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LINKS BETWEEN THE INVERSE AND THE DIRECT TULLY-FISHER RELATIONS

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Abstract – In this conference, R. Triay [9] has demonstrated the importance to define a statistical model describing the Tully-Fisher (TF) relation in the M-p plane. As long as the same model is used during the calibration step and the step of the determination of the distances of galaxies, standard statistical methods such as the maximum likelihood technic permits us to derive bias free estimators of the distances of galaxies. However in practice, it is convenient to use a different statistical model for calibrating the TF relation (because of its robustness, the Inverse TF (ITF) relation is prefered during this step) and for determinating the distances of galaxies (the Direct TF (DTF) relation is more accurate and robust in this case). So, is it possible to infer the calibration parameters of the DTF relation needed to determine the distances of galaxies from the calibration parameters of the ITF relation obtained during the calibration step ?. Assuming standard working hypothesis, we prove in Rauzy&Triay [5] (hereafter RT) that the ITF and DTF models are in fact mathematically equivalent (i.e. they describe the same physical data distibution in the TF diagram). Thus, it turns out that as long as the calibration parameters are obtained for a given model, we can deduce the corresponding parameters of the other model. Herein, we present this formulas of correspondance. In practice, the best suitable model will be choosen with regard to the selection effects in observation affecting the analysed sample during each of this 2 steps.

key-words – galaxies – distance scale – methods : statistical

1. INTRODUCTION

In a previous paper (Triay et *al.* [10], hereafter TLR) we have demonstrated the importance to define a statistical model describing the observed linear

correlation between the absolute magnitude M and the logarithm of the line width distance indicator p of galaxies (the TF relation). A random variable $\zeta = a\,p+b-M$ of zero mean was introduced to mimic the intrinsic scatter σ_ζ of the TF relation. In order to fully specify the statistical model, a second random variable ξ of mean ξ_0 and dispersion σ_ξ, statistically independent of ζ, has to be choosen. Herein in section 2, we generalize the results obtained in TLR by introducing a class of statistical models indexed by an angle parameter α. This class of α-models forms a continuous set of models including the ITF and DTF relation as boundary cases. We derive the maximum likelihood statistics for the 5 model dependent parameters a^α, b^α, σ_ζ^α, ξ_0^α and σ_ξ^α characterising an α-model and we illustrate their variations with respect to the angle parameter α. Assuming standard working hypothesis, we show in section 3 that all these α-models are indeed mathematically equivalent : i.e. they describe the same physical data distribution in the M-p plane. In particular, this result implies that there is no difference between the ITF and DTF models. It thus turns out that as long as the 5 parameters a^α, b^α, σ_ζ^α, ξ_0^α and σ_ξ^α are known for a given α-model (say the ITF model for example), we can deduce the corresponding 5 parameters for every α-models (in particular for the DTF model). These formulas of correspondance are derived in section 4. This property permits indeed to use a different statistical model for calibrating the TF relation and for determinating the distance of galaxies or the Hubble's constant.

2. THE SET OF THE α-MODELS

Regardless of selection effects in observation or measurment errors, the theoretical probability density (*pd*) describing the distribution of the absolute magnitude M and of the logarithm of the line width distance estimator p involved in the TF relation can be written as follows :

$$dP_{\rm th} = F(M,p)\; dM dp \tag{1}$$

The observed linear correlation between M and p (the TF relation) constrains the probability density function (*pdf*) $F(M,p)$ to adopt a specific form. In fact, it exists a straight line Δ_{TF} of equation $\widetilde{M}(p) = a\,p + b$ such that the data in the M-p plane are distributed about this line. The slope a and the zero point b of this line are unknown quantities which will be estimated during a preliminar calibration step. In TLR we have shown that it is convenient to express this intrinsic scatter about the line Δ_{TF} by introducing a random variable ζ of <u>zero mean</u> and of dispersion σ_ζ defined as follows :

$$\zeta = \widetilde{M}(p) - M = a\,p + b - M \tag{2}$$

A second random variable ξ statistically independent of ζ is required in order to fully specify the statistical model (i.e. the *pdf* $F(M,p)$) characterizing

the data distribution in the M-p plane [1]. Herein, we generalize the results obtained in TLR by introducing a set of models characterized by the choice of this second variable ξ. We define a family of model dependent variables ξ^α, linear combination of M and p and statistically independent of ζ^α, indexed by an angle parameter α varying continuously from 0 to $\pi/2$:

$$\xi^\alpha = \cos\alpha M + \sin\alpha\, a^\alpha p \tag{3}$$

where we rewrite Eq. (2) as follows (ζ is model dependent, see footnote 1) :

$$\zeta^\alpha = \widetilde{M}^\alpha(p) - M = a^\alpha p + b^\alpha - M \tag{4}$$

We have thus introduced a set of statistical α-models describing the TF diagram by the following *pd* :

$$dP_{\rm th} \approx dP^\alpha_{\rm th} = f_{\xi^\alpha}(\xi^\alpha)d\xi^\alpha\ g(\zeta^\alpha;0,\sigma^\alpha_\zeta)d\zeta^\alpha \tag{5}$$

In order to entirely characterize an α-model, we need to specify the form of the *pdf* $g(\zeta^\alpha;0,\sigma^\alpha_\zeta)$ and $f_{\xi^\alpha}(\xi^\alpha)$. Herein, we limit ourselves to the case of 2 gaussian *pdf*. Our working hypothesis are a gaussian (hereafter noted g_G) *pdf* of zero mean and of dispersion σ^α_ζ for the variable ζ^α characterizing the intrinsic scatter about the straight line Δ^α_{TF} ($g(\zeta^\alpha;0,\sigma^\alpha_\zeta) = g_G(\zeta^\alpha;0,\sigma^\alpha_\zeta)$) and a gaussian *pdf* of mean ξ^α_0 and of dispersion σ^α_ξ for the second random variable ξ^α ($f_{\xi^\alpha}(\xi^\alpha) = g_G(\xi^\alpha;\xi^\alpha_0,\sigma^\alpha_\xi)$). The *pd* describing an α-model reads finally as follows :

$$dP^\alpha_{\rm th} = g_G(\xi^\alpha;\xi^\alpha_0,\sigma^\alpha_\xi)d\xi^\alpha\ g_G(\zeta^\alpha;0,\sigma^\alpha_\zeta)d\zeta^\alpha \tag{6}$$

Note that the set of the α-models describes the Direct TF relation (p and ζ are statistically independent) and the Inverse TF relation (M and ζ are statistiscally independent) when the angle parameter α is equal to its boundaries values :

$$\text{DTF} : \begin{cases} \alpha = \pi/2 \\ dP^{\rm D}_{\rm th} = g_G(p;p_0,\sigma_p)dp\ g_G(\zeta^{\rm D};0,\sigma^{\rm D}_\zeta)d\zeta^{\rm D} \end{cases} \tag{7}$$

$$\text{ITF} : \begin{cases} \alpha = 0 \\ dP^{\rm I}_{\rm th} = g_G(M;M_0,\sigma_M)dM\ g_G(\zeta^{\rm I};0,\sigma^{\rm I}_\zeta)d\zeta^{\rm I} \end{cases} \tag{8}$$

The next step of the analysis is to derive the 5 model dependent parameters a^α, b^α, σ^α_ζ, ξ^α_0 and σ^α_ξ characterising an α-model from a calibration sample.

[1] In the absence of a better physical understanding of the TF relation, the parameters a and b have to be determined using a statistical process (the calibration step). Thus, these parameters a and b and so the random variables ζ and ξ depend on the statistical model used to describe the data distribution : they are model dependent.

We use the maximum likelihood technic to derive these statistics. Herein, we just present these statistics for the 2 following peculiar models (see RT for the general case). The statistics for the ITF model ($\alpha = 0$) :

$$a^{\mathrm{I}} = \frac{\Sigma(M)^2}{\mathrm{Cov}(p,M)} \tag{9}$$

$$b^{\mathrm{I}} = \langle M\rangle - \frac{\Sigma(M)^2}{\mathrm{Cov}(p,M)}\langle p\rangle \quad , \quad \xi_0^{\mathrm{I}} = \langle M\rangle \tag{10}$$

$$\sigma_\zeta^{\mathrm{I}\,2} = \Sigma(M)^2\left(\frac{1}{\rho(p,M)^2} - 1\right) \quad , \quad \sigma_\xi^{\mathrm{I}\,2} = \Sigma(M)^2 \tag{11}$$

and for the DTF model ($\alpha = \pi/2$) :

$$a^{\mathrm{D}} = \frac{\mathrm{Cov}(p,M)}{\Sigma(p)^2} \tag{12}$$

$$b^{\mathrm{D}} = \langle M\rangle - \frac{\mathrm{Cov}(p,M)}{\Sigma(p)^2}\langle p\rangle \quad , \quad \xi_0^{\mathrm{D}} = a^{\mathrm{D}}\langle p\rangle = \frac{\mathrm{Cov}(p,M)}{\Sigma(p)^2}\langle p\rangle \tag{13}$$

$$\sigma_\zeta^{\mathrm{D}\,2} = \Sigma(M)^2\left(1-\rho(p,M)^2\right) \ , \ \sigma_\xi^{\mathrm{D}\,2} = a^{\mathrm{D}\,2}\Sigma(p)^2 = \rho(p,M)^2\Sigma(M)^2 \tag{14}$$

with the standard notations : $\langle\rangle$ the average on the sample, Σ the variance, Cov the covariance and ρ the correlation coefficient.

3. EQUIVALENCE OF THE α-MODELS

In substituting the general statistics of the model dependent parameters a^α, b^α, σ_ζ^α, ξ_0^α and σ_ξ^α in the *pd* of Eq. (6), we find that, for every α belonging to $[0,\pi/2]$ (see RT for detailed calculations) :

$$dP_{\mathrm{th}}^\alpha = g_G(p;\langle p\rangle,\Sigma(p))\, g_G(M;\langle M\rangle + \frac{\mathrm{Cov}(p,M)}{\Sigma(p)^2}\left(p-\langle p\rangle\right), \Sigma(M)\sqrt{1-\rho^2})\, dM dp \tag{15}$$

It thus means that all the α-models are indeed mathematically equivalent and that they describe the <u>same</u> physical distribution of data in the M-p plane. Note that we can rewrite Eq. (15) as a binormal *pdf* in M and p, entirely characterized by its 5 moments of first and second order $\langle p\rangle$, $\langle M\rangle$, $\Sigma(p)$, $\Sigma(M)$ and $\mathrm{Cov}(p,M)$:

$$\forall\alpha\in[0,\pi/2] \ : \ dP_{\mathrm{th}}^\alpha = \frac{1}{2\pi\Sigma(M)\Sigma(p)\sqrt{1-\rho(p,M)^2}}$$

$$\times\exp\left\{-\frac{1}{2(1-\rho(p,M)^2)}\left(\frac{(p-\langle p\rangle)^2}{\Sigma(p)^2} - 2\frac{\mathrm{Cov}(p,M)(p-\langle p\rangle)(M-\langle M\rangle)}{\Sigma(M)^2\Sigma(p)^2} + \frac{(M-\langle M\rangle)^2}{\Sigma(M)^2}\right)\right\}\, dM dp \tag{16}$$

We now understand that our working hypothesis (2 gaussian *pdf* for ξ^α and ζ^α) imply that the knowledge of the 5 parameters a^α, b^α, σ_ζ^α, ξ_0^α and σ_ξ^α for a

given α-model is sufficient to entirely describe the data distribution of the TF diagram [2]. It thus turns out that if these 5 parameters are known for a given α-model, we can deduce the corresponding 5 parameters for every α-models.

4. LINK BETWEEN THE ITF AND DTF MODELS

We derive in RT the general formulas of correspondance between the estimates of the 5 model dependent parameters a^α, b^α, σ_ζ^α, ξ_0^α and σ_ξ^α characterizing different α-models. Herein, we just present these formulas of correspondance for 2 particular cases : the calibration parameters of the ITF relation are known and we want to infer the calibration parameters of the DTF relation,

$$a^{\mathrm{D}} = a^{\mathrm{I}} \frac{\sigma_\xi^{\mathrm{I}^2}}{\sigma_\xi^{\mathrm{I}^2} + \sigma_\zeta^{\mathrm{I}^2}} = \rho^{\mathrm{I}^2} a^{\mathrm{I}} \quad , \quad b^{\mathrm{D}} = \left(1 - \rho^{\mathrm{I}^2}\right) \xi_0^{\mathrm{I}} + \rho^{\mathrm{I}^2} b^{\mathrm{I}} \tag{17}$$

$$\sigma_\zeta^{\mathrm{D}^2} = \left(1 - \rho^{\mathrm{I}^2}\right)^2 \sigma_\xi^{\mathrm{I}^2} + \rho^{\mathrm{I}^4} \sigma_\zeta^{\mathrm{I}^2} = \rho^{\mathrm{I}^2} \sigma_\zeta^{\mathrm{I}^2} \tag{18}$$

$$\sigma_\xi^{\mathrm{D}^2} = \rho^{\mathrm{I}^4} \left(\sigma_\xi^{\mathrm{I}^2} + \sigma_\zeta^{\mathrm{I}^2}\right) = \rho^{\mathrm{I}^2} \sigma_\xi^{\mathrm{I}^2} \quad , \quad \xi_0^{\mathrm{D}} = \rho^{\mathrm{I}^2} \left(\xi_0^{\mathrm{I}} - b^{\mathrm{I}}\right) \tag{19}$$

or conversely the calibration parameters of the DTF relation are known and we want to deduce the calibration parameters of the ITF relation :

$$a^{\mathrm{I}} = a^{\mathrm{D}} \frac{\sigma_\xi^{\mathrm{D}^2} + \sigma_\zeta^{\mathrm{D}^2}}{\sigma_\xi^{\mathrm{D}^2}} = \frac{1}{\rho^{\mathrm{D}^2}} a^{\mathrm{D}} \quad , \quad b^{\mathrm{I}} = \left(1 - \frac{1}{\rho^{\mathrm{D}^2}}\right) \xi_0^{\mathrm{D}} + b^{\mathrm{D}} \tag{20}$$

$$\sigma_\zeta^{\mathrm{I}^2} = \left(1 - \frac{1}{\rho^{\mathrm{D}^2}}\right)^2 \sigma_\xi^{\mathrm{D}^2} + \frac{1}{\rho^{\mathrm{D}^4}} \sigma_\zeta^{\mathrm{D}^2} = \frac{1}{\rho^{\mathrm{D}^2}} \sigma_\zeta^{\mathrm{D}^2} \tag{21}$$

$$\sigma_\xi^{\mathrm{I}^2} = \sigma_\xi^{\mathrm{D}^2} + \sigma_\zeta^{\mathrm{D}^2} = \frac{1}{\rho^{\mathrm{D}^2}} \sigma_\xi^{\mathrm{D}^2} \quad , \quad \xi_0^{\mathrm{I}} = \xi_0^{\mathrm{D}} + b^{\mathrm{D}} \tag{22}$$

5. CONCLUSION

In order to mimic the Tully-Fisher diagram, we have introduced a continuous set of statistical models characterized by the straight line Δ_{TF}^α describing the observed linear correlation of M and p. This set of α-models include the ITF and DTF relation as boundaries cases. Assuming standard working hypothesis, we have shown that all these α-models describe indeed the same physical data distribution in the M-p plane. Thus, if the 5 calibration parameters a^α,

[2] Weaker hypothesis on the 2 *pdf* oblige us to take into account the higher order moments of the bivariate distribution in M and p. Thus, the α-models are no more strictly equivalent. However, the previous equations appear as sufficiently accurate approximations as long as the influence of the moments of higher order is small.

b^α, σ_ζ^α, ξ_0^α and σ_ξ^α are known for a given α-model, we can infer the calibration parameters of every α-models by using some formulas of correspondance. Practically, this property offers us the possibility to use a different statistical model during the calibration step of the TF relation and for determinating the distances of galaxies. The best suitable statistical model will thus be choosen with regard to the selection effects in observation affecting the samples during each of these 2 steps.

For example, the ITF model seems to be more adequate for calibrating the TF relation because of its robustness (the estimates of a^{I}, b^{I} and $\sigma_\zeta^{\mathrm{I}}$ don't depend on the luminosity function (Hendry et *al.* [3], TLR) but also because when calibrating the ITF relation in a cluster, the estimates of a^{I} and $\sigma_\zeta^{\mathrm{I}}$ don't depend on the distance of the cluster (Schechter [7], Tully [11], Lynden-Bell et *al.* [4],Teerikorpi [8],[3], Rauzy et *al.* [6]). Conversely, the use of the DTF relation is prefered to determine the distances of galaxies. It is more accurate (the intrinsic scatter of the DTF relation $\sigma_\zeta^{\mathrm{D}}$ is indeed smaller than the ITF one $\sigma_\zeta^{\mathrm{I}}$ ([11], TLR)), more robust (the DTF distance estimator doesn't depend on the luminosity function (TLR)) and more intuitive (an observed p gives directly a value fo M : $\widetilde{M}(p) = a^{\mathrm{D}}p + b^{\mathrm{D}}$ (Bottinelli et *al.* [1], Fouqué et *al.* [2])).

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