A higher-order multiscale method for second order elliptic equations

Aboubacar Konaté

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The aim of this work is to study a new discontinuous Galerkin (dG) discretization to the multiscale method introduced in [AB06] for solving a elliptic equation with parameters varying at a very small space scale. This is motivated by the fact that in some applications (for example in transport flow), particulary when the parameters are discontinuous or when the geometry is complex (non-conformities, faults, ...), dG discretizations are more suitable than those based on finite volume or continuous finite elements. Using standard methods when the parameters are varying at a very small space scale is demanding in term of computing times and in term of computer memory. Roughly speaking, multiscale methods consist in building basis functions which take into account the variation of parameters which leads to better balance between accuracy and computing times. We introduce a new Dirichlet-Neumann boundary condition to the so-called cell problems in order to reduce the resonance error (sometimes to remove it completely). An error estimate is established where the parameters are assumed to be periodic. Numerical illustrations are made both in periodic and non-periodic case.

1 Introduction

We consider, in this work, the following elliptic equation in a domain Ω

$$-\operatorname{div}\left(\mathbf{K}\left(\frac{x}{\epsilon}\right)\nabla p_\epsilon\right) = f \quad \text{in } \Omega \tag{1}$$

with homogeneous Dirichlet boundary condition. This equation is used to model for example the pressure p_ϵ in the porous media $\Omega \subset \mathbb{R}^d (d = 2, 3)$ where $\mathbf{K} \in \mathbb{R}^{d\times d}$ is the permeability and f is the term source.

Using classical methods (finite elements method, finite volume method, ...) to solve numerically the equation (1) is not satisfactory. Indeed, an accurate and efficient approximation is obtained with standard methods only if the mesh-size h is smaller than ϵ. This leads to tremendous amount of computer memory and CPU time. In order to overcome this difficulty, many methods are proposed and others are under investigation. Among these methods, there are the so-called multiscale methods.

The first multiscale finite element method was introduced by T. Hou and X. Wu in [HW97] in 1997 for solving elliptic problems with diffusion coefficients varying at a very small space scale. In these methods, we consider two meshes : a fine mesh on which the diffusion coefficients are defined and a coarse mesh. The main idea behind these methods consists in computing numerically basis functions which take into account the variations of the diffusion coefficients at the fine scale. These basis functions are used onto for assembling and solving a linear system on the coarse mesh. Then, in a second step, they are

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used for projecting the coarse solution onto the fine mesh. The reader can refer to [EH09, FW17] for comprehensive review of existing multiscale methods for elliptic problems.

In the same spirit, multiscale methods were introduced for convection-diffusion problems. We have the Heterogeneous Multiscale Method (HMM) proposed in [HO10, AE14]. A multiscale method for solving steady convection-diffusion problems was proposed in [Elf15]. This method is not based on homogenization and error estimates was established without periodic assumption on the coefficient. More recently, a new method based on the same idea that in [HW97] is proposed in [BLM15, Mad16]. A multiscale method based on the same idea that is studied in this work was introduced in [OADE15, Kon17]).

In this work, we study the multiscale method introduced for the first time in [AB06]. The main distinction of our approach from its original formulation in [AB06] is the use of a discontinuous Galerkin discretization.

The paper is organized as follows. In the next Section, we recall some results from the homogenization theory which allows us to justify the method. In Section 3, we present the idea behind the studied method. The Section 4 is devoted to our new multiscale discretization. Numerical examples are presented in Section 5. Finally, Section 6 is devoted to establish an error estimate before we conclude in Section 7.

2 Homogenization results

In this section, we recall some results from the homogenization theory which will be used to establish an error estimate. We assume that the permeability $\mathbf{K}$ is periodic with the unit cell $Y = [0, 1]^d$ as its period. The homogenized problem is defined as follows

$$-\operatorname{div}\left(\mathbf{K}^*\nabla p\right) = f \quad \text{in } \Omega, \tag{2}$$

with homogeneous Dirichlet boundary condition. The homogenized tensor $\mathbf{K}^*$ is defined as follows :

$$\mathbf{K}^*_{i,j} := \int_Y \mathbf{K}(y)\left(\nabla_y w_i(y) + \mathbf{e}_i\right)\cdot\left(\nabla_y w_j(y) + \mathbf{e}_j\right) dy. \tag{3}$$

where w_i is Y-periodic and solution of the following problem :

$$\begin{cases} \operatorname{div}_y\left(\mathbf{K}(y)\left(\nabla_y w_i(y) + \mathbf{e}_i\right)\right) = 0, & \forall y \in Y, \\ \int_Y w_i(y)\, dy = 0. \end{cases} \tag{4}$$

We define the fonction p_1 as follows :

$$p_1\left(x, \frac{x}{\epsilon}\right) := \sum_{i=1}^{d} w_i\left(\frac{x}{\epsilon}\right)\frac{\partial p}{\partial x_i}(x). \tag{5}$$

The following inequality holds

$$\left\| p_\epsilon(x) - p(x) - \epsilon p_1\left(x, \frac{x}{\epsilon}\right)\right\|_{H^1(\Omega)} \leqslant C_1\sqrt{\epsilon} \tag{6}$$

with C_1 a constant independant of ϵ. The rigorous justification of the above results can be found in [AB06].

3 The multiscale method

3.1 Setting

Here, we make assumptions under which a priori error estimate can be established. In the sequel, it will be assumed that the domain Ω is as regular as necessary. We denote by $\mathbf{n}_\Omega$ the unit outward vector normal to $\partial\Omega$.

When $W(\Omega)$ is a functional space, we say that $v \in W_\#(\Omega)$ if v and all of its derivatives, when they exist, belong to $W(\Omega)$ and are all Ω-periodic. We say that $v \in \dot{W} \in (\Omega)$ if $v \in W$ and $\int_\Omega v\, dx = 0$.
We shall have the following assumptions in the input data :

Assumptions 1. *1. The tensor $\mathbf{K} \in C^2(\overline{Y})^{d\times d}$. There exist constants $0 < \lambda_{min} < \lambda_{max}$ such that*

$$\lambda_{max} |\xi|^2 \geqslant \mathbf{K}(x)\xi \cdot \xi \geqslant \lambda_{min} |\xi|^2, \quad \forall \xi \in \mathbb{R}^d, \quad \forall x \in \Omega,$$

where $|\cdot|$ denotes the euclidian norm $\mathbb{R}^d$,

2. There exists a constant $K_{sta} > 0$ such that

$$\mathbf{K}(x) \leqslant K_{stat}, \quad \forall x \in \Omega,$$

3. $f \in H^3(\Omega)$.

Under the assumptions 1, we can prove by using classical results (see [Eva10]) that there exists a unique solution $p_\epsilon \in H^2(\Omega)$, a unique solution $p \in H^5(\Omega)$ and a unique solution of the cell problem $w_i \in \dot{W}^{1,\infty}_\#(\Omega)$. By Sobolev embedding theorem, we can prove that $p \in W^{3,\infty}(\Omega)$.

Since we will use a dG discretization, regularity more than $H^1(\Omega)$ is needed, namely we need error estimation in $H^s(\Omega)$ with $s \geqslant 3/2$. For this end, the following useful inequality has been established in [FW17] :

$$\left\| p_\epsilon(x) - p\,(x) - \epsilon p_1\left(x, \frac{x}{\epsilon}\right)\right\|_{H^2(\Omega)} \leqslant C_3\left(1 + \frac{1}{\sqrt{\epsilon}} + \epsilon\right) \tag{7}$$

where C_3 is a constant independent of ϵ.

3.2 The idea of the multiscale method

The idea of our method was introduced for the first time in [AB06]. Here, we describe its principe. Let us define the harmonic coordinates $\widehat{w}_i^\epsilon$ as follows :

$$\widehat{w}_i^\epsilon(x) := x_i + \epsilon w_i\left(\frac{x}{\epsilon}\right). \tag{8}$$

Let define $y := x/\epsilon$. We have $\nabla \widehat{w}_i^\epsilon(x) = \mathbf{e}_i + \nabla_y w_i\left(\frac{x}{\epsilon}\right)$. Replacing $\nabla \widehat{w}_i^\epsilon(x)$ by $\mathbf{e}_i + \nabla_y w_i\left(\frac{x}{\epsilon}\right)$ in (4) gives :

$$-\operatorname{div}\left(\mathbf{K}\left(\frac{x}{\epsilon}\right)\nabla \widehat{w}_i^\epsilon(x)\right) = 0 \quad \forall x \in \epsilon Y. \tag{9}$$

Remark that

$$p\,(x) + \sum_{i=1}^{d} (\widehat{w}_i^\epsilon(x) - x_i)\,\frac{\partial p}{\partial x_i}\,(x)$$

is the two first terms of Taylor expansion with integral rest of $p\,(\widehat{w}^\epsilon(x))$. Therefore, $p_\epsilon(x)$ can be approximated as follows :

$$p_\epsilon(x) \approx p_\epsilon(\widehat{w}^\epsilon(x)). \tag{10}$$

The approximation formula (10) is the starting point of the definition of new multiscale method adapted to the problem (1). This formula suggests to approximate the solution of problem (1) with basis functions which are polynomials composed with the function $\widehat{w}^\epsilon$. The main interest of this method rests on the fact that $\widehat{w}^\epsilon$ can be approximated numerically and very quickly.

4 Multiscale discretization

4.1 Partition of the domain

The finite element spaces that are used in the proposed method are defined below. They involve two differents meshes. Let $\mathcal{E}_h$ be a quasi-uniform conforming partition of Ω called the fine mesh on which are defined parameters of the equation (1) where h is the mesh-size of $\mathcal{E}_h$. It is chosen such that $h < \epsilon$. We denote by $\mathcal{E}_H = \{E_1, E_2, ... E_{n_c}\}$ the coarse mesh. We denote by Γ_H^i and Γ_H^b the collection of all interior edges and the collection of all boundary edges, respectively.

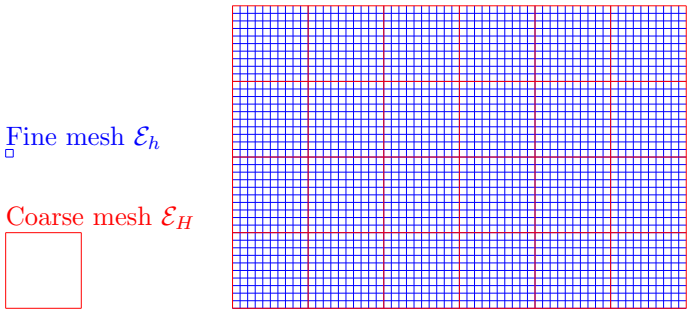


Figure 1: An illustration of a coarse mesh (in red) form a fine mesh (in blue).

Notation Here, we make a summary of some useful notation. We denote by

- $\mathcal{E}_h$ the fine mesh,
- $\mathcal{E}_H$ the coarse mesh,
- Γ_H^i the collection of all interior edges of $\mathcal{E}_H$,
- Γ_H^b the collection of boundary edges of $\mathcal{E}_H$,
- H_e the length of the edge e,
- $y = x/\epsilon$ the fast variable.

In the sequel, we use the notation $A \lesssim B$ when there exists a positive constant C with C independant of ϵ, h and H, such that $A \leqslant CB$. We denotes by $\mathbf{K}^\epsilon(x) = \mathbf{K}\left(\frac{x}{\epsilon}\right)$.

4.2 Cell problems approximation

We assume that each coarse element E of the mesh is convex. As mentioned above, the functions $\widehat{w}_i^\epsilon$ have to be approximated numerically and very quickly. For each coarse element $E \in \mathcal{E}_H$, $\widehat{w}_i^\epsilon$ is approximated

by the function $\widetilde{w}_i^{\epsilon,E}$ solution of the following problem :

$$\begin{cases} -\text{div}\left(\mathbf{K}^\epsilon(x)\nabla\widetilde{w}_i^{\epsilon,E}\right) &= 0 & \text{in} \quad E, \\ \widetilde{w}_i^{\epsilon,E}(x) &= x_i & \text{on} \quad \partial E, \end{cases} \tag{11}$$

We construct the vector $\widetilde{w}^{\epsilon,E}$ as follows : $\forall E \in \mathcal{E}_H, \quad \left(\widetilde{w}^{\epsilon,E}\right)_i = \widetilde{w}_i^{\epsilon,E}$. Since the coarse element E is convex, then $w_i^{\epsilon,E} \in H^2(\bar{E})$.

Remark 1. *1. Cell problems (11) are independent from each other, they can be solved in parallel.*

2. The boundary condition choice plays a crucial role in the quality of multiscale basis functions. The fact that the boundary conditions are defined independently of the variations of the permeability tensor $\mathbf{K}^\epsilon$, can result in large errors. Typically, if we consider the choice made in (11), then :

$$\left\|\widehat{w}_i^\epsilon - \widetilde{w}_i^{\epsilon,E}\right\|_{L^2(\partial E)} \leqslant \sqrt{\epsilon|\partial E|}\,\|w_i\|_{L^\infty(E)}\,,$$

$$\left\|\nabla\left(\widehat{w}_i^\epsilon - \widetilde{w}_i^{\epsilon,E}\right)\right\|_{L^2(\partial E)} \leqslant \sqrt{|\partial E|}\,\|\nabla_y w_i\|_{L^\infty(E)}\,.$$

where we use the fact that $w_i \in W^{1,\infty}(Y)$. Our boundary choice is justified by the fact when ϵ goes to 0, $\left\|\widehat{w}_i^\epsilon - \widetilde{w}_i^{\epsilon,E}\right\|_{L^2(\partial E)}$ goes to 0. However, it is not clear that $\left\|\nabla\left(\widehat{w}_i^\epsilon - \widetilde{w}_i^{\epsilon,E}\right)\right\|_{L^2(\partial E)}$ goes to 0 when ϵ goes to 0 and that is the main drawback of this approximation. Methods were introduced to overcome this difficulty :

- *oversampling method ([HW97, EHW00, CCSY08]),*
- *methods which consist in solving boundary problems ([HW97, JLT03, JLT05]),*
- *new other methods ([Glo11]).*

Dirichlet-Neumann boundary condition To reduce the resonance errors, we introduce a new strategy which consists in using a Dirichlet-Neumann condition. We will make the description of the strategy only for $d = 2$. The generalization of the strategy to higher dimension is straightformard. For each coarse element E, we make the following decompsition $\partial E = \Gamma_{up}^E \cup \Gamma_{down}^E \cup \Gamma_{left}^E \cup \Gamma_{right}^E$. We approximate $\widehat{w}_1^\epsilon$ by $\widetilde{w}_1^{\epsilon,E}$ solution of the following equation :

$$\begin{cases} -\text{div}\left(\mathbf{K}^\epsilon(x)\nabla\widetilde{w}_1^{\epsilon,E}\right) &= 0 & \text{in} \quad E, \\ \widetilde{w}_1^{\epsilon,E}(x) &= x_1 & \text{on} \quad \Gamma_{up}^E \cup \Gamma_{down}^E \\ \left(\mathbf{K}^\epsilon(x)\nabla\widetilde{w}_1^{\epsilon,E}(x)\right)\cdot\mathbf{n} &= \lambda_1 & \text{on} \quad \Gamma_{left}^E \cup \Gamma_{right}^E \end{cases} \tag{12}$$

Similarly, we approximate $\widehat{w}_1^\epsilon$ by $\widetilde{w}_2^{\epsilon,E}$ solution of the following equation :

$$\begin{cases} -\text{div}\left(\mathbf{K}^\epsilon(x)\nabla\widetilde{w}_2^{\epsilon,E}\right) &= 0 & \text{in} \quad E, \\ \widetilde{w}_2^{\epsilon,E}(x) &= x_2 & \text{on} \quad \Gamma_{left}^E \cup \Gamma_{right}^E \\ \left(\mathbf{K}^\epsilon(x)\nabla\widetilde{w}_2^{\epsilon,E}(x)\right)\cdot\mathbf{n} &= \lambda_2 & \text{on} \quad \Gamma_{up}^E \cup \Gamma_{down}^E \end{cases} \tag{13}$$

where λ_i are hand-user parameters.

Let us see how the hand-user variables λ_i may be choosen in practice. By recalling that

$$\widehat{w}_i^\epsilon(x) = x_i + \epsilon w_i\left(\frac{x}{\epsilon}\right),$$

we have $(\mathbf{K}^\epsilon(x)\nabla\widehat{w}_i^\epsilon(x))\cdot\mathbf{n} = \left(\mathbf{K}^\epsilon(x)\mathbf{e}_i + \nabla w_i\left(\frac{x}{\epsilon}\right)\right)\cdot\mathbf{n}_i$. Since w_i are unknown, a first candidate for λ_i is $(\mathbf{K}^\epsilon(x)\mathbf{e}_i)\cdot\mathbf{n}_i$.

Let us comment a little bit the intuition behind this new strategy for reducing the so-called resonance error. The function w_1 is designed to capture the variation of the permeability $\mathbf{K}^\epsilon$ in the direction $\mathbf{e}_1$. So, the idea consists in assuming that w_1 does not change on the portion of ∂E where the variation of the coordinate x_1 is very small. And, we try to capture the variation of the permeability $\mathbf{K}^\epsilon$ on the remainding portion by imposing a Neumann condition. The same can be said to w_2.

4.3 Discretization scheme on the coarse mesh

The methods we are interested in seek an approximation to p_ϵ by the discontinuous Galerkin finite element method. For this purpose, we need finite element spaces for these quantities consisting of piecewise polynomial functions. Namely, we introduce

$$X_H := \left\{v_h \in L^2(\Omega); \forall E \in \mathcal{E}_H,\, v_H \in \mathbb{P}_d^k(E)\right\},$$

where $\mathbb{P}_d^k$ is the space of polynomials of degree at most k. Let $e \in \Gamma_H^i$ such that $e = E_k \cap E_l$, we denote by $\mathbf{n}_e$ a unit normal vector oriented from E_k to E_l. We define the average and the jump for $v \in H^1(\mathcal{E}_H)$ as follows

$$[\psi]_e := \psi_{E_k} - \psi_{E_l}, \quad \{\psi\}_e := \frac{1}{2}(\psi_{E_k} + \psi_{E_l}).$$

When $e \in E \cap \Gamma_H^b$, we define the average and the jump for $v \in H^1(\mathcal{E}_H)$ as follows

$$[\psi]_e := \psi_E, \quad \{\psi\}_e := \psi_E.$$

The jump function is defined by $J_0(u,v) := \displaystyle\sum_{e\in\Gamma_H^i\cup\Gamma_{H,D}} \frac{\sigma_e}{H_e}\int_e [u]_e[v]_e\, d\sigma$ where σ_e is a constant on each face e. The multiscale space $X_{\epsilon,H}$ is defined as follows :

$$X_{\epsilon,H} := \left\{v_{\epsilon,H} := v_H \circ \widetilde{w}^\epsilon \text{ where } v_H \in X_H\right\}.$$

For each $E \in \mathcal{E}_H$, we denote $\{\varphi_j\}_{j=1,\dots,n_d}$ a basis of $\mathbb{P}_d^k(E)$. We define the interpolation operator π_H on $C^0(\bar{E})$ as follows :

$$\pi_H v(x) := \sum_{j=1}^{n_d} v(a_j)\varphi_j(x), \quad \forall v \in C^0(\bar{E}),$$

where a_j are such that $\varphi_i(a_j) = \delta_{ij}$. We define the interpolation operator $\pi_{\epsilon,H}$ on $C^0(\bar{E})$ as follows :

$$\pi_{\epsilon,H} v(x) := \sum_{j=1}^{n_d} v(a_j)\varphi_j \circ \widetilde{w}^{\epsilon,E}(x) = (\pi_H v)\circ\widetilde{w}^{\epsilon,E}(x), \quad \forall v \in C^0(\bar{E}). \tag{14}$$

Since $\widetilde{w}^{\epsilon,E} \in H^2(\overline{E})$, then for each $v_{\epsilon,H}$ in $X_{\epsilon,H}$, $v_{\epsilon,H|E} \in H^2(\bar{E})$. We define the broken norm in $H^1(\mathcal{E}_H)$ as follows :

$$\|v_H\|_{H^1(\mathcal{E}_H)} = \left(\sum_{E\in\mathcal{E}_H} \|(\mathbf{K}^\epsilon)^{1/2}\nabla v_H\|_{L^2(E)}^2 + J_{0,H}(v_H, v_H)\right)^{1/2}.$$

We approximate the solution of the problem (1) by $p_{\epsilon,H} \in X_{\epsilon,H}$ and solution of the following problem : find $p_{\epsilon,H} \in X_{\epsilon,H}$ such that for all $v_{\epsilon,H} \in X_{\epsilon,H}$:

$$\begin{cases} \displaystyle\sum_{E\in\mathcal{E}_H}\int_E (\mathbf{K}^\epsilon\nabla p_{\epsilon,H})\cdot\nabla v_{\epsilon,H}\,dx \quad - \sum_{e\in\Gamma_H^i\cup\Gamma_{H,D}}\int_e \{\mathbf{K}^\epsilon\nabla p_{\epsilon,H}\}_e\cdot\mathbf{n}_e[v_{\epsilon,H}]_e\,d\sigma \\ \displaystyle -\gamma\sum_{e\in\Gamma_H^i\cup\Gamma_{H,D}}\int_e \{\mathbf{K}^\epsilon\nabla v_{\epsilon,H}\}_e\cdot\mathbf{n}_e[p_{\epsilon,H}]_e\,d\sigma + J_{0,H}(p_{\epsilon,H},v_{\epsilon,H}) = \int_\Omega f v_{\epsilon,H}\,dx \end{cases} \tag{15}$$

where $\gamma = -1, 0, 1$. This scheme is obtained as follows. We replace p_ϵ by $p_{\epsilon,H}$ in (1) and multiply it by a test function $v_{\epsilon,H} \in X_{\epsilon,H}$ and proceed by integration by parts elementwise. The term

$$\gamma\sum_{e\in\Gamma_H^i\cup\Gamma_{H,D}}\int_e \{\mathbf{K}^\epsilon\nabla v_{\epsilon,H}\}_e\cdot\mathbf{n}_e[p_{\epsilon,H}]_e\,d\sigma$$

is added to have a symmetry or a non-symmetry. The term $J_{0,H}(p_{\epsilon,H},v_{\epsilon,H})$ is added to guarantee the stability the scheme. Following the same arguments as in [DPE12][Lemma 4.12], the penalty parameters σ_e can be chosen independently of ϵ and H_e such that :

$$A_H^\gamma(v_{\epsilon,H},v_{\epsilon,H}) + J_{0,H}(v_{\epsilon,H},v_{\epsilon,H}) \geqslant \frac{1}{2}\|v_{\epsilon,H}\|^2_{H^1(\mathcal{E}_H)}, \qquad \text{for all} \quad v_{\epsilon,H}\in X_{\epsilon,H}$$

where

$$\begin{aligned} A_H^\gamma(p_{\epsilon,H},v_{\epsilon,H}) = &\sum_{E\in\mathcal{E}_H}\int_E (\mathbf{K}^\epsilon\nabla p_{\epsilon,H})\cdot\nabla v_{\epsilon,H}\,dx - \sum_{e\in\Gamma_H^i\cup\Gamma_{H,D}}\int_e \{\mathbf{K}^\epsilon\nabla p_{\epsilon,H}\}_e\cdot\mathbf{n}_e[v_{\epsilon,H}]_e\,d\sigma \\ &-\gamma\sum_{e\in\Gamma_H^i\cup\Gamma_{H,D}}\int_e \{\mathbf{K}^\epsilon\nabla v_{\epsilon,H}\}_e\cdot\mathbf{n}_e[p_{\epsilon,H}]_e\,d\sigma \end{aligned}$$

$$J_{0,H}(p_{\epsilon,H},v_{\epsilon,H}) = \sum_{e\in\Gamma_H^i\cup\Gamma_{H,D}}\frac{\sigma_e}{H_e}\int_e [p_{\epsilon,H}]_e[v_{\epsilon,H}]_e\,d\sigma.$$

Finally, as in [DPE12][p. 135] we can prove that the scheme is consistent, in other words

$$A_H^\gamma(p_\epsilon - p_{\epsilon,H},v_{\epsilon,H}) + J_{0,H}(p_\epsilon - p_{\epsilon,H},v_{\epsilon,H}) = 0.$$

The main result Here is the main result of this paper.

Theorem 1. *Let p_ϵ be the solution of the problem (1). We assume that $\epsilon < H$. Let $p_{\epsilon,H}$ be the solution of the scheme (15). Under the assumptions 1, the following inequality holds :*

$$\|p_{\epsilon,H} - p_\epsilon\|_{H^1(\mathcal{E}_H)} \lesssim \sqrt{\epsilon} + H + \frac{\epsilon}{H} + \frac{H}{\sqrt{\epsilon}} + +\sqrt{R_{max}(\epsilon)}. \tag{16}$$

where $R_{max}(\epsilon) = \max_{e\in\Gamma_H^i}\|R(\epsilon)\|_{L^\infty(e)}$ with

$$\|R(\epsilon)\|_{L^\infty(e)} \lesssim \begin{cases} \epsilon & \text{if linear boundary condition is used and if we assume that } \|\nabla w^\epsilon\|_{L^\infty(e)} \leqslant \epsilon \\ \|\nabla w\|_{L^\infty(Y)} & \text{if no assumption is made} \end{cases}$$

Remark 2.

- *The error estimate (16) is the same one get where continuous finite elements are used except here we have the following additional term*

$$\frac{H}{\sqrt{\epsilon}} + \sqrt{R_{max}(\epsilon)}.$$

This is due to fact that where dG discretization is used, we need to estimate error of gradient at the interior edges.

- *Note that the convergence of the multiscale method deteriorates when ϵ and H are close because of the resonance term $\frac{\epsilon}{H}$. This is due to the choice of boundary conditions of the cell problems (11).*

5 Numerical results

To illustrate our method, we consider some examples in this section. The fine mesh is a uniform rectangular grids. Here, we denote by reference solution the numerical solution we get by classical dG scheme on the fine mesh $\mathcal{E}_h$. To measure the perfomance of our multiscale method, we compute the error between the multiscale solutions and the fine solutions in L^2-norm and in Broken norm. We implement our method on the software Deal.II (see [BHK]).

5.1 Example 1 : Periodic case :

In this first example, we consider The domain $\Omega = (0,\, 0.1)^2$ and the periodic permeability $\mathbf{K}^\epsilon$ introduced in [AB06] defined as follows :

$$\mathbf{K}^\epsilon(x_1, x_2) = a\left(\frac{x_1}{\epsilon}, \frac{x_2}{\epsilon}\right)\mathbb{I} \tag{17}$$

where

$$a\left(\frac{x_1}{\epsilon}, \frac{x_2}{\epsilon}\right) = \frac{1}{\left(1 + 1.8\sin(\frac{2\pi x_1}{\epsilon})\right)\left(1 + 1.8\sin(\frac{2\pi x_2}{\epsilon})\right)}$$

where $\mathbb{I}$ is the $d \times d$ identity matrix. The source term f is taken to be equal to -1 and ϵ is taken to be equal 0.005. The fine $\mathcal{E}_h$ is composed by 100 elements along each direction. Numerical solutions are represented on Figure 2. Errors are computed and represented in Figure 3. We observe that the multiscale method with the Dirichlet-Neumann boundary condition for the cell problems is more efficient than that when we use linear boundary condition. Observe that our new strategy removes completely the resonance error.

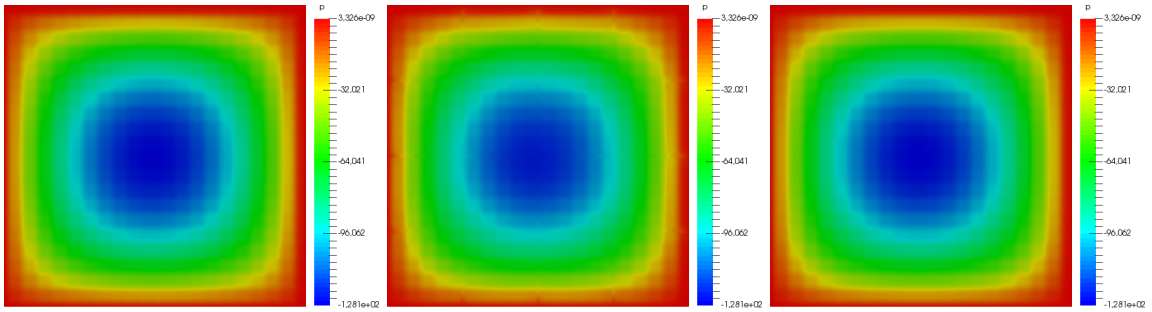
5.2 Example 2 : Non-periodic case :

In this example, we consider the domain $\Omega = (0, 128)^2$. We consider two permeabilities with different sizes of heterogeneities represented in Figure 4. We consider the following boundary conditions

$$\begin{cases} p(x) & = 1000 \times 6894.76 & \text{on} \quad \{x_1 = 0\} \\ p(x) & = 500 \times 6894.76 & \text{on} \quad \{x_1 = 128\} \\ (K\nabla p(x)) \cdot \mathbf{n}_\Omega & = 0 & \text{otherwise.} \end{cases}$$

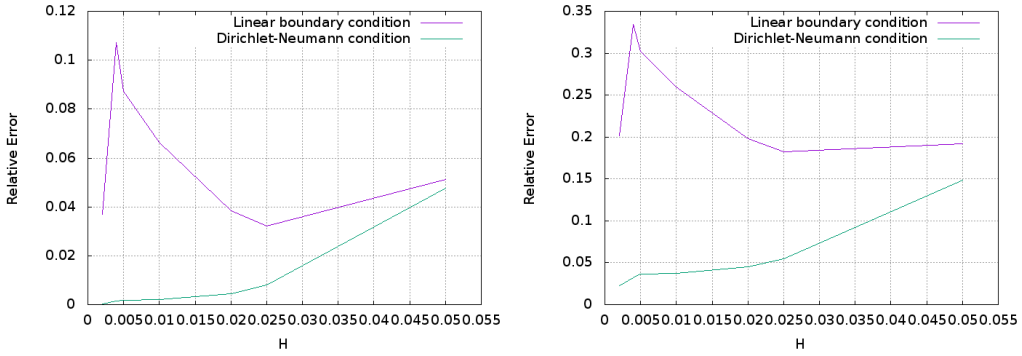
For the first permeability, numerical solutions are plotted in Figure 5 and errors are computed and reprsented in Figure 6.

Similarly, for the second permeability, numerical solutions are plotted in Figure 7 and errors are computed and reprsented in Figure 8.



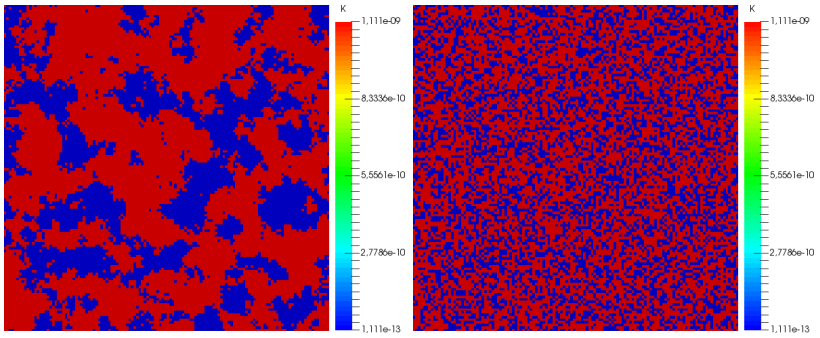
(a) Reference solution (b) Linear boundary (c) Dirichlet-Neumann.

Figure 2: Numerical solutions on 4×4 coarse grid.



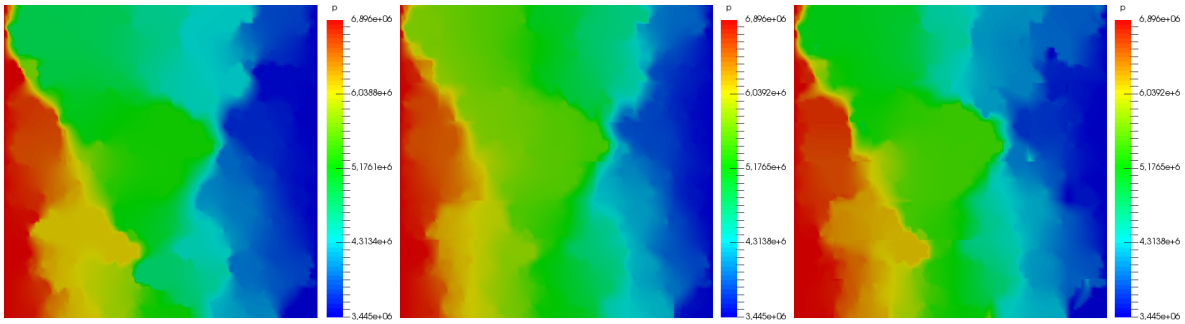
(a) L^2-norm (b) Broken norm

Figure 3: A graphical comparison between our new strategy for cell problems and that when linear boundary condition is used.



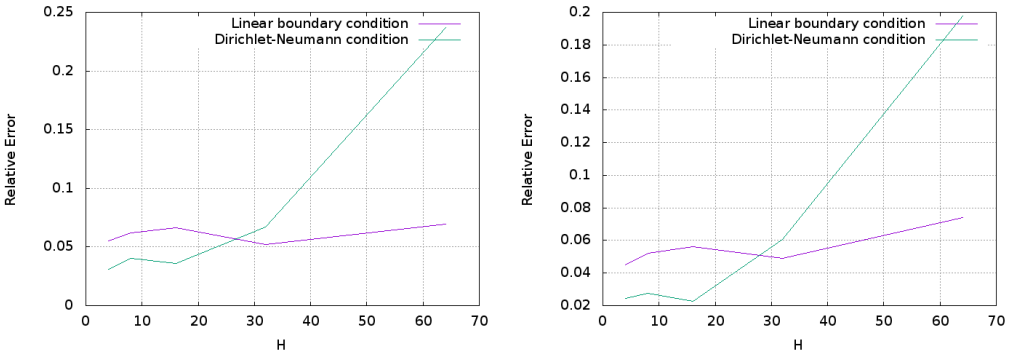
(a) Permeability (b) Permeability

Figure 4: Examples of permeability fields



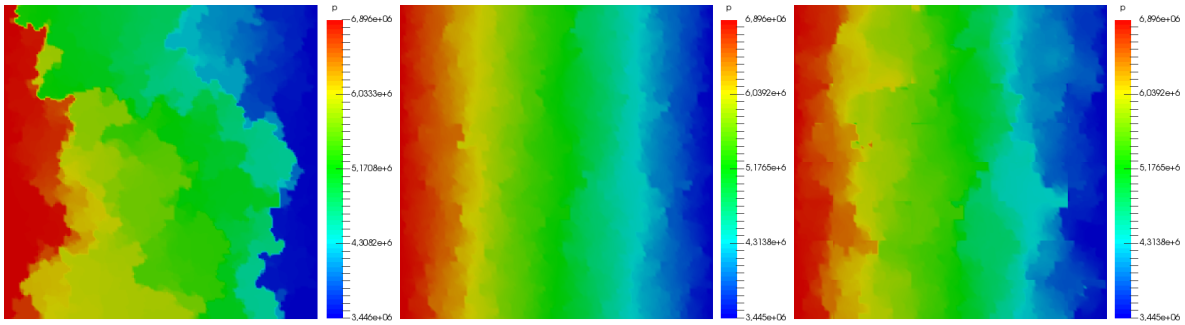
(a) Reference solution (b) Linear boundary (c) Dirichlet-Neumann

Figure 5: Numerical solutions on 4×4 coarse grid.



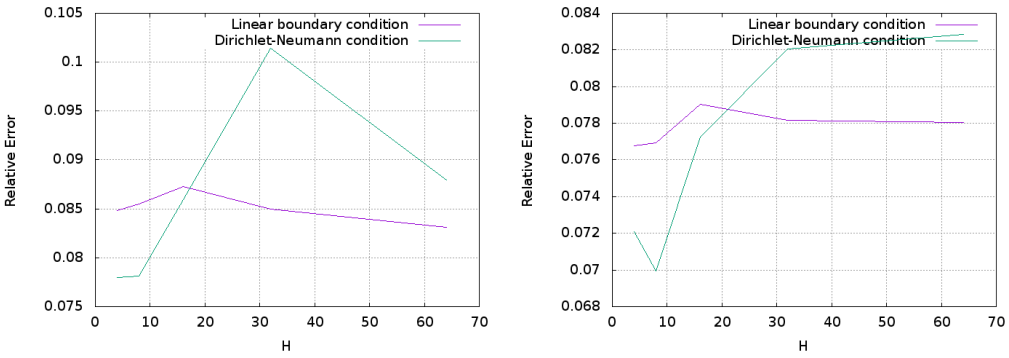
(a) L^2-norm (b) Broken norm

Figure 6: A graphical comparison between our new strategy for cell problems and that when linear boundary condition is used.



(a) Ref. solution (b) Linear boundary (c) Dirichlet-Neumann

Figure 7: Numerical solutions on 4×4 coarse grid.



(a) L^2-norm (b) Broken norm

Figure 8: A graphical comparison between our new strategy for cell problems and that when linear boundary condition is used.

6 Error analysis

The remainder of this paper will be devoted to prove the main result. Our strategy is similar in spirit to that used in [AB06]. We start with the following useful lemma.

Proposition 1. *The following inequality holds :*

$$\|p_\epsilon - p_{\epsilon,H}\|_{H^1(\mathcal{E}_H)} \lesssim \sum_{i=1}^{3} T_i \tag{18}$$

where

$$T_1^2 = \sum_{E\in\mathcal{E}_H} \left\| (\mathbf{K}^\epsilon)^{1/2} \nabla (p_\epsilon - p_{\epsilon,H}) \right\|^2_{L^2(E)},$$

$$T_2^2 = \sum_{e\in\Gamma_H^i} \frac{H_e}{\sigma_e} \|\{\mathbf{K}^\epsilon \nabla (p_\epsilon - p_{\epsilon,H})\}_e\|^2_{L^2(e)}, \quad T_3^2 = \sum_{e\in\Gamma_{H,D}} \frac{\sigma_e}{H_e} \|p_\epsilon - p_{\epsilon,H}\|^2_{L^2(e)},$$

Proof. The strategy is classical and follows from the consistency of the scheme. We first recall that

$$A_H^\gamma(p_{\epsilon,H}, v_{\epsilon,H}) = \sum_{E\in\mathcal{E}_H} \int_E (\mathbf{K}^\epsilon \nabla p_{\epsilon,H}) \cdot \nabla v_{\epsilon,H}\, dx - \sum_{e\in\Gamma_H^i\cup\Gamma_{H,D}} \int_e \{\mathbf{K}^\epsilon \nabla p_{\epsilon,H}\}_e \cdot \mathbf{n}_e [v_{\epsilon,H}]_e\, d\sigma$$
$$- \gamma \sum_{e\in\Gamma_H^i\cup\Gamma_{H,D}} \int_e \{\mathbf{K}^\epsilon \nabla v_{\epsilon,H}\}_e \cdot \mathbf{n}_e [p_{\epsilon,H}]_e\, d\sigma$$

$$J_{0,H}(p_{\epsilon,H}, v_{\epsilon,H}) = \sum_{e\in\Gamma_H^i\cup\Gamma_{H,D}} \frac{\sigma_e}{H_e} \int_e [p_{\epsilon,H}]_e [v_{\epsilon,H}]_e\, d\sigma.$$

The consistency of the scheme reads:

$$A_H^\gamma(p_\epsilon - p_{\epsilon,H}, v_{\epsilon,H}) + J_{0,H}(p_\epsilon - p_{\epsilon,H}, v_{\epsilon,H}) = 0.$$

Let us consider $A_H^\gamma(p_\epsilon - p_{\epsilon,H}, v_{\epsilon,H})$. Since $p_\epsilon - p_{\epsilon,H} \in H^1(\Omega)$, then we have $[p_\epsilon - p_{\epsilon,H}]_e = 0, \forall e \in \Gamma_H^i$. Using successively Schwartz and Young inequalities, yields :

$$\sum_{E\in\mathcal{E}_H}\int_E (\mathbf{K}^\epsilon\nabla(p_\epsilon - p_{\epsilon,H}))\cdot\nabla v_{\epsilon,H}\,dx \leqslant \left(\sum_{E\in\mathcal{E}_H}\|(\mathbf{K}^\epsilon)^{1/2}\nabla(p_\epsilon - p_{\epsilon,H})\|^2_{L^2(E)}\right)^{1/2}\|v_{\epsilon,H}\|_{H^1(\mathcal{E}_H)}, \tag{19}$$

$$\sum_{e\in\Gamma_H^i}\int_e \{\mathbf{K}^\epsilon\nabla(p_\epsilon - p_{\epsilon,H})\}_e\cdot\mathbf{n}_e[v_{\epsilon,H}]_e\,d\sigma \leqslant \left(\sum_{e\in\Gamma_H^i}\frac{H_e}{\sigma_e}\|\{\mathbf{K}^\epsilon\nabla(p_\epsilon - p_{\epsilon,H})\}_e\|^2_{L^2(e)}\right)^{1/2}\|v_{\epsilon,H}\|_{H^1(\mathcal{E}_H)},$$

Let consider $J_{0,H}(p_\epsilon - \pi_{\epsilon,H}(p), v_{\epsilon,H})$. By using the fact that $[\xi_\epsilon^I]_e = 0, \forall e \in \Gamma_H^i$, then we have :

$$J_{0,H}(p_\epsilon - p_{\epsilon,H}, v_{\epsilon,H}) = \sum_{e\in\Gamma_{H,D}}\frac{\sigma_e}{H_e}\int_e [p_\epsilon - p_{\epsilon,H}]_e[v_{\epsilon,H}]_e\,d\sigma.$$

By using the Cauchy-Schwarz's inequality, we have :

$$J_{0,H}(p_\epsilon - p_{\epsilon,H}, v_{\epsilon,H}) \leqslant \left(\sum_{e\in\Gamma_{H,D}}\frac{\sigma_e}{H_e}\|[p_\epsilon - p_{\epsilon,H}]_e\|^2_{L^2(e)}\right)^{1/2}\|v_{\epsilon,H}\|_{H^1(\mathcal{E}_H)}.$$

Using the fact that the penalty parameters σ_e can be chosen such that :

$$A_H^\gamma(p_\epsilon - p_{\epsilon,H}, p_\epsilon - p_{\epsilon,H}) + J_{0,H}(p_\epsilon - p_{\epsilon,H}, p_\epsilon - p_{\epsilon,H}) \geqslant \frac{1}{2}\|p_\epsilon - p_{\epsilon,H}\|^2_{H^1(\mathcal{E}_H)}$$

and by replacing $v_{\epsilon,H}$ by $p_\epsilon - p_{\epsilon,H}$, we have

$$\frac{1}{2}\|p_\epsilon - p_{\epsilon,H}\|_{H^1(\mathcal{E}_H)} \leqslant \left(\sum_{E\in\mathcal{E}_H}\|(\mathbf{K}^\epsilon)^{1/2}\nabla(p_\epsilon - p_{\epsilon,H})\|^2_{L^2(E)}\right)^{1/2} + \left(\sum_{e\in\Gamma_H^i}\frac{H_e}{\sigma_e}\|\{\mathbf{K}^\epsilon\nabla(p_\epsilon - p_{\epsilon,H})\}_e\|^2_{L^2(e)}\right)^{1/2}$$
$$+ \left(\sum_{e\in\Gamma_{H,D}}\frac{\sigma_e}{H_e}\|[p_\epsilon - p_{\epsilon,H}]_e\|^2_{L^2(e)}\right)^{1/2}.$$

□

It remains to bound each term of the above inequality (18).

6.1 Majoration of T_1

Let us start by giving the strategy to bound T_1. The main difficulty for bounding this term is due to the fact that $p_\epsilon - \pi_{\epsilon,H}p$ is not a interpolation error. Therefore, by remarking that there are three errors in the design of our multiscale method : homogenization error, error due to fact that $\hat{w}^E$ is approximated by $\widetilde{w}^E$ and discretization error, we make the following decomposition :

$$\sum_{E\in\mathcal{E}_H}\|\nabla(p_\epsilon - \pi_{\epsilon,H}p)\|^2_{L^2(E)} \leqslant 3\left(G^1_{hom} + G^1_{disc} + G^1_{cell}\right), \tag{20}$$

where

$$G^1_{hom} = \sum_{E\in\mathcal{E}_H} \|\nabla\left(p_\epsilon - p\circ\widehat{w}^\epsilon(\cdot)\right)\|^2_{L^2(E)},$$
$$G^1_{disc} = \sum_{E\in\mathcal{E}_H} \|\nabla\left(p - \pi_H p\right)\circ\widehat{w}^\epsilon(\cdot)\|^2_{L^2(E)},$$
$$\text{and}\quad G^1_{cell} = \sum_{E\in\mathcal{E}_H} \|\nabla\left((\pi_H p)\circ\widehat{w}^\epsilon(\cdot) - \pi_{\epsilon,H} p\right)\|^2_{L^2(E)}.$$

Lemma 1. *The following inequality holds :*

$$G^1_{hom} = \sum_{E\in\mathcal{E}_H} \|\nabla\left(p_\epsilon - p\circ\widehat{w}^\epsilon(\cdot)\right)\|^2_{L^2(E)} \lesssim \epsilon. \tag{21}$$

Proof. The term G^1_{hom} term measures the homogenization error. It will be bounded by making use of the homogenization error inequality (6). We first split the term in two as follows :

$$\|\nabla p_\epsilon - \nabla(p\circ\widehat{w}^\epsilon)\|_{L^2(E)} \leqslant \left\|\nabla p_\epsilon - \sum_{i=1}^d \nabla\widehat{w}^\epsilon_i \partial_{x_i} p\right\|_{L^2(E)} + \left\|\sum_{i=1}^d \nabla\widehat{w}^\epsilon_i\partial_{x_i}p - \nabla(p\circ\widehat{w}^\epsilon)\right\|_{L^2(E)}. \tag{22}$$

Let consider the first term of the right hand of (22). Simple manipulations give :

$$\nabla\left(p_\epsilon(x) - p(x) - \epsilon p_1\left(x,\frac{x}{\epsilon}\right)\right) = \nabla p_\epsilon(x) - \sum_{i=1}^d \nabla\widehat{w}^\epsilon_i(x)\partial_{x_i}p(x) - \epsilon\sum_{i=1}^d w_i\left(\frac{x}{\epsilon}\right)\nabla\partial_{x_i}p(x).$$

Using the inequality (6), the fact $p\in W^{3,\infty}(\Omega)$ and $w\in W^{1,\infty}(\Omega)$, we have :

$$\sum_{E\in\mathcal{E}_h}\left\|\nabla p_\epsilon - \sum_{i=1}^d \nabla\widehat{w}^\epsilon_i\partial_{x_i}p\right\|^2_{L^2(E)} \lesssim \epsilon^2\left(1+|\Omega|\right)\lesssim\epsilon^2. \tag{23}$$

Let consider the second term of the right hand of (22). Remark

$$\left\|\sum_{i=1}^d \nabla\widehat{w}^\epsilon_i(\cdot)\left(\partial_{x_i}p - (\partial_{x_i}p)\circ\widehat{w}^\epsilon\right)\right\|_{L^2(E)} \lesssim \|Id + \nabla_y w\|_{L^\infty(Y)}\|\nabla p - (\nabla p)\circ\widehat{w}^\epsilon\|_{L^2(E)}. \tag{24}$$

A Taylor expansion with integral rest yields :

$$\nabla(p\circ\widehat{w}^\epsilon(x)) = \nabla p(x) + \epsilon\int_0^1\sum_{i=1}^d w_i\left(\frac{x}{\epsilon}\right)\nabla\partial_{x_i}p\left(x+\epsilon s w\left(\frac{x}{\epsilon}\right)\right)ds.$$

Therefore

$$\sum_{E\in\mathcal{E}_H}\left\|\sum_{i=1}^d\nabla\widehat{w}^\epsilon_i\left(\partial_{x_i}p - (\partial_{x_i}p)\circ\widehat{w}^\epsilon\right)\right\|^2_{L^2(E)} \lesssim \sum_{E\in\mathcal{E}_H}\epsilon^2\|p\|^2_{W^{2,\infty}(\Omega)}\|w\|^2_{L^\infty(E)}|E| \lesssim \epsilon^2 \tag{25}$$

From the inequalities (23) and (25), we deduce the inequality (21).

□

Lemma 2. *The following inequalities hold :*

$$G^1_{disc} = \sum_{E\in\mathcal{E}_H} \|\nabla\left((p-\pi_H p)\circ \widehat{w}^\epsilon\right)\|^2_{L^2(E)} \lesssim H^6 + H^2\epsilon^2, \tag{26}$$

Proof. Note that

$$\|\nabla\left((p-\pi_H p)\circ \widehat{w}^\epsilon\right)\|_{L^2(E)} \lesssim \|Id + \nabla_y w\|_{L^\infty(Y)} \|\nabla\left(p-\pi_H p\right)\circ \widehat{w}^\epsilon\|_{L^2(E)}.$$

A Taylor expansion with integral rest yields :

$$\nabla\left((p-\pi_H p)\circ \widehat{w}^\epsilon(x)\right) = \nabla\left(p-\pi_H p\right)(x) + \epsilon\sum_{i=1}^d \int_0^1 w_i\left(\frac{x}{\epsilon}\right)\partial_{x_i}\left(\nabla\left(p-\pi_H p\right)\right)\left(t, x+\epsilon s w\left(\frac{x}{\epsilon}\right)\right) ds.$$

Using the regularity of p and interpolation errors, we get

$$\sum_{E\in\mathcal{E}_H} \|\nabla\left((p-\pi_H p)\circ \widehat{w}^\epsilon\right)\|^2_{L^2(E)} \lesssim H^6 + H^2\epsilon^2$$

□

Lemma 3. *The following inequality holds :*

$$G^1_{cell} = \sum_{E\in\mathcal{E}_H} \|\nabla\left(\pi_H p\circ \widehat{w}^\epsilon - \pi_{\epsilon,H} p\right)\|^2_{L^2(E)} \lesssim \frac{\epsilon}{H} + \epsilon H + \epsilon^2, \tag{27}$$

Proof. Note that

$$\begin{aligned}\|\nabla\left(\pi_H p\circ \widehat{w}^\epsilon - \pi_{\epsilon,H} p\right)\|_{L^2(E)} \leqslant & \left\|\sum_{i=1}^d \left(\nabla\widehat{w}_i^\epsilon - \nabla\widetilde{w}_i^{\epsilon,E}\right)\partial_{x_i}\left(\pi_H p\circ \widehat{w}^\epsilon\right)\right\|_{L^2(E)} \\ & + \left\|\sum_{i=1}^d \nabla\widetilde{w}_i^{\epsilon,E}\left(\partial_{x_i}\left(\pi_H p\circ \widehat{w}^\epsilon\right) - \partial_{x_i}\left(\pi_H p\circ \widetilde{w}^{\epsilon,E}\right)\right)\right\|_{L^2(E)}\end{aligned} \tag{28}$$

Let consider the first term of the right hand of (28). We obtain :

$$\left\|\sum_{i=1}^d \left(\nabla\widehat{w}_i^\epsilon - \nabla\widetilde{w}_i^{\epsilon,E}\right)\partial_{x_i}\left(\pi_H p\circ \widehat{w}^\epsilon\right)\right\|_{L^2(E)} \lesssim \left\|\sum_{i=1}^d \nabla\left(\widehat{w}_i^\epsilon - \widetilde{w}_i^{\epsilon,E}\right)\right\|_{L^2(E)} \left\|\sup_i \partial_{x_i}\left(\pi_H p\circ \widehat{w}^\epsilon\right)\right\|_{L^\infty(E)}.$$

Using the lemma 7 on each coarse element E, we obtain :

$$\sum_{E\in\mathcal{E}_H} \left\|\nabla\left(\widehat{w}^\epsilon - \widetilde{w}^{\epsilon,E}\right)\right\|^2_{L^2(E)} \lesssim \sum_{E\in\mathcal{E}_H} \epsilon\,|\partial E| \lesssim \epsilon H^{d-1}H^{-d} \lesssim \frac{\epsilon}{H}.$$

Therefore

$$\sum_{E\in\mathcal{E}_H} \left\|\sum_{i=1}^d \left(\nabla\widehat{w}_i^\epsilon - \nabla\widetilde{w}_i^{\epsilon,E}\right)\partial_{x_i}\left(\pi_H p\circ \widehat{w}^\epsilon\right)\right\|^2_{L^2(E)} \lesssim \frac{\epsilon}{H}\sup_{E\in\mathcal{E}_H}\left\|\sup_{i\in\{1,\dots,d\}} \partial_{x_i}\left(\pi_H p\circ \widehat{w}^\epsilon\right)\right\|^2_{L^\infty(E)}. \tag{29}$$

Using the fact that the term $\|\partial_{x_i}\pi_H p\|_{L^\infty(E)}$ is bounded as follows : there exists a constant C independent of H such that

$$\|\partial_{x_i}\pi_H p\|_{L^\infty(E)} \leqslant |p|_{L^\infty(0,T;W^{1,\infty}(K))} + CH^2|p|_{L^\infty(0,T;W^{3,\infty}(K))} \lesssim (1+H^2).$$

we obtain :

$$\sum_{E\in\mathcal{E}_H}\left\|\sum_{i=1}^{d}\nabla\left(\widehat{w}_i^{\epsilon}-\widetilde{w}_i^{\epsilon,E}\right)\partial_{x_i}\left(\pi_H p\circ\widehat{w}^{\epsilon}\right)\right\|^2_{L^2(E)}\lesssim\frac{\epsilon}{H}+\epsilon H. \tag{30}$$

Let consider the second term of the right hand of (28). Note that

$$\begin{aligned}&\left\|\sum_{i=1}^{d}\nabla\widetilde{w}_i^{\epsilon,E}\left(\partial_{x_i}\left(\pi_H p\circ\widehat{w}^{\epsilon}\right)-\partial_{x_i}\left(\pi_H p\circ\widetilde{w}^{\epsilon,E}\right)\right)\right\|_{L^2(E)}\\&\leqslant\left\|\sum_{i=1}^{d}\nabla\widetilde{w}_i^{\epsilon,E}\right\|_{L^\infty(E)}\sup_{i\in\{1,\dots,d\}}\left\|\partial_{x_i}\left(\pi_H p\circ\widehat{w}^{\epsilon}\right)-\partial_{x_i}\left(\pi_H p\circ\widetilde{w}^{\epsilon,E}\right)\right\|_{L^2(E)}.\end{aligned} \tag{31}$$

To bound $\left\|\partial_{x_i}\left(\pi_H p\right)\circ\left(\widehat{w}^{\epsilon}-\widetilde{w}^{\epsilon,E}\right)\right\|_{L^2(E)}$, we will use a Taylor inequality between $\widehat{w}^{\epsilon}(x)$ and $\widetilde{w}^{\epsilon,E}(x)$ which requires that $\pi_H p\in C^2(E)$. Since $p\in C^2(\Omega)$ we deduce that $\pi_H p\in C^2(E)$. It remains to find a subset of E in which $\widehat{w}^{\epsilon}(x)\in E$ and $\widetilde{w}^{\epsilon,E}(x)\in E$. By maximum principle, $\widehat{w}^{\epsilon}(x)\in E$ for all $x\in E$. Remark that

$$\left\|\widetilde{w}^{\epsilon,E}(x)-x\right\|_{L^\infty(E)}\leqslant 2\left\|w\right\|_{L^\infty(E)}.$$

We introduce the subset C_E on E as follows :

$$C_E=\left\{x\in E\mid B(x,2\epsilon\left\|w\right\|_{L^\infty(Y)})\subset E\right\}.$$

Clearly, for all $x\in C_E$, $\widetilde{w}^{\epsilon,E}(x)\in E$. Using the lemma 7 and the fact that

$$\left\|\nabla\partial_{x_i}\pi_H p\right\|_{L^\infty(K)}\lesssim 1,$$

we get

$$\left\|\partial_{x_i}\left(\pi_H p\circ\widehat{w}^{\epsilon}\right)-\partial_{x_i}\left(\pi_H p\circ\widetilde{w}^{\epsilon,E}\right)\right\|^2_{L^2(C_E)}\lesssim\epsilon^2H^d. \tag{32}$$

Using the fact that $\left|E\setminus C_E\right|\lesssim\left|\partial E\right|\epsilon$, we have

$$\left\|\partial_{x_i}\left(\pi_H p\circ\widehat{w}^{\epsilon}\right)-\partial_{x_i}\left(\pi_H p\circ\widetilde{w}^{\epsilon,E}\right)\right\|^2_{L^2(E\setminus C_E)}\lesssim\epsilon H^{d-1}. \tag{33}$$

Injecting the inequalities (33) and (32) in (31), we obtain

$$\sum_{E\mathcal{E}_H}\left\|\nabla\left(\pi_H p\circ\widehat{w}^{\epsilon}-\pi_{\epsilon,H}p\right)\right\|_{L^2(E)}\lesssim\frac{\epsilon}{H}+\epsilon H+\epsilon^2.$$

□

Proposition 2. *The following inequalities hold :*

$$T_1\lesssim\epsilon+H+\frac{\epsilon}{H}, \tag{34}$$

Proof. The proof follows directly from the lemmas 1, 2 and 3. □

6.2 Majoration of T_2 and T_3

In order to bound T_3, we make the following decomposition :

$$\sum_{e\in\Gamma_H^i\cup\Gamma_{H,D}} \|p-\pi_{\epsilon,H}p\|^2_{L^2(e)} \leqslant G^e_{hom}+G^e_{disc}+G^e_{cell},$$

where

$$\begin{aligned} G^e_{hom} &= \sum_{e\in\Gamma_H^i\cup\Gamma_{H,D}} \|p-p\circ\widehat{w}^\epsilon\|^2_{L^2(e)}\,,\\ G^e_{disc} &= \sum_{e\in\Gamma_H^i\cup\Gamma_{H,D}} \|p\circ\widehat{w}^\epsilon-\pi_H p\circ\widehat{w}^\epsilon\|^2_{L^2(e)}\,,\\ \text{and}\quad G^e_{cell} &= \sum_{e\in\Gamma_H^i\cup\Gamma_{H,D}} \|\pi_H p\circ\widehat{w}^\epsilon-\pi_{\epsilon,H}p\|^2_{L^2(e)}\,. \end{aligned}$$

In order to bound the term T_2, we make the following decomposition :

$$\sum_{e\in\Gamma_H^i} \|\nabla\left(p_\epsilon-\pi_{\epsilon,H}\tilde{p}_\epsilon\right)\|^2_{L^2(0,T;L^2(e))} \leqslant G^{e,2}_{hom}+G^{e,2}_{disc}+G^{e,2}_{cell}$$

where

$$\begin{aligned} G^{e,2}_{hom} &= \sum_{e\in\Gamma_H^i} \|\nabla\left(p-p\circ\widehat{w}^\epsilon\right)\|^2_{L^2(e)}\,,\\ G^{e,2}_{disc} &= \sum_{e\in\Gamma_H^i} \|\nabla\left(p\circ\widehat{w}^\epsilon-\pi_H p\circ\widehat{w}^\epsilon\right)\|^2_{L^2(e)}\\ \text{and}\quad G^{e,2}_{cell} &= \sum_{e\in\Gamma_H^i} \|\nabla\left(\pi_H p\circ\widehat{w}^\epsilon-\pi_{\epsilon,H}p\right)\|^2_{L^2(e)}\,, \end{aligned}$$

Lemma 4. *The following inequalities hold :*

$$G^e_{hom} = \sum_{e\in\Gamma_H^i\cup\Gamma_{H,D}} \|p-p\circ\widehat{w}^\epsilon\|^2_{L^2(e)} \lesssim \epsilon\left(H+\frac{1}{H}\right). \tag{35}$$

$$G^{e,2}_{hom} = \sum_{e\in\Gamma_H^i} \|\nabla\left(p-p\circ\widehat{w}^\epsilon\right)\|^2_{L^2(e)} \lesssim \frac{\epsilon^2}{H}+\left(H+\frac{H}{\epsilon}+H\epsilon^2+\frac{\epsilon}{H}\right) \tag{36}$$

Proof. Let $e\in\Gamma_H^i\cup\Gamma_{H,D}$. Let define $v(x)=p(x)-p\circ\widehat{w}^\epsilon(x)$. Using the following inequality

$$\|v\|_{L^2(e)} \lesssim \left(H^{-1}\|v\|^2_{L^2(E)}+H\|\nabla v\|^2_{L^2(E)}\right).$$

and summing over all faces yields :

$$\sum_{e\in\Gamma_H^i\cup\Gamma_{H,D}} \|p-p\circ\widehat{w}^\epsilon\|^2_{L^2(e)} \lesssim \sum_{e\in\Gamma_H^i\cup\Gamma_{H,D}} \frac{L}{H}\|p-p\circ\widehat{w}^\epsilon\|^2_{L^2(E)}+LH\|\nabla p-\nabla(p\circ\widehat{w}^\epsilon)\|^2_{L^2(E)}$$

where L is the maximun number of faces that belong to a coarse element. By using the inequality (21), we have

$$\sum_{e\in\Gamma_H^i\cup\Gamma_{H,D}} \|p-p\circ\widehat{w}^\epsilon\|^2_{L^2(e)} \lesssim \epsilon\left(H+\frac{1}{H}\right).$$

Let consider the term $\|\nabla\left(p-p\circ\widehat{w}^{\epsilon}\right)\|^2_{L^2(e)}$. We make the following decomposition :

$$\|\nabla p-\nabla(p\circ\widehat{w}^{\epsilon})\|_{L^2(e)} \quad\leqslant\quad \left\|\nabla p-\sum_{i=1}^{d}\nabla\widehat{w}_i^{\epsilon}\partial_{x_i}p\right\|_{L^2(e)} \quad+\quad \left\|\sum_{i=1}^{d}\nabla\widehat{w}_i^{\epsilon}\partial_{x_i}p-\nabla(p\circ\widehat{w}^{\epsilon})\right\|_{L^2(e)}. \tag{37}$$

Remark that

$$\nabla\left(p(x)+\epsilon p_1\left(x,\frac{x}{\epsilon}\right)\right)=\nabla p_{\epsilon}(x)-\sum_{i=1}^{d}\nabla\widehat{w}_i^{\epsilon}\partial_{x_i}p(x)-\epsilon\sum_{i=1}^{d}w_i\left(\frac{x}{\epsilon}\right)\nabla\partial_{x_i}p(x).$$

We deduce that :

$$\begin{aligned}\left\|\left(\nabla p-\sum_{i=1}^{d}\nabla\widehat{w}_i^{\epsilon}\partial_{x_i}p\right)\cdot\mathbf{n}_e\right\|_{L^2(e)} &\lesssim \left\|\left(\nabla\left(p-p\left(x\right)-\epsilon p_1\left(x,\frac{x}{\epsilon}\right)\right)\right)\cdot\mathbf{n}_e\right\|_{L^2(e)}\\ &\quad+\epsilon\sum_{i=1}^{d}\left\|\left(w_i\left(\frac{\cdot}{\epsilon}\right)\nabla\partial_{x_i}p\right)\cdot\mathbf{n}_e\right\|_{L^2(e)}.\end{aligned}$$

Then, by using the fact that $p\in W^{3,\infty}(\Omega)$ and $w_i\in W^{1,\infty}(\Omega)$, we get

$$\sum_{i=1}^{d}\left\|\left(w_i\left(\frac{\cdot}{\epsilon}\right)\nabla\partial_{x_i}p\right)\cdot\mathbf{n}_e\right\|_{L^2(e)}\lesssim|e|^{1/2}.$$

We obtain :

$$\begin{aligned}&\sum_{e\in\Gamma_H^i}\left\|\nabla p_{\epsilon}-\sum_{i=1}^{d}\nabla\widehat{w}_i^{\epsilon}\partial_{x_i}p\right\|^2_{L^2(e)}\\ &\lesssim 2\epsilon^2\sum_{e\in\Gamma_H^i}|e|+2\sum_{e\in\Gamma_H^i\cup\Gamma_{H,D}}\left\|\left(\nabla\left(p_{\epsilon}(x)-p\left(x\right)-\epsilon p_1\left(x,\frac{x}{\epsilon}\right)\right)\right)\cdot\mathbf{n}_e\right\|^2_{L^2(e)}\\ &\lesssim\frac{\epsilon^2}{H}+\sum_{e\in\Gamma_H^i}\left\|\left(\nabla\left(p_{\epsilon}-p\left(x\right)-\epsilon p_1\left(x,\frac{x}{\epsilon}\right)\right)\right)\cdot\mathbf{n}_e\right\|^2_{L^2(e)}.\end{aligned}$$

By using the inequality (51), we have

$$\sum_{e\in\Gamma_H^i}\left\|\nabla p_{\epsilon}-\sum_{i=1}^{d}\nabla\widehat{w}_i^{\epsilon}\partial_{x_i}p\right\|^2_{L^2(e)}\lesssim\frac{\epsilon^2}{H}+\left(H+\frac{H}{\epsilon}+H\epsilon^2+\frac{\epsilon}{H}\right).$$

□

Lemma 5. *The following inequalities hold*

$$G^e_{disc}=\sum_{e\in\Gamma_H^i\cup\Gamma_{H,D}}\|p\circ\widehat{w}^{\epsilon}-\pi_H p\circ\widehat{w}^{\epsilon}\|^2_{L^2(e)}\lesssim H^7+\epsilon^2H^2, \tag{38}$$

$$G^{e,2}_{disc}=\sum_{e\in\Gamma_H^i}\|\nabla\left(p\circ\widehat{w}^{\epsilon}-\pi_H p\circ\widehat{w}^{\epsilon}\right)\|^2_{L^2(e)}\lesssim H^5+\epsilon^2. \tag{39}$$

Proof. A Taylor expansion with integral rest yields :

$$p \circ \widehat{w}^{\epsilon}(x) - \pi_H p \circ \widehat{w}^{\epsilon}(x) = (p - \pi_H p)(t,x) + \epsilon \int_0^1 \sum_{i=1}^{d} w_i\left(\frac{x}{\epsilon}\right) \partial_{x_i}(p - \pi_H p)\left(t, x + \epsilon s w\left(\frac{x}{\epsilon}\right)\right) ds.$$

By using the fact that $p \in W^{3,\infty}(\Omega)$ and $w \in W^{1,\infty}(\Omega)$, we get :

$$\sum_{e \in \Gamma_H^i \cup \Gamma_{H,D}} \|p \circ \widehat{w}^{\epsilon} - \pi_H p \circ \widehat{w}^{\epsilon}\|^2_{L^2(e)} \lesssim H^7 + \epsilon^2 H^2.$$

In a similar way as above, we obtain

$$\sum_{e \in \Gamma_H^i} \|\nabla(p \circ \widehat{w}^{\epsilon} - \pi_H p \circ \widehat{w}^{\epsilon})\|^2_{L^2(e)} \lesssim H^5 + \epsilon^2.$$

□

Lemma 6. *The following inequalities hold*

$$G^e_{cell} = \sum_{e \in \Gamma_H^i \cup \Gamma_H^b} \|\pi_H p \circ \widehat{w}^{\epsilon} - \pi_{\epsilon,H} p\|^2_{L^2(e)} \lesssim \frac{\epsilon^2}{H}(1+H)^2, \tag{40}$$

$$G^{e,2}_{cell} = \sum_{e \in \Gamma_H^i} \|\nabla(\pi_H p \circ \widehat{w}^{\epsilon} - \pi_{\epsilon,H} p)\|^2_{L^2(e)} \lesssim \frac{\epsilon^2}{H} + \frac{R_{max}(\epsilon)}{H}(1+H)^2. \tag{41}$$

Proof. Remark that

$$\sum_{e \in \Gamma_H^i \cup \Gamma_{H,D}} \|\pi_H p \circ \widehat{w}^{\epsilon} - \pi_{\epsilon,H} p\|^2_{L^2(e)} \leqslant \epsilon^2 \|\nabla \pi_H p\|^2_{L^\infty(\Omega)} \|w\|^2_{L^\infty(Y)} \sum_{e \in \Gamma_H^i \cup \Gamma_{H,D}} |e|.$$

By using the following inequality $\|\nabla \pi_H p\|_{L^\infty(\Omega)} \lesssim (1+H)$ we deduce that :

$$\sum_{e \in \Gamma_H^i \cup \Gamma_{H,D}} \|\pi_H p \circ \widehat{w}^{\epsilon} - \pi_{\epsilon,H} p\|^2_{L^2(e)} \lesssim \frac{\epsilon^2}{H}(1+H)^2.$$

Let consider $\sum_{e \in \Gamma_H^i \cup \Gamma_{H,D}} \|\nabla(\pi_H p \circ \widehat{w}^{\epsilon} - \pi_{\epsilon,H} p)\|^2_{L^2(e)}$. By intercaling $\nabla \widetilde{w}_i^{\epsilon,E}(x) \partial_{x_i}(\pi_H p \circ \widehat{w}^{\epsilon}(x))$, we have :

$$\begin{aligned}\|\nabla(\pi_H p \circ \widehat{w}^{\epsilon} - \pi_{\epsilon,H} p)\|_{L^2(e)} \leqslant & \left\| \sum_{i=1}^{d} \left(\nabla \widehat{w}_i^{\epsilon} - \nabla \widetilde{w}_i^{\epsilon,E}\right) \partial_{x_i}(\pi_H p \circ \widehat{w}^{\epsilon}) \right\|_{L^2(e)} \\ & + \left\| \sum_{i=1}^{d} \nabla \widetilde{w}_i^{\epsilon,E} \left(\partial_{x_i} \pi_H p \circ \widehat{w}^{\epsilon} - \partial_{x_i} \pi_H p \circ \widetilde{w}^{\epsilon,E}\right) \right\|_{L^2(e)}\end{aligned}$$

Let consider $\left\| \sum_{i=1}^{d} \left(\nabla \widehat{w}_i^{\epsilon} - \nabla \widetilde{w}_i^{\epsilon,E}\right) \partial_{x_i}(\pi_H p \circ \widehat{w}^{\epsilon}) \right\|_{L^2(e)}$. We have :

$$\left\| \sum_{i=1}^{d} \left(\nabla \widehat{w}_i^{\epsilon} - \nabla \widetilde{w}_i^{\epsilon,E}\right) \partial_{x_i}(\pi_H p \circ \widehat{w}^{\epsilon}) \right\|_{L^2(e)} \leqslant \left\| \sum_{i=1}^{d} \nabla \left(\widehat{w}_i^{\epsilon} - \widetilde{w}_i^{\epsilon,E}\right) \right\|_{L^\infty(e)} |e|^{1/2} \left\| \sup_i \partial_{x_i}(\pi_H p) \right\|_{L^\infty(\Omega)}.$$

Let us denote by $R(\epsilon) = \left\| \sum_{i=1}^{d} \nabla \left(\widehat{w}_i^{\epsilon} - \widetilde{w}_i^{\epsilon,E} \right) \right\|_{L^\infty(e)}$. The function $R(\epsilon)$ depends on choice of the boundary used for solving the cell problems and bounded as follow

$$\|R(\epsilon)\|_{L^\infty(e)} \lesssim \begin{cases} \epsilon & \text{if linear boundary condition is used and if we assume that } \quad \|\nabla w^\epsilon\|_{L^\infty(e)} \leqslant \epsilon \\ \|\nabla w\|_{L^\infty(Y)} & \text{if no assumption is done} \end{cases}$$

By using the following inequality $\|\partial_{x_i} \pi_H p\|_{L^\infty(\Omega)} \lesssim (1+H)$ we obtain

$$\sum_{e \in \Gamma_H^i \cup \Gamma_{H,D}} \left\| \sum_{i=1}^{d} \left(\nabla \widehat{w}_i^{\epsilon} - \nabla \widetilde{w}_i^{\epsilon,E} \right) \partial_{x_i} \left(\pi_H p \circ \widehat{w}^{\epsilon} \right) \right\|_{L^2(e)}^2 \lesssim \frac{R_{max}(\epsilon)}{H} (1+H)^2 \tag{42}$$

where

$$R_{max}(\epsilon) = \max_{e \in \Gamma_H^i} \|R(\epsilon)\|_{L^\infty(e)}$$

Let consider $\left\| \sum_{i=1}^{d} \nabla \widetilde{w}_i^{\epsilon,E} \left(\partial_{x_i} (\pi_H p \circ \widehat{w}^{\epsilon}) - \partial_{x_i} (\pi_H p \circ \widetilde{w}^{\epsilon,E}) \right) \right\|_{L^2(e)}$. We have

$$\begin{aligned} &\left\| \sum_{i=1}^{d} \nabla \widetilde{w}_i^{\epsilon,E} \left(\partial_{x_i} \left(\pi_H p \circ \widehat{w}^{\epsilon} \right) - \partial_{x_i} \left(\pi_H p \circ \widetilde{w}^{\epsilon,E} \right) \right) \right\|_{L^2(e)} \\ &\qquad \leqslant \left\| \sum_{i=1}^{d} \nabla \widetilde{w}_i^{\epsilon,E} \right\|_{L^\infty(e)} \sup_i \left\| \partial_{x_i} (\pi_H p \circ \widehat{w}^{\epsilon}) - \partial_{x_i} (\pi_H p \circ \widetilde{w}^{\epsilon,E}) \right\|_{L^2(e)} \end{aligned}$$

We introduce the following subset

$$C_E^e = \left\{ x \in E \mid dist(x, 2\epsilon \|w\|_{L^\infty(Y)}) \subset E \right\} \cap e.$$

We have

$$\begin{aligned} \left\| \partial_{x_i} (\pi_H p \circ \widehat{w}^{\epsilon}) - \partial_{x_i} (\pi_H p \circ \widetilde{w}^{\epsilon,E}) \right\|_{L^2\left(C_E^e\right)} &\lesssim \|\nabla \partial_{x_i} (\pi_H p)\|_{L^\infty(E)} \left\| \widehat{w}^{\epsilon} - \widetilde{w}^{\epsilon,E} \right\|_{L^2(e)} \\ &\lesssim \|\nabla \partial_{x_i} (\pi_H p)\|_{L^\infty(E)} \|\epsilon w^{\epsilon}(x)\|_{L^2(e)} \\ &\lesssim \|\nabla \partial_{x_i} (\pi_H p)\|_{L^\infty(E)} \epsilon |e|^{1/2} \|w\|_{L^\infty(Y)}. \end{aligned}$$

Finally, we get

$$\sum_{e \in \Gamma_H^i} \left\| \partial_{x_i} (\pi_H p \circ \widehat{w}^{\epsilon}) - \partial_{x_i} (\pi_H p \circ \widetilde{w}^{\epsilon,E}) \right\|_{L^2\left(C_E^e\right)}^2 \lesssim \sum_{e \in \Gamma_H^i \cup \Gamma_{H,D}} \epsilon^2 |e| \qquad \lesssim \frac{\epsilon^2}{H}. \tag{43}$$

On $e \setminus C_E^e$, we use the fact that

$$\begin{cases} |e \setminus C_E^e| \lesssim \epsilon H & \text{if d} = 2, \\ |e \setminus C_E^e| \lesssim |\partial e| \, \epsilon & \text{if d} = 3. \end{cases}$$

Then, we get

$$\sum_{e \in \Gamma_H^i} \left\| \partial_{x_i} (\pi_H p \circ \widehat{w}^{\epsilon}) - \partial_{x_i} (\pi_H p \circ \widetilde{w}^{\epsilon,E}) \right\|_{L^2\left(e \setminus C_E^e\right)}^2 \lesssim \sum_{e \in \Gamma_H^i} \epsilon^2 \max(H, |\partial e|) \qquad \lesssim \frac{\epsilon^2}{H}. \tag{44}$$

By using the inequality (43) and (44), we get :

$$\sum_{e\in\Gamma_H^i}\left\|\sum_{i=1}^{d}\nabla\widetilde{w}_i^{\epsilon,E}\left(\partial_{x_i}\left(\pi_H p\circ\widehat{w}^{\epsilon}\right)-\partial_{x_i}\left(\pi_H p\circ\widetilde{w}^{\epsilon,E}\right)\right)\right\|^2_{L^2(e)}\lesssim\frac{\epsilon^2}{H}+\frac{R_{max}(\epsilon)}{H}(1+H)^2.$$

□

Proposition 3. *The following inequalities hold :*

$$T_2\lesssim\epsilon+\frac{H^2}{\epsilon}+H^2+H^6+\epsilon^2H^2+\epsilon H^2+R_{max}(1+H)^2, \tag{45}$$

$$T_3\lesssim\epsilon^2+\frac{\epsilon^2}{H^2}+H^6+\epsilon^2H, \tag{46}$$

Proof. The proof follows directly from the lemmas 4, 5 and 6. □

6.3 Putting all together

Proof. By putting together the inequalities (34), (45) and (46) and by using the fact that $\epsilon < H$ and the fact that when H and ϵ goes to 0, H^k is bounded by H and ϵ^2 is bounded by ϵ, we deduce that :

$$\|p_\epsilon - p_{\epsilon,H}\|_{H^1(\mathcal{E}_H)}\lesssim\sqrt{\epsilon}+H+\frac{\epsilon}{H}+\frac{H}{\sqrt{\epsilon}}++\sqrt{R_{max}(\epsilon)}$$

which is the desired result. □

7 Conclusion

In this work, we study and analyse a new multiscale method based discontinuous Galerkin discretization. We also introduce a new strategy to reduce the resonance error to the so-called cell problems. Numerical examples confirm the efficiency and the accuracy of the method.

In some applications, the quantity in interest is the velocity. In that case, method to recover the velocity from the solution of the pressure equation is introduced in [EV07]. This method cannot be apply here because of the unsatisfactory estimation (7). We have tested numerically the mentioned velocity recovery method introduced and it leads to very large errors. One way to overcome this difficulty consists in using our multiscale method as a preconditioner.

Appendices

Appendix A :

Lemma 7. *Let $\widehat{w}_i^{\epsilon}$ be the function defined in (8) and let $\widetilde{w}_i^{\epsilon,E}$ be the solution solution of the problem (11). Then, there exists a non-negative constant C_6 independent of ϵ and E such that*

$$\left|\widehat{w}^{\epsilon}-\widetilde{w}^{\epsilon,E}\right|_{H^1(E)}\leqslant C_6\sqrt{\epsilon\left|\partial E\right|}, \tag{47}$$

Proof. The proof is inspired from [Oua13](Annexe C).

Let us define $r_i^{\epsilon,E}(x) = \widehat{w}^{\epsilon}(x) - \widetilde{w}^{\epsilon,E}(x)$. We can easily verify that $r_i^{\epsilon,E}$ is solution to the following equation

$$\begin{cases} -\text{div}\left(\mathbf{K}^{\epsilon}(x)\nabla r_i^{\epsilon,E}\right) &= 0 \quad \forall x \in E, \\ r_i^{\epsilon,E}(x) &= \epsilon w_i^{\epsilon}(x) \quad \forall x \in \partial E, \end{cases} \tag{48}$$

Let c be a positive constant independent of ϵ and E. We define a regular function m_ϵ as follows

$$\begin{cases} m_\epsilon(x) &= 1 \quad \text{on} \quad \partial E \\ \|\nabla m_\epsilon\|_{L^\infty(E)} &\leqslant c/\epsilon \quad \text{in} \quad E \\ m_\epsilon(x) &= 0 \quad \text{if} \quad \text{dist}(x, \partial E) \geqslant c\epsilon \\ 0 \leqslant m_\epsilon(x) \leqslant 1 & \quad \text{in} \quad E. \end{cases}$$

We introduce the function t^ϵ such that

$$r^{\epsilon}(x) = -\epsilon w^{\epsilon}(x) m_\epsilon(x) + t^{\epsilon}(x). \tag{49}$$

The function t^ϵ satisfies the following equation

$$\begin{cases} -\text{div}\left(\mathbf{K}^{\epsilon}(x)\nabla t^{\epsilon}\right) &= -\text{div}\left(\mathbf{K}^{\epsilon}(x)\nabla\left(\epsilon w^{\epsilon}(x) m_\epsilon\right)\right) \quad \forall x \in E, \\ t^{\epsilon}(x) &= 0 \quad \forall x \in \partial E, \end{cases} \tag{50}$$

By remarking that the measure of the support of the function m_ϵ is bounded by $c\epsilon\,|\partial E|$, multiplying the equation (50) by t^ϵ and integrating by parts, we can prove that

$$|t^{\epsilon}|_{H^1(E)} \lesssim \sqrt{\epsilon\,\partial E}.$$

Finally, we get

$$\left|\nabla r_i^{\epsilon,E}\right|_{H^1(E)} \leqslant |\epsilon\, w^{\epsilon}(x) m_\epsilon(x)|_{H^1(E)} + |t^{\epsilon}|_{H^1(E)} \lesssim \sqrt{\epsilon\,\partial E}$$

which is the desired result. □

Appendix B :

Lemma 8. *Under the assumptions 1. There exists a constant C_9 independent of ϵ and H such that*

$$\sum_{e \in \Gamma_H^i \cup \Gamma_{H,D}} \left\|\nabla\left(p_\epsilon(x) - p\left(x\right) - \epsilon p_1\left(x, \frac{x}{\epsilon}\right)\right)\right\|_{L^2(e)}^2 \leqslant C_9\left(H + \frac{H}{\epsilon} + H\epsilon^2 + \frac{\epsilon}{H}\right). \tag{51}$$

Proof. Let denote by P_H^k the L^2-projector operator in $\mathbb{P}_d^k(\mathcal{E}_H)$ and

$$G = p_\epsilon(x) - p\left(x\right) - \epsilon p_1\left(x, \frac{x}{\epsilon}\right).$$

Let $E \in \mathcal{E}_H$ and $e \in \partial E$, then

$$\|\nabla G\|_{L^2(e)} \leqslant \left\|\nabla\left(G - P_H^k\left(G\right)\right)\right\|_{L^2(e)} + \left\|\nabla\left(P_H^k\left(G\right)\right)\right\|_{L^2(e)}.$$

Using the result proved in [DPE12][Lemma 1.59], yields

$$\left\|\nabla\left(G - P_H^k\left(G\right)\right)\right\|_{L^2(e)} \leqslant C_{tr} H^{1/2} |G|_{H^2(E)}, \quad \forall e \in \Gamma_H \cap E \tag{52}$$

where C_{tr} is a constant independent from E and H. Using the result proved in [Riv08][p.23], gives

$$\left\|\nabla\left(P_H^k\left(G\right)\right)\right\|_{L^2(e)} \leqslant C_{trb} H^{-1/2} \left\|\nabla P_H^k(G)\right\|_{L^2(E)}$$

where C_{trb} is a constant independent from E and H. It remains to evaluate $\left\|\nabla P_H^k(G)\right\|_{L^2(E)}$. Since P_H^k is stable in norm H^1, then

$$\left\|\nabla P_H^k(G)\right\|_{L^2(E)} \leqslant \left\|\nabla\left(P_H^k(G) - G\right)\right\|_{L^2(E)} + \left\|\nabla G\right\|_{L^2(E)}.$$

Passing by the reference element $\widehat{E}$, there exists a constant $\widehat{C}$ such that

$$\left\|\nabla P_H^k(G)\right\|_{L^2(E)} \leqslant \widehat{C}\left\|\nabla G\right\|_{L^2(E)} + \left\|\nabla G\right\|_{L^2(E)} \leqslant (1+\widehat{C})\left\|\nabla G\right\|_{L^2(E)}.$$

Again :

$$\left\|\nabla P_H^k(G)\right\|_{L^2(e)} \leqslant C_{trb}(1+\widehat{C}) H^{-1/2} \left\|\nabla G\right\|_{L^2(E)} \tag{53}$$

Using (52) and (53), we get

$$\begin{aligned}
&\sum_{e\in\Gamma_H^i\cup\Gamma_{H,D}} \left\|\nabla\left(p_\epsilon(x) - p\left(x\right) - \epsilon p_1\left(x, \frac{x}{\epsilon}\right)\right)\right\|_{L^2(e)}^2 \\
&\leqslant \sum_{e\in\Gamma_H^i\cup\Gamma_{H,D}} \left(2C_{tr}^2 H \left\|G\right\|_{H^2(E)}^2 + 2C_{trb}^2(1+\widehat{C})^2 H^{-1}\left\|\nabla G\right\|_{L^2(E)}^2\right) \\
&\leqslant 2LC_{tr}^2 H \sum_{E\in\mathcal{E}_H} \left\|G\right\|_{H^2(E)}^2 + 2C_{trb}^2(1+\widehat{C})^2 H^{-1} \sum_{E\in\mathcal{E}_H} \left\|\nabla G\right\|_{L^2(E)}^2
\end{aligned}$$

where L the maximum number of faces belonging to E. Finally, using (6), yields

$$\sum_{e\in\Gamma_H^i\cup\Gamma_{H,D}} \left\|\nabla\left(p_\epsilon(x) - p\left(x\right) - \epsilon p_1\left(x, \frac{x}{\epsilon}\right)\right)\right\|_{L^2(e)}^2 \leqslant C_9\left(H + \frac{H}{\epsilon} + H\epsilon^2 + H^{-1}\epsilon\right)$$

where $C_9 = \max\left(2LC_{tr}^2\,,\, 2LC_{trb}^2(1+\widehat{C})^2\right)$. □

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