{Z}} 2^{O(k+i, k-i, z)} GC(k+i, k-i, z) (q_1(k+i, k-i, z) - q_2(k+i, k-i, z)) \\
 & = \sum_{i \geq 0} 2^{O(k+i, k-i, z)} GC(k+i, k-i, z) q(k+i, k-i, z)
 \end{aligned}$$

Recall that $q(k+i, k-i, z)$ might be negative, see (3.4), and because of lemma 2.1 for all k , there exists an i_0 such that for all $0 \leq i \leq i_0$:

$$\begin{aligned}
 GC(k+i, k-i, z) & \leq GC(k+i_0, k-i_0, z) \\
 GC(k+i+1, k-i, z) & \leq GC(k+i_0+1, k-i_0, z) \\
 O(k+i, k-i, z) & \leq O(k+i_0, k-i_0, z) \\
 q(k+i, k-i, z) & \leq 0
 \end{aligned}$$

and for all $i \geq i_0$:

$$\begin{aligned}
 GC(k+i, k-i, z) & \geq GC(k+i_0, k-i_0, z) \\
 GC(k+i+1, k-i, z) & \geq GC(k+i_0+1, k-i_0, z) \\
 O(k+i, k-i, z) & \geq O(k+i_0, k-i_0, z) \\
 q(k+i, k-i, z) & \geq 0
 \end{aligned}$$

By lemma 3.3:

$$\begin{aligned} & \sum_{i \geq 0} 2^{O(k+i, k-i, z)} GC(k+i, k-i, z) q(k+i, k-i, z) \\ & \geq 2^{O(k+i_0, k-i_0, z)} GC(k+i_0, k-i_0, z) \sum_{i=z}^{k-z} q(k+i, k-i, z) \end{aligned}$$

And by lemma 3.4:

$$\begin{aligned} & \sum_{i \geq 0} 2^{O(k+i+1, k-i, z)} GC(k+i+1, k-i, z) q(k+i+1, k-i, z) \\ & \geq 2^{O(k+i_0+1, k-i_0, z)} GC(k+i_0+1, k-i_0, z) \sum_{i=z}^{k-z} q(k+i+1, k-i, z) \end{aligned}$$

This concludes the proof since:

$$\begin{aligned} & \mathbb{P}_{1/2}(s_0 \leftrightarrow s_n) - \mathbb{P}_{1/2}(s_0 \leftrightarrow s_{2n}) \\ & = 2^{-\#E} \sum_{k \geq 0} \sum_{z \geq 0} \sum_{i \geq 0} 2^{O(k+i, k-i, z)} GC(k+i, k-i, z) q(k+i, k-i, z) \\ & \quad + 2^{-\#E} \sum_{k \geq 0} \sum_{z \geq 0} \sum_{i \geq 0} 2^{O(k+i+1, k-i, z)} GC(k+i+1, k-i, z) q(k+i+1, k-i, z) \\ & \geq 0 \end{aligned}$$

□

The author thanks Cyril Roberto, Julien Bureaux and Florent Barret for the many insightful conversations about the topic as well as Joseba Dalmau and Anna Bonnet.

REFERENCES

- [1] Béla Bollobás and Graham Brightwell. Random walks and electrical resistances in products of graphs. *Discrete applied mathematics*, 73(1):69–79, 1997.
- [2] Philippe Flajolet and Robert Sedgewick. *Analytic combinatorics*. cambridge University press, 2009.
- [3] Geoffrey R. Grimmett and Harry Kesten. Percolation since Saint-Flour. In *Percolation theory at Saint-Flour*, Probab. St.-Flour, pages ix–xxvii. Springer, Heidelberg, 2012.
- [4] Olle Häggström. On a conjecture of bollobás and brightwell concerning random walks on product graphs. *Combinatorics, Probability and Computing*, 7(04):397–401, 1998.
- [5] Olle Häggström. Probability on bunkbed graphs. In *Proceedings of FPSAC*, volume 3, 2003.
- [6] Richard M Karp. The transitive closure of a random digraph. *Random Structures & Algorithms*, 1(1):73–93, 1990.
- [7] Madeleine Leander. Självständiga arbeten i matematik.
- [8] Svante Linusson. On percolation and the bunkbed conjecture. *Combinatorics, Probability and Computing*, 20(01):103–117, 2011.
- [9] Grigorii Aleksandrovich Margulis. Probabilistic characteristics of graphs with large connectivity. *Problemy peredachi informatsii*, 10(2):101–108, 1974.
- [10] Colin McDiarmid. General percolation and random graphs. *Advances in Applied Probability*, pages 40–60, 1981.
- [11] Lucio Russo. On the critical percolation probabilities. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, 56(2):229–237, 1981.
- [12] Jacob van den Berg, Olle Häggström, and Jeff Kahn. Some conditional correlation inequalities for percolation and related processes. *Random Structures & Algorithms*, 29(4):417–435, 2006.
- [13] Jacob van den Berg, Jeff Kahn, et al. A correlation inequality for connection events in percolation. *The Annals of Probability*, 29(1):123–126, 2001.

A PROOF OF THE BUNKBED CONJECTURE FOR THE COMPLETE GRAPH AT $p = \frac{1}{2}$ 11

(P. de Buyer) UNIVERSITÉ PARIS OUEST NANTERRE LA DÉFENSE - MODAL'X, 200 AVENUE
DE LA RÉPUBLIQUE 92000 NANTERRE, FRANCE
E-mail address: `debuyer@math.cnrs.fr`