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WHEN DOES A PERTURBED MOSER-TRUDINGER INEQUALITY ADMIT AN EXTREMAL?

PIERRE-DAMIEN THIZY

ABSTRACT. In this paper, we are interested in several questions raised mainly in [17] (see also [18, 20]). We consider the perturbed Moser-Trudinger inequality $I_\alpha^g(\Omega)$ below, at the critical level $\alpha = 4\pi$, where g , satisfying $g(t) \rightarrow 0$ as $t \rightarrow +\infty$, can be seen as a perturbation with respect to the original case $g \equiv 0$. Under some additional assumptions, ensuring basically that g does not oscillates too fast as $t \rightarrow +\infty$, we identify a new condition on g for this inequality to have an extremal. This condition covers the case $g \equiv 0$ studied in [3, 12, 23]. We prove also that this condition is sharp in the sense that, if it is not satisfied, $I_{4\pi}^g(\Omega)$ may have no extremal.

1. INTRODUCTION

Let Ω be a smooth, bounded domain of \mathbb{R}^2 and let $H_0^1 = H_0^1(\Omega)$ be the standard Sobolev space, obtained as the completion of the set of smooth functions with compact support in Ω , with respect to the norm $\|\cdot\|_{H_0^1}$ given by

$$\|u\|_{H_0^1}^2 = \int_{\Omega} |\nabla u(x)|^2 dx.$$

Throughout the paper, Ω is assumed to be connected. Let g be such that

$$g \in C^1(\mathbb{R}), \quad \lim_{s \rightarrow +\infty} g(s) = 0, \quad g(t) > -1 \text{ and } g(t) = g(-t) \text{ for all } t, \quad (1.1)$$

(see also Remark 1.2). Then, we have that

$$C_{g,\alpha}(\Omega) := \sup_{u \in H_0^1: \|u\|_{H_0^1}^2 \leq \alpha} \int_{\Omega} (1 + g(u)) \exp(u^2) dx \quad (I_\alpha^g(\Omega))$$

is finite for $0 < \alpha \leq 4\pi$ and equals $+\infty$ for $\alpha > 4\pi$. This result was first obtained by Moser [19] in the unperturbed case $g \equiv 0$. Still by [19], we easily extend the $g \equiv 0$ case to the case of g as in (1.1). At last, [19] gives also the existence of an extremal for $(I_\alpha^g(\Omega))$ if $0 < \alpha < 4\pi$ (see Lemma 3.1). If now $\alpha = 4\pi$, getting the existence of an extremal is more challenging; however Carleson-Chang [3], Struwe [23] and Flucher [12] were also able to prove that $(I_{4\pi}^0(\Omega))$ admits an extremal in the unperturbed case $g \equiv 0$. Yet, surprisingly, McLeod and Peletier [18] conjectured that there should exist a g as in (1.1) such that $(I_{4\pi}^g(\Omega))$ does not admit any extremal function. Through a nice but very implicit procedure, Pruss [20] was able to prove that such a g does exist. Observe that, since $g(u) \rightarrow 0$ as $u \rightarrow +\infty$ in (1.1), $(1 + g(u)) \exp(u^2)$ in $(I_\alpha^g(\Omega))$ sounds like a very mild perturbation of $\exp(u^2)$ as $u \rightarrow +\infty$ and then, this naturally raises the following question:

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Question 1. *To what extent does the existence of an extremal for the critical Moser-Trudinger inequality ($I_{4\pi}^0(\Omega)$) really depend on asymptotic properties of the function $t \mapsto \exp(t^2)$ as $t \rightarrow +\infty$?*

To investigate this question, we may rephrase it as follows: for what g satisfying (1.1) does ($I_{4\pi}^g(\Omega)$) admit an extremal? This is Open problem 2 in Mancini and Martinazzi [17], stated in this paper for $\Omega = \mathbb{D}^2$, the unit disk of \mathbb{R}^2 . In order to state our main general result, we introduce now some notations. For a first reading, one can go directly to Corollary 1.1, which aims to give a less general but more readable statement. We let $H : (0, +\infty) \rightarrow \mathbb{R}$ be given by

$$H(t) = 1 + g(t) + \frac{g'(t)}{2t}, \quad (1.2)$$

so that we have

$$[(1 + g(t)) \exp(t^2)]' = 2tH(t) \exp(t^2). \quad (1.3)$$

We set $tH(t) = 0$ for $t = 0$, so that $t \mapsto tH(t)$ is continuous at 0 by (1.1). This function H comes into play, since the Euler-Lagrange associated to ($I_\alpha^g(\Omega)$) reads as

$$\begin{cases} \Delta u = \lambda u H(u) \exp(u^2) \text{ in } \Omega, \\ u = 0 \text{ in } \partial\Omega, \end{cases} \quad (1.4)$$

where $\lambda \in \mathbb{R}$ is a Lagrange multiplier and $\Delta = -\partial_{xx} - \partial_{yy}$ (see also Lemma 3.1 below). Now, we make some further assumptions on the behavior of g at $+\infty$ and at 0. First, we assume that there exist $\delta_0 \in (0, 1)$ and a sequence of real numbers $A = (A(\gamma))_\gamma$ such that

$$\begin{aligned} \text{a) } & H\left(\gamma - \frac{t}{\gamma}\right) = H(\gamma) (1 + A(\gamma)t + o(|A(\gamma)| + \gamma^{-4})) \\ & \text{in } C_{loc}^0(\mathbb{R}_t), \text{ as } \gamma \rightarrow +\infty, \\ \text{b) } & \exists C > 0, \left| H\left(\gamma - \frac{t}{\gamma}\right) - H(\gamma) \right| \leq C|H(\gamma)|(|A(\gamma)| + \gamma^{-4})\exp(\delta_0 t) \\ & \text{for all } \gamma \gg 1 \text{ and all } 0 \leq t \leq \gamma^2, \\ \text{c) } & \lim_{\gamma \rightarrow +\infty} A(\gamma) = 0. \end{aligned} \quad (1.5)$$

In a) of (1.5) and of (1.6), γ is a parameter and the $C_{loc}^0([0, +\infty))$ convergence is in the t variable. We also assume that there exist $\delta'_0 \in (0, 1)$, $\kappa \geq 0$, $\tilde{\varepsilon}_0 \in \{-1, +1\}$, F given by $F(t) := \tilde{\varepsilon}_0 t^\kappa$, and a sequence $B = (B(\gamma))_\gamma$ of positive real numbers such that

$$\begin{aligned} \text{a) } & \frac{t}{\gamma} H\left(\frac{t}{\gamma}\right) = B(\gamma)F(t) + o(|B(\gamma)| + \gamma^{-1}) \\ & \text{in } C_{loc}^0((0, +\infty)_t), \text{ as } \gamma \rightarrow +\infty, \\ \text{b) } & \exists C > 0, \left| \frac{t}{\gamma} H\left(\frac{t}{\gamma}\right) \right| \leq C(|B(\gamma)| + \gamma^{-1})\exp(\delta'_0 t) \\ & \text{for all } \gamma \gg 1 \text{ and all } 0 \leq t \leq \gamma^2. \end{aligned} \quad (1.6)$$

Observe that we may have $B(\gamma) = o(\gamma^{-1})$ as $\gamma \rightarrow +\infty$, in which case the precise formula for F is not really significant. Since $t \mapsto (1 + g(t)) \exp(t^2)$ is an even C^1

function, we have that

$$\lim_{\gamma \rightarrow +\infty} B(\gamma) = 0, \quad (1.7)$$

in view of (1.3) and (1.6). Following rather standard notations, we may split the Green's function G of Δ , with zero Dirichlet boundary conditions in Ω , according to

$$G_x(y) = \frac{1}{4\pi} \left(\log \frac{1}{|x-y|^2} + \mathcal{H}_x(y) \right) \quad (1.8)$$

for all $x \neq y$ in Ω , where \mathcal{H}_x is harmonic in Ω and coincides with $-\log \frac{1}{|x-\cdot|^2}$ in $\partial\Omega$. Then the Robin function $x \mapsto \mathcal{H}_x(x)$ is smooth in Ω , and goes to $-\infty$ as $x \rightarrow \partial\Omega$, so that we may set

$$\begin{aligned} M &= \max_{x \in \Omega} \mathcal{H}_x(x), \\ K_\Omega &= \{y \in \Omega \text{ s.t. } \mathcal{H}_y(y) = M\} \text{ and} \\ S &= \max_{z \in K_\Omega} \int_\Omega G_z(y) F(4\pi G_z(y)) dy, \end{aligned} \quad (1.9)$$

where F is as in (1.6). For $N \geq 1$, we let g_N be given by

$$(1 + g_N(t)) \exp(t^2) = (1 + g(t))(1 + t^2) + (1 + g(t)) \left(\sum_{k=N+1}^{+\infty} \frac{t^{2k}}{k!} \right), \quad (1.10)$$

so that $g_N \leq g$, $g_N(0) = g(0)$ for all $N \geq 1$, while $g = g_N$ for $N = 1$. We also set

$$\Lambda_g(\Omega) := \max_{u \in H_0^1 : \|u\|_{H_0^1}^2 \leq 4\pi} \int_\Omega ((1 + g(u))(1 + u^2) - (1 + g(0))) dx. \quad (1.11)$$

We are now in position to state our main result, giving a new, very general and basically sharp picture about the existence of an extremal for the perturbed Moser-Trudinger inequality $(I_{4\pi}^g(\Omega))$.

Theorem 1.1 (Existence and non-existence of an extremal). *Let Ω be a smooth bounded domain of \mathbb{R}^2 . Let g be such that (1.1) and (1.5)-(1.6) hold true for H as in (1.2), and let A , B and F be thus given. Assume that*

$$l = \lim_{\gamma \rightarrow +\infty} \frac{\gamma^{-4} + A(\gamma)/2 + 4\gamma^{-3} \exp(-1 - M)B(\gamma)S}{\gamma^{-4} + |A(\gamma)| + \gamma^{-3}|B(\gamma)|} \quad (1.12)$$

exists, where M and S are given by (1.9). Then

- (1) *if $\boxed{l > 0 \text{ or } \Lambda_g(\Omega) \geq \pi \exp(1 + M)}$, $(I_{4\pi}^g(\Omega))$ admits an extremal, where $\Lambda_g(\Omega)$ is as in (1.11);*
- (2) *if $\boxed{l < 0 \text{ and } \Lambda_g(\Omega) < \pi \exp(1 + M)}$, there exists $N_0 \geq 1$ such that $(I_{4\pi}^{g_N}(\Omega))$ admits no extremal for all $N \geq N_0$, where g_N is given by (1.10).*

Observe that, for all given $N \geq 1$, g_N satisfies (1.1) and (1.5)-(1.6), with the same A , B and F as the original g , in view of $H(\gamma) \rightarrow 1$ as $\gamma \rightarrow +\infty$, see (3.3). Moreover it is clear that $\Lambda_{g_N}(\Omega) \leq \Lambda_g(\Omega)$. Then, this second assertion in Theorem 1.1 proves that the assumptions on g in the first assertion are basically sharp to get the existence of an extremal for $(I_{4\pi}^g(\Omega))$. As a remark, Pruss concludes in [20] that the existence of an extremal for the critical Moser-Trudinger inequality is in some sense accidental and relies on non-asymptotic properties of $\exp(u^2)$. Theorem 1.1 clarifies this tricky situation: the existence or nonexistence of an

extremal for $(I_{4\pi}^g(\Omega))$ may really depend on a balance of the asymptotic properties of g both at infinity (given by $A(\gamma)$) and at zero (given by $B(\gamma)$). Yet, it may also depend on the non-asymptotic quantity $\Lambda_g(\Omega)$ (see Corollary 1.2). Observe that $\Lambda_0(\Omega) = (4\pi)/\lambda_1(\Omega)$ in the unperturbed case $g \equiv 0$, where $\lambda_1(\Omega)$ is the first Dirichlet eigenvalue of Δ in Ω .

From now on, we illustrate Theorem 1.1 by two corollaries dealing with less general but more explicit situations. Let $c, c' \in \mathbb{R}$, $(a, b), (a', b') \in \mathcal{E}$, where

$$\mathcal{E} = \{(a, b) \in [0, +\infty) \times \mathbb{R} \mid b > 0 \text{ if } a = 0\}. \quad (1.13)$$

Let $R' > 0$ be a large positive constant. If one picks g such that

$$g(t) = \begin{cases} g_0(t) := g(0) + ct^{a+1} [\log \frac{1}{t}]^{-b} & \text{in } (0, 1/R'], \\ g_\infty(t) := c't^{-a'} [\log t]^{-b'} & \text{in } [R', +\infty), \end{cases} \quad (1.14)$$

l in (1.12) of Theorem 1.1 can be made more explicit. Indeed, we can then set

$$\begin{aligned} B(\gamma) &= \frac{1+g(0)}{\gamma} + \frac{c(a+1)}{2\gamma^a (\log \gamma)^b} \text{ and } F(t) = \begin{cases} t^{\min(a,1)} & \text{if } c \neq 0, \\ t & \text{otherwise,} \end{cases} \\ A(\gamma) &= c' \times \begin{cases} a'\gamma^{-(a'+2)} (\log \gamma)^{-b'} & \text{if } a' > 0, \\ b'\gamma^{-2} (\log \gamma)^{-(b'+1)} & \text{if } a' = 0, \end{cases} \end{aligned} \quad (1.15)$$

(see also Lemma 3.2). Theorem 1.1 is even more explicit in the particular case $\Omega = \mathbb{D}^2$. Indeed, in this case we have that $K_{\mathbb{D}^2} = \{0\}$ in (1.9) and $G_0(x) = \frac{1}{2\pi} \log \frac{1}{|x|}$. Still on the unit disk \mathbb{D}^2 , it is known that

$$\Lambda_0(\mathbb{D}^2) = \frac{4\pi}{\lambda_1(\mathbb{D}^2)} < \pi e, \quad (1.16)$$

($\lambda_1(\mathbb{D}^2) \simeq 5.78$). Property (1.16) shows in particular that the second assertion $\Lambda_0(\mathbb{D}^2) \geq \pi e$ of Theorem 1.1, Part (1), is not satisfied. In some sense, this is an additional motivation for the nice approach of [3], proving the existence of an extremal for $(I_{4\pi}^0(\mathbb{D}^2))$ via asymptotic analysis. As an illustration and a very particular case of Theorem 1.1, we get the following corollary.

Corollary 1.1 (Case $\Omega = \mathbb{D}^2$). *Assume that $\Omega = \mathbb{D}^2$. Let $c' \neq 0$ and $(a', b') \in \mathcal{E}$ be given, where \mathcal{E} is as in (1.13). Let g_∞ be as in (1.14).*

- (1) *If we assume $\boxed{a' > 2 \text{ or } c' > 0}$, then for all even function $g \in C^1(\mathbb{R})$ such that $g > -1$, such that*

$$(g - g(0))^{(i)}(t) = o(t^{2-i}) \quad (1.17)$$

as $t \rightarrow 0$ and such that

$$g^{(i)}(t) = g_\infty^{(i)}(t)(1 + o(1)) \quad (1.18)$$

as $t \rightarrow +\infty$ for all $i \in \{0, 1\}$, $(I_{4\pi}^g(\mathbb{D}^2))$ admits an extremal.

- (2) *If we assume $\boxed{a' < 2 \text{ and } c' < 0}$, there exists an even function $g \in C^1(\mathbb{R})$ such that $g > -1$, (1.17) and (1.18) hold true, while $(I_{4\pi}^g(\mathbb{D}^2))$ admits no extremal.*

Our main concern in Corollary 1.1 is to write a readable statement. In this result, the existence of an extremal in the unperturbed case $g \equiv 0$ is recovered for quickly decaying g 's, namely if $a' > 2$ (see [17]). But a threshold phenomenon appears

(only if $c' < 0$) and there are no more extremal for less decaying g 's, namely for $a' < 2$. Note that Theorem 1.1 also allows to point out the existence of a threshold $c' < 0$ in the border case $a' = 2, b' = 0$ (See Remark 1.1). Indeed, proving Corollary 1.1 basically reduces to give an explicit formula for l in (1.12), which only depends on Ω and on the asymptotics of g at $+\infty$ and at 0. On the contrary, we do not care about the precise asymptotics of g in the following corollary, thus illustrating the role of $\Lambda_g(\Omega)$ in Theorem 1.1.

Corollary 1.2 (Extremal for $\Lambda_g(\Omega)$ large). *Let Ω be a smooth bounded domain of \mathbb{R}^2 . Let $\lambda_1(\Omega) > 0$ be the first Dirichlet eigenvalue of Δ in Ω and M be given as in (1.9). Let \bar{A} be such that $4(1 + \bar{A}) > \lambda_1(\Omega) \exp(1 + M)$ and let $C > \bar{A}$ be given. Then there exists $R \gg 1$ such that $(I_{4\pi}^g(\Omega))$ admits an extremal for all g satisfying (1.1) and*

$$g(0) = \bar{A}, \quad g \geq g(0) \text{ in } [1/R, R] \quad \text{and} \quad |g| \leq C \text{ in } \mathbb{R}. \quad (1.19)$$

We give now an overview of the proof of Theorem 1.1, since it is a bit intricate. First, we comment on Part (1). For all $0 < \varepsilon \ll 1$ small, we start by picking an extremal function u_ε for $(I_{4\pi(1-\varepsilon)}^g(\mathbb{D}^2))$. Making our assumptions of Part (1), we only need to rule out the case where (2.1) holds true, as described in the *Proof of Theorem 1.1, Part (1)* of Section 2. Then we assume by contradiction that (2.1) holds true. By Lemma 3.3 in (**Case 2**), we get expansions of the u_ε 's, and then expansions both of the Moser-Trudinger functional (see (2.4)) and of the Dirichlet energy (see (2.5)). These results are gathered in Proposition 2.1 below, whose proof (see Section 4) amounts to show that not only M but also S in (1.9) may have to be attained at a blow-up point of our sequence of *maximizers* $(u_\varepsilon)_\varepsilon$ (see Lemma 4.1). Observe that this twofold maximization property is necessary to get a sharp picture in Theorem 1.1. Moreover, this is not seen when restricting to the case $\Omega = \mathbb{D}^2$, where K_Ω in (1.9) contains only the single point 0, so that expanding the Dirichlet energy of a blowing-up sequence of *critical points* $(u_\varepsilon)_\varepsilon$ is sufficient (see [17]). Theorem 1.1, Part (1), is eventually obtained by getting a contradiction with (2.1): either by comparing (2.4) with our assumption $\Lambda_g(\Omega) \geq \pi \exp(1 + M)$, or by comparing $\|u_\varepsilon\|_{H_0^1}^2 = 4\pi(1 - \varepsilon)$ (see (3.8) in Lemma 3.3) and (2.5) with our assumption $l > 0$.

Now we comment on Part (2). Making our assumptions of Part (2) and assuming also by contradiction that there exists an extremal function u_ε for $(I_{4\pi}^{g_{N_\varepsilon}}(\Omega))$ such that $N_\varepsilon \rightarrow +\infty$ as $\varepsilon \rightarrow 0$, we get from Lemma 3.3 in (**Case 1**) that our assumption $\Lambda_g(\Omega) < \pi \exp(1 + M)$ automatically implies (2.1) (see Step 3.2), so that we may get expansions of the u_ε 's and then (2.10). This gives a contradiction by comparing

$$\|u_\varepsilon\|_{H_0^1}^2 = 4\pi$$

and (2.10) with our assumption $l < 0$, as developed in the *Proof of Theorem 1.1, Part (2)* of Section 2. These key ingredients are gathered in Proposition 2.2. In comparison with the expansions of Part (1), the key observation is that the delicate N_ε -dependence generates additional terms which may only *reduce* the Dirichlet energy, as explained in the *Proof of Proposition 2.2* of Section 4.

Overall, the proof of Lemma 3.3 in (**Case 1**) is the most delicate part: we need to use first that the u_ε are *maximizers* to check that we are in a Moser-Trudinger critical regime (see Step 3.2 and Remark 3.2) and that the pointwise and global

gradient estimate (3.52) is true. In both cases (**Case 1**) and (**Case 2**), resuming the approach of [10], this last point is the key ingredient to be in position to use the radial model B_ε studied in Appendix A. To conclude, the case of a general domain Ω addressed by Theorem 1.1 requires sharp estimates, not only at small scales close to a blow-up point x_ε , as performed in the radial case by [17], but also in the whole Ω (see (3.99) or (4.1)). This allows in particular to get a useful accurate expansion of the Lagrange multiplier λ_ε in (4.12), when proving Proposition 2.1. As a remark, in the process of the proof below (see Remark 2.1), we answer the very interesting Open problem 6 of [17].

Remark 1.1 (Links between Theorem 1.1 and [3, 12, 17, 23]). *For $\Omega = \mathbb{D}^2$, Part (1) of Theorem 1.1 implies in general [17, Corollary 3], which gives back itself the existence of an extremal function for $(I_{4\pi}^0(\Omega))$ pioneered by [3] in the original case $g \equiv 0$. Even if both [17, Corollary 3] and Theorem 1.1 are much more general, we restrict there for simplicity to g 's satisfying (1.17) and coinciding with g_∞ for all $t \gg 1$ (see (1.14)). Then [17, Corollary 3] covers the fast decaying case $a' > 2$ (or $c' = 0$) on the disk. By (1.15), thanks to the explicit formulas above (1.16) for $\Omega = \mathbb{D}^2$ and since $\int_{\mathbb{D}^2} (\log |x|)^2 dx = \frac{\pi}{2}$, it is easy to check that we have in this latter case that $l > 0$ in (1.12), since we have*

$$\gamma^{-4} + A(\gamma)/2 + 4\gamma^{-3} \exp(-1 - M)B(\gamma)S = \gamma^{-4} \left(1 + \frac{2}{e}(1 + g(0)) \right) + o(\gamma^{-4}),$$

as $\gamma \rightarrow +\infty$. Pushing further their asymptotic analysis, Mancini-Martinazzi [17] cover also the case $a' = 2$ and then suspect (see [17, Theorem 4-Open problem 2]) that there could be no extremal function for $(I_{4\pi}^g(\mathbb{D}^2))$, if, in addition, c' is a sufficiently large negative constant. Corollary 1.2 claims that there can actually be an extremal for such a g , whatever c' is, and even independently from the precise behavior of g close to 0 or $+\infty$. However, Part (2) of Theorem 1.1 gives with (1.15) the following picture in this threshold case $a' = 2$:

$$\text{if } c' > - \left(1 + \frac{2}{e}(1 + g(0)) \right) \text{ or } \Lambda_g(\mathbb{D}^2) \geq \pi e, \text{ there is an extremal for } (I_{4\pi}^g(\mathbb{D}^2)),$$

$$\text{if } \begin{cases} c' < - \left(1 + \frac{2}{e}(1 + g(0)) \right), \\ \Lambda_g(\mathbb{D}^2) < \pi e, \\ \text{and } N \gg 1, \end{cases} \text{ there is no extremal for } (I_{4\pi}^{gN}(\mathbb{D}^2)).$$

Observe that there are many ways of building such g 's satisfying $\Lambda_g(\mathbb{D}^2) < \pi e$: one is given in the "Proof of Corollary (1.1)" of Section 2 (see also (1.16)). As observed just below Theorem 1.1, this gives a basically sharp picture about how far we can get the existence of an extremal function for $(I_{4\pi}^g(\Omega))$, **relying only on the asymptotic properties of g** (see Question 1). Theorem 1.1 gives a similar picture on any domain Ω , and then gives back (for $c' = 0$) the results of [12, 23]. Stronger perturbations, for instance $a' < 2$ or even $a' = 0$ and $b' > 0$, are also covered by Theorem 1.1.

We conclude this introductory section by the following remark about the relevance of the assumption (1.1) on g introduced by [17]. We also mention the nice and early result by de Figuereido-Ruf [6].

Remark 1.2 (About assumption (1.1)). *Indeed, assume that g is a C^1 , even function such that $1 + g > 0$ in \mathbb{R} . Assume also that $\bar{g} = \lim_{t \rightarrow +\infty} g(t) \in [-1, +\infty]$*

exists. Firstly, if $\bar{g} = +\infty$, it is easy to check with the test functions of Step 3.1 that $C_{g,4\pi}(\Omega) = +\infty$. Secondly, if $\bar{g} = -1$, it follows from standard integration theory (see for instance [17, Lemma 7]) and from Moser's result [19] that there exists an extremal function for $(I_{4\pi}^g(\Omega))$. Thus, up to replacing $1 + g$ by $(1 + g)/(1 + \bar{g})$, we have that (1.1) holds true in the remaining more sensitive case $\bar{g} \in (-1, +\infty)$. To end this remark, we mention that [6] already studied (1.4) in \mathbb{D}^2 , permitting to recover the existence of an extremal in some subcases where $\bar{g} = -1$. First, assuming that H given by (1.3) is positive in $(0, +\infty)$, it is clear that a nonnegative extremal for $(I_{4\pi}^g(\Omega))$ turns out to be a positive solution of (1.4) (for some $\lambda > 0$). Now following [6], assume also that $\Omega = \mathbb{D}^2$, that $t \mapsto tH(t)$ is C^2 and that, given $a > 0$, there exist $K, C, \sigma > 0$ such that $tH(t) = Kt^{-a}$ for all $t \gg 1$ and such that $H(t) \leq CKt^\sigma$ for all $t > 0$ close to 0. Then, [6, Theorem 1.1] allows to claim that there exists no positive solution of (1.4) for all $0 < \lambda \ll 1$ small enough if $a \geq 1$, while there exists a family of positive solutions of (1.4) blowing-up as $\lambda \rightarrow 0$ if $a < 1$. From by now standard arguments, this first property directly gives back the existence of an extremal in the subcase $a \geq 1$. However, observe that $\bar{g} = -1$ for all $a > 0$, since $1 + g(t) \sim 2Ke^{-t^2} \int_1^t s^{-a} e^{s^2} ds = O(t^{a+1}) \rightarrow 0$ as $t \rightarrow +\infty$, so that an extremal also exists in the subcase $a \in (0, 1)$. Actually we assert that a more precise analysis in the spirit of [17] allows to exclude that the aforementioned blow-up solutions of (1.4) were maximizers and to recover the existence of an extremal also in the subcase $a \in (0, 1)$ through this approach using the Euler-Lagrange equation.

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2. PROOF OF THE MAIN RESULTS

We begin by proving Corollary 1.1, assuming that Theorem 1.1 holds true.

Proof of Corollary 1.1. The first part of Corollary 1.1 is a direct consequence of the first part of Theorem 1.1: plugging the formulas of (1.15) in (1.12), we get that $l > 0$ for g as in Case (1) of Corollary 1.1. In order to prove the second part of Corollary 1.1, we apply the second part of Theorem 1.1. Let χ be a smooth nonnegative function in \mathbb{R} such that $\chi(t) = 0$ for all $t \leq 1/2$ and $\chi(t) = 1$ for all $t \geq 1$. By the Sobolev inequality and standard integration theory, we can check that $g_R := g_\infty \times \chi(\cdot/R)$ satisfies $\Lambda_{g_R}(\mathbb{D}^2) \rightarrow \Lambda_0(\mathbb{D}^2)$ as $R \rightarrow +\infty$. Then, by (1.15), (1.16), assuming $a' < 2, c' < 0$, the second part of Theorem 1.1 applies, starting from $g = g_R$, for $R \gg 1$ fixed sufficiently large. Observe that, for all given $N \gg 1$, $(g_R)_N$ (given by (1.10) for $g = g_R$) satisfies (1.17)-(1.18). Corollary 1.1 is proved. \square

Proof of Corollary 1.2. Let $\Omega, \bar{A}, \lambda_1(\Omega), C$ be as in the statement of the corollary. By Theorem 1.2, it is sufficient to prove that there exists $R \gg 1$ such that for all g satisfying (1.1) and (1.19), we have that $\Lambda_g(\Omega) \geq \pi \exp(1 + M)$, where $\Lambda_g(\Omega)$ is as in (1.11). Let $v > 0$ in Ω be the first eigenvalue of Δ normalized according to

$\|v\|_{H_0^1}^2 = 4\pi$. For all g satisfying (1.19), we have that

$$\begin{aligned}\Lambda_g(\Omega) &\geq \int_{\Omega} ((1+g(0))v^2 + (g(v) - g(0))(1+v^2)) dx, \\ &\geq (1+\bar{A}) \frac{4\pi}{\lambda_1(\Omega)} + \int_{\{v \notin [1/R, R]\}} (g(v) - g(0))(1+v^2) dx,\end{aligned}$$

and, since we have

$$\left| \int_{\{v \notin [1/R, R]\}} (g(v) - g(0))(1+v^2) dx \right| \leq (|\bar{A}| + C)(1 + \|v\|_{L^\infty}^2) |\{v \notin [1/R, R]\}| \rightarrow 0$$

as $R \rightarrow +\infty$, we get the result using that $4(1+\bar{A}) > \lambda_1(\Omega) \exp(1+M)$. \square

The following proposition is the core of the argument to get the existence of an extremal in Theorem 1.1, Part (1). Its proof is postponed in Section 4. It uses the tools developed in Druet-Thizy [10] that allow us to push the asymptotic analysis of a concentrating sequence of extremals $(u_\varepsilon)_\varepsilon$ further than in previous works. In the process of the proof of Proposition 2.1 (see Lemma 4.1), we show first that a concentration point \bar{x} of such u_ε 's realizes M in (1.9). But in the case where $|B(\gamma)|$ matters in (1.12) or, in other words, where $\gamma^3|A(\gamma)| + \gamma^{-1} \lesssim |B(\gamma)|$ as $\gamma \rightarrow +\infty$, we also show that S in (1.9) has to be attained at \bar{x} .

Proposition 2.1. *Let Ω be a smooth bounded domain of \mathbb{R}^2 . Let g be such that (1.1) and (1.5)-(1.6) hold true, for H as in (1.2), and let A , B and F be thus given. Let $(u_\varepsilon)_\varepsilon$ be a sequence of nonnegative functions such that u_ε is a maximizer for $(I_{4\pi(1-\varepsilon)}^g(\Omega))$, for all $0 < \varepsilon \ll 1$. Assume that*

$$u_\varepsilon \rightharpoonup 0 \text{ in } H_0^1 \quad (2.1)$$

as $\varepsilon \rightarrow 0$. Then, $\|u_\varepsilon\|_{H_0^1}^2 = 4\pi(1-\varepsilon)$, there exists a sequence $(\lambda_\varepsilon)_\varepsilon$ of real numbers such that u_ε solves in H_0^1

$$\begin{cases} \Delta u_\varepsilon = \lambda_\varepsilon u_\varepsilon H(u_\varepsilon) \exp(u_\varepsilon^2), & u_\varepsilon > 0 \text{ in } \Omega, \\ u_\varepsilon = 0 \text{ on } \partial\Omega, \end{cases} \quad (2.2)$$

$u_\varepsilon \in C^{1,\theta}(\bar{\Omega})$ ($0 < \theta < 1$) and we have that

$$\gamma_\varepsilon := \max_{y \in \Omega} u_\varepsilon \rightarrow +\infty. \quad (2.3)$$

Moreover, we have that

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} (1+g(u_\varepsilon)) \exp(u_\varepsilon^2) dx = |\Omega|(1+g(0)) + \pi \exp(1+M) \quad (2.4)$$

and that

$$\|u_\varepsilon\|_{H_0^1}^2 = 4\pi (1 + I(\gamma_\varepsilon) + o(\gamma_\varepsilon^{-4} + |A(\gamma_\varepsilon)| + \gamma_\varepsilon^{-3}|B(\gamma_\varepsilon)|)) \quad (2.5)$$

as $\varepsilon \rightarrow 0$, where

$$I(\gamma_\varepsilon) := \gamma_\varepsilon^{-4} + A(\gamma_\varepsilon)/2 + 4\gamma_\varepsilon^{-3} \exp(-1-M)B(\gamma_\varepsilon)S, \quad (2.6)$$

where $|\Omega|$ stands for the volume of the domain Ω and where M and S are as in (1.9).

Remark 2.1. Let g, H be such that (1.1), (1.2), (1.5)-(1.7) hold true. Let u_ε be a maximizer for $(I_{4\pi(1-\varepsilon)}^g)$ such that (2.1) holds true, as in Proposition 2.1. Then, for such a sequence $(u_\varepsilon)_\varepsilon$ satisfying in particular (2.2) and (2.3), we get in the process of the proof (see (3.16) below) that the term $I(\gamma_\varepsilon)$ in (2.5) is necessarily smaller than $o(\gamma_\varepsilon^{-2})$ as $\varepsilon \rightarrow 0$. Moreover this threshold $o(\gamma_\varepsilon^{-2})$ is sharp, in the sense that this term may be for instance of size $\gamma_\varepsilon^{-(2+a')}$, for all given $a' \in (0, 2]$. This can be seen by picking an appropriate g such that $I_{4\pi}^g(\Omega)$ has no extremal, as in Corollary 1.1, and by using Proposition 2.1. Observe that, for such a g , assumption (2.1) is indeed automatically true. This gives an answer to Open Problem 6 in [17].

Proof of Theorem 1.1, Part (1): existence of an extremal for $(I_{4\pi}^g(\Omega))$. We first prove the existence of an extremal stated in Part (1) of Theorem 1.1. Let g be such that (1.1) and (1.5)-(1.6) hold true, for H as in (1.2), and let A, B and F be thus given. Assume either that $l > 0$ in (1.12) or that $\Lambda_g(\Omega) \geq \pi \exp(1 + M)$. Using Lemma 3.1, let $(u_\varepsilon)_\varepsilon$ be a sequence of nonnegative functions such that u_ε is a maximizer for $(I_{4\pi(1-\varepsilon)}^g(\Omega))$, for all $0 < \varepsilon \ll 1$. Then, up to a subsequence, $(u_\varepsilon)_\varepsilon$ converges a.e. and weakly in H_0^1 to some u_0 . Independently, we check that

$$\lim_{\varepsilon \rightarrow 0} C_{g,4\pi(1-\varepsilon)}(\Omega) = C_{g,4\pi}(\Omega), \quad (2.7)$$

where $C_{g,\alpha}(\Omega)$ is as in $(I_\alpha^g(\Omega))$. Indeed, if one assumes by contradiction that the $C_{g,4\pi(1-\varepsilon)}(\Omega)$'s increase to some $\bar{l} < C_{g,4\pi}(\Omega)$ as $\varepsilon \rightarrow 0$, then we may choose some nonnegative u such that $\|u\|_{H_0^1}^2 \leq 4\pi$ and $\int_\Omega (1 + g(u)) \exp(u^2) dx > \bar{l}$. But, picking $v_\varepsilon = u\sqrt{1-\varepsilon}$, we have $\|v_\varepsilon\|_{H_0^1}^2 \leq 4\pi(1-\varepsilon)$ and

$$\lim_{\varepsilon \rightarrow 0} \int_\Omega (1 + g(v_\varepsilon)) \exp(v_\varepsilon^2) dx = \int_\Omega (1 + g(u)) \exp(u^2) dx,$$

by the dominated convergence theorem, using (1.1), $v_\varepsilon^2 \leq u^2$ and $\exp(u^2) \in L^1(\Omega)$. But this contradicts the definition of \bar{l} and concludes the proof of (2.7). Now, by (2.7) and since $\|u_0\|_{H_0^1}^2 \leq 4\pi$, in order to get that u_0 is the extremal for $(I_{4\pi}^g(\Omega))$ we look for, it is sufficient to prove that

$$\lim_{\varepsilon \rightarrow 0} \int_\Omega (1 + g(u_\varepsilon)) \exp(u_\varepsilon^2) dx = \int_\Omega (1 + g(u_0)) \exp(u_0^2) dx. \quad (2.8)$$

If $u_0 = 0$, then Proposition 2.1 gives a contradiction: either by (2.4) and (2.7) if $\Lambda_g(\Omega) \geq \pi \exp(1 + M)$, since it is clear that

$$C_{g,4\pi}(\Omega) > \Lambda_g(\Omega) + (1 + g(0))|\Omega|,$$

or by (2.5)-(2.6) if $l > 0$, since $\|u_\varepsilon\|_{H_0^1} \leq 4\pi$. Thus, we necessarily have that $u_0 \neq 0$. Then, noting that $\|u_\varepsilon - u_0\|_{H_0^1}^2 \leq 4\pi - \|u_0\|_{H_0^1}^2 + o(1)$, the standard Moser-Trudinger inequality $(I_{4\pi}^0(\Omega))$ and Vitali's theorem give that (2.8) still holds true, and Part (1) of Theorem 1.1 is proved in any case. \square

The following proposition is the core of the argument to get the non-existence of an extremal in Theorem 1.1, Part (2). Its proof is postponed in Section 4.

Proposition 2.2. Let Ω be a smooth bounded domain of \mathbb{R}^2 . Let g be such that (1.1) and (1.5)-(1.6) hold true, for H as in (1.2), and let A, B and F be thus given.

Assume that $\Lambda_g(\Omega) < \pi \exp(1+M)$, where M is as in (1.9) and $\Lambda_g(\Omega)$ as in (1.11). Assume that there exists a sequence of positive integers $(N_\varepsilon)_\varepsilon$ such that

$$\lim_{\varepsilon \rightarrow 0} N_\varepsilon = +\infty \quad (2.9)$$

and such that $(I_{4\pi}^{g_{N_\varepsilon}}(\Omega))$ admits a nonnegative extremal u_ε for all $\varepsilon > 0$, where g_{N_ε} is as in (1.10). Then we have (2.1) and that $\|u_\varepsilon\|_{H_0^1}^2 = 4\pi$ for all $0 < \varepsilon \ll 1$. Moreover, we have $u_\varepsilon \in C^{1,\theta}(\bar{\Omega})$ ($0 < \theta < 1$), (2.3) and that

$$\|u_\varepsilon\|_{H_0^1}^2 \leq 4\pi \left(1 + I(\gamma_\varepsilon) + o(\gamma_\varepsilon^{-4} + |A(\gamma_\varepsilon)| + \gamma_\varepsilon^{-3}|B(\gamma_\varepsilon)|)\right) \quad (2.10)$$

as $\varepsilon \rightarrow 0$, where $I(\gamma_\varepsilon)$ is given by (2.6).

Proof of Theorem 1.1, Part (2): non-existence of an extremal for $(I_{4\pi}^{g_N}(\Omega))$, $N \geq N_0$. Let g be such that (1.1) and (1.5)-(1.6) hold true, for H as in (1.2), and let A, B and F be thus given. Assume $l < 0$ and $\Lambda_g(\Omega) < \pi \exp(1+M)$, where l is as in (1.12), Λ_g as in (1.11) and M as in (1.9). In order to prove Part (2) of Theorem 1.1, we assume by contradiction that there exists a sequence $(N_\varepsilon)_\varepsilon$ of positive integers satisfying (2.9) and such that $(I_{4\pi}^{g_{N_\varepsilon}}(\Omega))$ admits an extremal, for g_{N_ε} as in (1.10). We let $(u_\varepsilon)_\varepsilon$ be a sequence of nonnegative functions such that u_ε is a maximizer for $(I_{4\pi}^{g_{N_\varepsilon}}(\Omega))$, for all $\varepsilon > 0$. But this is not possible by Proposition 2.2, since $\|u_\varepsilon\|_{H_0^1}^2 = 4\pi$ contradicts (2.10), since we also assume now $l < 0$. This concludes the proof of Part (2) of Theorem 1.1. \square

3. BLOW-UP ANALYSIS IN THE STRONGLY PERTURBED MOSER-TRUDINGER REGIME

In this section, we aim to prove the main blow-up analysis results that we need to get both Propositions 2.1 and 2.2. The following preliminary lemma deals with the existence of an extremal for the perturbed Moser-Trudinger inequality $(I_\alpha^g(\Omega))$ in the subcritical case $0 < \alpha < 4\pi$. Its proof relies on integration theory combined with $(I_{4\pi}^0(\Omega))$, and on standard variational techniques. It is omitted here and the interested reader may find more details in the proof of Proposition 6 of [17].

Lemma 3.1. *Let Ω be a smooth bounded domain of \mathbb{R}^2 . Let g be such that (1.1) holds true. Then, $(I_\alpha^g(\Omega))$ admits a nonnegative extremal u_α for all $0 < \alpha < 4\pi$. Moreover, we have the following alternative*

- (1) either $\|u_\alpha\|_{H_0^1}^2 < \alpha$ and $u_\alpha H(u_\alpha) = 0$ a.e. ,
- (2) or $\|u_\alpha\|_{H_0^1}^2 = \alpha$ and there exists $\lambda \in \mathbb{R}$ such that u_α solves in H_0^1 the Euler-Lagrange equation (1.4).

Remark 3.1. *The first alternative in Lemma 3.1 may occur in general, but does not if $t \mapsto (1 + g(t)) \exp(t^2)$ increases in $(0, +\infty)$.*

The following lemma investigates more precisely the behavior of g and H , when we assume (1.1) together with (1.5)-(1.6).

Lemma 3.2. *Let Ω be a smooth bounded domain of \mathbb{R}^2 . Let g be such that (1.1), (1.5) and (1.6) hold true, for H as in (1.2), and let A, B and $\delta_0, \delta'_0, F, \kappa$ be thus*

given. Then we have that

$$\begin{aligned}
 a) \quad & \left(1 + g\left(\frac{t}{\gamma}\right)\right) \exp\left(\frac{t^2}{\gamma^2}\right) = (1 + g(0)) + \frac{2B(\gamma)F(t)t}{\gamma(\kappa + 1)} \\
 & + o\left(\frac{|B(\gamma)|}{\gamma} + \frac{1}{\gamma^2}\right) \\
 & \text{in } C_{loc}^0((0, +\infty)_t), \text{ as } \gamma \rightarrow +\infty, \\
 b) \quad & \exists C > 0, \\
 & \left|\left(1 + g\left(\frac{t}{\gamma}\right)\right) \exp\left(\frac{t^2}{\gamma^2}\right) - (1 + g(0))\right| \leq C \left(\frac{|B(\gamma)|}{\gamma} + \frac{1}{\gamma^2}\right) t \exp(\delta'_0 t) \\
 & \text{for all } \gamma \gg 1 \text{ and all } 0 \leq t \leq 2\gamma, \\
 c) \quad & \|g\|_{L^\infty(\mathbb{R})} < +\infty,
 \end{aligned} \tag{3.1}$$

and that

$$\begin{aligned}
 a) \quad & 1 + g\left(\gamma - \frac{t}{\gamma}\right) = H(\gamma) \left(1 + A(\gamma) \left(t + \frac{1}{2}\right) + o(|A(\gamma)| + \gamma^{-4})\right) \\
 & \text{in } C_{loc}^0(\mathbb{R}_t), \text{ as } \gamma \rightarrow +\infty, \\
 b) \quad & \exists C > 0, \left|1 + g\left(\gamma - \frac{t}{\gamma}\right) - H(\gamma)\right| \leq C |H(\gamma)| (|A(\gamma)| + \gamma^{-4}) \exp(\delta_0 t) \\
 & \text{for all } \gamma \gg 1 \text{ and all } 0 \leq t \leq 2\gamma.
 \end{aligned} \tag{3.2}$$

In particular, we have that

$$H(\gamma) \rightarrow 1 \text{ as } \gamma \rightarrow +\infty. \tag{3.3}$$

Proof of Lemma 3.2. We first prove (3.3). Using (1.3), we write

$$(1 + g(r)) \exp(r^2) - (1 + g(0)) = 2 \int_0^r s H(s) \exp(s^2) ds \tag{3.4}$$

for all $r \geq 0$. Then, as $\gamma \rightarrow +\infty$, setting $r = \gamma$, we can write

$$\begin{aligned}
 & 1 + g(\gamma) \\
 &= \exp(-\gamma^2) (1 + g(0)) + 2 \int_0^{\gamma^2} \left(1 - \frac{u}{\gamma^2}\right) H\left(\gamma - \frac{u}{\gamma}\right) \exp\left(-2u + \frac{u^2}{\gamma^2}\right) du, \\
 &= O(\exp(-\gamma^2)) + 2H(\gamma) \int_0^{\gamma^2} \left(1 - \frac{u}{\gamma^2}\right) \exp\left(-2u + \frac{u^2}{\gamma^2}\right) du, \\
 &+ O\left(|H(\gamma)| (|A(\gamma)| + \gamma^{-4}) \int_0^{\gamma^2} \exp(-(1 - \delta_0)u) \exp\left(-u \left(1 - \frac{u}{\gamma^2}\right)\right) du\right), \\
 &= O(\exp(-\gamma^2)) + H(\gamma) (1 - \exp(-\gamma^2)) + o(H(\gamma)),
 \end{aligned}$$

using (1.5). This proves (3.3) since g satisfies (1.1). Observe that parts *a*) and *b*) of (3.1) follow from (1.6) and (3.4) with $r = t/\gamma$, while part *c*) of (3.1) is a straightforward consequence of (1.1). We prove now part *b*) of (3.2). As $\gamma \rightarrow +\infty$,

we write for all $0 \leq t \leq \gamma$

$$\begin{aligned}
& \left(1 + g\left(\gamma - \frac{t}{\gamma}\right)\right) \exp\left(\left(\gamma - \frac{t}{\gamma}\right)^2\right) - (1 + g(\gamma - 1)) \exp((\gamma - 1)^2) \\
&= 2 \int_{\gamma-1}^{\gamma-\frac{t}{\gamma}} r H(r) \exp(r^2) dr, \\
&= 2 \int_t^\gamma \left(1 - \frac{u}{\gamma^2}\right) H\left(\gamma - \frac{u}{\gamma}\right) \exp\left(\gamma^2 - 2u + \frac{u^2}{\gamma^2}\right) du, \\
&= H(\gamma) \left(\exp\left(\left(\gamma - \frac{t}{\gamma}\right)^2\right) - \exp((\gamma - 1)^2)\right) \\
&\quad + O\left(|H(\gamma)|(|A(\gamma)| + \gamma^{-4}) \int_t^\gamma \exp(\gamma^2 - (2 - \delta_0)u) du\right),
\end{aligned}$$

using *b*) in (1.5). Multiplying the above identity by $\exp(-(\gamma - (t/\gamma))^2)$, using $t \leq \gamma$, (1.1) and (3.3), part *b*) of (3.2) easily follows. Using now *a*) of (1.5) in the above before last inequality, we also get part *a*) of (3.2). \square

In the sequel, for all integer $N \geq 1$, we let φ_N be given by (see also (3.36) below)

$$\varphi_N(t) = \sum_{k=N+1}^{+\infty} \frac{t^k}{k!}. \quad (3.5)$$

The main results of this section are stated in the following lemma.

Lemma 3.3. *Let Ω be a smooth bounded domain of \mathbb{R}^2 . Let g be such that (1.1) and (1.5)-(1.6) hold true, for H as in (1.2), and let A , B and F be thus given. Let $(\alpha_\varepsilon)_\varepsilon$ be a sequence of numbers in $(0, 4\pi]$. Let $(N_\varepsilon)_\varepsilon$ be a sequence of positive integers. Assume that*

$$\lim_{\varepsilon \rightarrow 0} \alpha_\varepsilon = 4\pi \text{ and that } u_\varepsilon \geq 0 \text{ is an extremal for } (I_{\alpha_\varepsilon}^{g_{N_\varepsilon}}(\Omega)), \quad (3.6)$$

for all $0 < \varepsilon \ll 1$, where g_{N_ε} is as in (1.10). Assume in addition that we are in one of the following two cases:

$$\begin{aligned}
\text{(Case 1)} \quad & \lim_{\varepsilon \rightarrow 0} N_\varepsilon = +\infty, \quad \alpha_\varepsilon = 4\pi \text{ for all } \varepsilon, \text{ and} \\
& \Lambda_g(\Omega) < \pi \exp(1 + M),
\end{aligned} \quad (3.7)$$

where $\Lambda_g(\Omega)$ is as in (1.11) and M as in (1.9), or

$$\text{(Case 2)} \quad N_\varepsilon = 1 \text{ for all } \varepsilon \text{ and (2.1) holds true.}$$

Then, up to a subsequence,

$$\|u_\varepsilon\|_{H_0^1}^2 = \alpha_\varepsilon, \quad (3.8)$$

$u_\varepsilon \in C^{1,\theta}(\bar{\Omega})$ ($0 < \theta < 1$) solves

$$\begin{cases} \Delta u_\varepsilon = \lambda_\varepsilon u_\varepsilon H_{N_\varepsilon}(u_\varepsilon) \exp(u_\varepsilon^2), & u_\varepsilon > 0 \text{ in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \quad (3.9)$$

where $H_N(t) = 1 + g_N(t) + \frac{g'_N(t)}{2t}$. Moreover, we have (2.4), that

$$\lambda_\varepsilon = \frac{4 + o(1)}{\gamma_\varepsilon^2 \exp(1 + M)}, \quad (3.10)$$

that

$$A(\gamma_\varepsilon) - 2\xi_\varepsilon = o(\tilde{\zeta}_\varepsilon), \quad (3.11)$$

and that

$$x_\varepsilon \rightarrow \bar{x} \quad (\bar{x} \in K_\Omega) \quad (3.12)$$

as $\varepsilon \rightarrow 0$, where $x_\varepsilon, \gamma_\varepsilon$ satisfy

$$u_\varepsilon(x_\varepsilon) = \max_\Omega u_\varepsilon = \gamma_\varepsilon \rightarrow +\infty \quad (3.13)$$

as $\varepsilon \rightarrow 0$, where ξ_ε is given by

$$\xi_\varepsilon = \frac{\gamma_\varepsilon^{2(N_\varepsilon-1)}}{\varphi_{N_\varepsilon-1}(\gamma_\varepsilon^2)(N_\varepsilon-1)!}, \quad (3.14)$$

and where $\tilde{\zeta}_\varepsilon$ is given by

$$\tilde{\zeta}_\varepsilon = \max\left(\frac{1}{\gamma_\varepsilon^2}, |A(\gamma_\varepsilon)|, \xi_\varepsilon\right). \quad (3.15)$$

At last, (3.97)-(3.99) below hold true, for μ_ε as in (3.40) and t_ε as in (3.41).

Observe that $N_\varepsilon = 1$ in **(Case 2)** reduces to say that $g_{N_\varepsilon} = g$. From (3.30) obtained in the process of the proof below, we get that $\xi_\varepsilon = o(1/\gamma_\varepsilon^2)$ in **(Case 2)**, so that (3.11) is then equivalent to

$$A(\gamma_\varepsilon) = o\left(\frac{1}{\gamma_\varepsilon^2}\right), \quad (3.16)$$

as discussed in Remark 2.1.

Proof of Lemma 3.3. We start by several basic steps. First, a test function computation gives the following result.

Step 3.1. For all g such that (1.1) holds true, we have that

$$C_{g,4\pi}(\Omega) \geq |\Omega|(1+g(0)) + \pi \exp(1+M),$$

where $C_{g,4\pi}(\Omega)$ is as in $(I_\alpha^g(\Omega))$ ($\alpha = 4\pi$) and where M is as in (1.9).

Proof of Step 3.1. In order to get Step 3.1, it is sufficient to prove that there exist functions $f_\varepsilon \in H_0^1$ such that $\|f_\varepsilon\|_{H_0^1}^2 = 4\pi$ and such that

$$\int_\Omega (1+g(f_\varepsilon)) \exp(f_\varepsilon^2) dy \geq |\Omega|(1+g(0)) + \pi \exp(1+M) + o(1) \quad (3.17)$$

as $\varepsilon \rightarrow 0$. In order to reuse these computations later, we fix any sequence $(z_\varepsilon)_\varepsilon$ of points in Ω such that

$$\frac{\varepsilon^2}{d(z_\varepsilon, \partial\Omega)^2} = o\left(\left(\log \frac{1}{\varepsilon}\right)^{-1}\right). \quad (3.18)$$

For $0 < \varepsilon < 1$, we let v_ε be given by $v_\varepsilon(y) = \log \frac{1}{\varepsilon^2 + |y - z_\varepsilon|^2} + \mathcal{H}_{z_\varepsilon, \varepsilon}$, where $\mathcal{H}_{z_\varepsilon, \varepsilon}$ is harmonic in Ω and such that v_ε is zero on $\partial\Omega$. Then, by the maximum principle and (1.8), we have that

$$\mathcal{H}_{z_\varepsilon, \varepsilon}(y) = \mathcal{H}_{z_\varepsilon}(y) + O\left(\frac{\varepsilon^2}{d(z_\varepsilon, \partial\Omega)^2}\right) \text{ for all } y \in \Omega, \quad (3.19)$$

where $\mathcal{H}_{z_\varepsilon}$ is as in (1.8). Then, integrating by parts, we compute

$$\begin{aligned}
\|v_\varepsilon\|_{H_0^1}^2 &= \int_{\Omega} v_\varepsilon \Delta v_\varepsilon \, dy, \\
&= \int_{\Omega} \frac{4}{\varepsilon^2 \left(1 + \frac{|z_\varepsilon - y|^2}{\varepsilon^2}\right)^2} \left(\log \frac{1}{\varepsilon^2} + \log \frac{1}{1 + \frac{|y - z_\varepsilon|^2}{\varepsilon^2}} + \mathcal{H}_{z_\varepsilon, \varepsilon}(y) \right) dy, \\
&= 4\pi \left(\log \frac{1}{\varepsilon^2} + o(1) \right) - 4\pi (1 + o(1)) \\
&\quad + 4\pi (\mathcal{H}_{z_\varepsilon}(z_\varepsilon) + o(1)), \\
&= 4\pi \left(\log \frac{1}{\varepsilon^2} - 1 + \mathcal{H}_{z_\varepsilon}(z_\varepsilon) \right) + o(1),
\end{aligned} \tag{3.20}$$

where the change of variable $z = (y - z_\varepsilon)/\varepsilon$, (3.18), (3.19) and

$$\mathcal{H}_{z_\varepsilon}(z_\varepsilon + \varepsilon z) = \mathcal{H}_{z_\varepsilon}(z_\varepsilon) + O\left(\frac{\varepsilon|z|}{d(z_\varepsilon, \partial\Omega)}\right), \tag{3.21}$$

(see for instance Appendix B in [10]) are used. From now $\liminf_{\varepsilon \rightarrow 0} d(z_\varepsilon, \partial\Omega) > 0$ is assumed. Let f_ε be given by $4\pi v_\varepsilon^2 = f_\varepsilon^2 \|v_\varepsilon\|_{H_0^1}^2$. We can write

$$f_\varepsilon(y)^2 = \frac{\left(\log \frac{1}{|z_\varepsilon - y|^2 + \varepsilon^2}\right)^2 + 2\mathcal{H}_{z_\varepsilon, \varepsilon}(y) \log \frac{1}{|z_\varepsilon - y|^2 + \varepsilon^2} + \mathcal{H}_{z_\varepsilon, \varepsilon}(y)^2}{\log \frac{1}{\varepsilon^2} \left(1 + \frac{\mathcal{H}_{z_\varepsilon}(z_\varepsilon) - 1}{\log \frac{1}{\varepsilon^2}} + o\left(\frac{1}{\log \frac{1}{\varepsilon^2}}\right)\right)},$$

using (3.20). Then, writing $\log \frac{1}{|z_\varepsilon - y|^2 + \varepsilon^2} = \log \frac{1}{\varepsilon^2} + \log \frac{1}{1 + \frac{|z_\varepsilon - y|^2}{\varepsilon^2}}$, we get

$$\begin{aligned}
&\int_{B_{z_\varepsilon}(\check{r}_\varepsilon) \cap \Omega} (1 + g(f_\varepsilon)) \exp(f_\varepsilon^2) \, dy \\
&= \int_{B_{z_\varepsilon}(\check{r}_\varepsilon) \cap \Omega} (1 + o(1)) \frac{\exp(-2\check{t}_\varepsilon(y) + 2\mathcal{H}_{z_\varepsilon, \varepsilon}(y) - \mathcal{H}_{z_\varepsilon}(z_\varepsilon) + 1)}{\varepsilon^2} \times \\
&\quad \exp\left(\frac{\check{t}_\varepsilon^2}{\log \frac{1}{\varepsilon^2}} + O\left(\frac{1 + \check{t}_\varepsilon}{\log \frac{1}{\varepsilon^2}} + \frac{1 + \check{t}_\varepsilon^2}{(\log \frac{1}{\varepsilon^2})^2}\right)\right) dy, \\
&= \pi \exp(\mathcal{H}_{z_\varepsilon}(z_\varepsilon) + 1)(1 + o(1)),
\end{aligned} \tag{3.22}$$

as $\varepsilon \rightarrow 0$, using (1.1), (3.19) and (3.21), where $\check{t}_\varepsilon(y) = \log\left(1 + \frac{|z_\varepsilon - y|^2}{\varepsilon^2}\right)$ and where \check{r}_ε is given by $\log\left(1 + \frac{\check{r}_\varepsilon^2}{\varepsilon^2}\right) = \frac{1}{2} \log \frac{1}{\varepsilon^2}$. Now, we can check that

$$\begin{aligned}
f_\varepsilon(y)^2 &\leq \left(\log \frac{1}{\varepsilon^2} + O(1)\right)^{-1} \left(\log \frac{1}{|z_\varepsilon - y|^2} + O(1)\right)^2, \\
&\leq \left(\log \frac{1}{|z_\varepsilon - y|^2} + O(1)\right) \times \left(\frac{1}{2} + o(1)\right) \text{ for all } y \in \Omega \setminus B_{z_\varepsilon}(\check{r}_\varepsilon),
\end{aligned}$$

using (1.8), (3.19) and our definition of \check{r}_ε , so that we also get

$$\int_{\Omega \setminus B_{z_\varepsilon}(\check{r}_\varepsilon)} (1 + g(f_\varepsilon)) \exp(f_\varepsilon^2) \, dy \rightarrow (1 + g(0))|\Omega| \tag{3.23}$$

as $\varepsilon \rightarrow 0$, by the dominated convergence theorem, using (1.1). Property (3.17) and then Step 3.1 follow from (3.22) and (3.23), choosing $z_\varepsilon \in K_\Omega$ as in (1.9). \square

From now on, we make the assumptions of Lemma 3.3. In particular, we assume that either **(Case 1)**, or **(Case 2)** holds true. Given an integer $N \geq 1$, observe that Step 3.1 applies to g_N , since g_N satisfies (1.1), if g does. Then, using $\alpha_\varepsilon = 4\pi$ in **(Case 1)**, or (2.7) and $g_{N_\varepsilon} = g$ in **(Case 2)**, we get that

$$|\Omega|(1 + g(0)) + \pi \exp(1 + M) \leq \begin{cases} C_{g_{N_\varepsilon}, 4\pi} & \text{in } \mathbf{(Case 1)}, \\ C_{g, \alpha_\varepsilon} + o(1) & \text{in } \mathbf{(Case 2)}, \end{cases} \quad (3.24)$$

as $\varepsilon \rightarrow 0^+$, where $C_{g, \alpha}(\Omega)$ is as in formula $(I_\alpha^g(\Omega))$ and where M is as in (1.9). Let us rewrite now (3.9) in a more convenient way. Let Ψ_N be given by

$$\Psi_N(t) = (1 + g_N(t)) \exp(t^2). \quad (3.25)$$

Observe in particular that

$$(1 + g(t))(1 + t^2) \leq \Psi_N(t) \leq (1 + g(t)) \exp(t^2),$$

for all t and all N , by (1.1). Using (1.2), (1.3) and (1.10), we may rewrite (3.9) as

$$\begin{cases} \Delta u_\varepsilon = \frac{\lambda_\varepsilon}{2} \Psi'_{N_\varepsilon}(u_\varepsilon), & u_\varepsilon > 0 \text{ in } \Omega, \\ u_\varepsilon = 0 \text{ on } \partial\Omega, \end{cases} \quad (3.26)$$

with

$$\begin{aligned} \Psi'_N(t) &= 2tH(t)(1 + t^2 + \varphi_N(t^2)) + 2t(1 + g(t)) \left(\frac{t^{2N}}{N!} - t^2 \right), \\ &= 2tH(t)\varphi_N(t^2) + 2t \left(1 + \frac{t^{2N}}{N!} \right) (1 + g(t)) + g'(t)(1 + t^2). \end{aligned} \quad (3.27)$$

Indeed, in (3.9), it turns out that

$$H_N(t) = \frac{\Psi'_N(t) \exp(-t^2)}{2t}. \quad (3.28)$$

Observe that by (1.1) and (3.3), using the first line of (3.27), we clearly have that there exists $C > 0$ such that

$$|\Psi'_{N_\varepsilon}(t)| \leq Ct \exp(t^2) \quad (3.29)$$

for all $t \geq 0$ and all ε . In **(Case 2)**, (2.1) is assumed to be true. We prove now that (2.1) also holds true in **(Case 1)**.

Step 3.2. *Assume that we are in **(Case 1)**. Then (2.1) holds true. Moreover, if $\gamma_\varepsilon := \text{ess sup } u_\varepsilon < +\infty$ for all ε , we have that*

$$\liminf_{\varepsilon \rightarrow 0} \underbrace{\frac{\varphi_{N_\varepsilon}(\gamma_\varepsilon^2)}{\exp(\gamma_\varepsilon^2)}}_{:= \delta_\varepsilon \in (0,1)} > 0, \quad (3.30)$$

and, in other words, that

$$\liminf_{\varepsilon \rightarrow 0} \frac{\gamma_\varepsilon^2 - N_\varepsilon}{\sqrt{N_\varepsilon}} > -\infty, \quad (3.31)$$

where φ_N is as in (3.5).

Proof of Step 3.2. By (3.6) and (3.24), we get that

$$\int_{\Omega} \Psi_{N_{\varepsilon}}(u_{\varepsilon}) dy \geq (1 + g(0))|\Omega| + \pi \exp(1 + M). \quad (3.32)$$

Writing now

$$\Psi_N(t) = (1 + g(0)) + ((1 + g(t))(1 + t^2) - (1 + g(0))) + (1 + g(t))\varphi_N(t^2)$$

and using (1.1), we also get

$$\int_{\Omega} \Psi_{N_{\varepsilon}}(u_{\varepsilon}) dy \leq (1 + g(0))|\Omega| + \Lambda_g(\Omega) + \int_{\Omega} (1 + g(u_{\varepsilon}))\varphi_{N_{\varepsilon}}(u_{\varepsilon}^2) dy, \quad (3.33)$$

where Λ_g is as in (1.11). Then by (1.1) and (3.7), we get from (3.32) and (3.33) that

$$\liminf_{\varepsilon \rightarrow 0} \int_{\Omega} \varphi_{N_{\varepsilon}}(u_{\varepsilon}^2) dy > 0. \quad (3.34)$$

Up to a subsequence, $u_{\varepsilon} \rightharpoonup u_0$ in H_0^1 , for some $u_0 \in H_0^1$ such that $\|u_0\|_{H_0^1}^2 \leq 4\pi$. Let $0 < \beta \ll 1$ be given. First we have that

$$u_{\varepsilon}^2 \leq (1 + \beta)(u_{\varepsilon} - u_0)^2 + \left(1 + \frac{1}{\beta}\right) u_0^2.$$

Independently, by Moser-Trudinger's inequality, we have that

$$u \in H_0^1 \implies \forall p \in [1, +\infty), \quad \exp(u^2) \in L^p. \quad (3.35)$$

Therefore, if $u_0 \not\equiv 0$, $\lim_{\varepsilon \rightarrow 0} \|u_{\varepsilon} - u_0\|_{H_0^1}^2 < 4\pi$, there exists $p_0 > 1$ such that $(\exp(u_{\varepsilon}^2))_{\varepsilon}$ is bounded in L^{p_0} , by Moser's and Hlder's inequalities. Then, by Vitali's theorem, since $\varphi_{N_{\varepsilon}} \leq \exp$ in $[0, +\infty)$ and since $N_{\varepsilon} \rightarrow +\infty$ in **(Case 1)**, we get

$$u_0 \not\equiv 0 \implies \int_{\Omega} \varphi_{N_{\varepsilon}}(u_{\varepsilon}^2) dy = o(1)$$

as $\varepsilon \rightarrow 0$, which proves (2.1), in view of (3.34). Noting that the function $t \mapsto \varphi_N(t) \exp(-t)$ increases in $[0, +\infty)$, we can write

$$\int_{\Omega} \varphi_{N_{\varepsilon}}(u_{\varepsilon}^2) dy \leq \frac{\varphi_{N_{\varepsilon}}(\gamma_{\varepsilon}^2)}{\exp(\gamma_{\varepsilon}^2)} \int_{\Omega} \exp(u_{\varepsilon}^2) dy$$

and conclude that (3.30) holds true by (3.34) and Moser's inequality. Observe that

$$\varphi_N(\Gamma) = \exp(\Gamma) \int_0^{\Gamma} \exp(-s) \frac{s^N}{N!} ds. \quad (3.36)$$

Setting $\Gamma = \gamma_{\varepsilon}^2$, $N = N_{\varepsilon}$ and $s = N_{\varepsilon} + u\sqrt{N_{\varepsilon}}$, we get (3.31) from (3.30), using Stirling formula and

$$\left(1 + \frac{u}{\sqrt{N}}\right)^N e^{-u\sqrt{N}} \leq e^{-\frac{u^2}{2}}$$

for $-\sqrt{N} < u < 0$. □

The next steps applies in both **(Case 1)** and **(Case 2)**.

Step 3.3. We have that (3.8), (3.9) hold true, and that u_{ε} is in $C^{1,\theta}(\bar{\Omega})$.

Proof of Step 3.3. Assume by contradiction that (3.8) does not hold true, or in other words that $\|u_\varepsilon\|_{H_0^1}^2 < \alpha_\varepsilon$ for all $\varepsilon \ll 1$, up to a subsequence, then it follows from the fact that u_ε is an (unconstrained) critical point of our functional that $\Psi'_\varepsilon(u_\varepsilon) = 0$ a.e. in Ω . The key property is now that the Lebesgue measure of $\{t_0 < u_\varepsilon \leq t_1\}$ is positive, for all $0 \leq t_0 < t_1 \leq \gamma_\varepsilon$, as it follows by $\int_\Omega |\nabla T u_\varepsilon|^2 > 0$ where $T u_\varepsilon \in H_0^1$ is the truncation of $u_\varepsilon - t_0$ as 0 when $u_\varepsilon \leq t_0$ and as $t_1 - t_0$ when $u_\varepsilon > t_1$; this shows that $\Psi'_{N_\varepsilon} = 0$ in $(0, \gamma_\varepsilon)$ and then

$$(1 + g(t)) = \frac{1 + g(0)}{1 + t^2 + \varphi_{N_\varepsilon}(t^2)} \quad (3.37)$$

for all $t \in [0, \gamma_\varepsilon)$. If $\gamma_\varepsilon = +\infty$ a contradiction arises; then $\gamma_\varepsilon < +\infty$ and one can use Step 3.2 to show that $\gamma_\varepsilon \rightarrow +\infty$, still reaching a contradiction. Then (3.8) is proved, so that (3.9) holds true in H_0^1 . Thus for all given ε , u_ε is uniformly bounded and then in $C^{1,\theta}$ by (3.9) and elliptic theory. We also use there that g appearing in the formula (3.27) of Ψ'_{N_ε} is assumed to be C^1 in (1.1). \square

The previous steps give in particular that (3.13) makes sense and holds true.

Step 3.4. *There holds that $\lambda_\varepsilon > 0$ for all $0 < \varepsilon \ll 1$. Moreover*

$$\lambda_\varepsilon \rightarrow 0 \quad (3.38)$$

as $\varepsilon \rightarrow 0$, where λ_ε is as in (3.9).

Proof of Step 3.4. By (2.1), we have that $u_\varepsilon \rightarrow 0$ a.e. and in L^p , for all $p < +\infty$. Since $\int_{u_\varepsilon \leq M_0} \Psi_{N_\varepsilon}(u_\varepsilon) dx \rightarrow (1 + g(0))|\Omega|$, by (3.24) one has that

$$\liminf_{\varepsilon \rightarrow 0} \int_{u_\varepsilon > M_0} \Psi_{N_\varepsilon}(u_\varepsilon) dx \geq \pi \exp(1 + M)$$

for all given $M_0 > 0$; one can now use (3.27) with (1.1), (3.3) and some standard integration argument, to get that

$$\liminf_{\varepsilon \rightarrow 0} \int_\Omega [\Psi'_{N_\varepsilon}(u_\varepsilon) + 2(1 + g(u_\varepsilon))u_\varepsilon^3] u_\varepsilon dx = +\infty. \quad (3.39)$$

Then, multiplying (3.26) by u_ε and integrating by parts, we get that $\lambda_\varepsilon > 0$ and

$$4\pi + o(1) = \int_\Omega |\nabla u_\varepsilon|^2 dx \gg \lambda_\varepsilon,$$

which proves (3.38). \square

Then, using (3.3), we may let $\mu_\varepsilon > 0$ be given by

$$\lambda_\varepsilon H(\gamma_\varepsilon) \mu_\varepsilon^2 \gamma_\varepsilon^2 \varphi_{N_\varepsilon-1}(\gamma_\varepsilon^2) = 4, \quad (3.40)$$

where φ_N is as in (3.5). Before starting the core of the proof, we would like to make a parenthetical remark.

Remark 3.2. *Note that (Case 1) is particularly delicate to handle, since the nonlinearities $(\Psi'_{N_\varepsilon})_\varepsilon$ are not of uniform critical growth, even in the very general framework of [9, Definition 1]. A more intuitive way to see this is the following: if $(\tilde{\gamma}_\varepsilon)_\varepsilon$ is a sequence of positive real numbers such that $\tilde{\gamma}_\varepsilon \rightarrow +\infty$, but not too fast, in the sense that $\tilde{\gamma}_\varepsilon^2 \ll N_\varepsilon$, then it can be checked with (1.1) and (3.3) that*

$$\frac{\lambda_\varepsilon}{2} \Psi'_{N_\varepsilon}(\tilde{\gamma}_\varepsilon) = \tilde{\lambda}_\varepsilon (1 + o(1)) \tilde{\gamma}_\varepsilon^{2N_\varepsilon+1}$$

as $\varepsilon \rightarrow 0$, where $\tilde{\lambda}_\varepsilon = \lambda_\varepsilon / (N_\varepsilon!)$. Then, in the regime $0 \leq u_\varepsilon \leq \tilde{\gamma}_\varepsilon$, at least formally, (3.26) looks at first order like the Lane-Emden problem, namely

$$\begin{cases} \Delta u_\varepsilon = \tilde{\lambda}_\varepsilon u_\varepsilon^{2N_\varepsilon+1}, & u_\varepsilon > 0 \text{ in } \Omega, \\ u_\varepsilon = 0 \text{ on } \partial\Omega, \\ N_\varepsilon \rightarrow +\infty, \end{cases} \quad (\text{Lane-Emden problem})$$

for which very interesting, but very different concentration phenomena were pointed out (see for instance [2, 7, 8, 11, 21, 22]). A real difficulty to conclude the subsequent proofs is to extend the analysis developed in [1, 9, 10] for the Moser-Trudinger "purely critical" regime, in order to deal also with such other intermediate regimes. As a last remark, a much simpler version of the techniques developed here permits also to answer some open questions about the Lane-Emden problem, as performed in [24].

We let t_ε be given by

$$t_\varepsilon(x) = \log \left(1 + \frac{|x - x_\varepsilon|^2}{\mu_\varepsilon^2} \right). \quad (3.41)$$

Here and in the sequel, for a radially symmetric function f around of x_ε (resp. around 0), we will often write $f(r)$ instead of $f(x)$ for $|x - x_\varepsilon| = r$ (resp. $|x| = r$).

Step 3.5. We have that

$$\gamma_\varepsilon (\gamma_\varepsilon - u_\varepsilon(x_\varepsilon + \mu_\varepsilon \cdot)) \rightarrow T_0 := \log(1 + |\cdot|^2) \text{ in } C_{loc}^{1,\theta}(\mathbb{R}^2), \quad (3.42)$$

where $\gamma_\varepsilon, x_\varepsilon$ are as in (3.13) and μ_ε is as in (3.40). Moreover, we have that

$$\liminf_{\varepsilon \rightarrow 0} \lambda_\varepsilon \gamma_\varepsilon^2 > 0. \quad (3.43)$$

At this stage, by taking the log of (3.40), by estimating λ_ε with (3.38) and (3.43) we get from (3.3) and (3.30) that

$$\log \frac{1}{\mu_\varepsilon^2} = \gamma_\varepsilon^2 (1 + o(1)) \quad (3.44)$$

as $\varepsilon \rightarrow 0$. Observe in particular that (3.44) holds true in (**Case 1**).

Proof of Step 3.5. We first sketch the proof of (3.42). In (**Case 2**), (3.42) follows closely Step 1 of the proof of [9, Proposition 1]. Thus, we focus now on the the proof of (3.42) in (**Case 1**). Observe that

$$\sup_{t \in \mathbb{R}} \frac{t^{2N}}{N!} \exp(-t^2) = \frac{N^N}{N!} \exp(-N) \underset{N \rightarrow +\infty}{=} \frac{1 + o(1)}{\sqrt{2\pi N}}, \quad (3.45)$$

by Stirling's formula. Then, by (1.1), (3.3), (3.13), (3.27) and (3.30), we have that

$$\begin{aligned} \frac{\Psi'_{N_\varepsilon}(u_\varepsilon)}{2} &= u_\varepsilon H(u_\varepsilon) \varphi_{N_\varepsilon}(u_\varepsilon^2) + u_\varepsilon (1 + g(u_\varepsilon)) \frac{u_\varepsilon^{2N_\varepsilon}}{N_\varepsilon!} + O(\gamma_\varepsilon^3), \\ &\leq (1 + o(1)) \gamma_\varepsilon \varphi_{N_\varepsilon-1}(\gamma_\varepsilon^2). \end{aligned} \quad (3.46)$$

Observe that, by (3.13) and elliptic theory, we must have $\sup_\Omega \lambda_\varepsilon \Psi'_{N_\varepsilon}(u_\varepsilon) \rightarrow +\infty$ as $\varepsilon \rightarrow 0$. Then, (3.46) implies that $\lambda_\varepsilon \gamma_\varepsilon \varphi_{N_\varepsilon-1}(\gamma_\varepsilon^2) \rightarrow +\infty$ and then that $\mu_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$, by (3.40). Let τ_ε be given in $(\Omega - x_\varepsilon)/\mu_\varepsilon$ by

$$u_\varepsilon(x_\varepsilon + \mu_\varepsilon \cdot) = \gamma_\varepsilon - \frac{\tau_\varepsilon}{\gamma_\varepsilon}.$$

Then, since $\Delta \tau_\varepsilon = -\mu_\varepsilon^2 \gamma_\varepsilon (\Delta u_\varepsilon)(x_\varepsilon + \mu_\varepsilon \cdot)$, we get from (3.26), (3.40) and (3.46), that there exists $C > 0$ such that $|\Delta \tau_\varepsilon| \leq C$, while $\tau_\varepsilon \geq 0$, $\tau_\varepsilon(0) = 0$. As in

[9, p.231], we have that $\mu_\varepsilon = o(d(x_\varepsilon, \partial\Omega))$. Then, by standard elliptic theory, there exists τ_0 such that

$$\tau_\varepsilon \rightarrow \tau_0 \text{ in } C_{loc}^{1,\theta}(\mathbb{R}^2), \quad (3.47)$$

as $\varepsilon \rightarrow 0$. Note that for all $\Gamma, T > 0$ and all N , we have that

$$\varphi_N(T) = \varphi_N(\Gamma) \exp(-(\Gamma - T)) - \exp(T) \int_T^\Gamma \exp(-s) \frac{s^N}{N!} ds. \quad (3.48)$$

Writing the previous identity for $N = N_\varepsilon - 1$, $\Gamma = \gamma_\varepsilon^2$ and $T = u_\varepsilon^2 = \gamma_\varepsilon^2 - 2\tau_\varepsilon + \frac{\tau_\varepsilon^2}{\gamma_\varepsilon^2}$, noting from (3.45) and (3.47) that

$$\int_{u_\varepsilon^2}^{\gamma_\varepsilon^2} \exp(-s) \frac{s^{N_\varepsilon-1}}{(N_\varepsilon-1)!} ds = O\left(\frac{1}{\sqrt{N_\varepsilon}}\right)$$

in \mathbb{R}_{loc}^2 and resuming the arguments to get (3.46), we get that

$$\Delta(-\tau_0) = 4 \exp(-2\tau_0)$$

using also (3.26), (3.30) and (3.40). Now, choosing $R \gg 1$ such that $|g(t)| < 1$ and $H(t) > 0$ for all $t \geq R$, we easily see that there exists $C_R > 0$ such that

$$u_\varepsilon [\Psi'_{N_\varepsilon}(u_\varepsilon)]^- \leq C_R |u_\varepsilon| + 4u_\varepsilon^4, \quad (3.49)$$

by (1.1), (3.3) and (3.27), where $t^- = -\min(t, 0)$. Then, we have that

$$\frac{\lambda_\varepsilon}{2} \int_\Omega u_\varepsilon [\Psi'_{N_\varepsilon}(u_\varepsilon)]^+ dy = 4\pi + o(1),$$

by (3.8), (3.26), (3.38) and (3.49), where $t^+ = \max(t, 0)$. For all $A \gg 1$, we get that

$$4 \int_{B_0(A)} \exp(-2\tau_0) dy \leq \liminf_{\varepsilon \rightarrow 0} \frac{\lambda_\varepsilon}{2} \int_\Omega u_\varepsilon [\Psi'_{N_\varepsilon}(u_\varepsilon)]^+ dy,$$

by (3.47) and, since A is arbitrary, we get then that $\int_{\mathbb{R}^2} \exp(-2\tau_0) dy < +\infty$. Thus, by the classification result Chen-Li [4], since $\tau_0 \geq 0$ and $\tau_0(0) = 0$, we get that $\tau_0(y) = \log(1 + |y|^2)$. Thus (3.42) is proved by (3.47). Similarly, we may also choose some A_ε 's, such that $A_\varepsilon \rightarrow +\infty$ and such that

$$\frac{\lambda_\varepsilon}{2} \int_{B_{x_\varepsilon}(A_\varepsilon \mu_\varepsilon)} \Psi_{N_\varepsilon}(u_\varepsilon) dy = \frac{2\pi + o(1)}{\gamma_\varepsilon^2}.$$

We use for this (3.45) to write that

$$\frac{\varphi_{N_\varepsilon}(\gamma_\varepsilon^2)}{\varphi_{N_\varepsilon-1}(\gamma_\varepsilon^2)} = 1 - \frac{\gamma_\varepsilon^{2N_\varepsilon}}{N_\varepsilon! \varphi_{N_\varepsilon-1}(\gamma_\varepsilon^2)} = 1 + o(1)$$

as $\varepsilon \rightarrow 0$. Thus, since $0 < \Psi_{N_\varepsilon}(t) \leq (1 + g(t)) \exp(t^2)$ for all $t \geq 0$, and since $C_{g,4\pi}(\Omega) < +\infty$, we get (3.43) from (1.1). This concludes the proof of Step 3.5. \square

By Step 3.5 and estimates in its proof, since we assume $\|u_\varepsilon\|_{H_0^1}^2 \leq 4\pi$, we get that

$$\lim_{R \rightarrow +\infty} \lim_{\varepsilon \rightarrow 0} \int_{\Omega \setminus B_{x_\varepsilon}(R\mu_\varepsilon)} (\Delta u_\varepsilon(y))^+ u_\varepsilon dy = 0. \quad (3.50)$$

We let Ω_ε be given by

$$\Omega_\varepsilon = \begin{cases} \{y \in \Omega \text{ s.t. } \varphi_{N_\varepsilon-1}(u_\varepsilon(y)^2) \geq u_\varepsilon(y)^2 + 1\} & \text{in (Case 1),} \\ \Omega & \text{in (Case 2).} \end{cases}$$

Now, despite the difficulty pointed out in Remark 3.2, we are able to get the following weak, but global pointwise estimates.

Step 3.6. *There exists $C > 0$ such that*

$$|\cdot - x_\varepsilon|^2 |\Delta u_\varepsilon| u_\varepsilon \leq C \text{ in } \Omega_\varepsilon \quad (3.51)$$

and such that

$$|\cdot - x_\varepsilon| |\nabla u_\varepsilon| u_\varepsilon \leq C \text{ in } \Omega_\varepsilon \quad (3.52)$$

for all ε .

In **(Case 2)**, it is not so difficult to adapt the arguments of [9, 3, 4] to get Step 3.6. Thus, in the proof of Step 3.6 just below, we assume that we are in **(Case 1)**. Then observe that $\Omega_\varepsilon \neq \emptyset$ by Step 3.2. Given $\eta_0 \in (0, 1)$, writing

$$\varphi_{N_\varepsilon-1}(tN_\varepsilon) = \frac{t^{N_\varepsilon} N_\varepsilon^{N_\varepsilon}}{N_\varepsilon!} \left(\sum_{k=0}^{+\infty} t^k + o(1) \right) = \frac{(et)^{N_\varepsilon}}{\sqrt{2\pi N_\varepsilon}} \left(\frac{1}{1-t} + o(1) \right),$$

by Stirling's formula, where $o(1) \rightarrow 0$ as $\varepsilon \rightarrow 0$ uniformly in $|t| \leq \eta_0$, the unique positive solution Γ_ε of $\varphi_{N_\varepsilon-1}(\Gamma_\varepsilon) = \Gamma_\varepsilon + 1$ satisfies $\Gamma_\varepsilon = (1 + o(1)) \frac{N_\varepsilon}{e}$. Then, since $\varphi_{N_\varepsilon-1}/(1 + \cdot)$ increases in $(0, +\infty)$, we clearly get that

$$(1 + o(1)) \frac{N_\varepsilon}{e} \leq \min_{\Omega_\varepsilon} u_\varepsilon^2. \quad (3.53)$$

Observe also that (3.53) almost characterizes Ω_ε in the following sense: given $\delta > 0$, for all $\varepsilon \ll 1$ so that $(1 + \delta) \frac{N_\varepsilon}{e} \geq \Gamma_\varepsilon$, one has that $u_\varepsilon(y)^2 \geq (1 + \delta) \frac{N_\varepsilon}{e}$ implies $y \in \Omega_\varepsilon$.

Proof of Step 3.6, Formula (3.51). As aforementioned, we still assume that we are in **(Case 1)**. Thus, in particular, we assume that $N_\varepsilon \rightarrow +\infty$ as $\varepsilon \rightarrow 0$. Assume now by contradiction that

$$\max_{y \in \Omega_\varepsilon} |y - x_\varepsilon|^2 |\Delta u_\varepsilon(y)| u_\varepsilon(y) = |y_\varepsilon - x_\varepsilon|^2 |\Delta u_\varepsilon(y_\varepsilon)| u_\varepsilon(y_\varepsilon) \rightarrow +\infty \quad (3.54)$$

as $\varepsilon \rightarrow 0$, for some y_ε 's such that $y_\varepsilon \in \Omega_\varepsilon$. First for all sequence $(\check{z}_\varepsilon)_\varepsilon$ such that $\check{z}_\varepsilon \in \Omega_\varepsilon$, we have that $\Delta u_\varepsilon(\check{z}_\varepsilon) > 0$, that $g'(u_\varepsilon(\check{z}_\varepsilon)) = o(u_\varepsilon(\check{z}_\varepsilon))$ and that

$$\Psi'_{N_\varepsilon}(u_\varepsilon(\check{z}_\varepsilon)) = (1 + o(1)) 2 u_\varepsilon(\check{z}_\varepsilon) \varphi_{N_\varepsilon-1}(u_\varepsilon(\check{z}_\varepsilon)^2) \quad (3.55)$$

as $\varepsilon \rightarrow 0$, using (1.1), (3.3), (3.27) and (3.53). Besides, we have that

$$u_\varepsilon(y_\varepsilon) \rightarrow +\infty \quad (3.56)$$

as $\varepsilon \rightarrow 0$. Let $\nu_\varepsilon > 0$ be given by

$$\nu_\varepsilon^2 |\Delta u_\varepsilon(y_\varepsilon)| u_\varepsilon(y_\varepsilon) = 1.$$

Then, using also (3.54), we have that

$$\lim_{\varepsilon \rightarrow 0} \frac{|y_\varepsilon - x_\varepsilon|}{\nu_\varepsilon} = +\infty, \quad (3.57)$$

and, in view of Step 3.5, that

$$\lim_{\varepsilon \rightarrow 0} \frac{|y_\varepsilon - x_\varepsilon|}{\mu_\varepsilon} = +\infty. \quad (3.58)$$

For $R > 0$, we set $\Omega_{R,\varepsilon} = B_{y_\varepsilon}(R\nu_\varepsilon) \cap \Omega$ and $\tilde{\Omega}_{R,\varepsilon} = (\Omega_{R,\varepsilon} - y_\varepsilon)/\nu_\varepsilon$. Up to harmless rotations and since Ω is smooth, we may assume that there exists $B \in [0, +\infty]$ such

that $\tilde{\Omega}_{R,0} \rightarrow (-\infty, B) \times \mathbb{R}$ as $R \rightarrow +\infty$, where $\tilde{\Omega}_{R,\varepsilon} \rightarrow \tilde{\Omega}_{R,0}$ as $\varepsilon \rightarrow 0$. In this proof, for $z \in \tilde{\Omega}_{R,\varepsilon}$, we write $z_\varepsilon = y_\varepsilon + \nu_\varepsilon z \in \Omega_{R,\varepsilon}$. Let \tilde{u}_ε be given by

$$\tilde{u}_\varepsilon(z) = u_\varepsilon(y_\varepsilon) (u_\varepsilon(z_\varepsilon) - u_\varepsilon(y_\varepsilon)), \quad (3.59)$$

so that we get

$$(\Delta \tilde{u}_\varepsilon)(z) = \frac{(\Delta u_\varepsilon)(z_\varepsilon)}{(\Delta u_\varepsilon)(y_\varepsilon)} = \frac{\Psi'_{N_\varepsilon}(u_\varepsilon(z_\varepsilon))}{\Psi'_{N_\varepsilon}(u_\varepsilon(y_\varepsilon))}. \quad (3.60)$$

First, we prove that for all $R > 0$, there exists $C_R > 0$ such that

$$|\Delta \tilde{u}_\varepsilon| \leq C_R \text{ in } \tilde{\Omega}_{R,\varepsilon}, \quad (3.61)$$

for all $0 < \varepsilon \ll 1$. Otherwise, by (3.60), assume by contradiction that there exists $z_\varepsilon \in \Omega_{R,\varepsilon}$ such that

$$|\Psi'_{N_\varepsilon}(u_\varepsilon(z_\varepsilon))| \gg \Psi'_{N_\varepsilon}(u_\varepsilon(y_\varepsilon)) \quad (3.62)$$

as $\varepsilon \rightarrow 0$. If, still by contradiction, $z_\varepsilon \notin \Omega_\varepsilon$, we have that $u_\varepsilon(z_\varepsilon) < u_\varepsilon(y_\varepsilon)$, that

$$\varphi_{N_\varepsilon-1}(u_\varepsilon(z_\varepsilon)^2) < \varphi_{N_\varepsilon-1}(u_\varepsilon(y_\varepsilon)^2),$$

by definition of Ω_ε and since $\varphi_N/(1 + \cdot)$ increases in $[0, +\infty)$, and then that

$$|\Psi'_{N_\varepsilon}(u_\varepsilon(z_\varepsilon))| \lesssim u_\varepsilon(z_\varepsilon) (1 + u_\varepsilon(z_\varepsilon)^2 + \varphi_{N_\varepsilon-1}(u_\varepsilon(z_\varepsilon)^2)) \lesssim \Psi'_{N_\varepsilon}(u_\varepsilon(y_\varepsilon)),$$

using (1.1), (3.3), (3.27), (3.55) and $y_\varepsilon \in \Omega_\varepsilon$ again. This contradicts (3.62) and then it must be the case that $z_\varepsilon \in \Omega_\varepsilon$. Thus, since y_ε is a maximizer on Ω_ε in (3.54), we get from (3.57) and (3.62) that $u_\varepsilon(z_\varepsilon) \ll u_\varepsilon(y_\varepsilon)$. But this is not possible by (3.55) and (3.62), which proves (3.61). Now we prove that, for all $R > 0$,

$$\limsup_{\varepsilon \rightarrow 0} \sup_{z \in \tilde{\Omega}_{R,\varepsilon}} \tilde{u}_\varepsilon(z) \leq 0. \quad (3.63)$$

Until the end of this proof, we set $\tilde{\gamma}_\varepsilon := u_\varepsilon(y_\varepsilon)$. If (3.63) does not hold true, since $\tilde{u}_\varepsilon(0) = 0$ and by continuity, we may assume that there exist $z_\varepsilon \in \Omega_{R,\varepsilon}$ such that

$$\beta_\varepsilon := [\tilde{\gamma}_\varepsilon (u_\varepsilon(z_\varepsilon) - \tilde{\gamma}_\varepsilon)] \rightarrow \beta_0 \in (0, +\infty) \quad (3.64)$$

as $\varepsilon \rightarrow 0$. Since $u_\varepsilon(z_\varepsilon) > u_\varepsilon(y_\varepsilon)$ for $0 < \varepsilon \ll 1$ by (3.64), we have that $z_\varepsilon \in \Omega_\varepsilon$. Moreover, since y_ε is maximizing in (3.54), we then get from (3.55), (3.56) and (3.57) that

$$\varphi_{N_\varepsilon-1}(u_\varepsilon(z_\varepsilon)^2) \leq (1 + o(1)) \varphi_{N_\varepsilon-1}(\tilde{\gamma}_\varepsilon^2).$$

Independently, since φ_N is convex, we get that

$$\begin{aligned} \varphi_{N_\varepsilon-1}(u_\varepsilon(z_\varepsilon)^2) &\geq \varphi_{N_\varepsilon-1}(\tilde{\gamma}_\varepsilon^2) + \varphi'_{N_\varepsilon-1}(\tilde{\gamma}_\varepsilon^2) (u_\varepsilon(z_\varepsilon)^2 - \tilde{\gamma}_\varepsilon^2), \\ &\geq (1 + 2\beta_0(1 + o(1))) \varphi_{N_\varepsilon-1}(\tilde{\gamma}_\varepsilon^2), \end{aligned} \quad (3.65)$$

using (3.64) and $\varphi'_N(t) \geq \varphi_N(t)$ for $t \geq 0$. But (3.64)-(3.65) cannot hold true simultaneously, which proves (3.63). As in [9, p.231], $\tilde{u}_\varepsilon(0) = 0$, $u_\varepsilon = 0$ on $\partial\Omega$, (3.61) and (3.63) imply that

$$\lim_{\varepsilon \rightarrow 0} \frac{d(y_\varepsilon, \partial\Omega)}{\nu_\varepsilon} = +\infty. \quad (3.66)$$

Moreover, by standard elliptic theory, $\tilde{u}_\varepsilon(0) = 0$, (3.61), (3.63) and (3.66) give that

$$\tilde{u}_\varepsilon \rightarrow u_0 \text{ in } C^1_{loc}(\mathbb{R}^2) \quad (3.67)$$

as $\varepsilon \rightarrow 0$, for some $u_0 \in C^1(\mathbb{R}^2)$. Given $R > 0$, we prove now that

$$\liminf_{\varepsilon \rightarrow 0} \inf_{z \in \tilde{\Omega}_{R,\varepsilon}} (\Delta \tilde{u}_\varepsilon)(z) > 0. \quad (3.68)$$

Using (3.27), (3.56) and (3.67), we have that

$$\Psi'_{N_\varepsilon}(u_\varepsilon) = 2\tilde{\gamma}_\varepsilon \varphi_{N_\varepsilon-1}(u_\varepsilon^2)(1 + o(1)) + o(\tilde{\gamma}_\varepsilon^3),$$

uniformly in $\Omega_{R,\varepsilon}$. Then, coming back to (3.60), using (3.55) and $y_\varepsilon \in \Omega_\varepsilon$, we get that

$$(\Delta \tilde{u}_\varepsilon)(z) = (1 + o(1)) \frac{\varphi_{N_\varepsilon-1}(u_\varepsilon(z_\varepsilon)^2)}{\varphi_{N_\varepsilon-1}(\tilde{\gamma}_\varepsilon^2)} + o(1),$$

uniformly in $z \in \tilde{\Omega}_{R,\varepsilon}$. Now, we write (3.48) with $\Gamma = \tilde{\gamma}_\varepsilon^2$ and $T = u_\varepsilon^2$, where u_ε stands for $u_\varepsilon(z_\varepsilon)$ here and below. Then, in order to conclude the proof of (3.68), using also (3.36), it is sufficient to check that there exists $\eta_R < 1$ such that

$$I_\varepsilon := \frac{\exp(u_\varepsilon^2)}{\varphi_{\tilde{N}_\varepsilon}(\tilde{\gamma}_\varepsilon^2) \exp(-(\tilde{\gamma}_\varepsilon^2 - u_\varepsilon^2))} \int_{u_\varepsilon^2}^{\tilde{\gamma}_\varepsilon^2} \exp(-s) \frac{s^{\tilde{N}_\varepsilon}}{\tilde{N}_\varepsilon!} ds = \frac{\int_{u_\varepsilon^2}^{\tilde{\gamma}_\varepsilon^2} \exp(-s) \frac{s^{\tilde{N}_\varepsilon}}{\tilde{N}_\varepsilon!} ds}{\int_0^{\tilde{\gamma}_\varepsilon^2} \exp(-s) \frac{s^{\tilde{N}_\varepsilon}}{\tilde{N}_\varepsilon!} ds}, \quad (3.69)$$

$$\leq \eta_R,$$

for all $0 < \varepsilon \ll 1$, uniformly in $\Omega_{R,\varepsilon}$, where $\tilde{N}_\varepsilon = N_\varepsilon - 1$. If $u_\varepsilon \geq \tilde{\gamma}_\varepsilon$, the last inequality in (3.69) is obvious. If now $u_\varepsilon < \tilde{\gamma}_\varepsilon$, we write

$$\begin{aligned} I_\varepsilon &= \frac{\int_{u_\varepsilon^2 - \tilde{\gamma}_\varepsilon^2}^0 \exp(-t) \left(1 + \frac{t}{\tilde{\gamma}_\varepsilon^2}\right)^{\tilde{N}_\varepsilon} dt}{\int_{-\tilde{\gamma}_\varepsilon^2}^0 \exp(-t) \left(1 + \frac{t}{\tilde{\gamma}_\varepsilon^2}\right)^{\tilde{N}_\varepsilon} dt} \\ &\leq \frac{\int_{u_\varepsilon^2 - \tilde{\gamma}_\varepsilon^2}^0 \exp\left(t \left(\frac{\tilde{N}_\varepsilon}{\tilde{\gamma}_\varepsilon^2} - 1\right) + O\left(\frac{\tilde{N}_\varepsilon t^2}{\tilde{\gamma}_\varepsilon^4}\right)\right) dt}{\int_{2(u_\varepsilon^2 - \tilde{\gamma}_\varepsilon^2)}^0 \exp\left(t \left(\frac{\tilde{N}_\varepsilon}{\tilde{\gamma}_\varepsilon^2} - 1\right) + O\left(\frac{\tilde{N}_\varepsilon t^2}{\tilde{\gamma}_\varepsilon^4}\right)\right) dt} \\ &\leq \eta_R \end{aligned}$$

using (3.67), where I_ε is as in (3.69). We get the last inequality using (3.53) and $y_\varepsilon \in \Omega_\varepsilon$: (3.69) and then (3.68) are proved in any case. Let $R > 0$ be given. By (3.57), (3.58) and (3.68), we clearly get that

$$\int_{\Omega \setminus B_{x_\varepsilon}(R\mu_\varepsilon)} (\Delta u_\varepsilon(y))^+ u_\varepsilon dy \geq \int_{B_{y_\varepsilon}(\nu_\varepsilon)} \Delta u_\varepsilon(y) u_\varepsilon(y) dy$$

for all ε small enough. Using now (3.56) and (3.67), we write that $u_\varepsilon = \tilde{\gamma}_\varepsilon(1 + o(1))$ uniformly in $B_{y_\varepsilon}(\nu_\varepsilon)$, so that

$$\liminf_{\varepsilon \rightarrow 0} \int_{B_{y_\varepsilon}(\nu_\varepsilon)} \Delta u_\varepsilon(y) u_\varepsilon(y) dy = \liminf_{\varepsilon \rightarrow 0} \int_{B_0(1)} \Delta \tilde{u}_\varepsilon(z) (1 + o(1)) dz > 0,$$

by (3.68). Since this last term is independent of $R > 0$, this contradicts (3.50), which concludes the proof of (3.51). \square

Proof of Step 3.6, Formula (3.52). Remember that we assume that **(Case 1)** holds true. Assume then by contradiction that there exists $(y_\varepsilon)_\varepsilon$ such that $y_\varepsilon \in \Omega_\varepsilon$ and

$$\max_{y \in \Omega_\varepsilon} |y - x_\varepsilon| |\nabla u_\varepsilon(y)| u_\varepsilon(y) = |y_\varepsilon - x_\varepsilon| |\nabla u_\varepsilon(y_\varepsilon)| u_\varepsilon(y_\varepsilon) := C_\varepsilon \rightarrow +\infty \quad (3.70)$$

as $\varepsilon \rightarrow 0$. Then, by (3.53), (3.56) holds true. Let $\nu_\varepsilon > 0$ be given by

$$\nu_\varepsilon = \min(|x_\varepsilon - y_\varepsilon|, d(y_\varepsilon, \partial\Omega)). \quad (3.71)$$

For all $R > 1$ and all ε , we let $\Omega_{R,\varepsilon}$ and $\tilde{\Omega}_{R,\varepsilon}$ be given by the formulas above (3.59). Let w_ε be given by

$$w_\varepsilon(z) = u_\varepsilon(y_\varepsilon + \nu_\varepsilon z).$$

Since $\|u_\varepsilon\|_{H_0^1}^2 \leq 4\pi$, we get from Moser's inequality that $\int_\Omega \exp(u_\varepsilon^2) dy = O(1)$ and then that, for all given $p \geq 1$,

$$\|\nu_\varepsilon^{2/p} w_\varepsilon\|_{L^p(\tilde{\Omega}_{R,\varepsilon})} = O(1) \quad (3.72)$$

for all ε . Set $\tilde{x}_\varepsilon = \frac{x_\varepsilon - y_\varepsilon}{\nu_\varepsilon}$. Now, for any given $R > 1$ and all sequence $(z_\varepsilon)_\varepsilon$ such that $z_\varepsilon \in \Omega_{R,\varepsilon} \setminus \{x_\varepsilon\}$ (i.e. $\tilde{z}_\varepsilon := (z_\varepsilon - y_\varepsilon)/\nu_\varepsilon \in \tilde{\Omega}_{R,\varepsilon} \setminus \{\tilde{x}_\varepsilon\}$), we get that

$$|\Delta w_\varepsilon(\tilde{z}_\varepsilon)| = \nu_\varepsilon^2 |\Delta u_\varepsilon(z_\varepsilon)| \lesssim \begin{cases} \frac{1}{u_\varepsilon(z_\varepsilon) |\tilde{z}_\varepsilon - \tilde{x}_\varepsilon|^2} & \text{if } z_\varepsilon \in \Omega_\varepsilon, \\ \lambda_\varepsilon \nu_\varepsilon^2 |\Psi'_{N_\varepsilon}(u_\varepsilon(z_\varepsilon))| = O(\lambda_\varepsilon \nu_\varepsilon^2 (1 + u_\varepsilon(z_\varepsilon)^3)) & \text{if } z_\varepsilon \notin \Omega_\varepsilon, \end{cases}$$

using (3.51) for the first line, and (3.27) for the second one. Then, using either (3.53) or (3.38) with (3.72), we get that

$$\|\Delta w_\varepsilon\|_{L^p(\tilde{\Omega}_{R,\varepsilon} \setminus B_{\tilde{x}_\varepsilon}(1/R))} \rightarrow 0 \quad (3.73)$$

as $\varepsilon \rightarrow 0$. Independently, since $\|u_\varepsilon\|_{H_0^1} = O(1)$, we easily get that

$$\int_{\tilde{\Omega}_{R,\varepsilon}} |\nabla w_\varepsilon|^2 dz = O(1). \quad (3.74)$$

Observe that $|\tilde{x}_\varepsilon| \geq 1$. Now, we claim that up to a subsequence,

$$\nu_\varepsilon \rightarrow 0 \text{ and } \frac{d(y_\varepsilon, \partial\Omega)}{|x_\varepsilon - y_\varepsilon|} \rightarrow +\infty \quad (3.75)$$

as $\varepsilon \rightarrow 0$. In particular, by (3.71), this implies that $\nu_\varepsilon = |x_\varepsilon - y_\varepsilon|$. Now we prove (3.75). Indeed, if we assume by contradiction that (3.75) does not hold, for all $R \gg 1$ sufficiently large, we get that the $(w_\varepsilon/u_\varepsilon(y_\varepsilon))$'s converge locally out of $B_{\tilde{x}_\varepsilon}(1/2)$ to some C^1 function which is 1 at 0 and 0 on the non-empty and smooth boundary of $\lim_{R \rightarrow +\infty} \lim_{\varepsilon \rightarrow 0} \tilde{\Omega}_{R,\varepsilon}$ (maybe after a harmless rotation). We use here the Harnack inequality and elliptic theory with (3.56), (3.73) (with $p > 2$) and (3.74), since $u_\varepsilon = 0$ in $\partial\Omega$. This clearly contradicts (3.74) and (3.75) is proved. Up to a subsequence, we may now assume that

$$\tilde{x}_\varepsilon \rightarrow \tilde{x}, \quad |\tilde{x}| = 1, \quad (3.76)$$

as $\varepsilon \rightarrow 0$. By (3.56), (3.73), (3.74), and similar arguments including again Harnack's principle, we get that

$$\frac{w_\varepsilon}{u_\varepsilon(y_\varepsilon)} \rightarrow 1 \text{ in } C_{loc}^1(\mathbb{R}^2 \setminus \{\tilde{x}\}), \quad (3.77)$$

using also (3.75). By (3.72) and (3.77), we get that for all $p \geq 1$

$$\nu_\varepsilon^{2/p} u_\varepsilon(y_\varepsilon) = O(1) \quad (3.78)$$

as $\varepsilon \rightarrow 0$. Let now \tilde{w}_ε be given by $\tilde{w}_\varepsilon = \frac{w_\varepsilon - w_\varepsilon(0)}{\nu_\varepsilon |\nabla u_\varepsilon(y_\varepsilon)|}$, so that $|\nabla \tilde{w}_\varepsilon(0)| = 1$. For any given $R > 1$ and all sequence $(z_\varepsilon)_\varepsilon$ such that $\tilde{z}_\varepsilon := (z_\varepsilon - y_\varepsilon)/\nu_\varepsilon \in \tilde{\Omega}_{R,\varepsilon} \setminus B_{\tilde{x}}(1/R)$, we get that

$$|\Delta \tilde{w}_\varepsilon(\tilde{z}_\varepsilon)| = \frac{u_\varepsilon(y_\varepsilon)}{C_\varepsilon} |\Delta w_\varepsilon(\tilde{z}_\varepsilon)| \lesssim \begin{cases} \frac{1}{C_\varepsilon |\tilde{z}_\varepsilon - \tilde{x}_\varepsilon|^2} & \text{if } z_\varepsilon \in \Omega_\varepsilon, \\ \frac{\lambda_\varepsilon}{C_\varepsilon} \nu_\varepsilon^2 u_\varepsilon(y_\varepsilon)^4 & \text{if } z_\varepsilon \notin \Omega_\varepsilon, \end{cases}$$

for all ε , using (3.51), (3.70) and (3.77). Then, since $\lambda_\varepsilon = o(1)$, we get from (3.70), (3.75) and (3.78) (with $p \geq 4$) that

$$\Delta \tilde{w}_\varepsilon \rightarrow 0 \text{ in } L_{loc}^\infty(\mathbb{R}^2 \setminus \{\tilde{x}\}) \quad (3.79)$$

as $\varepsilon \rightarrow 0$. By (3.70), (3.76) and (3.77), given $R > 1$ and $\tilde{z}_\varepsilon \in \tilde{\Omega}_{R,\varepsilon} \setminus B_{\tilde{x}}(1/R)$, we get that

$$|\nabla \tilde{w}_\varepsilon(\tilde{z}_\varepsilon)| = \frac{|\nabla u_\varepsilon(z_\varepsilon)|}{|\nabla u_\varepsilon(y_\varepsilon)|} \leq \frac{u_\varepsilon(y_\varepsilon)}{u_\varepsilon(z_\varepsilon)} \frac{1}{|\tilde{x}_\varepsilon - \tilde{z}_\varepsilon|} \leq \frac{1 + o(1)}{|\tilde{x}_\varepsilon - \tilde{z}_\varepsilon|} \quad (3.80)$$

for all $0 < \varepsilon \ll 1$. Then, by (3.79), (3.80) and since $\tilde{w}_\varepsilon(0) = 0$, there exists a harmonic function \mathcal{H} in $\mathbb{R}^2 \setminus \{\tilde{x}\}$ such that $\lim_{\varepsilon \rightarrow 0} \tilde{w}_\varepsilon = \mathcal{H}$ in $C_{loc}^1(\mathbb{R}^2 \setminus \{\tilde{x}\})$. Now, for all given $\beta > 0$, integrating by parts, we get that

$$\begin{aligned} \int_{\partial B_{x_\varepsilon}(\beta \nu_\varepsilon)} u_\varepsilon \partial_\nu u_\varepsilon d\sigma &= C_\varepsilon \left(\int_{\partial B_{\tilde{x}}(\beta)} \partial_\nu \mathcal{H} d\sigma + o(1) \right), \\ &\leq \int_\Omega |\nabla u_\varepsilon|^2 dy + \int_\Omega u_\varepsilon (\Delta u_\varepsilon)^+ dy = O(1), \end{aligned}$$

using (3.70) and (3.77), as $\varepsilon \rightarrow 0$. Since $C_\varepsilon \rightarrow +\infty$, this implies that $\int_{\partial B_{\tilde{x}}(\beta)} \partial_\nu \mathcal{H} d\sigma = 0$. Then, also by (3.80), β being arbitrary, \mathcal{H} is bounded around \tilde{x} and then the singularity at \tilde{x} is removable. By the Liouville theorem, \mathcal{H} is constant in \mathbb{R}^2 , which is not possible since $|\nabla \tilde{w}_\varepsilon(0)| = |\nabla \mathcal{H}(0)| = 1$. This concludes the proof of (3.52). \square

Remark 3.3. Note that we do not assume that the continuous function Ψ'_{N_ε} is positive and increasing in $[0, +\infty)$. Then, standard moving plane techniques [1, 5, 13, 15] do not apply. We use in the proof below the variational characterization (3.6) of the u_ε 's to get that $\tilde{x} \in K_\Omega$, K_Ω as in (1.9), and that, in particular, $\tilde{x} \notin \partial\Omega$ in (3.12).

Let B_ε be the radial solution around x_ε of

$$\begin{cases} \Delta B_\varepsilon = \frac{\lambda_\varepsilon}{2} \Psi'_{N_\varepsilon}(B_\varepsilon), \\ B_\varepsilon(x_\varepsilon) = \gamma_\varepsilon, \end{cases} \quad (3.81)$$

where γ_ε is still given by (3.13). Let \bar{u}_ε be given by

$$\bar{u}_\varepsilon(z) = \frac{1}{2\pi|x_\varepsilon - z|} \int_{\partial B_{x_\varepsilon}(|x_\varepsilon - z|)} u_\varepsilon d\sigma, \quad (3.82)$$

for all $z \neq x_\varepsilon$ and $\bar{u}_\varepsilon(x_\varepsilon) = u_\varepsilon(x_\varepsilon) = \gamma_\varepsilon$. Let $\varepsilon_0 \in (\sqrt{1/e}, 1)$ be given. Let $\rho_\varepsilon > 0$ be given by

$$t_\varepsilon(\rho_\varepsilon) = (1 - \varepsilon_0)\gamma_\varepsilon^2. \quad (3.83)$$

By (3.44), we have that

$$\rho_\varepsilon^2 = \exp(-(\varepsilon_0 + o(1))\gamma_\varepsilon^2). \quad (3.84)$$

Let r_ε be given by

$$r_\varepsilon = \sup \left\{ r \in (0, \rho_\varepsilon] \text{ s.t. } |\bar{u}_\varepsilon - B_\varepsilon| \leq \frac{1}{\gamma_\varepsilon} \text{ in } B_{x_\varepsilon}(r) \right\}. \quad (3.85)$$

Observe that $r_\varepsilon \gg \mu_\varepsilon$ by Step 3.5 and Appendix A. Then, we state the following key result.

Step 3.7. *We have that*

$$\bar{u}_\varepsilon(r_\varepsilon) = B_\varepsilon(r_\varepsilon) + o\left(\frac{1}{\gamma_\varepsilon}\right) \quad (3.86)$$

and then that $r_\varepsilon = \rho_\varepsilon$ for all $0 < \varepsilon \ll 1$. Moreover, there exists $C > 0$ such that

$$|\nabla(B_\varepsilon - u_\varepsilon)| \leq \frac{C}{\rho_\varepsilon \gamma_\varepsilon} \text{ in } B_{x_\varepsilon}(\rho_\varepsilon) \quad (3.87)$$

for all $0 < \varepsilon \ll 1$, where $(x_\varepsilon)_\varepsilon$ is as in (3.13), B_ε as in (3.81), \bar{u}_ε as in (3.82), ρ_ε as in (3.83) and r_ε as in (3.85).

Since $B_\varepsilon(x_\varepsilon) = u_\varepsilon(x_\varepsilon) = \gamma_\varepsilon$, (3.87) obviously implies that

$$|B_\varepsilon - u_\varepsilon| \leq C \frac{|\cdot - x_\varepsilon|}{\rho_\varepsilon \gamma_\varepsilon} \text{ in } B_{x_\varepsilon}(\rho_\varepsilon) \quad (3.88)$$

for all $0 < \varepsilon \ll 1$. Then, combined with Appendix A, Step 3.7 provides pointwise estimates of the u_ε 's in $B_{x_\varepsilon}(\rho_\varepsilon)$.

Proof of Step 3.7. The proof of Step 3.7 follows the lines of [10, Section 3]. We only recall here the argument in the more delicate (**Case 1**). Let v_ε be given by

$$u_\varepsilon = B_\varepsilon + v_\varepsilon. \quad (3.89)$$

By Appendix A, we have that B_ε is well defined, radially decreasing in $B_{x_\varepsilon}(\rho_\varepsilon)$, and that

$$B_\varepsilon = \gamma_\varepsilon - \frac{t_\varepsilon}{\gamma_\varepsilon} + o\left(\frac{t_\varepsilon}{\gamma_\varepsilon}\right) \quad (3.90)$$

uniformly in $B_{x_\varepsilon}(\rho_\varepsilon)$ as $\varepsilon \rightarrow 0$, where t_ε is given by (3.41). Then, we get first from (3.83) and (3.90) the following lower bound:

$$\min_{B_{x_\varepsilon}(r_\varepsilon)} B_\varepsilon \geq \gamma_\varepsilon(\varepsilon_0 + o(1)).$$

Let us introduce now an intermediate radius \tilde{r}_ε given by

$$\tilde{r}_\varepsilon = \sup \left\{ r \in (0, r_\varepsilon] \text{ s.t. } \frac{\varepsilon_0 \gamma_\varepsilon}{2} |x_\varepsilon - \cdot| |\nabla u_\varepsilon| \leq C \text{ in } B_{x_\varepsilon}(r) \right\},$$

for C as in (3.52). We prove now that $\tilde{r}_\varepsilon = r_\varepsilon$ for all $\varepsilon \ll 1$. Indeed, by Wirtinger's inequality on $\partial B_0(r)$, $0 < r \leq \tilde{r}_\varepsilon$, we have that

$$|\bar{u}_\varepsilon - u_\varepsilon| \leq \frac{2C}{\varepsilon_0 \gamma_\varepsilon} \pi,$$

so that, by (3.85),

$$|v_\varepsilon| = |B_\varepsilon - u_\varepsilon| \leq \left(\frac{2\pi C}{\varepsilon_0} + 1 \right) \gamma_\varepsilon^{-1}$$

in $B_{x_\varepsilon}(\tilde{r}_\varepsilon)$. Then, we get a lower bound on u_ε as well, namely

$$\min_{B_{x_\varepsilon}(\tilde{r}_\varepsilon)} u_\varepsilon \geq \gamma_\varepsilon(\varepsilon_0 + o(1)), \quad (3.91)$$

so that, by (3.52), the condition in the definition of \tilde{r}_ε never saturates: $\tilde{r}_\varepsilon = r_\varepsilon$ for all $\varepsilon \ll 1$. Observe for this that (3.91) combined with (3.53) (see also the paragraph below (3.53)) and with our assumption $e\varepsilon_0^2 > 1$ implies $B_{x_\varepsilon}(\tilde{r}_\varepsilon) \subset \Omega_\varepsilon$. Observe in

particular that (3.31) provides $\gamma_\varepsilon^2 \geq N_\varepsilon(1 + o(1))$. Summarizing what we have just obtained in $B_{x_\varepsilon}(r_\varepsilon)$, we may write

$$\|x_\varepsilon - \cdot\| \|\nabla u_\varepsilon\|_{L^\infty(B_{x_\varepsilon}(r_\varepsilon))} = O\left(\frac{1}{\gamma_\varepsilon}\right),$$

and

$$\|v_\varepsilon\|_{L^\infty(B_{x_\varepsilon}(r_\varepsilon))} = O\left(\frac{1}{\gamma_\varepsilon}\right). \quad (3.92)$$

We also have that

$$B_\varepsilon \leq \gamma_\varepsilon \quad (3.93)$$

in $B_{x_\varepsilon}(r_\varepsilon)$. By combining (3.26) and (3.81), (3.92) allows to linearize (3.81) to control v_ε . More precisely, (1.5) and Lemma 3.2 permit to compute the variations of Ψ'_{N_ε} in (3.27), even if g is only C^1 in (1.1), so that Ψ'_{N_ε} is only continuous. Namely, we get from a) and c) in (1.5) and from Lemma 3.2 (for $\gamma = B_\varepsilon$) that

$$|\Delta v_\varepsilon| = |\Delta(u_\varepsilon - B_\varepsilon)| \leq C' \lambda_\varepsilon \gamma_\varepsilon^2 \varphi_{N_\varepsilon-2}(B_\varepsilon^2) \left[|v_\varepsilon| + o\left(\frac{1}{\gamma_\varepsilon}\right)\right] \text{ in } B_{x_\varepsilon}(r_\varepsilon)$$

for all ε , using (3.48), (3.91)-(3.93) and some computations. Then, (3.90) gives

$$|\Delta v_\varepsilon| \leq C'' \frac{\exp\left(-2t_\varepsilon(1 + o(1)) + \frac{t_\varepsilon^2}{\gamma_\varepsilon^2}\right)}{\mu_\varepsilon^2} \times \left[|v_\varepsilon| + o\left(\frac{1}{\gamma_\varepsilon}\right)\right] \text{ in } B_{x_\varepsilon}(r_\varepsilon) \quad (3.94)$$

using (3.30), (3.40) and (3.45). Starting now from (3.92)-(3.94), we can compute and argue as in [10, Section 3] in order to get (3.86)-(3.87). \square

Conclusion of the proof of Lemma 3.3. Let $\varepsilon'_0 \in (\varepsilon_0, 1)$ be fixed and let $\rho'_\varepsilon > 0$ be given by

$$t_\varepsilon(\rho'_\varepsilon) = (1 - \varepsilon'_0)\gamma_\varepsilon^2, \quad (3.95)$$

so that, by (3.44),

$$(\rho'_\varepsilon)^2 = \exp(-\varepsilon'_0(1 + o(1))\gamma_\varepsilon^2). \quad (3.96)$$

In order to conclude the proof of Lemma 3.3, by Steps 3.1-3.7, it remains to prove (2.4), (3.10)-(3.12), that

$$\left|u_\varepsilon(y) - \frac{4\pi G_{x_\varepsilon}(y)}{\gamma_\varepsilon}\right| = o\left(\frac{G_{x_\varepsilon}(y)}{\gamma_\varepsilon}\right) \quad (3.97)$$

uniformly in $B_{x_\varepsilon}(\rho'_\varepsilon)^c$, that

$$u_\varepsilon = \gamma_\varepsilon - \frac{t_\varepsilon}{\gamma_\varepsilon} + \frac{S_{0,\varepsilon}}{\gamma_\varepsilon^3} + \frac{S_{1,\varepsilon}}{\gamma_\varepsilon^5} + (A(\gamma_\varepsilon) - 2\xi_\varepsilon) \frac{S_{2,\varepsilon}}{\gamma_\varepsilon} + o\left(\frac{t_\varepsilon}{\gamma_\varepsilon}\right) \quad (3.98)$$

uniformly in $B_{x_\varepsilon}(\rho'_\varepsilon)$, where the $S_{i,\varepsilon}$'s are as in (A.5), and that

$$\begin{aligned} u_\varepsilon(y) &= G_{x_\varepsilon}(y) \left(\frac{4\pi}{\gamma_\varepsilon} + \sum_{i=0}^1 \frac{A_i}{\gamma_\varepsilon^{3+2i}} + \frac{A_2(A(\gamma_\varepsilon) - 2\xi_\varepsilon)}{\gamma_\varepsilon} \right) \\ &\quad + \frac{4B(\gamma_\varepsilon)}{\gamma_\varepsilon^2 \exp(1 + \mathcal{H}_{x_\varepsilon}(x_\varepsilon))} \int_{\Omega} G_y(x) F(4\pi G_{x_\varepsilon}(x)) dx \\ &\quad + o\left(\frac{\zeta_\varepsilon}{\gamma_\varepsilon} G_{x_\varepsilon}(y) + \frac{|B(\gamma_\varepsilon)|}{\gamma_\varepsilon^2}\right), \end{aligned} \quad (3.99)$$

uniformly in $B_{x_\varepsilon}(\rho'_\varepsilon)^c$, as $\varepsilon \rightarrow 0$, where F and $B(\gamma_\varepsilon)$ are given in (1.6), where the A_i 's are as in (A.3), and where ζ_ε is given in (A.8).

• (1) In this first point, we aim to get pointwise estimates of the u_ε 's out of $B_{x_\varepsilon}(\rho'_\varepsilon)$. Let G be the Green's function in (1.8). It is known that (see for instance [10, Appendix B]) there exists $C > 0$ such that

$$|\nabla_y G_x(y)| \leq \frac{C}{|x-y|} \text{ and } 0 < G_x(y) \leq \frac{1}{2\pi} \log \frac{C}{|x-y|} \quad (3.100)$$

for all $x, y \in \Omega$, $x \neq y$. By (3.87) and since $\|u_\varepsilon\|_{H_0^1}^2 \leq 4\pi$, it is possible to prove (see for instance the proof of [10, Claim 4.6]) that, given $p < 1/\varepsilon'_0$,

$$\|\exp(u_\varepsilon^2)\|_{L^p(B_{x_\varepsilon}(\rho'_\varepsilon/2)^c)} = O(1) \quad (3.101)$$

for all ε , where $B_{x_\varepsilon}(\rho'_\varepsilon/2)^c = \Omega \setminus B_{x_\varepsilon}(\rho'_\varepsilon/2)$. In the sequel, $p' > 1$ is chosen such that

$$\frac{1}{p} + \frac{1}{p'} < 1.$$

Let now $(z_\varepsilon)_\varepsilon$ be any sequence of points in $B_{x_\varepsilon}(\rho'_\varepsilon)^c$. By the Green's representation formula and (3.26), we can write that

$$u_\varepsilon(z_\varepsilon) = \frac{\lambda_\varepsilon}{2} \int_{\Omega} G_{z_\varepsilon}(y) \Psi'_{N_\varepsilon}(u_\varepsilon(y)) dy.$$

By (3.100), we have that there exists $C > 0$ such that

$$|G_{z_\varepsilon}(x_\varepsilon) - G_{z_\varepsilon}| \leq C \frac{|x_\varepsilon - \cdot|}{\rho'_\varepsilon} \quad (3.102)$$

in $B_{x_\varepsilon}(\rho'_\varepsilon/2)$ for all ε . Set $\bar{t}_\varepsilon = 1 + t_\varepsilon$. By (3.44) and (3.84), we have that

$$\frac{|\cdot - x_\varepsilon|}{\gamma_\varepsilon \rho_\varepsilon} = o\left(\frac{\bar{t}_\varepsilon}{\gamma_\varepsilon^5}\right) \text{ in } \tilde{\Omega}_\varepsilon := \{y \text{ s.t. } t_\varepsilon(y) \leq \gamma_\varepsilon\}$$

as $\varepsilon \rightarrow 0$, and then, by (3.88), (A.9) holds true for v_ε as in (3.89). Independently, using (3.29), (3.40), (3.88) and (A.3) with (A.7), we clearly get that there exists $C > 0$ such that

$$\lambda_\varepsilon |\Psi'_{N_\varepsilon}(u_\varepsilon)| \leq C \frac{\exp\left(-2t_\varepsilon + \frac{t_\varepsilon^2}{\gamma_\varepsilon^2}\right)}{\mu_\varepsilon^2 \gamma_\varepsilon} \text{ in } B_{x_\varepsilon}(\rho'_\varepsilon/2) \quad (3.103)$$

for all ε . Then, we get that

$$\begin{aligned} u_\varepsilon(z_\varepsilon) &= G_{z_\varepsilon}(x_\varepsilon) \int_{B_{x_\varepsilon}(\rho'_\varepsilon/2)} \frac{\lambda_\varepsilon \Psi'_{N_\varepsilon}(u_\varepsilon)}{2} dy \\ &+ O\left(\int_{B_{x_\varepsilon}(\rho'_\varepsilon/2)} \frac{\exp\left(-2t_\varepsilon + \frac{t_\varepsilon^2}{\gamma_\varepsilon^2}\right) |\cdot - x_\varepsilon|}{\mu_\varepsilon^2 \gamma_\varepsilon \rho'_\varepsilon} dy\right) + O(\lambda_\varepsilon \|u_\varepsilon\|_{L^{p'}}), \\ &= G_{z_\varepsilon}(x_\varepsilon) \frac{4\pi}{\gamma_\varepsilon} \left(1 + \frac{1}{\gamma_\varepsilon^2} + \frac{A(\gamma_\varepsilon) - 2\xi_\varepsilon}{2} + o(\tilde{\zeta}_\varepsilon)\right) \\ &+ o\left(\frac{1}{\gamma_\varepsilon}\right) + o(\|u_\varepsilon\|_{L^{p'}}), \end{aligned} \quad (3.104)$$

where $\tilde{\zeta}_\varepsilon$ is given by (3.15). We start by focusing on the first equality of (3.104): (3.102) and (3.103) are used to get the first two terms; the last term is obtained from (3.29), (3.100), (3.101) and Hlder's inequality. We focus now on the second

equality of (3.104), resuming the previous one term by term: the first term is easily computed by integrating (A.9) in $\tilde{\Omega}_\varepsilon$ and by plugging the values of the A_i 's from (A.2)-(A.4) on the one hand, and by estimating roughly in $B_{x_\varepsilon}(\rho'_\varepsilon) \setminus \tilde{\Omega}_\varepsilon$ with (3.103) on the other hand; the last term obviously follows from $\lambda_\varepsilon = o(1)$; as to the $o(1/\gamma_\varepsilon)$, we get first $O(\mu_\varepsilon/(\rho'_\varepsilon\gamma_\varepsilon))$ using $\varepsilon_0 > 1/2$, which clearly concludes by (3.95). Using first that $u_\varepsilon \leq \gamma_\varepsilon$ and (3.84) in $B_{x_\varepsilon}(\rho_\varepsilon)$, and then (3.104) with (3.100) in $\Omega \setminus B_{x_\varepsilon}(\rho_\varepsilon)$, we get that

$$\|u_\varepsilon\|_{L^{p'}} = o\left(\frac{1}{\gamma_\varepsilon} + \|u_\varepsilon\|_{L^{p'}}\right) + O\left(\frac{1}{\gamma_\varepsilon}\right).$$

This implies with (3.104) that

$$u_\varepsilon(z_\varepsilon) = \frac{4\pi G_{z_\varepsilon}(x_\varepsilon)}{\gamma_\varepsilon} \left(1 + \frac{1}{\gamma_\varepsilon^2} + \frac{A(\gamma_\varepsilon) - 2\xi_\varepsilon}{2} + o(\tilde{\zeta}_\varepsilon)\right) + o\left(\frac{1}{\gamma_\varepsilon}\right). \quad (3.105)$$

- (2) In this second point, we prove that

$$\lambda_\varepsilon \leq \frac{4 + o(1)}{\gamma_\varepsilon^2 \exp(1 + M)} \quad (3.106)$$

as $\varepsilon \rightarrow 0$, for M as in (1.9). Observe that (3.105) implies that

$$u_\varepsilon = (1 + o(1)) \frac{4\pi G_{x_\varepsilon} + o(1)}{\gamma_\varepsilon}$$

in $\Omega \setminus B_{x_\varepsilon}(\rho_\varepsilon)$. By (1.1) and (3.100), our definition of ρ_ε and the dominated convergence theorem, this implies that

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega \setminus B_{x_\varepsilon}(\rho_\varepsilon)} \Psi_{N_\varepsilon}(u_\varepsilon) dy = |\Omega|(1 + g(0)). \quad (3.107)$$

Independently, (A.7) and (3.88) give that

$$u_\varepsilon = \gamma_\varepsilon - \frac{(1 + o(1))t_\varepsilon}{\gamma_\varepsilon} \quad (3.108)$$

in $B_{x_\varepsilon}(\rho_\varepsilon)$, since $\mu_\varepsilon \ll \rho_\varepsilon$. Then, using (3.30), (3.45), $\varepsilon_0^2 > 1/e$ and resuming the arguments to get (3.55), we have that

$$\Psi_{N_\varepsilon}(u_\varepsilon) = (1 + o(1))\varphi_{N_\varepsilon-1}(u_\varepsilon^2) \text{ and } \Psi'_{N_\varepsilon}(u_\varepsilon) = 2(1 + o(1)) u_\varepsilon \varphi_{N_\varepsilon-1}(u_\varepsilon^2) \quad (3.109)$$

in $B_{x_\varepsilon}(\rho_\varepsilon)$. Independently, observe that, for all $\Gamma, \delta > 0$,

$$\varphi_N(\Gamma) = \delta \exp(\Gamma) \implies \forall T \in [0, \Gamma], \quad \varphi_N(T) \leq \delta \exp(T), \quad (3.110)$$

since $\varphi'_N \geq \varphi_N$ in $[0, +\infty]$. Then we get that

$$\int_{B_{x_\varepsilon}(\rho_\varepsilon)} \Psi_{N_\varepsilon}(u_\varepsilon) dy = \frac{4\pi(1 + o(1))}{\gamma_\varepsilon^2 \lambda_\varepsilon} \quad (3.111)$$

as $\varepsilon \rightarrow 0$, by (3.30), (3.40), (3.108), (3.109), with (3.48) for $|y - x_\varepsilon| \lesssim \mu_\varepsilon$, or with (3.110) and the dominated convergence theorem for $|y - x_\varepsilon| \gg \mu_\varepsilon$. Then, because of (3.6), we get that (3.106) holds true, by combining (3.107), (3.111) with (3.24).

- (3) In this point, we conclude the proof of (3.10), and prove (2.4) and (3.12). For $R > 1$, let $\chi_{\varepsilon,R}$ be given in $\Omega_{\varepsilon,R} := \Omega \setminus B_{x_\varepsilon}(R\mu_\varepsilon)$ by

$$\chi_{\varepsilon,R} = 4\pi \Lambda_{\varepsilon,R} G_{x_\varepsilon},$$

for $\Lambda_{\varepsilon,R} > 0$ to be chosen later such that

$$\chi_{\varepsilon,R} \leq u_\varepsilon \text{ on } \partial B_{x_\varepsilon}(R\mu_\varepsilon). \quad (3.112)$$

Integrating by parts, we can write that

$$\begin{aligned} \int_{\Omega_{\varepsilon,R}} |\nabla u_\varepsilon|^2 dy &= \int_{\Omega_{\varepsilon,R}} |\nabla \chi_{\varepsilon,R}|^2 dy - 2 \int_{\partial B_{x_\varepsilon}(R\mu_\varepsilon)} (\partial_\nu \chi_{\varepsilon,R})(u_\varepsilon - \chi_{\varepsilon,R}) d\sigma \\ &\quad + \int_{\Omega_{\varepsilon,R}} |\nabla(u_\varepsilon - \chi_{\varepsilon,R})|^2 dy, \\ &\geq \int_{\Omega_{\varepsilon,R}} |\nabla \chi_{\varepsilon,R}|^2 dy, \end{aligned} \quad (3.113)$$

where ν is the unit outward normal to the boundary of $B_{x_\varepsilon}(R\mu_\varepsilon)$, using (3.112). Indeed, by [10, Appendix B] for instance, since $d(x_\varepsilon, \partial\Omega) \gg \mu_\varepsilon$ by Step 3.5, we have that

$$\partial_\nu G_{x_\varepsilon} = -\frac{1}{2\pi R\mu_\varepsilon} + O\left(\frac{1}{d(x_\varepsilon, \partial\Omega)}\right) \text{ on } \partial B_{x_\varepsilon}(R\mu_\varepsilon). \quad (3.114)$$

Now, by (3.3), (3.40), (3.42), (3.45), (3.84), in order to have (3.112), we can choose $\Lambda_{\varepsilon,R}$ such that

$$\begin{aligned} \Lambda_{\varepsilon,R} &= \frac{1}{\gamma_\varepsilon} \left(1 - \frac{\log(1+R^2) + o(1)}{\gamma_\varepsilon^2}\right) \times \\ &\quad \left(1 + \frac{\log \frac{\delta_\varepsilon \lambda_\varepsilon \gamma_\varepsilon^2}{4R^2} + \mathcal{H}_{x_\varepsilon}(x_\varepsilon) + o(1)}{\gamma_\varepsilon^2}\right)^{-1}, \end{aligned} \quad (3.115)$$

with $\delta_\varepsilon \in (0, 1]$ as in (3.30). In (3.115), we use

$$|\mathcal{H}_{x_\varepsilon} - \mathcal{H}_{x_\varepsilon}(x_\varepsilon)| = O\left(\frac{\mu_\varepsilon}{d(x_\varepsilon, \partial\Omega)}\right) = o(1)$$

uniformly in $\partial B_{x_\varepsilon}(R\mu_\varepsilon)$, using Step 3.5 and computing as in (3.21). Now, by (1.8), (3.44), (3.84), and (3.114), we compute and get first that

$$\begin{aligned} &\int_{\Omega_{\varepsilon,R}} |\nabla \chi_{\varepsilon,R}|^2 dy \\ &\geq - \int_{\partial B_{x_\varepsilon}(R\mu_\varepsilon)} (\partial_\nu \chi_{\varepsilon,R}) \chi_{\varepsilon,R} d\sigma, \\ &\geq 4\pi \left(1 - \frac{2\log(1+R^2) + o(1)}{\gamma_\varepsilon^2}\right) \left(1 + \frac{\log \frac{\delta_\varepsilon \lambda_\varepsilon \gamma_\varepsilon^2}{4R^2} + \mathcal{H}_{x_\varepsilon}(x_\varepsilon) + o(1)}{\gamma_\varepsilon^2}\right)^{-1}, \end{aligned}$$

using also (3.115). Independently, we compute and get also that

$$\int_{B_{x_\varepsilon}(R\mu_\varepsilon)} |\nabla u_\varepsilon|^2 dy = \frac{4\pi}{\gamma_\varepsilon^2} \left(\log(1+R^2) - \frac{R^2}{1+R^2} + o(1)\right),$$

by (3.42). Thus, since $\|u_\varepsilon\|_{H_0^1}^2 \leq 4\pi$ and by (3.6) and (3.113), we eventually get

$$\frac{\log \delta_\varepsilon \lambda_\varepsilon + \mathcal{H}_{x_\varepsilon}(x_\varepsilon)}{\gamma_\varepsilon^2} \geq o(1).$$

Moreover, using also the definition (1.9) of M , (3.106), $\delta_\varepsilon \leq 1$ and that $R > 0$ may be arbitrarily large, we get together that

$$\delta_\varepsilon \rightarrow 1, \quad (3.116)$$

and that (3.10) and (3.12) hold true. As a remark, in **(Case 2)** where $N_\varepsilon = 1$, (3.116) is a direct consequence of the definition (3.30) of δ_ε . Then, (2.4) follows from (3.10), (3.107) and (3.111).

• (4) Now we prove (3.11). Since $\varepsilon'_0 > \varepsilon_0$, we get from (3.84), (3.88), (3.96) and (A.7) that

$$u_\varepsilon = \gamma_\varepsilon - \frac{t_\varepsilon}{\gamma_\varepsilon} - \frac{t_\varepsilon}{\gamma_\varepsilon^3} - (A(\gamma_\varepsilon) - 2\xi_\varepsilon) \frac{t_\varepsilon}{2\gamma_\varepsilon} + o\left(\frac{t_\varepsilon \tilde{\zeta}_\varepsilon}{\gamma_\varepsilon}\right) \quad (3.117)$$

uniformly in $\{y \in B_{x_\varepsilon}(\rho'_\varepsilon) \text{ s.t. } t_\varepsilon \geq \gamma_\varepsilon/4\}$, using also (A.3). Then, noting that the averages of (3.105) and (3.117) have to match on $\partial B_{x_\varepsilon}(\rho'_\varepsilon)$, we compute and get that

$$\lambda_\varepsilon = \frac{4}{\gamma_\varepsilon^2 \exp\left(1 + M + \frac{\gamma_\varepsilon^2(A(\gamma_\varepsilon) - 2\xi_\varepsilon)}{2} + o(\tilde{\zeta}_\varepsilon \gamma_\varepsilon^2)\right)}, \quad (3.118)$$

by (3.12), (3.116) and (3.40) with (3.3) and (3.45). Observe in particular that

$$1 \lesssim \gamma_\varepsilon^{-2} G_{x_\varepsilon} \lesssim 1, \quad 1 \lesssim \gamma_\varepsilon^{-2} t_\varepsilon \lesssim 1$$

on $\partial B_{x_\varepsilon}(\rho'_\varepsilon)$, by (3.95) and (3.96) with (1.8) and (3.12). By (3.10) and (3.118), (3.11) is proved.

• (5) Here, we conclude the proof of Lemma 3.3. As an immediate consequence of (3.105), we get that (3.97) holds true. Pushing now one step further the above computations with very similar arguments, we get that (3.98) holds true as well. At last, using in particular (3.10) with (1.6) to improve the estimates in Point (1) of this proof, we get (3.99). \square

Lemma 3.3 is proved. \square

4. PROOF OF PROPOSITION 2.1

Proof of Proposition 2.1. We make the assumptions of Lemma 3.3 in **(Case 2)** with $\alpha_\varepsilon = 4\pi(1 - \varepsilon)$. In particular, we assume that u_ε is a maximizer for $(I_{4\pi(1-\varepsilon)}^g(\Omega))$, for all $0 < \varepsilon \ll 1$, and that (2.1) holds true. Then, Lemma 3.3 in **(Case 2)** will be currently applied in the sequel. In particular, we may let $\lambda_\varepsilon, \gamma_\varepsilon, x_\varepsilon, \mu_\varepsilon$ be thus given and it only remains to prove (2.5)-(2.6) to get Proposition 2.1.

Let $z \in \Omega$ be given. In view of (3.99), for $\gamma, \mu > 0$, we let now $U_{\mu, \gamma, z}$ be given by

$$\begin{aligned} U_{\mu, \gamma, z}(x) &= \frac{1}{\gamma} \left(-\log \left(1 + \frac{|x - z|^2}{\mu^2} \right) + \log \frac{1}{\mu^2} + \mathcal{H}_{-1, \mu, z}(x) \right) \\ &\quad + \sum_{i=0}^1 \frac{1}{\gamma^{3+2i}} \left(S_i \left(\frac{x - z}{\mu} \right) + \frac{A_i}{4\pi} \left(\log \frac{1}{\mu^2} + \mathcal{H}_{i, \mu, z}(x) \right) - B_i \right) \\ &\quad + \frac{A(\gamma)}{\gamma} \left(S_2 \left(\frac{x - z}{\mu} \right) + \frac{A_2}{4\pi} \left(\log \frac{1}{\mu^2} + \mathcal{H}_{2, \mu, z}(x) \right) - B_2 \right) \\ &\quad + \frac{4B(\gamma)}{\gamma^2 \exp(1 + \mathcal{H}_z(z))} \int_{\Omega} G_x(y) F(4\pi G_z(y)) dy, \end{aligned} \quad (4.1)$$

where the S_i are given by (A.2), where the A_i, B_i are as in (A.3), where \mathcal{H} is as in (1.8), where the $\mathcal{H}_{j,\mu,z}$ are harmonic in Ω and given by

$$\mathcal{H}_{-1,\mu,z} = -\log\left(\frac{1}{\mu^2 + |z - \cdot|^2}\right) \text{ or by } \mathcal{H}_{j,\mu,z} = -\frac{4\pi}{A_j} \left(S_j \left(\frac{\cdot - z}{\mu} \right) - B_j \right) + \log \mu^2$$

on $\partial\Omega$, for $j \in \{0, 1, 2\}$. By the maximum principle and (A.3), we have that $\mathcal{H}_{j,\mu,z}(x) \rightarrow \mathcal{H}_z(x)$ and $|\partial_\mu \mathcal{H}_{j,\mu,z}(x)| \leq C\mu$ uniformly in $x \in \Omega$ as $\mu \rightarrow 0$, for all j . Then, setting $f_\gamma(\mu) = \gamma^{-1} U_{\mu,\gamma,z}(z) - 1$, using that $S_i(0) = 0$ and (4.1), it may be easily checked that $f_\gamma(\mu) = -\gamma^{-2} \log \mu^2 (1 + o(1)) - 1$, C^1 -uniformly in $\mu \in (0, \mu(\gamma))$ as $\gamma \rightarrow +\infty$, where $\mu(\gamma)$ is given by $-\log \mu(\gamma)^2 = \gamma^2/2$. In particular, there exists $\tilde{\gamma} \gg 1$ such that $\lim_{\mu \rightarrow 0} f_\gamma(\mu) = +\infty$, $f_\gamma(\mu(\gamma)) < 0$ and $f'_\gamma < 0$ in $(0, \mu(\gamma))$, so that there exists a unique $\tilde{\mu}(\gamma, z) \in (0, \mu(\gamma))$ such that $f_\gamma(\tilde{\mu}(\gamma, z)) = 0$ for all $\gamma \geq \tilde{\gamma}$. Fixing K a compact of Ω , it is clear that $\tilde{\gamma}$ can be chosen independent of $z \in K$; in particular, we may let $\tilde{\mu}_\varepsilon := \tilde{\mu}(\gamma_\varepsilon, z)$ be the unique $\mu \in (0, \mu(\gamma_\varepsilon))$ given by

$$U_{\mu,\gamma_\varepsilon,z}(z) = \gamma_\varepsilon \quad (4.2)$$

for all ε small. We denote from now on $\tilde{\mathcal{H}}_{j,\varepsilon,z} := \mathcal{H}_{j,\tilde{\mu}_\varepsilon,z}$ and $U_{\varepsilon,z} := U_{\tilde{\mu}_\varepsilon,\gamma_\varepsilon,z}$. The following result concludes the proof of Proposition 2.1.

Lemma 4.1. *We have that*

$$S = \int_{\Omega} G_{\bar{x}}(y) F(4\pi G_{\bar{x}}(y)) \, dy, \text{ if } \frac{\gamma_\varepsilon^{-3} B(\gamma_\varepsilon)}{\gamma_\varepsilon^{-4} + |A(\gamma_\varepsilon)|} \not\rightarrow 0 \quad (4.3)$$

as $\varepsilon \rightarrow 0$, where S is as in (1.9) and \bar{x} as in (3.12). Moreover, (2.5) holds true in any case.

Proof of Lemma 4.1. Let K be a compact subset of Ω and $(z_\varepsilon)_\varepsilon$ be a given sequence of points of K . For simplicity, we let in the proof below $\check{\zeta}_\varepsilon$ be given by

$$\check{\zeta}_\varepsilon = \max \left(\frac{1}{\gamma_\varepsilon^4}, |A(\gamma_\varepsilon)|, \frac{|B(\gamma_\varepsilon)|}{\gamma_\varepsilon^3} \right). \quad (4.4)$$

- (1) We first derive the following more explicit expression of the $\tilde{\mu}_\varepsilon$ from (4.2):

$$\begin{aligned} & \frac{4}{\tilde{\mu}_\varepsilon^2 \exp(\gamma_\varepsilon^2) \gamma_\varepsilon^2} \\ &= \frac{4}{\gamma_\varepsilon^2 \exp(1 + \mathcal{H}_{z_\varepsilon}(z_\varepsilon))} (1 + O(\check{\zeta}_\varepsilon + \gamma_\varepsilon^4 |A(\gamma_\varepsilon)|^2)) \times \\ & \quad \left(1 - \frac{\gamma_\varepsilon^2 A(\gamma_\varepsilon)}{2} - \frac{4B(\gamma_\varepsilon)}{\gamma_\varepsilon \exp(1 + \mathcal{H}_{z_\varepsilon}(z_\varepsilon))} \int_{\Omega} G_{z_\varepsilon}(y) F(4\pi G_{z_\varepsilon}(y)) \, dy \right) \end{aligned} \quad (4.5)$$

as $\varepsilon \rightarrow 0$. By the maximum principle and (A.3), we get that there exists $C_K > 0$ such that $|\tilde{\mathcal{H}}_{j,\varepsilon,z_\varepsilon}| \leq C_K$ in Ω , so that, by elliptic theory, the $\tilde{\mathcal{H}}_{j,\varepsilon,z_\varepsilon}$'s are also bounded in $C_{loc}^1(\Omega)$ for all ε and j . We get from (4.2) that $\left| \log \frac{1}{\tilde{\mu}_\varepsilon^2} - \gamma_\varepsilon^2 \right| \leq C'_K$, and then that

$$\left| \tilde{\mathcal{H}}_{j,\varepsilon,z_\varepsilon} - \mathcal{H}_{z_\varepsilon} \right| \leq C''_K \gamma_\varepsilon^8 \exp(-\gamma_\varepsilon^2) \text{ in } \Omega, \quad (4.6)$$

for all $0 < \varepsilon \ll 1$ and $j \in \{-1, \dots, 2\}$, by the maximum principle, (1.8) and (A.3). Rewriting then (4.2) as

$$\begin{aligned} \gamma_\varepsilon^2 = & \log \frac{1}{\tilde{\mu}_\varepsilon^2} \left(1 + \frac{A_0}{4\pi\gamma_\varepsilon^2} + \frac{A_1}{4\pi\gamma_\varepsilon^4} + \frac{A(\gamma_\varepsilon)A_2}{4\pi} \right) + \mathcal{H}_{z_\varepsilon}(z_\varepsilon) \left(1 + \frac{A_0}{4\pi\gamma_\varepsilon^2} \right) \\ & - \frac{B_0}{\gamma_\varepsilon^2} + \frac{4B(\gamma_\varepsilon)}{\gamma_\varepsilon \exp(1 + \mathcal{H}_{z_\varepsilon}(z_\varepsilon))} \int_\Omega G_{z_\varepsilon}(y) F(4\pi G_{z_\varepsilon}(y)) dy \\ & + O(\gamma_\varepsilon^{-4} + |A(\gamma_\varepsilon)|), \end{aligned}$$

we easily get (4.5), using (3.16) and (A.3) with $\frac{A_1}{4\pi} - \frac{A_0^2}{16\pi^2} - B_0 = 0$.

- (2) We prove now that

$$\int_\Omega |\nabla U_{\varepsilon, z_\varepsilon}|^2 dx = 4\pi (1 + I_{z_\varepsilon}(\gamma_\varepsilon) + o(\zeta_\varepsilon)) \quad (4.7)$$

as $\varepsilon \rightarrow 0$, where $I_{z_\varepsilon}(\gamma_\varepsilon)$ is given by

$$I_{z_\varepsilon}(\gamma_\varepsilon) = \gamma_\varepsilon^{-4} + \frac{A(\gamma_\varepsilon)}{2} + \frac{4B(\gamma_\varepsilon)}{\gamma_\varepsilon^3 \exp(1 + \mathcal{H}_{z_\varepsilon}(z_\varepsilon))} \int_\Omega G_{z_\varepsilon}(y) F(4\pi G_{z_\varepsilon}(y)) dy \quad (4.8)$$

and where $U_{\varepsilon, z_\varepsilon}$ is given by (4.1)-(4.2). By (1.6) and elliptic theory,

$$\left(x \mapsto \int_\Omega G_x(y) F(4\pi G_{z_\varepsilon}(y)) dy \right)_\varepsilon \text{ is a bounded sequence in } C^1(\bar{\Omega}). \quad (4.9)$$

By construction of the $\tilde{\mathcal{H}}_{j, \varepsilon, z_\varepsilon}$, we can write that

$$\begin{aligned} \int_\Omega |\nabla U_{\varepsilon, z_\varepsilon}(y)|^2 dy &= \int_\Omega \Delta U_{\varepsilon, z_\varepsilon}(y) U_{\varepsilon, z_\varepsilon}(y) dy, \\ &= \int_{\{y: \tilde{t}_\varepsilon(y) \leq \gamma_\varepsilon\}} \left(\frac{\Delta(-\tilde{t}_\varepsilon)}{\gamma_\varepsilon} + \frac{\Delta \tilde{S}_{0, \varepsilon}}{\gamma_\varepsilon^3} + \frac{\Delta \tilde{S}_{1, \varepsilon}}{\gamma_\varepsilon^5} + \frac{A(\gamma_\varepsilon) \Delta \tilde{S}_{2, \varepsilon}}{\gamma_\varepsilon} \right) \times \\ &\quad \left(\gamma_\varepsilon - \frac{\tilde{t}_\varepsilon}{\gamma_\varepsilon} + \frac{\tilde{S}_{0, \varepsilon}}{\gamma_\varepsilon^3} + O\left(\left(\frac{|A(\gamma_\varepsilon)|}{\gamma_\varepsilon} + \frac{1}{\gamma_\varepsilon^5}\right)(1 + \tilde{t}_\varepsilon) + \frac{|y - z_\varepsilon|}{\gamma_\varepsilon}\right) \right) dy \\ &\quad + o(\gamma_\varepsilon^{-4}) \\ &+ \int_{\{y: \tilde{t}_\varepsilon(y) \geq \gamma_\varepsilon(\gamma_\varepsilon - 1)\}} \left(O(\tilde{\mu}_\varepsilon^2 \gamma_\varepsilon^4) + \frac{4B(\gamma_\varepsilon)}{\gamma_\varepsilon^2 \exp(1 + \mathcal{H}_{z_\varepsilon}(z_\varepsilon))} F(4\pi G_{z_\varepsilon}(y)) \right) \times \\ &\quad \left(\frac{4\pi G_{z_\varepsilon}(y)}{\gamma_\varepsilon} + O\left(\frac{G_{z_\varepsilon}(y)}{\gamma_\varepsilon^3} + \frac{|B(\gamma_\varepsilon)|}{\gamma_\varepsilon^2}\right) \right) dy, \end{aligned} \quad (4.10)$$

where $\tilde{t}_\varepsilon(y) = \log(1 + |y - z_\varepsilon|^2 / \tilde{\mu}_\varepsilon^2)$ and $\tilde{S}_{i, \varepsilon} = S_i(|y - z_\varepsilon| / \tilde{\mu}_\varepsilon)$. We use also here (1.8) with (3.16), and the estimates of Point (1) of this proof, including (4.5)-(4.6). The integral on $\{\tilde{t}_\varepsilon \in (\gamma_\varepsilon, \gamma_\varepsilon(\gamma_\varepsilon - 1))\}$ gives a $o(\gamma_\varepsilon^{-4})$ term. Estimate (4.7) follows from (4.10), Appendix A and some computations that we do not develop here again (see also [17], 5).

- (3) We prove now that

$$\int_\Omega |\nabla u_\varepsilon|^2 dx = 4\pi (1 + I_{x_\varepsilon}(\gamma_\varepsilon) + o(\zeta_\varepsilon)) \quad (4.11)$$

as $\varepsilon \rightarrow 0$, where $I_{x_\varepsilon}(\gamma_\varepsilon)$ is given by (4.8), for $(x_\varepsilon)_\varepsilon$ as in (3.13). Now, we can push one step further the argument involving (3.118), writing now that both formulas

(3.98) and (3.99) must also coincide on $\partial B_{x_\varepsilon}(\rho'_\varepsilon)$, where $\rho'_\varepsilon > 0$ is as in (3.95). We compute and then get for μ_ε in (3.40) the analogue of (4.5) for $\tilde{\mu}_\varepsilon$

$$\begin{aligned} \lambda_\varepsilon H(\gamma_\varepsilon) &= \frac{4}{\mu_\varepsilon^2 \exp(\gamma_\varepsilon^2) \gamma_\varepsilon^2} \left(1 + o\left(\frac{1}{\gamma_\varepsilon^4}\right) \right) \\ &= \frac{4}{\gamma_\varepsilon^2 \exp(1 + \mathcal{H}_{x_\varepsilon}(x_\varepsilon))} (1 + o(\gamma_\varepsilon^2 \zeta_\varepsilon)) \times \\ &\quad \left(1 - \frac{\gamma_\varepsilon^2 A(\gamma_\varepsilon)}{2} - \frac{4B(\gamma_\varepsilon)}{\gamma_\varepsilon \exp(1 + \mathcal{H}_{x_\varepsilon}(x_\varepsilon))} \int_\Omega G_{x_\varepsilon}(y) F(4\pi G_{x_\varepsilon}(y)) dy \right), \end{aligned} \quad (4.12)$$

using (1.8), (3.16), (A.3)-(A.7). Independently, integrating by parts, resuming some computations in Appendix A and using (2.2), (3.12), (3.44), Point (1), and (3.97)-(3.99) (see also (3.89) and (A.9)), we get that

$$\begin{aligned} \int_\Omega |\nabla u_\varepsilon|^2 dx &= \int_\Omega u_\varepsilon (\lambda_\varepsilon H(u_\varepsilon) u_\varepsilon \exp(u_\varepsilon^2)) dx, \\ &= \int_\Omega U_{\varepsilon, x_\varepsilon} \Delta U_{\varepsilon, x_\varepsilon} dx + o(\zeta_\varepsilon). \end{aligned} \quad (4.13)$$

In order to get the second equality and to apply the dominated convergence theorem, it may be useful to split Ω according

$$\begin{aligned} \Omega &= \{y \text{ s.t. } t_\varepsilon(y) \leq \gamma_\varepsilon\} \cup \left\{ y \text{ s.t. } t_\varepsilon(y) > \gamma_\varepsilon \text{ and } \log \frac{1}{|x_\varepsilon - y|^2} \geq \frac{1 - \delta'_0}{2} \gamma_\varepsilon^2 \right\} \\ &\cup \left\{ y \text{ s.t. } \log \frac{1}{|x_\varepsilon - y|^2} < \frac{1 - \delta'_0}{2} \gamma_\varepsilon^2 \right\}, \end{aligned}$$

where δ'_0 is as in (1.6), and to use the first line of (4.12) with (1.5) (resp. with (3.29)) in the first region (resp. in the second region), or (1.6)-(1.7) in the last region. Observe that the argument here is to show that $U_{\varepsilon, x_\varepsilon}$ (resp. $\Delta U_{\varepsilon, x_\varepsilon}$) is in some sense the main part of the expansion of u_ε (resp. Δu_ε). Thus we get (4.11) from (4.7) and (4.13).

• (4) We prove now that, for any fixed sequence $(\eta_\varepsilon)_\varepsilon$ of real numbers such that $\eta_\varepsilon = o(\gamma_\varepsilon^{-2})$, we have that

$$\begin{aligned} &\int_\Omega (1 + g(V_{\varepsilon, z_\varepsilon})) \exp(V_{\varepsilon, z_\varepsilon}^2) dy \\ &= |\Omega| (1 + g(0)) + \pi \exp(1 + \mathcal{H}_{z_\varepsilon}(z_\varepsilon)) (1 - \eta_\varepsilon \gamma_\varepsilon^2) \times \\ &\quad H(\gamma_\varepsilon) \left(1 + \gamma_\varepsilon^2 I_{z_\varepsilon}(\gamma_\varepsilon) + \frac{1}{\gamma_\varepsilon^2} + o(\gamma_\varepsilon^2 (\zeta_\varepsilon + |\eta_\varepsilon|)) \right) \times \\ &\quad \left(1 + \frac{8B(\gamma_\varepsilon)}{\gamma_\varepsilon (\kappa + 1) \exp(1 + \mathcal{H}_{z_\varepsilon}(z_\varepsilon))} \int_\Omega G_{z_\varepsilon}(y) F(4\pi G_{z_\varepsilon}(y)) dy \right), \end{aligned} \quad (4.14)$$

where κ is as in (1.6) and where $V_{\varepsilon, z_\varepsilon} \geq 0$ is given by

$$V_{\varepsilon, z_\varepsilon}^2 = (1 - \eta_\varepsilon) U_{\varepsilon, z_\varepsilon}^2, \quad (4.15)$$

where $U_{\varepsilon, z_\varepsilon}$ is given in (4.1). Computations in the spirit of the proof of (4.13) give that

$$\int_\Omega (1 + g(U_{\varepsilon, x_\varepsilon})) \exp(U_{\varepsilon, x_\varepsilon}^2) dy = \int_\Omega (1 + g(u_\varepsilon)) \exp(u_\varepsilon^2) dy + o(\gamma_\varepsilon^2 \zeta_\varepsilon), \quad (4.16)$$

not only by combining (1.1), (1.5)-(1.6), Lemma 3.2, (3.12), (3.97)-(3.99) and Appendix A, and by splitting Ω as in (4.10), but also by using (4.5) and (4.12). In particular, once (4.14) is proved, choosing $\eta_\varepsilon = 0$ and $z_\varepsilon = x_\varepsilon$, we get from (4.16) that

$$\begin{aligned} \int_{\Omega} (1 + g(u_\varepsilon)) \exp(u_\varepsilon^2) dy &= |\Omega|(1 + g(0)) + \pi \exp(1 + \mathcal{H}_{x_\varepsilon}(x_\varepsilon)) H(\gamma_\varepsilon) \times \\ &\quad \left(1 + \gamma_\varepsilon^2 I_{x_\varepsilon}(\gamma_\varepsilon) + \frac{1}{\gamma_\varepsilon^2} + o(\gamma_\varepsilon^2 \check{\zeta}_\varepsilon) \right) \times \\ &\quad \left(1 + \frac{8B(\gamma_\varepsilon)}{\gamma_\varepsilon(\kappa + 1) \exp(1 + \mathcal{H}_{x_\varepsilon}(x_\varepsilon))} \int_{\Omega} G_{x_\varepsilon}(y) F(4\pi G_{x_\varepsilon}(y)) dy \right). \end{aligned} \quad (4.17)$$

It remains to prove (4.14). We compute and get that

$$U_{\varepsilon, z_\varepsilon}(y)^2 = \gamma_\varepsilon^2 - 2\tilde{t}_\varepsilon + \frac{\tilde{t}_\varepsilon^2}{\gamma_\varepsilon^2} + \frac{2\tilde{S}_{0,\varepsilon}}{\gamma_\varepsilon^2} + O(|A(\gamma_\varepsilon)| + \gamma_\varepsilon^{-4})(1 + \tilde{t}_\varepsilon(y)^2) + |y - z_\varepsilon| \quad (4.18)$$

for all y such that $\tilde{t}_\varepsilon(y) \leq \gamma_\varepsilon$, using (1.7), (4.1)-(4.2), (4.5), (4.9) and (A.3). Then we get

$$\begin{aligned} &\int_{\{\tilde{t}_\varepsilon \leq \gamma_\varepsilon\}} (1 + g(V_{\varepsilon, z_\varepsilon})) \exp(V_{\varepsilon, z_\varepsilon}^2) dy \\ &= \int_{\{\tilde{t}_\varepsilon \leq \gamma_\varepsilon\}} H(\gamma_\varepsilon) (1 + O(|A(\gamma_\varepsilon)| \exp(\delta_0 \tilde{t}_\varepsilon))) \exp(\gamma_\varepsilon^2) \exp(-2\tilde{t}_\varepsilon) \exp(-\eta_\varepsilon \gamma_\varepsilon^2) \times \\ &\quad \exp\left(\frac{\tilde{t}_\varepsilon^2 + 2\tilde{S}_{0,\varepsilon}}{\gamma_\varepsilon^2}\right) \exp(O(|\eta_\varepsilon| + |A(\gamma_\varepsilon)| + \gamma_\varepsilon^{-4})(1 + \tilde{t}_\varepsilon^2) + |y - z_\varepsilon|) dy, \end{aligned}$$

using (3.2) and (4.15) with (4.18). Then combining $\eta_\varepsilon = o(\gamma_\varepsilon^{-2})$, (3.16), (4.5), computing explicitly $\int_{\mathbb{R}^2} \exp(-2T_0) S_0 dy = 0$ and $\int_{\mathbb{R}^2} \exp(-2T_0) T_0^2 dy = 2\pi$ for T_0 as in (3.42), we get that

$$\begin{aligned} &\int_{\{\tilde{t}_\varepsilon \leq \gamma_\varepsilon\}} (1 + g(V_{\varepsilon, z_\varepsilon})) \exp(V_{\varepsilon, z_\varepsilon}^2) dy \\ &= \frac{(1 - \eta_\varepsilon \gamma_\varepsilon^2) H(\gamma_\varepsilon) \exp(\mathcal{H}_{z_\varepsilon}(z_\varepsilon) + 1)}{4} (1 + o(\gamma_\varepsilon^2(|A(\gamma_\varepsilon)| + |\eta_\varepsilon|) + \gamma_\varepsilon^{-2})) \\ &\quad \times \left(1 + \frac{\gamma_\varepsilon^2 A(\gamma_\varepsilon)}{2} + \frac{4B(\gamma_\varepsilon)}{\gamma_\varepsilon \exp(\mathcal{H}_{z_\varepsilon}(z_\varepsilon) + 1)} \times \right. \\ &\quad \left. \int_{\Omega} G_{z_\varepsilon}(x) F(4\pi G_{z_\varepsilon}(x)) dx + o\left(\frac{B(\gamma_\varepsilon)}{\gamma_\varepsilon}\right) \right) \times 4\pi \left(1 + \frac{2}{\gamma_\varepsilon^2} \right). \end{aligned} \quad (4.19)$$

Independently, we get from (1.6), (3.1) (parts *a*) and *b*) in $\{y, 4\pi G_{z_\varepsilon}(y) \leq \gamma_\varepsilon/2\}$, or part *c*) otherwise), (4.1), (4.5) and the dominated convergence theorem that

$$\begin{aligned} &\int_{\{\tilde{t}_\varepsilon \geq \gamma_\varepsilon\}} (1 + g(V_{\varepsilon, z_\varepsilon})) \exp(V_{\varepsilon, z_\varepsilon}^2) dy \\ &= |\Omega|(1 + g(0)) + \frac{8\pi B(\gamma_\varepsilon)}{\gamma_\varepsilon(\kappa + 1)} \int_{\Omega} G_{z_\varepsilon}(y) F(4\pi G_{z_\varepsilon}(y)) dy \\ &\quad + o\left(\frac{|B(\gamma_\varepsilon)|}{\gamma_\varepsilon} + \frac{1}{\gamma_\varepsilon^2}\right). \end{aligned} \quad (4.20)$$

Combining (4.19) and (4.20), we conclude that (4.14) holds true, using (3.3) and (4.5).

• (5) We are now in position to conclude the proof of Lemma 4.1. Let \bar{x}_0 be a point in the compact $K_\Omega \subset \subset \Omega$ where S is attained in the third equation of (1.9). Let η_ε be given by

$$(1 - \eta_\varepsilon) = \frac{4\pi(1 - \varepsilon)}{\|U_{\varepsilon, \bar{x}_0}\|_{H_0^1}^2}. \quad (4.21)$$

First, we can check that

$$\eta_\varepsilon = I_{\bar{x}_0}(\gamma_\varepsilon) - I_{x_\varepsilon}(\gamma_\varepsilon) + o(\zeta_\varepsilon), \quad (4.22)$$

so that the condition $\eta_\varepsilon = o(\gamma_\varepsilon^{-2})$ above (4.14) is satisfied, using (1.7), (3.6), (3.16), (4.7) and (4.11). Besides, we have that $\|V_{\varepsilon, \bar{x}_0}\|_{H_0^1}^2 = 4\pi(1 - \varepsilon)$, by our choice (4.21) of η_ε , and then, by (3.6), that

$$\int_\Omega (1 + g(u_\varepsilon)) \exp(u_\varepsilon^2) dy \geq \int_\Omega (1 + g(V_{\varepsilon, \bar{x}_0})) \exp(V_{\varepsilon, \bar{x}_0}^2) dy;$$

this implies, in view of (4.14), (4.17), (4.22) and of our choice of \bar{x}_0 , that (4.3) is true and then, by (4.11) again, that (2.5)-(2.6) are true as well. This concludes the proof of Lemma 4.1. \square

Proposition 2.1 is proved. \square

Proof of Proposition 2.2. Let Ω be a smooth bounded domain of \mathbb{R}^2 . Let g be such that (1.1) and (1.5)-(1.6) hold true, for H as in (1.2), and let A , B and F be thus given. Assume that $\Lambda_g(\Omega) < \pi \exp(1 + M)$, where M is as in (1.9) and $\Lambda_g(\Omega)$ as in (1.11). Assume that there exists a sequence of positive integers $(N_\varepsilon)_\varepsilon$ such that (2.9) holds true and such that $(I_{4\pi}^{g_{N_\varepsilon}}(\Omega))$ admits a nonnegative extremal u_ε for all $\varepsilon > 0$, where g_{N_ε} is as in (1.10). Then, by Lemma 3.3 in **(Case 1)**, we have (2.1) and that (3.8) holds true for $\alpha_\varepsilon = 4\pi$, for all $0 < \varepsilon \ll 1$. Moreover, we have $u_\varepsilon \in C^{1,\theta}(\bar{\Omega})$ ($0 < \theta < 1$) and (2.3) by (3.13). In order to conclude the proof of Proposition 2.2, it remains to prove (2.10). Still by Lemma 3.3 in **(Case 1)**, (3.97)-(3.99) and (A.9) (v_ε as in (3.89)) hold true. Concerning (3.97)-(3.99) and (A.9), observe that, contrary to **(Case 2)**, the term ξ_ε cannot be neglected in **(Case 1)** we are facing here. Indeed, using also now (3.30), (3.40), (3.110) and (A.9), we can resume computations of (4.10), (4.13) and Appendix A (now with (3.11)) to get that

$$\|u_\varepsilon\|_{H_0^1}^2 = 4\pi (1 + \check{I}(\gamma_\varepsilon) + o(\gamma_\varepsilon^{-4} + |A(\gamma_\varepsilon)| + \gamma_\varepsilon^{-3}|B(\gamma_\varepsilon)| + \xi_\varepsilon))$$

as $\varepsilon \rightarrow 0$, where

$$\check{I}(\gamma_\varepsilon) := \gamma_\varepsilon^{-4} + (A(\gamma_\varepsilon) - 2\xi_\varepsilon)/2 + 4\gamma_\varepsilon^{-3} \exp(-1 - M)B(\gamma_\varepsilon)S,$$

so that (2.10) holds true, which concludes. \square

APPENDIX A. RADIAL ANALYSIS

Let $(x_\varepsilon)_\varepsilon$ be a sequence of points in \mathbb{R}^2 and $(\gamma_\varepsilon)_\varepsilon$ be a sequence of positive real numbers such that $\gamma_\varepsilon \rightarrow +\infty$ as $\varepsilon \rightarrow 0$. Let g be such that (1.1) and (1.5) holds true for H as in (1.2), and let A be thus given. Let $(N_\varepsilon)_\varepsilon$ be a sequence of integers. We assume that we are in one of the following two cases:

$$N_\varepsilon \rightarrow +\infty \text{ as } \varepsilon \rightarrow 0, \text{ and (3.30)-(3.31) hold true,} \quad (\text{Case 1})$$

$$N_\varepsilon = 1 \text{ for all } \varepsilon. \quad (\text{Case 2})$$

Let B_ε be the radial solution around x_ε in \mathbb{R}^2 of (3.81), for Ψ_N as in (3.25), where $(\lambda_\varepsilon)_\varepsilon$ is any given sequence of positive real numbers. Let T_0 be given in \mathbb{R}^2 by

$$T_0(x) = \log(1 + |x|^2). \quad (\text{A.1})$$

Let S_i , $i = 0, 1, 2$, be the radially symmetric solutions around 0 in \mathbb{R}^2 of

$$\begin{aligned} \Delta S_0 - 8 \exp(-2T_0) S_0 &= 4 \exp(-2T_0) (T_0^2 - T_0), \\ \Delta S_1 - 8 \exp(-2T_0) S_1 &= 4 \exp(-2T_0) \left(S_0 + 2S_0^2 - 4T_0 S_0 + 2S_0 T_0^2 - T_0^3 + \frac{T_0^4}{2} \right), \\ \Delta S_2 - 8 \exp(-2T_0) S_2 &= 4 \exp(-2T_0) T_0, \end{aligned} \quad (\text{A.2})$$

such that $S_i(0) = 0$. In the sequel, we will use the following C^1 expansions of the S_i 's given by

$$\begin{aligned} S_0(r) &= \frac{A_0}{4\pi} \log \frac{1}{r^2} + B_0 + O(\log(r)^2 r^{-2}) \text{ where } A_0 = 4\pi, \quad B_0 = \frac{\pi^2}{6} + 2, \\ S_1(r) &= \frac{A_1}{4\pi} \log \frac{1}{r^2} + B_1 + O(\log(r)^4 r^{-2}) \text{ where } A_1 = 4\pi \left(3 + \frac{\pi^2}{6} \right), \quad B_1 \in \mathbb{R}, \\ S_2(r) &= \frac{A_2}{4\pi} \log \frac{1}{r^2} + B_2 + O(\log(r) r^{-2}) \text{ where } A_2 = 2\pi, \quad B_2 \in \mathbb{R}, \end{aligned} \quad (\text{A.3})$$

as $r = |x| \rightarrow +\infty$. Note that in particular

$$A_i = \int_{\mathbb{R}^2} \Delta S_i dx. \quad (\text{A.4})$$

The explicit formula for S_0

$$S_0(r) = -T_0(r) + \frac{2r^2}{1+r^2} - \frac{1}{2} T_0(r)^2 + \frac{1-r^2}{1+r^2} \int_1^{1+r^2} \frac{\log t}{1-t} dt,$$

and the expansions in (A.3) are derived in [16, 17]. Let $\varepsilon_0 \in (\sqrt{1/e}, 1)$ be given. Let μ_ε be given by (3.40) and t_ε by (3.41). Let $\rho_\varepsilon > 0$ be given by (3.83) and satisfying (3.84). Let $S_{i,\varepsilon}$ be then given by

$$S_{i,\varepsilon}(x) = S_i \left(\frac{|x - x_\varepsilon|}{\mu_\varepsilon} \right), \quad (\text{A.5})$$

for $i = 0, 1, 2$. Let $\xi_\varepsilon > 0$ be given by (3.14). In (Case 1) where $N_\varepsilon \rightarrow +\infty$ as $\varepsilon \rightarrow 0$, we get that $\xi_\varepsilon = O(N_\varepsilon^{-1/2})$ by (3.30) and (3.45). Then, in any case, we clearly have that

$$\xi_\varepsilon \rightarrow 0 \quad (\text{A.6})$$

as $\varepsilon \rightarrow 0$. Then we are in position to state the main result of this section.

Proposition A.1. *We have that*

$$B_\varepsilon = \gamma_\varepsilon - \frac{t_\varepsilon}{\gamma_\varepsilon} + \frac{S_{0,\varepsilon}}{\gamma_\varepsilon^3} + \frac{S_{1,\varepsilon}}{\gamma_\varepsilon^5} + (A(\gamma_\varepsilon) - 2\xi_\varepsilon) \frac{S_{2,\varepsilon}}{\gamma_\varepsilon} + o \left(t_\varepsilon \left(\frac{1}{\gamma_\varepsilon^5} + \frac{|A(\gamma_\varepsilon)| + \xi_\varepsilon}{\gamma_\varepsilon} \right) \right) \quad (\text{A.7})$$

uniformly in $[0, \rho_\varepsilon]$ as $\varepsilon \rightarrow 0$.

In particular, using also (1.1) and (3.3), it can be checked that B_ε is positive and radially decreasing in $[0, \rho_\varepsilon]$. Observe also that $\xi_\varepsilon \ll \gamma_\varepsilon^{-4}$ can be seen as a remainder term in (Case 2). Let $\zeta_\varepsilon > 0$ be given by

$$\zeta_\varepsilon = \max \left(\frac{1}{\gamma_\varepsilon^4}, |A(\gamma_\varepsilon)|, \xi_\varepsilon \right). \quad (\text{A.8})$$

Set $\bar{t}_\varepsilon = 1 + t_\varepsilon$. Resuming the computations below, we get as a by product of Proposition A.1 that, $v_\varepsilon = o(\bar{t}_\varepsilon \gamma_\varepsilon^{-5})$ implies that

$$\begin{aligned} \frac{\lambda_\varepsilon \Psi'_\varepsilon(B_\varepsilon + v_\varepsilon)}{2} &= \frac{4 \exp(-2t_\varepsilon)}{\mu_\varepsilon^2 \gamma_\varepsilon} \left[1 + \frac{(\Delta S_0) \left(\frac{\cdot - x_\varepsilon}{\mu_\varepsilon} \right)}{\gamma_\varepsilon^2} + \frac{(\Delta S_1) \left(\frac{\cdot - x_\varepsilon}{\mu_\varepsilon} \right)}{\gamma_\varepsilon^4} \right. \\ &\quad \left. + (A(\gamma_\varepsilon) - 2\xi_\varepsilon) (\Delta S_2) \left(\frac{\cdot - x_\varepsilon}{\mu_\varepsilon} \right) + o \left(\zeta_\varepsilon \exp(\tilde{\delta}_0 t_\varepsilon) \right) \right] \end{aligned} \quad (\text{A.9})$$

uniformly in $\{y \text{ s.t. } t_\varepsilon(y) \leq \gamma_\varepsilon\}$, for some given $\tilde{\delta}_0 \in (\delta_0, 1)$, for δ_0 as in (1.5).

Proof of Proposition A.1. Since both arguments are very similar to prove (Case 1) and (Case 2), for the sake of readability, we only write the proof of Claim A.1 in the more delicate (Case 1). Then, assume that we are in (Case 1). We let τ_ε be given by

$$B_\varepsilon = \gamma_\varepsilon - \frac{\tau_\varepsilon}{\gamma_\varepsilon}. \quad (\text{A.10})$$

Let \bar{w}_ε be given by

$$B_\varepsilon = \gamma_\varepsilon - \frac{t_\varepsilon}{\gamma_\varepsilon} + \frac{S_{0,\varepsilon}}{\gamma_\varepsilon^3} + \frac{S_{1,\varepsilon}}{\gamma_\varepsilon^5} + (A(\gamma_\varepsilon) - 2\xi_\varepsilon) \frac{S_{2,\varepsilon}}{\gamma_\varepsilon} + \frac{\zeta_\varepsilon \bar{w}_\varepsilon}{\gamma_\varepsilon}. \quad (\text{A.11})$$

Let $\bar{\delta} > 0$ be fixed and let $\bar{r}_\varepsilon \geq 0$ be given by

$$\bar{r}_\varepsilon = \sup \{ r > 0 \text{ s.t. } |\bar{w}_\varepsilon| \leq \bar{\delta} t_\varepsilon \text{ in } [0, r] \}. \quad (\text{A.12})$$

Now, since $\bar{\delta} > 0$ may be arbitrarily small, in order to get Claim A.1, it is sufficient to prove that $\bar{r}_\varepsilon = \rho_\varepsilon$, for all $0 < \varepsilon \ll 1$. Using (A.12), we perform computations in $[0, \bar{r}_\varepsilon]$ and the subsequent $o(1)$ are uniformly small in this set as $\varepsilon \rightarrow 0$. First, by (1.5), (A.3), (A.6) and (A.12), we have that

$$\tau_\varepsilon = t_\varepsilon(1 + o(1)). \quad (\text{A.13})$$

Observe that, as soon as we have $\Delta B_\varepsilon > 0$ in $[0, \bar{r}_\varepsilon]$, then B_ε is radially decreasing and (3.93) holds true in $[0, \bar{r}_\varepsilon]$. Let L_ε^H and L_ε^g be given by

$$H(B_\varepsilon) = H(\gamma_\varepsilon) (1 + L_\varepsilon^H) \text{ and then, } (1 + g(B_\varepsilon)) = H(\gamma_\varepsilon) (1 + L_\varepsilon^H + L_\varepsilon^g). \quad (\text{A.14})$$

In view of (A.10) and (A.13), estimates of $L_\varepsilon^H, L_\varepsilon^g$ are given by (1.5) and (3.2), respectively. We are now in position to expand the right-hand side of (3.81). From now on, it is convenient to denote

$$\tilde{N}_\varepsilon = N_\varepsilon - 1. \quad (\text{A.15})$$

Going back to (3.27), we have that

$$\frac{\Psi'_{N_\varepsilon}(B_\varepsilon)}{2} = B_\varepsilon H(\gamma_\varepsilon) \left[(1 + L_\varepsilon^H) (1 + \varphi_{\tilde{N}_\varepsilon}(B_\varepsilon^2)) + L_\varepsilon^g \left(\frac{B_\varepsilon^{2N_\varepsilon}}{N_\varepsilon!} - B_\varepsilon^2 \right) \right]. \quad (\text{A.16})$$

By (3.83), (A.10) and (A.13) and since $\bar{r}_\varepsilon \leq \rho_\varepsilon$, we have that

$$\min_{[0, \bar{r}_\varepsilon]} B_\varepsilon \geq (\varepsilon_0 + o(1)) \gamma_\varepsilon \rightarrow +\infty \quad (\text{A.17})$$

as $\varepsilon \rightarrow 0$. Thus, by Stirling's formula, we get that

$$B_\varepsilon^{2N_\varepsilon}/(N_\varepsilon!) \geq \exp\left(N_\varepsilon\left(\log\frac{\gamma_\varepsilon^2}{N_\varepsilon} + (\log\varepsilon_0^2 + 1) + o(1)\right)\right)$$

and then, for all given integer $k \geq 0$, that

$$B_\varepsilon^k = o(1) \times \frac{B_\varepsilon^{2N_\varepsilon}}{N_\varepsilon!} \quad (\text{A.18})$$

in $[0, \bar{r}_\varepsilon]$ as $\varepsilon \rightarrow 0$, using $\varepsilon_0^2 > 1/e$ with (3.31). Similarly, for all given integer $k \geq 0$, we have that

$$\frac{B_\varepsilon^k}{\varphi_{N_\varepsilon}(B_\varepsilon^2)} = o(1) \quad (\text{A.19})$$

in $[0, \bar{r}_\varepsilon]$ as $\varepsilon \rightarrow 0$. Then, by (3.40), (A.10), (A.19) and (A.18), we may rewrite (A.16) as

$$\begin{aligned} \frac{\lambda_\varepsilon \Psi'_{N_\varepsilon}(B_\varepsilon)}{2} &= \frac{4}{\mu_\varepsilon^2 \gamma_\varepsilon} \left(1 - \frac{\tau_\varepsilon}{\gamma_\varepsilon^2}\right) \left[O(\exp(-\gamma_\varepsilon^2)) + \frac{\varphi_{\tilde{N}_\varepsilon}(B_\varepsilon^2)}{\varphi_{\tilde{N}_\varepsilon}(\gamma_\varepsilon^2)} \times \right. \\ &\quad \left. \left(1 + L_\varepsilon^H + O\left(\frac{B_\varepsilon^{2N_\varepsilon}}{N_\varepsilon! \varphi_{\tilde{N}_\varepsilon}(B_\varepsilon^2)} L_\varepsilon^g\right)\right) \right] \end{aligned} \quad (\text{A.20})$$

in $[0, \bar{r}_\varepsilon]$, as $\varepsilon \rightarrow 0$. Indeed, by (A.17), we have that

$$L_\varepsilon^H = o(1) \text{ and } L_\varepsilon^g = o(1) \quad (\text{A.21})$$

in $[0, \bar{r}_\varepsilon]$ as $\varepsilon \rightarrow 0$, using (1.1), (3.3) and (A.14). In (A.20), the term $O(\exp(-\gamma_\varepsilon^2))$ equals $(1 + L_\varepsilon^H)/\varphi_{\tilde{N}_\varepsilon}(\gamma_\varepsilon^2)$ and we thus get this control by (3.30) and (A.21). In the following lines, we expand the terms of (A.20). By (3.48) with $\Gamma = \gamma_\varepsilon^2$ and $T = B_\varepsilon^2$, we get that

$$\frac{\varphi_{\tilde{N}_\varepsilon}(B_\varepsilon^2)}{\varphi_{\tilde{N}_\varepsilon}(\gamma_\varepsilon^2)} = \exp(B_\varepsilon^2 - \gamma_\varepsilon^2) - F_\varepsilon,$$

where F_ε satisfies in $[0, \bar{r}_\varepsilon]$

$$\begin{aligned} F_\varepsilon &= \frac{B_\varepsilon^{2\tilde{N}_\varepsilon}}{\tilde{N}_\varepsilon! \varphi_{\tilde{N}_\varepsilon}(\gamma_\varepsilon^2)} \int_0^{\gamma_\varepsilon^2 - B_\varepsilon^2} \exp(-u) \left(1 + \frac{u}{B_\varepsilon^2}\right)^{\tilde{N}_\varepsilon} du, \\ &= \frac{\exp(B_\varepsilon^2)}{\varphi_{\tilde{N}_\varepsilon}(\gamma_\varepsilon^2)} \int_{B_\varepsilon^2}^{\gamma_\varepsilon^2} \exp(-s) \frac{s^{\tilde{N}_\varepsilon}}{\tilde{N}_\varepsilon!} ds, \\ &= \xi_\varepsilon \exp(B_\varepsilon^2 - \gamma_\varepsilon^2) \int_{B_\varepsilon^2 - \gamma_\varepsilon^2}^0 \exp(-y) \left(1 + \frac{y}{\gamma_\varepsilon^2}\right)^{\tilde{N}_\varepsilon} dy. \end{aligned} \quad (\text{A.22})$$

By (A.10) and (A.11), we may write

$$\tau_\varepsilon = t_\varepsilon - \frac{S_{0,\varepsilon}}{\gamma_\varepsilon^2} - \frac{S_{1,\varepsilon}}{\gamma_\varepsilon^4} - (A(\gamma_\varepsilon) - 2\xi_\varepsilon) S_{2,\varepsilon} - \zeta_\varepsilon \bar{w}_\varepsilon.$$

Then, keeping in mind (A.3), (A.6), (A.12), (A.13) and $t_\varepsilon \leq \gamma_\varepsilon^2$, we may compute

$$\begin{aligned} & \exp(B_\varepsilon^2 - \gamma_\varepsilon^2) \\ &= \exp\left(-2\tau_\varepsilon + \frac{\tau_\varepsilon^2}{\gamma_\varepsilon^2}\right) \\ &= \exp\left[-2\tau_\varepsilon + \frac{1}{\gamma_\varepsilon^2}\left(t_\varepsilon^2 - \frac{2t_\varepsilon S_{0,\varepsilon}}{\gamma_\varepsilon^2} + O(\zeta_\varepsilon t_\varepsilon^2)\right)\right] \end{aligned} \quad (\text{A.23})$$

in $[0, \bar{r}_\varepsilon]$ as $\varepsilon \rightarrow 0$. Observe that

$$\left| \exp(y) - \sum_{j=0}^N \frac{y^j}{j!} \right| \leq \frac{|y|^{N+1}}{(N+1)!} \exp(|y|)$$

for all $y \in \mathbb{R}$ and all integer $N \geq 0$. Then we draw from (A.23) that

$$\begin{aligned} & \left(1 - \frac{\tau_\varepsilon}{\gamma_\varepsilon^2}\right) \exp(B_\varepsilon^2 - \gamma_\varepsilon^2) \\ &= \exp(-2t_\varepsilon) \left[1 + \frac{1}{\gamma_\varepsilon^2} (2S_{0,\varepsilon} + t_\varepsilon^2 - t_\varepsilon) + \right. \\ & \quad \frac{1}{\gamma_\varepsilon^4} \left(2S_{1,\varepsilon} + 2S_{0,\varepsilon}^2 + \frac{t_\varepsilon^4}{2} + 2S_{0,\varepsilon}t_\varepsilon^2 - 4S_{0,\varepsilon}t_\varepsilon - t_\varepsilon^3 + S_{0,\varepsilon} \right) \\ & \quad + 2(A(\gamma_\varepsilon) - 2\xi_\varepsilon)S_{2,\varepsilon} + 2\zeta_\varepsilon \bar{w}_\varepsilon \\ & \quad \left. + O\left(\left(\frac{\bar{t}_\varepsilon^6}{\gamma_\varepsilon^6} + \frac{\zeta_\varepsilon \bar{t}_\varepsilon^3}{\gamma_\varepsilon^2} + \zeta_\varepsilon^2 \bar{t}_\varepsilon^3\right) \exp\left(o(t_\varepsilon) + \frac{t_\varepsilon^2}{\gamma_\varepsilon^2}\right)\right) \right] \end{aligned} \quad (\text{A.24})$$

in $[0, \bar{r}_\varepsilon]$ as $\varepsilon \rightarrow 0$. Independently, by (3.30), (3.45), (A.10), (A.12), (A.13) and since $B_\varepsilon(x_\varepsilon) = \gamma_\varepsilon$, for all given $R > 0$, we have that

$$\left\| \frac{B_\varepsilon^{2\tilde{N}_\varepsilon}}{\tilde{N}_\varepsilon! \varphi_{\tilde{N}_\varepsilon}(B_\varepsilon^2)} + \frac{B_\varepsilon^{2N_\varepsilon}}{N_\varepsilon! \varphi_{N_\varepsilon}(B_\varepsilon^2)} \right\|_{L^\infty([0, \min(R\mu_\varepsilon, \bar{r}_\varepsilon)])} = O\left(\frac{1}{\sqrt{N_\varepsilon}}\right)$$

and that

$$\frac{B_\varepsilon^{2N_\varepsilon}}{N_\varepsilon! \varphi_{N_\varepsilon}(B_\varepsilon^2)} \leq 1$$

in $[0, \bar{r}_\varepsilon]$, the second inequality being obvious by (3.5) and (A.15). In the sequel, by (3.31), we may assume that

$$\beta_\varepsilon := \frac{\tilde{N}_\varepsilon}{\gamma_\varepsilon^2} \text{ satisfies } \lim_{\varepsilon \rightarrow 0} \beta_\varepsilon = \beta_0 \in [0, 1], \quad (\text{A.26})$$

up to a subsequence. Now, we give estimates for F_ε given in (A.22). Up to a subsequence, we can split our results according to the following two cases

$$\begin{aligned} \text{Case A: } & \lim_{\varepsilon \rightarrow 0} \frac{\gamma_\varepsilon^2 - \tilde{N}_\varepsilon}{\sqrt{N_\varepsilon}} = +\infty, \\ \text{Case B: } & \frac{\gamma_\varepsilon^2 - \tilde{N}_\varepsilon}{\sqrt{N_\varepsilon}} = O(1). \end{aligned} \quad (\text{A.27})$$

Observe that, since we assume (3.31), all the possible situations are considered in (A.27). Let $(r_\varepsilon)_\varepsilon$ be any sequence such that

$$r_\varepsilon \in [0, \bar{r}_\varepsilon] \quad (\text{A.28})$$

for all ε . We prove that, in (Case A):

$$F_\varepsilon(r_\varepsilon) = \begin{cases} O(\xi_\varepsilon \gamma_\varepsilon \exp(-2t_\varepsilon(r_\varepsilon)(\beta_0 + o(1)))) , & \text{if } B_\varepsilon(r_\varepsilon)^2 \geq \tilde{N}_\varepsilon + \sqrt{\tilde{N}_\varepsilon}, \\ O(\exp(-(1 + \varepsilon_0 + o(1))t_\varepsilon(r_\varepsilon))) , & \text{if } B_\varepsilon(r_\varepsilon)^2 < \tilde{N}_\varepsilon + \sqrt{\tilde{N}_\varepsilon}, \end{cases} \quad (\text{A.29})$$

while we get in (Case B):

$$F_\varepsilon(r_\varepsilon) = \begin{cases} 2t_\varepsilon(r_\varepsilon)\xi_\varepsilon \exp(-2t_\varepsilon(r_\varepsilon)(1 + o(1))) , & \text{if } t_\varepsilon(r_\varepsilon) = o(\gamma_\varepsilon), \\ O(t_\varepsilon(r_\varepsilon)\xi_\varepsilon \exp(-(1 + \varepsilon_0 + o(1))t_\varepsilon(r_\varepsilon))) & \text{if } \gamma_\varepsilon = O(t_\varepsilon(r_\varepsilon)). \end{cases} \quad (\text{A.30})$$

Now we prove (A.29). We start with the first estimate of (A.29). Then, we assume that $B_\varepsilon(r_\varepsilon)^2 \geq \tilde{N}_\varepsilon + \sqrt{\tilde{N}_\varepsilon}$, and thus in particular that

$$1 - \frac{\tilde{N}_\varepsilon}{B_\varepsilon(r_\varepsilon)^2} \geq \frac{1 + o(1)}{\sqrt{\tilde{N}_\varepsilon}}. \quad (\text{A.31})$$

Writing now F_ε according to the first formula of (A.22), using (3.93), (A.17) and

$$\log(1 + t) \leq t \text{ for all } t > -1, \quad (\text{A.32})$$

we get first that

$$F_\varepsilon(r_\varepsilon) \leq \xi_\varepsilon \exp(-2\tau_\varepsilon(r_\varepsilon)\beta_\varepsilon) \int_0^{\gamma_\varepsilon^2 - B_\varepsilon^2} \exp\left(-y \left(1 - \frac{\tilde{N}_\varepsilon}{B_\varepsilon(r_\varepsilon)^2}\right)\right) dy,$$

and conclude the proof of the first estimate of (A.29), by (3.31), (A.13) and (A.31). In order to prove the second estimate of (A.29), it is sufficient to write F_ε according to the second formula of (A.22), to check that

$$\int_{\mathbb{R}} \exp(-s) \frac{s^{\tilde{N}_\varepsilon}}{\tilde{N}_\varepsilon!} ds = 1,$$

that $r_\varepsilon \leq \bar{r}_\varepsilon \leq \rho_\varepsilon$ imply

$$t_\varepsilon(\bar{r}_\varepsilon) \leq (1 - \varepsilon_0)\gamma_\varepsilon^2, \quad (\text{A.33})$$

and to use (A.10), (A.13) and (3.30). Now we turn to the proof of (A.30). Then, we assume that (Case B) in (A.27) holds true and in particular that

$$1 - \beta_\varepsilon = O\left(\frac{1}{\gamma_\varepsilon}\right) \text{ in (Case B)}. \quad (\text{A.34})$$

Writing F_ε according to the third estimate of (A.22), we get that

$$F_\varepsilon = \xi_\varepsilon \exp\left(-\tau_\varepsilon\left(2 - \frac{\tau_\varepsilon}{\gamma_\varepsilon^2}\right)\right) (\gamma_\varepsilon^2 - B_\varepsilon^2) \times \int_0^1 \exp\left((\gamma_\varepsilon^2 - B_\varepsilon^2)y + \tilde{N}_\varepsilon \log\left(1 - \frac{(\gamma_\varepsilon^2 - B_\varepsilon^2)y}{\gamma_\varepsilon^2}\right)\right) dy \quad (\text{A.35})$$

at r_ε . Expanding the log, we easily get the first estimate of (A.30) from (A.13), (A.34), (A.35) and the assumption $t_\varepsilon(r_\varepsilon) = o(\gamma_\varepsilon)$. The second estimate of (A.30) can also be obtained from (A.35) by (A.13), (A.32), (A.33) and (A.34). This

concludes the proof of (A.30). Now, we prove that, in (Case A) of (A.27), we have that

$$\int_0^{\bar{r}_\varepsilon} F_\varepsilon(r) r dr = o\left(\frac{\mu_\varepsilon^2}{\gamma_\varepsilon^4}\right). \quad (\text{A.36})$$

Since $r_\varepsilon \leq \rho_\varepsilon$, we get from (3.14), (3.30), (3.31), (A.29) and by Stirling's formula that

$$\begin{aligned} & \int_{\left\{r \in [0, \bar{r}_\varepsilon], B_\varepsilon(r)^2 \geq \tilde{N}_\varepsilon + \sqrt{\tilde{N}_\varepsilon}\right\}} F_\varepsilon(r) r dr \\ & \lesssim \exp\left(\gamma_\varepsilon^2 [f(\beta_\varepsilon) + O((\log \gamma_\varepsilon)/\gamma_\varepsilon^2)]\right) \times \\ & \begin{cases} \mu_\varepsilon^2 & \text{if } \beta_0 > 1/2, \\ \mu_\varepsilon^2 \exp(\gamma_\varepsilon^2 (1 - \varepsilon_0)(1 - 2\beta_0 + o(1))) & \text{if } \beta_0 \leq 1/2, \end{cases} \end{aligned} \quad (\text{A.37})$$

where f is the continuous function in $[0, 1]$ given for $\beta \in (0, 1]$ by

$$f(\beta) = \beta \log \frac{1}{\beta} + \beta - 1.$$

Independently, since $\bar{r}_\varepsilon \leq \rho_\varepsilon$, if

$$r_\varepsilon \in J_\varepsilon := \left\{r \in [0, \bar{r}_\varepsilon], B_\varepsilon(r)^2 < \tilde{N}_\varepsilon + \sqrt{\tilde{N}_\varepsilon}\right\},$$

then $J_\varepsilon \neq \emptyset$ and $\gamma_\varepsilon^2 \lesssim \tilde{N}_\varepsilon$, by (A.10), (A.13) and (A.33). Thus we have that

$$\gamma_\varepsilon \lesssim \sqrt{\tilde{N}_\varepsilon} \ll t_\varepsilon(r_\varepsilon),$$

using that we are in (Case A) for the last estimate. Then, we get from (A.29) that

$$\int_{J_\varepsilon} F_\varepsilon(r) r dr \lesssim \int_{\{r \leq \rho_\varepsilon, t_\varepsilon \geq \gamma_\varepsilon\}} \exp(-(1 + \varepsilon_0 + o(1))t_\varepsilon(r)) r dr = o\left(\frac{\mu_\varepsilon^2}{\gamma_\varepsilon^4}\right). \quad (\text{A.38})$$

Observe that f and $\beta \mapsto f(\beta) + (1 - 2\beta)/2$ are negative in $[0, 1]$ and $[0, 1/2]$ respectively. Moreover, because of (Case A) and by (3.31), we can check that

$$\beta_\varepsilon = \frac{\tilde{N}_\varepsilon}{\gamma_\varepsilon^2} \leq \frac{1}{1 + \frac{1}{\sqrt{\tilde{N}_\varepsilon}}} \leq 1 - \frac{1 + o(1)}{\sqrt{\tilde{N}_\varepsilon}} \leq 1 - \frac{1 + o(1)}{\gamma_\varepsilon},$$

since $\gamma_\varepsilon^2 \geq \tilde{N}_\varepsilon + \sqrt{\tilde{N}_\varepsilon}$, and then that

$$0 < -f(\beta_\varepsilon) \lesssim 1/\gamma_\varepsilon. \quad (\text{A.39})$$

Thus, we get (A.36) from the first estimate of (A.37) with (A.39), from the second estimate of (A.37) with $1 - \varepsilon_0 < 1 - \sqrt{1/e} < 1/2$ and from (A.38). Computing as in (A.37), we get also that

$$\xi_\varepsilon = o\left(\frac{1}{\gamma_\varepsilon^4}\right) \quad (\text{A.40})$$

in (Case A) (see (A.39)). By (A.13) and the second part of (A.25), using that $\bar{r}_\varepsilon \leq \rho_\varepsilon$, we may rewrite (A.20) as

$$\begin{aligned} \frac{\lambda_\varepsilon \Psi'_{N_\varepsilon}(B_\varepsilon)}{2} = & \frac{4}{\mu_\varepsilon^2 \gamma_\varepsilon} \left[\left(1 - \frac{\tau_\varepsilon}{\gamma_\varepsilon^2} + L_\varepsilon^H \right) \exp(B_\varepsilon^2 - \gamma_\varepsilon^2) - F_\varepsilon \right. \\ & + O\left(\frac{t_\varepsilon}{\gamma_\varepsilon^2} |F_\varepsilon| + \exp(-\gamma_\varepsilon^2) \right) \\ & + O\left(\left(\frac{t_\varepsilon}{\gamma_\varepsilon^2} \exp(B_\varepsilon^2 - \gamma_\varepsilon^2) + |F_\varepsilon| \right) (|L_\varepsilon^H| + |L_\varepsilon^g|) \right) \\ & \left. + O\left(|L_\varepsilon^g| \exp(B_\varepsilon^2 - \gamma_\varepsilon^2) \frac{B_\varepsilon^{2N_\varepsilon}}{N_\varepsilon! \varphi_{N_\varepsilon}(B_\varepsilon^2)} \right) \right]. \end{aligned} \quad (\text{A.41})$$

By (3.84), we clearly have that

$$\int_0^{\rho_\varepsilon} \exp(-\gamma_\varepsilon^2) r dr = o\left(\frac{\mu_\varepsilon^2}{\gamma_\varepsilon^4} \right). \quad (\text{A.42})$$

Integrating by parts, observe that \bar{w}_ε given by (A.11) satisfies

$$\bar{w}_\varepsilon(0) = 0 \text{ and } -r_\varepsilon \bar{w}'_\varepsilon(r_\varepsilon) = \int_0^{r_\varepsilon} (\Delta \bar{w}_\varepsilon) r dr, \quad (\text{A.43})$$

where, still using radial notations, $\bar{w}_\varepsilon(r) = \frac{d\bar{w}_\varepsilon}{dr}(r)$. From now on, we estimate \bar{w}_ε in $[0, \bar{r}_\varepsilon]$ with (A.43). By (1.5) and (A.14), we may expand L_ε^H in (A.41). Now, since (3.81) holds true, by taking the laplacian of B_ε , we get from (A.11) and (A.41) an estimate of $\Delta \bar{w}_\varepsilon$ and then of the RHS of (A.43), for r_ε still as in (A.28). The key observation is that the precise form of the ODE's in (A.2) generates a cancellation, when plugging (A.24) in (A.41). The lower order terms when taking the laplacian of (A.11) are estimated thanks to (A.3). We are left with estimating the lower order terms in (A.41), in both Cases A and B of (A.27). Assume first that we are in (Case A). Estimating these lower order terms amounts to gather the appropriate previous estimates (see (A.21), (A.25), (A.29), (A.36), (A.40), (A.42)). This gives after some a bit long, but elementary computations that

$$\begin{aligned} \int_0^{r_\varepsilon} |(\Delta \bar{w}_\varepsilon)| r dr = & O\left(\|\bar{w}'_\varepsilon\|_{L^\infty([0, r_\varepsilon])} \int_0^{r_\varepsilon/\mu_\varepsilon} \frac{\mu_\varepsilon r^2 dr}{(1+r^2)^{1+\varepsilon_0+o(1)}} \right) \\ & + o\left(\int_0^{r_\varepsilon/\mu_\varepsilon} \frac{r dr}{(1+r^2)^{1+\varepsilon_0+o(1)}} \right). \end{aligned} \quad (\text{A.44})$$

We also use (1.5) and (3.2) to estimate L_ε^H and L_ε^g . The first term in the right hand side of (A.44) uses that, for all $r \in [0, r_\varepsilon]$,

$$|\bar{w}_\varepsilon(r)| \leq r \|\bar{w}'_\varepsilon\|_{L^\infty([0, r_\varepsilon])}.$$

Observe now that (A.44) still holds true in (Case B) of (A.27), replacing (A.29), (A.36) and (A.40) by (A.30) in the above argument. Since $\varepsilon_0 > 1/2$, we clearly get from (A.43) and (A.44) that, in (Case A) and in (Case B),

$$r_\varepsilon |\bar{w}'_\varepsilon(r_\varepsilon)| = O\left(\|\bar{w}'_\varepsilon\|_{L^\infty([0, r_\varepsilon])} \frac{\mu_\varepsilon (r_\varepsilon/\mu_\varepsilon)^3}{1 + (r_\varepsilon/\mu_\varepsilon)^3} \right) + o\left(\frac{(r_\varepsilon/\mu_\varepsilon)^2}{1 + (r_\varepsilon/\mu_\varepsilon)^2} \right). \quad (\text{A.45})$$

Now we prove that

$$\mu_\varepsilon \|\bar{w}'_\varepsilon\|_{L^\infty([0, \bar{r}_\varepsilon])} = o(1). \quad (\text{A.46})$$

If (A.46) does not hold true, then, by (A.45), there exists $s_\varepsilon \in [0, \bar{r}_\varepsilon]$ such that $s_\varepsilon = O(\mu_\varepsilon)$, $\mu_\varepsilon = O(s_\varepsilon)$,

$$|\bar{w}'_\varepsilon(s_\varepsilon)| = \|\bar{w}'_\varepsilon\|_{L^\infty([0, \bar{r}_\varepsilon])} \text{ and } \limsup_{\varepsilon \rightarrow 0} \mu_\varepsilon |\bar{w}'_\varepsilon(s_\varepsilon)| > 0. \quad (\text{A.47})$$

In particular, up to a subsequence, we may assume that there exists $\alpha_0 \in (0, +\infty]$ such that $\bar{r}_\varepsilon/\mu_\varepsilon \rightarrow \alpha_0$ as $\varepsilon \rightarrow 0$. Let \tilde{w}_ε be given by

$$\tilde{w}_\varepsilon(y) = \bar{w}_\varepsilon(\mu_\varepsilon y) / (\mu_\varepsilon \|\bar{w}'_\varepsilon\|_{L^\infty([0, \bar{r}_\varepsilon])}).$$

By (A.45) and (A.47), we get that $(\|(1 + \cdot)\tilde{w}'_\varepsilon\|_{L^\infty([0, \bar{r}_\varepsilon/\mu_\varepsilon])})_\varepsilon$ is a bounded sequence. Then, computing as in (A.44) and by radial elliptic theory with (3.81), we get that $\tilde{w}_\varepsilon \rightarrow \tilde{w}_0$ in $C^2([0, \alpha_0])$ if $\alpha_0 < +\infty$ or in $C^2_{loc}([0, \alpha_0])$ if $\alpha_0 = +\infty$, where \tilde{w}_0 solves

$$\begin{cases} \Delta \tilde{w}_0 = 8 \exp(-2T_0) \tilde{w}_0 \text{ in } B_0(\alpha_0), \\ \tilde{w}_0(0) = 0, \\ \tilde{w}_0 \text{ is radial around } 0 \in \mathbb{R}^2, \end{cases}$$

still making usual radial identifications, and where T_0 is given in (A.1). By standard theory of radial elliptic equation, this implies $\tilde{w}_0 \equiv 0$, which contradicts (A.47) and proves (A.46). Then, since $\bar{w}_\varepsilon(0) = 0$ and by the fundamental theorem of calculus, we get from (A.45) with (A.46) that $\bar{r}_\varepsilon = \rho_\varepsilon$ in (A.12). By the discussion just above (A.13), this concludes the proof of Proposition A.1. \square

REFERENCES

- [1] Adimurthi and O. Druet, *Blow-up analysis in dimension 2 and a sharp form of Trudinger-Moser inequality*, Comm. Partial Differential Equations **29** (2004), no. 1-2, 295–322. MR2038154
- [2] Adimurthi and Massimo Grossi, *Asymptotic estimates for a two-dimensional problem with polynomial nonlinearity*, Proc. Amer. Math. Soc. **132** (2004), no. 4, 1013–1019. MR2045416
- [3] Lennart Carleson and Sun-Yung A. Chang, *On the existence of an extremal function for an inequality of J. Moser*, Bull. Sci. Math. (2) **110** (1986), no. 2, 113–127. MR878016
- [4] Wen Xiong Chen and Congming Li, *Classification of solutions of some nonlinear elliptic equations*, Duke Math. J. **63** (1991), no. 3, 615–622. MR1121147
- [5] D. G. de Figueiredo, P.-L. Lions, and R. D. Nussbaum, *A priori estimates and existence of positive solutions of semilinear elliptic equations*, J. Math. Pures Appl. (9) **61** (1982), no. 1, 41–63. MR664341
- [6] D. G. de Figueiredo and B. Ruf, *Existence and non-existence of radial solutions for elliptic equations with critical exponent in \mathbb{R}^2* , Comm. Pure Appl. Math. **48** (1995), no. 6, 639–655. MR1338473
- [7] Francesca De Marchis, Isabella Ianni, and Filomena Pacella, *Asymptotic analysis for the Lane-Emden problem in dimension two* (2016). Preprint at arXiv:1602.06919.
- [8] ———, *Asymptotic profile of positive solutions of Lane-Emden problems in dimension two*, J. Fixed Point Theory Appl. **19** (2017), no. 1, 889–916. MR3625097
- [9] O. Druet, *Multibumps analysis in dimension 2: quantification of blow-up levels*, Duke Math. J. **132** (2006), no. 2, 217–269. MR2219258
- [10] Olivier Druet and Pierre-Damien Thizy, *Multi-bumps analysis for Trudinger-Moser nonlinearities I-Quantification and location of concentration points* (2018), 64. To appear in J. Eur. Math. Soc. (JEMS).
- [11] Pierpaolo Esposito, Monica Musso, and Angela Pistoia, *Concentrating solutions for a planar elliptic problem involving nonlinearities with large exponent*, J. Differential Equations **227** (2006), no. 1, 29–68. MR2233953
- [12] Martin Flucher, *Extremal functions for the Trudinger-Moser inequality in 2 dimensions*, Comment. Math. Helv. **67** (1992), no. 3, 471–497. MR1171306
- [13] B. Gidas, Wei Ming Ni, and L. Nirenberg, *Symmetry and related properties via the maximum principle*, Comm. Math. Phys. **68** (1979), no. 3, 209–243. MR544879

- [14] David Gilbarg and Neil S. Trudinger, *Elliptic partial differential equations of second order*, Classics in Mathematics, Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition. MR1814364 (2001k:35004)
- [15] Zheng-Chao Han, *Asymptotic approach to singular solutions for nonlinear elliptic equations involving critical Sobolev exponent*, Ann. Inst. H. Poincaré Anal. Non Linéaire **8** (1991), no. 2, 159–174. MR1096602
- [16] Andrea Malchiodi and Luca Martinazzi, *Critical points of the Moser-Trudinger functional on a disk*, J. Eur. Math. Soc. (JEMS) **16** (2014), no. 5, 893–908. MR3210956
- [17] Gabriele Mancini and Luca Martinazzi, *The Moser-Trudinger inequality and its extremals on a disk via energy estimates*, Calc. Var. Partial Differential Equations **56** (2017), no. 4, Art. 94, 26. MR3661018
- [18] J. B. McLeod and L. A. Peletier, *Observations on Moser’s inequality*, Arch. Rational Mech. Anal. **106** (1989), no. 3, 261–285. MR981664
- [19] J. Moser, *A sharp form of an inequality by N. Trudinger*, Indiana Univ. Math. J. **20** (1970/71), 1077–1092. MR0301504
- [20] Alexander R. Pruss, *Nonexistence of maxima for perturbations of some inequalities with critical growth*, Canad. Math. Bull. **39** (1996), no. 2, 227–237. MR1390360
- [21] Xiaofeng Ren and Juncheng Wei, *On a two-dimensional elliptic problem with large exponent in nonlinearity*, Trans. Amer. Math. Soc. **343** (1994), no. 2, 749–763. MR1232190
- [22] ———, *Single-point condensation and least-energy solutions*, Proc. Amer. Math. Soc. **124** (1996), no. 1, 111–120. MR1301045
- [23] Michael Struwe, *Critical points of embeddings of $H_0^{1,n}$ into Orlicz spaces*, Ann. Inst. H. Poincaré Anal. Non Linéaire **5** (1988), no. 5, 425–464. MR970849
- [24] Pierre-Damien Thizy, *Sharp quantization for Lane-Emden problems in dimension two* (2019). To appear in Pac. J. Math.

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