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Scaling limits of general population processes - Wright-Fisher and branching processes in random environment

Vincent Bansaye*, Maria-Emilia Caballero†, Sylvie Méléard‡

February 6, 2018

Abstract

Our motivation comes from the large population approximation of individual based models in population dynamics and population genetics. We propose a general method to investigate scaling limits of finite dimensional population size Markov chains to diffusion with jumps. The statements of tightness, identification and convergence in law are based on the convergence of suitable characteristics of the transition of the chain and strongly exploit the structure of the population processes defined recursively as sums of independent random variables. These results allow to reduce the convergence of characteristics of semimartingales to analytically tractable functional spaces.

We develop two main applications. First, we extend the classical Wright-Fisher diffusion approximation to independent and identically distributed random environment. Second, we obtain the convergence in law of generalized Galton-Watson processes with interactions and random environment to the solution of stochastic differential equations with jumps.

Key words: Tightness, diffusions with jumps, characteristics, semimartingales, Galton-Watson, Wright-Fisher, random environment

MSC 2010: 60J27, 60J75, 60F15, 60F05, 60F10, 92D25.

Contents

1	Introduction	2
2	A criterion for tightness and convergence in law	5
2.1	The functional spaces and specific functions	6
2.2	General statements	7
2.3	Proofs	8

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3	Wright-Fisher process with selection in Lévy environment	13
3.1	The discrete model	13
3.2	Tightness and identification	16
3.3	Pathwise uniqueness and convergence in law	19
3.4	Example	20
4	Continuous State Branching Process with interactions in Lévy environment	21
4.1	The discrete model	21
4.2	Tightness	24
4.3	Identification	25
4.4	Explosive CSBP with interactions and random environment	30
4.5	Conservative CSBP with interactions and random environment	33
5	Perspectives and multidimensional population models	35
6	Appendix	36
6.1	General construction of a discrete random variable satisfying A2	36
6.2	Taylor expansion for Galton-Watson process with cooperation	40
6.3	Some technical results	40
6.4	Stone-Weierstrass Theorem on locally compact space	42

1 Introduction

Our motivation in this paper is the study of the scaling limits of discrete population models. The scaling parameter is $N \in \mathbb{N}$ and the successive population sizes $(Z_n^N : n \in \mathbb{N})$ come from individual based models. We define a Markov chain recursively defined by an expression of the form

$$Z_{n+1}^N = \sum_{j=1}^{F_N(Z_n^N)} L_{i,n}^N(Z_n^N, E_n^N),$$

where F_N is a function estimating the number of individual events and for each z, e, N , $(L_{i,n}^N(z, e) : i, n \geq 1)$ is a family of independent identically distributed random variables and E_n^N is the environment in generation n . In particular, this class of processes includes well known processes in population dynamics and population genetics, such as Galton-Watson and Wright-Fisher processes. It also takes into account the effect of random environment and can include a lot of additional ecological forces as competition, cooperation, sexe, as will be seen in this paper.

For instance, in the Galton-Watson case, we have $Z_n^N \in \mathbb{N}$, $F_N(z) = z$ and $L_{i,n}^N(z, e) = L^N$. The criterion for the convergence in law of the renormalized process $(Z_{\lfloor v_N t \rfloor}^N / N : t \geq 0)$ with a time scale $v_N \rightarrow \infty$ is known from the works of Lamperti [22, 23] and Grimwall [15]. A necessary and sufficient condition can indeed be given in terms of the reproduction random variable L^N for $N \rightarrow \infty$, which is equivalent to the convergence of the associated random walk to a Lévy process (cf. [8]). The scaled process then converges as $N \rightarrow \infty$ to a Continuous State Branching Process (CSBP), which can be written as the unique strong solution of a Stochastic Differential Equation (SDE). The parameters of this SDE are given by the limiting characteristic triplet of L^N .

Our aim in this paper is to extend such criteria when the discrete process Z^N does not enjoy

the branching property. We obtain sufficient conditions on the asymptotic behavior of F^N and L^N , so that a suitable scaling of Z^N (space and time changes) converges in law. We develop here two main applications, which are our original motivations for this work. We generalize the convergence of Galton-Watson processes to CSBP both by adding independent and identically distributed random environments and by taking into account size dependent reproduction random variables. We also extend the Wright-Fisher diffusion approximation to the random environment framework, when the selection advantage is randomly fluctuating and can in particular provoke jumps in the frequencies of alleles.

Many works have been devoted to the approximation of Markov processes. They essentially rely on tightness argument and identification of the martingale problem, see [11, 18] for two references on the topic. Several difficulties arise when considering the general framework of renormalized Galton-Watson processes and in particular when the limit is non-continuous and explosive. The branching property allowed different elegant general proofs, using the Laplace exponent [15] to get the finite dimensional convergence, but also the relation with the convergence of random walks to Lévy processes via Lamperti's transform [22, 8]. Lamperti has also introduced a powerful transform in the stable framework, see e.g. [23] and [26] and [5]. Other time changes have been successfully used to obtain scaling limits of discrete processes, in particular for diffusion approximations, see for instance [19] for branching processes in random environment and [30] for controlled branching processes.

But such time change techniques are essentially restricted to branching processes or stable processes or diffusion approximations. We are here interested in processes where the branching property fails and major events may provoke jumps in the limit. The processes may even be explosive and are, in general, non-stable. To deal with such processes, we develop a general approach relying on the characteristic triplet of semimartingales Jacod-Shiryaev [18].

The law of a Markov process is determined by its transitions and thus the law of the scaled process $Z_{[v_N \cdot]}^N/N$ is determined by the family of functions

$$x \rightarrow \mathcal{G}_x^N(H) = v_N \mathbb{E} (H(Z_1^N/N - x))$$

for H continuous and bounded. Similarly, the asymptotic behavior of $\mathcal{G}_x^N(H)$ as $N \rightarrow \infty$ for a large enough class of functions H captures the convergence of the scaled process. The convergence as $N \rightarrow \infty$ of the characteristics of semimartingales defined by

$$\sum_{i \leq [v_N t]} \mathcal{G}_{Z_i^N/N}^N(H)$$

allows to get tightness and identification of the scaled process ([18] Chapter IX), under some regularity and domination assumptions. However controlling $\mathcal{G}_x^N(H)$ for such a family of functions is difficult in general, even for Galton-Watson processes. We prove in this paper that one just needs to get uniform convergence of the characteristics $\mathcal{G}_x^N(H)$ for H belonging to some (rich enough) functional space \mathcal{H} , which is dense in the set of regular functions null at zero for a norm equivalent to $\|H\| = \|u \rightarrow H(u)/(1 \wedge u^2)\|_\infty$. The choice of the space \mathcal{H} will depend on the assumptions on the model. In our applications, we exploit the independence property of the variables $(L_i^N(\cdot, \cdot) : i \geq 1)$. We plug at the level of the conditional increments the characterization of the law by the Laplace exponent, which is well adapted to the sum of non-negative independent random variables.

In the rest of the paper, we develop two applications to famous frameworks of population genetics and dynamics based on Wright-Fisher and Galton-Watson chains which were our original motivation and show the efficiency of our method. The first application studies a Wright-Fisher model with selection in a random environment which impacts the selective advantage. The environments are assumed to be independent and identically distributed and the associated random walk converges to a Lévy process. We obtain the convergence in law of the process evaluating the frequency of alleles A, jointly with the random walks of the environment, by using the functional space

$$\mathcal{H} = \{(u, w) \rightarrow 1 - e^{-ku - \ell w} ; k, \ell \geq 0\}.$$

We thus derive a diffusion with jumps in random environment, which generalizes the well known Wright-Fisher diffusion with selection and takes into account small random fluctuations and punctual dramatic advantages in the selective effects.

The second application focuses on generalized Galton-Watson processes with reproduction law both density dependent and environment dependent. We obtain a result of convergence in law to the so called *CSBP with interactions in Lévy environment* (introduced in [28, 16]). For such processes, characteristics are non-bounded and the result is deduced from the convergence of the compactified processes

$$\exp(-Z_k^N / N).$$

To deal with the joint laws of the latter and the environment random walks, we use the functional space

$$\mathcal{H} = \{(v, w) \rightarrow v^k \exp(-\ell w) : k \geq 1, \ell \geq 0\} \cup \{(v, w) \rightarrow 1 - \exp(-\ell w) : \ell \geq 1\}.$$

The results extend the criterion for the convergence of a sequence of Galton-Watson processes and the results we know in random environment [19, 4] or with interactions [10].

They are further applied to Galton-Watson processes with cooperation and to branching processes with logistic growth in random environment.

The paper is organized as follows. In Section 2, we give general results for the tightness, the identification and the convergence in law of a scaled Markov process to a diffusion with jumps in \mathbb{R}^d . The functional space \mathcal{H} is introduced in Section 2.1. Tightness and identification results are stated in Section 2.2 by assuming the uniform convergence and boundedness of characteristics $\mathcal{G}_t^N(H)$ for any $H \in \mathcal{H}$. Convergence requires an additional uniqueness assumption, which is here obtained from pathwise uniqueness in the applications, using standard techniques for non-negative SDE [17, 13]. Proofs of these general statements are given in Section 2.3. In Section 3, we apply our method to a Wright-Fisher model with selection in a random environment. We obtain in a suitable scaling limit a Wright-Fisher diffusion in random environment for which we prove uniqueness of solution. In Section 4 (sections 4.1, 4.2, 4.3), we apply our method to Galton-Watson processes with reproduction law both density dependent and environment dependent. Section 4.4 is devoted to explosive CSBP with interactions and random environment and an application to Galton-Watson processes with cooperatives effects is developed. Section 4.5 is dedicated to conservative CSBP with interactions and random environment and an application to Galton-Watson processes with logistic competition and small environmental fluctuations is studied in details. Section 5 develops some perspectives of applications of tightness and identification results. Indeed, we expect that the method can be

applied in various contexts and in particular for structured populations models with mating functions F_N , competition or cooperation, etc.

Notation

For $x \in \mathbb{R}^d$, we denote the euclidian norm by $|x|$.

The functional norms are denoted by $\|\cdot\|$. In particular the infinite norm of a bounded function f on a set \mathcal{U} is denoted by $\|f\|_{\mathcal{U},\infty}$. The sets $C_b(\mathcal{U},\mathbb{R})$ and $C_c(\mathcal{U},\mathbb{R})$ denote the spaces of continuous real functions respectively bounded and with compact support defined on \mathcal{U} .

2 A criterion for tightness and convergence in law

Let \mathcal{X} be a Borel subset of \mathbb{R}^d and \mathcal{U} be a closed subset of \mathbb{R}^d containing a neighborhood of 0.

Let us introduce a scaling parameter $N \in \mathbb{N}^*$. For any N , we consider a discrete time \mathcal{X} -valued Markov chain $(X_k^N : k \in \mathbb{N})$ satisfying for any $k \geq 0$,

$$\mathcal{L}(X_{k+1}^N | X_k^N = x) = \mathcal{L}(F_x^N),$$

where for any $N \in \mathbb{N}$, $(F_x^N, x \in \mathcal{X})$ denotes a measurable family of \mathcal{X} -valued random variables such that for any $x \in \mathcal{X}$, the random variable $F_x^N - x$ takes values in \mathcal{U} .

The natural filtration of the process X^N is denoted by $(\mathcal{F}_k^N)_k$. Note that the increments $X_{k+1}^N - X_k^N$ take values in \mathcal{U} .

Our aim is the characterization of the convergence in law of the sequence of processes $(X_{[v_N \cdot]}^N, N \in \mathbb{N})$, where $(v_N)_N$ is a given sequence of positive real numbers going to infinity when N tends to infinity. It is based on the criteria for tightness and identification of semimartingales by use of characteristics given in Jacod-Shiryaev [18], which consists in studying the asymptotic behavior of

$$\mathcal{G}_x^N(H) = v_N \mathbb{E}(H(F_x^N - x)) = v_N \mathbb{E}(H(X_{k+1}^N - X_k^N) | X_k^N = x), \quad (1)$$

for real valued bounded measurable functions H defined on \mathcal{U} .

Hypothesis (H0) *We first assume that the family of random variables $(F_x^N)_{N,x}$ satisfies*

$$\lim_{b \rightarrow \infty} \sup_{x \in \mathcal{X}, N \in \mathbb{N}^*} \mathcal{G}_x^N(\mathbb{1}_{B(0,b)^c}) = 0.$$

This assumption avoids to get infinite jumps in the limit. We will see in the examples developed in the next sections that this condition affects both the population and the environment dynamics.

Under Assumption **(H0)**, we prove that the study of (1) can be reduced to a rich enough and tractable subclass \mathcal{H} of functions H . The choice of \mathcal{H} depends on the models and is illustrated in the examples.

2.1 The functional spaces and specific functions

We consider a closed subset \mathcal{U} of \mathbb{R}^d containing a neighborhood of 0. As usual, we write $h(u) = o(g(u))$ (resp. $h(u) \sim g(u)$) when $h(u)/g(u)$ tends to 0 (resp. to 1) as u tends to 0. We introduce the functional space

$$C_{b,0}^2 = C_{b,0}^2(\mathcal{U}, \mathbb{R}) = \left\{ H \in C_b(\mathcal{U}, \mathbb{R}) : H(u) = \sum_{i=1}^d \alpha_i u_i + \sum_{i,j=1}^d \beta_{i,j} u_i u_j + o(|u|^2), \alpha_i, \beta_{i,j} \in \mathbb{R} \right\}.$$

The functions of $C_{b,0}^2$ can be decomposed in a similar way with respect to any smooth function which behaves like identity at 0, as stated in the next lemma. The proof comes from the uniqueness of the second order Taylor expansion in a neighborhood of 0.

Lemma 2.1. *Let $f = (f^1, \dots, f^d) \in (C_{b,0}^2)^d$ such that $f^i(u) \sim_{u \rightarrow 0} u_i$ for $i = 1, \dots, d$. For any $H \in C_{b,0}^2$, there exists a unique decomposition of the form*

$$H = \sum_{i=1}^d \alpha_i^f(H) f^i + \sum_{i,j=1}^d \beta_{i,j}^f(H) f^i f^j + \overline{H}^f,$$

where $\overline{H}^f = o(|f|^2)$ is a continuous and bounded function and $\alpha_i^f(H)$, $\beta_{i,j}^f(H)$, $i, j = 1 \cdots d$ are real coefficients and β is a symmetric matrix.

Let us now introduce a specific function h , which behaves as identity in a neighborhood of zero and plays a main role in our convergence results. In the whole paper, the function h satisfies

$$h = (h^1, \dots, h^d) \in (C_{b,0}^2)^d ; h^i(u) \sim_{u \rightarrow 0} u_i ; h^i(u) \neq 0 \text{ for } u \neq 0 \quad (i = 1, \dots, d). \quad (2)$$

Note that in general this function is not a truncation function in the sense of Jacod-Shiryaev [18] since it may not coincide with identity function in a neighborhood of 0. Its choice will be driven by the processes we are considering and we are giving different choices of functions h in the next sections, for instance $h(x) = 1 - \exp(-x)$ on $[-1, \infty)$ when $d = 1$.

We introduce a functional space \mathcal{H} which contains the coordinates of h , their square products and "generates" the continuous functions with compact support in \mathbb{R}^d . More precisely, we make the following assumptions.

Hypotheses (H1)

1. \mathcal{H} is a subset of $C_{b,0}^2$ and $h^i, h^i h^j \in \text{Vect}(\mathcal{H})$ for $i, j = 1, \dots, d$.
2. For any $g \in C_c(\mathcal{U}, \mathbb{R})$ with $g(0) = 0$, there exists a sequence $(g_n)_n \in C_{b,0}^2$ such that $\lim_{n \rightarrow \infty} \|g - g_n\|_{\infty, \mathcal{U}} = 0$ and $|h|^2 g_n \in \text{Vect}(\mathcal{H})$.

Remark 2.2. *In the examples of the next sections, Assumption (H1.2) is proved with the use of the locally compact version of the Stone-Weierstrass Theorem. We refer to the Appendix for a precise statement. Moreover, contrary to the "convergence determining class" of [18], the functions of \mathcal{H} will not be null (or $o(u^2)$) in a neighborhood of 0.*

2.2 General statements

Hypotheses (H'1) *There exists $(\mathcal{G}_x H, x \in \mathcal{X}, H \in \mathcal{H})$ such that for any $H \in \mathcal{H}$,*

1. $\lim_{N \rightarrow \infty} \sup_{x \in \mathcal{X}} |\mathcal{G}_x^N(H) - \mathcal{G}_x(H)| = 0.$
2. $\sup_{x \in \mathcal{X}} |\mathcal{G}_x(H)| < +\infty.$

Hence (H'1) implies that the map $x \in \mathcal{X} \rightarrow \mathcal{G}_x(H)$ is measurable and bounded for any $H \in \mathcal{H}$.

Theorem 2.3. *Assume that the sequence $(X_0^N)_N$ is tight in $\overline{\mathcal{X}}$ and that (H0), (H1) and (H'1) hold. Then the sequence of processes $(X_{[v_N, \cdot]}^N, N \in \mathbb{N})$ is tight in $\mathbb{D}([0, \infty), \overline{\mathcal{X}})$.*

We now introduce a truncation function in the sense of [18] :

$$h_0 = (h_0^1, \dots, h_0^d), \quad h_0 \in C_b(\mathcal{U}, \mathbb{R}^d), \quad h_0(x) = x \text{ in a neighborhood of } 0. \quad (3)$$

Obviously, $h_0^i h_0^j \in C_{b,0}^2$.

The next assumption (H2) is sufficient in addition to Hypotheses (H1) and (H'1) to get the identification of the limiting values and their representation as solutions of a stochastic differential equation. We use notations of Hypothesis (H'1) and Lemma 2.1.

Hypotheses (H2)

1. *For any $H \in \mathcal{H}$, the map $x \in \mathcal{X} \rightarrow \mathcal{G}_x(H)$ is continuous and extendable by continuity to $\overline{\mathcal{X}}$.*
2. *For any $x \in \overline{\mathcal{X}}$ and any $H \in \mathcal{H}$,*

$$\mathcal{G}_x(H) = \sum_{i=1}^d \alpha_i^{h_0}(H) b_i(x) + \sum_{i,j=1}^d \beta_{i,j}^{h_0}(H) c_{i,j}(x) + \int_V \overline{H}^{h_0}(K(x, v)) \mu(dv), \quad (4)$$

where $\alpha_i^{h_0}$, $\beta_{i,j}^{h_0}$ and \overline{H}^{h_0} have been defined in Lemma 2.1, b_i and $\sigma_{i,j}$ are measurable functions defined on $\overline{\mathcal{X}}$, V is a Polish space, μ is a σ -finite positive measure on V , K is a function from $\overline{\mathcal{X}} \times V$ with values in \mathcal{U} , $\int_V 1 \wedge |K(\cdot, v)|^2 \mu(dv) < +\infty$ and

$$c_{i,j}(x) = \sum_{k=1}^d \sigma_{i,k}(x) \sigma_{j,k}(x) + \int_V (h_0^i h_0^j)(K(x, v)) \mu(dv).$$

The elements (b, σ, V, μ, K) will be specified in the applications. The proof of the next theorem will explicitly give their relation with the associated stochastic differential equation (5).

Theorem 2.4. *Under the assumptions of Theorem 2.3 and the additional hypothesis (H2), any limiting value of $(X_{[v_N \cdot]}^N, N \in \mathbb{N})$ is a semimartingale solution of the well-defined stochastic differential system*

$$\begin{aligned} X_t = X_0 &+ \int_0^t b(X_s) ds + \int_0^t \sigma(X_s) dB_s + \int_0^t \int_V h_0(K(X_{s-}, v)) \tilde{N}(ds, dv) \\ &+ \int_0^t \int_V (Id - h_0)(K(X_{s-}, v)) N(ds, dv), \end{aligned} \quad (5)$$

where $X_0 \in \overline{\mathcal{X}}$, B is a d -dimensional Brownian motion, N is a Poisson Point measure on $\mathbb{R}_+ \times V$ with intensity $ds\mu(dv)$, X_0, B and N are independent and \tilde{N} is the compensated martingale measure of N , Id denotes the identity function.

Finally the next additional assumption allows to obtain the convergence in law of the sequence of processes $(X_{[v_N \cdot]}^N, N \in \mathbb{N})$ in $\mathbb{D}([0, \infty), \overline{\mathcal{X}})$.

Hypothesis (H3) *The law of the initial condition $X_0 \in \overline{\mathcal{X}}$ being given, the uniqueness in law of the solution of (5) holds in $\mathbb{D}([0, \infty), \overline{\mathcal{X}})$.*

Let us note that most often and at least in our applications, the uniqueness in law of (5) is deduced from the pathwise uniqueness.

We are now in position to state the convergence result.

Theorem 2.5. *Assume that the sequence $(X_0^N)_N$ converges in law in $\overline{\mathcal{X}}$ to X_0 and that (H0), (H1), (H'1), (H2) and (H3) hold. Then the sequence of processes $(X_{[v_N \cdot]}^N)_N$ converges in law in $\mathbb{D}([0, \infty), \overline{\mathcal{X}})$ to the solution of (5).*

2.3 Proofs

From now on, we assume that hypotheses (H0), (H1) and (H'1) hold. We write $\mathcal{U}^* = \mathcal{U} \setminus \{0\}$.

In the proofs, we use the space \mathcal{R}_b of continuous and bounded functions which are small enough close to 0 :

$$\mathcal{R}_b = \{H \in C_b(\mathcal{U}, \mathbb{R}), H(u) = o(|u|^2)\}.$$

Using Lemma 2.1 and (H1.1), we have

$$C_{b,0}^2 = Vect(\mathcal{H}) + \mathcal{R}_b. \quad (6)$$

We work with the norm

$$\|H\|_h = \sup_{u \in \mathcal{U}^*} \frac{|H(u)|}{|h(u)|^2},$$

defined for $H \in \mathcal{R}_b$ such that $\sup_{u \in \mathcal{U}^*} \frac{|H(u)|}{|h(u)|^2} < +\infty$. In that case, the positivity and linearity of \mathcal{G}_x^N for all $x \in \mathcal{X}$ and $N \geq 1$ imply that

$$|\mathcal{G}_x^N(H)| \leq \mathcal{G}_x^N(|h|^2) \|H\|_h \leq \alpha(|h|^2) \|H\|_h, \quad (7)$$

where $\alpha(|h|^2) = \sup_{N, x \in \mathcal{X}} |\mathcal{G}_x^N(|h|^2)| < \infty$ by (H'1) since $|h|^2 \in Vect(\mathcal{H})$ by (H1.1).

2.3.1 Proof of Theorem 2.3

We first extend the assumptions **(H'1)** to $C_{b,0}^2$ in order to prove the tightness. We note that **(H'1.1)** and **(H'1.2)** extend immediately to $H \in Vect(\mathcal{H})$ by linearity of $H \rightarrow \mathcal{G}_x^N(H)$ for any $x \in \mathcal{X}$ and $N \geq 1$.

Lemma 2.6. *For any $x \in \mathcal{X}$, there exists a linear extension $H \in C_{b,0}^2 \rightarrow \mathcal{G}_x(H)$ of \mathcal{G}_x such that **(H'1.1)** and **(H'1.2)** hold for any $H \in C_{b,0}^2$.*

As a consequence, writing $\alpha(H) = \sup_{N,x \in \mathcal{X}} |\mathcal{G}_x^N(H)|$, for any $H \in C_{b,0}^2$,

$$\sup_{x \in \mathcal{X}} |\mathcal{G}_x(H)| \leq \sup_{N,x \in \mathcal{X}} |\mathcal{G}_x^N(H)| = \alpha(H) < +\infty. \quad (8)$$

Proof. Using (6) and linearity, we only have to prove the extension to \mathcal{R}_b . Let us first prove the result for the compact supported functions of \mathcal{R}_b . Let us consider $H \in \mathcal{R}_b$ with compact support and show that the sequence $(\mathcal{G}_x^N(H))_N$ converges when N tends to infinity. The function $H/|h|^2$ defined on \mathcal{U}^* can be extended to a continuous function g on \mathcal{U} with compact support and $g(0) = 0$. Then by **(H1.2)**, there exists a sequence $(g_n)_n$ of functions of $C_{b,0}^2$ uniformly converging to g and such that $H_n = |h|^2 g_n \in Vect(\mathcal{H})$. Since $H = |h|^2 g$, $\|H_n - H\|_h \rightarrow 0$ when $n \rightarrow \infty$. Moreover the sequence $(\mathcal{G}_x^N(H_n))_N$ converges to $\mathcal{G}_x(H_n)$ when N tends to infinity for any fixed n and uniformly in x . Let us now consider two integers m and n . Using (7), we have

$$\sup_{N,x} |\mathcal{G}_x^N(H_m) - \mathcal{G}_x^N(H_n)| \leq \alpha(|h|^2) \|H_m - H_n\|_h$$

and letting N go to infinity, we obtain that $(\mathcal{G}_x(|h|^2 g_n))_n$ is a Cauchy sequence. Then it converges to a limit denoted by $\mathcal{G}_x(H)$, which satisfies $\sup_x |\mathcal{G}_x(H)| < \infty$. Moreover

$$|\mathcal{G}_x^N(H) - \mathcal{G}_x(H)| \leq |\mathcal{G}_x^N(H) - \mathcal{G}_x^N(H_n)| + |\mathcal{G}_x^N(H_n) - \mathcal{G}_x(H_n)| + |\mathcal{G}_x(H_n) - \mathcal{G}_x(H)|.$$

Using that $|\mathcal{G}_x^N(H) - \mathcal{G}_x^N(H_n)| \leq \alpha(|h|^2) \|g - g_n\|_\infty$ and choosing n and then N large enough ensure that the left hand side goes to 0 :

$$\sup_{x \in \mathcal{X}} |\mathcal{G}_x^N(H) - \mathcal{G}_x(H)| \xrightarrow{N \rightarrow \infty} 0.$$

Let us now consider $H \in \mathcal{R}_b$. We introduce a non-decreasing sequence $(\varphi_n)_n \in C^2(\mathbb{R}^d)$ such that $\|\varphi_n\|_\infty \leq 1$, $\varphi_n(x) = 1$ on $B(0, n)$ and $\varphi_n(x) = 0$ on $B(0, n+1)^c$. For any integer $m \geq n$ and $x \in \mathcal{X}$ and $N \geq 1$,

$$|\mathcal{G}_x^N(H\varphi_m) - \mathcal{G}_x^N(H\varphi_n)| \leq \|H\|_\infty \mathcal{G}_x^N(\mathbb{1}_{B(0,n)^c}) \leq \|H\|_\infty C_n,$$

where $C_n \rightarrow 0$ as $n \rightarrow \infty$ infinity by **(H0)**. Letting N tend to infinity, we obtain that for any $x \in \mathcal{X}$, the sequence $(\mathcal{G}_x(H\varphi_n))_n$ is Cauchy and converges to some real number $\mathcal{G}_x(H)$. Moreover $|\mathcal{G}_x(H) - \mathcal{G}_x(H\varphi_n)| \leq C_n \|H\|_\infty$. It follows that for any $H \in \mathcal{R}_b$,

$$\begin{aligned} |\mathcal{G}_x^N(H) - \mathcal{G}_x(H)| &\leq |\mathcal{G}_x^N(H) - \mathcal{G}_x^N(H\varphi_n)| + |\mathcal{G}_x^N(H\varphi_n) - \mathcal{G}_x(H\varphi_n)| + |\mathcal{G}_x(H\varphi_n) - \mathcal{G}_x(H)| \\ &\leq 2C_n \|H\|_\infty + |\mathcal{G}_x^N(H\varphi_n) - \mathcal{G}_x(H\varphi_n)| \end{aligned}$$

Adding that $H\varphi_n \in \mathcal{R}_b$ and has compact support, $\mathcal{G}_x^N(H\varphi_n) - \mathcal{G}_x(H\varphi_n)$ goes to 0 and then $\mathcal{G}_x^N(H) - \mathcal{G}_x(H)$ tends to 0 as N tends to infinity uniformly for $x \in \mathcal{X}$. It proves **(H'1.1)** and **(H'1.2)** and (8). \square

We now prove that a σ -finite measure can be associated to \mathcal{G}_x with measurability in the x -dependence. It describes the jumps of the limiting process.

Lemma 2.7. *There exists a measurable family of σ -finite measures $(\mu_x : x \in \mathcal{X})$ on \mathcal{U}^* such that for any $H \in \mathcal{R}_b$,*

$$\mathcal{G}_x(H) = \int_{\mathcal{U}^*} H(u) \mu_x(du). \quad (9)$$

For any $x \in \mathcal{X}$, \mathcal{G}_x is then extended by (9) to any measurable and bounded function H on $(\mathbb{R}^d)^*$ such that $H(u) = o(|u|^2)$. Moreover

$$\lim_{b \rightarrow \infty} \sup_{x \in \mathcal{X}} |\mathcal{G}_x(\mathbb{1}_{B(0,b)^c})| = 0. \quad (10)$$

Proof. For any $x \in \mathbb{R}^d$ and $H \in C_c(\mathcal{U}^*, \mathbb{R})$, the map $H \rightarrow \mathcal{G}_x(H)$ is a positive linear operator. Adding that \mathcal{U}^* is locally compact, Riesz Theorem leads to the existence of a σ -finite measure μ_x on \mathcal{U}^* such that for any $H \in C_c(\mathcal{U}^*, \mathbb{R})$, $\mathcal{G}_x(H) = \int_{\mathcal{U}^*} H(u) \mu_x(du)$. The extension of this identity to any $H \in \mathcal{R}_b$ follows again from an approximation procedure, using φ_n defined in the proof of Lemma 2.6. Indeed, on the one hand monotone convergence ensures that $\int_{\mathcal{U}^*} H \varphi_n \mu_x$ goes to $\int_{\mathcal{U}^*} H \mu_x$. On the other hand, $|\mathcal{G}_x(H \varphi_n) - \mathcal{G}_x(H)| \leq C_n \|H\|_\infty$ goes to 0. Finally (10) comes from (H0) with a monotone approximation of $\mathbb{1}_{B(0,b)^c}$ by elements of \mathcal{R}_b and the convergence of \mathcal{G}^N to \mathcal{G} . \square

Let us now turn to the proof of Theorem 2.3. We first prove the convergence of conditional increments functionals, defined for any function $H \in C_{b,2}^0$ and $t > 0$ by

$$\phi_t^N(H) = \sum_{k=1}^{\lfloor v_N t \rfloor} \mathbb{E}(H(X_k^N - X_{k-1}^N) | \mathcal{F}_{k-1}^N) = \frac{1}{v_N} \sum_{k=1}^{\lfloor v_N t \rfloor} \mathcal{G}_{X_{k-1}^N}^N(H), \quad (11)$$

where the last identity comes from the Markov property.

Proposition 2.8. *For any function $H \in C_{b,2}^0$ and $t > 0$,*

$$\lim_{N \rightarrow \infty} \sup_{t \leq T} \left| \phi_t^N(H) - \int_0^t \mathcal{G}_{X_{\lfloor v_N s \rfloor}^N}^N(H) ds \right| = 0 \quad a.s.$$

Proof. Using (8), we have

$$\frac{1}{v_N} \sum_{k=1}^{\lfloor v_N t \rfloor} \mathcal{G}_{X_{k-1}^N}^N(H) = \int_0^t \mathcal{G}_{X_{\lfloor v_N s \rfloor}^N}^N(H) ds - \int_{\frac{\lfloor v_N t \rfloor}{v_N}}^t \mathcal{G}_{X_{\lfloor v_N s \rfloor}^N}^N(H) ds = \int_0^t \mathcal{G}_{X_{\lfloor v_N s \rfloor}^N}^N(H) ds + \mathcal{O}\left(\frac{\alpha(H)}{v_N}\right).$$

Then

$$\sup_{t \leq T} \left| \phi_t^N(H) - \int_0^t \mathcal{G}_{X_{\lfloor v_N s \rfloor}^N}^N(H) ds \right| \leq T \sup_{x \in \mathcal{X}} |\mathcal{G}_x^N(H) - \mathcal{G}_x(H)| + \mathcal{O}\left(\frac{\alpha(H)}{v_N}\right)$$

and the conclusion follows from (H1'.1) proved in Lemma 2.6. \square

Let us now define on the canonical space $\mathbb{D}([0, \infty), \mathcal{X})$ a triplet which characterizes the limiting values of the sequence $(X_{[v_N \cdot]}^N, N \in \mathbb{N})$. Using the measurability and boundedness of $x \rightarrow \mathcal{G}_x(H)$ for $x \in \mathcal{X}$ and $H \in C_{b,2}^0$ and the truncation function h_0 introduced in (3), we define for any $\omega = (\omega_s, s \geq 0) \in \mathbb{D}([0, \infty), \mathcal{X})$ the functionals

$$\begin{aligned} B_t(\omega) &= \int_0^t \left(\mathcal{G}_{\omega_s}(h_0^1), \dots, \mathcal{G}_{\omega_s}(h_0^d) \right) ds, \\ \tilde{C}_t^{ij}(\omega) &= \int_0^t \mathcal{G}_{\omega_s}(h_0^i h_0^j) ds, \\ \nu_t(\omega, H) &= \int_0^t \mathcal{G}_{\omega_s}(H \mathbb{1}_U) ds = \int_0^t \int_{U^*} H(u) \mu_{\omega_s}(du) ds \end{aligned} \quad (12)$$

for any $H \in C_b(\mathbb{R}^d, \mathbb{R})$ such that $H(u) = o(|u|^2)$, where the last identity comes from (9).

Following Chapters II. 2 & 3 in [18] adapted to the arrival space $\bar{\mathcal{X}}$ (instead of \mathbb{R}^d), the characteristic triplet associated with the semimartingale X^N is given for $i, j \in \{1, \dots, d\}$ by

$$\begin{aligned} B_t^N &= \sum_{k \leq [v_N t]} \mathbb{E}(h_0(U_k^N) | \mathcal{F}_{k-1}^N) = (\phi_t^N(h_0^1), \dots, \phi_t^N(h_0^d)) \\ \tilde{C}_t^{N,ij} &= \sum_{k \leq [v_N t]} \left(\mathbb{E}(h_0^i(U_k^N) h_0^j(U_k^N) | \mathcal{F}_{k-1}^N) - \mathbb{E}(h_0^i(U_k^N) | \mathcal{F}_{k-1}^N) \mathbb{E}(h_0^j(U_k^N) | \mathcal{F}_{k-1}^N) \right) \\ \phi_t^N(H) &= \sum_{k \leq [v_N t]} \mathbb{E}(H(U_k^N) | \mathcal{F}_{k-1}^N), \end{aligned} \quad (13)$$

where $U_k^N = X_k^N - X_{k-1}^N$ and H is a continuous bounded function on \mathbb{R}^d null in a neighborhood of 0. Proposition 2.8 implies the convergence of the characteristics, as stated below.

Proposition 2.9. *For any $T > 0$ and any $i, j = 1, \dots, d$ and H a continuous bounded function null in a neighborhood of 0, we have the following almost-sure convergences :*

$$\sup_{t \leq T} \left| B_t^{N,i} - B_t^i \circ X_{[v_N \cdot]}^N \right| \xrightarrow{N \rightarrow \infty} 0; \quad (14)$$

$$\sup_{t \leq T} \left| \tilde{C}_t^{N,ij} - \tilde{C}_t^{ij} \circ X_{[v_N \cdot]}^N \right| \xrightarrow{N \rightarrow \infty} 0; \quad (15)$$

$$\sup_{t \leq T} \left| \phi_t^N(H) - \nu_t(X_{[v_N \cdot]}^N, H) \right| \xrightarrow{N \rightarrow \infty} 0. \quad (16)$$

Proof. From Proposition 2.8, we obtain immediately the first and last convergences and

$$\sup_{t \leq T} \left| \phi_t^N(h_0^i h_0^j) - \tilde{C}_t^{ij} \circ X_{[v_N \cdot]}^N \right| \rightarrow_{N \rightarrow \infty} 0 \quad \text{a.s.}$$

So it remains to replace $\phi_t^N(h_0^i h_0^j)$ by $\tilde{C}_t^{N,ij}$. We have

$$\left| \mathbb{E}(h_0^i(U_k^N) h_0^j(U_k^N) | \mathcal{F}_{k-1}^N) \right| \leq \frac{1}{v_N} \sup_{N, x \in \mathcal{X}} |\mathcal{G}_x^N(h_0^k)| \leq \alpha(h_0^k) \cdot \frac{1}{v_N}$$

and $\alpha(h_0^k) < \infty$ from (8). Hence the second term in $\tilde{C}_t^{N,ij}$ vanishes, which yields the result. \square

To apply Theorem 3.9 IX p543 in [18] and get the tightness, we need to check the required strong majoration hypothesis and condition on big jumps. For any process $(X_t, t \geq 0)$ with finite variation on any finite time interval, we denote by $Var(X)_t$ the total variation of X on $[0, t]$. First, using that for any $H \in C_{b,2}^0$, $\mathcal{G}H$ is bounded ensures that there exists a positive constant A such that for any $\omega \in \mathbb{D}([0, \infty), \mathcal{X})$,

$$\sum_{i=1}^d Var(B^i(\omega))_t + \sum_{i,j=1}^d \tilde{C}_t^{ij}(\omega) + \nu_t(w, u \rightarrow |u|^2 \wedge 1) \leq At. \quad (17)$$

Second, we can control the big jumps, using that $\nu_t(\cdot, \mathbb{1}_{B(0,b)^c}) \leq t \|\mathcal{G}(\mathbb{1}_{B(0,b)^c})\|_\infty$, which tends to 0 as b tends to infinity from (10). We obtain

$$\lim_{b \uparrow \infty} \sup_{w \in \mathbb{D}([0, \infty), \mathcal{X})} \nu_t(w, \mathbb{1}_{B(0,b)^c}) = 0. \quad (18)$$

The proof of Theorem 2.3 of $(X_{[v_N \cdot]}^N, N \in \mathbb{N})$ follows from (14)-(18) and the tightness of the initial condition, by an application of [18, Theorem 3.9, chapter IX, p543].

2.3.2 Proofs of Theorems 2.4 and 2.5

Let us now assume the additional Hypothesis **(H2)**. We wish to identify the limiting values of $(X_{[v_N \cdot]}^N, N \in \mathbb{N})$ as solutions of the stochastic differential system (5). We first need to extend continuously the limiting characteristic triplet to the boundary.

Lemma 2.10. *(i) For any $H \in \mathcal{R}_b$, the map $x \in \mathcal{X} \rightarrow \mathcal{G}_x(H)$ is continuous and extendable by continuity to $\overline{\mathcal{X}}$. Moreover*

$$\sup_{x \in \overline{\mathcal{X}}} |\mathcal{G}_x(H)| \leq \alpha(H) < +\infty. \quad (19)$$

(ii) For any $H \in \mathcal{R}_b$ and $x \in \overline{\mathcal{X}}$,

$$\mathcal{G}_x(H) = \int_V H(K(x, v)) \mu(dv) \quad (20)$$

and in particular $\int_V 1 \wedge |K(\cdot, v)|^2 \mu(dv)$ is bounded on $\overline{\mathcal{X}}$.

Proof. Let $H \in \mathcal{R}_b$. Using the sequence φ_n of the proof of Lemma 2.6 and approximating $\varphi_n H$ for $\|\cdot\|_h$ by $H_n \in Vect(\mathcal{H}) \cap \mathcal{R}_b$ as in the proof of Lemma 2.7, we obtain that

$$\sup_{x \in \mathcal{X}, N \geq 1} |\mathcal{G}_x^N(H) - \mathcal{G}_x^N(H_n)| \leq \|H\|_\infty C_n + \|\varphi_n H - H_n\|_h \alpha(|h|^2)$$

tends to 0 as $n \rightarrow \infty$. Letting $N \rightarrow \infty$ ensures that $\mathcal{G}_x H_n$ converges uniformly to $\mathcal{G}_x H$ as $n \rightarrow \infty$. Using **(H2.1)** for H_n , we obtain that $\mathcal{G}_x H$ is continuous on \mathcal{X} and extendable by continuity to $\overline{\mathcal{X}}$, which proves (i). Moreover (8) yields (19) by continuity.

For (ii), we first consider $H \in Vect(\mathcal{H}) \cap \mathcal{R}_b$. Then $\alpha^{h_0}(H) = \beta^{h_0}(H) = 0$, $\overline{H}^{h_0} = H$ and **(H2.2)** ensures that (20) holds for H . Let us now extend this identity to $H \in \mathcal{R}_b$ with compact support. We note that $H = |h|^2 g$ with $g \in C_c(\mathcal{U}, \mathbb{R})$ and using Assumption **(H1.2)**,

we know that g is uniformly approximated by a sequence g_n such that $|h|^2 g_n \in \text{Vect}(\mathcal{H}) \cap \mathcal{R}_b$. Identity (4) implies that (20) holds for any $|h|^2 g_n$ and

$$\mathcal{G}_x(|h|^2 g_n) = \int_V (|h|^2 g_n)(K(x, v)) \mu(dv).$$

We make n tend to infinity in both terms using (8) and the assumption $\int_V 1 \wedge |K(x, v)|^2 \mu(dv) < +\infty$. The extension to \mathcal{R}_b follows again from a monotone approximation by the compact supported functions $H\varphi_n$, which ends the proof. \square

This lemma allows us to extend the definitions of the characteristics and the identity of (12) to any $w \in \mathbb{D}([0, \infty), \bar{\mathcal{X}})$. Moreover (i) ensures that $w \in \mathbb{D}([0, \infty), \bar{\mathcal{X}}) \rightarrow (B_t(w), \tilde{C}_t(w), \nu_t(w, H))$ is continuous and that the dominations (17) and (18) extend from \mathcal{X} to $\bar{\mathcal{X}}$. We can then apply [18, Theorem 2.11, chapter IX, p530] on the closed set $\bar{\mathcal{X}}$ for the identification. We obtain that any limiting value of the law of $(X_{[v_N, \cdot]}^N)_N$ is a solution of the martingale problem on the canonical space $\mathbb{D}([0, \infty), \bar{\mathcal{X}})$ with characteristic triplet (B, C, ν) , where

$$C_t^{ij} = \tilde{C}_t^{ij} - \nu_t(\cdot, h_0^i h_0^j).$$

Finally, using **(H2.2)** for $H \in \{h_0^i, h_0^i h_0^j\}$ and (20), the characteristics in (12) write

$$\begin{aligned} B_t(w) &= \int_0^t b(w_s) ds \\ C_t^{ij}(w) &= \int_0^t \left(\sum_{k=1}^D \sigma_{i,k}(w_s) \sigma_{j,k}(w_s) \right) ds \\ \nu_t(w, H) &= \int_0^t \int_V H(K(w_s, v)) \mu(dv) ds, \end{aligned}$$

for any $w \in \mathbb{D}([0, \infty), \bar{\mathcal{X}})$. By [18, Chapter III, Theorem 2.26 p157], the set of solutions of the martingale problem with characteristic triplet (B, C, ν) then coincides with the set of weak solutions of the stochastic differential equation (5). That proves Theorem 2.4.

To conclude the proof of the convergence, we remark that uniqueness hypothesis **(H3)** guarantees (iii) in [18, Theorem 3.21, chapter IX, p546]. The other points (i – vi) of this theorem have been checked above and Theorem 2.5 follows.

3 Wright-Fisher process with selection in Lévy environment

3.1 The discrete model

Let us consider the framework of the Wright-Fisher model: at each generation, the alleles of a fixed size population are sampled from the previous generation. We consider a population of N individuals characterized by some allele which is either A or a . The number of individuals carrying allele A is a process $(Z_k^N, k \in \mathbb{N})$ whose dynamics depends on the environment. When N is fixed, we consider the coupled process describing both the discrete time dynamics

of the population process and the environment process. For each $N \geq 1$, it is recursively defined for $k \geq 0$ by

$$\begin{cases} Z_{k+1}^N = \sum_{i=1}^N \mathcal{E}_{k,i}^N(Z_k^N/N, E_k^N), \\ S_{k+1}^N = S_k^N + E_k^N, \end{cases} \quad (21)$$

and $S_0^N = 0, Z_0^N = [NZ_0]$, $Z_0 \in [0, 1]$ is a finite random variable, $(E_k^N)_k$ are independent and identically distributed with values in $(-1, +\infty)$ and the family of random variables $((\mathcal{E}_{k,i}^N(z, w), (z, w) \in [0, 1] \times (-1, \infty)); k \geq 1, i \geq 1)$ are independent and for each $(z, w) \in [0, 1] \times (-1, \infty)$. Moreover the random variables $(\mathcal{E}_{k,i}^N(z, w); k \geq 1, i \geq 1)$ are identically distributed as a Bernoulli random variable $\mathcal{E}^N(z, w)$ defined by

$$\mathbb{P}(\mathcal{E}^N(z, w) = 1) = p(z, w) ; \mathbb{P}(\mathcal{E}^N(z, w) = 0) = 1 - p(z, w).$$

We assume that $Z_0, ((\mathcal{E}_{k,i}^N(z, w), (z, w) \in [0, 1] \times (-1, \infty)); k \geq 1, i \geq 1)$ and $(E_k^N, k \geq 0)$ are independent.

We assume now that the random walk $S_{[N \cdot]}^N$ converges in law to a Lévy process with characteristics $(\alpha_E, \beta_E, \nu_E)$. Following [18, chap.VII Corollary 3.6] p.415, it can be stated as follows, by considering a truncation function h_E , i.e. h_E defined on $(-1, +\infty)$ continuous and bounded and $h_E(w) = w$ in a neighborhood of 0.

Assumption A. *There exist $\alpha_E \in \mathbb{R}$, $\sigma_E \geq 0$ and a measure ν_E on $(-1, +\infty)$ satisfying $\int_{(-1, +\infty)} (w^2 \wedge 1) \nu_E(dw) < +\infty$ such that*

$$\lim_{N \rightarrow \infty} N \mathbb{E}(h_E(E^N)) = \alpha_E ; \lim_{N \rightarrow \infty} N \mathbb{E}(h_E^2(E^N)) = \beta_E ; \lim_{N \rightarrow \infty} N \mathbb{E}(f(E^N)) = \int_{(-1, \infty)} f(w) \nu_E(dw),$$

for any f null in a neighborhood of 0, continuous and bounded and $\beta_E = \sigma_E^2 + \int_{(-1, \infty)} h_E^2(w) \nu_E(dw)$.

The small fluctuations of the environment are given by σ_E , while the dramatic events are given by the jump measure ν_E . Negative jumps will correspond to dramatic disadvantages of allele A and an usual set of selective advantages is $(-1, \infty)$, as illustrated in Section 3.4.

The limiting environment process Y can thus be defined by

$$Y_t = \alpha_E t + \int_0^t \sigma_E dB_s^E + \int_0^t \int_{(-1, +\infty)} h_E(w) \tilde{N}^E(ds, dw) + \int_0^t \int_{(-1, +\infty)} (w - h_E(w)) N^E(ds, dw), \quad (22)$$

where B^E is a Brownian motion and N^E is a Poisson point measure on $\mathbb{R}_+ \times (-1, +\infty)$ independent of B^E with intensity measure ν_E . By construction, this Lévy process has jumps larger than -1 .

Let us first prove a consequence of Assumption A which will be needed in the proof of the next theorem.

Lemma 3.1. *Let $g \in C^3([0, 1] \times (-1, \infty), \mathbb{R})$ bounded and satisfying $g(z, 0) = 0$ for any $z \in [0, 1]$. Then, under Assumption **A**,*

$$N\mathbb{E}(g(z, E^N)) \xrightarrow{N \rightarrow \infty} \mathcal{B}_z(g),$$

uniformly for $z \in [0, 1]$, with

$$\mathcal{B}_z(g) = \alpha_E \frac{\partial g}{\partial w}(z, 0) + \frac{\beta_E}{2} \frac{\partial^2 g}{\partial w^2}(z, 0) + \int_{(-1, \infty)} \widehat{g}(z, w) \nu_E(dw)$$

and $\widehat{g}(z, w) = g(z, w) - h_E(w) \frac{\partial g}{\partial w}(z, 0) - \frac{h_E(w)^2}{2} \frac{\partial^2 g}{\partial w^2}(z, 0)$.

Proof. Indeed, we can decompose as follows

$$N\mathbb{E}(g(z, E^N)) = \frac{\partial g}{\partial w}(z, 0) N\mathbb{E}(h_E(E^N)) + \frac{1}{2} \frac{\partial^2 g}{\partial w^2}(z, 0) N\mathbb{E}(h_E(E^N)^2) + N\mathbb{E}(\widehat{g}(z, E^N)).$$

The two first terms converge uniformly as $N \rightarrow \infty$ by a direct application of Assumption **A**. Moreover the last part of Assumption **A** can be extended to continuous function $f(w) = o(w^2)$ using a monotone approximation of f by functions null in a neighborhood of 0. Then the last term converges for fixed z and it remains to prove that the convergence is uniform on $[0, 1]$. First, let us consider a compact set $K = [0, 1] \times [-1 + \varepsilon_0, A]$ of $[0, 1] \times (-1, \infty)$. As g is $C^3([0, 1] \times (-1, \infty), \mathbb{R})$, the function

$$(z, w) \rightarrow \frac{\widehat{g}(z, w)}{h_E(w)^2} = \frac{g(z, w) - h_E(w) \frac{\partial g}{\partial w}(z, 0)}{h_E(w)^2} - \frac{1}{2} \frac{\partial^2 g}{\partial w^2}(z, 0)$$

and its first derivative with respect to z are well defined on $[0, 1] \times (-1, \infty) - [0, 1] \times \{0\}$ and extendable by continuity to $[0, 1] \times (-1, \infty)$. Thus the derivative of $\widehat{g}(z, w)/h_E(w)^2$ with respect to z is bounded on K and using that $(N\mathbb{E}(h_E(E^N)^2))_N$ is bounded by the second part of Assumption **A**, there exists $C > 0$ such that for any $N \geq 1$,

$$\left| N\mathbb{E} \left(\widehat{g}(z, E^N) 1_{E^N \in [-1 + \varepsilon_0, A]} \right) - N\mathbb{E} \left(\widehat{g}(z', E^N) 1_{E^N \in [-1 + \varepsilon_0, A]} \right) \right| \leq C |z - z'|.$$

Moreover, using that all functions involved in the definition of \widehat{g} are bounded, there exists $C' > 0$ such that

$$\left| N\mathbb{E} \left(|\widehat{g}(z, E^N)| 1_{E^N \notin [-1 + \varepsilon_0, A]} \right) \right| \leq C' N \mathbb{P}(E^N \notin [-1 + \varepsilon_0, A])$$

and by the last part of Assumption **A**,

$$\lim_{\varepsilon_0 \rightarrow 0, A \rightarrow \infty} \sup_N N \mathbb{P}(E^N \notin [-1 + \varepsilon_0, A]) = \lim_{\varepsilon_0 \rightarrow 0, A \rightarrow \infty} \nu_E((-1, -1 + \varepsilon_0) \cup (A, \infty)) = 0.$$

Combining the two last inequalities, we obtain that the family of functions $(N\mathbb{E}(\widehat{g}(\cdot, E^N)))_N$ is uniformly equicontinuous on $[0, 1]$ and the convergence is uniform by Ascoli Theorem. \square

We now assume that p is a C^3 -function from $[0, 1] \times (-1, \infty)$ to $[0, 1]$ and $p(z, 0) = z$ for any $z \in [0, 1]$. A main example is given by

$$p(z, w) = \frac{z(1+w)}{z(1+w) + 1 - z} \quad (23)$$

and describes a case where the environment w acts as a selection factor on the probability to get the allele A . By construction, this selection factor is larger than -1 . The particular case when the environment is non-random, i.e. $E_k^N = s/N$ a.s. for some real number $s \in (-1, +\infty)$, yields the classical Wright-Fisher process with weak selection. It is well known that in this case, the processes $(Z_{[N \cdot]}^N)_N$ converge in law to the Wright-Fisher diffusion with selection coefficient s whose equation is given by $dZ_t = \sqrt{Z_t(1-Z_t)}dB_t + sZ_t(1-Z_t)dt$. Here we generalize this result for a random environment.

3.2 Tightness and identification

For the statement, we introduce the drift coefficient inherited from the fluctuations of the environment:

$$b_1(z) = \alpha_E \frac{\partial p}{\partial w}(z, 0) + \frac{\sigma_E}{2} \frac{\partial^2 p}{\partial w^2}(z, 0) + \int_{(-1, \infty)} \left(p(z, w) - z - h_E(w) \frac{\partial p}{\partial w}(z, 0) \right) \nu_E(dw).$$

Theorem 3.2. *Under Assumption **A**, the sequence of processes $(Z_{[N \cdot]}^N/N, S_{[N \cdot]}^N)_N$ is tight in $\mathbb{D}([0, \infty), [0, 1] \times \mathbb{R})$ and any limiting value of this sequence is solution of the following well defined stochastic differential equation*

$$\begin{aligned} Z_t &= Z_0 + \int_0^t b_1(Z_s) ds + \int_0^t \sqrt{Z_s(1-Z_s)} dB_s^D + \sigma_E \int_0^t \frac{\partial p}{\partial w}(Z_s, 0) dB_s^E + \\ &\quad + \int_{(-1, \infty)} (p(Z_{t-}, w) - Z_{t-}) \tilde{N}(dt, dw); \\ Y_t &= \alpha_E t + \sigma_E B_t^E + \int_0^t \int_{(-1, \infty)} h_E(w) \tilde{N}(dt, dw) + \int_0^t \int_{(-1, \infty)} (w - h_E(w)) N(dt, dw), \end{aligned} \quad (24)$$

where B^D and B^E are Brownian motions; N is a point Poisson measure on $\mathbb{R}_+ \times (-1, \infty)$ with intensity $dt \nu_E(dw)$ and \tilde{N} is the compensated martingale measure of N ; Z_0, B^D, B^E and N are independent.

Proof. We apply our results to the Markov chain

$$X_k^N = \left(\frac{Z_k^N}{N}, S_k^N \right), \quad k \in \mathbb{N}$$

which takes values in $\mathcal{X} = [0, 1] \times \mathbb{R}$.

Let $x = (z, y) \in \mathcal{X}$, we set $F_x^N = F_{(z, y)}^N = \left(\frac{1}{N} \sum_{i=1}^N \mathcal{E}_i(z, E^N), y + E^N \right)$ and we have

$$F_x^N - x = \left(\frac{1}{N} \sum_{i=1}^N (\mathcal{E}_i(z, E^N) - z), E^N \right). \quad (25)$$

The state space of the random variables $F_x^N - x$ is $\mathcal{U} = [-1, 1] \times (-1, +\infty)$.

We first prove that the assumptions **(H0)**, **(H1)**, **(H'1)** and **(H2)** are satisfied with $v_N = N$.

(i) Let us first check **(H0)**. We take $b > 0$ and consider

$$\mathcal{G}_x^N(\mathbb{1}_{\mathcal{B}(0,b)^c}) = N \mathbb{E}(\mathbb{1}_{\mathcal{B}(0,b)^c}(F_x^N - x)).$$

Then

$$N \mathbb{E}(\mathbb{1}_{\mathcal{B}(0,b)^c}(F_x^N - x)) \leq N \mathbb{P}\left(\frac{1}{N} \left| \sum_{i=1}^N (\mathcal{E}_i(z, E^N) - z) \right| > b/\sqrt{2}\right) + N \mathbb{P}(|E^N| > b/\sqrt{2})$$

We observe that $\frac{1}{N} \left| \sum_{i=1}^N (\mathcal{E}_i(z, E^N) - z) \right| \in [-1, 1]$ a.s. Moreover the last part of Assumption **A** ensures that

$$\limsup_{N \rightarrow \infty} N \mathbb{P}(|E^N| > b/\sqrt{2}) \leq \nu[b/\sqrt{2} - 1, \infty),$$

which tends to 0 as $b \rightarrow +\infty$. Then $\sup_{N, x \in [0,1] \times (-1, \infty)} \mathcal{G}_x^N(\mathbb{1}_{\mathcal{B}(0,b)^c}) \rightarrow 0$ and **(H0)** is satisfied.

(ii) We define the function h on \mathcal{U} by

$$h(u, w) = (1 - e^{-u}, 1 - e^{-w})$$

and the space \mathcal{H} is the subset of real functions on \mathcal{U} defined as

$$\mathcal{H} = \{(u, w) \in \mathcal{U} \rightarrow H_{k,\ell}(u, w); k, \ell \geq 0\}, \quad \text{with } H_{k,\ell}(u, w) = 1 - e^{-ku - \ell w}.$$

We can apply the local Stone-Weierstrass Theorem on the locally compact Hausdorff space $\mathcal{U} - \{0, 0\}$ (see Appendix 6.4) to the subalgebra $Vect(\mathcal{H})$, which is thus dense in the set of functions of $C_c(\mathcal{U})$ null in zero. Moreover $Vect(\mathcal{H})$ is stable by multiplication by $|h|^2$. We deduce that **(H1.2)** is satisfied, while **(H1.1)** is obvious.

Let us now prove that **(H'1)** is satisfied. We need to study the limit of $\mathcal{G}_x^N(H_{k,\ell})$ as N tends to infinity. Recall that $\mathcal{G}_x^N(H_{k,\ell}) = N \mathbb{E}(H_{k,\ell}(F_x^N - x))$ with $x = (z, y)$ and $F_x^N - x$ given by (25). We have

$$\begin{aligned} \mathcal{G}_x^N(H_{k,\ell}) &= N \mathbb{E}\left(1 - e^{-\frac{k}{N} \sum_{i=1}^N (\mathcal{E}_i(z, E^N) - z)} e^{-\ell E^N}\right) \\ &= N \left(1 - \mathbb{E}\left(\mathbb{E}\left[e^{-\frac{k}{N} (\mathcal{E}(z, E^N) - z)} \mid E^N\right]^N e^{-\ell E^N}\right)\right) \\ &= N \left(1 - \mathbb{E}\left(\left[e^{-\frac{k}{N}(1-z)} p(z, E^N) + e^{\frac{k}{N}z} (1 - p(z, E^N))\right]^N e^{-\ell E^N}\right)\right). \end{aligned}$$

Using the following Taylor expansion

$$\log\left(e^{-\frac{k}{N}(1-z)} p + e^{\frac{k}{N}z} (1 - p)\right) = \frac{k}{N}(z - p) + \frac{k^2}{2N^2} p(1 - p) + \mathcal{O}(1/N^3),$$

with $N^3 \mathcal{O}(1/N^3)$ bounded uniformly in $p, z \in [0, 1]$, we obtain

$$\begin{aligned} \mathcal{G}_x^N(H_{k,\ell}) &= N \mathbb{E}\left(1 - e^{k(z - p(z, E^N)) - \ell E^N} \cdot e^{\frac{k^2}{2N} p(z, E^N)(1 - p(z, E^N))} \cdot e^{\mathcal{O}(1/N^2)}\right) \\ &= N \mathbb{E}\left(1 - [(1 - A_{k,\ell})(1 + B_{k,N})(1 + R_{k,N})](z, E^N)\right), \end{aligned}$$

where $N^2 R_{k,N}(z, w)$ is uniformly bounded for $z \in [0, 1], w \in (-1, \infty)$ and $N \geq 1$ and

$$A_{k,\ell}(z, w) = 1 - \exp(-k(p(z, w) - z) - \ell w); \quad B_{k,N}(z, w) = \frac{k^2}{2N} p(z, w)(1 - p(z, w)) + \mathcal{O}\left(\frac{1}{N^2}\right).$$

By expansion, we obtain

$$\begin{aligned} \mathcal{G}_x^N(H_{k,\ell}) &= N\mathbb{E}(A_{k,\ell}(z, E^N)) \left(1 + \mathcal{O}(1/N)\right) \\ &\quad - \frac{k^2}{2} \mathbb{E}\left(p(z, E^N)(1 - p(z, E^N))\right) + \mathcal{O}(1/N). \end{aligned} \quad (26)$$

Using Lemma 3.1 both for $(z, w) \rightarrow A_{k,\ell}(z, w)$ and $(z, w) \rightarrow p(z, w)(1 - p(z, w)) - z(1 - z)$, we obtain from (26) that

$$\mathcal{G}_x^N(H_{k,\ell}) \xrightarrow{N \rightarrow \infty} \mathcal{G}_x(H_{k,\ell}) = \mathcal{B}_z(A_{k,\ell}) - \frac{k^2}{2} z(1 - z),$$

uniformly for $x = (z, y) \in [0, 1] \times \mathbb{R}$. Then **(H'1.1)** is satisfied and for any $x \in \mathcal{X}$,

$$\mathcal{G}_x(H_{k,\ell}) = \alpha_E \frac{\partial A_{k,\ell}}{\partial w}(z, 0) + \frac{\beta_E}{2} \frac{\partial^2 A_{k,\ell}}{\partial w^2}(z, 0) + \int_{(-1, \infty)} \widehat{A}_{k,\ell}(z, w) \nu_E(dw) - \frac{k^2}{2} z(1 - z) \quad (27)$$

with

$$\frac{\partial A_{k,\ell}}{\partial w}(z, 0) = k \frac{\partial p}{\partial w}(z, 0) + \ell, \quad \frac{\partial^2 A_{k,\ell}}{\partial w^2}(z, 0) = k \frac{\partial^2 p}{\partial w^2}(z, 0) - \left(k \frac{\partial p}{\partial w}(z, 0) + \ell\right)^2$$

and

$$\widehat{A}_{k,\ell}(z, w) = A_{k,\ell}(z, w) - h_E(w) \frac{\partial A_{k,\ell}}{\partial w}(z, 0) - \frac{h_E^2(w)}{2} \frac{\partial^2 A_{k,\ell}}{\partial w^2}(z, 0).$$

The assumptions on the function p allow us to conclude that **(H'1.2)** is also satisfied.

(iii) We now check **(H2)**. First, the continuity of \mathcal{G} on $[0, 1] \times \mathbb{R}$ comes from the regularity of p and from the integrability assumption on ν_E (by Assumption **A**) with use of Lebesgue's Theorem.

Now, let us introduce the truncation function defined on $\mathcal{U} = [-1, 1] \times (-1, \infty)$ by

$$\forall (u, w) \in \mathcal{U}, \quad h_0(u, w) = (u, h_E(w)).$$

With the notation of Section 2, recall that $h_0^1(u, w) = u$ and $h_0^2(u, w) = h_E(w)$.

With the notation of Lemma 2.1, we have

$$\alpha_1^{h_0}(H_{k,\ell}) = k, \quad \alpha_2^{h_0}(H_{k,\ell}) = \ell, \quad \beta_{11}^{h_0}(H_{k,\ell}) = -\frac{k^2}{2}, \quad \beta_{12}^{h_0}(H_{k,\ell}) = \beta_{21}^{h_0}(H_{k,\ell}) = -\frac{k\ell}{2}, \quad \beta_{22}^{h_0}(H_{k,\ell}) = -\frac{\ell^2}{2}.$$

Moreover, setting $K(z, w) = (p(z, w) - z, w)$, we note that $A_{k,\ell} = H_{k,\ell} \circ K$ and

$$\widehat{A}_{k,\ell}(z, w) = \overline{H_{k,\ell}}^{h_0}(K(x, w)) + kp_1(z, w) - \frac{k^2}{2} p_2(z, w) - k\ell h_E(w) q(z, w),$$

with $p_1(z, w) = p(z, w) - z - h_E(w) \frac{\partial p}{\partial w}(z, 0) - \frac{h_E(w)^2}{2} \frac{\partial^2 p}{\partial w^2}(z, 0)$,
 $p_2(z, w) = (p(z, w) - z)^2 - h_E^2(w) \left(\frac{\partial p}{\partial w}(z, 0) \right)^2$ and $q(z, w) = p(z, w) - z - h_E(w) \frac{\partial p}{\partial w}(z, 0)$.

Now we set $V = (-1, \infty)$ and choose $\mu = \nu_E$ and for $x = (z, y) \in [0, 1] \times \mathbb{R}$, we define

$$b_1(x) = b_1(z) = \alpha_E \frac{\partial p}{\partial w}(z, 0) + \frac{\beta_E}{2} \frac{\partial^2 p}{\partial w^2}(z, 0) + \int_V p_1(z, w) \nu_E(dw) \quad ; \quad b_2(x) = \alpha_E$$

$$\sigma_{11}(x) = \sqrt{z(1-z)} \quad ; \quad \sigma_{22}(x) = \sigma_E \quad ; \quad \sigma_{21}(x) = 0 \quad ; \quad \sigma_{12}(x) = \sigma_E \frac{\partial p}{\partial w}(z, 0).$$

Then (27) rewrites

$$\mathcal{G}_x(H_{k,\ell}) = kb_1(x) + \ell b_2(x) - \frac{k^2}{2} c_{11}(x) - \frac{\ell^2}{2} c_{22}(x) - k\ell c_{12}(x) + \int_V \overline{H_{k,\ell}}^{h_0}(K(z, w)) \nu_E(dw),$$

where, recalling that $\beta_E = \sigma_E^2 + \int_{(-1, \infty)} h_E^2(w) \nu_E(dw)$,

$$c_{11}(x) = z(1-z) + \beta_E \left(\frac{\partial p}{\partial w}(z, 0) \right)^2 + \int_V p_2(z, w) \nu_E(dw)$$

$$= \sigma_{11}^2(x) + \sigma_{12}^2(x) + \int_V (h_0^1(K(z, w)))^2 \mu(dw),$$

$$c_{22}(x) = \beta_E = \sigma_{22}(x)^2 + \int_V (h_0^2(K(z, w)))^2 \mu(dw),$$

$$c_{12}(x) = \beta_E \frac{\partial p}{\partial w}(z, 0) + \int_{(-1, \infty)} h_E(w) q(z, w) \nu_E(dw) = \sigma_{12}(x) \sigma_{22}(x) + \int_V h_0^1 h_0^2(K(z, w)) \mu(dw).$$

Thus (H2) holds for any $H = H_{k,\ell} \in \mathcal{H}$.

We can now apply Theorems 2.3 and 2.4 for tightness and identification and conclude. \square

3.3 Pathwise uniqueness and convergence in law

To get uniqueness for (24) and convergence of the scaled process, we use here pathwise uniqueness results and need some additional regularity of the terms of (24). We follow Li-Pu [25] and assume that

Assumption B. For any $w \in (-1, +\infty)$, the function $z \rightarrow p(z, w)$ is non-decreasing. Moreover there exist functions r , b_r and b_d such that for any $z \in [0, 1]$,

$$b_1(z) = b_r(z) + b_d(z),$$

with b_d non-increasing and r non-negative, non-decreasing and concave, $\int_0^1 1/r(z) dz = +\infty$, and $|b_r(z) - b_r(z')| \leq r(|z - z'|)$ for any $z, z' \in [0, 1]$.

The monotonicity assumption on p is natural since $p(z, w)$ denotes the probability to get an offspring carrying allele A in a population where z individuals carry allele A . The regularity of b_1 may impose some moment conditions on the measure ν_E , as seen for an example in Section 3.4 below.

Corollary 3.3. Under Assumptions A and B, the sequence of processes $\left(Z_{[N,\cdot]}^N / N, S_{[N,\cdot]}^N \right)_N$ converges in law in $\mathbb{D}([0, \infty), [0, 1] \times \mathbb{R})$ to the unique strong solution (Z, Y) of (24).

Proof. We show that pathwise uniqueness holds for (24), which ensures **(H3)** and allows us to apply Theorem 2.5. The pathwise uniqueness of the process Y is well known. Let us focus on the first equation of (24) and prove the pathwise uniqueness of the process Z . We are in the conditions of applications of Theorem 3.2 in [25]. Indeed, we remark that the Brownian part of (24) writes

$$\sqrt{Z_t(1-Z_t)}dB_t^D + \sigma_E \frac{\partial p}{\partial w}(Z_t, 0) dB_t^E = \sqrt{Z_t(1-Z_t) + \sigma_E^2 \left(\frac{\partial p}{\partial w}(Z_t, 0) \right)^2} dW_t = \sigma(Z_t)dW_t,$$

with W Brownian motion since B^D and B^E are two independent Brownian motions. We prove easily that for any $z_1, z_2 \in [0, 1]$, $|\sigma(z_1) - \sigma(z_2)|^2 \leq L|z_1 - z_2|$ for some constant $L > 0$. Using $\nu_E(V - V_0) < \infty$ with $V_0 = (-1/2, 1)$ and $z \in [0, 1] \rightarrow (p(z, w) - z)/w$ is uniformly Lipschitz for $w \in V_0$ since its first derivative is bounded, all the required assumptions for [25] Theorem 3.2 are satisfied and we get the pathwise uniqueness of the solution of (24). \square

3.4 Example

A main example is given by

$$p(z, w) = \frac{z(1+w)}{z(1+w) + 1 - z} \quad (28)$$

and describes a case where the environment w acts as a selection factor on the probability to get the allele A . By construction, this selection factor is larger than -1 . The particular case when the environment is non-random, i.e. $E_k^N = s/N$ a.s. for some real number $s \in (-1, +\infty)$, yields the classical Wright-Fisher process with weak selection. It is well known that in this case, the processes $(Z_{[N,\cdot]}^N)_N$ converge in law to the Wright-Fisher diffusion with selection coefficient s whose equation is given by $dZ_t = \sqrt{Z_t(1-Z_t)}dB_t + sZ_t(1-Z_t)dt$. Here we generalize this result for a random environment.

Note that in this case, we have

$$\frac{\partial p}{\partial w}(z, 0) = z(1-z) ; \quad \frac{\partial^2 p}{\partial w^2}(z, 0) = -2z^2(1-z).$$

Then

$$b_1(z) = \alpha_E z(1-z) - \sigma_E z^2(1-z) + \int_{(-1, \infty)} \left(\frac{wz(1-z)}{zw+1} - h_E(w)z(1-z) \right) \nu_E(dw). \quad (29)$$

Under Assumption **A** and by Theorem 3.2, the sequence of processes $\left(Z_{[N,\cdot]}^N/N, S_{[N,\cdot]}^N \right)_N$ is tight in $\mathbb{D}([0, \infty), [0, 1] \times \mathbb{R})$ and the first coordinate of any limiting value is solution of

$$\begin{aligned} Z_t &= Z_0 + \int_0^t b_1(Z_s)ds + \int_0^t \sqrt{Z_s(1-Z_s)}dB_s^D + \sigma_E \int_0^t Z_s(1-Z_s)dB_s^E \\ &\quad + \int_0^t \int_{(-1, \infty)} \frac{wZ_{s-}(1-Z_{s-})}{1+wZ_{s-}} \tilde{N}(ds, dw). \end{aligned} \quad (30)$$

In particular if $\sigma_E = 0$ and $\nu_E = 0$, we recover the classical Wright-Fisher diffusion with deterministic selection α_E . Our extension allows us to consider small random fluctuations (asymptotically Brownian) and punctual dramatic advantage of the selective effects.

By Corollary 3.3, a sufficient condition for the convergence in law of $\left(Z_{[N,\cdot]}^N/N, S_{[N,\cdot]}^N\right)_N$ in $\mathbb{D}([0, \infty), [0, 1] \times \mathbb{R})$ to the unique strong solution of (30) is

$$\int_{\cdot}^{+\infty} w \nu_E(dw) < +\infty.$$

Indeed, writing

$$k(w, z) = \frac{\partial}{\partial z} \left(\frac{wz(1-z)}{zw+1} - h_E(w)z(1-z) \right) = \frac{1+w}{(1+wz)^2} - 1 - h_E(w)(1-2z),$$

the function $z \rightarrow (1+w)/(1+wz)^2 - 1$ is decreasing for small w , $k(w, z) = \mathcal{O}(w^2)$ on $[-1/2, 1]$ and $\sup_{z \in [0, 1], w \in [1, \infty)} k(w, z) \leq Cw$ for some $C > 0$. Therefore, the function b_1 defined in (29) satisfies Assumption B.

4 Continuous State Branching Process with interactions in Lévy environment

In this section, we are interested in large population approximations of population dynamics with random environment and interactions. We generalize in different directions the classical convergence of Galton-Watson processes to *Continuous State Branching processes* (CSBP), see for example [15, 22, 8]. We focus on models where the environment and the interaction mainly affect the mean of the reproduction law and thus modify the drift term of the CSBP by addition of stochastic and nonlinear terms. This allows us to construct a discrete population model approximating the so-called *CSBP in Lévy environment with interactions* (BPLEI) for large populations.

The CSBPs in random environment or with interaction have recently been subject of a large attention. More precisely, we refer to [28] for existence of the solution of the associated SDE with general assumptions, [4, 3, 26] for approximations and study of some classes of CSBP in random environment (without interactions), [2, 10, 29] for CSBP with interactions (without random environment), while [30] considered diffusion approximations in the continuous context. The method we develop here, relying on Section 2, allows to obtain new statements both for convergence of discrete population models and for existence of solutions of SDE with jumps, as detailed in the following Theorems.

4.1 The discrete model

Let us now describe our framework. We consider $N \in \mathbb{N}^*$ scaling the population size and a sequence $(v_N)_N$ which tends to infinity with N giving the time scale. As in Section 3, we introduce for any N a sequence of independent identically distributed real-valued random variables $(E_k^N)_{k \geq 0}$ with same law as E^N . The asymptotical behavior of $(E_k^N)_{k \geq 0}$ is similar as the one in Section 3 (Assumption A) but the scaling parameter is v_N . As in the previous section, h_E denotes a truncation function defined on $(-1, +\infty)$.

Assumption A1. Let us consider $\alpha_E \in \mathbb{R}$, $\sigma_E \in [0, \infty)$ and ν_E a measure on $(-1, \infty)$ such that

$$\int_{(-1, \infty)} (1 \wedge w^2) \nu_E(dw) < \infty. \quad (31)$$

Writing $\beta_E = \sigma_E^2 + \int_{(-1, \infty)} h_E^2(w) \nu_E(dw)$, we assume that

$$\lim_{N \rightarrow \infty} v_N \mathbb{E}(h_E(E^N)) = \alpha_E; \quad \lim_{N \rightarrow \infty} v_N \mathbb{E}(h_E^2(E^N)) = \beta_E; \quad \lim_{N \rightarrow \infty} v_N \mathbb{E}(f(E^N)) = \int_{(-1, \infty)} f(w) \nu_E(dw),$$

for any f nul in a neighborhood of zero.

We also consider the associated random walk defined by

$$S_0^N = 0, \quad S_{k+1}^N = S_k^N + E_k^N \quad (k \geq 0).$$

We recall as in Section 3 that **A1** is equivalent to the convergence of the random walk $S_{\lfloor v_N \cdot \rfloor}^N$ to the Lévy process Y with characteristics $(\alpha_E, \beta_E, \nu_E)$ defined in (22). We reduce the set of jumps to $(-1, \infty)$ to avoid degenerated cases when a catastrophe below -1 could kill all the population in one generation.

Let us fix N . We assume that each individual reproduces independently at generation k with the same reproduction law which depends both on the underlying environment and on the population size. More precisely, given a population size n and an environment w , this law is the law of a random variable $L^N(n, w)$. We thus introduce random variables $Z_0 \geq 0$ and $L_{i,k}^N(n, w)$ such that the family of random variables $(Z_0, (L_{i,k}^N(n, w), n \in \mathbb{N}, w \in (-1, +\infty)), E_j^N; i, k \in \mathbb{N}^*, j \in \mathbb{N})$ is independent and for each $n \in \mathbb{N}, w \in (-1, +\infty)$, the random variables $L_{i,k}^N(n, w)$ are all distributed as $L^N(n, w)$ for $i, k \in \mathbb{N}^*$. Moreover, we assume that the function $L_{i,k}^N$ defined on $\Omega \times \mathbb{N} \times (-1, +\infty)$ endowed by the product σ -field is measurable.

The population size Z_k^N at generation k is recursively defined as follows,

$$Z_0^N = \lfloor N Z_0 \rfloor, \quad Z_{k+1}^N = \sum_{i=1}^{Z_k^N} L_{i,k}^N(Z_k^N, E_k^N) \quad \forall k \geq 0, \quad (32)$$

with $Z_0 \in [0, \infty)$ a fixed integer valued random variable.

We investigate the convergence in law of the process $\left(\left(\frac{1}{N} Z_{\lfloor v_N t \rfloor}^N, S_{\lfloor v_N t \rfloor}^N \right), t \in [0, \infty) \right)$. We cannot apply directly our general result to $Z_{\lfloor v_N \cdot \rfloor}^N$. Indeed, the (associated) characteristics of the first component are not bounded. Moreover, scaling limits of Z^N can lead to explosive processes, as already happens in the Galton-Watson case. Therefore, we first study the convergence of the process

$$X_k^N = (\exp(-Z_k^N/N), S_k^N) \quad (k \in \mathbb{N}) \quad (33)$$

in $\mathbb{D}(\mathbb{R}_+, [0, 1] \times \mathbb{R})$ where the state space of the first coordinate has been compactified. Following notation of Section 2, we introduce for $x = (\exp(-z), y) \in (0, 1] \times \mathbb{R}$ the quantity

$$F_x^N = \left(\exp \left(-\frac{1}{N} \sum_{i=1}^{\lfloor Nz \rfloor} (L_i^N(\lfloor Nz \rfloor, E^N) - 1) - z \right), y + E^N \right), \quad (34)$$

and observe that for any $z \in \mathbb{N}/N$, conditionally on $X_k^N = (\exp(-z), y)$, the random variable X_{k+1}^N is distributed as F_x^N .

We apply now the theoretical framework developed in Section 2. Let us introduce $\chi = (0, 1] \times \mathbb{R}$ and $\mathcal{U} = [-1, 1] \times (-1, \infty)$ and for $u = (v, w) \in \mathcal{U}$, we define

$$h(u) = h(v, w) = (v, 1 - \exp(-w)), \quad h_0(u) = h_0(v, w) = (v, h_E(w)) \quad (35)$$

respectively as the specific function and the truncation function. We choose the functional space \mathcal{H} defined by

$$\mathcal{H} = \{H_{k,\ell} : k \geq 1, \ell \geq 0\} \cup \{H_\ell : \ell \geq 1\},$$

where for any $u = (v, w) \in \mathcal{U}$,

$$H_{k,\ell}(u) = v^k \exp(-\ell w) \quad \text{and} \quad H_\ell(u) = 1 - \exp(-\ell w).$$

The fact that \mathcal{H} satisfies **(H1)** is a consequence of the local Stone Weierstrass Theorem on $[-1, 1] \times [-1, \infty) \setminus \{(0, 0)\}$ (cf. Appendix 6.4). For any $k, \ell \geq 0$ and $x = (\exp(-z), y) \in (0, 1] \times \mathbb{R}$, we have

$$\begin{aligned} \mathcal{G}_x^N(H_{k,\ell}) &= \mathbb{E} \left(H_{k,\ell} \left(\exp \left(-\frac{1}{N} \sum_{i=1}^{[Nz]} (L_i^N([Nz], E^N) - 1) - z \right) - \exp(-z), E^N \right) \right) \\ &= e^{-kz} v_N \mathbb{E} \left(\left(e^{-\frac{1}{N} \sum_{i=1}^{[Nz]} (L_i^N([Nz], E^N) - 1)} - 1 \right)^k e^{-\ell E^N} \right). \end{aligned}$$

Using binomial expansion and setting

$$P_k^N(z, w) = \mathbb{E} \left(e^{-\frac{k}{N} (L^N([Nz], w) - 1)} \right)^{[Nz]} - 1 \quad (36)$$

and

$$A_{j,\ell}^N(z) = v_N \mathbb{E} \left(P_j^N(z, E^N) e^{-\ell E^N} \right). \quad (37)$$

we obtain by independence of the reproduction random variables conditionally on E^N that

$$\mathcal{G}_x^N(H_{k,\ell}) = e^{-kz} \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} A_{j,\ell}^N(z) \quad (38)$$

for $k \geq 1$, since $\sum_{j=0}^k \binom{k}{j} (-1)^{j-k} = 0$. Similarly, for $\ell \geq 1$,

$$\mathcal{G}_x^N(H_\ell) = v_N \mathbb{E}(1 - \exp(-\ell E^N)).$$

Using Section 2, the asymptotic study of the sequence $(X_{[v_N \cdot]}^N)_N$ can be determined via the convergence of the functions $A_{j,\ell}^N$. More precisely, the uniform convergence and boundedness of $\exp(-kz) A_{j,\ell}^N(z)$ ensure the tightness of $X_{[v_N \cdot]}^N$ using Theorem 2.3. The continuity of the limiting functions is involved in the identification of the characteristic triplet of the limiting semimartingales. Finally, their representation as solutions of a stochastic differential equation

and the associated uniqueness yield the convergence (Theorem 2.5).

Let us note that this characterization of the convergence of the process in terms of characteristics of the joint distribution of the offspring size and the environment applies in particular to Galton-Watson processes. In that case, $E^N = 0$, $L^N(z, w) = L^N$ and $A_{j,\ell}^N(z)$ becomes

$$A_j^N(z) = v_N P_j^N(z, 0) = v_N \left(\mathbb{E} \left(e^{-\frac{j}{N}(L^N - 1)} \right)^{[Nz]} - 1 \right).$$

We observe that $A_j^N(z) \sim v_N [Nz] \mathbb{E}(f_j((L^N - 1)/N))$ as $N \rightarrow \infty$. The uniform convergence of $e^{-jz} A_j^N(z)$ is equivalent to the classical necessary and sufficient condition for convergence in law of Galton-Watson processes [15], see Section 4.4 for details.

In the next sections, we generalize this convergence by considering a class of reproduction random variables taking into account interactions and random environment and converging to CSBP with interactions in Lévy environment.

4.2 Tightness

Using the process $(X_{[v_N \cdot]}^N)_N$ defined in (33), we first prove the tightness of the process by assuming uniform convergence of characteristics $A_{j,\ell}^N$ defined in (37).

Assumption A1'. For any $1 \leq j \leq k$ and $\ell \geq 0$, there exists a bounded function $A_{j,k,\ell}$ such that

$$e^{-kz} A_{j,\ell}^N(z) \xrightarrow{N \rightarrow \infty} A_{j,k,\ell}(z)$$

uniformly for $z \geq 0$.

We state a tightness criterion for the original scaled process in the state space $[0, \infty] \times \mathbb{R}$ endowed with a distance d which makes it compact and then Polish, say $d(z_1, z_2) = |\exp(-z_1) - \exp(-z_2)|$ for $z_1, z_2 \in [0, \infty]$ with the convention $\exp(-\infty) = 0$.

Theorem 4.1. Under Assumptions A1 and A1', the sequence of processes

$$\left(\left(\frac{1}{N} Z_{[v_N t]}^N, S_{[v_N t]}^N \right), t \in [0, \infty) \right)$$

is tight in $\mathbb{D}(\mathbb{R}_+, [0, \infty] \times \mathbb{R})$.

Proof. We first recall that for $\ell \geq 1$,

$$v_N \mathbb{E} \left(1 - e^{-\ell E^N} \right) \xrightarrow{N \rightarrow \infty} \gamma_\ell^E \quad (39)$$

for $1 - e^{-\ell w} = \ell h_E(w) - \frac{1}{2} \ell^2 h_E^2(w) + \kappa(w)$, where $\kappa(w) = o(w^2)$ is continuous bounded and using Assumption A1.

Then we can define \mathcal{G} on H_ℓ for $\ell \geq 1$ as

$$\mathcal{G}_x(H_\ell) = \gamma_\ell^E, \quad (40)$$

for any $x \in \mathcal{X} = (0, 1] \times \mathbb{R}$. Let us now define \mathcal{G} for $H_{k,\ell} \in \mathcal{H}$ and $k \geq 1, \ell \geq 0$. We set

$$\mathcal{G}_x(H_{k,\ell}) = \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} A_{j,k,\ell}(z). \quad (41)$$

for $x = (e^{-z}, y) \in \mathcal{X}$. Using Assumption **A1'** and (38), we obtain that

$$\lim_{N \rightarrow \infty} \sup_{x \in \mathcal{X}} |\mathcal{G}_x^N(H) - \mathcal{G}_x(H)| = 0$$

for any $H \in \mathcal{H}$. Moreover $\mathcal{G}(H)$ is bounded by **A1'** and Hypothesis **(H'1)** is satisfied. The tightness of $(X_{[v_N, \cdot]}^N)_N$ is then a consequence of Theorem 2.3 and yields the result. \square

4.3 Identification

We aim now at identifying the limiting values of $(X_{[v_N, \cdot]}^N)_N$ as diffusions with jumps. We are interested in models where the environment and the interactions affect the mean reproduction law and the limiting process will be *CSBP with interactions in a Lévy environment* (BPLEI).

We introduce a truncation function h_D on the state space $(0, +\infty)$, the parameters $\alpha_D \in \mathbb{R}$ and $\sigma_D \geq 0$ and a σ -finite measure ν_D on $(0, +\infty)$ such that

$$\int_0^\infty (1 \wedge z^2) \nu_D(dz) < +\infty. \quad (42)$$

We also consider a locally Lipschitz function g defined on \mathbb{R}^+ such that

$$e^{-z} z g(z) \xrightarrow{z \rightarrow \infty} 0. \quad (43)$$

The function g models the interaction between individuals. In the applications to population dynamics, the most relevant functions will be polynomial.

We provide now the scaling assumption on the reproduction random variable L^N so that the limiting values of Z^N/N can be identified to a BPLEI. This assumption will become more explicit and natural through the identification and examples of the next sections.

Assumption A2. Setting for $z \geq 0$,

$$\begin{aligned} \gamma_z^D &= \alpha_D z - \frac{1}{2} \sigma_D^2 z^2 + \int_{(0, +\infty)} (1 - e^{-zv} - zh_D(v)) \nu_D(dv), \\ \gamma_z^E &= \alpha_E z - \frac{1}{2} \sigma_E^2 z^2 + \int_{(-1, +\infty)} (1 - e^{-zw} - zh_E(w)) \nu_E(dw), \end{aligned}$$

we assume that for any $1 \leq j \leq k$ and $\ell \geq 0$,

$$\sup_{z \geq 0} e^{-kz} |A_{j,\ell}^N(z) + jz g(z) + \gamma_j^D z + \gamma_{jz+\ell}^E - \gamma_\ell^E| \xrightarrow{N \rightarrow \infty} 0, \quad (44)$$

where $A_{j,\ell}^N$ has been defined in (37).

Remark 4.2. (i) In Appendix 6.1, we provide an explicit construction of a family of random variables $L^N(z, e)$ satisfying **A2**, in the case $\beta_D = 0$ and α_D and ν_D and $(\alpha_E, \beta_E, \nu_E)$ and g satisfying the moment and regularity conditions given above.

(ii) We believe that the simple convergence in **A2** is actually necessary for the convergence of the process Z^N/N to a BPLEI. The uniformity can be obtained for many classes of reproduction laws via uniform continuity, using for example monotone or convex arguments, or boundedness of derivative on compact sets as in the previous section, see forthcoming examples.

(iii) Finally, let us remark that we only need to prove the previous convergence for $z \in \mathbb{N}/N$ in **A2**, using the definition of $A_{j,\ell}^N(z)$ and the uniform continuity of the limit. It will be more convenient for examples.

Note that under Assumption **A2**, Assumption **A1'** is satisfied with

$$A_{j,k,\ell}(z) = e^{-kz} (-jz g(z) - \gamma_j^D z + \gamma_\ell^E - \gamma_{jz+\ell}^E),$$

which is indeed bounded using (43) and the fact $\exp(-kz)\gamma_{jz+\ell}^E$ is bounded, since $|\gamma_{jz+\ell}^E| \leq C_{\ell,j}(z + z^2 + e^{jz/2}z^2\beta_E + e^{jz}\nu_E(-1, -1/2))$. Thus tightness holds.

Moreover we can simplify the expression (41) of the limiting characteristic \mathcal{G}_x , which writes

$$\mathcal{G}_x(H_{k,\ell}) = e^{-kz} \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} (-jz g(z) - \gamma_j^D z + \gamma_\ell^E - \gamma_{jz+\ell}^E) \quad (45)$$

for $x = (e^{-z}, y)$. For that purpose, we denote

$$f_z(u) = 1 - e^{-zu}$$

and use that $\sum_{j=0}^k \binom{k}{j} (-1)^{k-j} j = \delta_{1,k}$ and $\sum_{j=0}^k \binom{k}{j} (-1)^{k-j} j^2 = 2\delta_{2,k} + \delta_{1,k}$ and by a binomial expansion, $\sum_{j=0}^k \binom{k}{j} (-1)^{k-j} f_j(u) = (-1)^{k+1} f_1(u)^k$ and $\sum_{j=0}^k \binom{k}{j} (-1)^{k-j} f_{jz+\ell}(u) = (-1)^{k+1} e^{-\ell u} f_z(u)^k$. By straightforward computation, we obtain that for $k \geq 3$,

$$\mathcal{G}_x(H_{k,\ell}) = (-1)^k e^{-kz} \left(\int_{(-1,+\infty)} e^{-\ell w} (f_z(w))^k \nu_E(dw) + z \int_{(0,+\infty)} (f_1(v))^k \nu_D(dv) \right), \quad (46)$$

and for $k = 2$,

$$\begin{aligned} \mathcal{G}_x(H_{2,\ell}) = e^{-2z} & \left\{ z^2 \beta_E + \int_{(-1,+\infty)} (e^{-\ell w} (f_z(w))^2 - z^2 h_E^2(w)) \nu_E(dw) + z \beta_D \right. \\ & \left. + z \int_{(0,+\infty)} (f_1^2(v) - h_D^2(v)) \nu_D(dv) \right\}, \end{aligned} \quad (47)$$

and for $k = 1$,

$$\mathcal{G}_x(H_{1,\ell}) = e^{-z} \left\{ \gamma_\ell^E - \gamma_{z+\ell}^E - z g(z) - z \gamma_1^D \right\}. \quad (48)$$

To identify the limiting SDE, we have to find the drift and variance terms and the jump measures in (5), from the expressions (40), (46), (47) and (48).

We first remark that for $k \geq 3, \ell \geq 0$, $H_{k,\ell} = \overline{H_{k,\ell}}$. We work by identification for $x = (e^{-z}, y) \in (0, 1] \times \mathbb{R}$ using (46). We thus define the measure μ on $V = [0, +\infty) \times \mathbb{R}$ by

$$\mu(d\theta, du) = \mathbb{1}_{\theta \leq 1, u > -1} d\theta \nu_E(du) + \mathbb{1}_{\theta > 1, u > 0} d\theta \nu_D(du), \quad (49)$$

and the image function K by

$$K_1(x, \theta, u) = -e^{-z} \cdot \left(f_z(u) \mathbb{1}_{\theta \leq 1} + f_1(u) \mathbb{1}_{1 < \theta \leq 1+z} \right); \quad K_2(x, \theta, u) = u \mathbb{1}_{\theta \leq 1}. \quad (50)$$

Then $H_{k,\ell}$ satisfies **(H2.2)** for $k \geq 3, \ell \geq 0$.

Moreover it is easy to find b_2 and $\sigma_{2,2}$ so that H_ℓ satisfies **(H2.2)** for $\ell \geq 1$ using that

$$H_\ell(v, w) = \ell h_E(w) - \frac{\ell^2}{2} h_E(w)^2 + \overline{H}_\ell(v, w), \quad \overline{H}_\ell(v, w) = f_\ell(w) - \ell h_E(w) - \frac{\ell^2}{2} h_E(w)^2$$

and (40). Indeed, by identification and from the expression (4), we set for $x = (e^{-z}, y)$,

$$b_2(x) = \alpha_E; \quad \sigma_{2,2}(x) = \sigma_E. \quad (51)$$

Let us now consider the functions $H_{2,\ell}$ ($\ell \geq 0$). Note that for $u = (v, w)$, we have

$$H_{2,\ell}(v, w) = v^2 e^{-\ell w} = h_D^2(v) + \overline{H}_{2,\ell}, \quad \overline{H}_{2,\ell} = v^2(e^{-\ell w} - 1) + v^2 - h_D^2(v).$$

The fact that **(H2.2)** is satisfied for $H_{2,\ell}$ comes from (47) for the left hand side and for the right hand it is given by a direct computation of

$$\sigma_{1,1}(x)^2 + \sigma_{1,2}(x)^2 + \int_V K_1^2(x, \theta, u) \mu(d\theta, du) + \int_V \overline{H}_{2,\ell}(K(x, \theta, u)) \mu(d\theta, du),$$

where K is defined from (50). Using $\overline{H}_{2,\ell}(K(x, \theta, u)) = K_1(x, \theta, u)^2 (e^{-\ell K_2(x, \theta, u)} - 1)$, the condition writes for $x = (e^{-z}, y)$,

$$\sigma_{1,1}(x)^2 + \sigma_{1,2}(x)^2 = e^{-2z} (z\sigma_D^2 + z^2\sigma_E^2).$$

It remains to check **(H2.2)** for $H_{1,\ell}$, with

$$H_{1,\ell}(v, w) = v e^{-\ell w} = v(1 - \ell h_E(w)) + \overline{H}_{1,\ell}(v, w),$$

where $\overline{H}_{1,\ell}(v, w) = v(\ell h_E(w) - f_\ell(w)) = o(|w|^2)$. Using (48), we have

$$\begin{aligned} \mathcal{G}_x(H_{1,\ell}) &= e^{-z} \left(-\alpha_E z - z\gamma_1^D - z g(z) + \frac{z^2}{2} \sigma_E^2 + \ell z \sigma_E^2 \right. \\ &\quad \left. + \int_{(-1, +\infty)} f_z f_\ell(w) \nu_E(dw) + \int_{(-1, +\infty)} (z h_E(w) - f_z(w)) \nu_E(dw) \right). \end{aligned}$$

As a conclusion, both sides of (4) coincide for $H \in \mathcal{H}$ by setting for any $x = (e^{-z}, y) \in (0, 1] \times \mathbb{R}$,

$$b_1(x) = e^{-z} \left(-\alpha_E z - z\gamma_1^P - zg(z) + \frac{z^2}{2}\sigma_E^2 + \int_{(-1, +\infty)} (zh_E(w) - f_z(w))\nu_E(dw) \right) \quad (52)$$

and $b_2(x) = \alpha_E$ and K, μ defined by (50) and (49) and

$$\sigma_{1,1}(x) = -\sqrt{z}\sigma_D e^{-z}; \quad \sigma_{1,2}(x) = -z\sigma_E e^{-z}; \quad \sigma_{2,1}(x) = 0; \quad \sigma_{2,2}(x) = \sigma_E. \quad (53)$$

and for any $x \in \{0\} \times \mathbb{R}$ and $(\theta, u) \in V$,

$$b(x) = (0, \alpha_E), \quad \sigma_{11}(x) = \sigma_{21}(x) = \sigma_{12}(x) = 0, \quad \sigma_{22}(x) = \sigma_E, \quad (54)$$

$$K_1(x, \theta, u) = 0, \quad K_2(x, \theta, u) = u \mathbb{1}_{\theta \leq 1}. \quad (55)$$

The general identification result for the exponential transformation of the processes can then be stated as follows, with $h_0(v, w) = (v, h_E(w))$.

Theorem 4.3. *Under Assumptions A1 and A2, the sequence of processes*

$$\left(\left(\exp \left(-\frac{1}{N} Z_{[v_N t]}^N \right), S_{[v_N t]}^N \right) : t \in [0, \infty) \right)$$

is tight in $\mathbb{D}([0, \infty), [0, 1] \times \mathbb{R})$ and any limiting value $X \in \mathbb{D}([0, \infty), [0, 1] \times \mathbb{R})$ is a weak solution of the following two-dimensional stochastic differential equation

$$\begin{aligned} X_t = X_0 &+ \int_0^t b(X_s) ds + \int_0^t \sigma(X_s) dB_s + \int_0^t \int_V h_0(K(X_{s-}, v)) \tilde{N}(ds, dv) \\ &+ \int_0^t \int_V (Id - h_0)(K(X_{s-}, v)) N(ds, dv), \end{aligned} \quad (56)$$

where $X_0 = (\exp(-Z_0), 0)$, N is Poisson point measure with intensity $ds\mu(dv)$ on $\mathbb{R}^+ \times V = (0, +\infty)^2 \times \mathbb{R}$ and B is a two-dimensional Brownian motion and Z_0, B, N are independent. The functions $b = (b_1, b_2)$, the matrix σ , the measure μ and the image function K have been defined in (49)-(55).

Proof of Theorem 4.3. We already know that (H1) is a consequence A2. Let us check that (H2) is satisfied. We first prove the continuity (H2.1) of $x \rightarrow \mathcal{G}_x(H)$ for any $H \in \mathcal{H}$ and its extension to $\bar{\mathcal{X}}$. Recalling (45), we need to prove that $z \in [0, \infty) \rightarrow \gamma_\ell^E - \gamma_{jz+\ell}^E$ is continuous and $\exp(-jz)(\gamma_\ell^E - \gamma_{jz+\ell}^E) \rightarrow 0$ as $z \rightarrow \infty$. Indeed, the continuity can be obtained from the domination of $|1 - e^{-(jz+\ell)w} - (jz+\ell)h_E(w)| \leq C(1 \wedge w^2)$ for any $z \in [z_0, z_1] \subset [0, \infty)$, while the limit as $z \rightarrow \infty$ can be proved using Lemma 6.2 in Appendix and $\nu_E(-1, -1 + \varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. That allows us to prove that (H2.1) is satisfied.

Our choice of parameters in (49)-(55) ensures that (H2.2) is satisfied for any H_ℓ and $H_{k,\ell}$. Applying Theorem 2.4 to X^N allows us to conclude. \square

Let us now write explicitly the stochastic differential equation (56) for $X_t = (X_t^1, Y_t)$:

$$\begin{aligned}
dX_t^1 &= X_t^1 \log X_t^1 \left(\alpha_E + \frac{\sigma_E^2}{2} \log X_t^1 + g(-\log X_t^1) + \alpha_D - \frac{\sigma_D^2}{2} \right) dt \\
&\quad - X_t^1 \left(\int_{(-1, +\infty)} (1 - e^{w \log(X_t^1)} + \log X_t^1 h_E(w)) \nu_E(dw) \right. \\
&\quad \quad \left. - \log X_t^1 \int_{(0, +\infty)} (1 - e^{-v} - h_D(v)) \nu_D(dv) \right) dt \\
&\quad + \sigma_E X_t^1 \log X_t^1 dB_t^E - \sigma_D X_t^1 \sqrt{-\log X_t^1} dB_t^D - \int_{(-1, +\infty)} X_t^1 (1 - e^{w \log(X_t^1)}) \tilde{N}^E(dt, dw) \\
&\quad \quad - \int_{(0, +\infty)^2} \mathbb{1}_{\theta \leq -\log X_t^1} X_t^1 (1 - e^{-v}) \tilde{N}^D(dt, d\theta, dv) \\
dY_t &= \alpha_E dt + \sigma_E dB_t^E + \int_{(-1, +\infty)} h_E(w) \tilde{N}^E(dt, dw) + \int_{(-1, +\infty)} (w - h_E(w)) N^E(dt, dw),
\end{aligned}$$

where B^E and B^D are Brownian motions, N^D and N^E are Poisson Point measures respectively on $[0, \infty) \times (0, \infty)$ and $[0, \infty) \times (-1, \infty)$ with intensity $dt\nu_D(dv)$ and $dt\nu_E(dw)$ and Z_0, B^E, B^D, N^D and N^E are independent.

Using Itô's formula (see [17]), a straightforward computation leads to the equation satisfied by $Z_t = -\log X_t^1$. More precisely, we write T_{exp} the explosion time defined by :

$$T_{exp} = \lim_{\varepsilon \rightarrow 0^+} \inf\{t \geq 0; X_t^1 \leq \varepsilon\} = \lim_{a \rightarrow +\infty} \inf\{t \geq 0; Z_t \geq a\} \in [0, +\infty]$$

and we obtain

$$\begin{aligned}
Z_t &= Z_0 + \alpha_D \int_0^t Z_s ds + \int_0^t Z_{s-} dY_s + \int_0^t Z_s g(Z_s) ds + \sigma_D \int_0^t \sqrt{Z_s} dB_s^D + \\
&\quad \int_0^t \int_{(0, +\infty)^2} \mathbb{1}_{\theta \leq Z_{s-}} h_D(v) \tilde{N}^D(ds, d\theta, dv) + \int_0^t \int_{(0, +\infty)^2} \mathbb{1}_{\theta \leq Z_{s-}} (v - h_D(v)) N^D(dt, d\theta, dv).
\end{aligned} \tag{57}$$

on the time interval $[0, T_{exp})$. As expected, we set here $Z_t = +\infty$ for $t \geq T_{exp}$.

When $T_{exp} = +\infty$ almost surely, we say that the process is *conservative* (or non-explosive).

We have thus proved the tightness of the process and identified the limiting values of $(X_{[v_N]}^N)_N$ as weak solutions of a SDE. Uniqueness of the SDE (57) (Hypothesis **H3**) is now involved to conclude for the convergence. From the pioneering works of Yamada and Watanabe, several results have been obtained for pathwise uniqueness relaxing the Lipschitz conditions on coefficients. In particular, general results for positive processes with jumps have been obtained in [13, 25] and used in random environment, see in particular [28]. This technique allows us to conclude for strong uniqueness as soon as the process Z has not exploded. Here, the process may explode in finite time, which is already the case for classical CSBP and in our framework, explosion can also be due to cooperation or random environment. This leads us to consider two cases. First, we obtain a convergence in law on the state space $[0, \infty]$ under an additional regularity assumption on the drift term close to infinity. This result extends in particular the classical criterion for convergence of Galton-Watson processes, adding both random environment and interactions. It generalizes also statements of [4] to Lévy environment

with infinite variation and interactions. Second, we conclude for the convergence of $Z_{[v_N \cdot]}^N$ in $[0, \infty)$ when the limiting values of the sequence of processes are non-explosive.

The pathwise uniqueness of the SDE allows to capture limiting processes where infinity is either absorbing or non-accessible. Other situations are interesting, where infinity is regular and uniqueness in law could be invoked. In particular, we refer to [12] for a criterion for reflection at infinity of CSBP with quadratic competition and [20] and [5] for similar issues.

4.4 Explosive CSBP with interactions and random environment

In this section, the process may be non-conservative, i.e. T_{exp} may be finite. Following [13, 26], we make the following assumption for the regularity of the drift term at infinity to obtain the strong uniqueness.

Assumption A3. *There exist functions r , b_r and b_d such that for any $z \in [0, \infty)$,*

$$e^{-z} \left(zg(z) - \frac{\sigma_E^2}{2} z^2 + \int (1 - e^{-zw} - zh_E(w)) \nu_E(dw) \right) = b_r(z) + b_d(z), \quad (58)$$

with r non-negative, non-decreasing and concave, $\int_0^1 1/r(z) dz = \infty$, $|b_r(-\log(u)) - b_r(-\log(u'))| \leq r(|u - u'|)$ for any $u, u' \in (0, 1]$ and b_d is non-increasing and $b_r(z) + b_d(z) \rightarrow 0$ as $z \rightarrow \infty$.

Theorem 4.4. *We assume that **A1**, **A2** and **A3** hold.*

Then there exists a unique strong solution $(Z, Y) \in \mathbb{D}([0, \infty), [0, \infty] \times \mathbb{R})$ of (22) and (57) and

$$\left(\left(\frac{1}{N} Z_{[v_N t]}^N, S_{[v_N t]}^N \right) : t \in [0, \infty) \right) \Rightarrow ((Z_t, Y_t) : t \in [0, +\infty))$$

in $\mathbb{D}([0, \infty), [0, \infty] \times \mathbb{R})$, where $[0, \infty]$ is endowed with $d(z_1, z_2) = |\exp(-z_1) - \exp(-z_2)|$.

Proof of Theorem 4.4. We first remark that the convergence in law of $(X_{[v_N \cdot]}^N)_N$ in $\mathbb{D}([0, \infty), [0, 1] \times \mathbb{R})$ implies the weak convergence of $(Z_{[v_N \cdot]}^N/N, S_{[v_N \cdot]}^N)$ to $(-\log(X^1), Y)$ in $\mathbb{D}([0, \infty), [0, \infty] \times \mathbb{R})$, where $[0, \infty]$ is endowed with d and $-\log(0) = \infty$.

We recall from the previous section that X^N satisfies **(H1)**, **(H'1)** and **(H2)**. To apply Theorem 2.5 and conclude, it remains to check that X defined in (56) is unique in law.

Let us prove that under **A3**, pathwise uniqueness holds for X in $\mathbb{D}([0, T], [0, 1] \times \mathbb{R})$. First, the second component Y of X is a Lévy process and the pathwise uniqueness is well known. Second, the equation for the first component X^1 writes

$$X_t^1 = X_0^1 + \int_0^t b_1(X_s^1) ds + \int_0^t \sigma(X_s^1) dW_s + \int_0^t \int_V K^1(X_{s-}^1, v) \tilde{N}(ds, dv)$$

where $\sigma(u) = \sqrt{\sigma_{11}(u)^2 + \sigma_{12}(u)^2}$ and W is a Brownian motion independent of X_0^1 and N . We can now apply the pathwise uniqueness result of [25] and conclude. We introduce $V_0 = [0, 1] \times [-1/2, 1] \cup (1, \infty) \times (0, \infty)$ and observe that $\mu(V - V_0) < \infty$. First, combining (52) and (58), we have

$$b_1(x_1) = x_1 \log(x_1) (\alpha_E + \gamma_1^D) + b_r(x_1) + b_d(x_1) = \tilde{b}_r(x_1) + b_d(x_1),$$

where \tilde{b}_r satisfies $|\tilde{b}_r(x_1) - \tilde{b}_r(\tilde{x}_1)| \leq \tilde{r}(|x_1 - \tilde{x}_1|)$ for $x_1, \tilde{x}_1 \in [0, 1]$, with $\int_0^1 1/\tilde{r}(z) dz = \infty$ and \tilde{r} non-decreasing and concave. Indeed using Lemma 6.3 in Appendix, one can take

$\tilde{r}(u) = r(u) + Ku + K_1 r_1(u)$, with $r_1(x_1) = -x_1 \log(x_1)$.

Second, we easily check that σ^2 is Lipschitz and $|\sigma(u) - \sigma(u')|^2 \leq |\sigma(u)^2 - \sigma(u')^2|$ and $y \rightarrow y + K^1(y, v)$ is non-decreasing.

Finally,

$$\begin{aligned} \int_{V_0} (K^1(u, v) - K^1(u', v))^2 \mu(dv) &= g_1(u, u') \int_{\mathbb{R}^+} (e^{-v} - 1)^2 \nu_D(dv) \\ &\quad + \int_{[-1/2, 1]} (g_2(u, w) - g_2(u', w))^2 \nu_E(dw), \end{aligned}$$

where for any $u, u' \in (0, 1]$,

$$g_1(u, u') = \min(-\log(u), -\log(u'))(u - u')^2 + \min(u, u')^2 |\log(u) - \log(u')| \quad (59)$$

(with a nul extension at 0) and

$$g_2(u, w) = u(e^{\log(u)w} - 1). \quad (60)$$

Using now Lemma 6.4 in Appendix and the integrability assumptions on ν_D and ν_E , there exists $L > 0$ such that

$$\int_{V_0} (K^1(u, v) - K^1(u', v))^2 \mu(dv) \leq L|u - u'|.$$

Then we can apply Theorem 3.2 in [25] and conclude by observing that $X_t = 0$ for $t \geq T_{\text{exp}}$ by pathwise uniqueness. \square

Recently, Pardoux and Dramé [10] have proven the convergence of some continuous time and discrete space processes to CSBP with interactions, under the moment condition $\int_0^\infty (z^p \wedge z^2) \nu_D(dz) < +\infty$ for some $1 < p < 2$. Here we relax this moment assumption and deal with explosive processes. We also extend the convergence to random environment and general classes of reproduction laws, in a discrete time setting.

Application to Galton-Watson processes with cooperative effects. As an application of Theorem 4.4, we extend the convergence of a sequence of Galton-Watson processes by taking into account a cooperative effect. In this case, the interactions impede to use the classical generating function tool. We first consider the sequence of random variables $L^N \in \mathbb{N}$ such that

$$\begin{aligned} \lim_{N \rightarrow \infty} v_N N \mathbb{E}(h_D((L^N - 1)/N)) &= \alpha_D; & \lim_{N \rightarrow \infty} v_N N \mathbb{E}(h_D^2((L^N - 1)/N)) &= \beta_D; \\ \lim_{N \rightarrow \infty} v_N N \mathbb{E}(f((L^N - 1)/N)) &= \int_0^\infty f(v) \nu_D(dv), \end{aligned} \quad (61)$$

for any function f continuous bounded and null in a neighborhood of 0, where h_D is a truncation function, $\alpha_D \in \mathbb{R}$, $\int_{(0, \infty)} (1 \wedge v^2) \nu_D(dv) < \infty$, $\beta_D = \sigma_D^2 + \int_{(0, \infty)} h_D^2 \nu_D$ and $\sigma_D \geq 0$.

Note that Theorem 4.4 allows to recover the convergence in law of the Galton-Watson processes $(\widehat{Z}_{[v_N \cdot]}^N)_N$ defined as in (32) with these reproduction laws L^N to a (possibly explosive) CSBP

with characteristics $(\alpha_D, \beta_D, \nu_D)$ and solution of the stochastic differential equation

$$\begin{aligned} \widehat{Z}_t &= Z_0 + \alpha_D \int_0^t \widehat{Z}_s ds + \sigma_D \int_0^t \sqrt{\widehat{Z}_s} dB_s^D + \\ &+ \int_0^t \int_{(0, \infty)^2} \mathbf{1}_{\theta \leq \widehat{Z}_{s-}} h_D(z) \widetilde{N}^D(ds, dz, d\theta) + \int_0^t \int_{(0, \infty)^2} \mathbf{1}_{\theta \leq \widehat{Z}_{s-}} (z - h_D(z)) N^D(dt, dz, d\theta), \end{aligned} \quad (62)$$

where N^D is a Poisson measure with intensity $dt dz \nu_D(d\theta)$.

We add now a density-dependance effect on the reproduction law. The reproduction random variable $L^N(n)$ depends on the total population size n and is defined by

$$L^N(n) = L^N + \mathcal{E}^N(n), \quad (63)$$

where $\mathcal{E}^N(n) \in \{0, 1\}$ a.s. is a Bernoulli reproduction random variable defined for $n \in \mathbb{N}$ by

$$\mathbb{P}(\mathcal{E}^N(n) = 1) = \frac{g(n/N) \wedge v_N}{v_N} \quad (64)$$

and $g \in \mathcal{C}^1([0, \infty), [0, \infty))$. The process Z^N is defined as in (32) with the reproduction law $L^N(n)$.

We obtain the following convergence result.

Proposition 4.5. *We assume that $v_N \rightarrow \infty$ and that (61) and (63) and (64) hold. We also assume that $z \rightarrow \exp(-z)zg(z)$ is non-increasing for z large enough and goes to 0 as $z \rightarrow \infty$. Then $(Z_{[v_N \cdot]}^N / N : t \geq 0)$ converges in $\mathbb{D}([0, \infty), [0, \infty] \times \mathbb{R})$ to the unique strong solution Z of*

$$\begin{aligned} Z_t &= Z_0 + \alpha_D \int_0^t Z_s ds + \int_0^t Z_s g(Z_s) ds + \sigma_D \int_0^t \sqrt{Z_s} dB_s^D \\ &+ \int_0^t \int_{(0, \infty)^2} \mathbf{1}_{\theta \leq Z_{s-}} h_D(z) \widetilde{N}^D(ds, dz, d\theta) + \int_0^t \int_{(0, \infty)^2} \mathbf{1}_{\theta \leq Z_{s-}} (z - h_D(z)) N^D(dt, dz, d\theta) \end{aligned} \quad (65)$$

for $t < T_{exp}$ and $Z_t = +\infty$ for $t \geq T_{exp}$.

The monotonicity assumption on $z \rightarrow \exp(-z)zg(z)$ is chosen for sake of simplicity to obtain the pathwise uniqueness. It captures in particular simple cooperative functions as $g(z) = cz^\alpha$ ($c > 0, \alpha > 0$) or $g(z) = c + b(1 - 1/(1+z))$ ($c \geq 0, b > 0$).

We observe that the limiting process Z may be explosive, due to the heavy tails of the reproduction random variable L^N (i.e. the CSBP part is explosive) or due to cooperative effects (note for instance that $y'_t = y_t g(y_t)$ is explosive if $g(z) = z^\alpha$, $\alpha > 0$).

Proof. Let us introduce

$$C_j^N(z) = v_N \left(\mathbb{E} \left(e^{-j \Sigma^N} \right)^{Nz} \left(\frac{g(z) \wedge v_N}{v_N} e^{-j/N} + \left(1 - \frac{g(z) \wedge v_N}{v_N} \right) \right)^{Nz} - 1 \right). \quad (66)$$

By a Taylor expansion (developed in Appendix 6.2), one can prove that

$$\sup_{z; zN \in \mathbb{N}} e^{-kz} |C_j^N(z) + jz g(z) + \gamma_j^D z| \xrightarrow{N \rightarrow \infty} 0 \quad (67)$$

Assumption **A2** is fulfilled for $z \in \frac{\mathbb{N}}{N}$, which is enough as commented in Remark 4.2, while **A1** is trivial (no random environment). As $g \in \mathcal{C}^1([0, \infty), [0, \infty))$ and $\exp(-z)zg(z)$ is non-increasing for z large enough and goes to 0 as z goes to infinity, there exist b_r and b_d such that

$$e^{-z}zg(z) = b_r(z) + b_d(z),$$

with b_d non-increasing and $b_r(-\log(u))$ Lipschitz and Assumption **A3** is fulfilled. Indeed there exists z_0 such that $z \rightarrow e^{-z}zg(z)$ is non increasing for $z \geq z_0$ and one can take $b_d(z) = e^{-z}zg(z)$ for $z \geq z_0$ and b_d constant for $z \leq z_0$, and $b_r(z) = e^{-z}zg(z) - b_d(z)$.

We conclude using Theorem 4.4. □

4.5 Conservative CSBP with interactions and random environment

We focus on the conservative case. Now $+\infty$ is not accessible and the pathwise uniqueness is obtained without Assumption **A3**.

Theorem 4.6. *We assume that **A1** and **A2** hold and that any solution of (57) is conservative, i.e. $T_{exp} = +\infty$ a.s. Then there exists a unique strong solution $(Z, Y) \in \mathbb{D}([0, \infty), [0, \infty) \times \mathbb{R})$ of (22) and (57) and*

$$\left(\left(\frac{1}{N} Z_{[v_N t]}^N, S_{[v_N t]}^N \right) : t \in [0, \infty) \right) \Rightarrow ((Z_t, Y_t) : t \in [0, \infty))$$

in $\mathbb{D}([0, \infty), [0, \infty) \times \mathbb{R})$.

Theorem 4.6 allows to obtain various scaling limits to diffusions with jumps due either to the environment or to demographic stochasticity. The conditions for tightness and identification are very general. The conservativeness can be obtained by different methods as moment estimates or comparison with a conservative CSBP or conservative CSBP in random environment when the process is competitive or with bounded cooperation.

Proof. First, using that $T_{exp} = +\infty$ a.s., one can check that pathwise uniqueness holds for (57). It can be achieved by using the pathwise uniqueness for Z obtained in [28] before T_{exp} or adapting the proof of Theorem 4.4. We recall from Theorem 4.3 that weak existence also holds for (56) under **A1** and **A2**, so that both strong existence and weak uniqueness hold.

Then **(H3)** is fulfilled and we can apply Theorem 2.5 to X^N and get the weak convergence of $(\exp(-Z_{[v_N \cdot]}^N/N), S_{[v_N \cdot]}^N)$ to X in $\mathbb{D}([0, \infty), [0, 1] \times \mathbb{R})$. Since $T_{exp} = +\infty$, the weak convergence of $(Z_{[v_N \cdot]}^N/N, S_{[v_N \cdot]}^N)$ in $\mathbb{D}([0, \infty), [0, \infty) \times \mathbb{R})$ and the pathwise uniqueness of (Z, Y) follow, which ends up the proof. □

Application to logistic Feller diffusion in a Brownian environment. The next example illustrates the result. We consider a reproduction law which takes into account logistic competition and small fluctuations of the environment.

Corollary 4.7. *Assume that E^N are centered random variables such that $\sqrt{N}E^N$ is bounded with variance σ_E^2 . We define $L^N \in \{0, 1, 2\}$ for N large enough, $n \in \mathbb{N}$ and $e \in (-1, \infty)$ by*

$$\mathbb{P}(L^N(n, e) = 0) = \frac{1}{2}(\sigma_D^2 - e + g_N(n/N)), \quad \mathbb{P}(L^N(n, e) = 2) = \frac{1}{2}(\sigma_D^2 + e - g_N(n/N)), \quad (68)$$

where $\sigma_D \in (0, \sqrt{2})$, $g_N(z) = \alpha_D/N + c(z/N) \wedge (1/\sqrt{N})$ for $z \geq 0$ and $c \geq 0$ and $\alpha_D \in \mathbb{R}$. Then $(Z_t^N/N : t \in [0, \infty))$ converges in law in $\mathbb{D}([0, \infty), \mathbb{R} \times [0, \infty))$ to the unique strong solution Z of

$$Z_t = Z_0 + \alpha_D \int_0^t Z_s ds - c \int_0^t Z_s^2 ds + \sigma_E \int_0^t Z_s dB_s^E + \sigma_D \int_0^t \sqrt{Z_s} dB_s^D,$$

where B^E and B^D are two independent Brownian motions.

Proof. Assumption **A1** holds with $v_N = N$, $\alpha_E = 0$, $\nu_E = 0$ and $\beta_E = \sigma_E^2$. Let us now prove that **A2** holds.

First, from (68), we get

$$\mathbb{E} \left(e^{-\frac{j}{N}(L^N(n,e)-1)} \right) = 1 - \frac{j}{N}(e - g_N(n/N)) + \frac{j^2}{2N^2}\sigma_D^2 + o(1/N^2),$$

where $o(1/N^2)$ is uniform with respect to z and e . Then, for any $z \in \mathbb{N}/N$,

$$\begin{aligned} P_j^N(z, e) &= \mathbb{E} \left(e^{-\frac{j}{N}(L^N(Nz,e)-1)} \right)^{Nz} - 1 = e^{Nz \left(-\frac{j}{N}(e+g_N(z)) + \frac{j^2}{2N^2}\sigma_D^2 + o(1/N^2) \right)} - 1 \\ &= -jz(e - g_N(z)) + \frac{j^2}{2}z^2e^2 + \frac{j^2}{2N}\sigma_D^2 + o(e^{jz}/N) \end{aligned}$$

by considering the cases $z \leq \sqrt{N}$ and $z \geq \sqrt{N}$. We obtain for any $1 \leq j \leq k$ and $\ell \geq 0$,

$$\begin{aligned} e^{-kz} N \mathbb{E}(P_j^N(z, E^N) e^{-\ell E^N}) &= e^{-kz} \left(\left(jz N g_N(z) + \frac{j^2 z}{2} \sigma_D^2 \right) \mathbb{E} \left(e^{-\ell E^N} \right) \right. \\ &\quad \left. - jz N \mathbb{E} \left(E^N e^{-\ell E^N} \right) + \frac{j^2}{2} z^2 N \mathbb{E} \left((E^N)^2 e^{-\ell E^N} \right) \right) + o(1). \end{aligned}$$

Finally, $\sqrt{N}E^N$ is centered, bounded with variance 1, so $\mathbb{E} \left(e^{-\ell E^N} \right) \rightarrow 1$ and $N \mathbb{E}(f(E^N)) \rightarrow \sigma_E^2 f''(0)/2$ for $f \in C_0^{b,2}$ when N tends to infinity. In particular,

$$N \mathbb{E}(E^N e^{-\ell E^N}) \rightarrow -\ell \sigma_E^2, \quad N \mathbb{E}((E^N)^2 e^{-\ell E^N}) \rightarrow \sigma_E^2$$

and writing $g(z) = cz$ and using that $\gamma_j^D = j\alpha_D - \frac{j^2}{2}\sigma_D^2$ and $\gamma_v^E = \sigma_E^2 v^2/2$, we get

$$\sup_{z \in \mathbb{N}/N} e^{-kz} \left| \mathcal{C}_{j,\ell}^N(z) + \gamma_{jz+\ell}^E - \gamma_\ell^E - jz g(z) + \gamma_j^D z \right| \xrightarrow{N \rightarrow \infty} 0.$$

since $\gamma_{jz+\ell}^E - \gamma_\ell^E = \sigma_E^2(zj\ell + z^2j^2/2)$. We recall from Remark 4.1 (iii) that this uniform convergence then holds for $z \geq 0$ and **A2** is satisfied.

Finally, the fact that Z is conservative can easily be obtained by coupling with the Feller diffusion in Brownian environment ($c = 0$), which has been studied in particular in [6]. The result is then an application of Theorem 4.6. \square

5 Perspectives and multidimensional population models

The general results of Section 2 have been applied in the two previous sections to Wright-Fisher processes in a Lévy environment and Galton-Watson processes with interactions in a Lévy environment with jumps larger than -1 . These generalizations of historical population models were our original motivation for this work. The results of Section 2 can actually be applied in other interesting contexts. We mention here some hints in these directions and works in progress.

First, we could consider environments which are non independent and identically distributed or not restricted to $(-1, \infty)$.

This restriction to $(-1, \infty)$ allowed to consider a functional space generated by the functions $\exp(-k \cdot)$ ($k \geq 0$) which are bounded on $(-1, \infty)$. To extend the results to random walks converging to Lévy processes with a jump measure ν on \mathbb{R} such that $\int_{\mathbb{R}} (1 \wedge w^2) \nu_E(dw) < \infty$, one could consider the functional space of compactly supported functions

$$\mathcal{H} = \{(x, w) \rightarrow e^{-kx} f(w) : k \geq 1, f \in C_c^\infty(\mathbb{R})\} \cup \{(x, w) \rightarrow f(w) : f \in C_c^\infty(\mathbb{R}), f(0) = 0\}$$

for studying Wright Fisher in a Lévy environment and

$$\mathcal{H} = \{(u, w) \rightarrow u^k f(w) : k \geq 1, f \in C_c^\infty(\mathbb{R})\} \cup \{(u, w) \rightarrow f(w) : f \in C_c^\infty(\mathbb{R}), f(0) = 0\}$$

for studying branching processes with interactions and random environment. Indeed these sets satisfy **(H1)**. This would require to check that **(H1')** holds.

Such functional spaces could also help to study cases when the environment E_k^N depends on S_k^N and S^N converges to a diffusion with jumps.

Second, as explained in the introduction, we are more generally interested in multitype population models, where the population at generation n is described by a vector

$$Z_n^N = (Z_n^{1,N}, Z_n^{2,N}, \dots, Z_n^{k,N}),$$

where $Z_n^{i,N}$ counts the number of individuals of type i in generation n . The following processes have attracted lots of attention in population dynamics framework :

$$Z_{n+1}^{i,N} = \sum_{j=1}^{F_N(Z_n^N)} L_{i,j,n}^N(Z_n^N).$$

Such processes allow to model competition, prey predators interactions, sex, mutations Some examples have been well studied, as multitype branching processes, controlled branching processes or bisexual Galton-Watson processes, see e.g. respectively [27], [14] and [1].

One way to obtain the scaling limits is to consider the compactified proces

$$X^N = \left(\exp(-Z_n^{1,N}), \exp(-Z_n^{2,N}), \dots, \exp(-Z_n^{k,N}) \right)$$

and to use the functional space

$$\mathcal{H} = \{(u_1, \dots, u_k) \rightarrow u^{i_1} \times u^{i_2} : (i_1, \dots, i_k) \in \mathbb{N}^k \setminus (0, \dots, 0)\}.$$

Indeed \mathcal{H} satisfies Assumption **(H1)** and the exponential transformation combined with this functional space allows to exploit the independence structure of the model as for extended branching processes in Section 4. Some work will then be required to check that Assumption **(H1')** holds. Moreover uniqueness can be delicate. In a work in progress, we consider bisexual Galton-Watson processes and their scaling limits to bisexual CSBPs under general conditions. It is also worth noticing that in the scaling limits, the nonlinearity or the environment can also impact the diffusion or jump terms, and not only the drift as for BPLEI considered in Section 4. One could for instance also prove limits to CSBP with Lévy environment, where the jump measure associated to demographical stochasticity (large jumps coming from the offsprings of one single individual, at a rate proportional to the number of individuals) is impacted by the environment, see [4] for an example.

Finally, we observe that one may want to go beyond the boundeness assumptions on the characteristics \mathcal{G}^N . This seems to be challenging questions but the approach developed here may be extendable. Indeed, we here obtain the boundeness assumptions in Section 4 by a compactification of the state space using the function $z \rightarrow \exp(-z)$, which allows to consider explosive processes. In the case of conservative processes, one could prefer to adapt the results of Section 2 to non bounded characteristics using the localized version of the convergence criterion of semimartingales in [18]. Finally, we mention that these criteria deal with semimartingales in general and if the Markov setting allows to simplify the form of the characteristics \mathcal{G}^N and to reduce the problem to analytical approximations, one may try to deal with non Markovian processes with similar techniques.

6 Appendix

6.1 General construction of a discrete random variable satisfying A2

We first consider the case $\sigma_D = 0$ and we assume $E^N \in (-1 + 1/\sqrt{N}, \infty)$ for simplicity. We also introduce g_N which converges to g and assume

$$e^{-z} z g(z) \xrightarrow{z \rightarrow \infty} 0, \quad \sup_{z \geq 0} e^{-z} z |g_N(z) - g(z)| \xrightarrow{N \rightarrow \infty} 0, \quad \sup_{z \geq 0} \frac{|g_N(z)|}{N^{1/3}} < \infty. \quad (69)$$

One can take for instance $g_N(\cdot) = g(\cdot) \wedge N^{1/3}$. Let us define

$$m_N(n, e) = 1 + g_N(n/N)/N + \alpha_D/N + e$$

and observe that $m_N(n, E^N)$ is a.s. positive for N large enough. We consider the reproduction random variable $A^N(n, e) \in \{[m_N(n, e)], [m_N(n, e)] + 1\}$ defined by $\mathbb{E}(A^N(n, e)) = m_N(n, e)$, i.e.

$$\mathbb{P}(A^N(n, e) = [m_N(n, e)]) = p_N(n, e), \quad \mathbb{P}(A^N(n, e) = [m_N(n, e)] + 1) = 1 - p_N(n, e),$$

with $p_N(n, e) = [m_N(n, e)] + 1 - m_N(n, e)$. For the large reproductions events, we also introduce $\Sigma^N \in \mathbb{N}$ independent of $(A^N(n, e) : n \geq 0, e \in (-1, \infty))$ such that

$$\lim_{N \rightarrow \infty} N^2 \mathbb{E}(h_D(\Sigma^N)) = 0; \quad \lim_{N \rightarrow \infty} N^2 \mathbb{E}(h_D^2(\Sigma^N)) = 0; \quad \lim_{N \rightarrow \infty} N^2 \mathbb{E}(f(\Sigma^N)) = \int_0^\infty f(v) \nu_D(dv)$$

for f continuous bounded and null in a neighborhood of 0. The reproduction random variable L^N is then defined by

$$L^N(n, e) = A^N(n, e) + N\Sigma^N$$

for $n \in \mathbb{N}$ and $e > -1$ and writing $\kappa_j^N = -\log(1 - \mathbb{E}(f_j(\Sigma^N)))$, we have for $z \in \mathbb{N}/N$,

$$\begin{aligned} & \mathbb{E} \left(e^{-\frac{j}{N}(L^N(Nz, e) - 1)} \right) \\ &= \mathbb{E} \left(e^{-j\Sigma^N} \right) \mathbb{E} \left(e^{-\frac{j}{N}(A^N(Nz, e) - 1)} \right) \\ &= e^{-\kappa_j^N - je/N} \left(p(Nz, e) e^{-j([m_N(Nz, e)] - 1 - e)/N} + (1 - p(Nz, e)) e^{-j([m_N(Nz, e)] - e)/N} \right) \\ &= e^{-\kappa_j^N - je/N} \left(1 - \frac{j}{N} (m_N(Nz, e) - e - 1) + \frac{\phi_N(z, e)}{N^2} \right) \end{aligned}$$

where ϕ_N is bounded and by Taylor expansion

$$\begin{aligned} |\phi_N(z, e)| \leq c \cdot & \left(([m_N(Nz, e)] - e - g_N(z)/N - \alpha/N)(1 - 2([m_N(Nz, e)] - e)) \right. \\ & \left. + ([m_N(Nz, e)] - e)^2 \right). \end{aligned} \quad (70)$$

Moreover $m_N(Nz, e) - e - 1 = \mathcal{O}(N^{-2/3})$ uniformly for z, e and we obtain

$$\begin{aligned} P_j^N(z, e) &= e^{-zN\kappa_j^N - jez} \left(1 - \frac{j}{N} (m_N(Nz, e) - e - 1) + \frac{\phi_N(z, e)}{N^2} \right)^{Nz} - 1 \\ &= e^{-zN\kappa_j^N - jez - jz(m_N(Nz, e) - e - 1) + z\psi_N(z, e)} - 1 \\ &= e^{-zN\kappa_j^N - zj(g_N(z) + \alpha_D)/N} \cdot e^{-jze - z\psi_N(z, e)} - 1, \end{aligned}$$

where $N\psi_N(z, e)$ is continuous bounded and $N|\psi_N(z, e)| \leq c(1/N^{4/3} + |\phi_N(z, e)|)$ and c is here a constant which may change from line to line. Thus

$$\mathbb{E}(P_j^N(z, E^N) e^{-\ell E^N}) = A_1^N(z) + A_2^N(z) + A_1^N(z)A_2^N(z) + A_3^N,$$

where

$$A_1^N(z) = e^{-zN\kappa_j^N - zj(g_N(z) + \alpha_D)/N} - 1, \quad A_2^N(z) = \mathbb{E} \left(e^{-(jz + \ell)E^N - z\psi_N(z, E^N)} \right) - 1$$

and $A_3^N = \mathbb{E}(f_\ell(E^N))$. Assumption **A1** ensures that $v_N A_3^N$ converges to γ_ℓ^E when N tends to infinity (see (39) for details). To conclude and prove (44), we prove and combine the asymptotic results stated below.

Lemma 6.1. *For any $j \geq 1$,*

- (i) $\sup_{z \geq 0} e^{-jz} |NA_1^N(z) + z(\gamma_j^D + \alpha_D + g(z))| \xrightarrow{N \rightarrow \infty} 0$
- (ii) $\sup_{z \geq 0} e^{-jz} |NA_2^N(z) + \gamma_{jz + \ell}^E| \xrightarrow{N \rightarrow \infty} 0$
- (iii) $\sup_{z \geq 0} e^{-jz} |NA_1^N(z)A_2^N(z)| \xrightarrow{N \rightarrow \infty} 0$

Proof. (i) First, by Taylor expansion and using that $g_N(z)/N^{1/3}$ is bounded, there exists $c > 0$ such that for any $z \leq N^{2/3}$,

$$\begin{aligned} e^{-jz} & \left| N e^{-zN\kappa_j^N - zj(g_N(z) + \alpha_D)/N} + z(\gamma_j^D + \alpha_D + g(z)) \right| \\ & \leq c.e^{-jz} z \left(|N^2\kappa_j^N - \gamma_j^D| + |g_N(z) - g(z)| + N^{-1/3} \right). \end{aligned}$$

The right hand side goes to 0 uniformly as $N \rightarrow \infty$. Second,

$$e^{-jz} z(\gamma_j^D + \alpha_D + g(z)) \xrightarrow{z \rightarrow \infty} 0, \quad \sup_{z \geq N^{2/3}, N \geq 1} e^{-jz} \left| N e^{-zN\kappa_j^N - zj(g_N(z) + \alpha_D)/N} - 1 \right| \xrightarrow{A \rightarrow \infty} 0,$$

since for $z \geq N^{2/3}$,

$$e^{-jz} \left| N e^{-zN\kappa_j^N - zj(g_N(z) + \alpha_D)/N} - 1 \right| \leq e^{-jz} N (e^{zc/N^{2/3}} + 1) \leq N e^{-N^{1/3} \cdot (1-c/N^{2/3})} + N e^{-N^{1/3}},$$

which goes to 0. This proves (i).

Let us turn to (ii). We first prove the uniform convergence on compact sets using convexity and simple convergence. Indeed, recalling that $|N\psi_N(z, e)| \leq c(1/N^{4/3} + |\phi_N(z, e)|)$,

$$\left| A_2^N(z) + \mathbb{E}(f_{jz+\ell}(E^N)) \right| \leq \frac{c}{N} \left(1/N^{4/3} + \mathbb{E}(|\phi_N(z, E^N)|) \right)$$

for $z \in [0, A]$. Using now that E^N goes in probability to 0 from **A1** and ϕ_N bounded and $\phi_N(z, e) \rightarrow 0$ as $e \rightarrow 0$ uniformly with respect to $z \in [0, A]$ from (70), we obtain that $\sup_{z \in [0, A]} \mathbb{E}(|\phi_N(z, E^N)|) \rightarrow 0$ and

$$\sup_{z \in [0, A]} \left| N A_2^N(z) + N \mathbb{E}(f_{jz+\ell}(E^N)) \right| \xrightarrow{N \rightarrow \infty} 0.$$

Moreover, for any $z \geq 0$, $N \mathbb{E}(f_{jz+\ell}(E^N)) \rightarrow \gamma_{jz+\ell}^E$ by Assumption **A1** (see again (39) for details) and the convergence is uniform on $[0, A]$ by convexity of $z \rightarrow N \mathbb{E}(f_{jz+\ell}(E^N))$ and continuity of $z \rightarrow \gamma_{jz+\ell}^E$ (third Dini's theorem). It proves (ii) on compact sets and we now prove that $\sup_{z \geq A, N \geq 1} \exp(-jz) |N A_2^N(z)| \rightarrow 0$ as $A \rightarrow \infty$ to end the proof. Indeed, fix $\varepsilon > 0$ and

$$A_2^N(z) = B_\varepsilon^N(z) + C_\varepsilon^N(z) \tag{71}$$

where

$$B_\varepsilon^N(z) = \mathbb{E} \left(\mathbf{1}_{E^N \geq -1+\varepsilon} e^{-(jz+\ell)E^N - z\psi_N(z, E^N)} \right) - 1.$$

Recalling that $E^N \geq -1 + 1/\sqrt{N}$ and $N\psi_N$ bounded, we have

$$C_\varepsilon^N(z) = \mathbb{E} \left(\mathbf{1}_{E^N < -1+\varepsilon} e^{-(jz+\ell)E^N - z\psi_N(z, E^N)} \right) \leq \mathbb{P}(E^N < -1 + \varepsilon) e^{-(jz+\ell)(-1-1/\sqrt{N}+c/N)}.$$

Thus, the last part of Assumption **A1** ensures that

$$\lim_{\varepsilon \rightarrow 0} \sup_{N \geq c^2, z \geq 0} e^{-jz} N C_\varepsilon^N(z) = \lim_{\varepsilon \rightarrow 0} \sup_{N \geq 1} N \mathbb{P}(E^N < -1 + \varepsilon) = \lim_{\varepsilon \rightarrow 0} \nu_E(-1, -1 + \varepsilon) = 0$$

and we now deal with $B_\varepsilon^N(z)$. Writing

$$g_x(y) = f_x(y) - xh_E(y) = 1 - e^{-xy} - xh_E(y), \quad R_N(z, e) = e^{-(jz+\ell)e} \left(e^{-z\psi_N(z, e)} - 1 \right),$$

we have

$$B_\varepsilon^N(z) = \mathbb{P}(E^N < -1 + \varepsilon) - (jz + \ell)\mathbb{E}(h_E(E^N)1_{E^N \geq -1 + \varepsilon}) - \mathbb{E}(g_{jz+\ell}(E^N)1_{E^N \geq -1 + \varepsilon}) \\ + \mathbb{E}(R_N(z, E^N)1_{E^N \geq -1 + \varepsilon}).$$

First, we recall that $\sup_N \mathbb{P}(E^N < -1 + \varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$ and $N\mathbb{E}(h_E(E^N)1_{E^N \geq -1 + \varepsilon})$ is bounded (actually convergent from Assumption **A1**). Second, we prove that

$$\sup_{z \geq A, N \geq 1} N e^{-jz} \mathbb{E}(|g_{jz+\ell}(E^N)|1_{E^N \geq -1 + \varepsilon}) \xrightarrow{A \rightarrow \infty} 0$$

using that (see forthcoming Lemma 6.2 for details)

$$\sup_{y \geq -1 + \varepsilon, y \neq 0} \frac{e^{-jz}}{(1 - e^{-y})^2} |g_x(y)| \xrightarrow{x \rightarrow \infty} 0$$

and that $N\mathbb{E}((1 - \exp(-E^N))^2)$ is bounded from **A1**. Finally

$$e^{-jz} N |R_N(z, e)1_{e \geq -1 + \varepsilon}| \leq N \exp(-(\varepsilon - 1/\sqrt{N})jz) \cdot |\exp(cz/N) - 1|$$

ensures that

$$\sup_{z \geq A, N} N \mathbb{E}(R_N(z, E^N)1_{E^N \geq -1 + \varepsilon}) \xrightarrow{A \rightarrow \infty} 0,$$

using again $|\exp(cz/N) - 1| \leq c'z/N$ for $z \leq N$, while the right hand is bounded by $z \exp(-\varepsilon jz/2)$ for $z \geq N$ and $N \geq 16/\varepsilon^2$. Combing these estimates in (71) yields

$\sup_{z \geq A, N \geq 1} \exp(-jz) |NA_2^N(z)| \rightarrow 0$ as $A \rightarrow \infty$ and ends the proof of (ii) by recalling that $\exp(-jz)\gamma_{jz+\ell}^E \rightarrow 0$ as $z \rightarrow \infty$.

We finally prove (iii). First,

$$\sup_{z \geq 0} e^{-jz/2\sqrt{N}} |A_1^N(z)| \xrightarrow{A \rightarrow \infty} 0$$

using that $\exp(-jz/2\sqrt{N})|A_1^N(z)| \leq c \exp(-jz/2\sqrt{N})z/N^{2/3}$ for $z \leq N$ (and then one may use for N large that $z/N^{2/3}$ is small for $z \leq N^{7/12}$ and that $\exp(-z/2\sqrt{N})$ is small for $N^{7/12} \leq z \leq N$) and $\exp(-jz/2\sqrt{N})|A_1^N(z)| \leq \exp(-jz(1/2\sqrt{N} - c/N^{2/3}))$ for $z \geq N$. Second

$$\sup_{N \geq 1, z \in [0, \infty)} e^{-jz(1-1/2\sqrt{N})} N \mathbb{E}(A_2^N) < \infty,$$

since

$$\mathbb{E}(A_2^N) \leq \mathbb{P}(E^N \leq -1/2) e^{(jz+\ell)(1-1/\sqrt{N})+zc/N} + \mathbb{E}(A_2^N 1_{E^N > 1/2})$$

and $N\mathbb{P}(E^N \leq -1/2)$ is bounded from **A1** and $N\mathbb{E}(A_2^N 1_{E^N > 1/2})$ is bounded (following the point (ii) and using the following slight modification of Lemma 6.2

$$\sup_{y > -1/2, y \neq 0, N \geq N_0} \frac{e^{-x(1-2/\sqrt{N})}}{(1 - e^{-y})^2} |g_x(y)| \xrightarrow{x \rightarrow \infty} 0,$$

where N_0 is chosen such that $1 - 2/\sqrt{N_0} > 1/2$. □

6.2 Taylor expansion for Galton-Watson process with cooperation

Recalling (66) and (44),

$$\begin{aligned} C_j^N(z) &= v_N \left(\left(1 - \gamma_j^{N,D}/Nv_N\right)^{Nz} \left(1 - \frac{j}{N} \frac{g(z) \wedge v_N}{v_N} + \mathcal{O}\left(\frac{g(z) \wedge v_N}{N^2 v_N}\right)\right)^{Nz} - 1 \right) \\ &= v_N \left(e^{-z\gamma_j^{N,D}/v_N + \mathcal{O}(z/Nv_N^2) - jz(1 \wedge (g(z)/v_N))(1 + \mathcal{O}(1/N))} - 1 \right) \end{aligned}$$

for any $z \in \mathbb{N}/N$. For z such that $z + g(z)z \leq v_N$, we have $z/v_N \leq 1$ and $1 \wedge (g(z)/v_N) = g(z)/v_N$ for $z \geq 1$. We make a Taylor expansion and get

$$e^{-jz} |C_j^N(z) + z\gamma_j^{N,D} + jzg(z)| \leq e^{-jz} c \left(\frac{1}{N} + \frac{zg(z)}{N} \right)$$

for some constant $c > 0$. We obtain

$$\sup_{z+g(z)z \leq v_N} e^{-jz} |C_j^N(z) + z\gamma_j^{N,D} + jzg(z)| \xrightarrow{N \rightarrow \infty} 0.$$

To conclude, we observe that $\min\{z : z + g(z)z \geq v_N\} \rightarrow \infty$ as $N \rightarrow \infty$. Then $\sup_{z+g(z)z \geq v_N} e^{-jz} |z\gamma_j^{N,D} + jzg(z)| \rightarrow \infty$ and we prove now that

$$\sup_{z+g(z)z \geq v_N} e^{-jz} v_N |C_j^N(z)| \xrightarrow{N \rightarrow \infty} 0.$$

Indeed, $g \geq 0$ and either $z \geq v_N/2$ and

$$e^{-jz} v_N |C_j^N(z)| \leq e^{-jz} v_N e^{zc/v_N} \leq 2ze^{-zj/2}$$

for N such that $j - c/v_N \geq j/2$ or $z \leq v_N/2$ and $v_N \leq 2zg(z)$ and there exists $c > 0$ such that

$$e^{-jz} v_N |C_j^N(z)| \leq e^{-jz} 2zg(z) e^c.$$

Recalling that $\min\{z; z + g(z)z \geq v_N\} \rightarrow \infty$ as $N \rightarrow \infty$ and $zg(z) \exp(-z) \rightarrow 0$ as $z \rightarrow \infty$ ends the proof.

6.3 Some technical results

Lemma 6.2. *Let*

$$g_x(y) = 1 - e^{-xy} - xh_E(y),$$

then

$$\sup_{y > -1+\varepsilon, y \neq 0} \frac{e^{-x}}{(1 - e^{-y})^2} |g_x(y)| \xrightarrow{x \rightarrow \infty} 0.$$

Proof. Let \mathcal{V}_0 be an open finite interval containing 0 such that $h_E(y) = y$ for $y \in \mathcal{V}_0$. There exists $C > 0$ such that for any $y \notin \mathcal{V}_0$,

$$\frac{|g_x(y)|}{(1 - e^{-y})^2} \leq C(1 + x + e^{x(1-\varepsilon)})$$

since h_E and $1/(1 - \exp(-y))$ are bounded. The result follows on the complementary set of \mathcal{V}_0 . Let us now consider $y \in \mathcal{V}_0$. Assuming first $|xy| \leq 1$, we get $|g_x(y)| \leq C x^2 y^2$ and we conclude using that $y/(1 - \exp(-y))$ is bounded on $(-1, \infty)$.

Finally, if $y \in \mathcal{V}_0$ and $|xy| \geq 1$, we have

$$\frac{|g_x(y)|}{(1 - e^{-y})^2} \leq C \left(\frac{|1 - e^{-xy}|}{y^2} + \frac{x}{y} \right) \leq C x^2 \left(1 + e^{x(1-\varepsilon)} \right),$$

which ends the proof. \square

let us now prove the inequality (72).

Lemma 6.3. *Let $r_1(x) = -x \log(x)$. Then for any $x, x' \in [0, 1]$,*

$$|x \log(x) - x' \log(x')| \leq K(|x - x'| + r_1(|x - x'|)) \quad (72)$$

for some constant $K > 0$.

Proof. Let us first assume that $\min(x, x') \geq |x - x'|$. In this case, it is immediate that

$$|x \log x - x' \log x'| \leq |x - x'| (1 + \log(|x - x'|))$$

by the mean value theorem. We now assume that $0 \leq x \leq |x - x'| \leq x'$, which implies that $x' \leq 2|x - x'|$. We have

$$\begin{aligned} |x \log x - x' \log x'| &\leq |x \log(x/x') + (x - x') \log(x')| \\ &\leq |x \log(x/x')| + |\log(|x - x'|)| |x - x'| \\ &\leq C|x'| + |\log(|x - x'|)| |x - x'|, \end{aligned}$$

using that $x/x' \in [0, 1]$ and that the function $\alpha \in [0, 1] \rightarrow \alpha \log \alpha$ is bounded by some constant C . We obtain that $|x \log x - x' \log x'| \leq 2C|x - x'| + |\log(|x - x'|)| |x - x'|$, which ends the proof. \square

Lemma 6.4. *With notation (59) and (60), for any $x_1, \tilde{x}_1 \in [0, 1]$,*

$$g_1(x_1, \tilde{x}_1) \leq L|x_1 - \tilde{x}_1|$$

and for any $u \in [-1/2, 1]$,

$$(g_2(x_1, u) - g_2(\tilde{x}_1, u))^2 \leq C|x_1 - \tilde{x}_1|u^2.$$

Proof. For the first inequality, one can use that $-x \log x$ is bounded for the first term in (59) and the mean value theorem for the second one.

For the second inequality, we use

$$(g_2(x_1, u) - g_2(\tilde{x}_1, u))^2 \leq |g_2(x_1, u)^2 - g_2(\tilde{x}_1, u)^2| \leq \sup |(g_2^2(\cdot, u))'| |Z_s|$$

and

$$(g_2^2(\cdot, u))'(x_1) = 2x_1(e^{\log(x_1)u} - 1)^2 + ux_1 e^{\log(x_1)u} 2(e^{\log(x_1)u} - 1).$$

The results then come from the inequality $|e^{\log(x_1)u} - 1| \leq |\log(x_1)|u$. \square

6.4 Stone-Weierstrass Theorem on locally compact space

We recall here the local version of Stone-Weierstrass Theorem and assume that the space X is a locally compact Hausdorff space.

Let $C_0(X, \mathbb{R})$ the space of real-valued continuous functions on X which vanish at infinity, i.e. given $\varepsilon > 0$, there is a compact subset K such that $\|f(x)\| < \varepsilon$ whenever the point x lies outside K . In other words, the set $\{x, \|f(x)\| \geq \varepsilon\}$ is compact). Let us consider a subalgebra A of $C_0(X, \mathbb{R})$.

Then A is dense in $C_0(X, \mathbb{R})$ for the topology of uniform convergence if and only if it separates points and vanishes nowhere.

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