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# The expressiveness of quasiperiodic and minimal shifts of finite type\*

Bruno Durand and Andrei Romashchenko

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## Abstract

We study multidimensional *minimal* and *quasiperiodic* shifts of finite type. We prove for these classes several results that were previously known for the shifts of finite type in general, without restriction. We show that some quasiperiodic shifts of finite type admit only non-computable configurations; we characterize the classes of Turing degrees that can be represented by quasiperiodic shifts of finite type. We also transpose to the classes of minimal/quasiperiodic shifts of finite type some results on subdynamics previously known for the effective shifts without restrictions: every *effective* minimal (quasiperiodic) shift of dimension  $d$  can be represented as a projection of a subdynamics of a minimal (respectively, quasiperiodic) shift of *finite type* of dimension  $d + 1$ .

**Keywords:** minimal SFT; quasiperiodicity; tilings

## 1 Introduction

In this paper we study the multi-dimensional *shifts*, i.e., the shift-invariant and topologically closed sets of configurations in  $\mathbb{Z}^d$  over a finite alphabet. The *minimal shifts* are those shifts in which all configurations contain exactly the same finite patterns; the *quasiperiodic shifts* are the shifts where each configuration contains all of its finite patterns infinitely often, at least once in every large enough region.

Two classes of shifts play a prominent role in symbolic dynamics, in language theory, and in the theory of computability: the *shifts of finite type* (obtained by forbidding a finite number of finite patterns) and the *effective shifts* (obtained by forbidding a computable set of finite patterns).

The class of shifts of finite type is known to be very rich — even a finite set of simple local rules can induce a rather sophisticated global structure. It is known that some (non-empty) multi-dimensional shifts of finite type admit

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only aperiodic (see [1]) or even only non-computable ([3, 4]) configurations. The value of the combinatorial entropy of a shift of finite type can be any right computable real number  $h \geq 0$ , see [17]. Recent results (see [15, 19, 20]) show that if we look at the projective subdynamics of a shift of finite type of some dimension  $d > 1$ , we can obtain any effective shift of a dimension below  $d$ .

The proofs of the results mentioned above are based on embedding some computation in the structure of a shift of finite type. These embeddings use rather tricky combinatorial gadgets that permit marrying the ‘computational’ techniques with the framework of symbolic dynamics. Though the implied computational tricks may look natural to a computer scientist, they give in the end constructions that are rather complex and artificial if we look at them as objects of dynamical systems theory.

Thus, a natural question is whether the phenomena mentioned above hold for ‘simpler’ and ‘more natural’ shifts of finite type. A formal version of this problem is to transpose some results known for shifts of finite type *in general* to some sort of *primary* shifts. More technically, in this paper we deal with *minimal*, *quasiperiodic*, and *transitive* shifts. Our results can be subdivided into three groups:

- We prove that some quasiperiodic shifts of finite type admit only non-computable configurations. Moreover, we characterize the classes of Turing degrees that may correspond to a quasiperiodic shift of finite type.
- We transpose the characterization by Hochman and Meyerovitch of the entropies of multidimensional shifts of finite type to the class of transitive shifts. (The same result was recently proven by Gangloff and Sablik in [25] with a different technique.)
- We extend to the classes of minimal/quasiperiodic shifts of finite type some known results on subdynamics: every *effective* minimal (quasiperiodic) shift of dimension  $d$  can be represented as a projection of a subdynamics of a minimal (respectively, quasiperiodic) shift *of finite type* of dimension  $d + 1$ , which answers positively a question by E. Jeandel, [22]

All constructions in this paper involve the technique of self-simulating tilings developed in [19] (see also variants of this technique in [24, 28]).

## 1.1 Notation and basic definitions

**Shifts.** Let  $\Sigma$  be a finite set (an alphabet). Fix an integer  $d > 0$ . A  $\Sigma$ -*configuration* (or just a *configuration* if  $\Sigma$  is clear from the context) on  $\mathbb{Z}^d$  is a mapping  $\mathbf{f} : \mathbb{Z}^d \rightarrow \Sigma$ , i.e., a coloring of  $\mathbb{Z}^d$  by “colors” from  $\Sigma$ . A  $\mathbb{Z}^d$ -*shift* (or just a *shift*) is a set of configurations that is (i) translation invariant (with respect to the translations along each coordinate axis), and (ii) closed in Cantor’s topology.

If a  $\mathbb{Z}^d$ -shift  $\mathcal{S}_1$  is subset of a  $\mathbb{Z}^d$ -shift  $\mathcal{S}_2$ , we say that  $\mathcal{S}_1$  is a *subshift* of  $\mathcal{S}_2$ . The entire space  $\Sigma^{\mathbb{Z}^d}$  is itself a shift (where no pattern is forbidden) and is called *the full shift*. Thus, every  $\mathbb{Z}^d$ -shift is a subshift of the full shift.

A *pattern* is a mapping from a finite subset of  $\mathbb{Z}^d$  to  $\Sigma$  (a coloring of a finite set of  $\mathbb{Z}^d$ ); this set is called the support of the pattern. We say that a pattern  $P$  *appears* in a configuration  $\mathbf{f}(\bar{x})$  if for some  $\bar{c} \in \mathbb{Z}^d$  the pattern  $P$  coincides with the restriction of the shifted configuration  $\mathbf{f}_{\bar{c}}(\bar{x}) := \mathbf{f}(\bar{x} + \bar{c})$  to the support of this pattern. A pattern that appears in some configuration of a shift is called *globally admissible*.

Every shift is determined by the corresponding set of forbidden finite patterns  $\mathcal{F}$  (a configuration belongs to the shift if and only if no patterns from  $\mathcal{F}$  appear in this configuration).

A shift is called *effective* (or *effectively closed*) if it can be defined by a computably enumerable set of forbidden patterns. A shift is called a *shift of finite type* (SFT) if it can be defined by a finite set of forbidden patterns.

**Wang tilings.** A special class of two-dimensional SFT is defined in terms of *Wang tiles*. In this case, we interpret the alphabet  $\Sigma$  as a set of *tiles*, i.e., a set of unit squares with colored sides, assuming that all colors belong to some finite set  $C$  (we assign one color to each side of a tile, so technically  $\Sigma$  is a subset of  $C^4$ ). A (valid) *tiling* is a set of all configurations  $\mathbf{f} : \mathbb{Z}^2 \rightarrow \Sigma$  where every two neighboring tiles match, i.e., share the same color on adjacent sides. Wang tiles are powerful enough to simulate any SFT in a very strong sense: for each SFT  $\mathcal{S}$  there exists a set of Wang tiles  $\tau$  such that the set of all  $\tau$ -tilings is isomorphic to  $\mathcal{S}$ . In this paper we mainly use the formalism of tilings since Wang tiles are better adapted for explaining our techniques of self-simulation.

**Dynamics and subdynamics.** Every shift  $\mathcal{S} \subset \Sigma^{\mathbb{Z}^d}$  can be interpreted as a dynamical system. Indeed, there are  $d$  translations along the coordinate axes, and each of these translations maps  $\mathcal{S}$  to itself. Therefore, the group  $\mathbb{Z}^d$  naturally acts on  $\mathcal{S}$ .

Let  $\mathcal{S}$  be a shift on  $\mathbb{Z}^d$  and  $L$  be  $k$ -dimensional sub-lattice in  $\mathbb{Z}^d$  (i.e.,  $L$  must be an additive subgroup of  $\mathbb{Z}^d$  that is isomorphic to  $\mathbb{Z}^k$ ). Then the  *$L$ -projective subdynamics*  $\mathcal{S}_L$  of  $\mathcal{S}$  is the set of configurations of  $\mathcal{S}$  restricted on  $L$ . The  $L$ -projective subdynamics of a  $\mathbb{Z}^d$ -shift can be understood as a  $\mathbb{Z}^k$ -shift (note that  $L$  naturally acts on  $\mathcal{S}_L$ ). In particular, for every  $d' < d$  we have a  $\mathbb{Z}^{d'}$ -projective subdynamics on the shift  $\mathcal{S}$ , generated by the lattice on the first  $d'$  coordinate axis.

A configuration  $\mathbf{x}$  is called *recurrent* if every pattern that appears in  $\mathbf{x}$  at least once, must then appear in this configuration infinitely often.

A shift  $\mathcal{S}$  is called *transitive* if there exists a configuration  $x \in \mathcal{S}$  that contains every finite pattern that appears in at least one configuration  $y \in \mathcal{S}$ .

**Quasiperiodicity and minimality.** A configuration  $\mathbf{x}$  is called *quasiperiodic* (or *uniformly recurrent*) if every pattern  $P$  that appears in  $\mathbf{x}$  at least once must appear in every large enough cube  $Q$  in  $\mathbf{x}$ . Note that every periodic configuration is also quasiperiodic. A *quasiperiodic* shift is a shift that contains only quasiperiodic configurations.

Given a configuration  $\mathbf{x}$ , a *function of a quasiperiodicity* for  $\mathbf{x}$  is a mapping  $\varphi : \mathbb{N} \rightarrow \mathbb{N} \cup \{\infty\}$  such that every finite pattern of size (diameter)  $n$  either never appears in  $\mathbf{x}$  or appears in every cube of size  $\varphi(n)$  in  $\mathbf{x}$  (see [8]). We assume  $\varphi(n) = \infty$  if some pattern  $P$  of size  $n$  appears in  $\mathbf{x}$  but there exist arbitrarily large areas in  $\mathbf{x}$  that are free of  $P$ . By definition, for a quasiperiodic  $\mathbf{x}$  we have  $\varphi(n) < \infty$  for all  $n$ . We say that a shift  $\mathcal{S}$  has a function of quasiperiodicity if there exists a function  $\varphi(n)$  (finite for all  $n$ ) that is a function of a quasiperiodicity for every configuration in  $\mathcal{S}$ .

A shift is called *minimal* if it contains no non-trivial subshifts (except the empty set and itself). A shift  $\mathcal{S}$  is minimal if and only if all its configurations contain exactly the same patterns. If a shift is minimal, then it is quasiperiodic (the converse is not true).

Since each minimal shift  $\mathcal{S}$  is quasiperiodic, for every minimal shift  $\mathcal{S}$  there is a function of quasiperiodicity  $\varphi(n)$  that is finite for all  $n$ . Besides, for an effective minimal shift, the set of all finite patterns (that can appear in any configuration) is computable, see [15, 16]. From this fact it follows that every effective and minimal shift contains some computable configuration. Indeed, with an algorithm that checks whether a given pattern appears in every  $\mathbf{x} \in \mathcal{S}$ , we can incrementally (and algorithmically) increase a finite pattern, maintaining the property that this pattern appears in every configuration in  $\mathcal{S}$ .

**Topological entropy.** Let  $Q_k$  be the  $d$ -dimensional *cube* of size  $k$ ,

$$Q_k := \{0, 1, \dots, k-1\}^d.$$

For a shift  $\mathcal{S}$  in  $\Sigma^{\mathbb{Z}^d}$ , we denote by  $N_{\mathcal{S}}(k)$  the number of distinct  $\Sigma$ -colorings of  $Q_k$  that appear as a pattern in configurations from  $\mathcal{S}$ . The topological entropy of  $\mathcal{S}$  is defined by

$$h(\mathcal{S}) = \lim_{k \rightarrow \infty} \frac{\log N_{\mathcal{S}}(k)}{|Q_k|}$$

with the logarithm to base 2. (The limit above exists for any shift.)

## 1.2 The main results

The main results of this paper can be subdivided into three groups: constructions combining quasiperiodicity with high computational complexity, a construction of a transitive SFT with a given topological entropy, and subdynamics of minimal and quasiperiodic SFT. In what follows we state our results more precisely.

### 1.2.1 Quasiperiodicity is compatible with non-computability

A configuration  $\mathbf{f} : \mathbb{Z}^d \rightarrow \Sigma$ , is called *periodic* if there exists a non-zero  $c \in \mathbb{Z}^d$  such that  $\mathbf{f}(x+c) = \mathbf{f}(x)$  for all  $x$ . Otherwise a configuration is called *aperiodic*. In the classic paper [1], Berger came up with a shift of finite type where all configurations are aperiodic.

**Theorem 1** (Berger). *For every  $d > 1$  there exists a non-empty SFT on  $\mathbb{Z}^d$  where each configuration is aperiodic.*

This construction, as well as a simpler one proposed by Robinson in [2], is quite sophisticated, with tricky combinatorial gadgets embedded in the structure of all configurations of the shift. So the SFT obtained in these seminal papers (and many subsequent constructions elaborating the ideas of Berger and Robinson) look rather messy in terms of dynamical systems. The SFTs proposed by Berger and Robinson are not “simple” as a topological object, neither minimal nor transitive<sup>1</sup>.

Later it was shown that the property of aperiodicity can be enforced for *minimal* shifts of finite type. In other words, the combinatorial complexity (the property of aperiodicity of an SFT) can be combined with topological simplicity (the property of minimality), as in the following theorem.

**Theorem 2** (Ballier–Ollinger). *For every  $d > 1$  there exists a non-empty SFT on  $\mathbb{Z}^d$  where each configuration is aperiodic and quasiperiodic (and even minimal).*

(This result was proven in [14] for a tile set  $\tau$  constructed in [12]. Other examples of minimal aperiodic SFTs were suggested in [30, 29].) Our next result in some sense strengthens Theorem 2: we claim that there exists a quasiperiodic SFT that contains only non-computable configurations.

**Theorem 3.** *For every  $d > 1$  there exists a non-empty SFT on  $\mathbb{Z}^d$  where all configurations are non-computable and quasiperiodic.*

Note that we cannot strengthen Theorem 3 further and change *quasiperiodicity* to *minimality*: in every minimal SFT the set of globally admissible patterns is decidable, and therefore there exists a computable configuration, see [16] and [15].

With a similar technique we prove the following (more general) result, which characterizes the classes of Turing degrees that can be represented by quasiperiodic SFTs.

**Theorem 4.** *For every effectively closed set  $\mathcal{A}$  and for every  $d > 1$  there exists a non-empty SFT  $\mathcal{S}$  in  $\mathbb{Z}^d$  where all configurations are quasiperiodic, and the Turing degrees of all configurations in  $\mathcal{S}$  form exactly the upper closure of  $\mathcal{A}$  (defined as the set of all Turing degrees  $d$  such that  $d \geq_T \mathbf{x}$  for at least one  $\mathbf{x} \in \mathcal{A}$ ).*

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<sup>1</sup>The basic construction of an aperiodic SFT by Robinson admits configurations that consist of two half-planes which can be arbitrarily shifted one with respect to the other. Robinson referred to this phenomenon as a *fault* in a tiling, see [2, Section 3]. This makes the SFT non-minimal and even non-transitive.

The shifts obtained by enriching the generic construction of Robinson (tilings with embedded computations) may be even more irregular. If they admit configurations with potentially different computations (which is unavoidable in some applications), then it is difficult to enforce the appearance in *one single configuration* of all fragments of computations that potentially might appear in *at least one* valid configuration. Recently several authors proposed nontrivial regularizations of Robinson’s construction overcoming the aforementioned obstacles, see below.

*Remark 1.* The Turing degree spectrum of a non effective minimal shift can be much more complex than what we get in Theorem 4, see [27].

### 1.2.2 The Hochman–Meyerovitch theorem on the entropy of SFTs

Hochman and Meyerovitch showed in [17] that a real number  $h \geq 0$  is the topological entropy of some SFT if and only if  $h$  is right recursively enumerable. They raised the question whether the same property holds for the class of *transitive* shifts. Recently Gangloff and Sablik [25] answered this question positively using a construction based on Robinson’s aperiodic tilings. We prove the same result (Theorem 5 below) using a technique similar to the proof of Theorem 3.

**Theorem 5.** *For every integer  $d > 1$  and for every nonnegative right recursively enumerable real  $h \geq 0$  there exists a transitive SFT on  $\mathbb{Z}^d$  with the topological entropy  $h$ .*

### 1.2.3 Subdynamics of minimal and quasiperiodic shifts

Our next theorem claims that the subdynamics of an effective quasiperiodic SFT can be very rich. More specifically, we prove that every effective quasiperiodic  $\mathbb{Z}^d$ -shift can be simulated by a quasiperiodic SFT on  $\mathbb{Z}^{d+1}$ . By *simulation* we mean a factor of the subdynamics of a shift on  $\mathbb{Z}^{d+1}$  (where the subdynamics can be understood as the restriction of a shift on the first  $d$  coordinate axis). We proceed with a formal definition:

**Definition 1.** *We say that a shift  $\mathcal{A}$  on  $\mathbb{Z}^d$  is simulated by a shift  $\mathcal{B}$  on  $\mathbb{Z}^{d+1}$  if there exists a projection  $\pi : \Sigma_B \rightarrow \Sigma_A$  such that for every configuration*

$$\mathbf{f} : \mathbb{Z}^{d+1} \rightarrow \Sigma_B$$

*from  $\mathcal{B}$  and for all  $i_1, \dots, i_d, j, j'$  we have  $\pi(\mathbf{f}(i_1, \dots, i_d, j)) = \pi(\mathbf{f}(i_1, \dots, i_d, j'))$  (i.e., the projection  $\pi$  takes a constant value along each column  $(i_1, \dots, i_d, *)$ , see Fig. 1), and the resulting  $d$ -dimensional configuration*

$$\{\pi(\mathbf{f}(i_1, \dots, i_d, *))\}$$

*belongs to  $\mathcal{A}$ ; moreover, each configuration of  $\mathcal{A}$  can be represented in this way by some configuration of  $\mathcal{B}$ . Informally, we can say that each configuration from  $\mathcal{B}$  encodes a configuration from  $\mathcal{A}$ , and each configuration from  $\mathcal{A}$  is encoded by some configuration from  $\mathcal{B}$ .*

**Theorem 6.** (a) *Let  $\mathcal{A}$  be an effective quasiperiodic  $\mathbb{Z}^d$ -shift over some alphabet  $\Sigma_A$ . Then there exists a quasiperiodic SFT  $\mathcal{B}$  (over another alphabet  $\Sigma_B$ ) of dimension  $d + 1$  such that  $\mathcal{A}$  is simulated by  $\mathcal{B}$  in the sense of Definition 1.*

(b) *Let  $\mathcal{A}$  be a  $\mathbb{Z}^d$ -shift simulated in the sense of Definition 1 by some quasiperiodic SFT  $\mathcal{B}$  of dimension  $d + 1$ . Then  $\mathcal{A}$  is effective and quasiperiodic.*

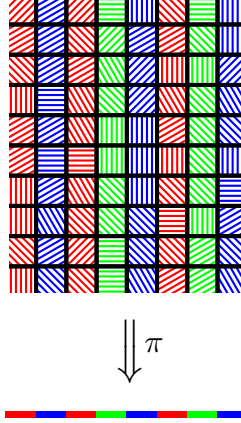


Figure 1: In this example, each cell of a two-dimensional configuration has two characteristics: the color and the direction of hatching. The color is maintained unchanged along each vertical line. The projection  $\pi$  maps each column to its color.

*Remark 2.* The parts (a) and (b) of Theorem 6 are the *if* and *only if* parts of the following characterization: an effective shift is quasiperiodic if and only if it is simulated by a quasiperiodic SFT of dimension higher by 1.

A similar result holds for effective minimal shifts:

**Theorem 7.** (a) *For every effective minimal  $\mathbb{Z}^d$ -shift  $\mathcal{A}$  there exists a minimal SFT  $\mathcal{B}$  in  $\mathbb{Z}^{d+1}$  such that  $\mathcal{A}$  is simulated by  $\mathcal{B}$  in the sense of Definition 1.*

(b) *Let  $\mathcal{A}$  be a  $\mathbb{Z}^d$ -shift simulated in the sense of Definition 1 by some minimal SFT  $\mathcal{B}$  of dimension  $d + 1$ . Then  $\mathcal{A}$  is effective and minimal.*

*Remark 3.* The *only if* part of this theorem (Theorem 7(b)) is due to the fact that the notion of *simulation* in Definition 1 is rather restrictive. We should keep in mind that in general a  $d$ -dimensional subdynamics of a minimal  $\mathbb{Z}^{d+1}$ -shift is not necessarily minimal.

Theorem 6 implies the following rather surprising corollary: a quasiperiodic  $\mathbb{Z}^2$ -SFT can have a highly “complex” language of patterns:

**Corollary 1.** *There exists a quasiperiodic SFT  $\mathcal{A}$  of dimension 2 such that the Kolmogorov complexity of every  $(N \times N)$ -pattern in every configuration of  $\mathcal{A}$  is  $\Omega(N)$ .*

In other words, a quasiperiodic  $\mathbb{Z}^2$ -SFT can have extremely “complex” languages of patterns.

*Remark 4.* A standalone pattern of size  $N \times N$  over an alphabet  $\Sigma$  (with at least two letters) can have Kolmogorov complexity up to  $\Theta(N^2)$ . However, this



density of information cannot be enforced by local rules, because in every SFT on  $\mathbb{Z}^2$  there exists a configuration such that the Kolmogorov complexity of all  $N \times N$ -patterns is bounded by  $O(N)$ , see [13]. Thus, the lower bound  $\Omega(N)$  in Corollary 1 is optimal in the class of all SFT on  $\mathbb{Z}^2$ .

*Remark 5.* Every effective (effectively closed) *minimal* shift is computable: given a pattern, we can algorithmically decide whether it belongs to the configurations of the shift, [15], which is in general not the case for effective quasiperiodic shift. On the other hand, it is known that patterns of high Kolmogorov complexity cannot be found algorithmically. Thus Corollary 1 cannot be extended to the class of minimal SFT.

### 1.3 General remarks and organization of this paper

In the theorems stated above, we claim something about (quasiperiodic, minimal, transitive) SFTs. In the proofs we deal mostly with tilings, which are a very special type of SFT. Since the principal results of the paper are positive statements (we claim that SFTs with some specific properties do exist), the focus on tilings does not restrict the generality. On the other hand, the formalism of Wang tiles perfectly matches the constructions of self-similar and self-simulating shifts of finite type, which are the main technique of this paper. To simplify the notation and make the argument more visual, in what follows we focus on the case  $d = 2$ . The proofs extend to any  $d > 1$  in a straightforward way, *mutatis mutandis*.

The central idea of our arguments is the notion of *self-simulation*, which goes back to [5] and which was extensively developed in the context of symbolic dynamics in [19]. The technique of hierarchical self-simulating tilings is quite generic and flexible. The drawback of this approach is that it is hard to isolate the core technique from features specific to some particular application. In every specific result we cannot just cite the *statement* of a previously known theorem about self-simulating tilings: rather, we need to reemploy the *constructions* from the proofs of a previously known theorem (embedding some new gadgets in the previously known general scheme). This makes the proofs long and somewhat cumbersome. To help the reader, we have tried to make this paper self-contained and explain here the entire machinery of hierarchical self-simulating tilings. The exposition of the main results becomes itself “hierarchical” and “self-similar.” We start with a general perspective of our technique (illustrating it by proofs of several classic results). Then in each succeeding section, we show how to adjust and extend this general construction to obtain this or that new result. We encourage the reader familiar with the concept of self-similar tilings to skip Section 2, which may look oversimplified, and go directly to Sections 3–7.

## 2 The general framework of self-simulating SFT

In this section we recall the principal ingredients of the technique of *self-simulating tile sets*. We start this section with a very basic version of a construction of

self-simulating tile sets from [19]. This part of the construction will be enough, in particular, to obtain a proof of the classic Theorem 1. Later, we will extend this construction and adapt it to prove much stronger statements.

## 2.1 The relation of simulation for tile sets

Let  $\tau$  be a tile set and  $N > 1$  be an integer. We call a *macro-tile* an  $N \times N$  square tiled by matching tiles from  $\tau$ . Every side of a  $\tau$ -macro-tile contains a sequence of  $N$  colors (of tiles from  $\tau$ ); we refer to this sequence as a *macro-color*. Further, let  $T$  be some set of  $\tau$ -macro-tiles (of size  $N \times N$ ). We say that  $\tau$  *implements*  $T$  with a *zoom factor*  $N$  if

- some  $\tau$ -tilings exist, and
- for every  $\tau$ -tiling there exists a unique lattice of vertical and horizontal lines that cuts this tiling into  $N \times N$  macro-tiles from  $T$ .

A tile set  $\tau$  *simulates* another tile set  $\rho$  if  $\tau$  implements a set of macro-tiles  $T$  (with a zoom factor  $N > 1$ ) that is isomorphic to  $\rho$ , i.e., there exists a one-to-one correspondence between  $\rho$  and  $T$  such that the matching pairs of  $\rho$ -tiles correspond exactly to the matching pairs of  $T$ -macro-tiles. A tile set  $\tau$  is called *self-similar* if it simulates itself.

If a tile set  $\tau$  is self-similar, then all  $\tau$ -tilings have a hierarchical structure. Indeed, each  $\tau$ -tiling can be uniquely split into  $N \times N$  macro-tiles from a set  $T$ , and these macro-tiles are isomorphic to the initial tile set  $\tau$ . Further, the grid of macro-tiles can be uniquely grouped into blocks of size  $N^2 \times N^2$ , where each block is a macro-tile of rank 2 (again, the set of all macro-tiles of rank 2 is isomorphic to the initial tile set  $\tau$ ), etc. It is not hard to deduce that a self-similar tile set  $\tau$  has only aperiodic tilings (for more details, see [19]). Below, we discuss a generic construction of self-similar tile sets.

## 2.2 Simulating a tile set defined by a Turing machine

Let us have a tile set  $\rho$ . In what follows, we present a general construction that allows to simulate  $\rho$  by some other tile set  $\tau$ , with a large enough zoom factor  $N$ . The number of tiles in the simulating tile set  $\tau$  will be  $O(N^2)$ , and the constant in the  $O(\cdot)$ -notation does not depend on the simulated  $\rho$ .

We assume that each color is a string of  $k$  bits (i.e., the set of colors  $C \subset \{0, 1\}^k$ ) and the set of tiles  $\rho \subset C^4$  is presented by a predicate  $P(c_1, c_2, c_3, c_4)$  (the predicate is true if and only if the quadruple  $(c_1, c_2, c_3, c_4)$  corresponds to a tile from  $\rho$ ). Suppose we have a Turing machine  $\mathcal{M}$  that computes  $P$ . (It might look wasteful to construct a Turing machine that computes a predicate with a finite domain, but we will see that this kind of abstraction is useful.) Now we construct in parallel a tile set  $\tau$  and a set of  $\tau$ -macro-tiles that simulate the given  $\rho$ .

When constructing a tile set  $\tau$ , we will keep in mind the desired structure of  $\tau$ -macro-tiles (that should simulate the given tile set  $\rho$ ). We require that

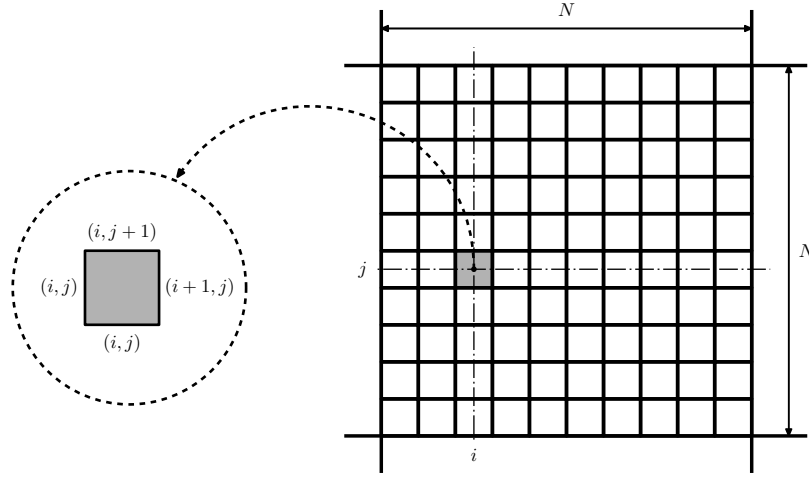


Figure 2: The basic structure of a macro-tile: A block of size  $N \times N$  consists of tiles, the coordinates of which in this block (a pair of integers between 0 and  $N - 1$ ) are written as “colors” of the left and the bottom side of each tile. On the right and on the bottom sides of every tile, one of the coordinates is incremented (modulo  $N$ ).

each tile in  $\tau$  “knows” its coordinates modulo  $N$  in the tiling. This information is included in the tile’s colors. More precisely, for a tile that is supposed to have coordinates  $(i, j)$  modulo  $N$ , the colors on the left and on the bottom sides should involve  $(i, j)$ , the color on the right side should involve  $(i + 1 \bmod N, j)$ , and the color on the top side involve  $(i, j + 1 \bmod N)$ , see Fig. 2.

This means that every  $\tau$ -tiling can be uniquely split into blocks (macro-tiles) of size  $N \times N$ , where the coordinates of the cells range from  $(0, 0)$  in the bottom-left corner to  $(N - 1, N - 1)$  in top-right corner, as shown in Fig. 2. Intuitively, each tile “knows” its position in the corresponding macro-tile. We will require that in addition to the coordinates, each tile in  $\tau$  has some supplementary information encoded in the colors on its sides. On the border of a macro-tile (where one of the coordinates of a cell is zero), we assign to the colors of each cell one additional bit of information. Thus, for each macro-tile of size  $N \times N$  the corresponding macro-colors (in the sense of the definition in Section 2.1, p. 9) can be represented as strings of  $N$  zeros and ones.

The number of bits encoding a macro-color (an  $N$ -bit string representing a macro-color of a macro-tile of size  $N \times N$ ) is excessively large for our future constructions. We choose an integer number  $k$  ( $k \ll N$ ) and allocate in the middle of a macro-tile’s sides  $k$  positions; we make them represent colors from  $C$ . The other  $(N - k)$  bits on the sides of a macro-tile are “dummy” (formally speaking, we set them all to zero), see Fig. 3.

Now we introduce additional restrictions on the tiles in  $\tau$  that will guarantee

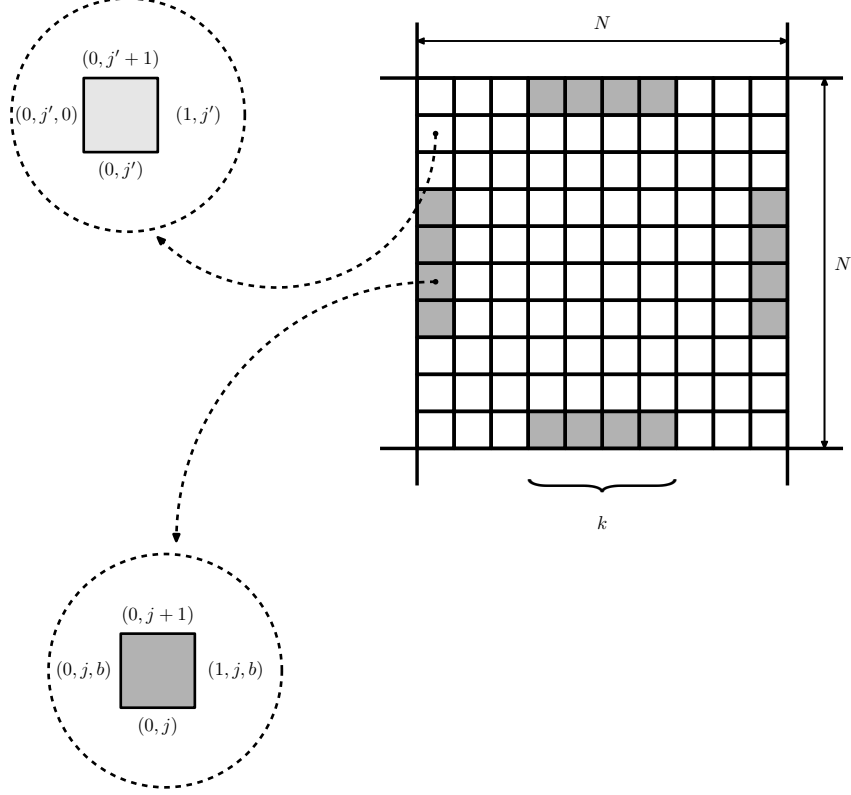


Figure 3: We require that tiles on the margins of an  $N \times N$  macro-tile carry one supplementary bit. For example, the  $j$ -th tile on the left margin of a macro-tile contains in the color of its left side a triple  $(0, j, b)$ , where  $(0, j)$  represent the coordinate of this tile in the macro-tile, and the bit  $b$  may be equal to 0 or 1. Thus, each macro-color can be now represented by a sequence of  $N$  bits embedded in the sides of the tiles on the corresponding margin of a macro-tile.

We assume that these supplementary bits are nontrivial only for the  $k$  tiles in the middle of each macro-tile's margin. In the figure, these tiles are shown in gray, e.g., the  $j$ -th tile on the left side of a macro-tile carries a bit  $b$ , which may be equal to 0 or 1. The other tiles on macro-tile's margin (e.g., the  $j'$ -th tile on the left side of a macro-tile) do not contribute in the macro-color — their supplementary bits are always set to 0. Thus, each macro-color is actually determined by a sequence of only  $k$  bits.

The bits representing the macro-colors are “propagated” in the macro-tile. In what follows we discuss how this information is “processed” inside.

Note that the value of  $k$  in our construction is usually *much* less than  $N$ , e.g.,  $k = O(\log N)$ .

that the macro-colors on the macro-tiles satisfy the “simulated” relation  $P$ . To this end, we ensure that bits from the macro-tile side are transferred to the central part of the tile, and the central part of a macro-tile is used to simulate a computation of the predicate  $P$ . We fix which cells in a macro-tile are “communication wires” and then require that these tiles carry the same (transferred) bit on two sides, see Fig. 4.

The central part of a macro-tile (of size, say,  $m \times m$ , where  $m \ll N$ ) should represent a space-time diagram of the machine  $\mathcal{M}$  (the tape is horizontal, and time goes up), see Fig. 5. This Turing machine processes the quadruple of inputs, which are the  $k$ -bit strings representing the macro-colors of this macro-tile.

Let us explain in more detail how we represent the computation of a Turing machine in a tiling. First of all, we assume that the machine has a single tape. We understand the *space-time diagram* of a Turing machine in a pretty standard way, as a table where each vertical column corresponds to one cell on the tape of the machine, and each horizontal row of the diagram represents an instance of a configuration of the Turing machine. For each row of the diagram, we

- specify for each cell (within a bounded part of the tape) the letter written in this position on the tape,
- specify with a special mark the position of the read head, and
- write the index of the internal state of the machine into the cell where the read head is currently located.

Each next row of the diagram represents the configuration of the machine at the next step of computation (once again, we assume that time goes up). Thus, the entire diagram is determined by its bottom line (with the input data of the machine).

The property of being a valid space-time diagram is defined locally, so we can easily represent such diagram of a given Turing machine by local matching rules for tiles. The details of this representation are not very important in the sequel; we may take, for example, the representation described in [7]. In what follows, we need only keep in mind some natural properties of the chosen representation (which trivially hold for the representations from [7]):

- a correct tiling of a frame  $m \times m$  represents a space-time diagram of the same<sup>2</sup> size  $m \times m$ ,
- a correct tiling of a frame  $m \times m$  with some specific bottom line can be formed if and only if the computation (with the corresponding input data) terminates in an accepting state in at most  $m$  steps and during this computation the read head never leaves the available finite part of the tape,

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<sup>2</sup>In fact, it is enough to assume that a tiling of size  $m \times m$  represents a space-time diagram of size  $\Omega(m) \times \Omega(m)$ .

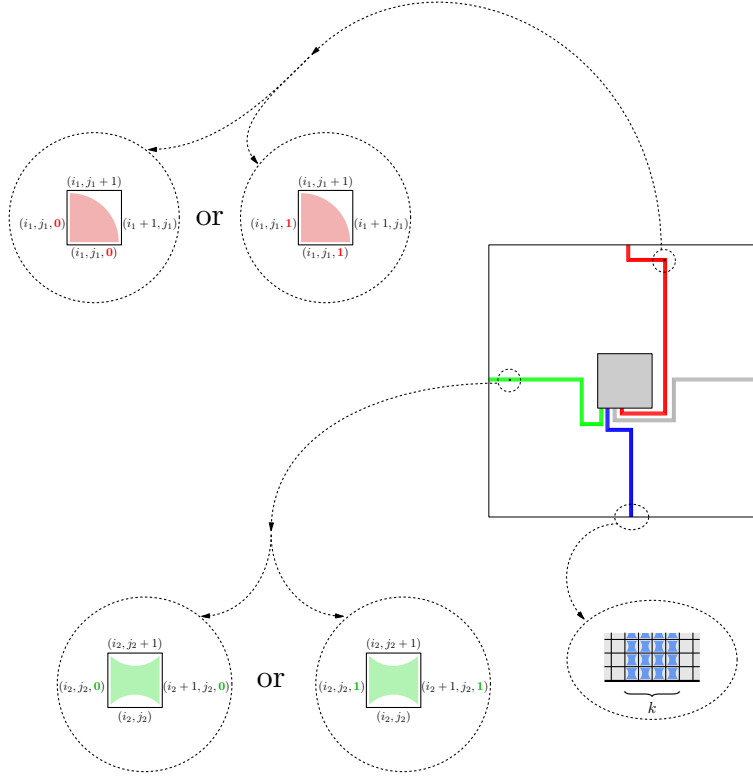


Figure 4: In the middle of each margin of a macro-tile we allocate  $k$  tiles that contain the bits representing a macro-color. These tiles are plugged into “communication wires” that transfer these data inside of a macro-tile. The value of  $k$  (which is the width of the red, green, blue, and gray stripes in the figure) is much less than the size of the macro-tile.

In the figure, we zoom in two tiles involved in the “communication wires.” The tile with coordinates  $(i_1, j_1)$  is a part of one of the communication wires shown in red, which transfer the bits of the top macro-color. As a part of a communication cable, this tile transfers a bit value *zero* or *one*. This tile is placed on a corner of a wire, so it “conducts” a bit value from its left side to its bottom side (the bit values embedded in its left and bottom sides must be both zeros or both ones). Similarly, the tile with coordinates  $(i_2, j_2)$  is a part of a communication wires shown in green, which transfer the bits of the left macro-color. This tile is involved in a horizontal part of a wire, and it “conducts” a bit value *zero* or *one* from the left to the right (so the bit values embedded in its left and right sides must be equal to each other).

The coordinates and the values of the transferred bits are embedded in the “colors” on the sides of the tiles. The colors inside individual tiles (red, green, blue, gray in this figure) are only illustrative and show the functional role of each tile in a macro-tile.

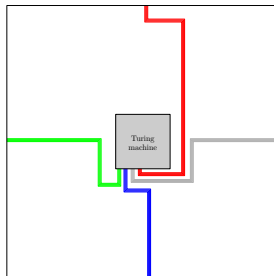


Figure 5: A macro-tile with a space-time diagram of a Turing machine in the middle part.

- every  $(k \times k)$ -fragment<sup>3</sup> of a correct tiling can be reconstructed by its borderline (this is the only property where we need the Turing machine to be deterministic).

The communication of the “computation zone” (the area representing a space-time diagram of a Turing machine) with the “outside world” is restricted to the bottom line (the input data of the computation), which must cohere with the bits representing the four macro-colors of the macro-tile.

To make all of this construction work, the size of a macro-tile (the integer  $N$ ) should be large enough: first, we need enough room to place the “communication wires” that transfer the bits of macro-colors to the “computation zone”; second, we need enough time and space in the computation zone of size  $m \times m$  so that all accepting computations of  $\mathcal{M}$  terminate in time  $m$  and on space  $m$ .

In this construction, the number of additional bits encoded in the colors of the tiles depends on the choice of machine  $\mathcal{M}$ . To avoid this dependency, we replace  $\mathcal{M}$  by a fixed universal Turing machine  $\mathcal{U}$  that runs a program simulating  $\mathcal{M}$ . Moreover, we prefer to separate the general program of a Turing machine (that involves a description of the predicate  $P$  corresponding to the simulated tile set  $\rho$ ) from the zoom factor  $N$ .

Technically, we assume that the tape of the universal Turing machine<sup>4</sup> has an additional read-only layer. Each cell of this layer carries a bit that never changes during the computation (so in the computation zone the columns carry unchanged bits). The construction of a tile set guarantees that these bits form two *read-only input fields*: (i) the program for  $\mathcal{M}$  and (ii) the binary expansion of an integer  $N$  (which is interpreted as the zoom factor). Accordingly, the computation zone of a macro-tile represents a view of an accepting computation for that program given  $N$  as one of the input, see Fig. 6.

<sup>3</sup>In what follows we use only a restricted version of this property. We need to be able to reconstruct every  $(2 \times 2)$ -block of tiles given the 12 tiles around this block, see Fig. 13 below.

<sup>4</sup>We assume that the reader is familiar with the basic notions of computability theory. For a detailed discussion of the notion of a universal computable function and universal Turing machines we refer the reader to the textbooks [6, Section II.1] and references therein, and to [10, Section 2.1].

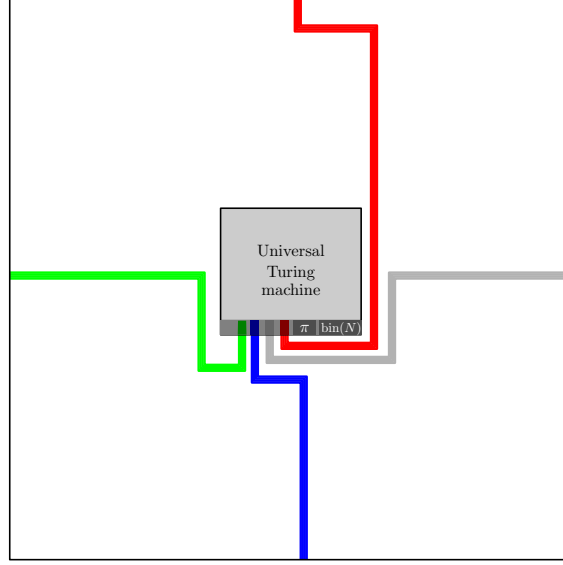


Figure 6: The computation zone represents a space-time diagram of the universal Turing machine. This machine simulates program  $\pi$ , which gets as input the binary codes of four macro-colors, the binary expansion of the zoom factor  $N$ , and its own text.

Thus, from now on we assume that the simulated program is given five inputs: the binary codes of four macro-colors (that are ‘transferred’ from the sides of this macro-tile) and the binary expansion of an integer  $N$  (interpreted as the zoom factor).

Without loss of generality, we assume that the positions of the “wires” and the size of the “computation zone” in a macro-tile are chosen in some simple and natural way, and can be effectively computed given the size of the macro-tile  $N$ . Moreover, we may assume that the “geometry” of a macro-tile (the positions of the communication wires and of the computation zone) can be computed in polynomial time. That is, given the binary expansions of  $N, i, j$ , we can compute in time  $\text{poly}(\log N)$  the ‘role’ played by a tile with coordinates  $(i, j)$  in a macro-tile of size  $N \times N$  (whether this tile is a part of a wire, or a computation zone, or none of these, and if it belongs to the computation zone, then what bits of the *read-only input fields* it should carry).

In this way we obtain an *explicit* construction of a tile set  $\tau$  that has  $O(N^2)$  tiles and simulates  $\rho$ . This construction works for all large enough  $N$ . Note that most steps of the construction of  $\tau$  do not depend on the program for  $\mathcal{M}$ : The tile set  $\tau$  does depend on the program simulated in the computation zone and on the choice of the zoom factor  $N$ . However, this dependency is very limited. The simulated program (and, implicitly, the predicate  $P$ ) affects only the rules for the tiles used in the bottom line of the computation zone. The colors on the



sides of all other tiles are generic and do not depend on the simulated tile set  $\rho$ .

The *explicitness* of the described construction can be understood quite formally as follows: there exists an algorithm that takes the binary expansions of  $N$ ,  $k$ ,  $m$  and a program for  $\mathcal{M}$  as an input, and returns the list of tiles in the tile set  $\tau$  described above. The algorithm halts with an appropriate warning if  $N$ ,  $k$ , or  $m$  is too small for our construction. Moreover, for every quadruple of colors we can verify in polynomial time (in time  $\text{poly}(\log N)$ ) whether this quadruple forms a tile in  $\tau$  or not.

### 2.3 Self-simulation with Kleene’s recursion technique

In the previous section we explained how to simulate any given tile set  $\rho$  by another tile set  $\tau$  with a large enough zoom factor  $N$ . Now we want to find a  $\rho$  such that the corresponding simulating tile set  $\tau$  is isomorphic to the simulated  $\rho$ . We achieve this by using an idea that comes from the proof of Kleene’s recursion theorem. Roughly speaking, we employ the idea that a program can somehow *access its own text* and use its bits in the computation. In most textbooks, the proof of Kleene’s recursion theorems involves the so-called s-m-n theorem, which explains how an algorithm can process the source codes of other programs (given as an input), see e.g. [6]. For a more informal discussion of the ideas behind Kleene’s recursion theorems we recommend [10].

Our goal is to construct a tile set  $\tau$  that simulates itself. At the end of the previous section we observed that our construction of a simulating tile set  $\tau$  has a very limited dependency on  $\mathcal{M}$  (and, therefore, on the simulated tile set  $\rho$ ). Let us fix a triple of parameters  $k, m, N$ ; we apply the construction from the previous section and produce the list of all tiles from the tile set  $\tau$  that simulates some tile set  $\rho$  (the simulated tiles may use at most  $2^k$  colors;  $N$  is the zoom factor and  $m$  is the size of the computation zone in  $\tau$ -macro-tiles). Since the simulated  $\rho$  is not fixed yet, we cannot produce the complete list of tiles in  $\tau$ ; we obtain only those tiles that *do not depend on the simulated tile set*, i.e., all tiles except for the tiles that appear in the bottom line of the computation zone. In what follows we choose the missing tiles in such a way that the resulting  $\tau$  simulates itself.

To complete the construction, we will add to  $\tau$  a few other tiles (exactly those tiles that appear in the bottom line of the computation zone). This construction will work for  $k = 2 \log N + O(1)$  (so that we can encode  $O(N^2)$  colors in strings of  $k$  bits) and  $m = \text{poly}(\log N)$  (so that we can put in the computation zone a space-time diagram of a polynomial-time computation of the Universal Turing machine).

Though the definition of  $\tau$  is not finished, we already know that every valid  $\tau$ -tiling (if such a tiling exists) consists of an  $N \times N$ -grid of macro-tiles, with  $k$ -bit macro-colors embedded in their sides, with communication wires and a computation zone, as shown in Fig. 6. We want to construct a program  $\pi$  that takes as an input the integers  $k, m, N$ , and four  $k$ -bit strings embodying macro-colors, and checks that a macro-tile with the given macro-colors represents one of the tiles in the would-be  $\tau$ . Then we embed this program in the tile set  $\tau$

and complete the construction.

Now let us focus on the program  $\pi$  that should perform the necessary checks (these checks should be simulated in the computation zone of every macro-tile). For the macro-tiles that represent the tiles that are already included in  $\tau$ , the required checks are straightforward (see the comment at the end of the previous section). It remains to implement (and encode in the program  $\pi$ ) the checks for the tiles of  $\tau$  that are not defined yet.

Let us remind that the tiles in  $\tau$  that are still missing are exactly the tiles that should represent the hardwired program. This might seem as a paradox: we have to write a text of a program that handles the tiles where this program will be hardwired. On the one hand, we need to know these tiles to write a program; on the other hand, we need to know the text of the program to produce the tiles. However, we can complete the description of the program  $\pi$  without knowing the missing tiles of  $\tau$ . Since in our construction of a macro-tile, the would-be program  $\pi$  (the list of instructions interpreted by the universal Turing machine) is written on the tape of the universal machine, this program can be instructed to access the bits of its own “text” and check that if a macro-tile plays a role of a tile in the computation zone, then this macro-tile carries the correct bit of the program.

This is the crucial point of the construction. The algorithm implemented in  $\pi$  can be explained as follows. The algorithm obtains as an input the bits encoding the four macro-colors of the macro-tile. We suppose that a macro-tile represents a tile from  $\tau$ , and we know that each  $\tau$ -tile contains a pair of coordinate  $(i, j)$  modulo  $N$ . We can extract these coordinates  $(i, j)$  from the macro-colors of the macro-tile. If this position does *not* belong to the bottom line of the computation zone of a macro-tile, then our task is simple: we use the algorithm outlines at the end of the previous section. But if the position  $(i, j)$  corresponds to the bottom line in the computational zone, then the task is subtle: in this case the  $\tau$ -tile represented by this macro-tile must involve some bit from the text of  $\pi$  (and it should be encoded in the color on the top side of the macro-tile). How to check that the corresponding bit embedded in the macro-color is correct? Where do we get the “correct” value of this bit? The answer is straightforward: the Universal Turing machine simulating  $\pi$  should move its reading head to the column  $j$  in its own computation zone and read there the required bit value. Note that there is no chicken-and-egg paradox, we can write the instructions for  $\pi$  before we know its full text.

Thus, we obtain the complete text (list of instructions) of the program  $\pi$ . This is exactly the program that should be simulated by the Universal Turing machine in the computational zone of each macro-tile, so it is the program whose text must be written on the bottom line of the computation zone. This program provides us with the missing part of the tile set  $\tau$  (we supplement the tile set  $\tau$  with the tiles that represent in the bottom line of the computation zone the text of  $\pi$ ).

It remains to choose the parameters  $N$  and  $m$ . We need them to be large enough so that the computation described above (which deals with inputs of size  $O(\log N)$ ) can fit in the computation zone. The computations are rather simple

(polynomial in the input size, i.e., polynomial in  $O(\log N)$ ), so they fit in the space and time bounded by  $m = \text{poly}(\log N)$ . Thus, we set  $m(N) = \text{poly}(\log N)$  for some specific polynomial that is not too small (e.g.,  $m := (\log N)^3$  is enough) and choose  $N$  large enough so that  $m(N) \ll N$ , and the geometry of a macro-tile shown on Fig. 6 can be realized. This completes the construction of a self-similar aperiodic tile set. Now, it is not hard to verify that the constructed tile set (i) allows a tiling of the plane, and (ii) is self-similar.

The construction described above works well for all large enough zoom factors  $N$ . In other words, for all large enough  $N$  we get a self-similar tile set  $\tau_N$ , and the tilings for all these  $\tau_N$  have very similar structure, with macro-tiles as shown in Fig. 6. Technically, the program  $\pi$  (simulated by the universal Turing machine) now takes as its input a tuple of six strings of bits: the bit strings of length  $k = k(N)$  representing the four macro-colors of a macro-tile, the binary expansion of the zoom factor  $N$ , and its own text. This program checks whether the given strings are coherent, i.e., whether the given quadruple of macro-colors in fact represents a quadruple of colors of one tile in our self-similar tile set  $\tau_N$  (corresponding to the given value  $N$  of the zoom factor).

The presented construction of a self-simulating tile set provides a proof of Theorem 1, see a comment at the end of Section 2.1. Indeed, if a tile set  $\tau$  simulates itself with a zoom factor  $N$ , then by definition every  $\tau$  tiling can be uniquely split into  $N \times N$  macro-tiles. Since these macro-tiles are isomorphic to the tiles of  $\tau$ , the grid of macro-tiles can be uniquely split into macro-macro-tiles of size  $N^2 \times N^2$ . The macro-macro-tiles are obviously also isomorphic to the  $\tau$ -tiles, so they also can be batched in macro-tiles of higher rank, and so on. We obtain a hierarchical structure of macro-tiles of rank  $k = 1, 2, \dots$ , see Fig. 7.

For discussion of the technique of self-simulating tilings and a motivation behind it we refer the reader to [19]. In what follows, we extend and generalize this construction step by step, and then apply it to prove much stronger statements.

*Notation.* We now introduce some useful terminology. In a hierarchical structure of macro-tiles, if a level- $k$  macro-tile  $M$  is a cell in a level- $(k+1)$  macro-tile  $M'$ , we refer to  $M'$  as the *father* of  $M$ . We refer to the level- $(k+1)$  macro-tiles neighboring  $M'$  as the *uncles* of  $M$ .

## 2.4 A more flexible construction: The choice of the zoom factor

For a large class of sufficiently “well-behaved” sequences of integers  $N_k$ , we can construct a family of tile sets  $\tau_k$  ( $i = 0, 1, \dots$ ) such that each  $\tau_{k-1}$  simulates the next  $\tau_k$  with the zoom factor  $N_k$  (and, therefore,  $\tau_0$  simulates each  $\tau_k$  with zoom factor  $L_k = N_1 \cdot N_2 \cdots N_k$ ).

The idea is to reuse the basic construction from the previous section and vary the sizes of the macro-tiles (the zoom factors) on the different levels of the hierarchy. While in the basic construction the macro-tiles (built of  $N \times N$  tiles), the macro-macro-tiles (built of  $N \times N$  macro-tiles), the macro-macro-macro-tiles

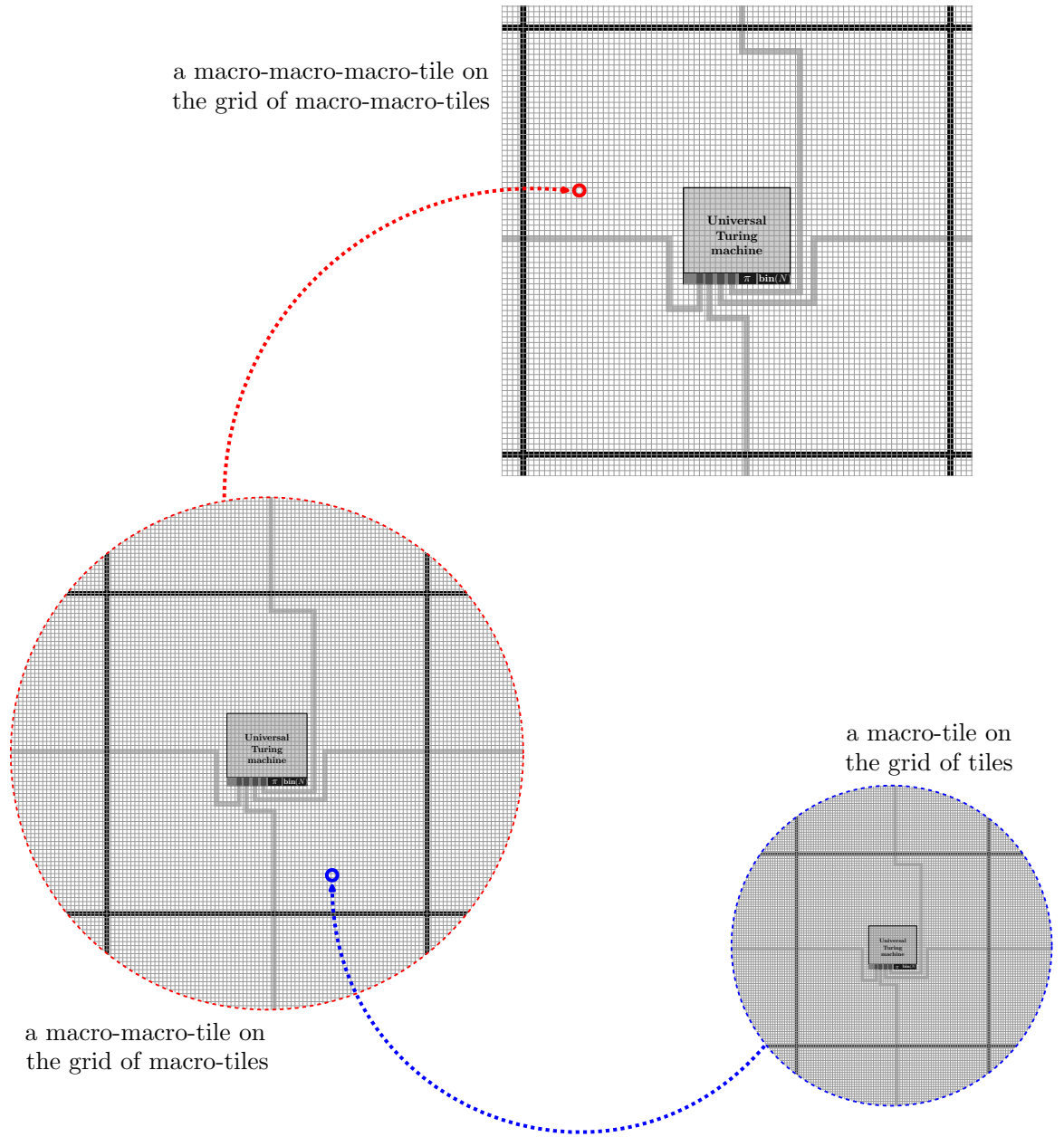


Figure 7: Hierarchical structure of macro-tiles. The level- $k$  macro-tiles are blocks in the level- $(k+1)$  macro-tiles, the level- $(k+1)$  macro-tiles are blocks in level- $(k+2)$  macro-tiles, etc. On all levels of the hierarchy, the structure of the macro-tiles is pretty much the same.

(built of  $N \times N$  macro-macro-tiles), and so on, behave in exactly the same way, in the revised construction the behavior of a level- $k$  macro-tile depends on  $k$ . We want to have macro-tiles built of  $N_1 \times N_1$  ground-level tiles, macro-macro-tiles built of  $N_2 \times N_2$  macro-tiles, macro-macro-macro-tiles built of  $N_3 \times N_3$  macro-macro-tiles, and so on. In this construction, the level- $k$  macro-tiles will be isomorphic to the tiles of  $\tau_k$ , and the idea of “self-simulation” should be understood less literally.

To implement this idea, we need only a minor revision of the construction from the previous section. Similar to our basic self-simulation construction, each tile of  $\tau_k$  “knows” its coordinates modulo  $N_k$  in the tiling: the colors on the left and on the bottom sides should involve  $(i, j)$ , the color on the right side should involve  $(i + 1 \bmod N_k, j)$ , and the color on the top side involves  $(i, j + 1 \bmod N_k)$ . Consequently, every  $\tau_k$ -tiling can be uniquely split into blocks (macro-tiles) of size  $N_k \times N_k$ , where the coordinates of the cells range from  $(0, 0)$  in the bottom-left corner to  $(N_k - 1, N_k - 1)$  in the top-right corner, similarly to Fig. 2. Again, intuitively, each macro-tile of level  $k$  “knows” its position in the corresponding macro-tile of level  $(k + 1)$ . For each  $k$ , the  $N_k \times N_k$ -macro-tile (built of tiles  $\tau_k$ ) should have the structure shown in Fig. 6, with communication wires, a computation zone, and auto-referential computation inside.

The difference with the basic construction is that now the computation simulated by a level- $k$  macro-tile gets, as an additional input, the value  $k$ , and the zoom factor  $N_k$  is computed as a function of  $k$ . In what follows, we always assume that  $N_k$  can be easily computed given the binary expansion of  $k$  (say, in time  $\text{poly}(\log N_k)$ ).

Technically, we assume now that the first line of the computation zone contains the following *fields* of the *input data*:

- (i) the program of a Turing machine  $\pi$  that verifies whether a quadruple of macro-colors corresponds to a valid macro-color,
- (ii) the binary expansion of the integer rank  $k$  of this macro-tile (the level in the hierarchy of macro-tiles),
- (iii) the bits encoding the macro-colors: each macro-color involves the position inside the father macro-tile of rank  $(k + 1)$  (two coordinates modulo  $N_{k+1}$ ) and  $O(1)$  bits of the supplementary information assigned to the macro-colors.

Note that now the zoom factor is not provided explicitly as one of the input fields. Instead, we have the binary expansion of  $k$ , so that a Turing machine can *compute* the value of  $N_k$ , see Fig. 8. The difference from Fig. 6 is that the computation in the macro-tile of rank  $k$  gets as an input the index  $k$  instead of the universal zoom factor  $N$ .

As before, we require that the simulated computation terminates in an accepting state if the macro-colors of the macro-tile form a valid quadruple (if not, no correct tiling can be formed). The simulated computation guarantees that the macro-tiles of level  $k$  are isomorphic to the tiles of  $\tau_{k+1}$ .

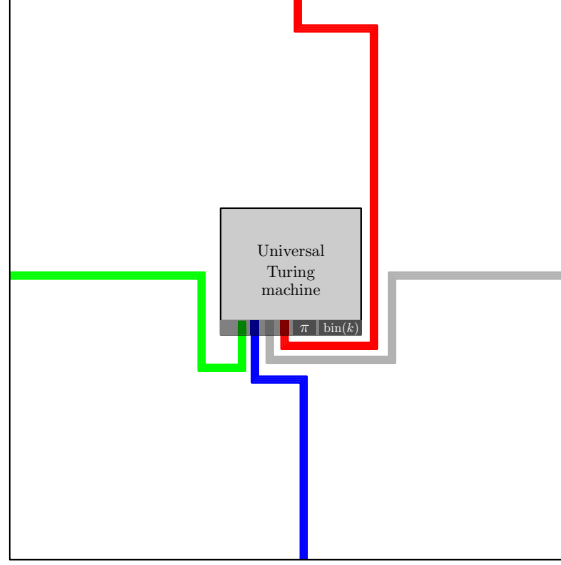


Figure 8: A macro-tile of level  $k$ . The computation zone represents the universal Turing machine that simulates a program  $\pi$ , which gets as input the binary codes of four macro-colors, the binary expansion of the level  $k$ , and the text of  $\pi$  itself.

Note that on each level  $k$  of the hierarchy, we simulate in macro-tiles a computation of one and the same Turing machine  $\pi$ . Only the inputs for this machine (including the binary expansion of the rank  $k$ ) vary from level to level.

This construction works well if  $N_k$  does not grow too slowly (so that the level- $k$  macro-tiles have enough room to keep the binary expansion of  $k$ ) yet not too fast (so that the computation zone in the level- $k$  macro-tiles can handle elementary arithmetic operations with  $N_{k+1}$ ). In what follows we assume that  $N_k = 3^{C^k}$  for some large enough constant  $C$ .

The growing zoom factor  $N_k$  allows embedding some *payload* in the computation zone: some “useful” computation that has nothing to do with self-simulation but affects the properties of a tiling. More precisely, we require that the program  $\pi$  (whose simulation by the Universal Turing machine is embedded in each macro-tile) performs two different tasks: its primary job is to perform the checks of consistency for the four macro-colors of the corresponding macro-tile, as explained above; the secondary job is to run some specific “useful” algorithm  $\mathcal{A}$ . In each proof based on this technique we use a particular algorithm  $\mathcal{A}$  (which is explicitly hardwired in the program  $\pi$  and, therefore, implicitly embedded in the constructed tile set). If necessarily, this algorithm may access as an input the macro-colors of the corresponding macro-tile (they are available in the computation zone). *A priori*, the computation of  $\mathcal{A}$  can be infinitely long. We assume that in each macro-tile the simulation of  $\mathcal{A}$  is performed within the limits of the allocated space and time (the size of the

“computation zone”), and the simulation is aborted when the Universal Turing machine runs out these limits. Though all macro-tiles (on all levels of the hierarchical structure) simulate one and the same algorithm  $\mathcal{A}$ , the available space depends on the rank of a macro-tiles. Since the zoom factor grows with the rank, on each subsequent level of the hierarchy of macro-tiles we can allocate to this secondary computation more and more space and time.

*Remark 6.* The presented construction of a self-simulating tile set is enough to prove the existence of an SFT where each configuration is non-computable (the result known from [3, 4]), for details, see [19].

### 3 Quasiperiodic self-simulating SFT

In this section, we revise once again the construction of a self-simulating tiling and enforce the property of quasi-periodicity or minimality. In particular, this construction will give a new proof of Theorem 2. To implement this construction, we have to superimpose some new properties of a self-simulating tiling.

#### 3.1 Supplementary features: Constraints that can be imposed on the self-simulating tiling

The tiles involved in our self-simulating tile set (as well as all macro-tiles of each rank) can be classified into three types:

- (a) the “skeleton” tiles, which keep no information except for their coordinates in the father macro-tile (the white area in Fig. 8; each of these tiles looks like the tile shown in Fig. 2): these tiles work as building blocks of the hierarchical structure;
- (b) the “communication wires,” which transmit the bits of the macro-colors from the borderline of the macro-tile to the computation zone (the colored lines in Fig. 8; each of these tiles looks like the tiles shown in Fig. 4);
- (c) the tiles of the computation zone (intended to simulate the space-time diagram of the Universal Turing machine, the gray area in Fig. 8).

Each pattern that includes only “skeleton” tiles (or “skeleton” macro-tiles of some rank  $k$ ) reappears infinitely often in all homologous positions inside all macro-tiles of higher rank. Unfortunately, this property is not true for the patterns that involve the “communication zone” or the “communication wires.” Thus, the basic construction of a self-simulating tiling does not imply the property of quasiperiodicity. To overcome this obstacle we need several new technical tricks.

First of all, we impose several restrictions on our construction of a self-simulating tiling. These restrictions in themselves do not make the tilings quasiperiodic, but they simplify the upcoming revision of the construction. More specifically, we enforce the following additional properties (p1)–(p4) of a tiling, with only a minor modification of the construction.

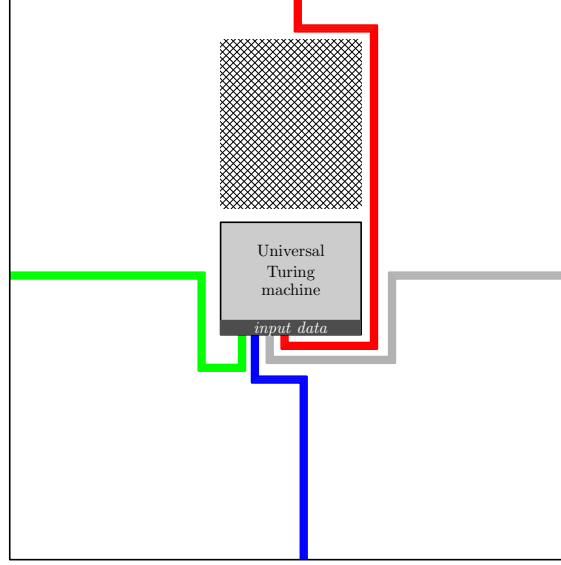


Figure 9: The “free” area reserved above the computation zone.

**(p1)** In our basic construction, each macro-tile contains a computation zone of size  $m_k$ , which is *much* less than the size of the macro-tile  $N_k$ . In what follows, we need to reserve free space in a macro-tile, in order to insert  $O(1)$  (some constant number) of copies of each  $(2 \times 2)$ -pattern from the computation zone (of this macro-tile), right above the computation zone. This requirement is easy to meet. We assume that the size of a level- $k$  macro-tile (measured in blocks that are themselves macro-tiles of level  $k - 1$ ) is  $N_k \times N_k$ , and the computation zone in this macro-tile is  $m_k \times m_k$  for  $m_k = \text{poly}(\log N_k)$ . Therefore, we can reserve an area of size  $\Theta(m_k)$  right above the computation zone, which is free of “communication wires” or any other functional gadgets, see the “empty” hatched area in Fig. 9. So far, this area consisted of only skeleton tiles; in what follows (Section 3.2 below), we will use this zone to place some new nontrivial elements of the construction.

**(p2)** We require that the tiling inside the computation zone satisfies the property of  $2 \times 2$ -*determinacy*. That is, if we know all of the colors on the borderline of a  $2 \times 2$ -pattern inside the computation zone (i.e., a tuple of 8 colors), then we can reconstruct the four tiles of this pattern. Again, we do not need any new ideas to implement this property. It is not hard to see that this requirement is met if we represent the space-time diagram of a Turing machine in a natural way (see the discussion on p. 14).

**(p3)** The communication channels in a macro-tile (the wires that transmit information from the macro-color on the borderline of this macro-tile to the bottom line of its computation zone) must be isolated from each other. The gap be-



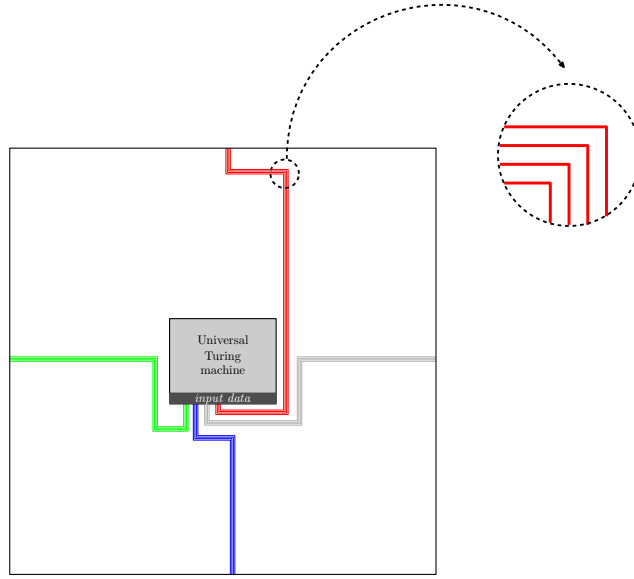


Figure 10: The wires in the “communication cables” (shown in red, blue, green, and gray in the figure) are separated by a gap (shown in white), so that the distance (measured in tiles) between every two wires is greater than two.

tween every two wires must be greater than two cells, as shown in Fig. 10. In other words, each group of cells of size  $2 \times 2$  can touch at most one communication wire. Since the number of wires in a level- $k$  macro-tile is only  $O(\log N_k)$ , we have enough free space to lay the “communication cables” maintaining the required safety gap, so this constraint is easy to satisfy.

**(p4)** In our construction, the macro-colors of a level- $k$  macro-tile are encoded by bit strings of length  $r_k = O(\log N_{k+1})$ . In the previous section we only assumed that this encoding is somewhat “natural” and easy to handle. So far, the choice of encoding was of small importance: we only required that some natural manipulations with macro-colors could be implemented in polynomial time.

We now add another (seemingly artificial) condition. We have decided that each macro-color is encoded in a string of  $r_k$  bits. We require now that each bit in this encoding takes both values 0 and 1 *quite often*. More precisely, we require that for each  $i = 1, \dots, r_k$  there are *quite many* macro-tiles where the  $i$ th bit of encoding of the top (bottom, left, right) macro-color is equal to 0, and there are *quite many* other macro-tiles where the  $i$ th bit of this encoding is equal to 1. In what follows we specify what the words *quite often* and *quite many* mean in this context.

Technically, we use the following property: for every position  $s = 1, \dots, r_k$  and for every  $i = 0, \dots, N_{k+1} - 1$  we require that

- there exists  $j_0$  such that the  $s$ th bit in the top, left, and right macro-colors of the level- $k$  macro-tile at the positions  $(i, j_0)$  in the level- $(k+1)$  father macro-tile are equal to 0, and
- there exists  $j_1$  such that the  $s$ th bit in the top, left, and right macro-colors of the level- $k$  macro-tile at the positions  $(i, j_1)$  in the level- $(k+1)$  father macro-tile are equal to 1.

There are many (more or less artificial) ways to implement this constraint. For example, we may subdivide the array of  $r_k$  bits encoding a macro-color into three equal zones of size  $r_k/3$  and require that for each macro-tile only one of these three zones contains the “meaningful” bits, and the two other zones contain only zeros and ones respectively; we require then that the “roles” of these three zones cyclically permute as we go upwards along a column of macro-tiles, see Fig. 11.

### 3.2 Enforcing minimality

To achieve the property of minimality of an SFT, we should guarantee that every finite pattern that appears *once* in at least one tiling must also appear in *every large enough square* in every tiling. In a tiling with a hierarchical structure of macro-tiles each finite pattern can be covered by at most four macro-tiles (by a  $2 \times 2$ -pattern) of an appropriate rank. Hence, to guarantee the property of minimality, it is enough to show that every  $(2 \times 2)$ -block of macro-tiles of any rank  $k$  that appears in at least one  $\tau$ -tiling actually reappears in this tiling in every large enough square. Let us classify all  $(2 \times 2)$ -block of macro-tiles (by their position in the father macro-tiles of higher rank) and discuss what revisions of the construction are needed.

*Case 1: Skeleton tiles.* For a  $(2 \times 2)$ -block of four “skeleton” macro-tiles of level  $k$ , there is nothing to do. Indeed, in our construction we have exactly the same blocks with every vertical translation by a multiple of  $L_{k+1}$  (we have there a similar block of level- $k$  “skeleton” macro-tiles contained in some other macro-tile of rank  $(k+1)$ ).

*Case 2: Communication wires.* Let us consider the case when a  $(2 \times 2)$ -block of level- $k$  macro-tiles involves a part of a communication wire. Due to property (p3) we may assume that only one wire is involved. The bit transmitted by this wire is either 0 or 1; in either case, due to property (p4), we can find another similar  $(2 \times 2)$ -block of level- $k$  macro-tiles (at the same position within the father macro-tile of rank  $(k+1)$  and with the same bit included in the communication wire) in every macro-tile of level  $(k+2)$ . In this case we can find a duplicate of the given block with a vertical translation of size  $O(L_{k+2})$ .

*Case 3: Computation zone.* Now we consider the most difficult case: when a  $(2 \times 2)$ -block of level- $k$  macro-tiles touches the computation zone. In this case we cannot obtain the property of quasiperiodicity for free, and so we have to make one more modification to our general construction of a self-simulating tiling.

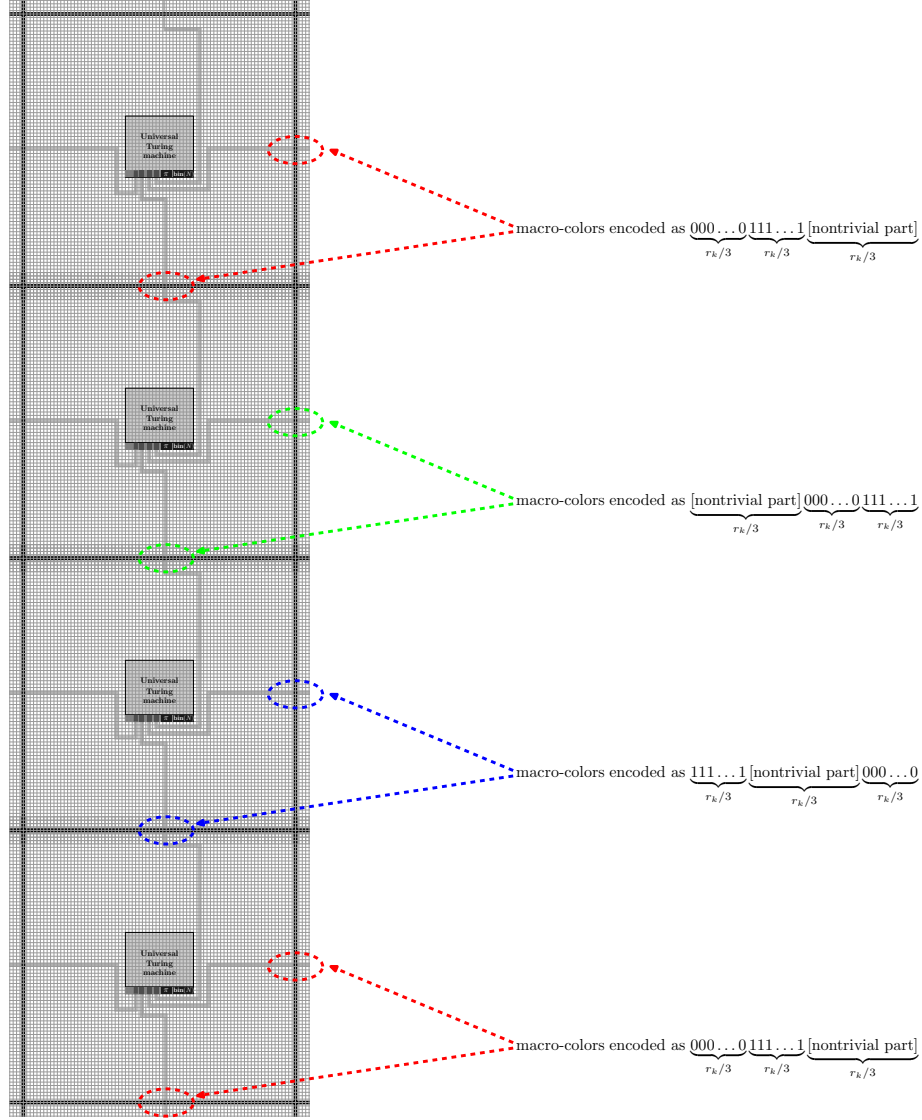


Figure 11: Encoding of macro-colors. The  $r_k$  bits of the code are split into three blocks of size  $r_k/3$ . One of them consists of all 0s, another of all 1s, and only the third one contains a nontrivial binary code. The roles of the three blocks change cyclically from one macro-tile to another when we move upwards.

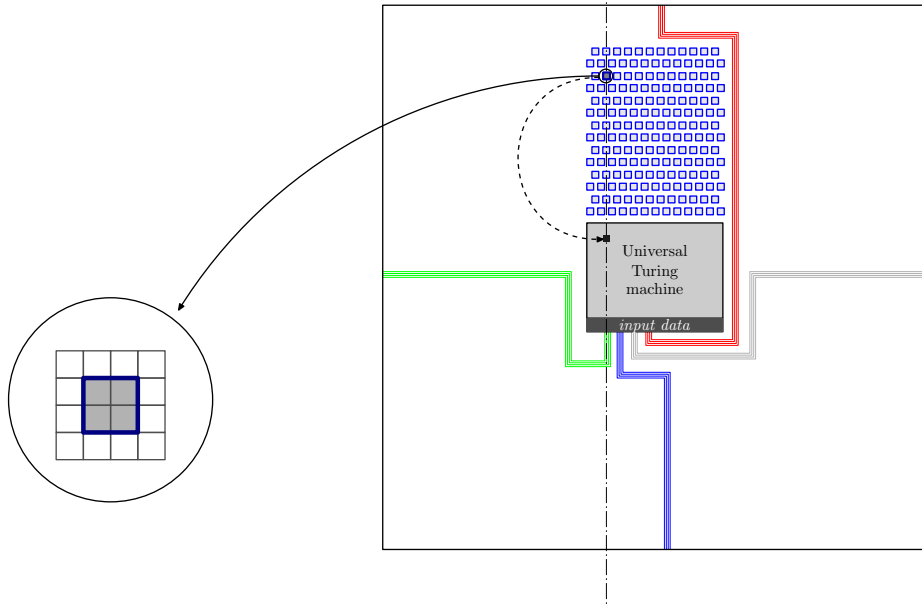


Figure 12: Positions of the *diversification slots* for patterns from the computation zone.

Note that for each  $2 \times 2$ -window that touches the computation zone of a macro-tile, there are only  $O(1)$  ways to tile them correctly. For each possible position of a  $2 \times 2$ -window in the computation zone and for each possible filling of this window by tiles, we reserve a special  $2 \times 2$ -slot in a macro-tile, which is essentially a block of size  $2 \times 2$  in the “free” zone of a macro-tile. We refer to this gadget as a *diversification slot*. These slots will enforce the property of “diversity”: for every small pattern that *could* appear in the computation zone, we will guarantee that it *must* appear in the corresponding diversity slot in every macro-tile of the same rank.

The diversification slots must be placed far away from the computation zone and from all communication wires. We prefer to place every diversification slot in the same vertical stripe as the “original” position of this block, as shown in Fig. 12 (this property of vertical alignment will be used in Section 6). We have enough free space to place all necessary diversification slots, due to property (p1). We define the neighbors around each diversification slot in such a way that only one specific  $(2 \times 2)$ -pattern can patch it (here we use the property (p2)).

In our construction, the tiles around this slot “know” their real coordinates in the bigger macro-tile, while the tiles inside the diversification slot do not (they “believe” they are tiles in the computation zone, while in fact they belong to an artificial isolated “diversity preserving” slot far outside of any real computation), see Fig. 12 and Fig. 13. The frame of the diversification slot consists of 12

$(i, j+4)$	$(i+1, j+4)$	$(i+2, j+4)$	$(i+3, j+4)$
$(i, j+3)$	$(i+1, j+3)$	$(i+1, j+3)$	$(i+2, j+3)$
$(i, j+3)$	$(s, t+2)$	$(s+1, t+2)$	$(i+3, j+3)$
$(i, j+3)$	$(s, t+2)$	$(s+1, t+2)$	$(i+3, j+3)$
$(i, j+2)$	$(s, t+1)$	$(s, t+1)$	$(s+1, t+1)$
$(i, j+2)$	$(s, t+1)$	$(s+1, t+1)$	$(i+3, j+2)$
$(i, j+2)$	$(s, t+1)$	$(s+1, t+1)$	$(i+3, j+2)$
$(i, j+1)$	$(s, t)$	$(s, t)$	$(s+1, t)$
$(i, j+1)$	$(s, t)$	$(s+1, t)$	$(i+3, j+1)$
$(i, j+1)$	$(s, t)$	$(s+1, t)$	$(i+3, j+1)$
$(i, j)$	$(i+1, j)$	$(i+1, j)$	$(i+2, j)$
$(i, j)$	$(i+1, j)$	$(i+2, j)$	$(i+3, j)$
$(i, j)$	$(i+1, j)$	$(i+2, j)$	$(i+3, j)$

Figure 13: A diversification slot for a  $2 \times 2$ -pattern from the computation zone.

“skeleton” tiles (the white squares in Fig. 13); they form a slot that involves inside a  $(2 \times 2)$ -pattern extracted from the computation zone (the gray squares in Fig. 13). In the picture, we show the “coordinates” encoded in the colors on the sides of each tile. Note that the colors of the bold lines (the blue lines between the white and gray tiles and the bold black lines between the gray tiles) should contain some information beyond the coordinates—these colors involve the bits used to simulate a space-time diagram of the universal Turing machine. In this picture, the “real” coordinates of the bottom-left corner of this slot are  $(i+1, j+1)$ , while the “natural” coordinates of the pattern inside the diversification slot (when this pattern appears in the computation zone) are  $(s, t)$ .

We choose the positions of the diversification slots in the macro-tile so that the coordinates can be computed by some simple algorithm in time polynomial in  $\log N_k$ . We require that all diversification slots be detached from each other in space, so they do not damage the general structure of the “skeleton” tiles building the macro-tiles.

Now it is not hard to see that for the revised tile set, every pattern that appears *at least once* in *at least one* tiling must in fact appear in *every large enough* pattern in *every* tiling. Thus, the revised construction of a self-simulating tiling guarantees that

- every tiling is aperiodic (the argument from the previous section remains valid), and
- every pattern that appears at least once in at least one configuration must appear in every large enough square in every tiling.

Hence, our construction implies Theorem 2.

## 4 Quasiperiodicity and non-computability

So far, we have used the method of self-simulating tilings to combine the properties of quasiperiodicity (and even minimality) and aperiodicity. Now we are going to go further and combine quasiperiodicity with non-computability. In this section, we extend the construction discussed above and prove Theorem 3 and Theorem 4. To this end, we extensively use the technique of a self-simulating tiling with a *variable zoom factor*, introduced in Section 2.4. As we mentioned above, we assume that the size of a macro-tile of rank  $k$  is equal to  $N_k \times N_k$ , for  $N_k = 3^{C^k}$ ,  $k = 1, 2, \dots$ , and the size of the computation zone  $m_k$  grows as  $m_k = \text{poly}(\log N_k)$ .

We take as the starting point the generic scheme of a quasi-periodic self-simulating tiling explained in the previous section, and adjust it with some new features. From now on we require that all macro-tiles of rank  $k$  contain, in their computation zone, the prefix (e.g., of length  $\lceil \log k \rceil$ ) of some infinite sequence  $X = x_0x_1x_2\dots$ . We want that all macro-tiles of rank  $k$  contain the same prefix  $x_0x_1x_2\dots k_{\lceil \log k \rceil}$ , so we will embed these bits in the macro-colors of all macro-tiles. To make things more pictorial, we require also that these bits be provided in the bottom line of the computation zone as one supplementary input field, as shown in Fig. 14. But we stress again that the bits  $x_0x_1x_2\dots k_{\lceil \log k \rceil}$  make part of each macro-color. So in Fig. 14 these bits are repeated five times: they appear four times implicitly as a part the codes of macro-colors (green, blue, gray, and red fields in the picture) and one more time explicitly (the rightmost part of the computation zone). The computation hidden in each macro-tile should verify that these five copies of  $x_0x_1x_2\dots k_{\lceil \log k \rceil}$  provided to the computation zone are identical.

Since every two neighboring macro-tiles must have matching macro-colors, we can guarantee every two neighboring level- $k$  macro-tiles contain one and the same prefix  $x_0x_1x_2\dots k_{\lceil \log k \rceil}$  (this remains true even for neighboring level- $k$  macro-tiles that belong to different father macro-tiles of level  $(k + 1)$ ). Thus, the construction implies that in each valid configuration all level- $k$  macro-tiles involve one and the same prefix  $x_0x_1x_2\dots k_{\lceil \log k \rceil}$ .

We also need coherence between bits  $x_i$  embedded in macro-tiles of different ranks (the string embedded in a macro-tile of higher rank should extend the string embedded in macro-tiles of lower rank). So we suppose that the computation embedded in the computational zones of macro-tiles verifies whether the input data are coherent, i.e., that the bits embedded in each of the four macro-tiles match the bits  $x_0x_1x_2\dots k_{\lceil \log k \rceil}$  given explicitly in this new input field. Using the usual self-simulation, we can guarantee that the bits of  $X$  embedded in a macro-tile of rank  $k + 1$  extend the prefix embedded in a macro-tile of rank  $k$ . Since the size of the computation zone increases as a function of  $k$ , the entire tiling of the plane determines an infinite sequence of bits  $X$  (whose prefixes are encoded in the macro-tiles of all ranks).

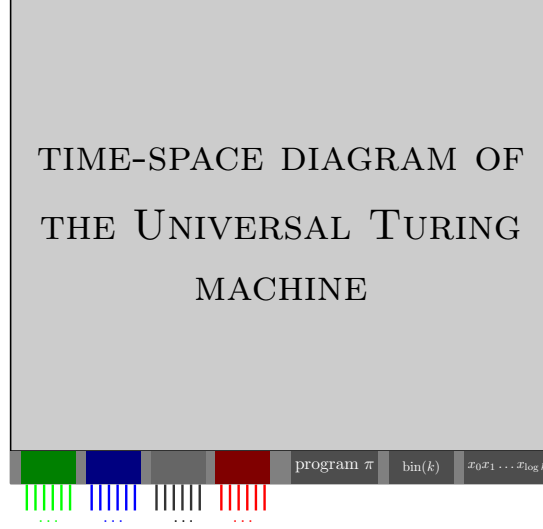


Figure 14: The *computation zone* of a macro-tile in the revised construction. The input data consist of the codes of the macro-colors, the simulated program  $\pi$ , the binary expansion of the rank  $k$ , and the first  $\log k$  bits of the embedded sequence  $X = x_0 x_1 x_2 \dots$ .

*Remark 7.* The new feature allows embedding an infinite sequence  $X$  in a tiling. This embedding is *highly distributed* in the following sense: we can extract the first  $\log k$  bits of the sequence from any level- $k$  macro-tile. Note that the tile set does not determine the embedded sequence  $X$  (different tilings of the same tile set can represent different sequences  $X$ ). However we are going to control the class of sequences  $X$  that can be embedded in a tiling.

Since the embedded sequence  $X$  is not uniquely defined by the tile set, we lose the property of *minimality* (different  $\tau$ -tilings involve different embedded sequences  $X$ , and therefore different finite patterns). However, we still have the property of *quasiperiodicity*. Indeed, every valid tiling contains a well-defined embedded sequence  $X$ . Let us fix one infinite sequence  $X$  and restrict ourselves to the class of tilings  $\mathcal{T}(X)$  that represent this specific sequence. Then, all level- $k$  macro-tiles in every tiling in  $\mathcal{T}(X)$  involve (one and the same) prefix of  $X$ . Now the argument from the previous section, repeated word for word, gives the following property.

*Every  $(2 \times 2)$ -block of level- $k$  macro-tiles that appears at least once in at least one tiling in  $\mathcal{T}(X)$  must reappear in every large enough pattern of every tiling in  $\mathcal{T}(X)$ .*

It follows immediately that every tiling in this construction is quasiperiodic.

When the class of embedded sequences of  $X$  is not restricted, this construction is of no interest. It becomes meaningful if we can enforce some special properties of the embedded sequence  $X$ .

As we explained in Section 2.2, we can include in the computation performed by each macro-tile a simulation of some specific algorithm  $\mathcal{A}$  (see the discussion on p. 22). We may require that  $\mathcal{A}$  verifies some particular property of the embedded sequence  $X$ . Each macro-tile accesses a prefix of the embedded sequence, so we may test some computable properties of the available prefix of  $X$ . As usual, the computation in each macro-tile is limited by the size of the computation zone. However, we may assume that the space and time allocated to the computation grow with the levels of the hierarchy.

We start with a rather simple example of a property of  $X$  that can be implemented. We can guarantee that for every tiling the embedded sequence  $X$  is *not computable*. This is not completely trivial: our aim is to implement a *computable* procedure which guarantees that a sequence of bits  $X$  is *not computable*. More technically, we need to find a computable set of constraints (forbidden finite patterns) such that each  $X$  that satisfies these constraints (i.e., contains none of the forbidden patterns) is not computable. From the computability theory we know many examples of sets with the required properties (co-recursively enumerable sets with no computable points, see, e.g., [6, Proposition V.5.25]).

Our construction can be combined with any example of a co-recursively enumerable non-empty set  $A \subset \{0,1\}^{\mathbb{N}}$  with no computable points (i.e., each sequence  $x \in A$  must be incomputable). We can embed such a set  $A$  in a quasi-periodic self-simulating tiling, so that the resulting SFT satisfies the statement of Theorem 3. For the sake of self-containedness, we prefer to be more specific and describe below in some detail one particular construction of such a set.

*Short digression: a co-recursively enumerable set of bit sequences with no computable points.* Arguably the simplest classic construction of a set with this property is based on a pair of recursively inseparable enumerable sets. Let us remind that there exists a pair of disjoint recursively enumerable sets  $S_1, S_2 \subset \mathbb{N}$  such that there is no decidable “separator”  $W$  satisfying  $S_1 \subset W$  and  $S_2 \subset \mathbb{N} \setminus W$  (see, e.g., [6, Section II.2] and [10, Section 2.4]). In order to implement the outlined plan, we embed in the computation zone of macro-tiles an algorithm  $\mathcal{A}$  that performs the following job: it enumerates two non-separable enumerable sets (on each level  $k$  we run these two enumerations for the number of steps that fits the computation zone available in a macro-tile of rank  $k$ ). Then,  $\mathcal{A}$  checks that the bits of  $X$  represent a separator between these two sets. Technically, in each macro-tile the computation verifies that these (partially) enumerated sets are indeed separated by the given prefix of  $X$ . In every valid tiling, the embedded sequence  $X$  must pass the checks performed in all macro-tiles on all levels of the hierarchy. Hence, this  $X$  must be a separator between two non-separable enumerable sets. Therefore, this sequence  $X$  must be non-computable. It follows that the tiling (which contain this sequence) must be also non-computable.

Combining all the ingredients together, we obtain a tile set  $\tau$  that enjoys two nontrivial properties: all  $\tau$ -tilings are non-computable and quasiperiodic. This gives a new proof of Theorem 3.

Thus, we constructed a tile set  $\tau$  such that all  $\tau$ -tilings are quasiperiodic



and non-computable. Up to now, we could not say much more about the degree of unsolvability (Turing degrees) of  $\tau$ -tilings. Now we are going to enhance this construction by implementing some more precise control on the class of embeddable sequences  $X$ , and therefore on the class of possible Turing degrees of  $\tau$ -tilings. We start with a proposition that characterizes the no-go zones for this technique.

**Theorem 8.** (a) *Every SFT is effectively closed.* (b) *For every infinite minimal SFT  $\mathcal{S}$ , the class of the Turing degrees representable by configurations in  $\mathcal{S}$  is upper-closed: if there exists a  $\tau$ -tiling that has a Turing degree  $T$ , then every Turing degree  $T' > T$  is also represented by some  $\tau$ -tiling.*

*Proof.* (a) is trivial, (b) is proven in [21]. □

*Remark 8.* Observe that we cannot guarantee that the Turing degrees of all  $\tau$ -tilings are *very* high. More specifically, we cannot guarantee that all  $\tau$ -tilings are *not low* or *not hyperimmune-free*. Indeed, due to the low basis theorems, for every tile set  $\tau$ , some  $\tau$ -tilings are not low and not hyperimmune-free.

Theorem 4 essentially claims that a class of Turing degrees which is not forbidden by Theorem 8 (i.e., a class that is upwards closed and corresponds to an effectively closed set) can be implemented by a suitable tile set.

*Proof of Theorem 4.* To prove this theorem, we again employ the idea of embedding an infinite sequence  $X$  in a tiling, and control more precisely the properties of the embedded sequence. Similarly to the construction discussed above, we require that all macro-tiles of rank  $k$  involve the same finite sequence of  $\log k$  bits on their computation zone, which is understood as a prefix of  $X$ . We can guarantee that the prefix embedded in macro-tiles of rank  $k$  is compatible with the prefix available to the macro-tiles of the next rank  $(k + 1)$ .

Further, since  $\mathcal{A}$  is in  $\Pi_1^0$ , we can enumerate the (potentially infinite) list of patterns that should not appear in  $X$ . On each level, the macro-tiles run this enumeration for the available space and time (limited by the size of the computation zone available on this level), and verify that the discovered forbidden patterns do not appear in the prefix of  $X$  accessible to the macro-tiles of this level. Since the computation zone becomes bigger and bigger with each level, the enumeration extends longer and longer. Thus, a sequence  $X$  can be embedded in an infinite tiling if and only if this sequence does not contain any forbidden pattern (i.e., if this  $X$  belongs to  $\mathcal{A}$ ).

What are the Turing degrees of the tilings in the described tile set? In our tile set, every tiling is uniquely defined by the following information: the sequence  $X$  embedded in this tiling and the sequences of integers  $\sigma_h, \sigma_v$  that specify the shifts (the vertical and the horizontal ones) of macro-tiles of each level relative to the origin of the plane. Indeed, on each level  $k$  we split the macro-tiles of the previous rank into blocks of size  $N_k \times N_k$ . These blocks make level- $k$  macro-tiles, and there are  $N_k^2$  ways to choose the grid of horizontal and vertical lines that define this splitting. Given these sequences  $\sigma_h, \sigma_v$  and an  $X \in \mathcal{A}$ , we can reconstruct the entire tiling. It remains to note that  $\sigma_h$  and  $\sigma_v$

can be absolutely arbitrary. Thus, the Turing degree of a tiling is the Turing degree of  $(X, \sigma_h, \sigma_v)$ , which can be an arbitrary degree not less than  $X$ . That is, the set of degrees of tilings is exactly the closure of  $\mathcal{A}$ , i.e., the set of all  $Y$  that are not less than some  $X \in \mathcal{A}$ . So we get the statement of Theorem 4.  $\square$

## 5 Transitive version of the Hochman–Meyerovitch theorem about possible entropies of SFT

In this section, we prove Theorem 5. As usual, we consider only the case  $d = 2$ . Before proceeding with the proof, we establish the following proposition.

**Proposition 1.** *For every integer  $d > 1$  and for every nonnegative right recursively enumerable real  $h$  there exists a tile set  $\tau$  split into two disjoint subsets,  $\tau = \tau_R \cup \tau_B$  (we will say that the tiles in  $\tau_R$  are red and the tiles in  $\tau_B$  are blue) that admits a tiling of the plane, and*

- *the set of  $\tau$ -tilings is a minimal shift,*
- *$\limsup$  of the fraction of red tiles in globally admissible  $n \times n$  patterns (i.e., in correctly tiled squares of size  $n \times n$  that can appear in a tiling of the plane) is equal to  $h$ .*

*Proof.* We take as the starting point the construction of a tile set from Section 3, which enforces a tiling with growing zoom factors and guarantees the minimality of the corresponding SFT. We upgrade it in the following way.

*Red and blue tiles and macro-tiles.* We want that each tile and each macro-tile (of each rank) “knows” its color, which can be red or blue. On the ground level, we basically make two copies of every tile (one copy is red and another is blue). Below we specify the local constraints imposed on the “colors” assigned to the neighboring tiles. We also slightly revise the structure of the macro-tiles (which implicitly requires, of course, some minor modification of the ground level tiles).

In the new construction, every macro-tile can be “red” or “blue.” First of all, *the color of a macro-tile* (represented by a single bit of information) is written as a new input field in the bottom line of the computation zone. (Thus, we add one more input field besides the macro-colors and the binary expansion of macro-tile’s rank, similarly to Fig. 14.) The color of the cells composing a level- $k$  macro-tile (as usual, the cells are macro-tiles of level  $(k - 1)$ ) are subject to the following rule:

- the cells in the bottom-left corner of size  $\alpha_k \times \alpha_k$  are always red (whatever is the “color” of the macro-tile in whole);
- the cells in the top-right corner of size  $\beta_k \times \beta_k$  are always blue (whatever is the “color” of the macro-tile in whole);
- the color in diversification slots (that guarantee quasiperiodicity) can be red or blue, see below;

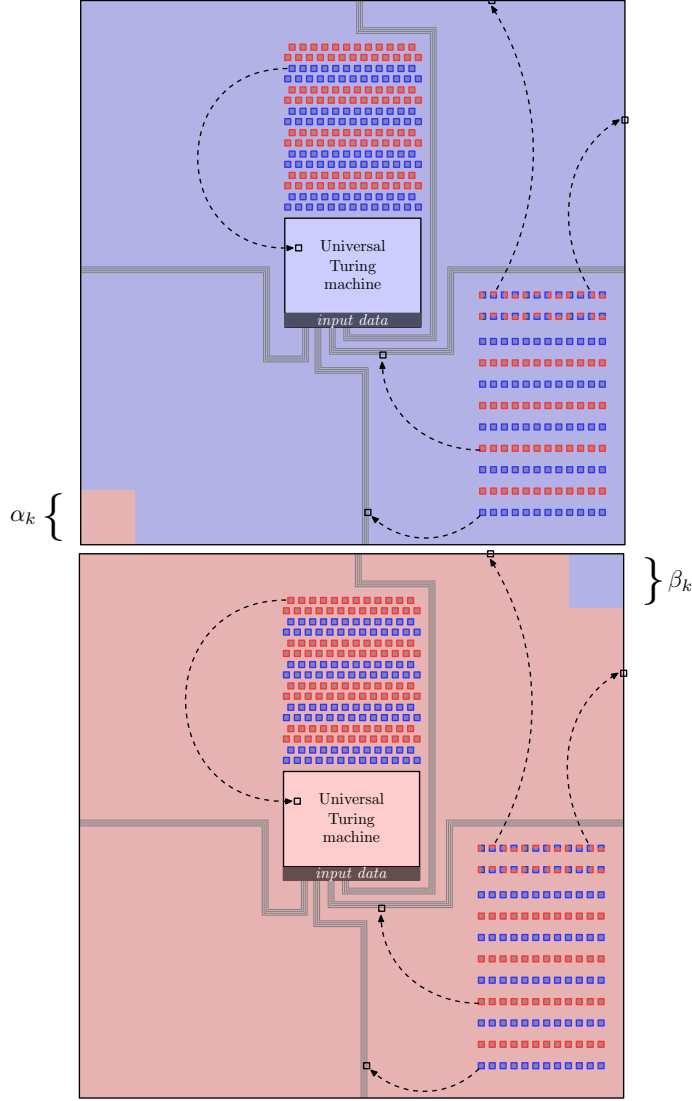


Figure 15: A blue macro-tile of rank  $k$  consists of mostly blue blocks of rank  $(k - 1)$ , with only an  $(\alpha_k \times \alpha_k)$  corner of red blocks. For a red macro-tile the situation is dual: most blocks are red, with a  $(\beta_k \times \beta_k)$  corner of blue blocks. In comparison with Fig. 12, here we have diversification slots for  $2 \times 2$ -pattern not only in the computation zone but also in the communication wires and on the borderline of the macro-tile. In both types of macro-tiles (red and blue) we keep the diversification slots with both red and blue versions of the “cloned” patterns.

- the other cells (skeleton cells as well as communication wires and the computation zone) are red in a red macro-tile and blue in a blue macro-tile.

The structure of “red” and “blue” macro-tiles is shown in Fig. 15.

*Remark 9.* (a) The parameters  $\alpha_k$  and  $\beta_k$  are small enough so that the “red” and “blue” corners in a macro-tile neither intersect the computation zone nor the communication wires. We fix the values of these parameters later.

(b) The coherence of the “color” of most cells in a macro-tile is guaranteed by local rules: neighboring cells inside a macro-tile must have the same “color” (except for the two special corners mentioned above and the diversification slots).

(c) Since the cells in the computation zone “know” their colors, it is easy to guarantee that the color of the macro-tile provided as an input of the Turing machine (simulated in the computation zone) is coherent with the color of the cells in this macro-tile.

It remains to explain the policy for the color of the diversification slots (which reproduce all  $(2 \times 2)$  patterns from the computation zone). For each of the  $(2 \times 2)$ -patterns touching the computation zone, we reserve twice as many slots as we did in Section 3. For each of these configurations, we prepare both “red” and “blue” clones, no matter what the real color of the entire macro-tile is.

In addition to the diversification slots duplicating the patterns in the computation zone, we embed, in the macro-tile, similar slots (with both possible colors) for all  $(2 \times 2)$ -patterns that intersect the communication wires, see Fig. 15. (In a level- $k$  macro-tile, we have only  $O(\log N_k)$  wires, and the length of each wire is at most  $O(N_k)$ . Hence, we have enough free space to place all necessary diversification slots.)

Last, we also add similar diversification slots to the  $(2 \times 2)$ -patterns intersecting the borderline of the macro-tile. Note that neighboring cells that belong to different macro-tiles may have different colors. So in this case we build diversification slots with multi-colored  $2 \times 2$ -patterns inside. Since the length of the borderline of a macro-tile is  $O(N_k)$ , we have enough free space in a macro-tile to make an isolated diversification slot for each of these patterns. This family of gadgets concludes the construction of the new tile set.

*Claim 1.* The set of all tilings for the described tile set is a minimal SFT.

*Proof of Claim 1.* We need to show that every finite pattern that appears in at least one tiling must appear in every tiling (in every large enough area). As usual, we profit from the structure of a hierarchical self-simulating tiling: it is enough to prove this property for patterns built of  $(2 \times 2)$ -group of level- $k$  macro-tiles. For a  $(2 \times 2)$ -pattern that touches either the computation zone or a communication wire or the borderline of a macro-tile of level  $(k + 1)$ , this property is simple to establish: we can find a clone of such a pattern inside the corresponding diversification slot (which exists in every macro-tile of rank  $k + 1$ ).

For a  $(2 \times 2)$ -pattern that consists of only ordinary skeleton cells, things are somewhat trickier. We cannot say that exactly the same pattern can be found at the homologous position in *every* other macro-tile of rank  $(k+1)$ . In fact, we only find the same pattern at the same position in macro-tiles of rank  $(k+1)$  with the same color. Thus, it remains to explain why in every tiling, for every  $k$ , there exist both red and blue macro-tiles of rank  $k+1$ . Here we use the existence of red and blue corners in our macro-tiles. Indeed, by construction, in every macro-tile of rank  $(k+2)$  there exist some red cells and some blue cells (which are themselves macro-tiles of rank  $(k+1)$ ). This argument works for any reasonable choice of  $\alpha_k$  and  $\beta_k$ : we only need to assume that  $\alpha_k$  and  $\beta_k$  are strictly positive for all  $k$ .  $\square$

*Claim 2.* The values of  $\alpha_k$  and  $\beta_k$  can be chosen so that (a) these parameters are computable (as functions of  $k$ ) in the space and time available in the computation zone of macro-tiles of rank  $(k-1)$ , and (b) limsup of the fraction of red tiles in the globally admissible blocks of size  $n \times n$  is equal to  $h$ .

*Proof.* To prove property (b), it is enough to estimate the fractions of red and blue tiles inside a macro-tile (of growing rank). It is clear that the fraction of red tiles in level- $k$  red and blue macro-tiles (denote this by  $\nu_R(k)$ , and that for blue by  $\nu_B(k)$ ) can be computed recursively by

$$\begin{aligned} \nu_R(k) &:= \left[ \begin{array}{c} \text{fraction of red level-}(k-1) \text{ macro-tiles} \\ \text{in each red level-}k \text{ macro-tile} \end{array} \right] \times \nu_R(k-1) \\ &\quad + \left[ \begin{array}{c} \text{fraction of blue level-}(k-1) \text{ macro-tiles} \\ \text{in a red level-}k \text{ macro-tile} \end{array} \right] \times \nu_B(k-1), \\ \nu_B(k) &:= \left[ \begin{array}{c} \text{fraction of red level-}(k-1) \text{ macro-tiles} \\ \text{in a blue level-}k \text{ macro-tile} \end{array} \right] \times \nu_R(k-1) \\ &\quad + \left[ \begin{array}{c} \text{fraction of blue level-}(k-1) \text{ macro-tiles} \\ \text{in a blue level-}k \text{ macro-tile} \end{array} \right] \times \nu_B(k-1). \end{aligned}$$

Recall that the zoom factor  $N_k$  grows very fast (as  $3^{C^k}$ ). The fractions of red and blue level- $(k-1)$  macro-tiles in a red level- $k$  macro-tile depend on the choice of  $\alpha_k$  and  $\beta_k$ , and on the fraction of diversification slots (by construction, for each slot the “color” of the involved pattern is uniquely defined, and does not depend on the color of the entire macro-tile). The fraction of a macro-tile occupied by the diversification slots is only  $O(\frac{\log N_{k+1}}{N_k})$ , and the choices of  $\alpha_k$  and  $\beta_k$  are under our control.

The construction works well for a very wide range of parameters. For example, we can set  $\alpha_k = 1$  (or  $\alpha_k = \text{const}$  for any other constant) for all  $k$ , and vary  $\beta_k$  between 1 and, say,  $N_k/10$  (as a function of  $k$ ).

Note that if  $\alpha_k = \text{const}$  and  $\beta_k = \text{const}$  for all  $k$ , then the value of  $\nu_R(k)$  converges to some (computable) limit in the interval  $(0, 1)$ . Moreover, we can make this limit arbitrarily close to 1 by choosing a large enough constant  $C$  in the definition of the zoom factor  $N_k = 3^{C^k}$ .

It remains to explain how to reduce the limit of  $\nu_R(k)$  to the given right recursively enumerable number  $h < 1$ . To this end we increase the values of  $\beta_k$  (for some indices  $k$ ). There is a difficulty: we cannot compute the exact value of  $h$  in finite time, unless  $h$  is rational. Moreover, in general, we cannot even compute an  $\epsilon$ -approximation of  $h$  for every given precision  $\epsilon$ . We only know that this number is right recursively enumerable. That is, by definition, there is an algorithm which never halts and enumerate an infinite sequence of rational numbers that converge to the limit  $h$  from the right. We embed in the computation zone this algorithm and simulate it within the space and time available on the computation zone (the simulation is aborted when the computation runs out of the allocated space and time, see the discussion on p. 22). As the level of a macro-tile grows, the size of the computation zone becomes greater, and we have more and more space and time to simulate this computation. Hence, on each next level of the hierarchy of macro-tiles we obtain better and better approximations of  $h$ . (We have no tool to estimate the precision of the current approximation, but we know that the limit of the enumerated sequence is equal to  $h$ .)

Thus, each level- $k$  macro-tile obtains some approximation of  $h$  (from above). Accordingly, given the current approximation of  $h$ , we compute a suitable value of  $\beta_k$ , so that  $\nu_R(k)$  converges to  $h$  as  $k$  tends to infinity.  $\square$

Combining Claim 1 and Claim 2 completes the proof of the theorem.  $\square$

*Proof of Theorem 5.* At first we consider the case  $h < 1$ . Let us take the SFT  $\mathcal{S}$  from the proof of Proposition 5. Now we make two copies of each red proto-tile and denote the resulting shift by  $\mathcal{S}'$ . We claim that  $\mathcal{S}'$  has entropy  $h$ . Indeed, the hierarchical structure of macro-tiles gives no contribution to the entropy. Indeed, in the initial SFT  $\mathcal{S}$  every level- $k$  macro-tile of size  $L_k \times L_k$  can be reconstructed given its color (red or blue) and the macro-colors on its borderline, which requires only  $O(L_k)$  bits of information. Thus, the positive entropy of the new shift  $\mathcal{S}'$  results from the choices between two copies of each red tile (at every position where such a tile is used). Hence, the entropy is equal to the limsup of the density of red tiles, which is guaranteed to be  $h$ .

In case  $h > 1$ , we can superimpose the same construction with a trivial shift (without any local constraints) with an alphabet of size  $2^{\lfloor h \rfloor}$ .

Let us prove that  $\mathcal{S}'$  is transitive. The minimality of  $\mathcal{S}$  means that every configuration in this shift contains all globally admissible patterns. Let us fix any configuration  $x \in \mathcal{S}$ . Now we want to make a choice for each position with a red tile in  $x$  so that the resulting configuration involves all possible instantiations of every globally admissible pattern. In fact, this property is true with probability 1 if we choose an instance of each red tile in  $x$  at random.

Indeed, for every finite pattern (with finitely many red tiles) we can find in  $x$  infinitely many disjoint copies. Hence, if we choose a version of every red tile at random, then with probability 1 we obtain infinitely many copies of every instantiation of every finite pattern.  $\square$

*Remark 10.* Theorem 5 guarantees the existence of a transitive SFT with a given right recursively enumerable entropy. The construction in the proof implies that this SFT enjoys also a (weak) version of irreducibility: every two globally admissible patterns can be combined in one common configuration; moreover, every two globally admissible patterns can be positioned rather close to each other (roughly speaking, every two globally admissible macro-tiles of level  $k$  can be placed inside one and the same globally admissible macro-tile of level  $(k+2)$ ). However, the relative arrangement of two globally admissible patterns must be coherent with the global hierarchical grid of macro-tiles. For example, the vertical and the horizontal translations of a pattern  $P_1$  with respect to another pattern  $P_2$  in a common infinite configuration modulo the first rank zoom factor  $N_1$  is uniquely defined (by the this pair of patterns). Gangloff and Sablik in [25] proposed a construction with a somewhat stronger property of irreducibility, where the relative arrangement of any two globally admissible patterns is rather flexible.

## 6 On subdynamics of co-dimension 1 for self-simulating SFT

In Section 4 we used a sort of embedding of one-dimensional sequences in a two-dimensional SFT. That embedding was highly distributed: the first  $\log k$  bits of the sequence embedded in a configuration could be found in every macro-tile of rank  $k$  of this configuration. In this section we discuss a different way of embedding a (bi-infinite) one-dimensional sequence in two-dimensional configurations of an SFT; this version of embedding is less distributed and more local. With this technique we will be able to control the subdynamics of a two-dimensional shift, and as a result we will prove Theorems 6–7.

### 6.1 The general scheme of letter delegation

We are going to embed a bi-infinite sequence  $\mathbf{x} = (x_i)$  over an alphabet  $\Sigma$  into our tiling. To this end we assume that each individual  $\tau$ -tile “keeps in mind” a letter from  $\Sigma$  that propagates without change in the vertical direction. Formally speaking, a letter from  $\Sigma$  should be a part of the top and bottom colors of every  $\tau$ -tile (the letters assigned to both sides of a tile must be equal to each other), see Fig. 16 and Fig. 1. We want to guarantee that a  $\Sigma$ -sequence can be embedded in a  $\tau$ -tiling if and only if this sequence belongs to a fixed given effective shift  $\mathcal{A}$ . (We postpone for a while the discussion of *quasiperiodicity* of this embedding.)

The general plan is to “delegate” the factors of the embedded sequence to the computation zones of macro-tiles, where these factors can be validated

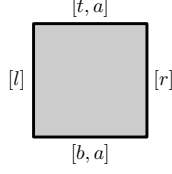


Figure 16: A tile propagating a letter  $a \in \Sigma$  in the vertical direction. Formally speaking, this tile is a quadruple of *colors*, the left side has color  $[l]$ , the right side has color  $[r]$ , the top and the bottom sides have colors  $[t, a]$  and  $[b, a]$ , respectively. The colors for the top and bottom sides involve a letter from  $\Sigma$ . We allow only tiles where the colors of the top and bottom sides involve one and the same letter.

(that is, the simulated Turing machine can verify whether these factors do not contain any forbidden subwords). By using tilings with growing zoom factors, we can guarantee that the size of the computation zone of a  $k$ -rank macro-tile grows with  $k$ . So we have at our disposal the computational resources required to run all necessary validation tests on the embedded sequence. It remains to organize the propagation of the letters of the embedded sequence to the “conscious memory” (the computation zones) of the macro-tiles of all ranks. In what follows we explain how this propagation is organized.

*The zone of responsibility of a macro-tile.* In our construction, a macro-tile of level  $k$  is a square of size  $L_k \times L_k$ , with  $L_k = N_1 \cdot N_2 \cdot \dots \cdot N_k$  (where  $N_i$  is the zoom factor on level  $i$  of the hierarchy of macro-tiles). We say that a level- $k$  macro-tile is *responsible* for the letters of the embedded sequence  $\mathbf{x}$  assigned to the columns of the (ground level) tiles of this macro-tile as well as to the columns of macro-tiles of the same rank on its left and on its right. That is, the *zone of responsibility* of a level- $k$  macro-tile is a factor of length  $3L_k$  from the embedded sequence, see Fig. 17. (The zones of responsibility of two vertically aligned macro-tiles are the same; the zones of responsibility of two horizontally neighboring macro-tiles overlap.)

*Letter assignment:* The computation zone of a level- $k$  macro-tile (of size  $m_k \times m_k$ ) is too small to contain all the letters from its zone of responsibility. So we require that the computation zone obtains as an input a (short enough) chunk of letters from its zone of responsibility. Let us say that it is a factor of length  $l_k := \log \log L_k$  from the stripe of  $3L_k$  columns constituting the zone of responsibility of this macro-tile. We will say that this chunk is *assigned* to this macro-tile.

The infinite stripe of vertically aligned level- $k$  macro-tiles share the same zone of responsibility. However, different macro-tiles in such a stripe will obtain different assigned chunks. The choice of the assigned chunk varies from 0 to  $(3L_k - l_k)$ . Therefore, we need to choose for each level- $k$  macro-tile a position of a factor of length  $l_k$  in its zone of responsibility of length  $3L_k$ . This choice is quite arbitrary. Let us say, for definiteness, that for a macro-tile  $M$  of rank  $k$



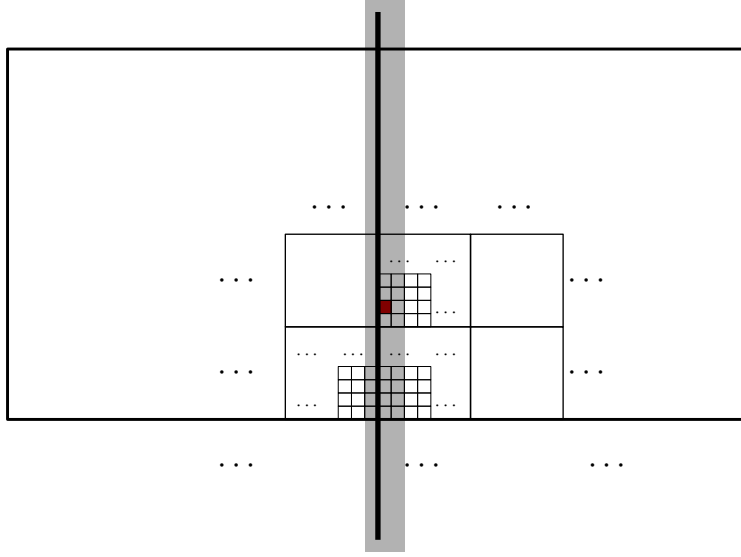


Figure 17: The zone of responsibility (the gray vertical stripe) of a macro-tile (the red square) is three times wider than the macro-tile itself.

the first position of the assigned chunk (in the stripe of length  $3L_k$ ) is defined as the vertical position of  $M$  in the father macro-tile of rank  $(k+1)$  (taken modulo  $(3L_k - l_k)$ ).

*Remark 11.* We have chosen zoom factors  $N_k$  growing doubly exponentially in  $k$ , so  $N_{k+1} \gg 3L_k$ . Hence, every chunk of length  $l_k$  from a stripe of width  $3L_k$  is assigned to some (actually, to infinitely many) of the macro-tiles “responsible” for these  $3L_k$  letters.

*Remark 12.* Since the zones of responsibility of neighboring level- $k$  macro-tiles overlap by more than  $l_k$ , every finite factor of length  $l_k$  in the embedded sequence  $\mathbf{x}$  is assigned to some level- $k$  macro-tile (even if the  $l_k$  columns containing the letters of this factor are not covered by any single level- $k$  macro-tile and touch two horizontally neighboring level- $k$  macro-tiles), see Fig. 18.

*Implementing the letter assignment by self-simulation.* In the *letter assignment* paragraph above, we formulated several requirements: how the data should be propagated from the ground level (individual tiles) to level- $k$  macro-tiles. That is, for each level- $k$  macro-tile  $\mathcal{M}$  we specified which chunk of the embedded sequence should be a part of the data fields on the computation zone of  $\mathcal{M}$ . So far we have not explained how this propagation can be implemented, i.e., how the assigned chunks can arrive at the high-level data fields. Now, we will explain how to implement the required scheme of letter assignment in a self-simulating tiling. Technically, we append to the input data of the computation zones of macro-tiles some supplementary data fields:

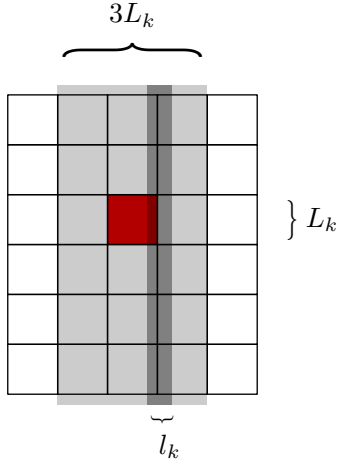


Figure 18: The macro-tile of size  $L_k$  (shown in red) is *responsible* for the vertical stripe of width  $3L_k$  shown in light gray (three times wider than the macro-tile itself). Such a macro-tile can handle a factor of length  $l_k$  of the embedded sequence that corresponds to the group of columns that touch this macro-tile as well as its neighbor on the right or on the left (an example is shown in dark gray).

- (iv) a block of  $l_k$  letters from the embedded sequence assigned to this macro-tile,
- (v) three blocks of bits of  $l_{k+1}$  letters of the embedded sequence assigned to this father macro-tile, and two uncle macro-tiles (the left and the right neighbors of the father),
- (vi) the coordinates of the father macro-tile in the “grandfather” (of rank  $(k + 2)$ ).

(In other words, the first line of the computation zone still looks similar to Fig. 14, but now it contains more input data in the first line of the computation zone.) Speaking informally, the computation in each level- $k$  macro-tile must check the consistency of the data in fields (iv), (v) and (vi). That is, if some letters from the fields (iv) and (v) correspond to the same vertical column in the zone of responsibility, then these letters must be equal to each other. Also, if a level- $k$  macro-tile plays the role of a cell in the computation zone of the level- $(k + 1)$  father, it should check the consistency of its (v) and (vi) with the bits displayed in the father’s computation zone. Lastly, we must ensure the coherence of the fields (v) and (vi) for each pair of neighboring level- $k$  macro-tiles; so this data should also be a part of the macro-colors (with a minor exception, see Remark 13 below).

Note that the data from the “uncle” macro-tiles are necessary to deal with the letters from the columns that physically belong to the neighboring macro-

tiles. So the consistency of the fields (v) is imposed also on neighboring level- $k$  macro-tiles that belong to different level- $(k+1)$  fathers (the borderline between these level- $k$  macro-tiles is also the borderline between their fathers).

The coherence of fields (iv), (v), (vi) on every level of the hierarchy implies that for each macro-tile the content of field (iv) fairly represents the assigned factor of the embedded sequence. If we want to be formal, this statement can be proven by induction on the level  $k$  of the hierarchy of macro-tiles. Indeed, by construction, the chunk of  $l_k$  letters assigned to a level- $k$  macro-tile  $M$  is “known” to the “children” and “nephews” of this macro-tile — to those level- $(k-1)$  macro-tiles that constitute  $M$  itself and its “brothers” on the left and on the right; these level- $(k-1)$  macro-tiles make sure that their father’s (or uncle’s) chunk of  $l_k$  letters is consistent with their own chunks of  $l_{k-1}$  letters. The letters assigned to the level- $(k-1)$  macro-tiles must be consistent with the letters assigned to their own children, which are level- $(k-2)$  macro-tiles, and which in turn keep on their computation zones the assigned chunks of  $l_{k-2}$  letters, and so on. On the ground level we explicitly make sure that the assigned letters are consistent with the symbols of the embedded sequence.

This scheme works properly since the chosen zoom factors  $N_k$  grow very fast, so that  $N_k \gg L_{k-1}$ . This guarantees for every level- $k$  macro-tile  $M$  that each factor of the embedded sequence of length  $l_{k-1}$  that appears in the zone of responsibility of  $M$ , is assigned to several level- $(k-1)$  macro-tiles  $M'$  that are children or nephews of  $M$ . So every letter assigned to  $M$  is validated by macro-tiles of the previous level.

This procedure of validation of letter assignment is even redundant — each letter from the zone of responsibility of a level- $k$  macro-tile  $M$  is validated at once by *many* level- $(k-1)$  macro-tiles  $M'$  inside  $M$ . The delegation still works correctly even if we exclude some level- $(k-1)$  macro-tiles from the procedure of validation of the data of their level- $k$  father. More precisely, the construction still works properly, if (a) the excluded macro-tiles occupy only  $O(1)$  successive positions in each column, and (b) the non-excluded macro-tiles constitute a connected component and, therefore, are coherent with each other. Thus, in what follows we may revise our construction and exclude, for example, the cells of the *communication wires* from validation of their father’s data, see Remark 13 below.

The computation that verifies the coherence of the data in fields (iv)–(vi) is pretty simple. It can be performed in polynomial time, and the required revision of the construction fits the usual constraints on the parameter (the size of the computation zone in a level- $k$  macro-tile is  $\text{poly} \log(N_k)$ ). For a detailed discussion of the hierarchical schema of “letter delegation” we refer the reader to [19, Section 7].

*Remark 13.* As mentioned above, the defined construction is somewhat excessive: to ensure the correct “information propagation” from the level  $(k-1)$  to the level  $k$  of the hierarchy, we do not need to keep the content of the auxiliary fields (v) and (vi) in *each* macro-tiles of level  $(k-1)$ .

We take advantage of this observation and make a minor (seemingly arti-

cial) revision of our construction. We assume that the content of the field (vi) is *empty* for the macro-tiles that play in their fathers the role of a communication wire, as well as the neighbors of the communication wires. (Every macro-tile “knows” its position in the father macro-tiles, so it knows whether it is a communication wire or not). Observe that the macro-tiles with a non-empty field (vi) form a connected component in their father macro-tile, so they must be coherent with each other. The purpose of this modification will become clear in the proof of Theorem 6 and Theorem 7 (see Remark 21 below).

*Concluding remarks: Testing against forbidden factors.* To guarantee that the embedded sequence  $\mathbf{x}$  contains no forbidden patterns, each level- $k$  macro-tile should allocate some part of its computation zone to enumerate (within the limits of available space and time) the forbidden pattern, and verify that the block of  $l_k$  letters assigned to this macro-tile contains none of the forbidden factors found.

The time and space allocated to enumerating the forbidden words grow as functions of  $k$ . To ensure that the embedded sequence contains no forbidden patterns, it is enough to guarantee that each forbidden pattern is found by macro-tiles of high enough rank, and every factor of the embedded sequence is compared (on some level of the hierarchy) with every forbidden factor.

Thus, we get a construction of a two-dimensional tiling that simulates a given effective one-dimensional shift: for a given effective one-dimensional shift  $\mathcal{A}$ , we can construct a tile set  $\tau$  such that the bi-infinite sequences that can be embedded in  $\tau$ -tilings are exactly the sequences of  $\mathcal{A}$ . In the next sections we explain how to make these tilings quasiperiodic in the case when the simulated one-dimensional shift is also quasiperiodic.

## 6.2 The letter delegation scheme combined with quasiperiodicity

In Section 3 we described a very general construction of a self-simulating tile set, and showed that the corresponding SFT enjoys the properties of quasiperiodicity or even minimality. In the previous section we upgraded this construction and superimposed on the generic scheme of self-simulation a new technique: the scheme of embedding in a tiling a sequence from some effective one-dimensional shift. In general, the new “upgraded” SFT may lose the property of quasiperiodicity. To maintain it, we will need some additional effort. The following lemma is a useful technical tool: it helps control the properties of quasiperiodicity and minimality of the tilings with embedded sequences.

**Lemma 1.** *For a tile set defined in Section 6.1, two globally admissible macro-tiles of rank  $k$  are equal to each other if these macro-tiles*

- (a) *contain the same bits in fields (i)–(vi) in the input data on the computation zone, and*
- (b) *the factors of the encoded sequence corresponding to the zones of responsibility of these macro-tiles (in the corresponding vertical stripes of width*

$3L_k$ ) are equal to each other.

*Proof.* The proof is by induction on rank  $k$ . For a macro-tile of rank 1, the statement follows directly from the construction. In the inductive step, we are given a pair of macro-tiles  $M_1$  and  $M_2$  of rank  $(k + 1)$  that hold identical data in fields (i)–(vi), and the factors (of length  $3L_k$ ) from the encoded sequences in the zones of responsibility of  $M_1$  and  $M_2$  are also equal to each other. We observe that the corresponding cells in  $M_1$  and  $M_2$  (which are macro-tiles of rank  $k$ ) contain the same data in their own fields (i)–(vi), since the communication wires of  $M_1$  and  $M_2$  carry the same information bits, their computation zones represent exactly the same computations, etc. Therefore, we can apply the inductive assumption.  $\square$

The statement of Lemma 1 is chosen in such a way that the inductive proof is simple. However, to apply this lemma, it is useful to separate the data contained in a macro-tile into two parts: the data relevant to the construction of the hierarchical structure of macro-tiles, and the symbols of the embedded sequence:

**Corollary 2.** *For a tile set defined in Section 6.1, two globally admissible level- $k$  macro-tiles  $M_1$  and  $M_2$  are equal to each other if the following conditions hold true*

- $M_1$  and  $M_2$  have the same position (modulo  $N_{k+1}$ ) with respect to their fathers, which are level- $(k + 1)$  macro-tiles;
  - the fathers of  $M_1$  and  $M_2$  have the same position (modulo  $N_{k+2}$ ) with respect to the grandfathers of  $M_1$  and  $M_2$ , which are in turn level- $(k + 2)$  macro-tiles;
  - if  $M_1$  and  $M_2$  play the role of communication wires in their fathers, then these wires communicate the same value (i.e, both of them communicate 0 or both of them communicate 1);
  - if  $M_1$  and  $M_2$  are involved in the computation zone in their fathers, then they contain identical finite patterns of the space-time diagram;
- (1)

and

- the factors of length  $3L_k$  of the embedded sequence for which  $M_1$  and  $M_2$  are responsible are equal to each other; moreover, the factors of length  $3L_{k+1}$  of the embedded sequence for which the fathers of  $M_1$  and  $M_2$  are responsible, are also equal to each other.
- (2)

*Proof.* Conditions (1) and (2) imply that the fields (i)–(vi) in the input data on the computation zones of  $M_1$  and  $M_2$  are equal to each other. Besides, (2) implies that  $M_1$  and  $M_2$  contain in their zones of responsibility the same factors of the embedded sequence. Thus, we can apply Lemma 1.  $\square$

*Remark 14.* We made Condition (2) very strict (we could be less restrictive on the symbols in the zones of responsibility of the fathers of  $M_1$  and  $M_2$ ) in order to simplify the future applications of this corollary.

To control the property of quasiperiodicity of self-simulating tilings, we will use two simple lemmas concerning quasiperiodic sequences. One of them (Lemma 2, which is known from [9, 18]) is purely combinatorial. In the other one (Lemma 3, which to the best of our knowledge is new), we combine the combinatorial properties with an algorithmic twist. For the sake of self-containedness, we give the proofs of both of these lemmas in the next section.

**Lemma 2.** *Let  $\mathbf{x} = (\dots x_{-1}x_0x_1x_2\dots)$  be a bi-infinite recurrent sequence,  $v = x_sx_{s+1}\dots x_{s+N-1}$  be an  $N$ -letter factor in  $\mathbf{x}$ , and  $q$  a positive integer. Then there exists an integer  $t > 0$  such that another copy of  $v$  appears in  $\mathbf{x}$  with a translation  $q \cdot t$ , i.e.,*

$$x_sx_{s+1}\dots x_{s+N-1} = x_{s+qt}x_{s+qt+1}\dots x_{s+qt+N-1}. \quad (3)$$

*In other words, in  $\mathbf{x}$  there exists another instance of the same factor  $v$  with a translation divisible by  $q$ .*

*Moreover, if  $\mathbf{x}$  is quasiperiodic, then for all integers  $q$  and  $N$  there exists an integer  $L = L(\mathbf{x}, N, q)$  such that the absolute value of  $(qt)$  in (3) can be chosen less than  $L$ . (Note that  $L$  does not depend on  $s$ , i.e., it does not depend on a specific instance of the pattern  $v$  in  $\mathbf{x}$ .)*

*Remark 15.* From the symmetry, it follows that a similar statement can be proven with an integer  $t < 0$ . Thus, we can find in  $\mathbf{x}$  two copies of  $v$ , one on the left and the other one on the right of the originally given instance of this factor, both copies with translations divisible by  $q$ . Clearly, we can iterate this procedure and obtain a bi-infinite (infinite to the left and to the right) sequence of copies of  $v$ , where each copy has a translation (with respect to the original instance of the factor) divisible by  $q$ .

*Remark 16.* The property of quasiperiodicity guarantees by definition that there is a uniform bound for the gaps between neighboring appearances of each  $N$ -letter factor  $v$  in  $\mathbf{x}$ . Lemma 2 claims that there is also a uniform bound for the gaps between neighboring appearances of each  $N$ -letter factor  $v$  in  $\mathbf{x}$ , for appearances at positions that are congruent to each other modulo  $q$ .

*Remark 17.* For higher dimensions, Lemma 2 can be generalized as follows. Let  $\mathbf{x}$  be a  $d$ -dimensional recurrent configurations on  $\mathbb{Z}^d$  (over a finite alphabet  $\Sigma$ ),  $v$  be a finite pattern in  $\mathbf{x}$ , and  $q$  be a positive integer. Then in  $\mathbf{x}$  there exists another instance of the same pattern  $v$  such that the translation between these two copies of  $v$  is a non-zero vector  $\bar{t} = (t_1, \dots, t_d)$ , where each component  $t_i$  is divisible by  $q$ . Moreover, if  $\mathbf{x}$  is quasiperiodic, then the size of each  $t_i$  can be bounded by some number  $L$  that depends only on  $\mathbf{x}$ ,  $q$ , and  $v$  (but not on a specific instance of the pattern  $v$  in  $\mathbf{x}$ ).

Below we prove Lemma 2 for  $d = 1$ ; our argument can be easily extended to the general case (for any  $d > 1$ ).

In the next lemma we use the following notation. For a configuration  $\mathbf{x}$  (over some finite alphabet) we denote with  $\mathcal{S}(\mathbf{x})$  the shift that consists of all configurations  $\mathbf{x}'$  containing only patterns from  $\mathbf{x}$ . If a shift  $\mathcal{T}$  is minimal, then  $\mathcal{S}(\mathbf{x}) = \mathcal{T}$  for all configurations  $\mathbf{x} \in \mathcal{T}$ .

**Lemma 3.** (a) *Let  $\mathcal{T}$  be an effective minimal shift. Then for every  $\mathbf{x} = (x_i)$  from  $\mathcal{T}$  and every periodic configuration  $\mathbf{y} = (y_i)$ , the direct product  $\mathbf{x} \otimes \mathbf{y}$  (the bi-infinite sequence of pairs  $(x_i, y_i)$  for  $i \in \mathbb{Z}$ ) generates a minimal shift, i.e.,  $\mathcal{S}(\mathbf{x} \otimes \mathbf{y})$  is minimal. (b) *If, in addition, the sequence  $\mathbf{x}$  is computable, then the set of patterns in  $\mathcal{S}(\mathbf{x} \otimes \mathbf{y})$  is also computable.**

*Remark 18.* In general, different configurations  $\mathbf{x} \in \mathcal{T}$  in the product with one and the same periodic  $\mathbf{y}$  can result in different shifts  $\mathcal{S}(\mathbf{x} \otimes \mathbf{y})$ .

In the next section we employ these lemmas in the proofs of Theorem 6 and Theorem 7.

### 6.3 Proof of Theorems 6–7

*The proof of Theorem 6.* The proof of statement (b) of Theorem 6 is simple. Let a one-dimensional configuration  $\mathbf{x} \in \mathcal{A}$  be a projection of a two-dimensional configuration  $\mathbf{y} \in \mathcal{B}$ . Let  $w$  be a factor of  $\mathbf{x}$ . Then  $w$  is obtained as a projection of a vertical stripe of width  $|w|$  in  $\mathbf{y}$ . Let us take any finite part  $P$  of this stripe. Since  $\mathbf{y}$  is quasiperiodic, the pattern  $P$  reappears in every large enough region in  $\mathbf{y}$ . The projections of all these patterns result in copies of the same factor  $w$  in  $\mathbf{x}$ . Hence,  $w$  reappears in every large enough subword of  $\mathbf{x}$ .

Now we prove statement (a) which is much more difficult. In this proof we combine arguments from Sections 6.1–6.2 and show that the defined embedding of a quasiperiodic one-dimensional shift in a two-dimensional tiling results in a quasiperiodic two-dimensional SFT. The main technical tools in the argument are Lemma 1 (and Corollary 2) and Lemma 2.

When we prove quasiperiodicity of our tiling, we use its hierarchical structure: instead of looking for copies of all pattern  $P$  in the tiling, we need only to show the existence of copies for each  $(2 \times 2)$ -block of level- $k$  macro-tiles (for every integer  $k$ ). To prove this property, we use Corollary 2. To explain the general scheme of the proof, we introduce a technical notion of a *grid of siblings*.

**Definition 2.** *We say that a configuration  $\mathbf{x}$  (a tiling) contains a grid of siblings for a  $(2 \times 2)$ -block  $P$  of level- $k$  macro-tiles, if in  $\mathbf{x}$  there is an infinite grid of patterns  $P_{ij}$ ,  $i, j \in \mathbb{Z}$  (with a horizontal step of  $q_x$  and a vertical step of  $q_y$ ) such that every  $P_{ij}$  is also a  $(2 \times 2)$ -block of level- $k$  macro-tiles, and the four macro-tiles in  $P_{ij}$  play the same roles in the hierarchical structure (in the sense of Condition (1) in Corollary 2) as the corresponding four macro-tiles in the original pattern  $P$ .*

*Moreover, we say that a grid of siblings is horizontally aligned with  $P$ , if the grid includes a column of patterns  $P_{i0j}$ ,  $j \in \mathbb{Z}$  that are aligned with  $P$  in the  $x$ -coordinate (i.e., they share with  $P$  the same  $(2L_k)$  columns of the tiling).*

*Remark 19.* The initial block  $P$  does not necessarily belong to the grid of its siblings (even if this grid is horizontally aligned with  $P$ ).

The patterns in a *grid of siblings* are not necessarily equal to each other and to the original pattern  $P$ . Indeed, they all play similar roles in the hierarchical structure of macro-tiles, but they may involve different parts of the embedded sequence. However, we show that in a horizontally aligned grid of siblings there are *some* patterns that are equal to the original  $P$ .

**Lemma 4.** *If a  $(2 \times 2)$ -block  $P$  of level- $k$  macro-tiles in our tiling has a horizontally aligned grid of siblings  $P_{ij}$  (with a horizontal step of  $q_x$  and a vertical step of  $q_y$ ), then some elements of this grid are equal to the pattern  $P$ . Moreover, copies of  $P$  in the grid of  $P_{ij}$  can be found in every large enough  $(M \times M)$ -pattern of the tiling (the value of  $M$  can be computed as a function of  $k$ ,  $q_x$ , and  $q_y$ ).*

*Proof of lemma:* We are going to apply Corollary 2. By definition, for all patterns  $P_{ij}$  in the grid of siblings, we have Condition (1) from Corollary 2. It remains to find one  $P_{ij}$  that satisfies Condition (2) of this corollary.

To this end, we use Lemma 2. More specifically, we focus on the factor of the embedded sequence which covers the zones of responsibility of the four macro-tiles in the pattern  $P$  and their fathers, and find with the help of Lemma 2 a copy of this factor in some other place of the embedded sequence, with a non-zero horizontal translation divisible by  $q_x$  (the horizontal step of the grid).

Since the grid of siblings is horizontally aligned with  $P$ , the found copy of the factor in the embedded sequence will be aligned with some  $P_{ij}$ . Due to Corollary 2, such a pattern must be equal to  $P$ .

The *moreover* part of Lemma 2 (see also Remark 16) guarantees that the copies of the required factor of the embedded sequence (taken with translations divisible by  $q_x$ ) are dense: they appear in every large enough segment of the embedded sequence. Hence, we can find a block  $P_{ij}$  that is equal to  $P$  in every large enough part of the grid and, therefore, in every large enough  $(M \times M)$ -pattern of the tiling. (The minimal size of  $M$  depends on the size of  $k$ , and on the steps of the grid  $P_{ij}$ , but not on a specific instance of  $P$ .)  $\square$

*Remark 20.* In the proof of the lemma we have shown a slightly stronger statement: if one pattern  $P_{ij}$  in the grid is equal to  $P$ , then the entire column of patterns  $P_{ij'}$  in the grid consists of copies of  $P$  (since all vertically aligned patterns  $P_{ij'}$  share the same factor from the embedded sequence). However, we will only use the fact that the copies of  $P$  are dense (appear in every large enough pattern in the tiling).

In what follows we systematically apply Lemma 4 to different patterns  $P$ . In each case, to apply the lemma we only need to find a *horizontally aligned grid of siblings*, with uniformly bounded steps of  $q_x$  and  $q_y$ .

*Case 1: Assume all macro-tiles in a  $(2 \times 2)$ -block are skeletons.* In this case we can immediately include the given block  $P$  in a grid of  $(2 \times 2)$ -blocks of level- $k$  macro-tiles, with the vertical and horizontal steps of  $L_{k+2}$ . Each block in this



grid has exactly the same position with respect to its father of rank  $(k + 1)$  and its grandfather of rank  $(k + 2)$ . Thus, all blocks in this grid are similar to each other in the sense of Condition (1), see the grid of siblings in Fig. 19. Now we can apply Lemma 4.

*Case 2: Computation zone.* Let us consider now the case when the  $(2 \times 2)$ -block of level- $k$  macro-tiles touches the computation zone. In this case we employ the trick introduced in Section 3.2. Recall that for each  $(2 \times 2)$ -window that touches the computation zone, there are only  $O(1)$  admissible patterns from the space-time diagram. For each possible position of a  $(2 \times 2)$ -window in the computation zone and for each possible  $(2 \times 2)$ -pattern in the space-time diagram, we reserved a special  $(2 \times 2)$ -*diversification slot* in a macro-tile, which is essentially a block of size  $2 \times 2$  in the “free” zone of the macro-tile, as shown in Fig. 12.

Note that this diversification slot is placed far away from the computation zone and from all communication wires, but in the same vertical stripe as the “original” position of this block. Further, we defined the neighbors around each diversification slot in such a way that “conscious memory” (i.e., the content of input data fields (i)–(vi)) of the macro-tiles inside this slot is uniquely defined (here we use property (p2), see Section 3.2).

Hence, the sibling that we found for the original block of level- $k$  macro-tiles (the sibling which is placed in one of the diversification slots) is similar to the original block in the sense of the content of its computational zone and also in the sense of the involved factor of the embedded sequence.

Observe that there are infinitely many “homologues” of the found sibling — there are similar diversification slots in each level- $(k + 1)$  macro-tile that take the same position with respect to their fathers and grandfathers, see Fig. 20. All these blocks are similar to each other in the sense of Condition (1). Thus, we obtain a regular grid (with the horizontal and vertical steps of  $L_{k+2}$ ) of blocks that all satisfy Condition (1), and we can apply Lemma 4.

*Case 3: Communication wires.* Now we consider the case when a  $(2 \times 2)$ -block of level- $k$  macro-tiles involves a part of a communication wire. Due to property (p3) (see p. 24) we may assume that only one wire is involved. The bit transmitted by this wire is either 0 or 1; in both cases, due to property (p4) we can find another similar  $(2 \times 2)$ -block of level- $k$  macro-tiles (at the same position within the father macro-tile of rank  $(k + 1)$  and with the same bit included in the communication wire) in every macro-tile of level  $(k + 2)$  (see the grid of siblings in Fig. 21).

We can immediately find blocks that are similar to ours in the sense of Condition (2), with a vertical translation of size  $O(L_{k+2})$ . These blocks are obviously vertically aligned with the original block.

*Remark 21.* In this case, we find for the given block of level- $k$  macro-tiles a sibling which has the same position with respect to their father macro-tile, but possibly different position with respect to the grandfather. However, due to Remark 13 (p. 42), for a block of macro-tiles involving a cell from a communication wire, this minor displacement does not affect the field (vi) of the computation

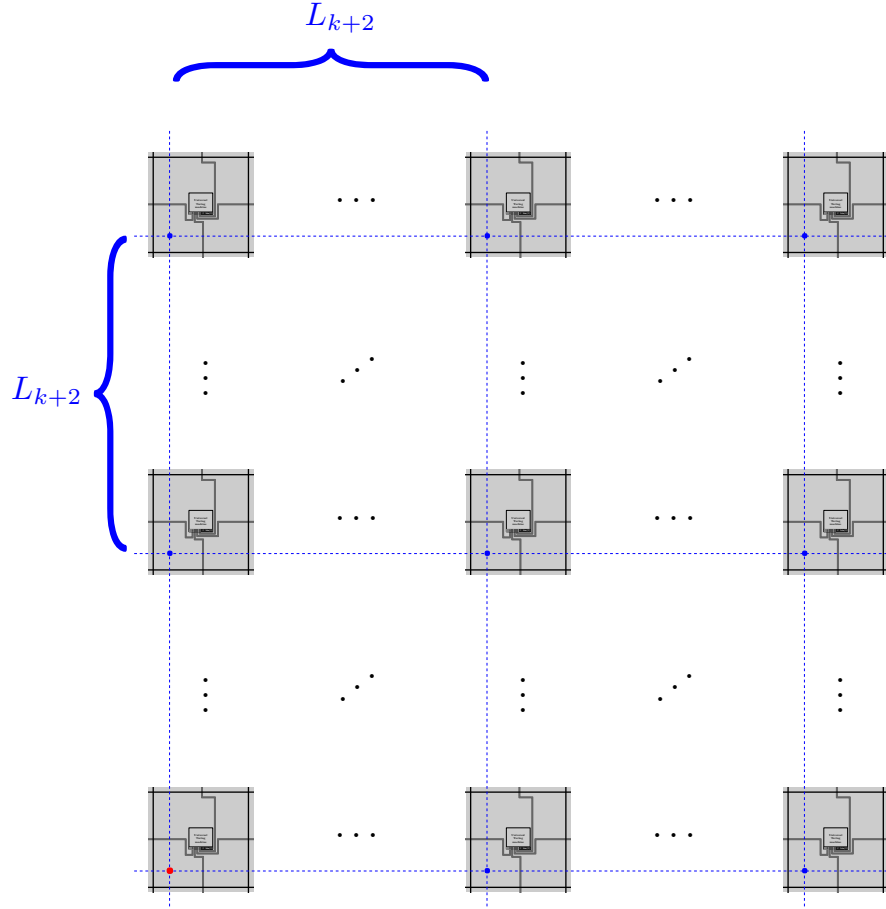


Figure 19: We are looking for a copy of a block of level- $k$  macro-tiles, which is shown as a red spot. We take its “siblings” — blocks at the same position in the father level- $(k+1)$  macro-tiles (shown as gray squares), which in turn must have the same position with respect to their level- $(k+2)$  fathers. These siblings form a regular grid (shown in blue). The step of the grid of blue siblings is equal to  $L_{k+2}$ , i.e., to the size of the grandfather of the initial block.

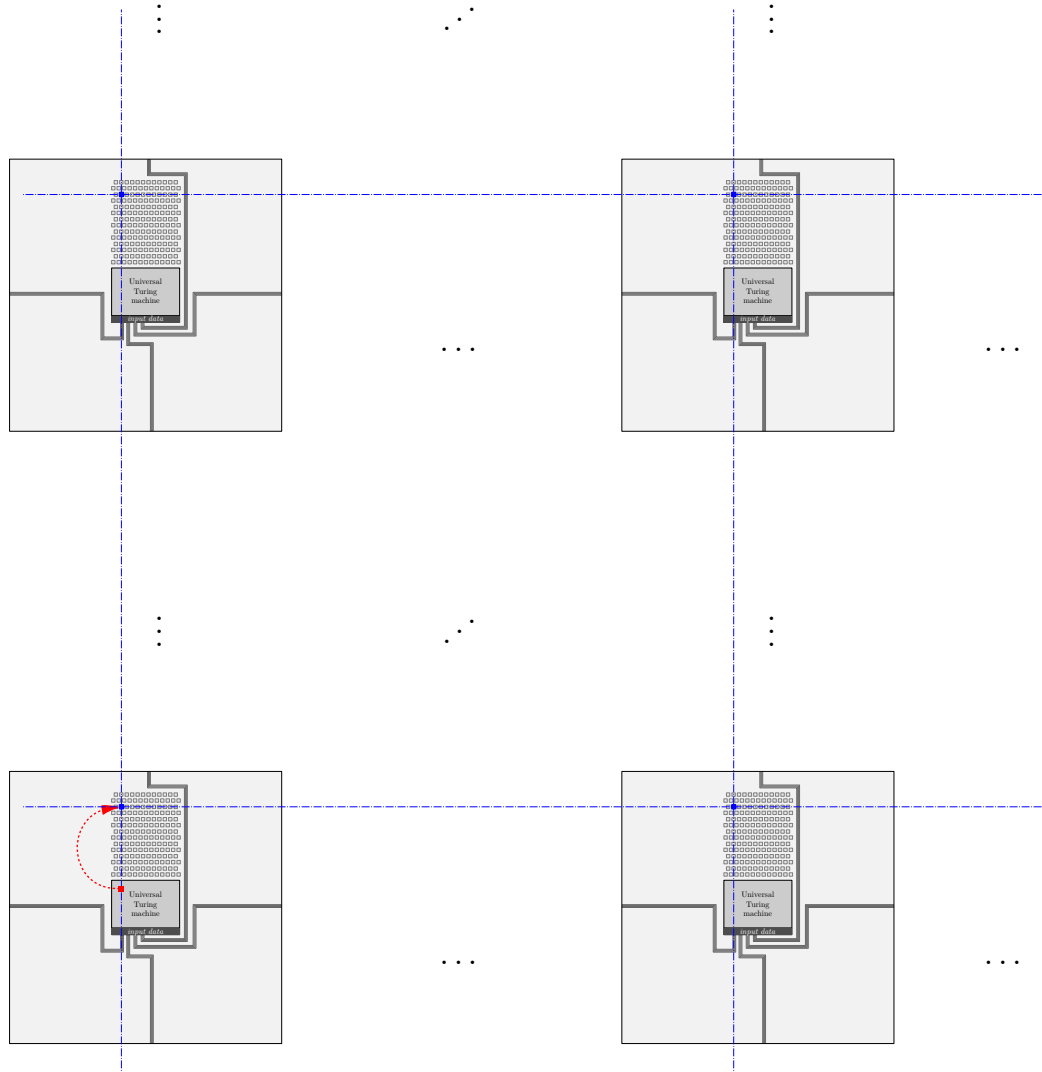


Figure 20: We are looking for a copy of a block of level- $k$  macro-tiles (shown as a red spot) touching the computation zone of its father. We find a periodic grid of *diversification slots* (shown as blue spots) that contain “siblings” of the original block in the sense of Condition (1). The step of this grid is  $L_{k+2}$ . Note that not all these siblings are similar to the original block in the sense of Condition (2).

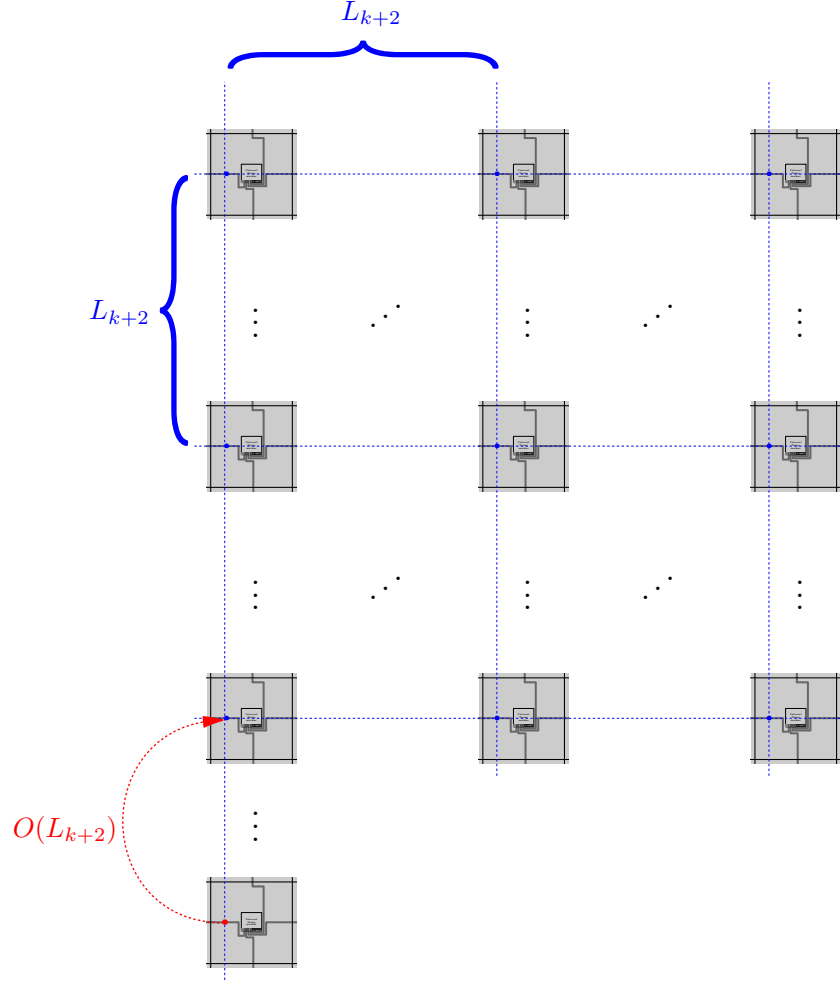


Figure 21: We are looking for a copy of a block of level- $k$  macro-tiles (shown as a red spot) touching its father’s communication wire. We find a periodic grid with step  $L_{k+2}$  (shown as blue spots), which consists of “siblings” of the original block in the sense of Condition (1). Notice that the original block does not belong to this grid: it is aligned with one of the grid column but not with the rows.

zone (which is empty). Therefore, the found siblings have exactly the same data in all fields (i)-(vi) as the initial  $(2 \times 2)$ -block of macro-tiles.

It remains to extend the found column of siblings of  $P$  and obtain a two-dimensional grid of similar siblings. We can take the number  $L_{k+2}$  as the horizontal step of the grid since a horizontal translation of size  $L_{k+2}$  does not change the position of  $(2 \times 2)$ -blocks of level- $k$  macro-tiles with respect to their fathers and grandfathers, see Fig. 21. Then, we can again apply Lemma 4.

Thus, we have constructed a tile set  $\tau$  such that every  $L_k \times L_k$  pattern that appears in a  $\tau$ -tiling (and which is necessarily covered by a  $(2 \times 2)$ -block of level- $k$  macro-tiles) must also appear in every large enough square in this tiling. So, the constructed tile set satisfies the requirements of Theorem 6.  $\square$

*The proof of Corollary 1.* To prove Corollary 1 we combine Theorem 6 with a fact from [11]: there exists a one-dimensional shift  $\mathcal{S}$  that is quasiperiodic, and for every configuration  $\mathbf{x} \in \mathcal{S}$  the Kolmogorov complexity of all factors is linear, i.e.,  $C(x_i x_{i+1} \dots x_{i+n}) = \Omega(n)$  for all  $i$  (here  $C(w)$  denotes the Kolmogorov complexity of a string  $w$ ).  $\square$

*The proof of Theorem 7.* The proof of Theorem 7(b) is very similar to the proof of Theorem 6(b) and rather simple. We focus on the proof of Theorem 7(a).

First of all, we stress that the proof of Theorem 6(a) discussed above does not imply Theorem 7(a). If we take an effective minimal one-dimensional shift  $\mathcal{A}$  and plug it into the construction from the proof of Theorem 6, we obtain a tile set  $\tau$  (simulating  $\mathcal{A}$ ) which is quasiperiodic but not necessarily minimal.

The property of minimality can be lost even for a periodic shift  $\mathcal{A}$ . Indeed, assume that the minimal period  $t > 0$  of the configurations in  $\mathcal{A}$  is a factor of  $L_k$  (i.e., of the size of each level- $k$  macro-tile in our self-simulating tiling). Then we can extract from the resulting SFT  $\tau$  nontrivial subshifts  $T_i$ ,  $i = 0, 1, \dots, t-1$  corresponding to the position of the embedded one-dimensional configuration with respect to the grid of macro-tiles.

To overcome this obstacle, we superimpose some additional constraints on the embedding of the simulated  $\mathbb{Z}$ -shift in a  $\mathbb{Z}^2$ -tiling. Roughly speaking, we will enforce only “standard” positioning of the embedded 1D sequences with respect to the grid of macro-tiles. This will not change the class of the one-dimensional sequences that can be embedded in a tiling (we still get all configurations from a given minimal shift  $\mathcal{A}$ ), but the classes of all valid tilings will reduce to some minimal SFT on  $\mathbb{Z}^2$ .

*The standardly aligned grid of macro-tiles:* In general, the hierarchical structure of macro-tiles permits uncountably many ways of cutting the plane into macro-tiles of different ranks. We fix one particular variant of this hierarchical structure and say that a grid of macro-tiles is *standardly aligned* if for each level  $k$  the point  $(0, 0)$  is the bottom-left corner of a level- $k$  macro-tile, see Fig. 22. This means that the tiling is cut into level- $k$  macro-tiles of size  $L_k \times L_k$  by vertical lines with abscissae  $x = L_k \cdot t'$  and ordinates  $y = L_k \cdot t''$ , with  $t', t'' \in \mathbb{Z}$ . This structure has a kind of degeneration: the vertical line  $(0, *)$  and the horizontal

line  $(*, 0)$  serve as separating lines for macro-tiles of all ranks. This specific structure of macro-tiles is obviously computable.

*The canonical representative of a minimal shift:* A minimal effectively closed 1D-shift  $\mathcal{A}$  is always computable, i.e., the set of finite patterns that appear in configurations of this shift is computable, see [16, Corollary 4.9] or [15, Proposition 9.6]. It follows immediately that  $\mathcal{A}$  contains some computable configuration: we can incrementally increase a pattern maintaining the property that it is globally admissible, i.e., appears in configurations of the shift. Let us fix one (arbitrary) computable configuration  $\mathbf{x}$ ; in what follows we call it *canonical*.

*The standard embedding of the canonical configuration:* We superimpose the canonical configuration  $\mathbf{x}$  on the standardly aligned grid of macro-tiles: we take the direct product of the hierarchical structures of the standardly aligned grid of macro-tiles with the canonical configuration  $\mathbf{x}$  from  $\mathcal{A}$  (that is, each tile with coordinates  $(i, j)$  “contains” in itself the letter  $x_i$  from the canonical configuration). We split the rest of the proof into five claims.

*Claim 1 (purely combinatorial):* Let  $m, L$  be integers and  $w$  be a word. If the factor  $w$  appears at least once in the standard embedding at the position  $m \bmod L$ , then this happens infinitely often:  $w$  reappears in the standard embedding at infinitely many positions  $i$  congruent to  $m \bmod L$ . Moreover, such positions  $i$  can be found in every large enough factor of the standard embedding. This claim follows from Lemma 3(a) applied to the product of the canonical configuration  $\mathbf{x}$  with the periodic sequence

$$\dots 1\ 2\ 3\ \dots\ L\ 1\ 2\ 3\ \dots\ L\ 1\ 2\ \dots$$

This claim implies that the combinatorial structure of the standard embedding of the canonical configuration is regular: if a factor  $w$  appears at least once in the standard embedding with a horizontal coordinate  $(m \bmod L_k)$  (where  $L_k$  is the size of the level- $k$  macro-tiles), then  $w$  reappears at the same position with respect to the grid of level- $k$  macro-tiles in every large enough pattern of this configuration.

*Remark 22.* We will say that an infinite configurations is *quasi-standard* if it contains only finite patterns that appear in the standard embedding of the canonical configuration. The comment above applies to all quasi-standard configurations: if a factor  $w$  appears at least once in the standard embedding at the position  $m \bmod L_k$ , then  $w$  reappears at the same position (with respect to the grid of level- $k$  macro-tiles) in all quasi-standard configurations. On the other hand, if a factor  $w$  never appears at the position  $m \bmod L_k$  in the standard embedding, it cannot appear at this position in any quasi-standard configuration.

*Claim 2:* Given a pattern  $w$  of size  $n \leq L_k$  and an integer  $i$ , we can algorithmically verify whether the factor  $w$  appears in the standard embedding of the canonical representative at the position  $(i \bmod L_k)$  relative to the grid of level- $k$  macro-tiles. This follows from Lemma 3(b) applied to the superposition of the canonical configuration  $\mathbf{x}$  with the periodical grid of squares of size  $L_k \times L_k$ .

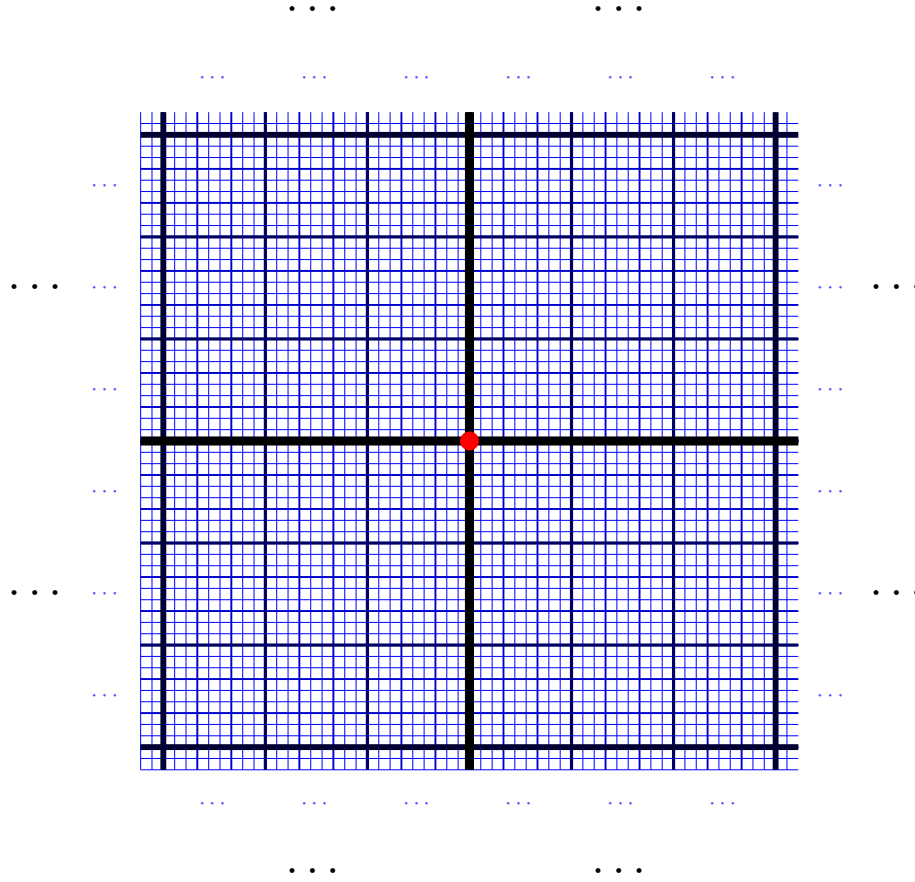


Figure 22: An example of a standardly aligned grid of macro-tiles (in this example we use on each level the same zoom factor  $N = 3$ , so every square of rank  $k$  consists of  $3 \times 3$  squares of the previous rank). The central point (marked red) is a corner of four squares (macro-tiles) of each rank  $k = 1, 2, 3, \dots$

*Remark 23.* This verification procedure is computable, but its computational complexity can be very high. To perform the necessary computation, the space and time needed might be much bigger than the length of  $w$  and  $L_k$ .

*Upgrade of the main construction:* Before we go further, we need to update the construction of the self-simulating tiling from the proof of Theorem 6. Up to now, we have required that every macro-tile (of every level  $k$ ) performs in its computation zone the standard *verification procedure*, which checks whether the delegated factor of the embedded sequences contains no patterns forbidden for the shift  $\mathcal{A}$ . Now we make the verified property stronger: we require that the delegated factor contains only factors allowed in the shift  $\mathcal{A}$ , and these factors must be located at the positions (relative to the grid of macro-tiles) permitted for factors in the standard embedding of the canonical configuration  $\mathbf{x}$ . This property is computable (due to Lemma 3(b)), so every forbidden pattern, or a pattern in a forbidden position, will be discovered in a computation in a macro-tile of high enough rank.

The computational complexity of this procedure can be very high (see Remark 23), and we cannot guarantee that the forbidden patterns of small length are discovered by the computations in macro-tiles of small size. However, we do guarantee that each forbidden pattern, or a pattern in a forbidden position, is discovered by a computation in some macro-tile of high enough rank.

*Claim 3: The new tile set admits valid tilings of the plane.* By construction, there exists at least one valid tiling. Indeed, the standard embedding of the canonical representative corresponds to a valid tiling of the plane: in this specific tiling, the macro-tiles of all rank never find any forbidden placement of patterns of the embedded sequence.

*Remark 24.* Our tile set admits many (in fact, infinitely many) different tilings. We cannot guarantee that all valid tilings represent the *standard embedding of the canonical configuration*  $\mathbf{x}$  defined above. However, we can guarantee a weaker property: *locally* all valid tilings look similar to each other. More specifically, all valid configurations are *quasi-standard* in the sense of Remark 22.

*Claim 4: The new tile set simulates the shift  $\mathcal{A}$ .* This follows immediately from the construction: the embedded sequence must be a configuration without factors forbidden for  $\mathcal{A}$ .

*Claim 5: For the constructed tile set  $\tau$  the set of all tilings is a minimal shift.* We need to show that every  $\tau$ -tiling contains all patterns that can appear in at least one  $\tau$ -tiling. Similarly to the proof of Theorem 6, it is enough to prove this property for  $(2 \times 2)$ -blocks of level- $k$  macro-tiles.

The argument is similar to the proof in Theorem 6. Let us fix a  $(2 \times 2)$ -block of level- $k$  macro-tiles that appears in a tiling  $T$  and denote it  $B$ . We need to show that  $B$  reappears in every valid tiling. That is, in every other tiling  $T'$  we must find a similar  $(2 \times 2)$ -block of level- $k$  macro-tiles that is identical to  $B$ . The main technical tools are Lemma 1 and Corollary 2. More specifically, we need to find in  $T'$  a  $(2 \times 2)$ -block of level- $k$  macro-tiles to which we can apply Conditions (1) and (2), see p. 44. We will find such a block in  $T'$  in two steps.



At first we focus on Condition (1). We re-employ the argument from the proof of Theorem 6 and study separately three cases: our block of macro-tiles either appears in the “skeletons” area, or it involves part of the computation zone of the father level- $(k+1)$  macro-tile, or it touches one of the communication wires of the father level- $(k+1)$  macro-tile. In each of these three cases we can find a periodic grid of blocks with exactly the same position with respect to their fathers and grandfathers, and which are similar to the original block in the sense of Condition (1) (see Cases 1-3 of the proof of Theorem 6). In each of these cases, the  $(2 \times 2)$ -block of level- $k$  macro-tiles that look similar to the original block  $B$  in the sense of Condition (1) form in every tiling  $T'$  a regular grid with the horizontal and vertical steps  $L_{k+2}$ . (Note that similar macro-tiles may appear also in other positions, outside of this regular grid; but we focus only on the regular grid of blocks with the required properties.) However, the blocks that appear at nodes of this grid are not necessarily equal to each other: the involved macro-tiles may contain different factors of the embedded sequence.

We need to find in the infinite grid of  $(2 \times 2)$ -blocks a position where the block is equal to the original block  $B$ . Due to Corollary 2, it is enough to find a block where we can apply Condition (2). Denote  $w$  the factor of the embedded sequence that covers the zones of responsibility of the macro-tiles in  $B$ . Let  $m$  be the horizontal coordinate of this factor with respect to the grid of level- $(k+2)$  macro-tiles. By construction, if a factor  $w$  of the embedded sequence may appear in a valid tiling at the position  $m$  modulo  $L_{k+2}$ , than it appears at least once (and, therefore, infinitely often, see Claim 1) at this specific position in the standard embedding of the canonical configuration.

Now we use Remark 22 above. We know that the factor  $w$  appears at least once at the position  $m \bmod L_{k+2}$  in every large enough pattern of the standard embedding of the canonical configuration. Therefore, it appears at the same position in every other valid configuration of our tile set. This concludes the proof of minimality.  $\square$

## 7 The proofs of the combinatorial lemmas

*Proof of Lemma 2.* Denote by  $N$  the length of  $v$ . We are given that  $v$  is a factor of  $\mathbf{x}$ . W.l.o.g., we may assume that  $v = \mathbf{x}_{[0:N-1]}$ . We need to prove that  $v$  reappears again in  $\mathbf{x}$  with a shift  $t \cdot q$ , i.e.,  $v = \mathbf{x}_{[tq:tq+N-1]}$  for some  $t > 0$ .

Since  $\mathbf{x}$  is recurrent, there exists an integer  $l_1 > 0$  such that the pattern  $v$  appears once again in  $\mathbf{x}$  with the shift of size  $l_1 > l_0$  to the right,

$$v = \mathbf{x}_{[l_1+1:l_1+N-1]}.$$

If  $q$  is a factor of  $l_1$ , then we are done. Otherwise (if  $q$  is not a factor of  $l_1$ ), we use the recurrency of  $\mathbf{x}$  once again (now for a bigger pattern). From recurrence it follows that there exists an  $l_2 > 0$  such that  $\mathbf{x}_{[l_0:l_0+l_1+N-1]}$  appears again in  $\mathbf{x}$  with some shift of size  $l_2$  to the right,

$$\mathbf{x}_{[l_0:l_0+l_1+N-1]} = \mathbf{x}_{[l_2:l_2+l_1+N-1]}.$$

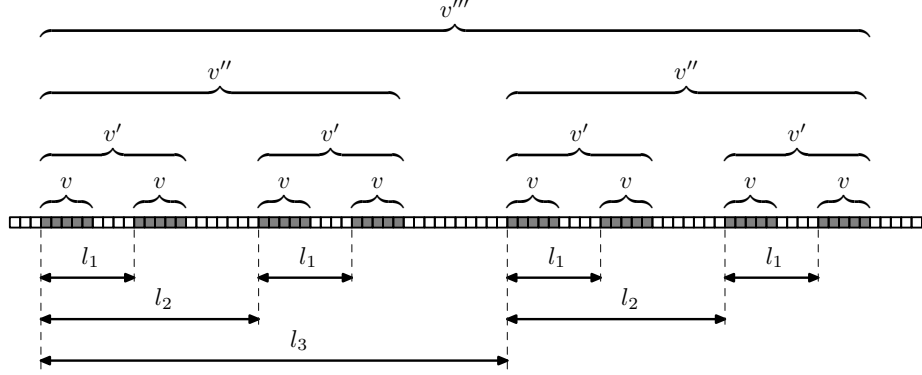


Figure 23: Reappearance of factors in a quasiperiodic sequence. At first we find a reappearance of the factor  $v$ , then we find a reappearance of the factor  $v'$  involving two copies of  $v$ , then a reappearance of  $v''$  involving two copies of  $v'$  (and four copies of  $v$ ), etc.

Now we have two new occurrences of  $v$  in  $\mathbf{x}$ ,

$$v = \mathbf{x}_{[l_1:l_1+N-1]} = \mathbf{x}_{[l_2:l_2+N-1]} = \mathbf{x}_{[l_1+l_2:l_1+l_2+N-1]}.$$

If  $q$  is a factor of  $l_2$  or  $l_1+l_2$ , we get a subword in  $\mathbf{x}$  (starting at the position  $l_2$  or, respectively,  $l_2+l_1$ ) that is equal to  $v$ . Otherwise, we repeat the same argument again, and find in  $\mathbf{x}$  a copy of an even greater pattern  $\mathbf{x}_{[l_0:l_0+l_1+l_2+N-1]}$ . Repeating this argument  $k$  times, we obtain a sequence of positive integers  $l_1, \dots, l_k$  such that the word  $v$  reappears in  $\mathbf{x}$  with all shifts composed of terms  $l_i$  (all possible sums of several different  $l_i$ ). That is, for each integer

$$\sigma = l_{j_1} + l_{j_2} + \dots + l_{j_r} \quad (4)$$

(composed of a family of  $r \leq k$  pairwise different  $j_1, \dots, j_r$ ) we have  $v = \mathbf{x}_{[\sigma:\sigma+N-1]}$ , see Fig. 6. If  $k$  is large enough, then  $q$  is a factor of at least one of the shifts (4). Indeed, if  $k > q(q-1)$  then we can find  $q$  different  $l_j$  congruent to each other modulo  $q$  (the pigeon hole principle). Then the sum of these  $l_j$  must be equal to 0 modulo  $q$ .

By iterating the same argument we obtain that every factor  $v$  of a recurrent  $\mathbf{x}$  reappears in the sequence with infinitely many shifts divisible by  $q$  (if  $v = \mathbf{x}_{[i:i+n-1]}$ , then there are infinitely many  $t$  such that  $v = \mathbf{x}_{[qt+i:qt+i+n-1]}$ ).

So far we have used only the fact that  $\mathbf{x}$  is recurrent. To prove the *moreover* part of the lemma, we note that for a uniformly recurrent  $\mathbf{x}$ , all integers  $l_j$ ,  $j = 1, \dots, k$  in the argument above can be majorized by some uniform upper bound that depends only on  $\mathbf{x}$  and  $n$  (but not on the specific position of a factor  $w$  in  $\mathbf{x}$  chosen in the first place). This observation concludes the proof.  $\square$

*Proof of lemma 3.* (a) Denote by  $q$  the period of  $\mathbf{y}$ . Since  $\mathcal{T}$  is a minimal shift, the configuration  $\mathbf{x}$  is recurrent and even quasiperiodic. By Lemma 2, every factor  $w = \mathbf{x}_{[m:n]}$  reappears in  $\mathbf{x}$  with a shift divisible by  $q$ . Moreover, a copy of  $w$  can be found with a shift divisible by  $q$  in every large enough pattern  $\mathbf{x}_{[i:i+L]}$  (where the value of  $L$  depends on  $w$  but not on the specific position of  $m$  in  $\mathbf{x}$ ). It follows that the corresponding factor of the product  $\tilde{w} = (\mathbf{x} \otimes \mathbf{y})_{[m:n]}$  reappears in every large enough pattern of all sequences in  $\mathcal{S}(\mathbf{x} \otimes \mathbf{y})$ .

(b) We need to verify algorithmically whether a given factor  $\tilde{v}$  appears in  $\mathbf{x} \otimes \mathbf{y}$ . In other words, for a given word  $v$  and for a given integer  $i$  we need to find out whether  $v$  appears in  $\mathbf{x}$  in a position congruent to  $i \bmod q$ .

From Lemma 2 it follows that there exists an  $L = L(v)$  such that for every appearance of  $v$  in  $\mathbf{x}$ , this factor reappears in  $\mathbf{x}$  in the same position modulo  $q$  with a translation at most  $L$  to the right. More precisely, if  $v = \mathbf{x}_{[i:i+|v|-1]}$  for some  $i$ , then there exists a positive  $\sigma < L$  divisible by  $q$  such that  $\mathbf{x}_{[i:i+|v|-1]} = \mathbf{x}_{[i+\sigma:i+\sigma+|v|-1]}$ .

Since  $\mathcal{T}$  is minimal, the language of the finite factors of all configurations in  $\mathcal{T}$  is computable. It follows that the bound  $L = L(v)$  defined above is computable. Indeed, by the brute force search we can find the maximal possible gap between two neighboring appearances (in this shift) of the word  $v$  in positions congruent to each other modulo  $q$ . Thus, to find out whether  $v$  appears in  $\mathbf{x}$  in a position congruent to  $i \bmod q$ , it is enough to compute the first  $L(v)$  letters of the sequence  $\mathbf{x}$ .  $\square$

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## References

- [1] R. Berger, *The Undecidability of the Domino Problem*. *Mem. Amer. Math. Soc.*, vol. 66 (1966).
- [2] R.M. Robinson, Undecidability and nonperiodicity for tilings of the plane. *Inventiones Mathematicae* 12, 177–209 (1971).
- [3] W. Hanf. Nonrecursive tilings of the plane. I. *The Journal of Symbolic Logic* 39, 283–285 (1974).
- [4] D. Myers. Nonrecursive tilings of the plane. II. *The Journal of Symbolic Logic* 39, 286–294 (1974).
- [5] P. Gács. Reliable computation with cellular automata. *Journal of Computer and System Sciences* 32(1), 15–78 (1986).

- [6] P. Odifreddi. *Classical recursion theory: The theory of functions and sets of natural numbers*. (1992) Elsevier.
- [7] C. Allauzen and B. Durand. Tiling problems. In E. Borger, E. Gradel, Y. Gurevich, *The Classical Decision Problem*. Springer-Verlag (1997).
- [8] B. Durand, Tilings and quasiperiodicity. *Theoretical Computer Science* 221(1), 61–75 (1999).
- [9] S. V. Avgustinovich, D. G. Fon-Der-Flaass, and A. E. Frid. Arithmetical Complexity of Infinite Words. In: *Words, Languages & Combinatorics III. 2003*. pp. 51–62.
- [10] A. Shen and N. Vereshchagin. *Computable functions*. American Mathematical Soc., 2003.
- [11] A. Rumyantsev and M. Ushakov. Forbidden substrings, Kolmogorov complexity and almost periodic sequences. In *Proc. Annual Symposium on Theoretical Aspects of Computer Science* (2006) pp. 396–407.
- [12] N. Ollinger, Two-by-two substitution systems and the undecidability of the domino problem. *Logic and Theory of Algorithms* 476–485 (2008).
- [13] B. Durand, L. Levin, and A. Shen. Complex tilings. *The Journal of Symbolic Logic* 73(2), 593–613 (2008).
- [14] A. Ballier, *Propriétés structurelles, combinatoires et logiques des pavages*. PhD thesis, Marseille, November 2009.
- [15] M. Hochman. On the dynamics and recursive properties of multidimensional symbolic systems. *Inventiones Mathematicae* 176(1), 131–167 (2009)
- [16] A. Ballier and E. Jeandel. Tilings and model theory. In *Proc. Journées Automates Cellulaires* (JAC 2008), pp. 29–39.
- [17] M. Hochman and T. Meyerovitch. A characterization of the entropies of multidimensional shifts of finite type. *Annals of Mathematics* 2011–2038 (2010).
- [18] P. Salimov, On uniform recurrence of a direct product. *Discrete Mathematics and Theoretical Computer Science* 12(4), 1 (2010).
- [19] B. Durand, A. Romashchenko, and A. Shen, Fixed-point tile sets and their applications. *Journal of Computer and System Sciences* 78(3), 731–764 (2012).
- [20] N. Aubrun, M. Sablik. Simulation of effective subshifts by two-dimensional subshifts of finite type. *Acta Applicandae Mathematicae* 128(1), 35–63 (2013)

- [21] E. Jeandel and P. Vanier. Turing degrees of multidimensional SFTs. *Theoretical Computer Science* 505, 81–92 (2013).
- [22] E. Jeandel, Personal communication. 2015.
- [23] B. Durand and A. Romashchenko. Quasiperiodicity and non-computability in tilings. In *Proc. International Symposium on Mathematical Foundations of Computer Science*, pp. 218–230 (2015).
- [24] C. Zinoviadis. *Hierarchy and expansiveness in two-dimensional subshifts of finite type*. In proc. International Conference on Language and Automata Theory and Applications 2015 Mar 2 (pp. 365–377). The full version: arXiv:1603.05464 (2016).
- [25] S. Gangloff and M. Sablik. Block gluing intensity of bidimensional SFT: Computability of the entropy and periodic points. arXiv preprint arXiv:1706.01627 (2017).
- [26] B. Durand and A. Romashchenko. On the Expressive Power of Quasiperiodic SFT. In *Proc. International Symposium on Mathematical Foundations of Computer Science*. (2017) LIPIcs-Leibniz International Proceedings in Informatics, vol. 83. Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik.
- [27] M. Hochman and P. Vanier, Turing degree spectra of minimal subshifts. In *Proc. International Computer Science Symposium in Russia*. pp. 154–161 (2017).
- [28] L. B. Westrick. Seas of squares with sizes from a  $\Pi_1^0$ . *Israel Journal of Mathematics* 222(1), 431–462 (2017).
- [29] S. Labbé. *A self-similar aperiodic set of 19 Wang tiles*. arXiv:1802.03265 (2018).
- [30] C. Goodman-Strauss. *Lots of aperiodic sets of tiles*. Journal of Combinatorial Theory, Series A, 160, 409–445 (2018).