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## THE COMPARED COSTS OF DOMINATION LOCATION-DOMINATION AND IDENTIFICATION

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### Abstract

Let  $G = (V, E)$  be a finite graph and  $r \geq 1$  be an integer. For  $v \in V$ , let  $B_r(v) = \{x \in V : d(v, x) \leq r\}$  be the ball of radius  $r$  centered at  $v$ . A set  $C \subseteq V$  is an  $r$ -dominating code if for all  $v \in V$ , we have  $B_r(v) \cap C \neq \emptyset$ ; it is an  $r$ -locating-dominating code if for all  $v \in V$ , we have  $B_r(v) \cap C \neq \emptyset$ , and for any two distinct non-codewords  $x \in V \setminus C$ ,  $y \in V \setminus C$ , we have  $B_r(x) \cap C \neq B_r(y) \cap C$ ; it is an  $r$ -identifying code if for all  $v \in V$ , we have  $B_r(v) \cap C \neq \emptyset$ , and for any two distinct vertices  $x \in V$ ,  $y \in V$ , we have  $B_r(x) \cap C \neq B_r(y) \cap C$ . We denote by  $\gamma_r(G)$  (respectively,  $ld_r(G)$  and  $id_r(G)$ ) the smallest possible cardinality of an  $r$ -dominating code (respectively, an  $r$ -locating-dominating code and an  $r$ -identifying code). We study how small and how large the three differences  $id_r(G) - ld_r(G)$ ,  $id_r(G) - \gamma_r(G)$  and  $ld_r(G) - \gamma_r(G)$  can be.

**Keywords:** graph theory, dominating set, locating-dominating code, identifying code, twin-free graph.

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## 1. INTRODUCTION

## 1.1. Definitions and notation

For graph theory, we refer to, e.g., [1, 2] or [8]; for the vast topic of domination in graphs, see [13]. For locating-dominating codes, see the first papers [7] and [18], for identifying codes, see the seminal paper [14]; for both, see also the large bibliography at [15].

We shall denote by  $G = (V, E)$  a finite, simple, undirected graph with vertex set  $V$  and edge set  $E$ , where an *edge* between  $x \in V$  and  $y \in V$  is indifferently denoted by  $xy$  or  $yx$ . The *order* of the graph is its number of vertices,  $|V|$ . Our graphs will generally be connected. The *distance* between two vertices  $x \in V$ ,  $y \in V$ , will be denoted by  $d_G(x, y)$ , or  $d(x, y)$  when there is no ambiguity.

For an integer  $k \geq 2$ , the *k-th transitive closure*, or *k-th power*, of  $G = (V, E)$  is the graph  $G^k = (V, E^k)$  defined by  $E^k = \{uv : u \in V, v \in V, 0 < d_G(u, v) \leq k\}$ . For a given graph  $G^*$ , any graph  $G$  such that  $G^k = G^*$  is called a *k-th root* of  $G^*$ ; such roots do not always exist.

For any integer  $r \geq 1$ , and for every vertex  $x \in V$ , we denote by  $B_{G,r}(x)$  (and  $B_r(x)$  when there is no ambiguity) the *ball of radius  $r$  centered at  $x$* , i.e., the set of vertices at distance at most  $r$  from  $x$ :

$$B_r(x) = \{y \in V : d(x, y) \leq r\}.$$

Whenever  $x \in B_r(y)$  (which is equivalent to  $y \in B_r(x)$ ), we say that  $x$  and  $y$  *r-dominate* or *r-cover* each other. A vertex  $x \in V$  is said to be *r-universal* if it *r-dominates* all the vertices, i.e., if  $B_r(x) = V$ . When three vertices  $x, y, z$  are such that  $z \in B_r(x)$  and  $z \notin B_r(y)$ , we say that  $z$  *r-separates*  $x$  and  $y$  in  $G$  (note that  $z = x$  is possible). A set of vertices is said to *r-separate*  $x$  and  $y$  if at least one of its element does.

Let  $C \subseteq V$  be a set of vertices; the set  $C$  is called a *code*, and its elements *codewords*.

A code  $C$  is said to be an *r-dominating set* or an *r-dominating code* (*r-D* code for short) if for all  $x \in V$ , we have  $B_r(x) \cap C \neq \emptyset$ . One can also find the terminology *dominating set at distance  $r$* , or *distance  $r$  dominating set*.

A code  $C$  is said to be *r-locating-dominating* (*r-LD* for short) if for all  $x \in V$ , we have  $B_r(x) \cap C \neq \emptyset$ , and for any two distinct non-codewords  $x \in V \setminus C$ ,  $y \in V \setminus C$ , we have  $B_r(x) \cap C \neq B_r(y) \cap C$ .

A code  $C$  is said to be *r-identifying* (*r-ID* for short) if for all  $x \in V$ , we have  $B_r(x) \cap C \neq \emptyset$ , and for any two distinct vertices  $x \in V$ ,  $y \in V$ , we have  $B_r(x) \cap C \neq B_r(y) \cap C$ .

In other words: every vertex must be *r-dominated* by at least one code-word for the three definitions; in addition, every pair of distinct non-codewords (respectively, vertices) must be *r-separated* by an *r-LD* (respectively, *r-ID*) code.

Two vertices  $x \in V$ ,  $y \in V$ ,  $x \neq y$ , are said to be  $r$ -twins if  $B_r(x) = B_r(y)$ . Dominating and locating-dominating codes exist for all graphs. On the other hand, it is easy to see that a graph  $G$  admits an  $r$ -identifying code if and only if

$$(1) \quad \forall x \in V, \forall y \in V, x \neq y : B_r(x) \neq B_r(y).$$

A graph satisfying (1) is called  $r$ -identifiable or  $r$ -twin-free.

## 1.2. Aim of the paper

For all three concepts, we are often interested in finding the minimum sized codes. We denote by  $\gamma_r(G)$  (respectively,  $ld_r(G)$  and  $id_r(G)$ ) the smallest possible cardinality of an  $r$ -dominating code (respectively, an  $r$ -locating-dominating code and an  $r$ -identifying code when  $G$  is  $r$ -twin-free). We call  $\gamma_r(G)$  the  $r$ -domination number of  $G$ . Since obviously an  $r$ -ID code (when it exists) is an  $r$ -LD code which in turn is an  $r$ -D code, the following inequalities hold:

$$\gamma_r(G) \leq ld_r(G) \leq id_r(G).$$

In other words, location-domination is more “expensive” than domination, and identification is more expensive than location-domination. In this paper, we compare the respective “costs” for these three definitions.

More precisely, denoting

$$\mathcal{G}_{r,n} = \{G : G \text{ is } r\text{-twin-free, connected, with order } n \geq 2\},$$

$$\text{and } \mathcal{G}_{r,n}^{tw} = \{G : G \text{ has } r\text{-twins and is connected, with order } n \geq 2\},$$

we study the following maximum and minimum differences:

- $F_{id,ld}(r, n) = \max\{id_r(G) - ld_r(G) : G \in \mathcal{G}_{r,n}\},$
- $f_{id,ld}(r, n) = \min\{id_r(G) - ld_r(G) : G \in \mathcal{G}_{r,n}\},$
- $F_{id,\gamma}(r, n) = \max\{id_r(G) - \gamma_r(G) : G \in \mathcal{G}_{r,n}\},$
- $f_{id,\gamma}(r, n) = \min\{id_r(G) - \gamma_r(G) : G \in \mathcal{G}_{r,n}\}.$

For D- and LD-codes, we have two cases, (a) and (b), which study graphs which are without or with twins, respectively:

- (a) •  $F_{ld,\gamma}(r, n) = \max\{ld_r(G) - \gamma_r(G) : G \in \mathcal{G}_{r,n}\},$   
 •  $f_{ld,\gamma}(r, n) = \min\{ld_r(G) - \gamma_r(G) : G \in \mathcal{G}_{r,n}\};$

these two functions are considered on the same set of graphs (the twin-free graphs) as the four functions involving identification, unlike the two functions below:

- (b) •  $F_{ld,\gamma}^{tw}(r, n) = \max\{ld_r(G) - \gamma_r(G) : G \in \mathcal{G}_{r,n}^{tw}\},$   
 •  $f_{ld,\gamma}^{tw}(r, n) = \min\{ld_r(G) - \gamma_r(G) : G \in \mathcal{G}_{r,n}^{tw}\}.$

Finally, if we want to consider all the connected graphs of order  $n$ , twin-free or not, the result is obviously obtained by taking  $\max \{F_{ld,\gamma}(r, n), F_{ld,\gamma}^{tw}(r, n)\}$  and  $\min \{f_{ld,\gamma}(r, n), f_{ld,\gamma}^{tw}(r, n)\}$ .

## 2. SOME EARLIER RESULTS

The following easy four lemmas are as old as the definitions of dominating, locating-dominating or identifying codes. We give the proofs only for the first two.

**Lemma 1.** (a) *For any graph  $G = (V, E)$  of order  $n$  and any integer  $r \geq 1$ , we have*

$$(2) \quad ld_r(G) \geq \lceil \log_2(n - ld_r(G) + 1) \rceil.$$

(b) *For any integer  $r \geq 1$  and any  $r$ -twin-free graph  $G = (V, E)$  of order  $n$ , we have*

$$(3) \quad id_r(G) \geq \lceil \log_2(n + 1) \rceil.$$

**Proof.** (a) Let  $C$  be any  $r$ -LD code in  $G$ . All the  $n - |C|$  non-codewords  $v \in V \setminus C$  must be given nonempty and distinct sets  $B_r(v) \cap C$  constructed with the  $|C|$  codewords, so  $2^{|C|} - 1 \geq n - |C|$ , from which (2) follows when  $C$  is optimal;

(b) the argument is the same, but we have to consider all the  $n$  vertices  $v \in V$ , so  $2^{|C|} - 1 \geq n$ . ■

**Lemma 2.** *Let  $r \geq 2$  be any integer and  $G = (V, E)$  be a graph.*

- (a) *A code  $C$  is 1-locating-dominating in  $G^r$ , the  $r$ -th power of  $G$ , if and only if it is  $r$ -locating-dominating in  $G$ .*
- (b) *A code  $C$  is 1-identifying in  $G^r$  if and only if it is  $r$ -identifying in  $G$ .*
- (c) *A code  $C$  is 1-dominating in  $G^r$  if and only if it is  $r$ -dominating in  $G$ .*

**Proof.** (a) For every vertex  $v \in V$ , we have

$$\{c \in C : d_G(v, c) \leq r\} = \{c \in C : d_{G^r}(v, c) \leq 1\},$$

so if for all  $v \in V \setminus C$ , the sets on the left-hand side of the equality are nonempty and distinct, then the sets on the right side also are, and *vice-versa*; (b) the same proof, for all  $v \in V$ ; (c) the same proof, for all  $v \in V$ , with only nonemptiness to be checked. ■

**Lemma 3.** (a) *For any integer  $r \geq 1$ , if  $G$  is a connected graph of order  $n$ , then*

$$(4) \quad ld_r(G) \leq n - 1.$$

- (b) If  $G$  is an  $r$ -twin-free graph of order  $n$ , then  $n \geq 2r + 1$ , and the only  $r$ -twin-free graph of order  $2r + 1$  is the path.
- (c) If  $G$  is an  $r$ -twin-free cycle of order  $n$ , then  $n \geq 2r + 2$ .

The following obvious lemma is often used implicitly.

**Lemma 4.** Let  $r \geq 1$  be any integer and  $G = (V, E)$  be a graph.

- (a) If  $C$  is  $r$ -dominating in  $G$ , so is any set  $S \supset C$ .
- (b) If  $C$  is  $r$ -locating-dominating in  $G$ , so is any set  $S \supset C$ .
- (c) If  $C$  is  $r$ -identifying in  $G$ , so is any set  $S \supset C$ .

**Proposition 5.** (a) [16], [13, p. 41] If  $G$  has no isolated vertices (in particular, if  $G$  is connected) and has order  $n$ , then  $\gamma_1(G) \leq \frac{n}{2}$ .

- (b) [11] If  $G$  is a 1-twin-free graph, then  $id_1(G) \leq 2ld_1(G)$ .

The following result is from [3], but a shorter proof can be found in [12].

**Proposition 6.** If  $G$  is a connected 1-twin-free graph of order  $n$ , then  $id_1(G) \leq n - 1$ .

**Corollary 7.** Let  $r \geq 1$  be any integer.

- (i) If  $G$  is a connected graph of order  $n$ , then  $\gamma_r(G) \leq \frac{n}{2}$ .
- (ii) If  $G$  is an  $r$ -twin-free graph, then  $id_r(G) \leq 2ld_r(G)$ .
- (iii) If  $G$  is a connected  $r$ -twin-graph of order  $n$ , then

$$(5) \quad id_r(G) \leq n - 1.$$

**Proof.** Use the  $r$ -th power of  $G$ , together with the previous two propositions. ■

Both lower bounds (2), (3) and upper bounds (4), (5) for  $r$ -LD and  $r$ -ID codes can be reached [6], as well as all intermediate values [4], [5].

The graphs  $G$  of order  $n$  such that  $id_1(G) = n - 1$  have been characterized in [10], but the case  $r \geq 2$  remains open.

### 3. SOME IMPORTANT GRAPHS

The following three lemmas describe three useful graphs, which have been used in previous papers. The first graph is the “star”.

**Lemma 8.** For  $n \geq 3$ , let  $G_n$  be the tree consisting of  $n$  vertices  $v_0, v_1, \dots, v_{n-1}$ , and  $n - 1$  edges  $v_0v_i$ ,  $1 \leq i \leq n - 1$ . Then

$$\gamma_1(G_n) = 1, ld_1(G_n) = n - 1 \text{ and } id_1(G_n) = n - 1.$$

**Proof.** (a) Since  $v_0$  is a 1-universal vertex, we have  $\gamma_1(G_n) = 1$ .

(b) It is quite straightforward to check that taking for codewords any set of  $n - 1$  vertices is necessary and sufficient to obtain a 1-LD or 1-ID code, except for  $n = 3$ , when only  $\{v_1, v_2\}$  is a 1-ID code. ■

The second graph, denoted  $G_{2p}^*$ , has even order and is the complete graph (or clique) minus a perfect matching; see Figure 1.

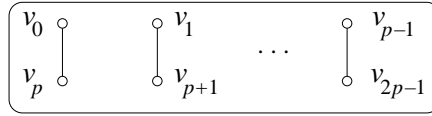


Figure 1. The complement of  $G_{2p}^*$ : only the missing edges  $v_0v_p, \dots, v_{p-1}v_{2p-1}$  are represented.

**Lemma 9.** Let  $p \geq 2$  and  $G_{2p}^* = (V_{2p}^*, E_{2p}^*)$ , with  $V_{2p}^* = \{v_0, v_1, \dots, v_{2p-1}\}$ ,  $E_{2p}^* = \{v_iv_j : v_i \in V_{2p}^*, v_j \in V_{2p}^*, i \neq j, i \neq j + p \bmod 2p\}$ . Then

$$\gamma_1(G_{2p}^*) = 2, \text{ld}_1(G_{2p}^*) = p \text{ and } \text{id}_1(G_{2p}^*) = 2p - 1.$$

**Proof.** For every  $v_i \in V_{2p}^*$ , we have  $B_1(v_i) = V_{2p}^* \setminus \{v_{i+p \bmod 2p}\}$ , and for every pair of distinct vertices  $v_i \in V_{2p}^*, v_j \in V_{2p}^*$ , we have  $B_1(v_i) \Delta B_1(v_j) = \{v_{i+p \bmod 2p}, v_{j+p \bmod 2p}\}$ , where  $\Delta$  stands for the symmetric difference.

(a) The fact that  $\gamma_1(G_{2p}^*) = 2$  is easy to check.

(b) Obviously,  $C = \{v_0, \dots, v_{p-1}\}$  is a 1-LD code, of size  $p$ . Assume that there is a minimum 1-LD code  $C$  with fewer than  $p$  elements. Then there is at least one  $j$  such that  $v_j \notin C$  and  $v_{j+p \bmod 2p} \notin C$ . Without loss of generality, we may assume that  $v_0 \notin C, v_p \notin C$ . Then  $B_1(v_0) \Delta B_1(v_p) = \{v_0, v_p\}$  leads to  $C \cap (B_1(v_0) \Delta B_1(v_p)) = \emptyset$ , contradicting the definition of a 1-LD code:  $v_0$  and  $v_p$  are non-codewords not 1-separated by any codeword.

(c) We know that at most  $2p - 1$  codewords are necessary in any minimum 1-ID code  $C$ ; therefore, assume, without loss of generality, that  $v_0 \notin C$ . Then for all  $j \neq p$ ,  $B_1(v_p) \Delta B_1(v_j) = \{v_0, v_{j+p \bmod 2p}\}$ , and, since  $v_p$  and  $v_j$  are 1-separated by at least one codeword, we have  $\emptyset \neq (B_1(v_p) \Delta B_1(v_j)) \cap C \subseteq \{v_{j+p \bmod 2p}\}$ . So for all values of  $j$  but one, the  $2p - 1$  distinct vertices  $v_{j+p \bmod 2p}$  are codewords, and  $|C| \geq 2p - 1$ , i.e.,  $|C| = 2p - 1$ . ■

The third graph is obtained from the previous one by adding one 1-universal vertex, and its order is odd.

**Lemma 10.** Let  $p \geq 2$  and  $G_{2p+1}^* = (V_{2p+1}^*, E_{2p+1}^*)$ , with  $V_{2p+1}^* = \{v_0, v_1, \dots, v_{2p}\}$ ,  $E_{2p+1}^* = \{v_iv_j : v_i \in V_{2p+1}^* \setminus \{v_{2p}\}, v_j \in V_{2p+1}^* \setminus \{v_{2p}\}, i \neq j, i \neq j + p \bmod 2p\} \cup \{v_{2p}v_j : v_j \in V_{2p+1}^* \setminus \{v_{2p}\}\}$ . Then

$$\gamma_1(G_{2p+1}^*) = 1, ld_1(G_{2p+1}^*) = p \text{ and } id_1(G_{2p+1}^*) = 2p.$$

**Proof.** (a) The fact that  $v_{2p}$  is 1-universal shows that  $\gamma_1(G_{2p+1}^*) = 1$ .

(b) For 1-LD codes, the argument of the Case (b) of the previous proof can be applied *mutatis mutandis*, because the 1-universal vertex does not change anything when considering symmetric differences of balls of radius one.

(c) For  $i \in \{0, \dots, 2p-1\}$ , we have  $B_1(v_{2p}) \Delta B_1(v_i) = \{v_{i+p \bmod 2p}\}$ , therefore all vertices but  $v_{2p}$  must be codewords. ■

Now, what is more difficult and interesting is that the two graphs  $G_{2p}^*$  and  $G_{2p+1}^*$  just described in Lemmas 9 and 10 admit  $r$ -th roots for any  $r$ , if  $p$  is sufficiently large [6]. More precisely:

**Proposition 11.** *Let  $r \geq 2$  and  $p \geq 2$  be integers.*

- (a) [6, Theorem 5] *If  $2p \geq 3r^2$ , then there exists a graph  $G_{2p}$  of order  $2p$  such that  $(G_{2p})^r = G_{2p}^*$ .*
- (b) [6, Theorem 6] *If  $2p \geq 3r^2$ , then there exists a graph  $G_{2p+1}$  of order  $2p+1$  such that  $(G_{2p+1})^r = G_{2p+1}^*$ .*
- (c) *For  $n \geq 3r^2$ , there exists a graph  $G_n$  of even order  $n$  such that  $\gamma_r(G_n) = 2$ ,  $ld_r(G_n) = \frac{n}{2}$  and  $id_r(G_n) = n-1$ .*
- (d) *For  $n \geq 3r^2+1$ , there exists a graph  $G_n$  of odd order  $n$  such that  $\gamma_r(G_n) = 1$ ,  $ld_r(G_n) = \frac{n-1}{2}$  and  $id_r(G_n) = n-1$ .*

**Proof.** (c)–(d). Use the properties of  $r$ -th powers of graphs (Lemma 2). ■

See also the constructions presented and discussed immediately after Proposition 23 in Section 7.1.

#### 4. THE VERY SMALL CASES: $n \leq 4$

Here, we denote by  $T_r(G)$  the triple  $(\gamma_r(G), ld_r(G), id_r(G))$ , with the convention that  $id_r(G) = ?$  if  $G$  is not  $r$ -twin-free. Figure 2 gives all the nonisomorphic unlabeled connected graphs with two, three or four vertices, together with their triples for  $r = 1$ .

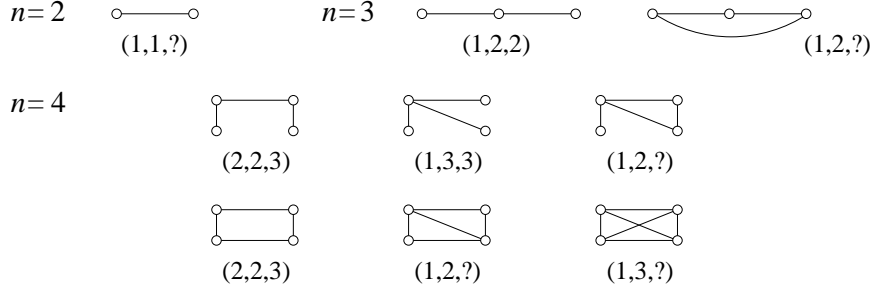
For  $r = 2$ , the triples are, for the nine graphs of Figure 2, respectively:  $(1, 1, ?)$ ;  $(1, 2, ?)$  and  $(1, 2, ?)$ ;  $(1, 2, ?)$ ,  $(1, 3, ?)$ ,  $(1, 3, ?)$ ,  $(1, 3, ?)$ , and  $(1, 3, ?)$ . For  $r \geq 3$ , the triples are  $(1, n-1, ?)$  for all nine graphs. From this, we have the following result.

**Proposition 12.** *We have*

- (a)  $r = 1$
- $n = 2 : F_{ld, \gamma}^{tw}(1, 2) = f_{ld, \gamma}^{tw}(1, 2) = 0;$



- $n = 3 : F_{id,ld}(1, 3) = f_{id,ld}(1, 3) = 0; F_{id,\gamma}(1, 3) = f_{id,\gamma}(1, 3) = 1;$   
 $F_{ld,\gamma}(1, 3) = f_{ld,\gamma}(1, 3) = 1; F_{ld,\gamma}^{tw}(1, 3) = f_{ld,\gamma}^{tw}(1, 3) = 1;$   
 $n = 4 : F_{id,ld}(1, 4) = 1, f_{id,ld}(1, 4) = 0; F_{id,\gamma}(1, 4) = 2, f_{id,\gamma}(1, 4) = 1;$   
 $F_{ld,\gamma}(1, 4) = 2, f_{ld,\gamma}(1, 4) = 0; F_{ld,\gamma}^{tw}(1, 4) = 2, f_{ld,\gamma}^{tw}(1, 4) = 1;$   
 (b)  $r = 2$   
 $n = 2 : F_{ld,\gamma}^{tw}(2, 2) = f_{ld,\gamma}^{tw}(2, 2) = 0;$   
 $n = 3 : F_{ld,\gamma}^{tw}(2, 3) = f_{ld,\gamma}^{tw}(2, 3) = 1;$   
 $n = 4 : F_{ld,\gamma}^{tw}(2, 4) = 2, f_{ld,\gamma}^{tw}(2, 4) = 1;$   
 (c)  $r \geq 3$   
 $n \in \{2, 3, 4\} : F_{ld,\gamma}^{tw}(r, n) = f_{ld,\gamma}^{tw}(r, n) = n - 2.$

Figure 2. Small graphs,  $r = 1$ .

## 5. IDENTIFICATION VS DOMINATION

First, we construct an infinite family of graphs  $G_n^*$  such that  $G_n^*$  has order  $n$  and satisfies  $id_r(G_n^*) = \gamma_r(G_n^*)$ .

These graphs will have order  $n = k(r + 1)$  and consist of one cycle of order  $k$  and  $k$  strings with  $r$  vertices each:  $G_n^* = (V_n^*, E_n^*)$ , with  $V_n^* = V_0 \cup (\bigcup_{1 \leq i \leq k} V_i)$  and  $E_n^* = E_0 \cup (\bigcup_{1 \leq i \leq k} E_i)$ , where  $V_0 = \{v_{1,0}, v_{2,0}, \dots, v_{k,0}\}$ ,  $V_i = \{v_{i,j} : 1 \leq j \leq r\}$  for  $i \in \{1, 2, \dots, k\}$ ,  $E_0 = \{v_{i,0}v_{i+1,0} : 1 \leq i \leq k-1\} \cup \{v_{k,0}v_{1,0}\}$  and  $E_i = \{v_{i,j}v_{i,j+1} : 0 \leq j \leq r-1\}$  for  $i \in \{1, 2, \dots, k\}$  (see Figure 3(a)).

**Proposition 13.** *For all  $r \geq 1$  and  $k \geq 2r + 2$ , the graph  $G_n^*$  is such that*

$$\gamma_r(G_n^*) = id_r(G_n^*).$$

**Proof.** The  $k$  leaves  $v_{i,r}$  must be  $r$ -dominated by at least one codeword, and no vertex can  $r$ -dominate two leaves, so  $\gamma_r(G_n^*) \geq k$ . On the other hand, the code  $C = V_0$  represented by the black vertices in Figure 3(a) has cardinality  $k$ , and it is

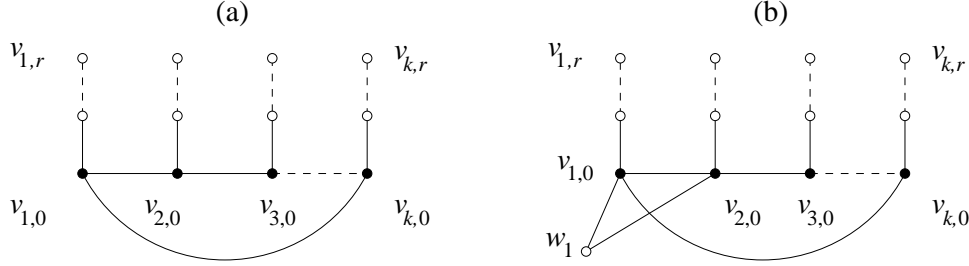


Figure 3. (a) The graph  $G_n^*$ . (b) The graph  $G_{n+1}$ . The  $k$  black vertices represent codewords.

straightforward to check that it is  $r$ -identifying. Note in particular that vertices in  $V_0$  are  $r$ -dominated by exactly  $2r + 1$  codewords (this is where the assumption  $k \geq 2r + 2$  is crucial, cf. Lemma 3(c)), and vertices  $v_{i,j} \in V_i$  are  $r$ -dominated by exactly  $2r - 2j + 1$  codewords. See also the proof of Proposition 28 for  $r$ -LD codes, which is analogous but more intricate.

So  $k \leq \gamma_r(G_n^*) \leq id_r(G_n^*) \leq k$ . ■

Second, if we want  $n$  to reach all intermediate values between  $k(r + 1)$  and  $(k + 1)(r + 1) - 1$ , we can do so by adding  $p \in \{0, \dots, r\}$  vertices to  $G_n^*$  in the following way: since  $p < \frac{k}{2}$ , we can add the set of  $p$  vertices  $W_p = \{w_1, \dots, w_p\}$  together with the set of edges  $X_p = \{w_1 v_{1,0}, w_1 v_{2,0}, w_2 v_{3,0}, w_2 v_{4,0}, \dots, w_p v_{2p-1,0}, w_p v_{2p,0}\}$ , see Figure 3(b) for  $p = 1$ . Setting  $G_{n+p} = (V_n^* \cup W_p, E_n^* \cup X_p)$ , we obtain a graph of order  $n + p$ , for which, due to the assumption  $k \geq 2r + 2$  and the remark in the proof of Proposition 13 stating that all vertices in  $G_n^*$  are  $r$ -dominated by an odd number of codewords, it is again straightforward to check that  $C = V_0$  is still a (minimum)  $r$ -ID code. Therefore we have the following.

**Proposition 14.** *For all  $r \geq 1$ ,  $k \geq 2r + 2$  and  $p \in \{0, \dots, r\}$ , the graph  $G_{k(r+1)+p}$  is such that  $\gamma_r(G_{k(r+1)+p}) = id_r(G_{k(r+1)+p})$ . As a consequence, for all  $r \geq 1$  and  $n \geq (2r + 2)(r + 1)$ , we have*

$$f_{id,\gamma}(r, n) = 0.$$

In advance of the next sections, we have the following obvious consequence.

**Corollary 15.** *For all  $r \geq 1$  and  $n \geq (2r + 2)(r + 1)$ , we have*

$$f_{id,ld}(r, n) = f_{ld,\gamma}(r, n) = 0.$$

For  $r = 1$ , the construction for Propositions 13 and 14 works for  $n \geq 8$ ; however, we have the exact value of  $f_{id,\gamma}(1, n)$  for all  $n$ , due to an alternative construction.

Proposition 12(a) has already settled the cases  $n = 3$ ,  $n = 4$ . Lemma 1(b) and Proposition 5(a) establish that any (1-twin-free) graph  $G$  with five vertices

is such that  $id_1(G) \geq 3$  and  $\gamma_1(G) \leq 2$ ; on the other hand it is easy to find graphs  $G$  of order five with  $id_1(G) = 3$  and  $\gamma_1(G) = 2$ , e.g., the path, so that  $f_{id,\gamma}(1, 5) = 1$ . For even  $n$ ,  $n \geq 6$ , and odd  $n$ ,  $n \geq 7$ , it is easy to see that Figure 4 gives graphs  $G$  such that  $id_1(G) = \gamma_1(G) = k$ .

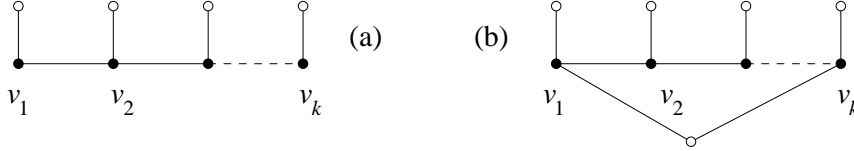


Figure 4. (a)  $n$  even. (b)  $n$  odd. The  $k$  black vertices represent codewords constituting both a minimum 1-identifying and 1-dominating code.

**Proposition 16.** (a) For all  $n \geq 6$ , we have  $f_{id,\gamma}(1, n) = 0$ ; consequently,  $f_{id,ld}(1, n) = f_{ld,\gamma}(1, n) = 0$ .

(b) For  $n \in \{3, 4, 5\}$ , we have  $f_{id,\gamma}(1, n) = 1$ .

Now how large can the difference  $id_r(G) - \gamma_r(G)$  be? By Corollary 7(iii), it is at most  $n - 2$ , obtained by graphs  $G$  with  $id_r(G) = n - 1$  and  $\gamma_r(G) = 1$ .

We first treat the case  $r = 1$ , which is easy: the star on  $n$  vertices (Lemma 8) is an example of a graph  $G$  with  $id_1(G) = n - 1$  and  $\gamma_1(G) = 1$ .

**Proposition 17.** For all  $n \geq 3$ , we have  $F_{id,\gamma}(1, n) = n - 2$ .

We now turn to the case  $r \geq 2$ . When  $n$  is odd, the answer is given by Proposition 11(d). Again, we can reach  $n - 2$  for the difference  $id_r(G) - \gamma_r(G)$ . When  $n$  is even, the study of all the graphs  $G$  of even order  $n$  such that  $id_1(G) = n - 1$  [10] shows that none of them contains a 1-universal vertex, i.e., none of them is such that  $\gamma_1(G) = 1$ , except the star; but the star cannot be the power of any graph. Therefore, for  $r \geq 2$ , there can exist no graph  $G$  with even order  $n$  such that  $id_r(G) = n - 1$  and  $\gamma_r(G) = 1$ , since the  $r$ -th power of this graph would contradict the characterization from [10]; consequently the difference  $id_r(G) - \gamma_r(G)$  is at most  $n - 3$ . On the other hand, Proposition 11(c) gives an example achieving  $n - 3$ , and we have proved the following.

**Proposition 18.** (a) For all  $r \geq 2$  and even  $n \geq 3r^2$ , we have  $F_{id,\gamma}(r, n) = n - 3$ .

(b) For all  $r \geq 2$  and odd  $n \geq 3r^2 + 1$ , we have  $F_{id,\gamma}(r, n) = n - 2$ .

## 6. IDENTIFICATION VS LOCATION-DOMINATION

We have already seen in Corollary 15 that, for  $r \geq 1$  and  $n \geq (2r + 2)(r + 1)$ , we have  $f_{id,ld}(r, n) = 0$ .

For  $r = 1$ , and for all values of  $n$ , Propositions 12(a) and 16(a) completely settle this case except when  $n = 5$ , where  $f_{id,ld}(1, 5) = 0$  thanks to the graph  $G_5$  with vertex set  $\{v_1, v_2, v_3, v_4, v_5\}$  and edge set  $\{v_1v_2, v_2v_3, v_3v_4, v_4v_1, v_1v_5\}$ ,  $id_1(G_5) = ld_1(G_5) = 3$ . Therefore we have the following.

**Proposition 19.** *For all  $n \geq 3$ , we have  $f_{id,ld}(1, n) = 0$ .*

What about  $F_{id,ld}(r, n)$ ? We can use Corollary 7(ii) and obtain that any (connected)  $r$ -twin-free graph  $G$  is such that  $id_r(G) \leq 2ld_r(G)$ . Therefore,  $-ld_r(G) \leq -\frac{id_r(G)}{2}$ , and  $id_r(G) - ld_r(G) \leq id_r(G) - \frac{id_r(G)}{2} \leq \frac{n-1}{2}$ , leading to  $id_r(G) - ld_r(G) \leq \lceil \frac{n}{2} \rceil - 1$ . On the other hand, Proposition 11(c)–(d) gives examples of graphs reaching  $\lceil \frac{n}{2} \rceil - 1$ .

**Proposition 20.** *For all  $r \geq 1$  and  $n \geq 3r^2 + 1$ , we have  $F_{id,ld}(r, n) = \lceil \frac{n}{2} \rceil - 1$ .*

**Proposition 21.** (a) *For all  $n \geq 4$ , we have  $F_{id,ld}(1, n) = \lceil \frac{n}{2} \rceil - 1$ .*

(b)  $F_{id,ld}(1, 3) = 0$ .

**Proof.** Proposition 12(a) settles the case  $n = 3$ . ■

## 7. LOCATION-DOMINATION VS DOMINATION

### 7.1. Twin-free graphs

We have already seen in Corollary 15 that, for  $r \geq 1$  and  $n \geq (2r + 2)(r + 1)$ , we have  $f_{ld,\gamma}(r, n) = 0$ . Moreover, for  $r = 1$ , Propositions 12(a) and 16(a) treat all values of  $n$  but  $n = 5$ , for which the path shows that  $f_{ld,\gamma}(1, 5) = 0$ .

**Proposition 22.** (a) *For all  $n \geq 4$ , we have  $f_{ld,\gamma}(1, n) = 0$ .*

(b)  $f_{ld,\gamma}(1, 3) = 1$ .

(c) *For all  $r \geq 1$  and  $n \geq (2r + 2)(r + 1)$ , we have  $f_{ld,\gamma}(r, n) = 0$ .*

We know, using the example of the star (Lemma 8), that  $F_{ld,\gamma}(1, n) = n - 2$ . What about  $F_{ld,\gamma}(r, n)$  for general  $r$ ?

On the one hand, Proposition 11(c)–(d) immediately gives examples proving that  $F_{ld,\gamma}(r, n) \geq \lceil \frac{n}{2} \rceil - 2$ , for all  $r \geq 2$  and  $n \geq 3r^2 + 1$ . On the other hand, the characterization [10] of the graphs  $G$  of order  $n$  such that  $id_1(G) = n - 1$  gives graphs which, apart from the star which is not the power of any graph, are such that  $ld_1(G) \leq n - 2$ . This allows to conclude that  $F_{ld,\gamma}(r, n) \leq n - 3$ . Indeed,  $F_{ld,\gamma}(r, n) = n - 2$  is possible only if a graph  $G$  of order  $n$  satisfies  $\gamma_r(G) = 1$  and  $ld_r(G) = n - 1$ , which implies  $\gamma_1(G^r) = 1$  and  $ld_1(G^r) = n - 1 = id_1(G^r)$ , contradicting the previous sentence.

**Proposition 23.** (a) *For all  $n \geq 3$ , we have  $F_{ld,\gamma}(1, n) = n - 2$ .*

- (b) For all  $r \geq 2$  and  $n \geq 3r^2 + 1$ , we have  $F_{ld,\gamma}(r, n) \geq \lceil \frac{n}{2} \rceil - 2$ .  
(c) For all  $r \geq 2$  and  $n \geq 2r + 1$ , we have  $F_{ld,\gamma}(r, n) \leq n - 3$ .

We now present a general framework using Theorem 5 in [6], and, to a lesser extent, Theorem 6 in [6], cf. Section 3, Proposition 11(a)–(b). We shall use it in the case  $r = 2$ , when this gives a lower bound for  $F_{ld,\gamma}(2, n)$  which is better than  $\lceil \frac{n}{2} \rceil - 2$ , for all  $n \geq 24$ ; for  $r = 3$ ,  $n = 30$ , this gives no improvement, and we shall informally explain why for  $r = 3$  and larger  $n$ , or for larger  $r$ , this method is doomed to fail.

Let  $m = 2p \geq 3r^2 + 1$ . We consider the Euclidean division of  $p$  by  $r$ :  $p = rQ + R$ ,  $0 \leq R \leq r - 1$ , and set  $k = Q + 1$ ,  $A = r - R$ , so that  $p = rk - A$  with  $A \in \{1, 2, \dots, r\}$ . We build  $G_m = (V_m, E_m)$  in the following way:

$$(6) \quad V_m = \{v_i : 0 \leq i \leq m - 1\},$$

$$(7) \quad E_m = \{v_i v_{i+j \bmod m} : 0 \leq i \leq m - 1, j \in J = \{1, 2, \dots, k - A - 1, k\}\}.$$

The graph  $G_m$  can be seen as a cycle with chords added according to the set  $J$ , where every vertex plays the same role, see Figure 5(a). Theorem 5 from [6] states that the  $r$ -th power of  $G_m$  is the graph  $G_m^*$  of Lemma 9, with  $m = 2p$ . We now need to be more specific with respect to  $r$ .

In the case  $r = 2$ , we can improve on Proposition 23(b) and build, for  $n$  large enough, graphs of order  $n$  proving that  $\frac{F_{ld,\gamma}(2,n)}{n}$  tends to  $\frac{5}{8}$  when  $n$  goes to infinity. The first step is the study of graphs with order a multiple of eight.

**Proposition 24.** *For  $n = 8t \geq 24$ , there exists a 2-twin-free graph  $G_n$  of order  $n$ , with  $\gamma_2(G_n) = 2$  and  $ld_2(G_n) = 5t - 1$ .*

**Proof.** Let  $m = 6t = 2p \geq 18$  and  $p = 2k - A$  with  $A \in \{1, 2\}$ . Because  $A \in \{1, 2\}$  and  $p \geq 9$ , we have  $p \geq 3A + 3 \Rightarrow p - A - 1 \geq \frac{2p}{3} = 2t \Rightarrow 2k - 2A - 1 \geq 2t$ , and finally

$$(8) \quad k - A \geq 2t - (k - A - 1).$$

When  $p = 9$ , then  $A = 1$  and  $\frac{p}{3} \geq A + 2$ , which also holds whenever  $p \geq 12$ . Therefore,  $\frac{2k-A}{3} \geq A + 2 \Rightarrow k \geq 2A + 3 \Rightarrow 3k - 3A - 3 \geq 2k - A \Rightarrow k - A - 1 \geq \frac{p}{3}$ , and finally

$$(9) \quad 2t + k - A - 1 = \frac{2p}{3} + k - A - 1 \geq p.$$

These two inequalities, (8) and (9), will be used later on. We have already seen that the graph  $G_m = (V_m, E_m)$  defined by (6) and (7) is such that the square power of  $G_m$  is the graph  $G_m^*$  of Lemma 9, with  $m = 2p$ . This means that

$B_{G_m,2}(v_i) = V_m \setminus \{v_{i+p \bmod m}\}$ , for all  $i \in \{0, \dots, m-1\}$ ; in other words, from any  $v_i$  we can go in two steps to any vertex in  $V_m$  but  $v_{i+p \bmod m}$ . Next, we diverge from Theorem 6 in [6]: we add  $\frac{m}{3} = 2t$  vertices  $Z_j$ ,  $0 \leq j \leq 2t-1$ ; every  $Z_j$  is linked to three vertices in  $V_m$ , namely  $v_j$ ,  $v_{j+2t}$  and  $v_{j+4t}$ . By setting  $G_n = (V_n, E_n)$  with  $V_n = V_m \cup \{Z_j : 0 \leq j \leq 2t-1\}$  and  $E_n = E_m \cup \{Z_j v_j, Z_j v_{j+2t}, Z_j v_{j+4t} : 0 \leq j \leq 2t-1\}$ , we obtain a graph of order  $8t$ , see Figure 5(b).

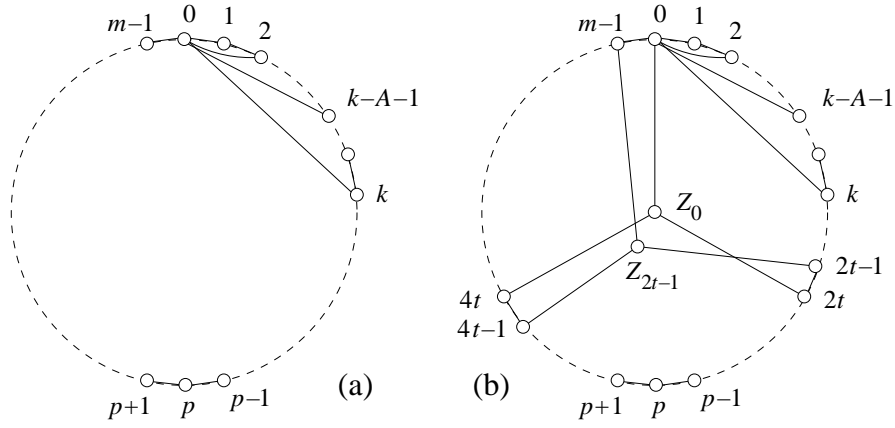


Figure 5. (a) The graph  $G_m$ . (b) The graph  $G_n$  ( $r = 2$ ). Not all vertices nor edges are represented. Only the indices of the vertices  $v_i$  are given.

We claim that

- (a)  $B_{G_n,2}(Z_j) = V_m \setminus \{Z_\ell : 0 \leq \ell \leq 2t-1, \ell \neq j\}$ , for all  $j \in \{0, \dots, 2t-1\}$ ;
- (b)  $B_{G_n,2}(v_i) = V_m \setminus \{v_{i+p \bmod m}\}$ , for all  $i \in \{0, \dots, m-1\}$ .

(a) That we cannot go in two steps from  $Z_j$  to  $Z_\ell$  is obvious since  $B_{G_n,1}(Z_j) \cap B_{G_n,1}(Z_\ell) = \emptyset$ . Note already that this could not be directly transposed to the case  $r \geq 3$ , since then the existence of paths such as  $Z_j v_j, v_j v_{j+1}, v_{j+1} Z_{j+1}$  would lead to a contradiction; see the discussion below for  $r = 3$ .

Next, we show that we can go in two moves from any  $Z_j$  to any  $v_i$ ; because of the symmetries of the graph, we need to do it only for, say,  $Z_0$ , and the vertices going from  $v_0$  to  $v_p$ . Thanks to the edge  $Z_0 v_0$ ,  $Z_0$  can reach  $v_1, v_2, \dots, v_{k-1-A}$  (and  $v_k$ , but we do not need it) in two moves; thanks to the edge  $Z_0 v_{2t}$ ,  $Z_0$  can also reach  $v_{2t+1}, \dots, v_{2t+k-1-A}$  (and  $v_{2t+k}$ ), as well as  $v_{2t-1}, \dots, v_{2t-(k-1-A)}$  (and  $v_{2t-k}$ ). Using (8), we can see that in the worst case,  $v_{2t-(k-1-A)} = v_{k-A}$  and all the vertices between  $v_0$  and  $v_{2t}$  can be reached, including in particular  $v_{k-1-A}$  and  $v_{k-A}$ . In other words, the areas reached in one move by going clockwise from  $v_0$  or anticlockwise from  $v_{2t}$  do meet. Similarly, by (9), we have in the worst case  $v_{2t+k-A-1} = v_p$  and all the vertices between  $v_{2t+1}$  and  $v_p$  can be reached in two moves from  $Z_0$ . Claim (a) is proved.

(b) The proof is the same as for the graph  $G_m$ ; we just have to check that the additional vertices  $Z_j$  and their edges do not make it possible to go in two moves from  $v_0$  to  $v_p$  (this is sufficient for reasons of symmetry).

Claims (a) and (b) show that  $G_n$  is 2-twin-free; they also show that the square power of  $G_n$  is the following graph.  $(G_n)^2$  has vertex set  $V_n$  and edge set all the possible edges except the edges  $v_i v_{i+p \bmod m}$ ,  $0 \leq i \leq p$ , and  $Z_j Z_\ell$ ,  $\{j, \ell\} \subset \{0, \dots, 2t-1\}$ ,  $j \neq \ell$ , see Figure 6.

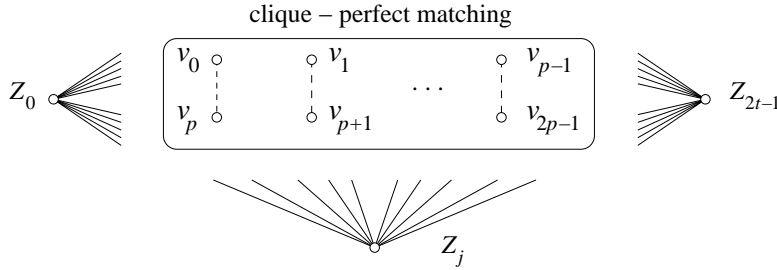


Figure 6. A partial representation of  $(G_n)^2$ . Dotted lines are non-edges.

Now what are  $\gamma_1((G_n)^2)$  and  $ld_1((G_n)^2)$  (or equivalently,  $\gamma_2(G_n)$  and  $ld_2(G_n)$ )? Obviously,  $\gamma_1((G_n)^2) = 2$ . Next, the argument of Case (b) of the proofs of Lemmas 9 and 10 can be used to show that it is necessary to take half of the vertices  $v_i$  in  $V_m$  for a 2-LD code. Then, for  $j \neq \ell$ , we have  $B_{(G_n)^2,1}(Z_j) \Delta B_{(G_n)^2,1}(Z_\ell) = \{Z_j, Z_\ell\}$ , which implies that we have to take all vertices  $Z_j$  but one as codewords, together with  $p$  vertices in  $V_m$ , and this is sufficient,  $ld_1((G_n)^2) = p + (2t-1) = 5t-1$ . ■

In order to reach the values of  $n$  other than multiples of eight, we might consider  $m = 6t+2$  or  $6t+4$  instead of  $m = 6t$ , but it is more efficient to stick to  $m = 6t$  and simply add a number of vertices  $Z_j$  smaller (by a number between 1 and 7) than  $2t$ . From  $m = 6t \geq 18$  we constructed a graph with  $8t$  vertices; now, we start from  $6(t+1)$ , and, instead of building a graph with order  $6(t+1) + 2(t+1) = 8(t+1)$ , we build a graph with  $6(t+1) + [2(t+1) - q] = 8t + (8-q)$  vertices, by adding only  $2(t+1) - q$  vertices  $Z_j$ , with  $1 \leq q \leq 7$ . The resulting graph has its 2-domination number equal to 1 (in the unique case when  $t = 3$ ,  $q = 7$  and we add only one vertex,  $Z_0$ ) or 2; any minimum 2-LD code has size  $3(t+1) + [2(t+1) - q - 1] = 5(t+1) - q - 1$ , including when  $2(t+1) - q = 1$ .

So, letting  $i = 8-q$ ,  $1 \leq i \leq 7$ , we obtain graphs  $G_n$  with order  $n = 8t+i$  and  $ld_2(G_n) = 5t+i-4$  (the borderline case  $i = 0$ , i.e., dropping eight vertices  $Z_j$ , logically leads to a worse result, namely  $5t-4$ , than if we start from  $6t$  to reach  $8t$ , in which case we have just seen that we obtain  $5t-1$ ). Since the 2-domination number of these graphs is at most 2, we have the following result.

**Proposition 25.** (a) *Let  $n = 8t \geq 24$ . Then  $F_{ld,\gamma}(2, n) \geq 5t - 3$ .*  
 (b) *Let  $n = 8t + i \geq 24$ , with  $1 \leq i \leq 7$ . Then  $F_{ld,\gamma}(2, n) \geq 5t + i - 6$ .*

The least favorable case is when  $i = 1$ , which leads to

$$(10) \quad F_{ld,\gamma}(2, n) \geq \frac{5n - 45}{8}.$$

The case  $m = 6t$  works best because we have a miraculously large number of  $Z_j$ 's, namely  $2t$ , which is advantageous when we look for a “large” LD-code, since we have to take all of them but one in a 1-LD code in  $(G_n)^2$ . If we consider  $m = 6t + 2$  or  $m = 6t + 4$ , we cannot take as many vertices  $Z_j$ ; yet, if we can take a number of  $Z_j$ 's which is only a fraction  $\frac{m}{\beta}$  with  $\beta > 3$ , then we obtain a graph  $G_n$  with order  $n = m + \frac{m}{\beta}$  and  $ld_2(G_n) = ld_1((G_n)^2) = \frac{m}{2} + \frac{m}{\beta} - 1$ , leading to the ratio  $\frac{F_{ld,\gamma}(2,n)}{n}$  greater than  $\frac{ld_2(G_n)-2}{n} \approx \frac{\beta+2}{2\beta+2}$ , which is not as good as  $\frac{5}{8}$ .

For  $r = 3$ , we consider  $m = 30 = 2p = 2(3k - A)$ , leading to  $p = 15$ ,  $k = 6$ ,  $A = 3$ ,  $J = \{1, 2, 6\}$ . To the graph  $G_{30}$  defined by (6) and (7), whose third power, by [6, Theorem 5], is  $G_{30}^*$ , we add first the vertex  $Z_0$  together with the edges  $Z_0v_0$ ,  $Z_0v_{10}$  and  $Z_0v_{20}$ . Then we add the vertex  $Z_1$ . Because we want no path of the type  $Z_0v_0v_iZ_1$  for some  $i$ , among the vertices  $\{v_1, \dots, v_{15}\}$ ,  $Z_1$  cannot be linked to  $v_1$ ,  $v_2$  nor  $v_6$ ; because of  $Z_0v_{10}$ , this also excludes  $v_4$ ,  $v_8$ ,  $v_9$ ,  $v_{11}$  and  $v_{12}$  as neighbours of  $Z_1$ . Finally we can take, e.g.,  $Z_1$  with the edges  $Z_1v_3$ ,  $Z_1v_{13}$ ,  $Z_1v_{23}$ ,  $Z_2$  with  $Z_2v_7$ ,  $Z_2v_{17}$ ,  $Z_2v_{27}$ , and no more. Exactly as before, this leads to a graph  $G_{33}$  whose third power is a graph of the type given by Figure 6, with 33 vertices,  $ld_3(G_{33}) = 15 + (3 - 1) = 17$ , 3-domination number equal to 2, and  $ld_3(G_{33}) - \gamma_3(G_{33}) = 15 = \lceil \frac{33}{2} \rceil - 2$ , i.e., not better than Proposition 23(b).

It is impossible to take fewer than three neighbours for each vertex  $Z_\ell$ . On the other hand, as discussed above when studying the possible neighbours of  $Z_1$ , if  $v_0$  is the neighbour of  $Z_0$ , the “first” neighbour of  $Z_1$  will be  $v_i$  with  $i \geq k - A$ , for  $Z_2$  it will be  $v_j$  with  $j \geq i + (k - A) \geq 2(k - A)$ , ... So, roughly speaking, the total number of possible neighbours for the vertices  $Z_\ell$  is at most

$$(11) \quad \frac{m}{k - A} = \frac{m}{\frac{p}{3} - \frac{2A}{3}} = \frac{6p}{p - 2A},$$

and therefore, the number of vertices  $Z_\ell$  is at most  $\frac{2p}{p-2A}$ . When  $p = 15$ , this leads to at most three vertices  $Z_\ell$ , and things only worsen when  $m$ , hence  $p$  increases. Anyway, with only three additional vertices  $Z_\ell$ , all we can reach is a graph  $G_n$  with  $n = m + 3$  vertices and

$$ld_3(G_n) - \gamma_3(G_n) = (p + (3 - 1)) - 2 = p = \frac{n - 3}{2}.$$



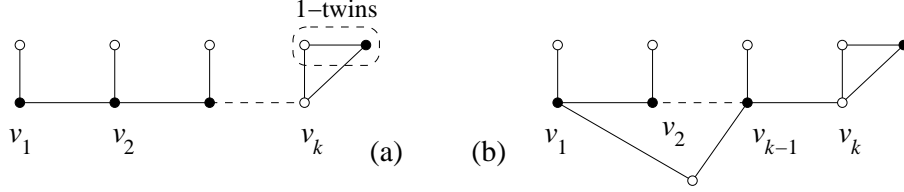


Figure 7. (a)  $n$  odd:  $n = 2k + 1 \geq 5$ . (b)  $n$  even:  $n = 2k + 2 \geq 8$ . The  $k$  black vertices represent codewords constituting both a minimum 1-dominating and 1-locating-dominating code.

When we place ourselves again in the general case for  $r$ , we must have the “first” neighbour of  $Z_1$ , say  $v_i$ , such that  $i \geq (r - 2)(k - A)$ , in order to avoid a path of length  $r$  between  $Z_0$  and  $Z_1$ , and equalities (11) now read

$$\frac{m}{(r - 2)(k - A)} = \frac{m}{(r - 2) \left( \frac{p}{r} - \frac{(r-1)A}{r} \right)} = \frac{2rp}{(r - 2)(p - (r - 1)A)}.$$

Even with  $p = \frac{3r^2}{2}$  and  $A = r$ , this can lead only to approximately  $\frac{3r^3}{(r-2)\frac{r^2}{2}} \approx 6$ , hence at most two vertices  $Z_\ell$ , and again, things only worsen when  $p$  increases. Therefore, other constructions should be invented—that is, if improvements do exist in Proposition 23(b).

**Open Problem.** Reduce the gap between lower and upper bounds for  $F_{ld,\gamma}(r, n)$ , when  $r > 1$ .

## 7.2. Graphs with twins

The study of  $F_{ld,\gamma}^{tw}(r, n)$  is trivial, because of the clique, or complete graph on  $n$  vertices,  $K_n$ , which obviously contains  $r$ -twins, and is such that  $\gamma_r(K_n) = 1$  and  $ld_r(K_n) = n - 1$ .

We are going to prove that (i) for  $r = 1$  and  $n \in \{2, 5\}$  or  $n \geq 7$  (Proposition 26) and (ii) for any  $r \geq 2$  and  $n$  large enough (Proposition 28), we have  $f_{ld,\gamma}^{tw}(r, n) = 0$ .

**Proposition 26.** (a) For  $n = 2$ ,  $n = 5$  and all  $n \geq 7$ , we have  $f_{ld,\gamma}^{tw}(1, n) = 0$ ;  
 (b) For  $n \in \{3, 4, 6\}$ , we have  $f_{ld,\gamma}^{tw}(1, n) = 1$ .

**Proof.** We already know by Proposition 12(a) that

$$f_{ld,\gamma}^{tw}(1, 2) = 0; \quad f_{ld,\gamma}^{tw}(1, 3) = 1; \quad f_{ld,\gamma}^{tw}(1, 4) = 1.$$

For  $n = 6$ , Lemma 1(a) and Proposition 5(a) state that for any connected graph  $G$  with six vertices,  $ld_1(G) \geq 3$  and  $\gamma_1(G) \leq 3$ ; but a study of the graphs

with 1-twins shows that for them,  $\gamma_1(G) \leq 2$  (alternatively, one can use the characterization of the graphs with even order and 1-domination number half their order [13, p.42], [9, 17]), and eventually  $f_{ld,\gamma}^{tw}(1,6) = 1$ . For  $n = 5$  and  $n \geq 7$ , we consider the graphs in Figure 7, obtained from the graphs in Figure 4 by a slight modification, intended to create one pair of 1-twins. The study of these graphs is straightforward and gives the desired result. ■

We now turn to the case  $r \geq 2$  (even if the results below are also valid for  $r = 1$ ); first, we give an analogue of Proposition 13 for  $r$ -LD codes. We take the graphs  $G_n^* = (V_n^*, E_n^*)$  represented in Figure 3(a) and described just before Proposition 13, and transform them into graphs  $G_{n+1}^y$  by applying the same type of modification just performed for  $r = 1$ . We simply add one vertex  $y$  which is the  $r$ -twin of  $v_{k,r}$ , see Figure 8(a). The order of  $G_{n+1}^y$  is  $n + 1 = k(r + 1) + 1$ .

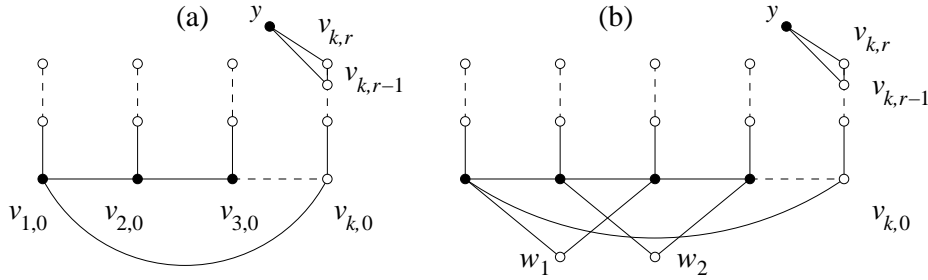


Figure 8. The  $k$  black vertices represent codewords.

**Observation 27.** *Because here we deal with  $r$ -LD codes, not  $r$ -ID codes like in Proposition 13, the bound for  $k$  could be lowered, down to  $k \geq 2r$ . For simplicity and because this does not represent a significant improvement, we keep the bound  $k \geq 2r + 2$ .*

**Proposition 28.** *For all  $r \geq 2$  and  $k \geq 2r + 2$ , the graphs  $G_{n+1}^y$  are such that  $\gamma_r(G_{n+1}^y) = ld_r(G_{n+1}^y)$ .*

**Proof.** Obviously,  $k \leq \gamma_r(G_{n+1}^y)$  (and  $k \leq ld_r(G_{n+1}^y)$ ), and the code  $C = \{v_{1,0}, v_{2,0}, \dots, v_{k-1,0}, y\}$ , with  $k$  codewords, is an  $r$ -D code. We are going to prove that  $C$  is also  $r$ -LD. In spite of the fact that all we have to check is that any two distinct non-codewords are  $r$ -separated by  $C$ , the proof is a little more intricate than the proof of Proposition 13 for  $r$ -ID codes, because of the “missing” codeword  $v_{k,0}$ , so we present it in detail.

(a) The non-codewords  $v_{k,j}$ ,  $0 \leq j \leq r$ , are the only non-codewords  $r$ -dominated by  $y$ , so they all are  $r$ -separated by  $y \in C$  from other non-codewords; each of them is  $r$ -dominated by a different number of codewords, because  $k$  is large enough, and therefore they are pairwise  $r$ -separated by  $C$ .

(b) Consider any two non-codewords  $v_{i,j}$ ,  $v_{i,t}$  on the same string  $i$ ,  $1 \leq i \leq k-1$ ,  $\{j, t\} \subseteq \{1, \dots, r\}$ ,  $j < t$ ; then  $v_{i,j}$  is  $r$ -dominated by at least one codeword more than  $v_{i,t}$  (it would be at least two if we had all  $k$  elements of the cycle in the code), and so these two vertices are  $r$ -separated by  $C$ .

(c) Let us consider two non-codewords  $v_{i,j}$  and  $v_{s,t}$  belonging to two different strings, other than the  $k$ -th string:  $\{i, s\} \subset \{1, \dots, k-1\}$ ,  $i \neq s$ ,  $\{j, t\} \subseteq \{1, \dots, r\}$ ; without loss of generality, we may assume that  $j \leq t$ .

If  $j < t$ , then again,  $v_{i,j}$  is  $r$ -dominated by at least one codeword more than  $v_{s,t}$ ; so from now on, we assume that  $j = t$ . The set of codewords  $r$ -dominating  $v_{i,j}$  has cardinality  $2r - 2j + 1$  or  $2r - 2j$ , and consists, with computations performed modulo  $k$ , of  $v_{i,0}$ ,  $v_{i-1,0}, \dots, v_{i-r+j,0}$ ,  $v_{i+1,0}, \dots, v_{i+r-j,0}$ , or of the same set without  $v_{k,0}$ , which is not a codeword. In both cases, it cannot be the same as the set of codewords  $r$ -dominating  $v_{s,j}$ .

We have just proved that  $C$   $r$ -separates the non-codewords  $v_{k,j}$  between themselves and from the other non-codewords; the non-codewords belonging to the same string; the non-codewords belonging to different strings. Therefore  $C$  is an  $r$ -LD code. ■

Now, like in Proposition 14, we want to reach all intermediate values between  $k(r+1) + 1$  and  $(k+1)(r+1)$ . We do so by adding a set  $W_p = \{w_1, \dots, w_p\}$  of  $p$  vertices,  $p \in \{0, \dots, r\}$ . However, if we proceed exactly as for Proposition 14 by creating the edge set  $X_p = \{w_1v_{1,0}, w_1v_{2,0}, w_2v_{3,0}, w_2v_{4,0}, \dots, w_pv_{2p-1,0}, w_pv_{2p,0}\}$  but now considering the code  $C = \{v_{1,0}, \dots, v_{k-1,0}\} \cup \{y\}$ , we might have one or two pairs of vertices not  $r$ -separated by  $C$ . We show one such pair ( $v_{4,1}$  and  $w_2$ ) in Figure 9 when  $r = 4$ ,  $k = 11$ ,  $p = 4$ ; more generally, this may occur whenever  $r$  is even. Moreover, a symmetrical situation appears when  $k - r$  is odd, see the same Figure with  $w_4$  and  $v_{7,1}$ . The existence of both pairs is due to the fact that  $v_{k,0} \notin C$ .

So we choose another way of linking the vertices  $w_i$  to the vertices  $v_{s,0}$ :  $X_p = \{w_1v_{1,0}, w_1v_{3,0}, w_2v_{2,0}, w_2v_{4,0}, \dots\}$ , see Figure 8(b) for  $p = 2$ . Setting  $G_{n+1+p}^y = (V_n^* \cup \{y\} \cup W_p, E_n^* \cup \{yv_{k,r}, yv_{k,r-1}\} \cup X_p)$ , we obtain a graph of order  $n+1+p = k(r+1) + 1 + p$ .

**Proposition 29.** *For all  $r \geq 2$ ,  $k \geq 2r + 2$  and  $p \in \{0, \dots, r\}$ , the graph  $G_{k(r+1)+1+p}^y$  is such that  $\gamma_r(G_{k(r+1)+1+p}^y) = ld_r(G_{k(r+1)+1+p}^y)$ .*

**Proof.** Again, we take  $C = \{v_{i,0} : 1 \leq i \leq k-1\} \cup \{y\}$ . Using anew the proof of the previous proposition, we can see that we have only to prove in addition the following two assertions about the  $w_i$ 's.

(a) If  $p \geq 2$ , any two non-codewords  $w_i$  and  $w_s$ ,  $\{i, s\} \subseteq \{1, 2, \dots, p\}$ ,  $i < s$ , are  $r$ -separated by  $C$ . If  $w_i$  is linked to  $v_{\ell,0}$  and  $v_{\ell+2,0}$ , then the set of codewords  $r$ -dominating  $w_i$  has size  $3 + 2(r-1)$  or  $2 + 2(r-1)$ , and consists (with computations

modulo  $k$ ) of  $v_{\ell,0}, v_{\ell+1,0}, v_{\ell+2,0}, v_{\ell-1,0}, \dots, v_{\ell-r+1,0}, v_{\ell+3,0}, \dots, v_{\ell+2+r-1,0}$ , or of the same set without  $v_{k,0}$ . In both cases, it cannot be the same as the set of codewords  $r$ -dominating  $w_s$ .

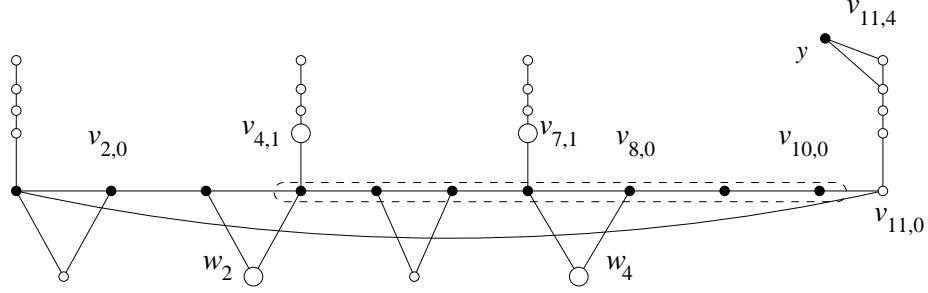


Figure 9.  $r = 4$ ,  $n = 60$ . The eleven black vertices represent codewords. Not all strings are shown. The vertices  $w_2$  and  $v_{4,1}$  are not 4-separated by  $C$ ; neither are  $w_4$  and  $v_{7,1}$ , which are both 4-dominated by  $v_{i,0}$ ,  $4 \leq i \leq 10$ , as indicated by the dotted-line box.

(b) Two non-codewords  $w_i$ ,  $i \in \{1, \dots, p\}$ , and  $v_{s,t}$ ,  $1 \leq s \leq k-1$ ,  $1 \leq t \leq r$ , are  $r$ -separated by  $C$ . If  $w_i$  is linked to  $v_{\ell,0}$  and  $v_{\ell+2,0}$ , the most crucial cases are when  $s \in \{\ell, \ell+1, \ell+2\}$  and  $t = 1$ , but even here,  $w_i$  is  $r$ -dominated by more codewords than  $v_{s,1}$  (note that this “W-construction” would not have worked for  $r$ -ID codes, because then  $w_i$  and  $v_{\ell+1,0}$  would not be  $r$ -separated by the code). ■

**Corollary 30.** *For all  $r \geq 2$  and  $n \geq (2r+2)(r+1)+1$ , we have  $f_{ld,\gamma}^{tw}(r, n) = 0$ .*

## 8. CONCLUSION

In the following tables, we recapitulate our results on the different minimum and maximum differences between cardinalities of minimum dominating, locating-dominating or identifying codes in connected graphs, first for  $r = 1$ , then for  $r \geq 2$ . For  $r = 1$ , we have exact values for all  $n$  and all functions.

$n$	2	3	4	5	6	$\geq 7$	Proposition
$f_{id,\gamma}(1, n)$	$\times$	1	1	1	0	0	16
$F_{id,\gamma}(1, n)$	$\times$	1	2	3	4	$n-2$	17
$f_{id,ld}(1, n)$	$\times$	0	0	0	0	0	19
$F_{id,ld}(1, n)$	$\times$	0	1	2	2	$\lceil \frac{n}{2} \rceil - 1$	21
$f_{ld,\gamma}(1, n)$	$\times$	1	0	0	0	0	22(a)–(b)
$F_{ld,\gamma}(1, n)$	$\times$	1	2	3	4	$n-2$	23(a)
$f_{ld,\gamma}^{tw}(1, n)$	0	1	1	0	1	0	26
$F_{ld,\gamma}^{tw}(1, n)$	0	1	2	3	4	$n-2$	(clique)

For  $r \geq 2$ , most results are valid for  $n$  large (typically,  $n$  is in  $r^2$ ).

$id$ vs $\gamma$	$\forall r \geq 2, f_{id,\gamma}(r, n) = 0$ [Proposition 14] $n$ even, $\forall r \geq 2, F_{id,\gamma}(r, n) = n - 3$ [Proposition 18(a)] $n$ odd, $\forall r \geq 2, F_{id,\gamma}(r, n) = n - 2$ [Proposition 18(b)]
$id$ vs $ld$	$\forall r \geq 2, f_{id,ld}(r, n) = 0$ [Corollary 15] $\forall r \geq 2, F_{id,ld}(r, n) = \lceil \frac{n}{2} \rceil - 1$ [Proposition 20]
$ld$ vs $\gamma$ (twin-free graphs)	$\forall r \geq 2, f_{ld,\gamma}(r, n) = 0$ [Corollary 15, Proposition 22(c)] $F_{ld,\gamma}(2, n) \geq \frac{5n-45}{8} \approx \frac{5n}{8}$ [Prop. 25(b), case $i = 1$ , ineq. (10)] $\forall r \geq 3, F_{ld,\gamma}(r, n) \geq \lceil \frac{n}{2} \rceil - 2$ [Proposition 23(b)] $\forall r \geq 2, F_{ld,\gamma}(r, n) \leq n - 3$ [Proposition 23(c)]
$ld$ vs $\gamma$ (with twins)	$\forall r \geq 2, f_{ld,\gamma}^{tw}(r, n) = 0$ [Corollary 30] $\forall r \geq 2, F_{ld,\gamma}^{tw}(r, n) = n - 2$ (clique)

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