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# A test for assessing the goodness-of-fit of luminosity function models to magnitude–redshift data

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## ABSTRACT

We propose a test which allows us to check whether a luminosity function model can account for the intrinsic luminosity distribution of a magnitude–redshift sample complete in apparent magnitude. No a priori assumptions are required concerning the redshift-space distribution of the sources, so neither the clustering nor an eventual evolution of the mean number density of the galaxies affects the conclusions of the goodness-of-fit test. The statistical efficiency of the test, if used as a fitting technique for estimating the best-fitting solution of a parametric luminosity function model, is comparable to the efficiency of standard maximum likelihood fitting techniques. The goodness-of-fit test presents however a major improvement compared with fitting techniques in general: the capacity to assess the adequacy of the proposed parametric model to the data. The computational implementation of this new test is straightforward. Its potential is illustrated on the Southern Sky Redshift Survey of da Costa et al.

**Key words:** methods: statistical – galaxies: distances and redshifts – galaxies: luminosity function, mass function – large-scale structure of Universe.

## 1 INTRODUCTION

Recovering the intrinsic luminosity distribution (i.e. the luminosity function) of the galaxy population from a magnitude–redshift catalogue remains one of the major concerns of observational cosmology. Knowledge of the luminosity function puts strong constraints on galaxy formation and evolution models (see for example Ellis 1997) and is a crucial prerequisite for large-scale structure analyses (see for example Strauss & Willick 1995). Considering the rapid instrumental progress made recently in high quality data acquisition, it appears as an urgent matter to develop at the same time sophisticated statistical tools for analysing the large redshift surveys already available or in progress (see for example Colless 2000). Estimating the galaxy luminosity function is not a trivial task and is plagued in practice by numerous observational and statistical difficulties.

A first observational difficulty lies in the very definition of the intrinsic luminosity of a galaxy. Because galaxies are extended objects, various measures of their apparent magnitude can be adopted (e.g. isophotal, visual, total magnitudes) which do not behave equivalently owing to the variation of surface brightness with redshift (see for example Sandage & Perelmuter 1990, Petrosian 1976, Driver 1999). Moreover the galaxy distance is inferred from its redshift, which presupposes that the geometry of

the Universe is known and that the contribution of peculiar velocities to observed redshifts is well understood. Furthermore, accurate models for the  $k$ -correction term and for the galactic extinction correction are required.

The presence of selection effects in observation, e.g. a detection threshold in apparent fluxes, is also a source of problems. Because a part of the population is, in fact, not observed standard statistical recipes lead in general to biased estimates of the luminosity function. Most of the methods proposed in the literature for recovering an unbiased estimate of the luminosity function are restricted to the case of samples complete in apparent magnitude (i.e. magnitude–redshift data sharply truncated by a lower flux limit). Because additional selection effects in observation are often at work, this completeness assumption may in practice be a crude oversimplification for most of the surveys found in the literature. However, as a preliminary step, the completeness of a redshift survey can be checked, for example by applying the simple test recently proposed in Rauzy (2001), which permits robust determination of the completeness limit in apparent magnitude.

The various methods proposed for estimating the luminosity function from a sample complete in apparent magnitude can be roughly classified as follows.

The  $1/V_{\max}$  estimator of Schmidt (1968) and its extensions (e.g. Felten 1977, Eales 1993, van Waerbeke et al. 1996, Qin & Xie 1997) provide a non-parametric reconstruction and fitting technique for the luminosity function. The estimator makes use of information from the whole sample, as opposed to volume limited analyses for example. A major drawback of the  $1/V_{\max}$

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estimator is nevertheless that it does not account for the observed clustering of the galaxies. In practice, the Schmidt estimator leads thus to a biased estimate of the luminosity function (see Willmer 1997 and Takeuchi, Yoshikawa & Ishii 2000 for a practical comparison of the various luminosity function estimators proposed in the literature). Note that if the clustering can be ignored, the null-correlation approach of Fliche & Souriau (1979), which does not require to assume the completeness of the sample, presents a real improvement over the  $1/V_{\max}$  statistic (see Bigot, Fliche & Triay 1988, Bigot & Triay 1991, Bigot, Rauzy & Triay 1991).

The  $C^-$  method of Lynden-Bell (1971) provides a discrete, non-parametric reconstruction of the cumulative luminosity function, independent of clustering and of evolution of the mean galaxy density (see also Cholonewski 1987 and 1986, and for an enlightening review Petrosian 1992). The  $C^-$  method is therefore a very robust tool that gives a clear visual idea of the shape of the cumulative luminosity function. One minor problem with the method is however that the reconstructed function is not smooth and thus cannot be differentiated. The problem can be overcome in practice by introducing a smoothing kernel function (Caditz & Petrosian 1993) – a standard approach in inverse problem theory. A more serious drawback concerns the error analysis. Because the  $C^-$  reconstructed function is not characterized by any parameters, there is no standard way for performing an error analysis on the reconstruction.

The maximum likelihood fitting technique proposed by Sandage, Tammann & Yahil (1979) (see also Turner 1979), which is also insensitive to clustering and density evolution, allows the estimation of the best-fitting solution of a parametric luminosity function model. Since it is a maximum likelihood fitting method, a standard error analysis can be easily achieved. On the other hand, as was pointed out by Efstathiou, Ellis & Peterson (1988), the goodness-of-fit (i.e. the adequacy to the data) of the proposed parametric luminosity function cannot be assessed.

An intermediate step between the  $C^-$  method and the technique of Sandage et al. (1979) is the so-called non-parametric stepwise maximum likelihood method proposed by Efstathiou et al. (1988). It consists of proposing as the luminosity function model a multiparametric function general enough to allow the description of any reasonable shape of luminosity function. The absolute magnitude axis is divided in piecewise intervals, the values of the luminosity function on each interval being the parameters of the model. The goodness-of-fit of this luminosity function model is obviously satisfied as long as the number of intervals is large enough.

The error analysis is however not trivial in this case since the estimates of the amplitudes of the luminosity function on each interval are highly correlated (i.e. the covariance matrix of the likelihood function is not diagonal). In particular, allocating individual error bars to the values of the luminosity function on each interval of luminosity (see the plots in Efstathiou et al. 1988, Lin et al. 1996, Heyl et al. 1997 for example) can be misleading. This problem persists even when using the new parametrization introduced by Springel & White (1998). A second drawback of the method is that the minimum number of piecewise intervals necessary for describing the data is in practice difficult to assess (see however Takeuchi et al. 2000).

In the present paper, we propose a new test which allows to check the goodness-of-fit of any luminosity function model to the data. The test of fit can be used firstly to assess whether a given parametric model is suitable for describing the luminosity function of the sampled population. If a region of parameter space is

acceptable, the goodness-of-fit test then provides the standard sampling errors on the estimated parameters. It also allows to compare two different luminosity function models and therefore permits to decide how many parameters are, in fact, required to account for the distribution function of the sample. The importance of providing a goodness-of-fit test for luminosity function models has been previously addressed in Qin & Xie (1999) within the context of the  $1/V_{\max}$  analysis. The goodness-of-fit test we propose herein presents the additional advantages of being independent of the clustering and the evolution of the mean galaxies density with time.

The statistical background of the method as well as the goodness-of-fit test are presented Section 2. An example of application is given in Section 3, where we investigate the luminosity function of the spiral galaxies of the Southern Sky Redshift Survey of da Costa et al. (1998). The properties of the new test are finally summarized in Section 4.

## 2 THE GOODNESS-OF-FIT TEST

### 2.1 Assumptions and statistical model

The luminosity function of the galaxy population is herein defined as the probability distribution function  $f_i(M)$  of the absolute magnitude  $M$  of the galaxies depending in general on the epoch  $t$  (see for example Bingelli, Sandage & Tammann 1988). At any epoch, the luminosity function is by definition normalized (i.e.  $\int f_i(M) dM = 1$ ). It is assumed hereafter in this section that the luminosity function of the population does not depend on the 3D redshift-space position  $z = (z, l, b)$  of the galaxies. The general case is treated in Appendix B. Without accounting for selection effects in observation, the probability density describing the population splits under this assumption as

$$dP_{zM} \propto dP_z \times dP_M = \rho(z, l, b) dl db dz \times f(M) dM. \quad (1)$$

where  $\rho(z, l, b)$  is the 3D redshift-space distribution function of the sources along the past light-cone.

The present model accounts for the observed spatial fluctuations of the galaxy density (i.e. galaxy clustering and large-scale structure) and for a pure density evolution scenario (i.e. variation of the mean galaxy density with redshift or equivalently with time). On the other hand, equation (1) fails to describe environmental effects (i.e. the luminosity function of the sampled objects depending systematically on the local environment). The case of evolution in the specific characteristics of the luminosity function with time (e.g. mean absolute magnitude, shape) has been delayed for clarity until Appendix B.

The selection function  $\psi$  describing observational selection effects can be expressed in general in terms of the observable quantities, namely the line-of-sight direction  $(l, b)$ , the redshift  $z$  and the raw apparent magnitude  $m$ , i.e.  $\psi \equiv \psi(m, z, l, b)$ . It is assumed hereafter that the sample is complete in raw apparent magnitude up to a given magnitude limit  $m_{\text{lim}}$ , or in other words the selection function in apparent magnitude is well described by a sharp cut-off, i.e.

$$\psi(m, z, l, b) \equiv \theta(m_{\text{lim}} - m) \times \phi(z, l, b) \quad (2)$$

with  $\theta(x)$  the Heaviside or ‘step’ function. The function  $\phi(z, l, b)$  describes some eventual selection effects in angular position and observed redshift. For example, it could account for a mask in angular position as well as pure selection or subsampling in redshift (e.g. a lower and upper limit). Note that the selection

function of equation (2) describes also samples selected following a sparse-sampling strategy, at least as long as the galaxies selected for redshift measurement have been chosen at random and not on the basis of apparent magnitude criteria. How to account for an additional upper cut-off  $m_{\text{inf}}$  in raw apparent magnitude is described in Appendix B. Accounting for selection effects in observation, the probability density describing the sample may be written as

$$dP = \frac{1}{A} \phi(z, l, b) \rho(z, l, b) dl db dz f(M) dM \theta(m_{\text{lim}} - m) \quad (3)$$

with  $A$  the normalization factor satisfying  $\int dP = 1$ .

The absolute magnitude  $M$  is obtained from the raw apparent magnitude following

$$M = m_{\text{cor}} - \mu(z) \quad (4)$$

where the (luminosity) distance modulus  $\mu(z)$  can be evaluated from the redshift given a cosmological world model  $\{H_0, \Omega_0, \Lambda_0\}$  (see for example Weinberg 1972). Note that it has been implicitly assumed herein that the contribution of peculiar velocities to the observed redshifts is negligible. The corrected apparent magnitude  $m_{\text{cor}}$  is expressed as

$$m_{\text{cor}} = m - k_{\text{cor}}(z) - A_g(l, b) \quad (5)$$

where expressions for the  $k$ -correction term  $k_{\text{cor}}(z)$  and the Galactic extinction correction  $A_g(l, b)$  are required.

At this stage, it is convenient to introduce the quantity  $Z$  defined by

$$Z = m - M = \mu(z) + k_{\text{cor}}(z) + A_g(l, b), \quad (6)$$

which can be computed from the observables  $z$  and  $(l, b)$  assuming a cosmological world model. Thanks to the introduction of the quantity  $Z$ , the maximum absolute magnitude  $M_{\text{lim}}(Z)$  for which a galaxy at a given  $Z$  would be visible in the sample is uniquely defined, i.e.

$$M_{\text{lim}}(Z) = m_{\text{lim}} - Z. \quad (7)$$

The cut-off in raw apparent magnitude can then be expressed using this notation as  $\theta(m_{\text{lim}} - m) \equiv \theta(M_{\text{lim}}(Z) - M)$  and the probability density of equation (3) may be rewritten as

$$dP = \frac{1}{A} h(Z, l, b) dl db dZ f(M) dM \theta[M_{\text{lim}}(Z) - M], \quad (8)$$

where the distribution function  $h(Z, l, b)$  may be expressed, if required, as a function of the 3D redshift-space distribution  $\rho(z, l, b)$ , the selection function  $\phi(z, l, b)$  introduced in equation (2) and using the definition of  $Z$  given equation (6).

## 2.2 The random variable $\zeta$

The new method relies on the introduction of the random variable  $\zeta$  defined as follows

$$\zeta = \zeta(M, Z) = \frac{F(M)}{F[M_{\text{lim}}(Z)]}, \quad (9)$$

where  $F(M)$  stands for the cumulative luminosity function, i.e.

$$F(M) = \int_{-\infty}^M f(x) dx \quad (10)$$

Note that given the function  $F$ , the random variable  $\zeta$  is computable from the observable quantities  $M$  and  $Z$ . The volume element of

equation (8) may be rewritten as

$$dl db dZ d\zeta = \frac{f(M)}{F[M_{\text{lim}}(Z)]} dl db dZ dM \quad (11)$$

and by definition the random variable  $\zeta$  for a sampled galaxy belongs to the interval  $[0, 1]$ , i.e. because from equation (8)  $-\infty < M \leq M_{\text{lim}}(Z)$ , the random variable  $\zeta$  introduced equation (9) takes its value between  $\zeta(-\infty, Z) = 0$  and  $\zeta(M_{\text{lim}}(Z), Z) = 1$ . The probability density of equation (8) reduces therefore to

$$dP = \frac{1}{A} h(Z, l, b) F[M_{\text{lim}}(Z)] dl db dZ \times \theta(\zeta) \theta(1 - \zeta) d\zeta, \quad (12)$$

with  $A = \int h(Z, l, b) F[M_{\text{lim}}(Z)] dl db dZ$ . Two properties follow from equation (12),

- (i) P1:  $\zeta$  is uniformly distributed between 0 and 1,
- (ii) P2:  $\zeta$  and  $(Z, l, b)$  are statistically independent.

It is worthwhile mentioning that no assumptions have been made concerning the distribution function  $h(Z, l, b)$  introduced in equation (8). This means that properties P1 and P2 hold for any 3D redshift-space distribution  $\rho(z, l, b)$  (allowing for the presence of clustering and the evolution of the mean galaxy number density with time), and for any selection function  $\phi(z, l, b)$  (e.g. subsampling in redshift bins will not alter the two properties).

We have shown previously (Rauzy & Hendry 2000; Rauzy 2001) that the random variable  $\zeta$  can be estimated directly from the data without any prior knowledge of the cumulative luminosity function  $F(M)$ . This leads to a robust method for fitting peculiar velocity field models (Rauzy & Hendry 2000) and to a useful tool for assessing the completeness in apparent magnitude of the sample (Rauzy 2001). In the present paper, we devise a method which makes use of properties P1 and P2 for testing the goodness-of-fit to the data of a proposed luminosity function model.

## 2.3 The goodness-of-fit test $T_f$

Our test of fit is based on the following remark. Any acceptable candidate function for describing the cumulative luminosity function of the population must necessarily verify properties P1 and P2. We show in Appendix A that this condition is also sufficient, i.e. the only function jointly satisfying properties P1 and P2 (and which enters the definition of  $\zeta$ ) is the cumulative luminosity function  $F(M)$ . For a given luminosity function model, the associated random variable  $\zeta$  can be computed for each galaxy of the sample. The test of fit consists in checking whether or not that  $\zeta$  distribution satisfies simultaneously properties P1 and P2:

- (i) If it does not, the necessary condition allows to reject the proposed function as an acceptable model for the luminosity function of the sample.
- (ii) If it does, the sufficiency condition ensures that there is no means using that particular sample to reject the proposed luminosity function model, unless some additional information is supplemented (e.g. more data points, a priori hypotheses on the spatial distribution of the sources, etc.).

In the following subsections we present methods for checking properties P1 and P2 in practice, accounting for the effects of sampling fluctuations.

### 2.3.1 The $T_\zeta$ test: $\zeta$ is uniformly distributed between 0 and 1

Property P1 is tested by investigating the deviation of the  $\zeta$  distribution (associated with the proposed luminosity function model) from a uniform distribution between 0 and 1. This is achieved in practice by performing a Kolmogorov–Smirnov test on the  $\zeta$  distribution. The statistic  $T_\zeta$  is hereafter defined as the confidence level for rejecting the statement ‘the  $\zeta$  distribution is drawn from a uniform distribution between 0 and 1’ on the basis of the Kolmogorov–Smirnov test (the closer  $T_\zeta$  to 1, the more confident we are in rejecting this statement). If the model tested corresponds to the genuine luminosity function of the population  $f(M)$ , the  $T_\zeta$  statistic is therefore expected to be uniformly distributed between 0 and 1.

In practice, if the number of galaxies is large enough (say  $N_{\text{gal}} > 10$ ), the confidence level of rejection is given by

$$T_\zeta \approx 1 - Q_{\text{KS}} \left( \left[ \sqrt{N_{\text{gal}}} + 0.12 + \frac{0.11}{\sqrt{N_{\text{gal}}}} \right] D_{\text{KS}} \right) \quad (13)$$

where the function  $Q_{\text{KS}}$  is defined as

$$Q_{\text{KS}}(\lambda) = 2 \sum_{j=1}^{+\infty} (-1)^{j-1} \exp(-2j^2 \lambda^2) \quad (14)$$

and with  $D_{\text{KS}}$  the Kolmogorov–Smirnov distance between the observed (i.e. sample) cumulative distribution function of the  $\zeta$ s,  $\text{CDF}(\zeta)$ , and the straight line of equation  $y = \zeta$  corresponding to the cumulative distribution of the uniform distribution, i.e.

$$D_{\text{KS}} = \max_{i=1, N_{\text{gal}}} (|\text{CDF}(\zeta_i) - \zeta_i|) \quad (15)$$

See for example the Numerical recipes book by Press, Flannery & Teukolsky (1986) for a computer implementation of the Kolmogorov–Smirnov test.

### 2.3.2 The $T_\rho$ test: $\zeta$ and $Z$ are not correlated

Property P2 is tested by analysing the coefficient of correlation  $\rho(Z, \zeta)$ <sup>1</sup> of the random variables  $Z$  and  $\zeta$ . The independence between  $Z$  and  $\zeta$  will be rejected when the absolute value of the observed coefficient of correlation  $\rho_{\text{obs}} \equiv \rho_{\text{obs}}(Z, \zeta)$  is too high to be as a result of sampling fluctuations (i.e. a double-sided rejection test). We define the  $T_\rho$  statistic as the confidence level for rejecting the statement ‘the deviation of  $\rho_{\text{obs}}$  from 0 is owing to sampling fluctuations’. The closer  $T_\rho$  to 1, the less likely it is that the variables  $Z$  and  $\zeta$  are independent. For the genuine luminosity function of the population, the  $T_\rho$  statistic is expected to be uniformly distributed between 0 and 1.

For a number of galaxies large enough (say  $N_{\text{gal}} > 30$ ), a good approximation to the confidence level of rejection is in that case

$$T_\rho \approx \text{erf} \left( \frac{1}{\sqrt{2}} \frac{|\rho_{\text{obs}}|}{\sqrt{1 - \rho_{\text{obs}}^2}} \sqrt{N_{\text{gal}} - 2} \right) \quad (16)$$

where  $\text{erf}(x) = (2/\sqrt{\pi}) \int_0^x \exp(-t^2) dt$  is the error function. The

<sup>1</sup> Any other suitable measure of correlation, such as Spearman’s coefficient of rank correlation, could also be used here as the independence test statistic, with the rejection criterion based on the appropriate sampling distribution.

observed correlation coefficient  $\rho_{\text{obs}}$  is given by

$$\rho_{\text{obs}}(Z, \zeta) = \frac{\text{Cov}(Z, \zeta)}{\sqrt{\text{Cov}(Z, Z)} \sqrt{\text{Cov}(\zeta, \zeta)}} \quad (17)$$

where the covariance  $\text{Cov}(X, Y)$  on the sample of two random variables  $X$  and  $Y$  is

$$\text{Cov}(X, Y) = \frac{1}{N_{\text{gal}} - 1} \sum_{i=1}^{N_{\text{gal}}} (X_i - \bar{X})(Y_i - \bar{Y}) \quad (18)$$

with  $\bar{X}$  and  $\bar{Y}$  respectively the means of the  $X_i$  and  $Y_i$  on the sample, i.e.

$$\bar{X} = \frac{1}{N_{\text{gal}}} \sum_{i=1}^{N_{\text{gal}}} X_i, \quad \bar{Y} = \frac{1}{N_{\text{gal}}} \sum_{i=1}^{N_{\text{gal}}} Y_i. \quad (19)$$

Note that  $T_\rho$  is not strictly speaking a sufficient statistic for testing property P2 ( $\zeta$  and  $(Z, l, b)$  are independent). First,  $\rho(Z, \zeta) = 0$  does not imply systematically that  $Z$  and  $\zeta$  are independent. However, this problem can be overcome, if required, by using more sophisticated statistical tools such as rank order statistics (see for example Efron & Petrosian 1992). Secondly, the independence between the random variable  $\zeta$  and the angular direction  $(l, b)$  is not tested. We choose however to neglect this information since  $\zeta$  should depend on  $(l, b)$  only through the Galactic extinction correction, which is a second-order effect for the present analysis.

### 2.3.3 The $T_f$ test: $T_f = \text{Sup}(T_\rho, T_\zeta)$

The test of fit  $T_f$ , designed to test the goodness-of-fit of the proposed luminosity function model to the data, requires the two properties P1 and P2 to be satisfied conjointly. In other words, the luminosity function model can be rejected if one of the  $T_\zeta$  or  $T_\rho$  statistics allows to do so. A convenient way to define the  $T_f$  statistic is, therefore, to take the greatest of  $T_\zeta$  and  $T_\rho$  values as the value of  $T_f$ , i.e.

$$T_f = \text{Sup}(T_\rho, T_\zeta). \quad (20)$$

The goodness-of-fit test consists, therefore, in rejecting the proposed luminosity function model when the value of the  $T_f$  statistic is high (i.e. close to 1). The confidence level of rejection associated to this test can be evaluated as follows.

If  $x$  and  $y$  are two independent random variables uniformly distributed between 0 and 1, the probability that the quantity  $z = \text{Sup}(x, y)$  is less than  $c$  ( $c \in [0, 1]$ ) is given by  $\text{Prob}(z \leq c) = c^2$ . If the random variables  $x$  and  $y$  are not independent, this result transforms in  $c \geq \text{Prob}(z \leq c) \geq c^2$  (see for example Kendall & Stuart 1979). We have seen previously that if the genuine luminosity function is tested, the  $T_\zeta$  and  $T_\rho$  statistics are by definition uniformly distributed between 0 and 1.  $T_\zeta$  and  $T_\rho$  are not independent in general, which means that the probability that  $T_f$  is less than a given value  $c$  satisfies the constraint  $c \geq \text{Prob}(T_f \leq c) \geq c^2$ . The value of the confidence level of rejection associated to the  $T_f$  test is therefore  $T_f^2$  (since we have to consider the lower bound of the inequality in that case).

## 2.4 The $T_f$ test used as a fitting method

It is worthwhile to mention that the goodness-of-fit test proposed here can also be used to fit a parametric model of luminosity function to the data. Let us consider a luminosity function model  $f_\alpha(M)$  parametrized by an  $N$ -dimensional vector

$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N)$ . For each point  $\alpha$  of the parameter space, we can apply the  $T_f$  test on  $f_\alpha(M)^2$  and therefore evaluate the confidence level of rejection  $T_f^2(\alpha)$  associated to the vector  $\alpha$ . Scanning the parameter space allows to find the best-fitting solution [say the lowest value of  $T_f^2(\alpha)$ ] and draw the  $1\sigma$ ,  $2\sigma$ , ... error ellipsoids [i.e. the curve respectively defined by  $T_f^2(\alpha) = 0.68$ ,  $T_f^2(\alpha) = 0.95$ , etc.], as usual.

However, the great advantage of the  $T_f$  test over standard fitting techniques, such as maximum likelihood for example, is that the  $T_f$  test allows to assess the adequacy of the parametric luminosity function model to the data. In particular, the  $T_f$  test will reject any parametric models unable to describe the luminosity function of the population.

### 3 EXAMPLE OF APPLICATION

As an illustration, the method is applied to the Southern Sky Redshift Survey (SSRS2) sample of da Costa et al. (1998). The completeness of the sample is addressed in Section 3.2. Three different two-parameter luminosity function models are afterwards investigated: a Schechter luminosity model, a Gaussian model and a toy model of luminosity evolution. These examples allow us to emphasize the major advantages of the goodness-of-fit test  $T_f$  compared to the standard maximum likelihood fitting technique. A variety of alternative parametric luminosity function models could be found in Yahil et al. (1991), Santiago et al. (1996) and Bingelli et al. (1988) for example.

#### 3.1 The SSRS2 sample

The SSRS2 sample contains 5369 galaxies with measured  $B$ -band magnitude and redshift. It has been drawn primarily from the list of non-stellar objects identified in the *Hubble Space Telescope* Guide Star Catalog (Lasker et al. 1990). The redshift survey is more than 99 per cent complete up to the magnitude limit  $m_{\text{SSRS2}}$  of 15.5 mag (da Costa et al. 1998).

The redshifts are hereafter transformed in the CMB rest frame and the distance modulus is computed adopting an Hubble constant of  $H_0 = 100 \text{ km s}^{-1} \text{ Mpc}^{-1}$  in a flat universe with no cosmological constant (i.e.  $\Omega_0 = 1$  and  $\Lambda_0 = 0$ ). Galaxies not belonging to the redshift range  $[2500, 15\,000] \text{ km s}^{-1}$  are discarded. The lower limit in redshift has been introduced in order to minimize the impact of peculiar velocities on the luminosity function analysis. In particular, the kinematical influence of the Virgo cluster will be considerably reduced by removing nearby galaxies. The upper bound in redshift sets some limits on the interval of cosmic time spanned by the data and thus reduces the role of an eventual evolution of the luminosity function ( $cz = 15\,000 \text{ km s}^{-1}$  corresponds to a look-back time of 0.5 Gyr for  $H_0 = 100 \text{ km s}^{-1} \text{ Mpc}^{-1}$ , and 1 Gyr if  $H_0 = 50 \text{ km s}^{-1} \text{ Mpc}^{-1}$ ). Note that such a subsampling in redshift will not affect the result of our goodness-of-fit test.

We restrict furthermore our analysis to the sample of spiral galaxies (from type  $T = 1$  to  $T = 8$ ), having in mind that the study of the E/SO+spirals sample could be affected by some

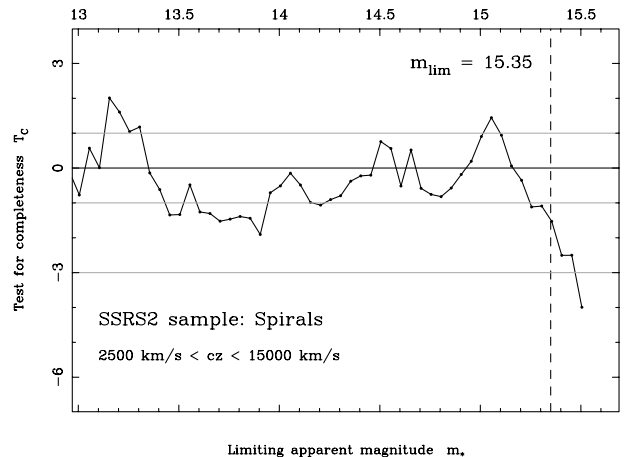
environmental effects which are not described by the formalism presented Section 2 (see Rauzy 2001 for details).

Galactic extinctions are obtained as  $A_g(l, b) = 4.325E(B - V)$  by use of the dust maps of Schlegel, Finkbeiner & Davis (1998) for the reddening correction. We apply a type-dependent  $k$ -correction calculated following Pence (1976), i.e.  $k_{\text{cor}}(z) = K_B(T) \times cz / (10\,000 \text{ km s}^{-1})$  with  $K_B(T) = 0.15 - 0.025T$  for  $3 \geq T \geq 0$  and  $K_B(T) = 0.075 - 0.010(T - 3)$  for  $3 \leq T$ . The formalism presented in Section 2 does not account explicitly for that type dependency. More specifically, if the luminosity function of the population is also type-dependent, a type-by-type analysis would in this case be required. However the upper bound in redshift limits the magnitude of the  $k$ -correction term, which in turn moderates the amplitude of this effect, as well as the amplitude of the errors introduced by an uncertain  $k$ -correction.

#### 3.2 Completeness of the sample

The completeness of the sample is a crucial prerequisite for any luminosity function analysis. Indeed, if the sample is not complete in apparent magnitude the number of faint galaxies will be systematically underestimated, introducing a bias in the determination of the sought luminosity function. We apply herein the test for completeness proposed in Rauzy (2001) for determining the completeness limit of the SSRS2 spirals sample.

The completeness test presented in Rauzy (2001) makes use of the fact that, in the case of incompleteness in apparent magnitude, the distribution of the random variable  $\zeta$  introduced in equation (9) is systematically shifted towards 0, invalidating property P1. The  $T_C$  statistic is constructed from the observed values of  $\zeta$  in such a way that  $T_C$  is normally distributed up to the effective completeness limit of the sample and is systematically negative beyond that limiting apparent magnitude. The  $T_C$  test for completeness presents a real improvement compared to standard completeness tests, such as the analysis of the variation of galaxy number counts as a function of the limiting apparent magnitude (Hubble 1926). Namely, no a priori assumptions are required concerning the redshift-space distribution of the sources. It means in particular that neither clustering nor evolution of the mean number density of the galaxies affect the result of the test.



**Figure 1.** The test for completeness in apparent magnitude  $T_C$  (see Rauzy 2001) is applied to the 2780 spirals galaxies of the SSRS2 sample with redshifts between 2500 and 15 000  $\text{km s}^{-1}$ . The systematic decline of the  $T_C$  statistic from 0 to negative values indicates incompleteness in apparent magnitude beyond  $m_{\text{SSRS2}} = 15.35$  mag.

<sup>2</sup>Note that both  $T_\zeta$  and  $T_\rho$  are functions of the parameters  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N)$ . Although, for each galaxy, the random variable  $Z$  is independent of the luminosity function parameters, the dependence of  $\zeta$  on  $\alpha$  means that  $\rho_{\text{obs}}(Z, \zeta)$  – and hence  $T_\rho$  – also depend on  $\alpha$ .

The result of the test for completeness applied to the 2780 spirals galaxies of the SSRS2 sample with redshifts between 2500 and  $15\,000\text{ km s}^{-1}$  is shown in Fig. 1. It suggests to adopt a value of  $m_{\text{lim}} = 15.35\text{ mag}$  as a reasonable completeness limit (for a full discussion on the completeness of the SSRS2 sample, see Rauzy 2001). After discarding galaxies fainter than  $m_{\text{SSRS2}} = 15.35\text{ mag}$ , we are left with a sample of 2341 spirals.

### 3.3 Schechter luminosity function

The goodness-of-fit test  $T_f$  is now applied in order to check whether the Schechter model is a good candidate for describing the luminosity function of the SSRS2 spirals. Schechter luminosity function models are characterized by two parameters  $\alpha$  and  $M_*$  and follow the shape

$$f(M) \propto \exp\{c_1(\alpha + 1)(M_* - M) - \exp[c_1(M_* - M)]\} \quad (21)$$

with  $c_1 = 0.4 \ln 10$ .

For each pair  $(\alpha, M_*)$ , the random variable  $\zeta$  is computed following equation (9) with  $F \equiv F(M; M_*, \alpha)$  the corresponding cumulative Schechter luminosity function. The two statistics  $T_\rho$  and  $T_\zeta$  are afterwards calculated using respectively equations (16) and (13). The confidence levels of rejection associated with these tests as a function of the parameters  $\alpha$  and  $M_*$  are shown in Fig. 2. We can observe that each of the two tests allows to reject a different portion of parameter space. They will, therefore, be both informative for the goodness-of-fit test  $T_f$ .

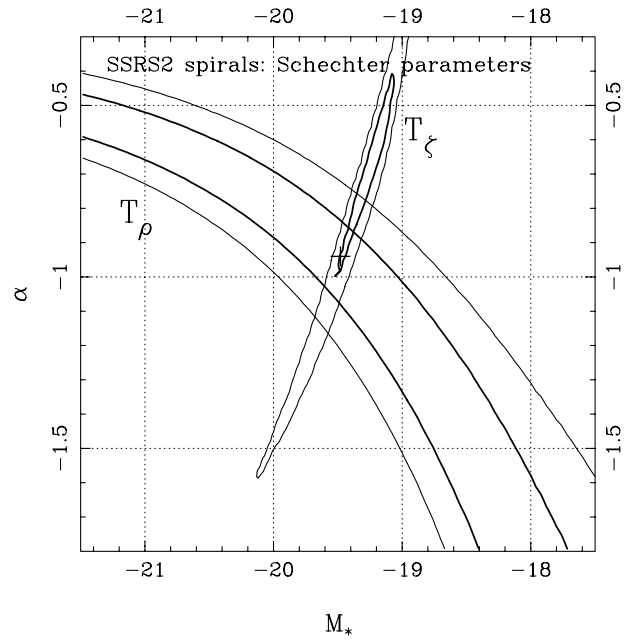
The statistic  $T_f$  is obtained from  $T_\rho$  and  $T_\zeta$  following equation (20) and the confidence level of rejection associated to the goodness-of-fit test  $T_f$  is computed for each pair  $(\alpha, M_*)$  as  $T_f^2$ . The results are displayed in Fig. 3. We observe that there exists a region of the parameter space which cannot be rejected on the basis of the test of fit, at least using the sample under consideration. Or in other words, the Schechter luminosity function model is a good candidate for describing the luminosity function of the population of the sampled galaxies. Fig. 3 also clearly illustrates how to make use of the  $T_f$  test as a fitting technique, by choosing for example the range of the admissible Schechter parameters within the  $1\sigma$  confidence level region and by taking as the best-fitting solution the minimum value of the  $T_f$  test, herein  $(\alpha = -0.94, M_* = -19.48)$ .

For comparison, we also applied to our sample the maximum likelihood fitting technique of Sandage et al. (1979). The method consists in adopting as the best-fitting solution of the luminosity function model  $f_\alpha(M)$ , the value of the parameter  $\alpha$  corresponding to the maximum  $\mathcal{L}_{\text{max}}$  of the efficient part of the logarithm of the maximum likelihood function

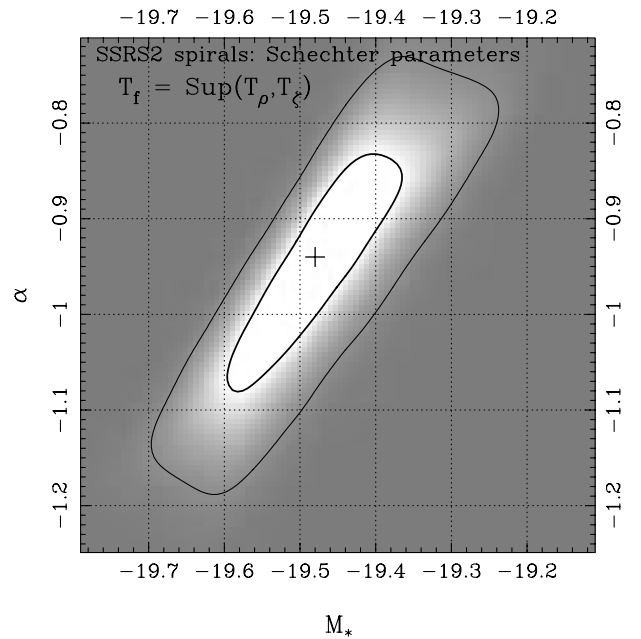
$$\mathcal{L}(\alpha) = \sum_{i=1}^{N_{\text{gal}}} [\ln f_\alpha(M_i) - \ln F_\alpha(M_{\text{lim}}(Z_i))] \quad (22)$$

If the luminosity function model is a good descriptor of the genuine luminosity function of the population, the quantity  $2(\mathcal{L}_{\text{max}} - \mathcal{L}(\alpha))$  is expected to follow asymptotically a  $\chi^2$  probability law with  $N$  degrees of freedom ( $N$  is the dimension of the parameters space). The error ellipsoids associated with the maximum likelihood estimate can then be computed on this basis. Note, however, that this error analysis rests on the assumption that the luminosity function model is a good descriptor, whereas our method *also* allows to test this goodness-of-fit at the same time as providing best-fitting parameters.

The maximum likelihood estimate of the Schechter parameters is shown in Fig. 4. The best-fitting solution corresponds to  $(\alpha = -0.99, M_* = -19.50)$ , within the  $1\sigma$  error ellipsoid of the  $T_f$

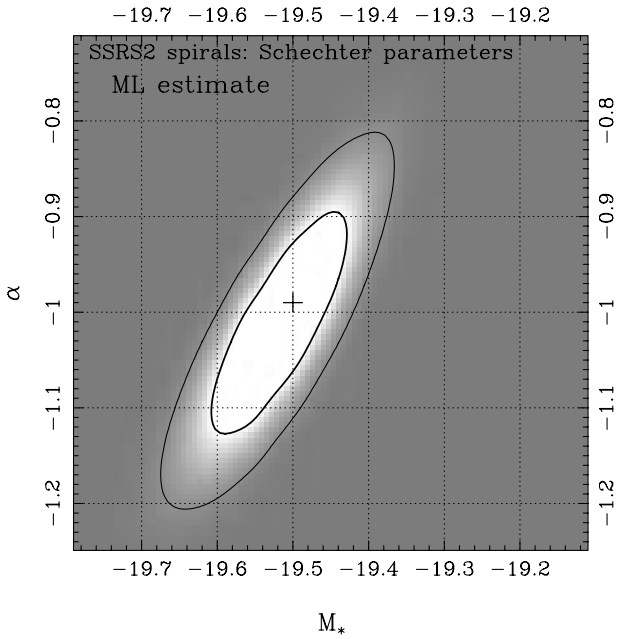


**Figure 2.** Confidence levels of rejection associated to the tests  $T_\rho$  and  $T_\zeta$  as a function of the parameters  $\alpha$  and  $M_*$  characterizing the Schechter luminosity function model. Bold and normal contours indicate respectively confidence levels of 68 and 95 per cent (i.e.  $1\sigma$  and  $2\sigma$ ).



**Figure 3.** Confidence levels of rejection associated to the goodness-of-fit test  $T_f$  as a function of the parameters  $\alpha$  and  $M_*$  characterizing the Schechter luminosity function models. Bold and normal contours indicate respectively confidence levels of 68 and 95 per cent (i.e.  $1\sigma$  and  $2\sigma$ ) and the shading represents increasing values of the confidence level. Within that range of parameters, the Schechter models cannot be ruled out by the goodness-of-fit test. The minimum of the test indicated by the + sign occurs for  $(\alpha = -0.94, M_* = -19.48)$ .

test. The maximum likelihood technique is, statistically speaking, very efficient for estimating the best-fitting parameters of a model in the sense that it provides the estimator with the smallest variance (see for example Kendall & Stuart 1979). We have shown in Section 2.3 that the goodness-of-fit test  $T_f$  is close to being a



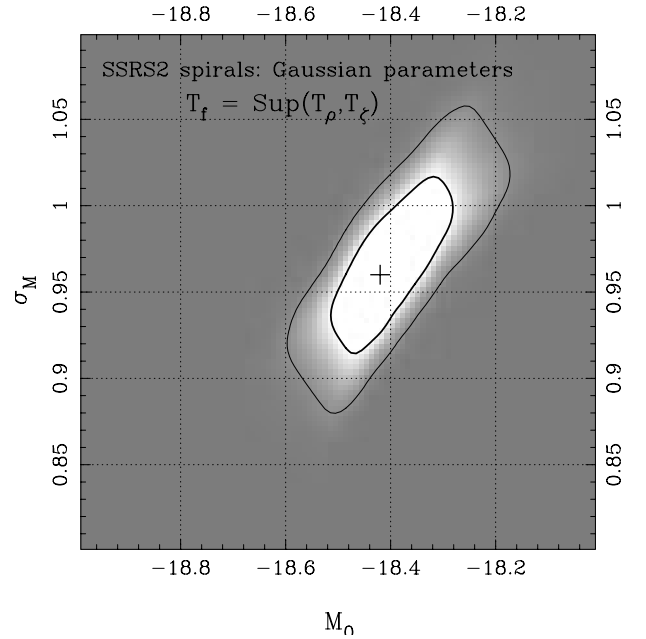
**Figure 4.** Maximum Likelihood estimate of the Schechter parameters  $\alpha$  and  $M_*$ . The best-fitting solution indicated by the + sign is ( $\alpha = -0.99$ ,  $M_* = -19.50$ ). Bold and normal contours give respectively the  $1\sigma$  and  $2\sigma$  errors ellipsoid associated to the maximum likelihood estimate. The shading represents increasing values of the confidence level.

sufficient test, which means in particular that its efficiency is expected to be close to the optimal efficiency of a maximum likelihood estimator. That property is observed in Figs 3 and 4. The regions of parameter space associated with a given confidence level of rejection for the  $T_f$  test have indeed a comparable area to the corresponding maximum likelihood error ellipsoids. This implies that the goodness-of-fit test  $T_f$  used as a fitting technique is also a very efficient estimator.

That property can be checked more thoroughly by means of numerical simulations. We have obtained 100 samples from Monte Carlo simulations assuming a Schechter luminosity function of parameters ( $\alpha = -0.95$ ,  $M_* = -19.50$ ) and conserving the same spatial distribution and the same cut-off in apparent magnitude as for our selection of the SSRS2 spirals sample. For each of these simulated samples, we estimated the minimum of the  $T_f$  test as well as the best-fitting solution of the maximum likelihood technique. The average over the 100 simulations gives ( $\langle\alpha\rangle = -0.942$ ,  $\langle M_*\rangle = -19.496$ ) with a standard deviation of ( $\sigma_\alpha = -0.099$ ,  $\sigma_{M_*} = 0.084$ ) for the  $T_f$  test and ( $\langle\alpha\rangle = -0.941$ ,  $\langle M_*\rangle = -19.497$ ) with a standard deviation of ( $\sigma_\alpha = -0.089$ ,  $\sigma_{M_*} = 0.067$ ) for the maximum likelihood technique. This shows that the  $T_f$  test as a fitting technique is slightly less accurate, but still very close to the optimal efficiency of the maximum likelihood technique. Note that our modest number of simulations does not allow us to detect the small bias in the maximum likelihood technique, previously reported by Efstathiou et al. (1988) for example, which tends to underestimate  $M_*$ . The amplitude of this bias is however substantially smaller than  $1\sigma$  and tends towards zero when the number of sampled objects increases (i.e. maximum likelihood estimators are asymptotically unbiased).

### 3.4 Gaussian luminosity function

The Schechter function has become, during the past decades, the



**Figure 5.** Confidence levels of rejection associated with the goodness-of-fit test  $T_f$  for the case of a Gaussian luminosity function of mean  $M_0$  and dispersion  $\sigma_M$ . Bold and normal contours indicate respectively confidence levels of 68 and 95 per cent. The shading represents increasing values of the confidence level. Therefore, Gaussian models can account as well as Schechter models for the luminosity function of our selection of SSRS2 spirals sample. The minimum of the  $T_f$  test indicated by the + sign is ( $\sigma_M = 0.96$ ,  $M_0 = -18.42$ ).

most popular candidate for describing the shape of the optical luminosity function of the local population of galaxies (see for example Marzke et al. 1998). However, the question of whether alternative luminosity function models can also be admitted has not been addressed so often in the literature. Here, we make use of our test of fit to test the goodness-of-fit of a Gaussian luminosity function model, i.e.

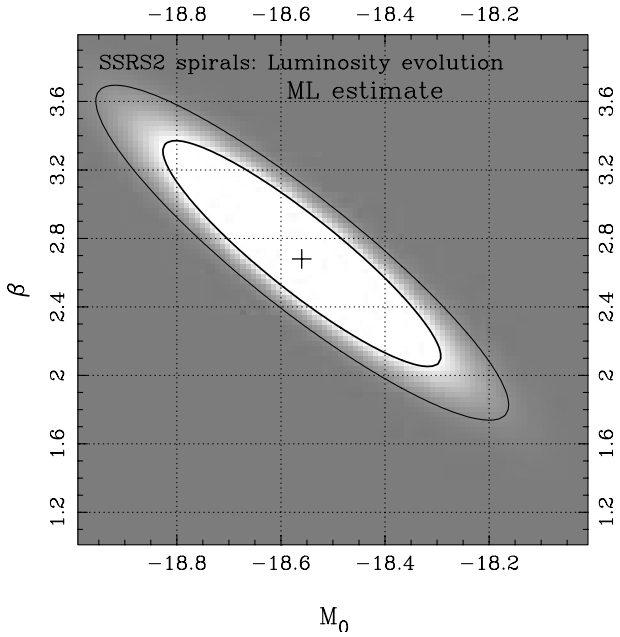
$$f(M) = \frac{1}{\sigma_M \sqrt{2\pi}} \exp\left(-\frac{(M - M_0)^2}{2\sigma_M^2}\right) \quad (23)$$

with  $M_0$  the mean absolute magnitude and  $\sigma_M$  the intrinsic dispersion.

The results of the test of fit applied to our selection of SSRS2 spirals are presented in Fig. 5. A region of parameter space cannot be rejected on the basis of the goodness-of-fit test  $T_f$ . We have seen in the previous subsection that the efficiency of the  $T_f$  statistic is close to being optimal. It turns out that a Gaussian function is effectively as good a model as a Schechter function to account for the luminosity function of our selection of SSRS2 spirals, and that to discriminate between the two models would essentially require a larger sample.

In the present analysis, we have however discarded the nearby galaxies of the SSRS2 sample in order to avoid severe contamination of the redshifts by peculiar velocities. This subsampling preferentially removes the very faint galaxies which prove to be of importance for analysing the faint end of the luminosity function, where the difference between a Gaussian and Schechter function is more marked. In effect, our selection spans five orders of magnitudes, from  $M \approx -22$  to  $M \approx -17$ . We have thus performed the goodness-of-fit test on the spirals of the SSRS2 sample with redshifts between 2000 and  $15\,000 \text{ km s}^{-1}$ , pushing





**Figure 6.** Maximum likelihood estimate of the parameters  $\beta$  and  $M_0$  characterizing the toy model of luminosity evolution (see equation 21 for description). The best-fitting solution marked by the + sign occurs for ( $\beta = 2.68$ ,  $M_0 = -18.56$ ). Bold and normal contours give respectively the  $1\sigma$  and  $2\sigma$  error ellipsoids. The shading represents increasing values of the confidence level.

the faint limit in absolute magnitude to  $M \approx -16.5$  (the number of galaxies increases to 2471 in that case). The Schechter model with ( $\alpha = -0.94$ ,  $M_* = -19.48$ ) gives a confidence level of rejection of  $T_f^2 = 0.46$  while the Gaussian model with ( $\sigma_M = 0.96$ ,  $M_0 = -18.42$ ) can be rejected with a confidence level of  $T_f^2 = 0.99$ . Therefore, the inclusion of more nearby galaxies of the sample favours in some way the Schechter model against the Gaussian one.

### 3.5 Luminosity evolution

It remains to illustrate the major advantage of the goodness-of-fit test  $T_f$  over fitting techniques in general. We consider below a toy model characterized by the evolution of the luminosity function with time. At a given redshift  $z$ , the conditional luminosity function is a Gaussian of fixed dispersion  $\sigma_M = 1.25$  mag with a mean absolute magnitude  $\bar{M}(z)$  linearly evolving with the redshift, i.e.

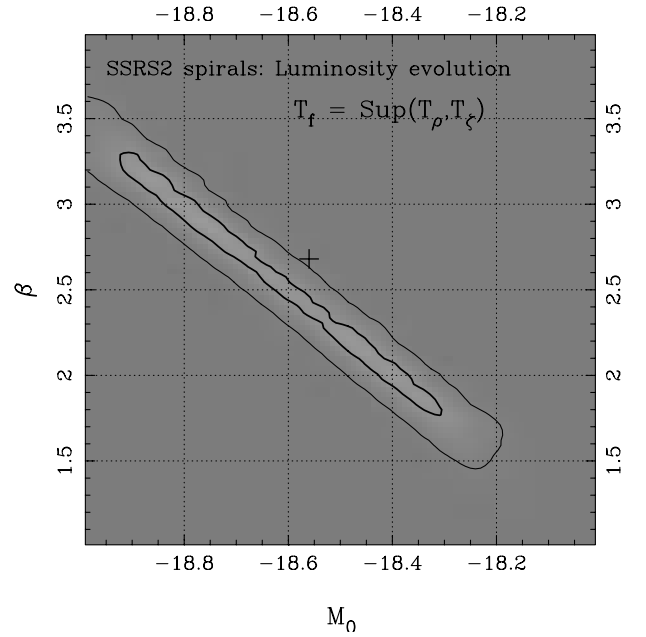
$$f(M|z) = \frac{1}{\sigma_M \sqrt{2\pi}} \exp\left\{-\frac{[M - \bar{M}(z)]^2}{2\sigma_M^2}\right\}$$

with

$$\bar{M}(z) = M_0 + \beta \frac{z}{0.05} \quad \text{and} \quad \sigma_M = 1.25 \text{ mag} \quad (24)$$

The two parameters of our toy model are thus  $M_0$  and the evolution parameter  $\beta$ .

We want to emphasize that our goal is not herein to detect some evolution in the luminosity function of the SSRS2 spirals. First we have seen in the previous section that a Schechter function (and to some lesser extent a Gaussian function) with no evolution can perfectly account for the luminosity function of the sample. Secondly since we have discarded far away galaxies (i.e.  $cz < 15000 \text{ km s}^{-1}$ , which corresponds to a look-back time of 1 Gyr if  $H_0 = 50 \text{ km s}^{-1} \text{ Mpc}^{-1}$ ), we do not expect any drastic



**Figure 7.** Confidence levels of rejection associated to the goodness-of-fit test  $T_f$  for the toy model of luminosity evolution. The bold contour indicates a confidence level of rejection of 95 per cent and the normal contour a level of 99 per cent. The shading represents increasing values of the confidence level. The + sign is the best-fitting estimate obtained by the maximum likelihood technique. None of the  $\beta$  and  $M_0$  parameters values gives a confidence level of rejection below 90 per cent. That luminosity evolution model can therefore be rejected on the basis of the  $T_f$  results.

evolution of the luminosity function within this range of cosmic time. As a matter of fact, we choose our time evolving toy model exactly with the intention of showing that it may be rejected by the  $T_f$  test as a good descriptor of the sample luminosity function.

We applied first the maximum likelihood fitting technique for obtaining the best-fitting solution of the evolution model. The error ellipsoids are shown in Fig. 6. The best-fitting solution ( $\beta = 2.68$ ,  $M_0 = -18.56$ ) corresponds to strong evolution of the mean absolute magnitude, about 3 mag between  $z = 0$  (i.e.  $t = 0$ ) and  $z = 0.05$  (i.e.  $t = 1$  Gyr if  $H_0 = 50 \text{ km s}^{-1} \text{ Mpc}^{-1}$ ). The maximum likelihood fitting technique is, however, unable to assess the adequacy of that solution to the data.

We show in Appendix B that the formalism presented Section 2 can be easily generalized to account for variation of the specific characteristics of the luminosity function with time (e.g. mean absolute magnitude, shape). The goodness-of-fit test  $T_f$  can therefore be used to check the adequacy of any luminosity function evolution model, as long as this time dependency is of course specified. The results of the  $T_f$  test for our toy model of evolution are shown in Fig. 7. As expected, our evolution model is rejected as a whole (none of the  $\beta$  and  $M_0$  parameters values gives a confidence level of rejection below 90 per cent).

## 4 SUMMARY

We have proposed a goodness-of-fit test  $T_f$  which allow to check whether a luminosity function model can account for the intrinsic luminosity distribution of the galaxies of a magnitude–redshift sample complete in apparent magnitude. The model can include luminosity evolution of the population. No a priori assumptions are required concerning the redshift-space distribution of the sources.

This means in particular that neither clustering nor evolution of the mean number density of the galaxies with time affects the conclusions of the goodness-of-fit test. The statistical efficiency of the  $T_f$  test, if used as a fitting technique for estimating the best-fitting solution of a parametric luminosity function model, is comparable to the optimal efficiency of the standard maximum likelihood fitting technique of Sandage et al. (1979). However, the major advantage of the goodness-of-fit test lies in its capacity to assess the adequacy of the proposed parametric model to the data.

The computational implementation of the  $T_f$  statistic is straightforward and is comparable in CPU time with the maximum likelihood technique proposed by Sandage et al. (1979). We reiterate finally that the results of the test are reliable to the extent that the analysed sample meet the following criteria.

(i) The various corrections entering the definition of the absolute magnitude of the galaxies (e.g.  $k$ -correction, galactic extinction) are reasonably accurate.

(ii) The (luminosity) distances of the galaxies are known, which implies that the cosmological world model  $\{H_0, \Omega_0, \Lambda_0\}$  has to be specified and that the contribution of peculiar velocities to observed redshifts can be safely neglected.

(iii) The sample must be strictly complete in apparent magnitude. This criterion can be checked independently by using the test for completeness proposed in Rauzy (2001). Note that surveys selected following a sparse-sampling strategy meet this criterion.

(iv) The luminosity function of the population can evolve with time but environmental effects are not present.

## ACKNOWLEDGMENTS

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## REFERENCES

- Bigot G., Triay R., 1991, Phys. Lett. A, 159, 201  
 Bigot G., Fliche H.-H., Triay R., 1988, A&A, 206, 1  
 Bigot G., Rauzy S., Triay R., 1991, Phys. Lett. A, 158, 282  
 Bingeli B., Sandage A., Tammann G. A., 1988, ARA&A, 26, 509  
 Caditz D., Petrosian V., 1993, ApJ, 416, 450  
 Cholonewski J., 1986, MNRAS, 223, 1  
 Cholonewski J., 1987, MNRAS, 226, 273  
 Colless M., 2000, PASA, 17, 215  
 da Costa L. N. et al., 1998, AJ, 116, 1  
 Driver S. M., 1999, ApJ, 526, L69  
 Eales S., 1993, ApJ, 404, 51  
 Efron B., Petrosian V., 1992, ApJ, 399, 345  
 Efstathiou G., Ellis R. S., Peterson B. A., 1988, MNRAS, 232, 431  
 Ellis R. S., 1997, ARA&A, 35, 389  
 Felten J. E., 1977, AJ, 82, 861  
 Fliche H.-H., Souriau J.-M., 1979, A&A, 78, 87  
 Heyl J., Colless M., Ellis R. S., Broadhurst T., 1997, MNRAS, 285, 613  
 Hubble E., 1926, ApJ, 64, 321  
 Kendall M. G., Stuart A., 1979, The advanced theory of Statistics. Charles Griffin, London  
 Lasker B. M., Sturch C. R., McLean B. M., Russel J. L., Jenker H., Shara M., 1990, AJ, 99, 2019

- Lin H., Kirshner R. P., Schectman S. A., Landy S. D., Oemler A., Tucker D. L., Schechter P. L., 1996, ApJ, 464, 60  
 Lynden-Bell D., 1971, MNRAS, 155, 95  
 Marzke R. O., da Costa L. N., Pellegrini P. S., Willmer N. A., Geller M. J., 1998, ApJ, 503, 617  
 Pence W., 1976, ApJ, 203, 39  
 Petrosian V., 1976, ApJ, 209, L1  
 Petrosian V., 1992, in Feigelson E. D., Babu G. J., eds, Statistical challenges in modern Astronomy. Springer-Verlag, New York, p. 173  
 Press W. H., Flannery B. P., Teukolsky S. A., 1986, Numerical recipes: the art of scientific computing. Cambridge University Press, Cambridge  
 Qin Y.-P., Xie G.-Z., 1997, ApJ, 486, 100  
 Qin Y.-P., Xie G.-Z., 1999, A&A, 341, 693  
 Rauzy S., 2001, MNRAS, 324, 51  
 Rauzy S., Hendry M. A., 2000, MNRAS, 316, 621  
 Sandage A., Perelmuter J.-M., 1990, ApJ, 350, 481  
 Sandage A., Tammann G. A., Yahil A., 1979, ApJ, 232, 352  
 Santiago B. X., Strauss M. A., Lahav O., Davis M., Dressler A., Huchra J. P., 1996, ApJ, 461, 38  
 Schlegel D. J., Finkbeiner D. P., Davis M., 1998, ApJ, 500, 525  
 Schmidt M., 1968, AJ, 151, 393  
 Springel V., White S. D. M., 1998, MNRAS, 298, 143  
 Strauss M. A., Willick J. A., 1995, Phys. Rep., 261, 271  
 Takeuchi T. T., Yoshikawa K., Ishii T. T., 2000, ApJS, 129, 1  
 Turner E. L., 1979, ApJ, 231, 645  
 van Waerbeke L., Mathez G., Mellier Y., Bonnet H., Lachièze-Rey M., 1996, A&A, 316, 1  
 Wienberg S., 1972, Gravitation and Cosmology. J. Wiley, New York  
 Willmer C. N. A., 1997, AJ, 114, 898  
 Yahil A., Strauss M. A., Davis M., Huchra J. P., 1991, ApJ, 372, 380

## APPENDIX A: SUFFICIENCY OF THE TEST OF FIT

We show below that the only function entering the definition of  $\zeta$  given in equation (9) which verifies properties P1 and P2 is the cumulative luminosity function  $F(M)$ .

Let assume that, for the sample described by the probability distribution function  $dP$  of equation (8), we have found a normalized function  $g(M)$  (and its associated cumulative distribution function  $G(M) = \int_{-\infty}^M g(x) dx$  with  $G(+\infty) = 1$ ) such that the random variable  $\xi$  defined as

$$\xi = \xi(M, Z) = \frac{G(M)}{G[M_{\text{lim}}(Z)]} \quad (\text{A1})$$

verifies P1 ( $\xi$  is uniformly distributed between 0 and 1) and P2 [ $\xi$  and  $(Z, l, b)$  are statistically independent]. Property P2 implies that  $dP$  may be written as

$$dP = k(Z, l, b) dl db dZ \times j(\xi) d\xi \quad (\text{A2})$$

with  $k(Z, l, b)$  and  $j(\xi)$  two unknown functions. Property P1 implies that  $j(\xi) d\xi = \theta(\xi)\theta(1 - \xi) d\xi$ , i.e.

$$dP = k(Z, l, b) dl db dZ \times \theta(\xi)\theta(1 - \xi) d\xi \quad (\text{A3})$$

From the definition of  $\xi$ , we have

$$dl db dZ d\xi = \frac{g(M)}{G[M_{\text{lim}}(Z)]} dl db dZ dM \quad (\text{A4})$$

which allows to rewrite  $dP$  as

$$dP = k(Z, l, b) dl db dZ \frac{g(M)}{G[M_{\text{lim}}(Z)]} dM \theta[M_{\text{lim}}(Z) - M] \quad (\text{A5})$$

The identification of equation (A5) with equation (8) implies that

$$\frac{g(M)}{f(M)} = \frac{1}{A} \frac{h(Z, l, b)G[M_{\text{lim}}(Z)]}{k(Z, l, b)}. \quad (\text{A6})$$

Because the left-hand term of this equation depends only on  $M$  and the right-hand term only on  $(Z, l, b)$ , it follows that the ratio  $g(M)/f(M)$  is constant. Adding to that the normalization condition  $G(+\infty) = F(+\infty) = 1$ , we obtain

$$g \equiv f, \quad (\text{A7})$$

which proves that the only function verifying properties P1 and P2 is the cumulative luminosity function  $F(M)$ . Note that no assumptions have been made concerning the distribution function  $h(Z, l, b)$  introduced in equation (8). This implies the existence of a sufficient statistic for recovering the luminosity function without specifying the 3D redshift-space distribution of the population  $\rho(z, l, b)$  and the selection function  $\phi(z, l, b)$  introduced in equation (2).

## APPENDIX B: LUMINOSITY EVOLUTION

In this appendix we generalize the result obtained in Section 2 for the case of time-dependent luminosity functions. We also include an additional upper cut-off  $m_{\text{inf}}$  in raw apparent magnitude for the selection effects in observation.

The luminosity function is now defined as the probability distribution function  $f_t(M)$  of the absolute magnitude  $M$  depending on the epoch  $t$  or equivalently on the redshift  $z$  (so herein we consider that the cosmological world model is given and that the influence of peculiar velocities on the redshifts can be neglected), i.e.

$$f_t(M) = f(M|t) \equiv f(M|z) \quad (\text{B1})$$

The above notation emphasizes that  $f(M|z)$  is defined as the conditional probability density function of  $M$  given  $z$ , which implies that at any redshift  $z$  the luminosity function is normalized (i.e.  $\int f(M|z) dM = 1$ ). Without accounting for selection effects in observation, the probability density describing the population may be written as

$$dP_{zM} \propto \rho(z, l, b) dl db dz f(M|z) dM \quad (\text{B2})$$

where  $\rho(z, l, b)$  is the 3D redshift-space distribution function of the sources along the past light-cone. Note that the term  $\rho(z, l, b)$  then includes the density evolution part of the luminosity function.

The model now accounts for the spatial fluctuations of the galaxy density (clustering, large-scale structure, etc.), for the variation of the mean galaxy density with redshift or equivalently with time, and through  $f(M|z)$  for the evolution of the specific characteristics of the luminosity function with time (e.g. mean absolute magnitude, shape). On the other hand, equation (B2) fails to describe environmental effects.

The selection function  $\psi$  describing observational selection effects is taken as

$$\psi(m, z, l, b) \equiv \theta(m_{\text{lim}} - m)\theta(m - m_{\text{inf}}) \times \phi(z, l, b) \quad (\text{B3})$$

We thus assume that the sample is strictly complete in raw apparent magnitude up to a given magnitude limit  $m_{\text{lim}}$ , and we allow for an upper cut-off  $m_{\text{inf}}$  in magnitude, discarding the brightest galaxies of the sample. The function  $\phi(z, l, b)$  describes some eventual mask in angular position and pure selection or subsampling in redshift, as well as sparse-sampling strategy. Accounting for selection effects

in observation, the probability density describing the sample may be written as

$$dP = \frac{1}{A} \phi(z, l, b) \rho(z, l, b) dl db dz f(M|z) dM S_{m_{\text{inf}}}^{m_{\text{lim}}}(m) \quad (\text{B4})$$

with  $S_{m_{\text{inf}}}^{m_{\text{lim}}}(m) = \theta(m_{\text{lim}} - m)\theta(m - m_{\text{inf}})$  the selection in raw apparent magnitude and  $A$  the normalization factor warranting  $\int dP = 1$ .

The absolute magnitude  $M$ , corrected apparent magnitude  $m_{\text{cor}}$  and variable  $Z$  are obtained following equations (4, 5, 6). The maximum  $M_{\text{lim}}(Z)$  and minimum  $M_{\text{inf}}(Z)$  absolute magnitudes for which a galaxy at a given  $Z$  would be visible in the sample are uniquely defined, i.e.

$$M_{\text{lim}}(Z) = m_{\text{lim}} - Z, \quad M_{\text{inf}}(Z) = m_{\text{inf}} - Z. \quad (\text{B5})$$

The selection in raw apparent magnitude can be expressed using this notation as  $S_{m_{\text{inf}}}^{m_{\text{lim}}}(m) \equiv S_{M_{\text{inf}}}^{M_{\text{lim}}}(M, Z)$  with

$$S_{M_{\text{inf}}}^{M_{\text{lim}}}(M, Z) = \theta[M_{\text{lim}}(Z) - M]\theta[M - M_{\text{inf}}(Z)] \quad (\text{B6})$$

and the probability density of equation (B4) rewritten as

$$dP = \frac{1}{A} h(Z, l, b) dl db dz f(M|z) dM S_{M_{\text{inf}}}^{M_{\text{lim}}}(M, Z), \quad (\text{B7})$$

where the distribution function  $h(Z, l, b)$  depends on the functions  $\rho(z, l, b)$ ,  $\phi(z, l, b)$  and on the definition of  $Z$  given equation (6).

The random variable  $\zeta$  is now defined as

$$\zeta = \zeta(M, Z, z) = \frac{F(M|z) - F(M_{\text{inf}}(Z)|z)}{F(M_{\text{lim}}(Z)|z) - F(M_{\text{inf}}(Z)|z)}, \quad (\text{B8})$$

where  $F(M|z)$  is the conditional cumulative luminosity function of  $M$  given  $z$ , i.e.

$$F(M|z) = \int_{-\infty}^M f(x|z) dx. \quad (\text{B9})$$

Note that providing the two-variable function  $F(M|z)$  is known, the random variable  $\zeta$  is computable from the observable quantities  $(M, Z, z)$ . By definition the random variable  $\zeta$  of a sampled galaxy belongs to the interval  $[0, 1]$ , i.e.

$$S_{M_{\text{inf}}}^{M_{\text{lim}}}(M, Z) \equiv \theta(\zeta)\theta(1 - \zeta). \quad (\text{B10})$$

Because  $\partial Z/\partial M = \partial l/\partial M = \partial b/\partial M = 0$ , the Jacobian of the variable transformation  $(\zeta, Z, l, b) \mapsto (M, Z, l, b)$  reduces to

$$\left| \frac{\partial \zeta}{\partial M} \right| = \frac{f(M|z)}{F[M_{\text{lim}}(Z)|z] - F[M_{\text{inf}}(Z)|z]} \quad (\text{B11})$$

and the volume element of equation (B7) rewrites as

$$dl db dZ d\zeta = \left| \frac{\partial \zeta}{\partial M} \right| dl db dZ dM. \quad (\text{B12})$$

Finally by introducing the function

$$k(Z, l, b) = \frac{1}{A} h(Z, l, b)[F(M_{\text{lim}}(Z)|z) - F(M_{\text{inf}}(Z)|z)]. \quad (\text{B13})$$

The probability density of equation (B7) reduces therefore to

$$dP = k(Z, l, b) dl db dZ \times \theta(\zeta)\theta(1 - \zeta) d\zeta, \quad (\text{B14})$$

which implies that

- (i) P1:  $\zeta$  is uniformly distributed between 0 and 1,
- (ii) P2:  $\zeta$  and  $(Z, l, b)$  are statistically independent.

We thus have shown that any candidate function for describing the conditional cumulative distribution function  $F(M|z)$  of the population must necessarily verify properties P1 and P2. By following the same calculations as presented in Appendix A, it is straightforward to check the sufficiency of this condition. It thus turns out that the goodness-of-fit test  $T_f$  presented Section 2.3 allows to assess the adequacy to the data of a proposed time dependent luminosity function model  $f(M|z)$ .

During the analysis, no assumptions have been made concerning the distribution function  $h(Z, l, b)$  introduced equation (B7). This

means in particular that the 3D redshift-space distribution  $\rho(z, l, b)$ , which also describes the evolution of the mean galaxy density with time, does not require to be modelled in order to apply the goodness-of-fit test  $T_f$ . It implies that the evolution of the shape of the luminosity function can thus be detected independently of density evolution.

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