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The finite Larmor radius regime for the Vlasov-Poisson equations. The three dimensional setting with non uniform magnetic field

Mihaï BOSTAN ^{*}, Aurélie FINOT [†]

Abstract

The subject matter of this paper concerns the mathematical analysis of tokamak plasmas. We focus on the finite Larmor radius regime with general magnetic shapes. We concentrate on the asymptotic analysis for the Vlasov-Poisson equations under strong external magnetic fields, by averaging with respect to the fast cyclotronic motion around the magnetic lines. The main properties of the limit model are emphasized : mass and energy balances, Hamiltonian structure.

Keywords: Vlasov-Poisson system, Hamiltonian formulation, Finite Larmor radius regime.

AMS classification: 78A35, 82D10.

1 Introduction

We consider a population of charged particles of mass m , charge q , whose presence density is denoted by $f(t, x, v)$. Motivated by the study of tokamak plasmas, we analyse

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the dynamics of this population under the effect of a given magnetic field $\mathbf{B}(x) = B(x)e(x)$, with $B > 0, |e| = 1$. We use the Vlasov-Poisson equations, that is, the particles are transported under the action of the electro-magnetic force $q(E(t, x) + B(x)v \wedge e(x))$, where E is the self-consistent electric field corresponding to the charge density $\rho(t, x) = q \int_{\mathbb{R}^3} f(t, x, v) dv, (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3$. The evolution of the presence density f and the electric field E is given by the Vlasov-Poisson system :

$$\partial_t f + v \cdot \nabla_x f + \frac{q}{m} (E(t, x) + B(x)v \wedge e(x)) \cdot \nabla_v f = 0, \quad (t, x, v) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3 \quad (1)$$

$$E(t, x) = -\nabla_x \phi, \quad -\varepsilon_0 \Delta_x \phi = q \int_{\mathbb{R}^3} f(t, x, v) dv, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3 \quad (2)$$

supplemented by the initial condition

$$f(0, x, v) = f^{\text{in}}(x, v) \geq 0, \quad (x, v) \in \mathbb{R}^3 \times \mathbb{R}^3. \quad (3)$$

Here ε_0 stands for the electric permittivity of the vacuum. By introducing the fundamental solution of the Laplace operator in $\mathbb{R}^3, x \rightarrow \frac{1}{4\pi|x|}, x \in \mathbb{R}^3 \setminus \{0\}$, the Poisson equation (2) becomes

$$\phi(t, x) = \frac{1}{4\pi\varepsilon_0} \int_{\mathbb{R}^3} \frac{\rho(t, y)}{|x - y|} dy = \frac{q}{\varepsilon_0} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{f(t, y, w)}{4\pi|x - y|} dw dy$$

and therefore the electric field writes

$$E(t, x) = \frac{q}{4\pi\varepsilon_0} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f(t, y, w) \frac{x - y}{|x - y|^3} dw dy, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3.$$

The well posedness of the Vlasov-Poisson problem (1), (2), (3) is well understood. Weak solutions have been constructed in [3], strong solutions have been investigated in [13]. For the propagation of the moments and regularity results we refer to [4, 15]. Clearly, as the field $v \cdot \nabla_x + \frac{q}{m} (E(t, x) + B(x)v \wedge e(x)) \cdot \nabla_v$ is divergence free, we have the mass conservation

$$\frac{d}{dt} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f(t, x, v) dv dx = 0, \quad t \in \mathbb{R}_+.$$

The total energy is conserved as well. Indeed, the balance of the kinetic energy gives

$$\frac{d}{dt} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} m \frac{|v|^2}{2} f(t, x, v) dv dx - \int_{\mathbb{R}^3} E(t, x) \cdot j(t, x) dx = 0 \quad (4)$$

where $j(t, x) = q \int_{\mathbb{R}^3} f(t, x, v) v \, dv$ is the current density. Combining the continuity equation and the Poisson equation $\varepsilon_0 \operatorname{div}_x E = \rho$, we obtain

$$\operatorname{div}_x \{ \varepsilon_0 \partial_t E + j \} = 0.$$

After multiplication by the potential ϕ and integration by parts one gets the balance for the electric energy

$$\frac{d}{dt} \frac{\varepsilon_0}{2} \int_{\mathbb{R}^3} |E(t, x)|^2 \, dx + \int_{\mathbb{R}^3} E(t, x) \cdot j(t, x) \, dx = 0. \quad (5)$$

The total energy conservation follows by (4), (5). We assume that the initial presence density has finite mass and total energy

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f^{\text{in}}(x, v) \, dv dx < +\infty$$

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} m \frac{|v|^2}{2} f^{\text{in}}(x, v) \, dv dx + \frac{q^2}{2\varepsilon_0} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{f^{\text{in}}(x, v) f^{\text{in}}(y, w)}{4\pi|x-y|} \, dw dy dv dx < +\infty.$$

We focus on the asymptotic regimes concerning the magnetic fusion : the charged particles are confined under the effect of strong magnetic fields. It is very instructive to consider first a uniform magnetic field $\mathbf{B} = {}^t(0, 0, B)$, $B > 0$ and to neglect the electric field. In this case, the characteristic equations of (1), or the motion equations of the charged particles in the phase space writes

$$\frac{d\mathcal{X}}{dt} = \mathcal{V}(t; x, v), \quad \frac{d\mathcal{V}}{dt} = \frac{qB}{m} {}^t(\mathcal{V}_2(t; x, v), -\mathcal{V}_1(t; x, v), 0)$$

together with the conditions $\mathcal{X}(0; x, v) = x$, $\mathcal{V}(0; x, v) = v$. It is easily seen that

$${}^t(\mathcal{V}_1(t; x, v), \mathcal{V}_2(t; x, v)) = \mathcal{R}(-\omega t) {}^t(v_1, v_2), \quad \mathcal{V}_3(t; x, v) = v_3, \quad \mathcal{X}_3(t; x, v) = x_3 + tv_3.$$

The space coordinates in the orthogonal directions with respect to the magnetic field come by observing that

$$\frac{d}{dt} \left\{ (\mathcal{X}_1, \mathcal{X}_2) + \frac{(\mathcal{V}_2, -\mathcal{V}_1)}{\omega} \right\} = (0, 0), \quad \text{with } \omega = \frac{qB}{m}.$$

As $\sqrt{(\mathcal{V}_1^2 + \mathcal{V}_2^2)(t; x, v)} = \sqrt{v_1^2 + v_2^2}$, we deduce that the particle trajectories in the orthogonal plane are circles of center $(x_1, x_2) + \frac{(v_2, -v_1)}{\omega}$ and radius $\sqrt{v_1^2 + v_2^2}/|\omega|$. When the magnetic field is high, this radius (called the Larmor radius) becomes small and

the plasma remains confined around the magnetic lines. Notice also that the dynamics in the orthogonal directions evolves on a smaller time scale than in the parallel direction. Indeed, $\mathcal{X}_1, \mathcal{X}_2, \mathcal{V}_1, \mathcal{V}_2$ are periodic in time, of cyclotronic period $T_c = \frac{2\pi}{\omega}$, which is small when B is large. We are face to a multi-scale problem (see [1, 16]), and solving numerically for both slow and fast dynamics would require a huge amount of computation efforts. Alternatively, we can search for homogenization procedure : determine the effective particle trajectories, after averaging with respect to the fast dynamics. Several mathematical analysis have been performed in the framework of uniform magnetic fields [11, 12, 14, 5, 8, 9].

In this paper we intend to investigate the setting of general magnetic fields, by taking into account the curvature of the magnetic lines. Our paper is organized as follows. In Section 2 we present the main results : the limit model and its conservations. Section 3 is devoted to the homogenization process with respect to fast rotation. In Section 4 we extend the previous analysis to general transport operators. We appeal to the hamiltonian formalism. The finite Larmor radius regime is investigated in Section 5. We present formal arguments, by following the rigorous analysis in [9]. We refer to Appendix A for a classical result concerning the hamiltonian flows (the invariance of the symplectic structure).

2 Presentation of the main results

Based on the analysis in the case of uniform magnetic fields, we intend to extend the asymptotic study to non uniform magnetic fields. Therefore we have to average with respect to the fast motion in the orthogonal directions. This corresponds to a separation between the evolution time scales of the advections entering the vector field

$$v \cdot \nabla_x + \frac{q}{m} (E(t, x) + B(x) v \wedge e(x)) \cdot \nabla_v. \quad (6)$$

Such a separation is obtained when splitting the kinetic energy of the Hamiltonian

$$H = m \frac{|v|^2}{2} + q\phi$$

into the parallel and orthogonal contributions

$$H_a = m \frac{(v \cdot e(x))^2}{2} + q\phi, \quad H_b = m \frac{|v \wedge e(x)|^2}{2}.$$

To this decomposition of H it corresponds the following decomposition of the vector field in (6)

$$a(t, x, v) \cdot \nabla_{x,v} = (v \cdot e(x))e(x) \cdot \nabla_x + \left[\frac{q}{m}E(t, x) - (v \cdot e(x)) {}^t\partial_x e v \right] \cdot \nabla_v$$

$$b(x, v) \cdot \nabla_{x,v} = [v - (v \cdot e(x))e(x)] \cdot \nabla_x + [\omega(x)v \wedge e(x) + (v \cdot e(x)) {}^t\partial_x e v] \cdot \nabla_v$$

where $\omega(x) = qB(x)/m$. The dynamics generated by the advection $b \cdot \nabla_{x,v}$ corresponds to the cyclotronic motion. It preserves the modulus of the perpendicular velocity and evolves at a time scale related to the cyclotronic period $T_c = 2\pi/\omega$. The dynamics generated by the advection $a \cdot \nabla_{x,v}$ conserves the sum between the parallel kinetic energy and the potential energy. It contains oscillations in the parallel direction at the plasma frequency. We are interested on the asymptotic limit such that the dynamics generated by the advection $a \cdot \nabla_{x,v}$, characterizing the parallel motion, occurs at a much larger time scale with respect to the cyclotronic period. Therefore we are looking to the asymptotic behavior of the presence densities $(f^\varepsilon)_{\varepsilon>0}$ solving the Vlasov-Poisson problems (1), (2), (3) when the ratio between the time scales characterizing the dynamics of $b \cdot \nabla_{x,v}$ and $a \cdot \nabla_{x,v}$ becomes small (the parameter ε refers to this ratio). We proceed by smoothing out the oscillations due to the fast cyclotronic motion. We search for a new family of presence densities $(F^\varepsilon)_{\varepsilon>0}$ such that at any time t , $f^\varepsilon(t)$ appears as the composition between $F^\varepsilon(t)$ and the fast oscillating characteristic flow of $b \cdot \nabla_{x,v}$

$$f^\varepsilon(t, x, v) = F^\varepsilon(t, \mathcal{X}(-t; x, v), \mathcal{V}(-t; x, v)), \quad (t, x, v) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3, \quad \varepsilon > 0$$

where

$$\frac{d\mathcal{X}}{dt} = [I_3 - e(\mathcal{X}(t; x, v)) \otimes e(\mathcal{X}(t; x, v))] \mathcal{V}(t; x, v)$$

$$\frac{d\mathcal{V}}{dt} = \omega(\mathcal{X}(t; x, v)) \mathcal{V}(t; x, v) \wedge e(\mathcal{X}(t; x, v)) + (\mathcal{V}(t; x, v) \cdot e(\mathcal{X}(t; x, v))) {}^t\partial_x e(\mathcal{X}) \mathcal{V}(t; x, v)$$

and $\mathcal{X}(0; x, v) = x, \mathcal{V}(0; x, v) = v$. We expect that the family $(F^\varepsilon)_{\varepsilon>0}$ is stable, and determine the problem satisfied by the limit density $F = \lim_{\varepsilon \searrow 0} F^\varepsilon$, giving us the asymptotic behavior for small ε

$$f^\varepsilon(t, x, v) = F(t, \mathcal{X}(-t; x, v), \mathcal{V}(-t; x, v)) + o(1), \quad \text{as } \varepsilon \searrow 0.$$

The derivation of the equation satisfied by the limit density F relies on the averaging techniques for hamiltonian vector fields cf. Proposition 4.2. The key point is to appeal

to the classical result saying that any hamiltonian vector field preserves the symplectic structure, see Proposition A.1.

Theorem 2.1 *Let f^{in} be a non negative smooth enough presence density, with finite mass and total (kinetic and electric) energy*

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f^{\text{in}}(x, v) \, dv dx < +\infty$$

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} m \frac{|v|^2}{2} f^{\text{in}}(x, v) \, dv dx + \frac{q^2}{2\varepsilon_0} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{f^{\text{in}}(x, v) f^{\text{in}}(y, w)}{4\pi|x-y|} \, dw dy dv dx < +\infty.$$

We denote by $(f^\varepsilon, \phi^\varepsilon)_{\varepsilon>0}$ the solutions of the Vlasov-Poisson problems (1), (2), (3).

Let $(\mathcal{X}, \mathcal{V})$ be the characteristic flow of the vector field

$$b(x, v) \cdot \nabla_{x,v} = [v - (v \cdot e(x))e(x)] \cdot \nabla_x + [\omega(x)v \wedge e(x) + (v \cdot e(x)) {}^t \partial_x e v] \cdot \nabla_v$$

and \mathcal{E} the function given by

$$\mathcal{E}(X, V, Y, W) = \lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T \frac{dt}{4\pi|\mathcal{X}(t; X, V) - \mathcal{X}(t; Y, W)|}, \quad (X, V, Y, W) \in \mathbb{R}^{12}.$$

We denote by F the solution of the problem

$$\partial_t F + (F(t), \mathcal{H}[F(t)]) = 0, \quad (t, X, V) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3 \quad (7)$$

$$F(0, X, V) = f^{\text{in}}(X, V), \quad (X, V) \in \mathbb{R}^3 \times \mathbb{R}^3 \quad (8)$$

where

$$\mathcal{H}[F(t)](X, V) = m \frac{\langle (v \cdot e)^2 \rangle}{2}(X, V) + \frac{q^2}{\varepsilon_0} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \mathcal{E}(X, V, Y, W) F(t, Y, W) \, dW dY \quad (9)$$

and the Poisson bracket (\cdot, \cdot) is given by

$$m(F, \mathcal{H}[F]) = \nabla_V \mathcal{H}[F] \cdot \nabla_X F - \nabla_X \mathcal{H}[F] \cdot \nabla_V F + (\nabla_V F \wedge \nabla_V \mathcal{H}[F]) \cdot \frac{q\mathbf{B}}{m}. \quad (10)$$

Then we have the asymptotic behavior

$$f^\varepsilon(t, x, v) = F(t, \mathcal{X}(-t; x, v), \mathcal{V}(-t; x, v)) + o(1), \quad \text{as } \varepsilon \searrow 0.$$

The notation $\langle \cdot \rangle$ stands for the average along the flow $(\mathcal{X}, \mathcal{V})$, see Proposition 4.2.

The above limit model conserves the mass and the total energy. Actually, the perpendicular kinetic energy is preserved as well.

Theorem 2.2 *Assume that the hypotheses in Theorem 2.1 hold true. Let us denote by F the solution of the problem (7), (8), (9), (10). Then we have*

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} F(t, X, V) \, dV dX &= 0, \quad t \in \mathbb{R}_+ \\ \frac{d}{dt} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} m \frac{|V \wedge e(X)|^2}{2} F(t, X, V) \, dV dX &= 0, \quad t \in \mathbb{R}_+ \\ \frac{d}{dt} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} F(t, X, V) \left\{ m \frac{\langle (v \cdot e)^2 \rangle}{2} + \frac{q^2}{2\varepsilon_0} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \mathcal{E}(X, V, Y, W) F(t, Y, W) \, dW dY \right\} dV dX &= 0. \end{aligned}$$

3 Effective transport under fast rotation

Before analyzing the non linear Vlasov-Poisson system, it is very instructive to consider the linear transport problem

$$\partial_t g + a(t, z) \cdot \nabla_z g + \omega^\perp z \cdot \nabla_z g = 0, \quad (t, z) \in \mathbb{R}_+ \times \mathbb{R}^2 \quad (11)$$

$$g(0, z) = g^{\text{in}}(z), \quad z \in \mathbb{R}^2 \quad (12)$$

where $a(t, z)$ is a given smooth, divergence free vector field of amplitude $\mathbf{a} := \|a\|_{L^\infty(\mathbb{R}_+ \times \mathbb{R}^2)}$, $\omega > 0$ and $^\perp z = (z_2, -z_1)$, for any $z = (z_1, z_2) \in \mathbb{R}^2$. We assume that the variations of the initial condition g^{in} occurs on a typical length L , that is $\left| \frac{\nabla_z g^{\text{in}}}{g^{\text{in}}} \right| \sim \frac{1}{L}$, and that

$$\frac{L}{\mathbf{a}} \gg \frac{1}{\omega}. \quad (13)$$

The above condition says that the dynamics generated by the advection $a \cdot \nabla_z$ evolves on a much longer time scale than the dynamics generated by the advection $\omega^\perp z \cdot \nabla_z$, whose period is $2\pi/\omega$. In this case it is legitimate to approximate the global dynamics, by averaging with respect to the fast rotation corresponding to the transport operator $\omega^\perp z \cdot \nabla_z$, see also [9, 7]. The idea is to filter out the fast rotation and after that to pass to the limit when $\varepsilon = \frac{\mathbf{a}}{L\omega}$ becomes small. More exactly, let us introduce the flow

$$\frac{dZ}{dt} = \omega^\perp Z(t; z), \quad Z(0; z) = z$$

and the new unknowns $(G^\varepsilon)_{\varepsilon>0}$ given by

$$g^\varepsilon(t, z) = G^\varepsilon(t, Z), \quad Z = Z(-t; z) = \mathcal{R}(\omega t)z$$

where $(g^\varepsilon)_{\varepsilon>0}$ solve (11), (12) with $\frac{\mathbf{a}}{L\omega} = \varepsilon$, and $\mathcal{R}(s)$ stands for the rotation of angle $s \in \mathbb{R}$. The family $(G^\varepsilon)_{\varepsilon>0}$ satisfy

$$\partial_t G^\varepsilon + \tilde{\varphi}(\omega t) a(t) \cdot \nabla_Z G^\varepsilon = 0, \quad (t, Z) \in \mathbb{R}_+ \times \mathbb{R}^2 \quad (14)$$

$$G^\varepsilon(0, Z) = g^{\text{in}}(Z), \quad Z \in \mathbb{R}^2$$

where for any vector field $c = c(z)$, the notation $\tilde{\varphi}(s)c$ stands for the vector field

$$(\tilde{\varphi}(s)c)(Z) = \mathcal{R}(s)c(\mathcal{R}(-s)Z), \quad Z \in \mathbb{R}^2, \quad s \in \mathbb{R}.$$

The family $(G^\varepsilon)_{\varepsilon>0}$ contains no fast rotation, and therefore we expect stability as $\varepsilon \searrow 0$. In order to identify the limit model, we appeal to the following two-scale Hilbert development

$$G^\varepsilon(t, Z) = G(t, s = \omega t, Z) + G^1(t, s = \omega t, Z) + G^2(t, s = \omega t, Z) + \dots \quad (15)$$

where $G^1(t, s, Z) \sim \varepsilon G(t, s, Z)$, $G^2(t, s, Z) \sim \varepsilon^2 G(t, s, Z)$ Plugging the Ansatz (15) in (14) we obtain

$$\begin{aligned} \partial_t G(t, \omega t, Z) + \omega \partial_s G(t, \omega t, Z) + \partial_t G^1(t, \omega t, Z) + \omega \partial_s G^1(t, \omega t, Z) + \dots \\ + \tilde{\varphi}(\omega t) a(t) \cdot (\nabla_Z G(t, \omega t, Z) + \nabla_Z G^1(t, \omega t, Z) + \dots) = 0. \end{aligned}$$

Identifying the contributions of the same orders, one gets

$$\partial_s G(t, s, Z) = 0, \quad (t, s, Z) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^2$$

$$\partial_t G(t, s, z) + \omega \partial_s G^1(t, s, Z) + \tilde{\varphi}(s) a(t) \cdot \nabla_Z G(t, s, Z) = 0, \quad (t, s, Z) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^2. \quad (16)$$

Therefore G depends only on t and Z and its evolution comes by (16), after eliminating G^1 . As $\tilde{\varphi}(s)$ is 2π -periodic, we expect that all dependencies with respect to s are 2π -periodic and therefore, averaging for $s \in [0, 2\pi]$ leads to the limit problem

$$\partial_t G(t, Z) + \frac{1}{2\pi} \int_0^{2\pi} \tilde{\varphi}(s) a(t) \, ds \cdot \nabla_Z G(t, Z) = 0, \quad (t, Z) \in \mathbb{R}_+ \times \mathbb{R}^2 \quad (17)$$

$$G(0, Z) = g^{\text{in}}(Z), \quad Z \in \mathbb{R}^2. \quad (18)$$

We point out that in the non periodic case, it is possible to replace the mean over one period by the ergodic mean. Indeed, when $S \rightarrow +\infty$, we have by (13)

$$\frac{1}{S} \int_0^S \omega \partial_s G^1 \, ds = \left[\frac{\omega G^1}{S} \right]_0^S \sim \left[\frac{\omega \varepsilon G}{S} \right]_0^S \sim \frac{1}{S} \left[\frac{\mathbf{a}}{L} G \right]_0^S \ll \frac{1}{S} \int_0^S \tilde{\varphi}(s) a(t) \, ds \cdot \nabla_Z G$$

and therefore in the general case (not necessarily periodic), the equation (17) becomes

$$\partial_t G + \lim_{S \rightarrow +\infty} \frac{1}{S} \int_0^S \tilde{\varphi}(s) a(t) ds \cdot \nabla_Z G = 0, \quad (t, Z) \in \mathbb{R}_+ \times \mathbb{R}^2$$

provided that the family $(\frac{1}{S} \int_0^S \tilde{\varphi}(s) a(t) ds)_{S>0}$ admits a limit, when $S \rightarrow +\infty$, for any $t \in \mathbb{R}_+$. The leading term in the development (15) satisfies a transport equation, whose advection vector field at any time t appears as the average of $a(t)$ along the characteristic flow of $\omega^\perp z \cdot \nabla_z$. We call it the effective advection vector field under fast rotation

$$\langle a(t) \rangle := \lim_{S \rightarrow +\infty} \frac{1}{S} \int_0^S \tilde{\varphi}(s) a(t) ds.$$

At this stage, the argument are completely formal. Nevertheless we will prove that the family $(G^\varepsilon)_{\varepsilon>0}$ converges strongly in $L_{\text{loc}}^\infty(\mathbb{R}_+, L^2(\mathbb{R}^2))$ toward G when $\varepsilon \searrow 0$, under suitable hypotheses. For doing that, let us analyse the family of transformations $(\tilde{\varphi}(s))_{s \in \mathbb{R}}$ in $L^2(\mathbb{R}^2)^2$.

Proposition 3.1 *The family of linear transformations $c \rightarrow \tilde{\varphi}(s)c = \mathcal{R}(s)c(\mathcal{R}(-s)\cdot)$, $s \in \mathbb{R}$ is a C^0 -group of unitary operators on $L^2(\mathbb{R}^2)^2$. If $c \in L^2(\mathbb{R}^2)^2$ is divergence free, then so is $\tilde{\varphi}(s)c$, for any $s \in \mathbb{R}$*

Proof. We only check that $\text{div}_Z \tilde{\varphi}(s)c = 0$ for any divergence free vector field $c \in L^2(\mathbb{R}^2)^2$. Let $c \in L^2(\mathbb{R}^2)^2$ be a vector field such that $\text{div}_z c = 0$ in $\mathcal{D}'(\mathbb{R}^2)$. For any test function $\Psi(Z) \in C_c^1(\mathbb{R}^2)$ and any $s \in \mathbb{R}$ we have

$$\begin{aligned} \int_{\mathbb{R}^2} \tilde{\varphi}(s)c \cdot \nabla_Z \Psi dZ &= \int_{\mathbb{R}^2} \mathcal{R}(s)c(\mathcal{R}(-s)Z) \cdot \nabla_Z \Psi(Z) dZ \\ &= \int_{\mathbb{R}^2} c(\mathcal{R}(-s)Z) \cdot (\nabla_z(\Psi \circ \mathcal{R}(s)))(\mathcal{R}(-s)Z) dZ \\ &= \int_{\mathbb{R}^2} c(z) \cdot (\nabla_z(\Psi \circ \mathcal{R}(s)))(z) dz = 0 \end{aligned}$$

saying that $\text{div}_Z \tilde{\varphi}(s)c = 0$ in $\mathcal{D}'(\mathbb{R}^2)$. □

We denote by \mathcal{L} the infinitesimal generator of $(\tilde{\varphi}(s))_{s \in \mathbb{R}}$

$$\mathcal{L} : \text{dom} \mathcal{L} \subset L^2(\mathbb{R}^2)^2 \rightarrow L^2(\mathbb{R}^2)^2, \quad \text{dom} \mathcal{L} = \left\{ c \in L^2(\mathbb{R}^2)^2 : \exists \lim_{s \rightarrow 0} \frac{\tilde{\varphi}(s)c - c}{s} \text{ in } L^2(\mathbb{R}^2)^2 \right\}$$

and

$$\mathcal{L}c = \lim_{s \rightarrow 0} \frac{\tilde{\varphi}(s)c - c}{s} = \perp z \cdot \nabla_z c - \perp c, \quad c \in \text{dom} \mathcal{L}.$$

The properties of the operator \mathcal{L} are summarized below.

Proposition 3.2

1. The domain of \mathcal{L} is dense in $L^2(\mathbb{R}^2)^2$ and \mathcal{L} is closed.
2. The operator \mathcal{L} is skew-adjoint.
3. The average operator $c \in L^2(\mathbb{R}^2)^2 \rightarrow \langle c \rangle := \frac{1}{2\pi} \int_0^{2\pi} \tilde{\varphi}(s)c \, ds$ coincides with the orthogonal projection on $\ker \mathcal{L} = \{c \in L^2(\mathbb{R}^2)^2 : \tilde{\varphi}(s)c = c, s \in \mathbb{R}\}$. If the vector field c is divergence free, then so is the vector field $\langle c \rangle$.
4. We have the Poincaré inequality

$$\|c\|_{L^2(\mathbb{R}^2)} \leq \frac{\pi}{2} \|\mathcal{L}c\|_{L^2(\mathbb{R}^2)}$$

for any $c \in \text{dom}\mathcal{L} \cap (\ker \mathcal{L})^\perp$ and $\text{Range}\mathcal{L} = (\ker \mathcal{L})^\perp$.

Proof.

1. The operator \mathcal{L} is the infinitesimal generator of a C^0 -group, and therefore $\text{dom}\mathcal{L}$ is dense in $L^2(\mathbb{R}^2)^2$ and \mathcal{L} is closed.
2. The skew-adjointness of \mathcal{L} is a consequence of the fact that $(\tilde{\varphi}(s))_{s \in \mathbb{R}}$ is a C^0 -group of unitary operators.
3. It is easily seen that for any $c \in L^2(\mathbb{R}^2)^2$, the average $\langle c \rangle$ is left invariant by the group $(\tilde{\varphi}(s))_{s \in \mathbb{R}}$. Indeed, by periodicity, we have for any $h \in \mathbb{R}$

$$\tilde{\varphi}(h) \langle c \rangle = \tilde{\varphi}(h) \frac{1}{2\pi} \int_0^{2\pi} \tilde{\varphi}(s)c \, ds = \frac{1}{2\pi} \int_0^{2\pi} \tilde{\varphi}(s+h)c \, ds = \frac{1}{2\pi} \int_h^{2\pi+h} \tilde{\varphi}(s)c \, ds = \langle c \rangle$$

and therefore $\langle c \rangle \in \ker \mathcal{L}$. For any $a \in \ker \mathcal{L}$ we have

$$\int_{\mathbb{R}^2} a \cdot c \, dz = \int_{\mathbb{R}^2} \tilde{\varphi}(s)a \cdot \tilde{\varphi}(s)c \, dz = \int_{\mathbb{R}^2} a \cdot \tilde{\varphi}(s)c \, dz, \quad s \in \mathbb{R}.$$

Averaging with respect to s over one period, one gets

$$\int_{\mathbb{R}^2} a \cdot c \, dz = \int_{\mathbb{R}^2} a \cdot \langle c \rangle \, dz$$

saying that $c - \langle c \rangle \perp \ker \mathcal{L}$. Therefore we have the equality $\langle c \rangle = \text{Proj}_{\ker \mathcal{L}} c, c \in L^2(\mathbb{R}^2)^2$. Assume that $c \in L^2(\mathbb{R}^2)^2$ is a divergence free vector field. By Proposition 3.1 we know that $\tilde{\varphi}(s)c$ is divergence free for any $s \in \mathbb{R}$. For any test function $\Psi \in C_c^1(\mathbb{R}^2)$ we have

$$\int_{\mathbb{R}^2} \langle c \rangle \cdot \nabla_Z \Psi \, dZ = \int_{\mathbb{R}^2} \frac{1}{2\pi} \int_0^{2\pi} \tilde{\varphi}(s)c \, ds \cdot \nabla_Z \Psi \, dZ = \frac{1}{2\pi} \int_0^{2\pi} \int_{\mathbb{R}^2} \tilde{\varphi}(s)c \cdot \nabla_Z \Psi \, dZ ds = 0$$

saying that $\operatorname{div}_Z \langle c \rangle = 0$.

4. Let $a \in L^2(\mathbb{R}^2)^2$ be a vector field such that $\langle a \rangle = 0$. We consider the vector field

$$c = \frac{1}{2\pi} \int_0^{2\pi} s \tilde{\varphi}(s) a \, ds.$$

We have

$$\tilde{\varphi}(h)c = \frac{1}{2\pi} \int_0^{2\pi} s \tilde{\varphi}(s+h) a \, ds = \frac{1}{2\pi} \int_h^{2\pi+h} (\sigma-h) \tilde{\varphi}(\sigma) a \, d\sigma = \frac{1}{2\pi} \int_h^{2\pi+h} \sigma \tilde{\varphi}(\sigma) a \, d\sigma$$

and we deduce that $\lim_{h \rightarrow 0} (\tilde{\varphi}(h)c - c)/h = a$, saying that $c \in \operatorname{dom} \mathcal{L}$ and $\mathcal{L}c = a$.

Observe also that c has zero average

$$4\pi^2 \langle c \rangle = \int_0^{2\pi} \int_0^{2\pi} s \tilde{\varphi}(s+h) a \, ds dh = \int_0^{2\pi} s \underbrace{\int_0^{2\pi} \tilde{\varphi}(s+h) a \, dh}_{2\pi \langle a \rangle = 0} ds = 0.$$

Moreover, we have the inequality

$$\|c\|_{L^2} = \left\| \frac{1}{2\pi} \int_0^{2\pi} (s-\pi) \tilde{\varphi}(s) a \, ds \right\|_{L^2} \leq \frac{\|a\|_{L^2}}{2\pi} \int_0^{2\pi} |s-\pi| \, ds = \frac{\pi}{2} \|a\|_{L^2}.$$

The range of \mathcal{L} is closed, since we have

$$(\ker \mathcal{L})^\perp \subset \operatorname{Range} \mathcal{L} \subset \overline{\operatorname{Range} \mathcal{L}} = (\ker \mathcal{L}^*)^\perp = (\ker \mathcal{L})^\perp.$$

□

Remark 3.1

1. The group $(\tilde{\varphi}(s))_{s \in \mathbb{R}}$ acts also on $L^\infty(\mathbb{R}^2)^2$. More exactly, if $c \in L^\infty(\mathbb{R}^2)^2$, then $\tilde{\varphi}(s)c \in L^\infty(\mathbb{R}^2)^2$ and $\|\tilde{\varphi}(s)c\|_{L^\infty} = \|c\|_{L^\infty}$, for any $s \in \mathbb{R}$.
2. We define the average operator on $L^\infty(\mathbb{R}^2)^2$ by the same formula

$$\langle c \rangle = \frac{1}{2\pi} \int_0^{2\pi} \tilde{\varphi}(s) c \, ds, \quad c \in L^\infty(\mathbb{R}^2)^2$$

and we have $\|\langle c \rangle\|_{L^\infty} \leq \|c\|_{L^\infty}$ for any $c \in L^\infty(\mathbb{R}^2)^2$.

3. For any $a \in L^2(\mathbb{R}^2)^2 \cap L^\infty(\mathbb{R}^2)^2 \cap (\ker \mathcal{L})^\perp$, there is a unique $c \in \operatorname{dom} \mathcal{L} \cap L^\infty(\mathbb{R}^2)^2 \cap (\ker \mathcal{L})^\perp$ such that $\mathcal{L}c = a$. Moreover we have

$$\|c\|_{L^\infty} \leq \frac{\pi}{2} \|a\|_{L^\infty}.$$

In order to check the asymptotic behavior $G^\varepsilon - G = \mathcal{O}(\varepsilon)$ in $L_{\text{loc}}^\infty(\mathbb{R}_+, L^2(\mathbb{R}^2))$, as $\varepsilon \searrow 0$, we need to introduce a corrector. Thanks to the fourth statement of Proposition 3.2, for any $t \in \mathbb{R}_+$ we consider $c(t) \in \text{dom}\mathcal{L} \cap (\ker \mathcal{L})^\perp$ such that $a(t) - \langle a(t) \rangle = \mathcal{L}c(t)$. Observe that

$$\begin{aligned} \partial_t G + \tilde{\varphi}(s)a(t) \cdot \nabla_Z G &= \tilde{\varphi}(s)a(t) \cdot \nabla_Z G - \langle a(t) \rangle \cdot \nabla_Z G \\ &= \tilde{\varphi}(s)(a(t) - \langle a(t) \rangle) \cdot \nabla_Z G \\ &= \tilde{\varphi}(s)\mathcal{L}c(t) \cdot \nabla_Z G \\ &= \frac{d}{ds} \{ \tilde{\varphi}(s)c(t) \cdot \nabla_Z G \} \end{aligned}$$

and therefore (16) is equivalent to

$$\omega G^1(t, s, Z) + \tilde{\varphi}(s)c(t) \cdot \nabla_Z G = \omega G^1(t, 0, Z) + c(t) \cdot \nabla_Z G.$$

The choice $G^1(t, 0, Z) = 0$, $(t, Z) \in \mathbb{R}_+ \times \mathbb{R}^2$, leads to the corrector

$$\omega G^1(t, s, Z) = (c(t) - \tilde{\varphi}(s)c(t)) \cdot \nabla_Z G. \quad (19)$$

If the initial condition g^{in} is smooth enough, we prove that G remains smooth and thanks to the inequalities

$$\|c(t) - \tilde{\varphi}(s)c(t)\|_{L^\infty(\mathbb{R}^2)} \leq 2\|c(t)\|_{L^\infty(\mathbb{R}^2)} \leq \pi\|\mathcal{L}c(t)\|_{L^\infty(\mathbb{R}^2)} \leq 2\pi\|a(t)\|_{L^\infty(\mathbb{R}^2)}$$

we expect that

$$\begin{aligned} \sup_{s \in \mathbb{R}_+} \|G^1(t)\|_{L^2(\mathbb{R}^2)} &\leq \frac{2\pi\mathbf{a}}{\omega} \frac{\|\nabla_Z G(t)\|_{L^2(\mathbb{R}^2)}}{\|G(t)\|_{L^2(\mathbb{R}^2)}} \|G(t)\|_{L^2(\mathbb{R}^2)} \\ &\sim \frac{2\pi\mathbf{a}}{\omega} \frac{\|\nabla_Z g^{\text{in}}\|_{L^2(\mathbb{R}^2)}}{\|g^{\text{in}}\|_{L^2(\mathbb{R}^2)}} \|G(t)\|_{L^2(\mathbb{R}^2)} \sim \varepsilon \|G(t)\|_{L^2(\mathbb{R}^2)}. \end{aligned}$$

The idea will be to estimate $G^\varepsilon(t, Z) - G(t, Z) - G^1(t, \omega t, Z)$ and to combine with a triangular inequality, by taking into account that $G^1 \sim \varepsilon G$. Notice that

$$\begin{aligned} \frac{d}{dt} \{G(t, Z) + G^1(t, \omega t, Z)\} + \tilde{\varphi}(\omega t)a(t) \cdot \nabla_Z \{G(t, Z) + G^1(t, \omega t, Z)\} &= \partial_t G(t, Z) \\ &\quad + \partial_t G^1(t, \omega t, Z) + \omega \partial_s G^1(t, \omega t, Z) + \tilde{\varphi}(\omega t)a(t) \cdot \nabla_Z \{G + G^1\} \\ &= \partial_t G^1(t, \omega t, Z) + \tilde{\varphi}(\omega t)a(t) \cdot \nabla_Z G^1(t, \omega t, Z) \end{aligned}$$

which together with (14), gives

$$\begin{aligned} \frac{d}{dt} \{G^\varepsilon(t, Z) - G(t, Z) - G^1(t, \omega t, Z)\} + \tilde{\varphi}(\omega t)a(t) \cdot \nabla_Z \{G^\varepsilon(t, Z) - G(t, Z) - G^1(t, \omega t, Z)\} \\ = -\partial_t G^1(t, \omega t, Z) - \tilde{\varphi}(\omega t)a(t) \cdot \nabla_Z G^1(t, \omega t, Z). \end{aligned}$$

As at any time t , the vector field $a(t)$ is divergence free, we know by Proposition 3.1 that $\tilde{\varphi}(\omega t)a(t)$ is also divergence free, and by standard computations one gets

$$\frac{d}{dt} \|G^\varepsilon(t) - G(t) - G^1(t, \omega t)\|_{L^2} \leq \sup_{s \in \mathbb{R}_+} \|\partial_t G^1(t, s)\|_{L^2} + \sup_{s \in \mathbb{R}_+} \|\tilde{\varphi}(s)a(t) \cdot \nabla_Z G^1(t, s)\|_{L^2}.$$

Integrating with respect to t and taking into account that $G^\varepsilon(0) - G(0) - G^1(0, 0) = g^{\text{in}} - g^{\text{in}} - 0 = 0$, we obtain

$$\begin{aligned} \|G^\varepsilon(t) - G(t)\|_{L^2(\mathbb{R}^2)} &\leq \sup_{s \in \mathbb{R}_+} \|G^1(t, s)\|_{L^2(\mathbb{R}^2)} \\ &+ \int_0^T \left\{ \sup_{s \in \mathbb{R}_+} \|\partial_t G^1(t, s)\|_{L^2(\mathbb{R}^2)} + \sup_{s \in \mathbb{R}_+} \|\tilde{\varphi}(s)a(t) \cdot \nabla_Z G^1(t, s)\|_{L^2(\mathbb{R}^2)} \right\} ds, t \in [0, T]. \end{aligned} \quad (20)$$

It remains to estimate the L^2 norms of $G^1(t, s)$, $\partial_t G^1(t, s)$, $\tilde{\varphi}(s)a(t) \cdot \nabla_Z G^1(t, s)$ locally in t and uniformly in $s \in \mathbb{R}_+$. This is a little bit tedious, but can be done under suitable smoothness assumptions. It comes by appealing to the explicit expression (19) of the corrector G^1 and the regularity of G , the solution of the limit model (17), (18). The estimates are uniform with respect to s thanks to the fact that $(\tilde{\varphi}(s))_{s \in \mathbb{R}}$ is a C^0 -group of unitary transformations in $L^2(\mathbb{R}^2)$. For details we refer to [10]. Finally, under the condition (13), we obtain

$$\sup_{(t,s) \in [0,T] \times \mathbb{R}_+} \left\{ \|G^1(t, s)\|_{L^2} + \|\partial_t G^1(t, s)\|_{L^2} + \|\tilde{\varphi}(s)a(t) \cdot \nabla_Z G^1(t, s)\|_{L^2} \right\} \leq C(T, G)\varepsilon$$

and (20) becomes

$$\sup_{t \in [0, T], \varepsilon > 0} \|G^\varepsilon(t) - G(t)\|_{L^2(\mathbb{R}^2)} \leq \tilde{C}(T, G)\varepsilon$$

for some constants C, \tilde{C} depending on T and G . Coming back to the unknowns $g^\varepsilon(t, z) = G^\varepsilon(t, \mathcal{R}(\omega t)z)$, we deduce that

$$g^\varepsilon(t, z) - G(t, \mathcal{R}(\omega t)z) = \mathcal{O}(\varepsilon) \quad \text{in } L_{\text{loc}}^\infty(\mathbb{R}_+, L^2(\mathbb{R}^2)), \quad \text{as } \varepsilon \searrow 0.$$

4 Effective transport under fast advection

We generalize the previous analysis to the case of two vector fields $a(t, z) \cdot \nabla_z$, $b(z) \cdot \nabla_z$ whose associated phase flows evolve on different time scales T_a, T_b . We only indicate the main lines, following the arguments developed in Section 3. For a complete rigorous

study we refer to [10]. We investigate the asymptotic behavior of the solutions $(g^\varepsilon)_{\varepsilon>0}$ solving

$$\partial_t g^\varepsilon + a(t, z) \cdot \nabla_z g^\varepsilon + b(z) \cdot \nabla_z g^\varepsilon = 0, \quad (t, z) \in \mathbb{R}_+ \times \mathbb{R}^m \quad (21)$$

$$g^\varepsilon(0, z) = g^{\text{in}}(z), \quad z \in \mathbb{R}^m \quad (22)$$

when $\varepsilon = T_b/T_a$ becomes small. We suppose that the initial density belongs to $L^2(\mathbb{R}^m)$ and that the total advection vector field is divergence free

$$\text{div}_z \{a(t) + b\} = 0, \quad t \in \mathbb{R}_+$$

which guarantees the L^2 norm conservation

$$\int_{\mathbb{R}^m} (g^\varepsilon(t, z))^2 dz = \int_{\mathbb{R}^m} (g^{\text{in}}(z))^2 dz, \quad t \in \mathbb{R}_+, \quad \varepsilon > 0.$$

If the vector field $b(z) \cdot \nabla_z$ is smooth, with growth at most linear at infinity, it possesses a smooth global flow $\mathcal{Z}(t; z)$

$$\frac{d\mathcal{Z}}{dt} = b(\mathcal{Z}(t; z)), \quad \mathcal{Z}(0; z) = z, \quad (t, z) \in \mathbb{R} \times \mathbb{R}^m.$$

Filtering out the fast motion along b , and taking into account the mass constraint, we are led to the new unknown given by

$$g^\varepsilon(t, z) dz = G^\varepsilon(t, Z) dZ, \quad Z = \mathcal{Z}(-t; z)$$

that is

$$G^\varepsilon(t, Z) = g^\varepsilon(t, z) J(t, Z), \quad z = \mathcal{Z}(t; Z), \quad J(t, Z) = \det \frac{\partial \mathcal{Z}}{\partial Z}(t; Z).$$

Recall that the Jacobian determinant satisfies $J(0, \cdot) = 1$ and

$$\partial_t J(t, Z) = J(t, Z) (\text{div}_z b)(\mathcal{Z}(t; Z)) = -J(t, Z) (\text{div}_z a(t))(\mathcal{Z}(t; Z)), \quad (t, Z) \in \mathbb{R} \times \mathbb{R}^m.$$

In the next proposition we identify the transport problems satisfied by the densities $(G^\varepsilon)_{\varepsilon>0}$.

Proposition 4.1 *The new densities $(G^\varepsilon)_{\varepsilon>0}$ given by $G^\varepsilon(t, Z) dZ = g^\varepsilon(t, z) dz$, $Z = \mathcal{Z}(-t; z)$ verify*

$$\partial_t G^\varepsilon + \text{div}_Z \{G^\varepsilon \varphi(t) a(t)\} = 0, \quad (t, Z) \in \mathbb{R}_+ \times \mathbb{R}^m \quad (23)$$

$$G^\varepsilon(0, Z) = g^{\text{in}}(Z), \quad Z \in \mathbb{R}^m \quad (24)$$

where for any vector field $c = c(z)$, the notation $\varphi(t)c$ stands for the vector field

$$(\varphi(t)c)(Z) = \partial \mathcal{Z}(-t; \mathcal{Z}(t; Z))c(\mathcal{Z}(t; Z)), \quad (t, Z) \in \mathbb{R} \times \mathbb{R}^m.$$

Proof. Let us pick a compactly supported smooth test function $\Psi(t, Z)$ in $\mathbb{R}_+ \times \mathbb{R}^m$.

Using the weak formulation of (21), (22) with $\psi(t, z) = \Psi(t, \mathcal{Z}(-t; z))$, one gets

$$\begin{aligned} 0 &= \int_{\mathbb{R}^m} g^{\text{in}}(z)\psi(0, z) \, dz + \int_{\mathbb{R}_+} \int_{\mathbb{R}^m} (\partial_t \psi + a(t, z) \cdot \nabla_z \psi + b(z) \cdot \nabla_z \psi) g^\varepsilon(t, z) \, dz dt \\ &= \int_{\mathbb{R}^m} g^{\text{in}}(Z)\Psi(0, Z) \, dZ + \int_{\mathbb{R}_+} \int_{\mathbb{R}^m} a(t, \mathcal{Z}(t; Z)) \cdot {}^t \partial \mathcal{Z}(-t; \mathcal{Z}(t; Z)) \nabla_Z \Psi(t, Z) G^\varepsilon(t, Z) \, dZ dt \\ &\quad + \int_{\mathbb{R}_+} \int_{\mathbb{R}^m} \{(\partial_t \psi)(t, \mathcal{Z}(t; Z)) + b(\mathcal{Z}(t; Z)) \cdot (\nabla_z \psi)(t, \mathcal{Z}(t; Z))\} G^\varepsilon(t, Z) \, dZ dt \\ &= \int_{\mathbb{R}^m} g^{\text{in}}(Z)\Psi(0, Z) \, dZ + \int_{\mathbb{R}_+} \int_{\mathbb{R}^m} \{\partial_t \Psi(t, Z) + \varphi(t)a(t) \cdot \nabla_Z \Psi\} G^\varepsilon(t, Z) \, dZ dt \end{aligned}$$

and therefore the density G^ε satisfies (23), (24). \square

Remark 4.1 If $g^{\text{in}} \in L^p(\mathbb{R}^m)$, $1 \leq p < +\infty$, then we have

$$\int_{\mathbb{R}^m} |g^\varepsilon(t, z)|^p \, dz = \int_{\mathbb{R}^m} |g^{\text{in}}(z)|^p \, dz, \quad t \in \mathbb{R}_+, \quad \varepsilon > 0$$

implying that

$$\int_{\mathbb{R}^m} \frac{|G^\varepsilon(t, Z)|^p}{J^{p-1}(t, Z)} \, dZ = \int_{\mathbb{R}^m} |g^{\text{in}}(Z)|^p \, dZ, \quad t \in \mathbb{R}_+, \quad \varepsilon > 0.$$

Remark 4.2 For any vector field c , we have the formula

$$\text{div}_Z \{J(t, Z)\varphi(t)c\} = J(t, Z)(\text{div}_z c)(\mathcal{Z}(t; Z)). \quad (25)$$

Indeed, let us pick a compactly supported smooth test function $\Psi(Z)$ and consider

$\psi(z) = \Psi(Z)$, $z = \mathcal{Z}(t; Z)$. By the chain rule, we obtain

$$\begin{aligned} \langle \text{div}_Z \{J(t, Z)\varphi(t)c\}, \Psi \rangle_{\mathcal{D}', \mathcal{D}} &= - \int_{\mathbb{R}^m} J(t, Z)\varphi(t)c \cdot \nabla_Z \Psi \, dZ \\ &= - \int_{\mathbb{R}^m} \partial \mathcal{Z}(-t; z)c(z) \cdot (\nabla_Z \Psi)(\mathcal{Z}(-t; z)) \, dz \\ &= - \int_{\mathbb{R}^m} c(z) \cdot {}^t \partial \mathcal{Z}(-t; z)(\nabla_Z \Psi)(\mathcal{Z}(-t; z)) \, dz \\ &= - \int_{\mathbb{R}^m} c(z) \cdot \nabla_z (\Psi \circ \mathcal{Z}(-t; \cdot)) \, dz \\ &= - \int_{\mathbb{R}^m} c(z) \cdot \nabla_z \psi(z) \, dz \\ &= \langle \text{div}_z c, \psi \rangle_{\mathcal{D}', \mathcal{D}} \end{aligned}$$

saying that (25) holds true. In particular, if the vector field b is divergence free, we have $J = 1$, and in that case we obtain

$$\operatorname{div}_Z \varphi(t)c = (\operatorname{div}_z c) \circ \mathcal{Z}(t; \cdot).$$

Moreover, if the vector field c is divergence free, then so is the vector field $\varphi(t)c$, see also Proposition 3.1.

Motivated by the example in Section 3, we introduce the average operator

$$\langle c \rangle = \lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T \varphi(t)c \, dt = \lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T \partial \mathcal{Z}(-t; \mathcal{Z}(t; \cdot))c(\mathcal{Z}(t; \cdot)) \, dt$$

and we claim that $G = \lim_{\varepsilon \searrow 0} G^\varepsilon$ solves the problem

$$\partial_t G + \operatorname{div}_Z(G \langle a(t) \rangle) = 0, \quad (t, Z) \in \mathbb{R}_+ \times \mathbb{R}^m$$

$$G(0, Z) = g^{\text{in}}(Z), \quad Z \in \mathbb{R}^m.$$

Moreover, under suitable hypotheses, we expect the asymptotic behavior

$$g^\varepsilon(t, \cdot) = \frac{G(t, \mathcal{Z}(-t; \cdot))}{J(t, \mathcal{Z}(-t; \cdot))} + \mathcal{O}(\varepsilon) \quad \text{in } L_{\text{loc}}^\infty(\mathbb{R}_+, L^2(\mathbb{R}^m)), \quad \text{as } \varepsilon \searrow 0.$$

The average operator behaves nicely with respect to hamiltonian vector fields. Consider a symplectic structure on \mathbb{R}^m (with m an even integer), that is, let σ be a differential 2-form on \mathbb{R}^m , which is non degenerate and closed. We denote by Σ the matrix field corresponding to the differential 2-form σ .

Proposition 4.2 *Let b and c be two hamiltonian vector fields on the symplectic manifold (\mathbb{R}^m, σ) , corresponding to the Hamiltonians H_b, H_c respectively. Then the average vector field*

$$\langle c \rangle = \lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T \partial \mathcal{Z}(-t; \mathcal{Z}(t; \cdot))c(\mathcal{Z}(t; \cdot)) \, dt$$

is hamiltonian, and corresponds to the average Hamiltonian

$$\langle H_c \rangle := \lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T H_c(\mathcal{Z}(t; \cdot)) \, dt.$$

Proof. The differential 2-form is left invariant by any hamiltonian flow cf. [2] pp. 204, see also Appendix A. In particular we have (here \mathcal{Z} is the flow of b)

$$\mathcal{Z}(t; \cdot)^* \sigma = \sigma, \quad t \in \mathbb{R}$$

or equivalently

$${}^t\partial\mathcal{Z}(t; \cdot)\Sigma(\mathcal{Z}(t; \cdot))\partial\mathcal{Z}(t; \cdot) = \Sigma(\cdot), \quad t \in \mathbb{R}.$$

Taking into account that

$$\partial\mathcal{Z}(-t; \mathcal{Z}(t; \cdot))\partial\mathcal{Z}(t; \cdot) = I_m$$

we obtain

$$\partial\mathcal{Z}(-t; \mathcal{Z}(t; \cdot))\Sigma^{-1}(\mathcal{Z}(t; \cdot)){}^t\partial\mathcal{Z}(-t; \mathcal{Z}(t; \cdot)) = \Sigma^{-1}(\cdot), \quad t \in \mathbb{R}.$$

As the vector field c is hamiltonian, we have $c = -\Sigma^{-1}\nabla H_c$ and therefore

$$\begin{aligned} \varphi(t)c &= \partial\mathcal{Z}(-t; \mathcal{Z}(t; \cdot))c(\mathcal{Z}(t; \cdot)) \\ &= -\partial\mathcal{Z}(-t; \mathcal{Z}(t; \cdot))\Sigma^{-1}(\mathcal{Z}(t; \cdot))(\nabla H_c)(\mathcal{Z}(t; \cdot)) \\ &= -\Sigma^{-1}(\cdot){}^t\partial\mathcal{Z}(t; \cdot)(\nabla H_c)(\mathcal{Z}(t; \cdot)) \\ &= -\Sigma^{-1}(\cdot)\nabla(H_c \circ \mathcal{Z}(t; \cdot)). \end{aligned}$$

Finally the average of c is given by

$$\langle c \rangle = \lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T \varphi(t)c \, dt = -\Sigma^{-1} \nabla \lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T H_c(\mathcal{Z}(t; \cdot)) \, dt = -\Sigma^{-1} \nabla \langle H_c \rangle$$

saying that the average field $\langle c \rangle$ is hamiltonian, and corresponds to the average Hamiltonian $\langle H_c \rangle$ (see [6] for more details on function averages). \square

5 The finite Larmor radius regime

Let us come back to the Vlasov-Poisson equations (1), (2), (3). In order to apply the previous formalism, we need to identify the advection vector fields acting on different time scales. Clearly, when the magnetic field is uniform, that is $\mathbf{B} = {}^t(0, 0, B)$, the advection vector field in the Vlasov equation writes $(a(t, x, v) + b(x, v)) \cdot \nabla_{x,v}$, where

$$a(t, x, v) \cdot \nabla_{x,v} = v_3 \partial_{x_3} + \frac{q}{m} E(t, x) \cdot \nabla_v \quad (26)$$

$$b(x, v) \cdot \nabla_{x,v} = v_1 \partial_{x_1} + v_2 \partial_{x_2} + \omega(v_2 \partial_{v_1} - v_1 \partial_{v_2}), \quad \omega = \frac{qB}{m}. \quad (27)$$

Notice that when the magnetic field is strong, $b \cdot \nabla_{x,v}$ generates a periodic group, whose period $2\pi/\omega$ is much smaller than the time scale on which evolves the group generated

by $a \cdot \nabla_{x,v}$. Therefore it is legitimate to average $a \cdot \nabla_{x,v}$ with respect to $b \cdot \nabla_{x,v}$. In order to better understand the decomposition of the particle dynamics into slow and fast motions, let us recall that the charged particle flows under electro-magnetic fields are hamiltonians. More exactly, if ϕ, A are the scalar and vector potentials of the electro-magnetic field (E, B) , that is

$$E = -\nabla_x \phi, \quad \mathbf{B} = \nabla_x \wedge A$$

let us consider the differential 2-form θ on \mathbb{R}^6 , whose matrix field is

$$\Theta = \begin{pmatrix} qM[\mathbf{B}] & mI_3 \\ -mI_3 & O_3 \end{pmatrix}.$$

Here, for any vector $u \in \mathbb{R}^3$, the notation $M[u]$ stands for the matrix of the endomorphism $v \in \mathbb{R}^3 \rightarrow u \wedge v \in \mathbb{R}^3$, that is

$$M[u] = \begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix}, \quad u \in \mathbb{R}^3.$$

By straightforward computations we obtain the formula

$$\theta = d[(qA_1 + mv_1)dx_1 + (qA_2 + mv_2)dx_2 + (qA_3 + mv_3)dx_3]$$

and therefore the differential 2-form θ is closed, *i.e.*, $d\theta = 0$. Notice also that θ is non degenerate

$$\Theta^{-1} = \begin{pmatrix} O_3 & -\frac{I_3}{m} \\ \frac{I_3}{m} & \frac{q}{m^2}M[\mathbf{B}] \end{pmatrix}$$

and therefore θ is a symplectic structure on \mathbb{R}^6 . Observe that the vector field $v \cdot \nabla_x + \frac{q}{m}(E + v \wedge \mathbf{B}) \cdot \nabla_v$ is hamiltonian, and corresponds to the Hamiltonian

$$H = m\frac{|v|^2}{2} + q\phi.$$

Indeed, we have

$$\nabla_{x,v}H = -\Theta^{-1} \left(v, \frac{q}{m}(E + v \wedge \mathbf{B}) \right)$$

or equivalently

$$dH(\eta) = \theta \left(\eta, \Theta^{-1} \left(v, \frac{q}{m}(E + v \wedge \mathbf{B}) \right) \right), \quad \eta \in \mathbb{R}^6.$$

The Poisson bracket of the functions f and H , that is, the derivative of the function f in the direction of the characteristic flow with Hamiltonian H , is given by

$$\begin{aligned} (f, H) &= \nabla_{x,v} f \cdot {}^t \left(v, \frac{q}{m} (E + v \wedge \mathbf{B}) \right) \\ &= -\nabla_{x,v} f \cdot \Theta^{-1} \nabla_{x,v} H \\ &= \frac{1}{m} \left\{ \nabla_v H \cdot \nabla_x f - \nabla_x H \cdot \nabla_v f + (\nabla_v f \wedge \nabla_v H) \cdot \frac{q\mathbf{B}}{m} \right\}. \end{aligned}$$

Therefore, the Vlasov-Poisson equations (1), (2) write

$$\begin{aligned} \partial_t f + (f(t), H[f(t)]) &= 0, \quad (t, x, v) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3 \\ H[f(t)](x, v) &= m \frac{|v|^2}{2} + \frac{q^2}{\varepsilon_0} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{f(t, y, w)}{4\pi|x-y|} dw dy, \quad (x, v) \in \mathbb{R}^3 \times \mathbb{R}^3. \end{aligned}$$

We distinguish between the parallel and orthogonal kinetic energy, leading to the decomposition

$$H = H_a + H_b, \quad H_a = m \frac{(v \cdot e(x))^2}{2} + q\phi, \quad H_b = m \frac{|v \wedge e(x)|^2}{2}.$$

To these Hamiltonians it corresponds the vector fields $a \cdot \nabla_{x,v}$ and $b \cdot \nabla_{x,v}$, given by

$$\nabla_{x,v} H_a = -\Theta a, \quad \nabla_{x,v} H_b = -\Theta b.$$

Obviously, these hamiltonian vector fields represent a decomposition of $v \cdot \nabla_x + \frac{q}{m} (E + v \wedge \mathbf{B}) \cdot \nabla_v$

$$-\Theta {}^t \left(v, \frac{q}{m} (E + v \wedge \mathbf{B}) \right) = \nabla_{x,v} H = \nabla_{x,v} (H_a + H_b) = -\Theta (a + b).$$

This decomposition writes

$$a(t, x, v) \cdot \nabla_{x,v} = (v \cdot e(x)) e(x) \cdot \nabla_x + \left[\frac{q}{m} E(t, x) - (v \cdot e(x)) {}^t \partial_x e v \right] \cdot \nabla_v \quad (28)$$

$$b(x, v) \cdot \nabla_{x,v} = [v - (v \cdot e(x)) e(x)] \cdot \nabla_x + [\omega(x) v \wedge e(x) + (v \cdot e(x)) {}^t \partial_x e v] \cdot \nabla_v \quad (29)$$

where $\omega(x) = qB(x)/m$. Notice that both $a \cdot \nabla_{x,v}$, $b \cdot \nabla_{x,v}$ are divergence free

$$\begin{aligned} \operatorname{div}_{x,v} a &= \operatorname{div}_x [(v \cdot e) e] + \operatorname{div}_v \left[\frac{q}{m} E - (v \cdot e) {}^t \partial_x e v \right] \\ &= (v \cdot e) \operatorname{div}_x e + e(x) \cdot {}^t \partial_x e v - e(x) \cdot {}^t \partial_x e v - (v \cdot e) \operatorname{trace}({}^t \partial_x e) = 0 \end{aligned}$$

$$\operatorname{div}_{x,v} b = \operatorname{div}_{x,v} (a + b) - \operatorname{div}_{x,v} a = 0.$$

In the case of uniform magnetic fields $\mathbf{B} = {}^t(0, 0, B)$, the decomposition (28), (29) is exactly that in (26), (27). Motivated by that, in the case of non uniform magnetic fields, we consider the asymptotic regime where the dynamics generated by the advection (28) evolves on a much larger time scale than the dynamics generated by the advection (29). Therefore we need to average $a \cdot \nabla_{x,v}$ with respect to $b \cdot \nabla_{x,v}$, when considering the Vlasov equation

$$\partial_t f^\varepsilon + a \cdot \nabla_{x,v} f^\varepsilon + b \cdot \nabla_{x,v} f^\varepsilon = 0, \quad (t, x, v) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3.$$

The characteristic flow of $b \cdot \nabla_{x,v}$ writes

$$\frac{d\mathcal{X}}{dt} = [I_3 - e(\mathcal{X}(t; x, v)) \otimes e(\mathcal{X}(t; x, v))] \mathcal{V}(t; x, v), \quad t \in \mathbb{R}$$

$$\frac{d\mathcal{V}}{dt} = \omega(\mathcal{X}(t; x, v)) (\mathcal{V} \wedge e(\mathcal{X})) (t; x, v) + (\mathcal{V} \cdot e(\mathcal{X})) (t; x, v) {}^t \partial_x e(\mathcal{X}(t; x, v)) \mathcal{V}(t; x, v), \quad t \in \mathbb{R}$$

with the conditions $(\mathcal{X}, \mathcal{V})(0; x, v) = (x, v)$. At any time $t \in \mathbb{R}_+$, we perform the change of coordinates

$$(X, V) = (\mathcal{X}(-t; x, v), \mathcal{V}(-t; x, v))$$

and we introduce the new presence densities

$$F^\varepsilon(t, X, V) = f^\varepsilon(t, x, v) J(t, X, V), \quad (x, v) = (\mathcal{X}, \mathcal{V})(t; X, V), \quad J(t, X, V) = \det \frac{\partial(\mathcal{X}, \mathcal{V})(t)}{\partial(X, V)}. \quad (30)$$

Since $\operatorname{div}_{x,v} b = 0$, the jacobian determinant equals 1 and (30) reduces to

$$F^\varepsilon(t, X, V) = f^\varepsilon(t, x, v), \quad (X, V) = (\mathcal{X}(-t; x, v), \mathcal{V}(-t; x, v))$$

The new presence densities $(F^\varepsilon)_{\varepsilon>0}$ satisfy cf. Proposition 4.1

$$\partial_t F^\varepsilon + \operatorname{div}_Z \{F^\varepsilon \varphi(t) a[F^\varepsilon(t)]\} = 0, \quad t \in \mathbb{R}_+, \quad Z = (X, V) \in \mathbb{R}^6 \quad (31)$$

$$F^\varepsilon(0, Z) = f^{\text{in}}(Z), \quad Z = (X, V) \in \mathbb{R}^6 \quad (32)$$

where

$$-\Theta(x, v) a[F^\varepsilon(t)] = \nabla_{x,v} \left\{ m \frac{(v \cdot e)^2}{2} + \frac{q^2}{\varepsilon_0} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{F^\varepsilon(t, Y, W)}{4\pi|x - \mathcal{X}(t; Y, W)|} dW dY \right\} \quad (33)$$

and

$$(\varphi(t) a[F^\varepsilon(t)])(X, V) = \partial \mathcal{Z}(-t; \mathcal{Z}(t; Z)) a[F^\varepsilon(t)](\mathcal{Z}(t; Z))$$

$$z = (x, v), \quad Z = (X, V), \quad \mathcal{Z}(t; Z) = (\mathcal{X}(t; X, V), \mathcal{V}(t; X, V)).$$

The behavior of $(F^\varepsilon)_{\varepsilon>0}$, as ε becomes small, that is, when the characteristic time of the perpendicular dynamics is negligible with respect to the characteristic time of the parallel dynamics, follows by passing to the limit in the weak formulation of (31), (32)

$$\begin{aligned} & \int_{\mathbb{R}_+} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} F^\varepsilon(t, X, V) \partial_t \Psi \, dV dX dt + \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f^{\text{in}}(X, V) \Psi(0, X, V) \, dV dX \\ & + \int_{\mathbb{R}_+} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} F^\varepsilon(t, X, V) (\varphi(t) a[F^\varepsilon(t)])(X, V) \cdot \nabla_{X, V} \Psi \, dV dX dt = 0 \end{aligned} \quad (34)$$

for any $\Psi \in C_c^1(\mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3)$. Obviously, the main difficulty is the treatment of the non linear term

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} F^\varepsilon(t, X, V) (\varphi(t) a[F^\varepsilon(t)])(X, V) \cdot \nabla_{X, V} \Psi \, dV dX dt \quad (35)$$

due to the fact that the advection field $a \cdot \nabla_{x, v}$ depends on the unknown presence density F^ε , cf. (33). The linear case is now well understood [10]. The mathematical study is much more elaborated in the non linear case. Two-scale analysis arguments should be combined to compactness arguments, and the formal derivation can be completely justified, at least in the case of strong uniform magnetic fields, see [8, 9]. We concentrate on the formal derivation of the limit model (with non uniform magnetic field) and the study of its properties, the rigorous justification being out of the scope of this paper.

In order to pass to the limit in (35), we distinguish between the slow and fast time

dependencies. Following the computations in Proposition 4.2, we obtain

$$\begin{aligned}
& \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}_+} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} F^\varepsilon(t, X, V) (\varphi(t) a[F^\varepsilon(t)])(X, V) \cdot \nabla_{X,V} \Psi \, dV dX dt \\
&= \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}_+} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} F^\varepsilon(t, X, V) \partial \mathcal{Z}(-t; \mathcal{Z}(t; X, V)) a[F^\varepsilon(t)](\mathcal{Z}(t; X, V)) \cdot \nabla_{X,V} \Psi \, dV dX dt \\
&= - \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}_+} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} F^\varepsilon(t, X, V) \partial \mathcal{Z}(-t; \mathcal{Z}(t; X, V)) \Theta^{-1}(\mathcal{Z}(t; X, V)) \\
&\quad \nabla_{x,v} \left\{ m \frac{(v \cdot e)^2}{2} + \frac{q^2}{\varepsilon_0} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{F^\varepsilon(t, Y, W)}{4\pi |x - \mathcal{X}(t; Y, W)|} \, dW dY \right\} (\mathcal{Z}(t; X, V)) \cdot \nabla_{X,V} \Psi \, dV dX dt \\
&= - \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}_+} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} F^\varepsilon(t, X, V) \Theta^{-1}(X, V) {}^t \partial \mathcal{Z}(t; X, V) \\
&\quad \nabla_{x,v} \left\{ m \frac{(v \cdot e)^2}{2} + \frac{q^2}{\varepsilon_0} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{F^\varepsilon(t, Y, W)}{4\pi |x - \mathcal{X}(t; Y, W)|} \, dW dY \right\} (\mathcal{Z}(t; X, V)) \cdot \nabla_{X,V} \Psi \, dV dX dt \\
&= - \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}_+} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} F^\varepsilon(t, X, V) \Theta^{-1}(X, V) \\
&\quad \nabla_{X,V} \left\{ m \frac{(\mathcal{V} \cdot e(\mathcal{X}))^2}{2} + \frac{q^2}{\varepsilon_0} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{F^\varepsilon(t, Y, W)}{4\pi |\mathcal{X}(t; X, V) - \mathcal{X}(t; Y, W)|} \, dW dY \right\} \cdot \nabla_{X,V} \Psi \, dV dX dt.
\end{aligned}$$

We denote by F the limit of the presence densities $(F^\varepsilon)_{\varepsilon>0}$, when ε goes to 0. Notice that $F = F(t, X, V)$ varies slowly with respect to t . Actually F is the limit of the densities $(F^\varepsilon)_{\varepsilon>0}$, obtained from the densities $(f^\varepsilon)_{\varepsilon>0}$, after filtering the fast oscillations corresponding to the fast dynamics associated to the advection field $b \cdot \nabla_{x,v}$. All the other time dependencies, through the flow $(\mathcal{X}, \mathcal{V})$, are fast and thus, averaging with respect to the fast time variable, one gets

$$\begin{aligned}
& \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}_+} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} F^\varepsilon(t, X, V) (\varphi(t) a[F^\varepsilon(t)])(X, V) \cdot \nabla_{X,V} \Psi \, dV dX dt \\
&= - \int_{\mathbb{R}_+} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} F(t, X, V) \Theta^{-1}(X, V) \nabla_{X,V} \mathcal{H}[F(t)] \cdot \nabla_{X,V} \Psi \, dV dX dt
\end{aligned}$$

where for any density $G = G(X, V)$, the notation $\mathcal{H}[G]$ stands for

$$\begin{aligned}
\mathcal{H}[G](X, V) &= \lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T \left\{ m \frac{(\mathcal{V} \cdot e(\mathcal{X}))^2}{2} + \frac{q^2}{\varepsilon_0} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{G(Y, W)}{4\pi |\mathcal{X}(t; X, V) - \mathcal{X}(t; Y, W)|} \, dW dY \right\} dt \\
&= m \frac{\langle (v \cdot e)^2 \rangle}{2} + \frac{q^2}{\varepsilon_0} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \mathcal{E}(X, V, Y, W) G(Y, W) \, dW dY
\end{aligned}$$

with

$$\mathcal{E}(X, V, Y, W) = \lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T \frac{1}{4\pi |\mathcal{X}(t; X, V) - \mathcal{X}(t; Y, W)|} \, dt. \quad (36)$$

Finally, passing to the limit in the weak formulations (34) yields

$$\begin{aligned} & \int_{\mathbb{R}_+} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} F(t, X, V) \partial_t \Psi \, dV dX dt + \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f^{\text{in}}(X, V) \Psi(0, X, V) \, dV dX \\ & - \int_{\mathbb{R}_+} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} F(t, X, V) \Theta^{-1}(X, V) \nabla_{X,V} \mathcal{H}[F(t)] \cdot \nabla_{X,V} \Psi \, dV dX dt = 0 \end{aligned}$$

for any $\Psi \in C_c^1(\mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3)$. It is easily seen that for any smooth function $\mathcal{H} = \mathcal{H}(X, V)$, the hamiltonian vector field $-\Theta^{-1}(X, V) \nabla_{X,V} \mathcal{H}$ is divergence free. Indeed, we have

$$\begin{aligned} \text{div}_{X,V} \{ -\Theta^{-1}(X, V) \nabla_{X,V} \mathcal{H} \} &= \frac{1}{m} \{ \text{div}_X \nabla_V \mathcal{H} - \text{div}_V \nabla_X \mathcal{H} - \omega(X) \text{div}_V (e(X) \wedge \nabla_V \mathcal{H}) \} \\ &= -\frac{\omega(X)}{m} \text{div}_V (e(X) \wedge \nabla_V \mathcal{H}) = 0. \end{aligned} \quad (37)$$

Therefore the limit density F solves the problem

$$\partial_t F - \Theta^{-1}(X, V) \nabla_{X,V} \mathcal{H}[F(t)] \cdot \nabla_{X,V} F = 0, \quad (t, X, V) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3 \quad (38)$$

$$F(0, X, V) = f^{\text{in}}(X, V), \quad (X, V) \in \mathbb{R}^3 \times \mathbb{R}^3$$

which ends the proof of Theorem 2.1, by observing that

$$\begin{aligned} & -\Theta^{-1}(X, V) \nabla_{X,V} \mathcal{H}[F(t)] \cdot \nabla_{X,V} F(t) \\ &= \frac{1}{m} \left\{ \nabla_V \mathcal{H}[F(t)] \cdot \nabla_X F(t) - \nabla_X \mathcal{H}[F(t)] \cdot \nabla_V F(t) + (\nabla_V F(t) \wedge \nabla_V \mathcal{H}[F(t)]) \cdot \frac{q\mathbf{B}}{m} \right\} \\ &= (F(t), \mathcal{H}[F(t)]). \end{aligned}$$

We focus now on the mass and energy balances for the limit model (7), (8), (9), (10).

We establish first the following properties of the function \mathcal{E} .

Proposition 5.1 *The function $\mathcal{E} = \mathcal{E}(X, V, Y, W)$ given in (36) satisfies*

1.

$$\mathcal{E}(X, V, Y, W) = \mathcal{E}(Y, W, X, V), \quad (X, V, Y, W) \in \mathbb{R}^{12}.$$

2.

$$\mathcal{E}(\mathcal{Z}(h; X, V), \mathcal{Z}(h; Y, W)) = \mathcal{E}(X, V, Y, W), \quad h \in \mathbb{R}, \quad (X, V, Y, W) \in \mathbb{R}^{12}.$$

3.

$$(\nabla_Z \mathcal{E})(X, V, Y, W) \cdot b(X, V) + (\nabla_Z \mathcal{E})(Y, W, X, V) \cdot b(Y, W) = 0, \quad (X, V, Y, W) \in \mathbb{R}^{12}$$

where ∇_Z stands for the gradient with respect to the first two arguments (X, V) of \mathcal{E} .

Proof.

1. The first statement comes immediately, by the definition of \mathcal{E} .

2. It is easily seen that for any $h \in \mathbb{R}$, $(X, V, Y, W) \in \mathbb{R}^{12}$ we have

$$\begin{aligned} \mathcal{E}(\mathcal{Z}(h; X, V), \mathcal{Z}(h; Y, W)) &= \lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T \frac{dt}{|\mathcal{X}(t; \mathcal{Z}(h; X, V)) - \mathcal{X}(t; \mathcal{Z}(h; Y, W))|} \\ &= \lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T \frac{dt}{|\mathcal{X}(t+h; X, V) - \mathcal{X}(t+h; Y, W)|} \\ &= \lim_{T \rightarrow +\infty} \frac{1}{T} \int_h^{T+h} \frac{dt}{|\mathcal{X}(t; X, V) - \mathcal{X}(t; Y, W)|} \\ &= \lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T \frac{dt}{|\mathcal{X}(t; X, V) - \mathcal{X}(t; Y, W)|} \\ &\quad - \lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^h \frac{dt}{|\mathcal{X}(t; X, V) - \mathcal{X}(t; Y, W)|} \\ &\quad + \lim_{T \rightarrow +\infty} \frac{1}{T} \int_T^{T+h} \frac{dt}{|\mathcal{X}(t; X, V) - \mathcal{X}(t; Y, W)|} \\ &= \lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T \frac{dt}{|\mathcal{X}(t; X, V) - \mathcal{X}(t; Y, W)|} \\ &= \mathcal{E}(X, V, Y, W). \end{aligned}$$

3. It follows by combining 1. et 2. If we denote by $\nabla_{Z'}$ the gradient with respect to the last two arguments Y, W of \mathcal{E} , we obtain by the first statement

$$(\nabla_Z \mathcal{E})(X, V, Y, W) = (\nabla_{Z'} \mathcal{E})(Y, W, X, V). \quad (39)$$

Taking the derivative with respect to h , at $h = 0$, in the second statement, one gets

$$(\nabla_Z \mathcal{E})(X, V, Y, W) \cdot b(X, V) + (\nabla_{Z'} \mathcal{E})(X, V, Y, W) \cdot b(Y, W) = 0. \quad (40)$$

Putting together (39), (40) yields the last statement. \square

Proof. (of Theorem 2.2)

Since the hamiltonian vector field $-\Theta^{-1}(X, V) \nabla_{X, V} \mathcal{H}[F]$ is divergence free cf. (37),

see also Remark 4.2, it is easily seen by (38) that the mass is conserved. Multiplying (38), written in conservative form, by $H_b(X, V) = m|V \wedge e(X)|^2/2$, and integrating with respect to (X, V) , we obtain

$$\begin{aligned}
\frac{d}{dt} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} F H_b(X, V) \, dV dX &= - \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} F \Theta^{-1}(X, V) \nabla_{X,V} \mathcal{H}[F] \cdot \nabla_{X,V} H_b \, dV dX \quad (41) \\
&= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} F(t, X, V) \nabla_{X,V} \mathcal{H}[F(t)] \cdot \Theta^{-1} \nabla_{X,V} H_b \, dV dX \\
&= - \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} F(t, X, V) \nabla_{X,V} \mathcal{H}[F(t)] \cdot b(X, V) \, dV dX \\
&= - \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} F(t, X, V) \frac{q^2}{\varepsilon_0} \nabla_{X,V} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \mathcal{E}(X, V, Y, W) F(t, Y, W) \, dW dY \cdot b(X, V) \, dV dX.
\end{aligned}$$

For the last equality we have used the fact that $m \langle (v \cdot e)^2 \rangle / 2$ is constant along the flow of b . We claim that the integral

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} F(t, X, V) F(t, Y, W) b(X, V) \cdot (\nabla_Z \mathcal{E})(X, V, Y, W) \, dW dY dV dX$$

vanishes. Indeed, thanks to the last statement in Proposition 5.1, we can write

$$\begin{aligned}
&\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} F(t, X, V) F(t, Y, W) b(X, V) \cdot (\nabla_Z \mathcal{E})(X, V, Y, W) \, dW dY dV dX \\
&= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} F(t, Y, W) F(t, X, V) b(Y, W) \cdot (\nabla_Z \mathcal{E})(Y, W, X, V) \, dW dY dV dX \\
&= \frac{1}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} F(t, X, V) F(t, Y, W) \\
&\quad \times \{b(X, V) \cdot (\nabla_Z \mathcal{E})(X, V, Y, W) + b(Y, W) \cdot (\nabla_Z \mathcal{E})(Y, W, X, V)\} \, dW dY dV dX \\
&= 0.
\end{aligned}$$

We inquire now about the conservation of the sum between the parallel kinetic energy and the electric energy. Multiplying (38) by $m \langle (v \cdot e)^2 \rangle / 2$ one gets, thanks to the anti-symmetry of Θ^{-1}

$$\begin{aligned}
\frac{d}{dt} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} F \frac{m \langle (v \cdot e)^2 \rangle}{2} \, dV dX &= - \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} F \Theta^{-1}(X, V) \nabla_{X,V} \mathcal{H}[F] \cdot \nabla_{X,V} \frac{m \langle (v \cdot e)^2 \rangle}{2} \, dV dX \\
&= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} F \nabla_{X,V} \left\{ \frac{m \langle (v \cdot e)^2 \rangle}{2} + \frac{q^2}{\varepsilon_0} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \mathcal{E} F \, dW dY \right\} \cdot \Theta^{-1}(X, V) \nabla_{X,V} \frac{m \langle (v \cdot e)^2 \rangle}{2} \, dV dX \\
&= \frac{q^2}{\varepsilon_0} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} F(t, X, V) \nabla_{X,V} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \mathcal{E} F \, dW dY \cdot \Theta^{-1}(X, V) \nabla_{X,V} \frac{m \langle (v \cdot e)^2 \rangle}{2} \, dV dX.
\end{aligned} \tag{42}$$

Multiplying (38) by $\frac{q^2}{\varepsilon_0} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \mathcal{E} F \, dW dY$ we deduce, thanks to the symmetry of \mathcal{E} (see the first statement of Proposition 5.1) and to the anti-symmetry of Θ^{-1}

$$\begin{aligned}
& \frac{d}{dt} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} F(t, X, V) \frac{q^2}{2\varepsilon_0} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \mathcal{E}(X, V, Y, W) F(t, Y, W) \, dW dY \, dV dX \\
&= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \partial_t F(t, X, V) \frac{q^2}{\varepsilon_0} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \mathcal{E}(X, V, Y, W) F(t, Y, W) \, dW dY \, dV dX \\
&= - \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} F(t, X, V) \Theta^{-1}(X, V) \nabla_{X, V} \mathcal{H}[F(t)] \cdot \frac{q^2}{\varepsilon_0} \nabla_{X, V} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \mathcal{E} F \, dW dY \, dV dX \\
&= - \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} F(t, X, V) \Theta^{-1}(X, V) \nabla_{X, V} \frac{m \langle (v \cdot e)^2 \rangle}{2} \cdot \frac{q^2}{\varepsilon_0} \nabla_{X, V} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \mathcal{E} F \, dW dY \, dV dX.
\end{aligned} \tag{43}$$

Combining (42), (43) yields the conservation

$$\frac{d}{dt} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} F(t, X, V) \left\{ \frac{m \langle (v \cdot e)^2 \rangle}{2} + \frac{q^2}{2\varepsilon_0} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \mathcal{E}(X, V, Y, W) F(t, Y, W) \, dW dY \right\} dV dX = 0 \tag{44}$$

Putting together (41), (44) we deduce also the conservation of the total (kinetic and electric) energy. Indeed, as $H_b(x, v) = m|v \wedge e(x)|^2/2$ is left invariant along the flow of b , we have

$$m \frac{\langle |v|^2 \rangle}{2} = m \frac{\langle |v \wedge e|^2 \rangle}{2} + m \frac{\langle (v \cdot e)^2 \rangle}{2} = m \frac{|V \wedge e(X)|^2}{2} + m \frac{\langle (v \cdot e)^2 \rangle}{2}$$

and therefore

$$\frac{d}{dt} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} F(t, X, V) \left\{ \frac{m \langle |v|^2 \rangle}{2} + \frac{q^2}{2\varepsilon_0} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \mathcal{E}(X, V, Y, W) F(t, Y, W) \, dW dY \right\} dV dX = 0$$

□

We end our study by analyzing the case of well prepared initial conditions. We assume that the initial presence density satisfies the constraint

$$b(x, v) \cdot \nabla_{x, v} f^{\text{in}} = 0, \quad (x, v) \in \mathbb{R}^3 \times \mathbb{R}^3 \tag{45}$$

where the vector field $b \cdot \nabla_{x, v}$ is given by (29). We will prove that at any time $t > 0$, the limit presence density $F = F(t, X, V)$ satisfies the previous constraint and we will relate the corresponding limit potential

$$\Phi[F(t)](X, V) := \frac{q}{\varepsilon_0} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \mathcal{E}(X, V, Y, W) F(t, Y, W) \, dW dY, \quad (t, X, V) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3 \tag{46}$$

to the solution of the Poisson equation associated to the right hand side term $\frac{q}{\varepsilon_0} \int_{\mathbb{R}^3} F dV$. The crucial point is the invariance of the limit model (38). We will establish first the following lemma.

Lemma 5.1 For any presence density $F = F(X, V)$ and $h \in \mathbb{R}$ we have

$$\mathcal{H}[F_h] = (\mathcal{H}[F])_h.$$

Here, for any function $G = G(X, V)$ and $h \in \mathbb{R}$, the notation G_h stands for the composition product $G \circ \mathcal{Z}(h; \cdot)$.

Proof. It is easily seen that any average function is constant along the flow of $b \cdot \nabla_{x,v}$, and therefore we have

$$\left(m \frac{\langle (v \cdot e)^2 \rangle}{2} \right)_h = m \frac{\langle (v \cdot e)^2 \rangle}{2}.$$

Thanks to the second statement in Proposition 5.1 we write

$$\begin{aligned} & \left(\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \mathcal{E}(\cdot, \cdot, Y, W) F(Y, W) \, dW dY \right)_h (X, V) \\ &= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \mathcal{E}(\mathcal{Z}(h; X, V), Y, W) F(Y, W) \, dW dY \\ &= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \mathcal{E}(\mathcal{Z}(h; X, V), \mathcal{Z}(h; Y, W)) F(\mathcal{Z}(h; Y, W)) \, dW dY \\ &= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \mathcal{E}(X, V, Y, W) F_h(Y, W) \, dW dY \\ &= \left(\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \mathcal{E}(\cdot, \cdot, Y, W) F_h(Y, W) \, dW dY \right) (X, V), \quad (h, X, V) \in \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3. \end{aligned}$$

Finally we obtain

$$\begin{aligned} (\mathcal{H}[F])_h(X, V) &= \left(m \frac{\langle (v \cdot e)^2 \rangle}{2} + \frac{q^2}{\varepsilon_0} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \mathcal{E}(\cdot, \cdot, Y, W) F(Y, W) \, dW dY \right)_h (X, V) \\ &= \left(m \frac{\langle (v \cdot e)^2 \rangle}{2} + \frac{q^2}{\varepsilon_0} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \mathcal{E}(\cdot, \cdot, Y, W) F_h(Y, W) \, dW dY \right) (X, V) \\ &= \mathcal{H}[F_h](X, V), \quad (h, X, V) \in \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3. \end{aligned}$$

□

Proposition 5.2 Let f^{in} be a non negative smooth enough presence density, with finite mass and total (kinetic and electric) energy. We denote by F the solution of (7), (8), (9), (10). Then, for any $h \in \mathbb{R}$, the presence density $F(t)_h = F(t, \mathcal{Z}(h; X, V))$ solves (7), (9), (10) with the initial condition f_h^{in} . In particular, if the initial presence density f^{in} satisfies the constraint (45), then, at any time $t > 0$, the presence density $F(t)$ satisfies the same constraint.

Proof. The limit model (7) also writes, cf. (38)

$$\partial_t F - \Theta^{-1}(X, V) \nabla_{X, V} \mathcal{H}[F(t)] \cdot \nabla_{X, V} F = 0, \quad (t, X, V) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3.$$

Recall that the differential 2-form θ is left invariant by the hamiltonian flow \mathcal{Z}

$${}^t \partial \mathcal{Z}(h; \cdot) \Theta(\mathcal{Z}(h; \cdot)) \partial \mathcal{Z}(h; \cdot) = \Theta, \quad h \in \mathbb{R}$$

or equivalently

$$\partial \mathcal{Z}^{-1}(h; \cdot) \Theta^{-1}(\mathcal{Z}(h; \cdot)) {}^t \partial \mathcal{Z}^{-1}(h; \cdot) = \Theta^{-1}, \quad h \in \mathbb{R}.$$

By Lemma 5.1, we deduce that

$$\begin{aligned} \Theta^{-1} \nabla \mathcal{H}[F(t)_h] \cdot \nabla F(t)_h &= \Theta^{-1} \nabla(\mathcal{H}[F(t)])_h \cdot \nabla F(t)_h \\ &= \Theta^{-1} {}^t \partial \mathcal{Z}(h; \cdot) (\nabla \mathcal{H}[F(t)])_h \cdot {}^t \partial \mathcal{Z}(h; \cdot) (\nabla F(t))_h \\ &= \underbrace{\partial \mathcal{Z}(h; \cdot) \Theta^{-1} {}^t \partial \mathcal{Z}(h; \cdot)}_{\Theta^{-1}(\mathcal{Z}(h; \cdot))} (\nabla \mathcal{H}[F(t)])_h \cdot (\nabla F(t))_h \\ &= \Theta^{-1}(\mathcal{Z}(h; \cdot)) (\nabla \mathcal{H}[F(t)])_h \cdot (\nabla F(t))_h \\ &= (\Theta^{-1} \nabla \mathcal{H}[F(t)] \cdot \nabla F(t))_h. \end{aligned}$$

Therefore we obtain

$$\partial_t F(t)_h - \Theta^{-1} \nabla \mathcal{H}[F(t)_h] \cdot \nabla F(t)_h = (\partial_t F - \Theta^{-1} \nabla \mathcal{H}[F(t)] \cdot \nabla F(t))_h = 0$$

saying that $F(t)_h$ solves (7), (9), (10) with the initial condition

$$F(0)_h(X, V) = F(0, \mathcal{Z}(h; X, V)) = f^{\text{in}}(\mathcal{Z}(h; X, V)), \quad (X, V) \in \mathbb{R}^3 \times \mathbb{R}^3.$$

If the initial condition verifies $f^{\text{in}} = f_h^{\text{in}}, h \in \mathbb{R}$, then $F(t), F(t)_h$ satisfy the same problem, for any $h \in \mathbb{R}$, and thus

$$F(t) = F(t)_h, \quad t \in \mathbb{R}_+, \quad h \in \mathbb{R}.$$

□

When the initial condition is well prepared, the limit potential (46) appears as the average of the solution of the Poisson equation

$$-\varepsilon_0 \Delta \tilde{\phi}(t) = q \int_{\mathbb{R}^3} F(t, x, v) dv, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3. \quad (47)$$

Proposition 5.3 *Let f^{in} be a non negative smooth enough presence density, with finite mass and total (kinetic and electric) energy, satisfying the constraint (45). We denote by F the solution of (7), (8), (9), (10). The notations $\Phi[F(t)](X, V), \tilde{\phi}(t, x)$ stand for the limit potential (46) and the solution of (47) respectively. Then we have*

$$\Phi[F(t)] = \left\langle \tilde{\phi}(t) \right\rangle, \quad t \in \mathbb{R}_+.$$

Proof. By Proposition 5.2 we know that $F(t) = F(t)_h$, $t \in \mathbb{R}_+, h \in \mathbb{R}$ and therefore we write

$$\begin{aligned} \Phi[F(t)](X, V) &= \frac{q}{\varepsilon_0} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \mathcal{E}(X, V, Y, W) F(t, Y, W) \, dW dY \\ &= \frac{q}{\varepsilon_0} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T \frac{dh}{4\pi |\mathcal{X}(h; X, V) - \mathcal{X}(h; Y, W)|} F(t, Y, W) \, dW dY \\ &= \frac{q}{\varepsilon_0} \lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{F(t, \mathcal{Z}(h; Y, W))}{4\pi |\mathcal{X}(h; X, V) - \mathcal{X}(h; Y, W)|} \, dW dY dh \\ &= \frac{q}{\varepsilon_0} \lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{F(t, y, w)}{4\pi |\mathcal{X}(h; X, V) - y|} \, dw dy dh \\ &= \frac{q}{\varepsilon_0} \lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T \int_{\mathbb{R}^3} \frac{1}{4\pi |\mathcal{X}(h; X, V) - y|} \int_{\mathbb{R}^3} F(t, y, w) \, dw dy dh \\ &= \lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T \tilde{\phi}(t, \mathcal{X}(h; X, V)) \, dh \\ &= \left\langle \tilde{\phi}(t) \right\rangle (X, V), \quad (t, X, V) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3. \end{aligned}$$

□

A Invariance of the symplectic structure

For the sake of the presentation, we indicate here a proof for the invariance of the symplectic differential 2-form along hamiltonian vector fields, as stated in Proposition 4.2.

Proposition A.1 *Consider a symplectic structure on \mathbb{R}^m (with m an even integer), that is, let σ be a differential 2-form on \mathbb{R}^m , which is non degenerate and closed. Then the differential 2-form σ is left invariant by any hamiltonian flow.*

Proof. The notation Σ stands for the matrix field corresponding to the differential 2-form σ . Let H be a Hamiltonian, whose hamiltonian vector field is denoted by b

$$\nabla H = -\Sigma b.$$

We are done if we prove that

$${}^t\partial\mathcal{Z}(t; \cdot)\Sigma(\mathcal{Z}(t; \cdot))\partial\mathcal{Z}(t; \cdot) = \Sigma, \quad t \in \mathbb{R}$$

or equivalently

$$\partial\mathcal{Z}(-t; \mathcal{Z}(t; \cdot))\Sigma^{-1}(\mathcal{Z}(t; \cdot)){}^t\partial\mathcal{Z}(-t; \mathcal{Z}(t; \cdot)) = \Sigma^{-1}, \quad t \in \mathbb{R} \quad (48)$$

where \mathcal{Z} is the flow of b . We introduce the family of transformations $(G(t))_{t \in \mathbb{R}}$, acting on matrix fields, given by

$$G(t)C = \partial\mathcal{Z}(-t; \mathcal{Z}(t; \cdot))C(\mathcal{Z}(t; \cdot)){}^t\partial\mathcal{Z}(-t; \mathcal{Z}(t; \cdot)), \quad t \in \mathbb{R}.$$

We claim that $(G(t))_{t \in \mathbb{R}}$ is a group. Indeed, for any matrix field C we have

$$\begin{aligned} G(t)G(h)C &= \partial\mathcal{Z}(-t; \mathcal{Z}(t; \cdot))(G(h)C)(\mathcal{Z}(t; \cdot)){}^t\partial\mathcal{Z}(-t; \mathcal{Z}(t; \cdot)) \\ &= \partial\mathcal{Z}(-t; \mathcal{Z}(t; \cdot))\partial\mathcal{Z}(-h; \mathcal{Z}(h; \mathcal{Z}(t; \cdot)))C(\mathcal{Z}(h; \mathcal{Z}(t; \cdot))) \\ &{}^t\partial\mathcal{Z}(-h; \mathcal{Z}(h; \mathcal{Z}(t; \cdot))){}^t\partial\mathcal{Z}(-t; \mathcal{Z}(t; \cdot)) \\ &= \partial\mathcal{Z}^{-1}(t; \cdot)\partial\mathcal{Z}(-h; \mathcal{Z}(h+t; \cdot))C(\mathcal{Z}(h+t; \cdot)){}^t\partial\mathcal{Z}(-h; \mathcal{Z}(h+t; \cdot)){}^t\partial\mathcal{Z}^{-1}(t; \cdot). \end{aligned}$$

Notice that

$$\mathcal{Z}(-h; \mathcal{Z}(h+t; \cdot)) = \mathcal{Z}(t; \cdot)$$

and by differentiation one gets

$$\partial\mathcal{Z}(-h; \mathcal{Z}(h+t; \cdot))\partial\mathcal{Z}(h+t; \cdot) = \partial\mathcal{Z}(t; \cdot)$$

saying that

$$\partial\mathcal{Z}^{-1}(t; \cdot)\partial\mathcal{Z}(-h; \mathcal{Z}(h+t; \cdot)) = \partial\mathcal{Z}^{-1}(h+t; \cdot) = \partial\mathcal{Z}(-h-t; \mathcal{Z}(h+t; \cdot)).$$

Therefore we deduce that

$$\begin{aligned} G(t)G(h)C &= \partial\mathcal{Z}(-h-t; \mathcal{Z}(h+t; \cdot))C(\mathcal{Z}(h+t; \cdot)){}^t\partial\mathcal{Z}(-h-t; \mathcal{Z}(h+t; \cdot)) \\ &= G(t+h)C, \quad t, h \in \mathbb{R}. \end{aligned}$$

The condition (48) writes $G(t)\Sigma^{-1} = \Sigma^{-1}, t \in \mathbb{R}$, and therefore, thanks to the group property, it is enough to check that the matrix field Σ^{-1} belongs to the kernel of the infinitesimal generator of $(G(t))_{t \in \mathbb{R}}$

$$\left. \frac{d}{dt} \right|_{t=0} G(t)\Sigma^{-1} = 0 \quad (49)$$

Notice that for any smooth matrix field C , we have

$$\begin{aligned} \frac{d}{dt} \Big|_{t=0} G(t)C &= \frac{d}{dt} \Big|_{t=0} \partial \mathcal{Z}^{-1}(t; \cdot) C(\mathcal{Z}(t; \cdot)) {}^t \partial \mathcal{Z}^{-1}(t; \cdot) \\ &= -\partial b C + (b \cdot \nabla) C - C {}^t \partial b \end{aligned}$$

and therefore (49) becomes

$$(b \cdot \nabla) \Sigma^{-1} = \partial b \Sigma^{-1} + \Sigma^{-1} {}^t \partial b$$

or equivalently

$$(b \cdot \nabla) \Sigma + \Sigma \partial b + {}^t \partial b \Sigma = 0. \quad (50)$$

As b is hamiltonian vector field we can write, by using the Einstein summation convention

$$-\partial_{z_j} (\Sigma_{ik} b_k) = \partial_{z_j} (\partial_{z_i} H) = \partial_{z_i} (\partial_{z_j} H) = -\partial_{z_i} (\Sigma_{jk} b_k), \quad i, j \in \{1, \dots, m\}.$$

Therefore one gets

$$\partial_{z_j} \Sigma_{ik} b_k + \Sigma_{ik} \partial_{z_j} b_k = \partial_{z_i} \Sigma_{jk} b_k + \Sigma_{jk} \partial_{z_i} b_k, \quad i, j \in \{1, \dots, m\}$$

implying that

$$(\Sigma \partial b)_{ij} + ({}^t \partial b \Sigma)_{ij} = b_k (\partial_{z_i} \Sigma_{jk} + \partial_{z_j} \Sigma_{ki}), \quad i, j \in \{1, \dots, m\}.$$

In the above equality we have used the anti-symmetry of Σ . Clearly, for justifying (50), it is enough to check that

$$\partial_{z_k} \Sigma_{ij} + \partial_{z_i} \Sigma_{jk} + \partial_{z_j} \Sigma_{ki} = 0, \quad i, j, k \in \{1, \dots, m\}$$

which is also equivalent to

$$\partial_{z_k} \Sigma_{ij} + \partial_{z_i} \Sigma_{jk} + \partial_{z_j} \Sigma_{ki} = 0, \quad 1 \leq i < j < k \leq m. \quad (51)$$

But (51) is exactly the closedness of σ , since the condition $d\sigma = 0$ writes

$$\begin{aligned} 0 &= -2 d\sigma = d(\Sigma_{ij} dz_i \wedge dz_j) = \partial_{z_k} \Sigma_{ij} dz_i \wedge dz_j \wedge dz_k \\ &= 2 \sum_{1 \leq i < j < k \leq m} (\partial_{z_k} \Sigma_{ij} + \partial_{z_i} \Sigma_{jk} + \partial_{z_j} \Sigma_{ki}) dz_i \wedge dz_j \wedge dz_k. \end{aligned}$$

□

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