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SOME INEQUALITIES INVOLVING THE EXTENDED GAMMA AND BETA FUNCTIONS

KOTTAKKARAN SOOPPY NISAR, FENG QI, GAUHAR RAHMAN, SHAHID MUBEEN,
AND MUHAMMAD ARSHAD

ABSTRACT. In the paper, the authors present some inequalities involving the extended gamma and beta functions via some classical inequalities such as Chebychev's inequality for synchronous (or asynchronous respectively) mappings, give a new proof of the log-convexity of the extended gamma and beta functions by using the Hölder inequality, and introduce a Turán type mean inequality for the Kummer confluent k -hypergeometric function.

1. INTRODUCTION

The gamma function Γ can be defined [26, 30, 33, 34] by

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt, \quad \Re(z) > 0.$$

Alternatively, it can also be defined [22] by

$$\Gamma(z) = \lim_{n \rightarrow \infty} \frac{n! n^{z-1}}{(z)_n},$$

where $(z)_n$ for $z \neq 0$ is the Pochhammer symbol defined [29] as

$$(z)_n = \begin{cases} z(z+1)(z+2) \cdots (z+n-1), & n \geq 1; \\ 1, & n = 0. \end{cases}$$

The relation between $(z)_n$ and $\Gamma(z)$ is

$$(z)_n = \frac{\Gamma(z+n)}{\Gamma(z)}.$$

The beta function $B(x, y)$ can be defined [19, 23, 24] by

$$B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt, \quad \Re(x), \Re(y) > 0$$

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and can be expressed by

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad \Re(x), \Re(y) > 0.$$

In 1994, Chaudhry and Zubair [4] introduced the extended gamma function

$$\Gamma_b(z) = \int_0^\infty t^{z-1} e^{-t-bt^{-1}} dt, \quad \Re(z) > 0, \quad b \geq 0. \quad (1.1)$$

If $b = 0$, then Γ_b becomes the classical gamma function Γ .

In 1997, Chaudhry et al [3] introduced the extended beta function

$$B_b(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} e^{-b/t(1-t)} dt, \quad \Re(b), \Re(x), \Re(y) > 0. \quad (1.2)$$

It is clear that $B_0(x, y) = B(x, y)$.

In 2009, Barnard et al [1] established three inequalities

$$\begin{aligned} [\phi(a, a+b, x)]^2 &> \phi(a+v, a+b, x)\phi(a-v, a+b, x), \\ [\phi(a, c, x)]^2 &> \phi(a+v, c, x)\phi(a-v, c, x), \end{aligned}$$

and

$$\begin{aligned} A(\phi(a+v, a+b, x), \phi(a-v, a+b, x)) &> \phi(a, a+b, x) \\ &> G(\phi(a+v, a+b, x), \phi(a-v, a+b, x)), \end{aligned}$$

where $A(\alpha, \beta) = \frac{\alpha+\beta}{2}$ and $G(\alpha, \beta) = \sqrt{\alpha\beta}$ are the arithmetic and geometric means and

$$\phi(z) = \sum_{n=0}^{\infty} \frac{(a)_n x^n}{(b)_n n!}$$

is the Kummer confluent hypergeometric function [27, 30].

The Kummer confluent hypergeometric k -function is defined by

$$\phi_k(a, b, x) = \sum_{n=0}^{\infty} \frac{(a)_{n,k} x^n}{(b)_{n,k} n!},$$

where

$$(a)_{n,k} = a(a+k)(a+2k)\cdots[a+(n-1)k]$$

for $n \geq 1$ and $k > 0$ with $(a)_{0,k} = 1$ is the Pochhammer k -symbol which can also be rewritten as

$$(a)_{n,k} = \frac{\Gamma_k(a+nk)}{\Gamma_k(a)}$$

and the gamma k -function $\Gamma_k(a)$ is defined [6] by

$$\Gamma_k(a) = \int_0^\infty t^{a-1} e^{-t^k/k} dt$$

In 2012, Mubeen [17] introduced the k -analogue of Kummer's transformation as

$$\phi_k(a, b, x) = e^x \phi_k(a, b - a, -x). \quad (1.3)$$

In Section 2, we prepare two lemmas. In Section 3, we discuss applications of some integral inequalities such as Chebychev's integral inequalities. In Section 4, we prove the logarithmic convexity of the extended gamma and beta functions. In the last section, we introduce a mean inequality of Turán type for the Kummer k -function.

2. LEMMAS

In order to obtain our main results, we need the following lemmas.

Lemma 2.1 (Chebychev's integral inequality [7, 8, 12, 25]). *Let $f, g, h : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be mappings such that $h(x) \geq 0$, $h(x)f(x)g(x)$, $h(x)f(x)$, and $h(x)g(x)$ are integrable on I . If $f(x)$ and $g(x)$ are synchronous (or asynchronous respectively) on I , that is,*

$$[f(x) - f(y)][g(x) - g(y)] \gtrless 0$$

for all $x, y \in I$, then

$$\int_I h(x) dx \int_I h(x)f(x)g(x) dx \gtrless \int_I h(x)f(x) dx \int_I h(x)g(x) dx.$$

Lemma 2.2 (Hölder's inequality [31, 32]). *Let p and q be positive real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$ and $f, g : [c, d] \rightarrow \mathbb{R}$ be integrable functions. Then*

$$\left| \int_c^d f(x)g(x) dx \right| \leq \left[\int_c^d |f(x)|^p dx \right]^{1/p} \left[\int_c^d |g(x)|^q dx \right]^{1/q}.$$

3. INEQUALITIES VIA CHEBYCHEV'S INTEGRAL INEQUALITY

In this section, we prove some inequalities involving extended gamma and beta functions via Chebychev's integral inequality in Lemma 2.1.

Theorem 3.1. *For positive numbers r, s, p , and q satisfying*

$$(p - r)(q - s) \gtrless 0, \quad (3.1)$$

we have

$$B_b(r, s)B_b(p, q) \gtrless B_b(p, s)B_b(r, q), \quad b \geq 0. \quad (3.2)$$

Proof. Consider mappings $f, g, h : [0, 1] \rightarrow [0, \infty)$ given by

$$f(x) = x^{p-r}, \quad g(x) = (1-x)^{q-s}, \quad \text{and} \quad h(x) = x^{r-1}(1-x)^{s-1}e^{-b/x(1-x)}.$$

It is easy to see that

$$f'(x) = (p-r)x^{p-r-1}, \quad g'(x) = (s-q)(1-x)^{q-s-1}, \quad x \in (0, 1).$$

By virtue of Lemma 2.1 and the condition (3.1), we observe that the mappings f and g are synchronous (asynchronous) on $[0, 1]$ and h is non-negative on $[0, 1]$. Thus, using Chebychev's integral inequality for the function f, g , and h arrives at

$$\begin{aligned} & \int_0^1 x^{r-1}(1-x)^{s-1}e^{-b/x(1-x)} dx \int_0^1 x^{p-r}(1-x)^{q-s}x^{r-1}(1-x)^{s-1}e^{-b/x(1-x)} dx \\ & \geq \int_0^1 x^{r-1}x^{p-r}(1-x)^{s-1}e^{-b/x(1-x)} dx \int_0^1 x^{r-1}(1-x)^{s-1}(1-x)^{q-s}e^{-b/x(1-x)} dx \end{aligned}$$

which implies

$$\begin{aligned} & \int_0^1 x^{r-1}(1-x)^{s-1}e^{-b/x(1-x)} dx \int_0^1 x^{p-1}(1-x)^{q-1}e^{-b/x(1-x)} dx \\ & \geq \int_0^1 x^{p-1}(1-x)^{s-1}e^{-b/x(1-x)} dx \int_0^1 x^{r-1}(1-x)^{q-1}e^{-b/x(1-x)} dx. \end{aligned}$$

Further applying (1.2) leads to the desired inequality (3.2). \square

Corollary 3.1. *For $r > 0$, we have*

$$B_b(r, p) \geq \sqrt{B_b(p, p)B_b(r, r)}.$$

Proof. This follows immediately from setting $q = p$ and $s = r$ in Theorem 3.1. \square

Remark 3.1. When setting $b = 0$ in Theorem 3.1 and Corollary 3.1, we can obtain those corresponding results for Euler's beta function.

Theorem 3.2. *Let m, p and r be positive real numbers such that $p > r > -m$. If $r(p - m - r) \geq 0$, then*

$$\Gamma_b(m)\Gamma_b(p) \geq \Gamma_b(p-r)\Gamma_b(m+r). \quad (3.3)$$

Proof. Let us define the mappings $f, g, h : [0, \infty) \rightarrow [0, \infty)$ given by

$$f(t) = t^{p-r-m}, \quad g(t) = t^r, \quad \text{and} \quad h(t) = t^{m-1}e^{-t-bt^{-1}}.$$

If $r(p - m - r) \geq 0$, then we can claim that the mappings f and g are synchronous (asynchronous) on $(0, \infty)$. Thus, by applying Chebychev's inequality on $I = (0, \infty)$ to the functions f, g and h defined above, we can write

$$\begin{aligned} & \int_0^\infty t^{m-1}e^{-t-bt^{-1}} dt \int_0^\infty t^{p-r-m}t^r t^{m-1}e^{-t-bt^{-1}} dt \\ & \geq \int_0^\infty t^{p-r-m}t^{m-1}e^{-t-bt^{-1}} dt \int_0^\infty t^r t^{m-1}e^{-t-bt^{-1}} dt. \end{aligned}$$

This implies that

$$\int_0^\infty t^{m-1}e^{-t-bt^{-1}} dt \int_0^\infty t^{p-1}e^{-t-bt^{-1}} dt$$

$$\geq \int_0^\infty t^{p-r-1} e^{-t-bt^{-1}} dt \int_0^\infty t^{m+r-1} e^{-t-bt^{-1}} dt.$$

By (1.1), we acquire the required inequality (3.3). \square

Corollary 3.2. *If $p > 0$ and $q \in \mathbb{R}$ with $|q| < p$, then*

$$\Gamma_b(p) \leq [\Gamma_b(p-q)\Gamma_b(p+q)]^{1/2}.$$

Proof. By setting $m = p$ and $r = q$ in Theorem 3.2, we obtain $r(p-m-r) = -q^2 \leq 0$ and then the inequality (3.3) provides the desired Corollary 3.2. \square

Theorem 3.3. *If $m, n > 0$ are similarly (oppositely) unitary, then*

$$\Gamma_b(m+n+b) \geq \frac{\Gamma_b(m+b+1)\Gamma_b(n+b+1)}{\Gamma_b(b+2)}.$$

Proof. Consider the mappings $f, g, h : [0, \infty) \rightarrow [0, \infty)$ defined by

$$f(t) = t^{m-1}, \quad g(t) = t^{n-1}, \quad \text{and} \quad h(t) = t^{b+1} e^{-t-bt^{-1}}.$$

Now if the condition $(m-1)(n-1) \geq 0$ holds, then Chebychev's integral inequality applied to the functions f, g , and h means

$$\begin{aligned} \int_0^\infty t^{b+1} e^{-t-bt^{-1}} dt \int_0^\infty t^{m-1} t^{n-1} t^{b+1} e^{-t-bt^{-1}} dt \\ \geq \int_0^\infty t^{m-1} t^{b+1} e^{-t-bt^{-1}} dt \int_0^\infty t^{n-1} t^{b+1} e^{-t-bt^{-1}} dt. \end{aligned}$$

This implies that

$$\begin{aligned} \int_0^\infty t^{b+1} e^{-t-bt^{-1}} dt \int_0^\infty t^{m+n+b-1} e^{-t-bt^{-1}} dt \\ \geq \int_0^\infty t^{m+b} e^{-t-bt^{-1}} dt \int_0^\infty t^{n+b} e^{-t-bt^{-1}} dt. \end{aligned}$$

By the definition of extended gamma function, we have

$$\Gamma_b(b+2)\Gamma_b(m+n+b) \geq \Gamma_b(m+b+1)\Gamma_b(n+b+1),$$

or

$$\Gamma_b(m+n+b) \geq \frac{\Gamma_b(m+b+1)\Gamma_b(n+b+1)}{\Gamma_b(b+2)}.$$

The required proof is complete. \square

Corollary 3.3. *If $b = 0$, then*

$$\Gamma(m+n) \geq \frac{m\Gamma(m)\Gamma(n)}{\Gamma(2)}.$$

Theorem 3.4. *If m and n are positive real numbers such that m and n are similarly (oppositely) unitary, then*

$$\Gamma_b(b+1)\Gamma_b(m+n+b+1) \geq \Gamma_b(m+b+1)\Gamma_b(n+b+1), \quad b \geq 0.$$

Proof. Consider the mappings $f, g, h : [0, \infty) \rightarrow [0, \infty)$ defined by

$$f(t) = t^m, \quad g(t) = t^n, \quad \text{and} \quad h(t) = t^b e^{-t-bt^{-1}}.$$

If the conditions of Theorem 3.2 hold, then the mappings f and g are synchronous (asynchronous) on $[0, \infty)$. Thus, by applying Chebychev's integral inequality in Lemma 2.1 to the functions f, g and h defined above, we have

$$\begin{aligned} \int_0^\infty t^b e^{-t-bt^{-1}} dt \int_0^\infty t^m t^n t^b e^{-t-bt^{-1}} dt \\ \geq \int_0^\infty t^m t^b e^{-t-bt^{-1}} dt \int_0^\infty t^n t^b e^{-t-bt^{-1}} dt. \end{aligned}$$

This implies that

$$\begin{aligned} \int_0^\infty t^b e^{-t-bt^{-1}} dt \int_0^\infty t^{m+n+b} e^{-t-bt^{-1}} dt \\ \geq \int_0^\infty t^{m+b} e^{-t-bt^{-1}} dt \int_0^\infty t^{n+b} e^{-t-bt^{-1}} dt. \end{aligned}$$

Thus by the definition of extended gamma function, we have

$$\Gamma_b(b+1)\Gamma_b(m+n+b+1) \geq \Gamma_b(m+b+1)\Gamma_b(n+b+1).$$

The required proof is complete. \square

Corollary 3.4. *If $b = 0$, then*

$$\Gamma(m+n) \geq \frac{mn\Gamma(m)\Gamma(n)}{(m+n)}.$$

4. LOG-CONVEXITY OF THE EXTENDED GAMMA AND BETA FUNCTION

It is well known that, if $f > 0$ and $\ln f$ is convex, then f is said to be a logarithmically convex function. Every logarithmically convex must be convex. See [18] and [21, Remark 1.9]. In this section, we verify the log-convexity of extended gamma and extended beta functions.

Theorem 4.1. *Let p and q be positive numbers such that $\frac{1}{p} + \frac{1}{q} = 1$. Then the extended gamma function $\Gamma_b : (0, \infty) \rightarrow \mathbb{R}$ is logarithmically convex.*

Proof. Since

$$\Gamma_b\left(\frac{x}{p} + \frac{y}{q}\right) \leq [\Gamma_b(x)]^{1/p} [\Gamma_b(y)]^{1/q},$$

see [5], letting $\lambda = \frac{1}{p}$ and $(1-\lambda) = \frac{1}{q}$ leads to

$$\Gamma_b[\lambda x + (1-\lambda)y] \leq [\Gamma_b(x)]^\lambda [\Gamma_b(y)]^{(1-\lambda)}.$$

As a result, the function Γ_b is logarithmically convex. \square

Theorem 4.2 ([16, Theorem 2]). *As a function of two variables, the extended beta function B_b is logarithmically convex on $(0, \infty) \times (0, \infty)$.*

Proof. Let $(p, q), (r, s) \in (0, \infty) \times (0, \infty)$ and $c, d \geq 0$ with $c + d = 1$. Then

$$B_b(c(p, q) + d(r, s)) = B_b(cp + dm, cq + dn). \quad (4.1)$$

Applying the definition of extended beta function on the right hand side of (4.1) shows

$$\begin{aligned} B_b(c(p, q) + d(r, s)) &= \int_0^1 t^{cp+dm-1} (1-t)^{cq+dn-1} e^{-b/t(1-t)} dt \\ &= \int_0^1 t^{cp+dm-(c+d)} (1-t)^{cq+dn-(c+d)} e^{-b(c+d)/t(1-t)} dt \\ &= \int_0^1 t^{c(p-1)} t^{d(r-1)} (1-t)^{c(q-1)} (1-t)^{d(s-1)} e^{-bc/t(1-t)} e^{-bd/t(1-t)} dt \\ &= \int_0^1 [t^{p-1} (1-t)^{q-1} e^{-b/t(1-t)}]^c [t^{r-1} (1-t)^{s-1} e^{-b/t(1-t)}]^d dt. \end{aligned}$$

Thus, it follows that

$$B_b(c(p, q) + d(r, s)) \leq [B_b(p, q)]^c [B_b(r, s)]^d$$

which implies the logarithmic convexity of B_b on $(0, \infty) \times (0, \infty)$. \square

5. A MEAN INEQUALITY OF THE TURÁN TYPE FOR THE KUMMER k -FUNCTION

In this section, we present a mean inequality which involving the confluent hypergeometric k -function. For this, we consider the relation

$$\phi_k(a+k, b, x) - \phi_k(a, b, x) = \frac{kx}{b} \phi_k(a+k, b+k, x), \quad k > 0. \quad (5.1)$$

Theorem 5.1. *For $a, b, k > 0$ and $v \in \mathbb{N}$ with $a, b \geq v - k$, the inequality*

$$[\phi_k(a, a+b, x)]^2 > \phi_k(a+v, a+b, x) \phi_k(a-v, a+b, x) \quad (5.2)$$

is valid for all nonzero $x \in \mathbb{R}$.

Proof. Assume that $x > 0$. For $c \neq 0, -1, -2, \dots$, define

$$f_{v,k}(x) = [\phi_k(a, c, x)]^2 - \phi_k(a+v, c, x) \phi_k(a-v, a+b, x)$$

and

$$f_{v+k,k}(x) = \phi_k(a, c, x)^2 - \phi_k(a+v+k, c, x) \phi_k(a-v-k, a+b, x).$$

From (5.1), it follows that

$$\begin{aligned} f_{v+k,k}(x) - f_{v,k}(x) &= \phi_k(a+v, c, x) \phi_k(a-v, c, x) \\ &\quad - \phi_k(a+v+k, c, x) \phi_k(a-v-k, c, x) \end{aligned}$$

$$\begin{aligned}
&= \phi_k(a-v, c, x)[\phi_k(a+v, c, x) - \phi_k(a+v+k, c, x)] \\
&\quad + \phi_k(a+v+k, c, x)[\phi_k(a-v, c, x) - \phi_k(a-v-k, c, x)] \\
&= \phi_k(a-v, c, x) \left(\frac{-kx}{c} \right) \phi_k(a+v+k, c+k, x) \\
&\quad + \phi_k(a+v+k, c, x) \left(\frac{kx}{c} \right) \phi_k(a-v, c+k, x) \\
&= \frac{kx}{c} g_{v,k}(x),
\end{aligned}$$

where

$$\begin{aligned}
g_{v,k}(x) &= \phi_k(a+v+k, c, x)\phi_k(a-v, c+k, x) \\
&\quad - \phi_k(a-v, c, x)\phi_k(a+v+k, c+k, x).
\end{aligned}$$

Accordingly, by the Cauchy product, we have

$$\begin{aligned}
g_{v,k}(x) &= \sum_{s=0}^{\infty} \sum_{r=0}^s \frac{(a+v+k)_{s,k}(a-v)_{s-r,k}}{r!(s-r)!} \\
&\quad \times \left[\frac{1}{(c)_{s,k}(c+k)_{s-r,k}} - \frac{1}{(c)_{s-r,k}(c+k)_{s,k}} \right] x^s \\
&= \sum_{s=0}^{\infty} \sum_{r=0}^s \frac{(a+v+k)_{s,k}(a-v)_{s-r,k}}{r!(s-r)!} \left[\frac{(c+mk) - (c+nk-mk)}{c(c+k)_{s,k}(c+k)_{s-r,k}} \right] x^s \\
&= \frac{k}{c} \sum_{s=0}^{\infty} \sum_{r=0}^s T_{s,r,k}(2r-s)x^s,
\end{aligned}$$

where

$$T_{s,r,k} = \frac{(a+v+k)_{s,k}(a-v)_{s-r,k}}{r!(s-r)!(c+k)_{s,k}(c+k)_{s-r,k}}.$$

If s is even, then

$$\begin{aligned}
\sum_{r=0}^s T_{s,r,k}(2r-s) &= \sum_{r=0}^{s/2-1} T_{s,r,k}(2r-s) + \sum_{r=s/2+1}^s T_{s,r,k}(2r-s) \\
&= \sum_{r=0}^{s/2-1} T_{s,r,k}(2r-s) + \sum_{r=0}^{s/2-1} T_{s,s-r,k}(2(s-r)-s) \\
&= \sum_{r=0}^{\lceil (s-1)/2 \rceil} (T_{s,s-r,k} - T_{s,r,k})(s-2r),
\end{aligned}$$

where $\lceil x \rceil$ denotes the ceiling function whose value is the greatest integer not more than x . Similarly, if s is odd,

$$\sum_{r=0}^s T_{s,r,k}(2r-s) = \sum_{r=0}^{\lceil (s-1)/2 \rceil} (T_{s,s-r,k} - T_{s,r,k})(s-2r).$$

Accordingly,

$$\begin{aligned} f_{v+k,k}(x) - f_{v,k}(x) &= \frac{kx}{c} g_{v,k}(x) \\ &= \frac{k^2 x}{c^2} \sum_{s=1}^{\infty} \sum_{r=0}^{\lceil (s-1)/2 \rceil} (T_{s,s-r,k} - T_{s,r,k})(s-2r)x^s. \end{aligned} \quad (5.3)$$

Carefully simplifying gives

$$\begin{aligned} T_{s,s-r,k} - T_{s,r,k} &= \frac{(a+v+k)_{s,k}(a-v)_{s,k} - (a+v+1)_{s,k}(a-v)_{s,k}}{r!(s-r)!(c+k)_{s-r,k}(c+k)_{s,k}} \\ &= \frac{(a+v+k)_{s,k}(a-v)_{s,k}}{r!(s-r)!(c+k)_{s-r,k}(c+k)_{s,k}} \left[\frac{(a+v+k)_{s-r,k}}{(a+v+k)_{s,k}} - \frac{(a-v)_{s-r,k}}{(a-v)_{s,k}} \right] \\ &= \frac{(a+v+k)_{s,k}(a-v)_{s,k}}{r!(s-r)!(c+k)_{s-r,k}(c+k)_{s,k}} [h_k(a+v+k) - h_k(a-v)], \end{aligned} \quad (5.4)$$

where $h_k(x) = \frac{\Gamma'_k(x)}{\Gamma_k(x)}$. For $x > 0$ and $s-r > r$, that is, $\lceil \frac{s-1}{2} \rceil \geq r$, the logarithmic derivatives of h_k is

$$\frac{h'_k(x)}{h_k(x)} = \psi_k(x + (s-r)k) - \psi_k(x + nk) > 0,$$

where $\psi_k = \frac{\Gamma'_k}{\Gamma_k}$ is the digamma k -function (see [6, 11, 18]). Hence, the function h_k is increasing under the condition stated. This fact together with the aid of (5.3) and (5.4) yields

$$\begin{aligned} f_{v+k,k}(x) - f_{v,k}(x) &= \frac{kx}{c} g_{v,k}(x) \\ &= \frac{k^2 x}{c^2} \sum_{s=1}^{\infty} \sum_{r=0}^{\lceil (s-1)/2 \rceil} (T_{s,s-r,k} - T_{s,r,k})(s-2r)x^s > 0, \end{aligned} \quad (5.5)$$

where $a \geq v \geq 0$, $x > 0$, $c+k > 0$, and $c \neq 0$. Consequently, from (5.5), it follows that

$$f_{v+k,k}(x) = [f_{v+k,k}(x) - f_{v,k}(x)] + [f_{v,k}(x) - f_{v-k,k}(x)] + \cdots + [f_{1,k}(x) - f_{0,k}(x)]$$

is positive for $a \geq v \geq v-k \geq v-2k \geq \cdots \geq 0$ and $f_{0,k}(x) = 0$. Now replacing v by $v-k$ gives that

$$f_{v,k}(x) > 0, \quad x > 0, v \in \mathbb{N}, a \geq v-k. \quad (5.6)$$

Therefore, the function $f_{v,k}$ is absolutely monotonic on $(0, \infty)$, that is, $f_{v,k}^{(\ell)}(x) > 0$ for $\ell = 0, 1, 2, \dots$. This proves Theorem 5.1 for the case $x > 0$.

Now suppose that $x < 0$, $a, b > 0$, and $v \in \mathbb{N}$ with $a, b \geq v-k$. Since $\phi_k(a, c, x)$ is symmetric in a and b , by interchanging a and b in Theorem 5.1, we obtain

$$\phi_k(b, a+b, -x)^2 - \phi_k(b+v, a+b, -x)\phi_k(b-v, a+b, -x) > 0.$$

By using Kummer's transformation (1.3), we have

$$e^{-2x}\phi_k(a, a+b, x)^2 - e^{-2x}\phi_k(a-v, a+b, x)\phi_k(a+v, a+b, x) > 0.$$

Thus, Theorem 5.1 also holds for $x < 0$. \square

Corollary 5.1. *If $a > 0$ and $c+k > 0$ with $c \neq 0$, then the inequality*

$$\phi_k(a, c, x)^2 \geq \phi_k(a-v, c, x)\phi_k(a+v, c, x)$$

holds for any $v \in \mathbb{N}$ with $a \geq v-k$.

Proof. This follows directly from the proof of Theorem 5.1 and the fact that the equation (5.6) holds under the conditions $c+k > 0$ and $c \neq 0$. \square

Corollary 5.2. *If $v \in \mathbb{N}$ and $a, b \geq v$, then*

$$\begin{aligned} A(\phi_k(a+v, a+b, x), \phi_k(a-v, a+b, x)) &> \phi_k(a, a+b, x) \\ &> G(\phi_k(a+v, a+b, x), \phi_k(a-v, a+b, x)) \end{aligned} \quad (5.7)$$

for all nonzero $x \in \mathbb{R}$, where A and G are respectively the arithmetic and geometric means.

Proof. First assume $x \geq 0$ and $a, b \geq v$ for $v \in \mathbb{N}$. Then the left hand side inequality in (5.7) is a direct consequence of the facts that

$$A((a+v)_{s,k}, (a-v)_{s,k}) = (a)_{s,k}$$

for $s = 0, 1$ and

$$A((a+v)_{s,k}, (a-v)_{s,k}) > (a)_{s,k}$$

for $s \geq 2$. Hence, by induction, we have

$$\begin{aligned} A(\phi_k(a+v, a+b, x), \phi_k(a-v, a+b, x)) &= \sum_{s=0}^{\infty} \frac{A((a+v)_{s,k}, (a-v)_{s,k})x^s}{(a+b)_{s,k}s!} \\ &> \sum_{s=0}^{\infty} \frac{(a)_{s,k}x^s}{(a+b)_{s,k}s!} = \phi_k(a, a+b, x). \end{aligned}$$

For $x \geq 0$, the right hand side inequality in (5.7) follows from taking square root of (5.2). The proof of Corollary 5.2 for $x \geq 0$ is thus complete.

Now assume $x < 0$ with $a, b \geq v$. Interchanging a and b in (5.7) arrives at

$$\begin{aligned} A(\phi_k(b+v, a+b, -x), \phi_k(b-v, a+b, -x)) &> \phi_k(b, a+b, -x) \\ &> G(\phi_k(b+v, a+b, -x), \phi_k(b-v, a+b, -x)). \end{aligned}$$

Making use of the k -analogue of Kummer's transformation and the homogeneity of A and G acquires

$$\begin{aligned}
e^{-x}A(\phi_k(a-v, a+b, x), \phi_k(a+v, a+b, x)) &> e^{-x}\phi_k(a, a+b, x) \\
&> e^{-x}G(\phi_k(a-v, a+b, x), \phi_k(a+v, a+b, x)).
\end{aligned}$$

Consequently, Theorem (5.7) also follows for $x < 0$. □

Remark 5.1. In Section 5, we have established a Turán type and mean inequality for k -analogue of Kummer confluent hypergeometric function. If we letting $k \rightarrow 1$, then we can conclude corresponding inequalities of confluent hypergeometric function.

Remark 5.2. In [2], some inequalities of the Turán type for confluent hypergeometric functions of the second kind were also discovered.

Remark 5.3. By the way, we note that the papers [9, 10, 13, 28, 34, 35] belong to the same series in which inequalities and complete monotonicity for functions involving the gamma function $\Gamma(x)$ and the logarithmic function $\ln(1+x)$ were discussed.

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