On Inventory Control For Perishable Inventory Systems Subject To Uncertainties On Customer Demands

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Abstract:
This paper deals with the inventory controller design for constrained production systems subject to uncertainties on the customer demands. The case study focuses on the inventory regulation problem in production systems where contain perishable finite products. Such systems are characterized by the presence of delays due to production processes, and constraints from the instantaneous inventory level, production level and the finite capacities of stocks. To do that, we propose a management strategy based on the inventory control, using either a linear control law or a bang-bang control. A design method is proposed to determine the parameters of an admissible control law. The design method is based on the invariance principle, and our proof is based on the exact identification of the admissible region on the space of the parameters of the system and the control specifications.

Keywords: Inventory control design, Perishable finite products, Time-delay systems, predictor-feedback structure, $\mathcal{D}$-invariance properties.

1. INTRODUCTION

In this paper, we are interested on the inventory regulation problem in perishable inventory systems that must satisfy the customer demand the we suppose unknown but bounded by defined values. Also, the production system is characterized by the presence of delay due to the process time, and the positivity constraints due to the specifications of the system, such as production and storage capacities. Another characteristic of the system is the presence of a perishability factor which is related to the stock. The main difficulty is developing control laws for perishable inventories stems from the necessity of conducting an exact analysis of product lifetimes. The difficulty to obtain a robust control is more complicated in the situation when the customer demand is subject to significant uncertainty and the inventories are replenished with non-negligible delay. Our objective is to maintain high service level without interruption and at the same time to minimize the cost inventory. For that, we must define a control law which is useful to satisfy the demand during procurement latency but also to compensate the stock deterioration in that time.

In this paper, we carried out a study on the inventory control problem of logistical systems subject to delays and perishable finite products using an approach based on control theory. Those systems are subject to positivity and saturation constraints mainly related to the physical characteristics of the processes and should be taken into account in the modelling of systems and the design of control laws to stabilize them with respect to the delay. We have introduced the basic model for each node of a logistic system, Which is a delay system on the input. This delay is interpreted as the duration of production operations. The logistic system considered must satisfy a customer demand while respecting constraints induced by the positivity of the variables and the finite capacity of the processes.

We proposed two permissible command structures: the first is an affine law and the second is a bang-bang type law, which serve to stabilize the logistical system under consideration, whilst respecting the constraints of saturation and positivity, and for any unknown but bounded demand $d(t)$. We studied a delayed logistic system for zero initial conditions, that is to say without taking into account its dynamics between the instants 0 and *theta* . We have obtained necessary conditions for the existence of the admissible laws according to the intrinsic parameters of the system, such as *sigma*, *theta*, etc.

This article is organised as follows. The second section is dedicated to the problem statement and the characteristics of the considered problem. description of the problem. In the third section, some backgrounds about the $\mathcal{D}$-invariance properties are briefly described. In section 4, the inventory control problem with perishable finite products and under unknown demand is developed. The section 5 is dedicated to the controller design, where the conditions of the existence of the controller are given. The paper concludes with some discussions, as well as the directions of the future work.

2. PROBLEM STATEMENT

2.1 Model Description

In this study, we consider an elementary production system composed of a supplying unit and a storage one. The supplying unit is characterized by a supplying order rate denoted $u(t)$, which is limited by a minimum value denotes u_{min} and a maximum supplying order rate denoted u_{max}. Furthermore, the production system is characterizer by a delay θ, $\theta \geq 0$, which corresponds to the time needed to complete the finite products. The storage unit is characterized by the inventory level denoted $y(t)$ bounded by a minimum and maximum storage capacity denoted, respectively, y_{min} and y_{max}. In this work, the customer demand denoted $d(t)$ is supposed to be unknown but assumed to be bounded by two values denoted d_{min} and d_{max}, known in advance, and corresponding to, respectively, minimum and maximum demand rates.

This production system is as time-delay system with perishable products defined by an expiration rate named σ. Indeed, $u(t)$ is the control input, $d(t)$ is an external perturbation, and $y(t)$ is the output to be controlled. The generic model for the inventory level dynamics is then described by the following first order delayed equation.

$$\dot{y}(t) = \begin{cases} -\sigma y(t) + u(t-\theta) - d(t) & \text{for } t \geq \theta\,, \\ -\sigma y(t) - d(t) & \text{for } 0 \leq t < \theta\,. \end{cases} \quad (1)$$

$u(t)$, $y(t)$ and $d(t)$ are non negative variables and they represent, respectively, the production level, the inventory level and the customer demand. In this paper, the study of the production system is focused on the horizon time t, $t \geq \theta$.

2.2 Constraints And Objective

The objective of this study consists first to define necessary and sufficient conditions of existence of an admissible control law $u(t)$ in order to fulfil the demand $d(t)$ taking into account the different constraints. related to supplying units and inventories which are limited resources, and they can take only non-negative values. These constraints are formulates as follows.

The controller should be designed such that, for all $t \geq 0$:

$$y_{min} \leq y(t) \leq y_{max}\ , \quad (2)$$

with

$$u_{min} \leq u(t) \leq u_{max}\ , \quad (3)$$

and for every demand function satisfying

$$d_{min} \leq d(t) \leq d_{max}\ . \quad (4)$$

3. PROPOSED CONTROL STRATEGY

3.1 Prediction and invariance

As developed in (Moussaoui, 2014) and extended in (Abbou et al.,, 2015a) and (Abbou *et al.*, 2015b), our proposed approach to control systems with delayed inputs is based on a **predictor based feedback structure**. This structure permits to stabilize the system and to compensate the delay effects present in the loop. The specifications of the production system are introduced as constraints imposed to the controller, so as to forbid any overruns on the production rates or on the inventory levels, which can cause the saturation of the production unit. The role of the controller is then to keep the production rate and so, the inventory level, as far as possible within their limits.

Using the feedback-predictor structure, also known as model reduction or Arstein reduction (Artstein, 1982), the basic idea of state prediction is to compensate the time delay θ by generating a control law that enables one to directly use the corresponding delay-free system, thanks to the prediction expressed by

$$z(t) = \mathrm{e}^{-\sigma\theta} y(t) + \int_{t-\theta}^{t} \mathrm{e}^{-\sigma(t-\tau)} u(\tau) d\tau\ . \quad (5)$$

By time derivation of the equation (5), one can see that the resulting system

$$\dot{z}(t) = -\sigma z(t) + u(t) - \mathrm{e}^{-\sigma\theta} d(t)\ . \quad (6)$$

is free-delay. Then we can applicate the invariance theory explained in the next paragraph.

3.2 $\mathcal{D}$-invariance concept

We consider a function f defining a system $\dot{x}(t) = f(x(t)) - d(t)$. The interval $\mathcal{Z}$ defined as $\mathcal{Z} = [z_{min}, z_{max}]$ is $\mathcal{D}$-invariant for this system, with $\mathcal{D} = [d_{min}, d_{max}]$, if and only if the following conditions are fulfilled.

$$f(z_{min}, d_{max}) \geq 0 \quad (7)$$

$$f(z_{max}, d_{min}) \leq 0 \quad (8)$$

If we consider the system defined by the equation (6), and we applicate the $\mathcal{D}$-invariance results, we deduce the following expressions for the maximum and the minimum values of $z(t)$, for $z(t) = z_{min}$ and for $z(t) = z_{max}$.

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For $z(t) = z_{min}$, the expression is

$$-\sigma z_{min} + u(t) - \mathrm{e}^{-\sigma\theta} d_{max} \geq 0 \quad (9)$$

and for $z(t) = z_{max}$, we get

$$-\sigma z_{max} + u(t) - \mathrm{e}^{-\sigma\theta} d_{min} \leq 0 \quad (10)$$

3.3 Types of control laws

We can consider two forms of control laws that allow the stability of the system (1) in closed loop taking into account positivity and saturation constraints (2) et (3). For each form of the control law, we consider two values u_1 and u_2 which fulfil the constraint (3) expressed by

$$u_1 \in [u_{min}, u_{max}] \quad (11)$$

and

$$u_2 \in [u_{min}, u_{max}]\ . \quad (12)$$

Affine control law

The first control law is an affine one, such as, for all $d(t) \in \mathcal{D}$, and for $z(t) \in \mathcal{Z}$, the affine control law is defined as

$$u(t) = \begin{cases} u_1 & \text{if } z(t) = z_{min} \,, \\ u_2 & \text{if } z(t) = z_{max} \,. \end{cases} \quad (13)$$

and structured as follows:

$$u(t) = \begin{cases} K(z_0 - z(t)) & \text{if } u_1 \neq u_2 \,, \\ u_1 = u_2 & \text{if } u_1 = u_2 \,. \end{cases} \quad (14)$$

with $K = \frac{u_2 - u_1}{z_{max} - z_{min}}$ and $z_0 = \frac{(u_2 z_{max} - u_1 z_{min})}{u_2 - u_1}$.

Bang-bang control law

The another law which can be applied is the so called bang-bang. This law belongs to the class of well-known optimal control laws. The only both values that the law can take are the minimum and a maximum as described in Fig. 1 and given as follows.

$$u(t) = \begin{cases} u_1 & \text{if } z(t) \leq z_{min} \,, \\ u_2 & \text{if } z(t) \geq z_{max} \,. \end{cases} \quad (15)$$

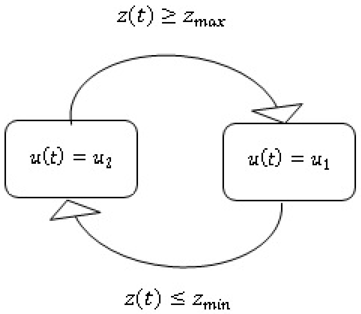

Fig. 1. Bang-bang control automaton

Application of the $\mathcal{D}$-invariance concept

By applying the $\mathcal{D}$-invariance concept, and from the inequalities (9) and (10), we suppose that there are two values of the control $u(t)$, which are u_1 et u_2, such as

$$u_1 \geq \sigma z_{min} + e^{-\sigma\theta} d_{max}, \quad (16)$$

and

$$u_2 \leq \sigma z_{max} + e^{-\sigma\theta} d_{min} \,. \quad (17)$$

We deduce from these expressions (16), (17), (11) and (12) that the following conditions are fulfilled

$$u_{max} \geq \sigma z_{min} + e^{-\sigma\theta} d_{max} \quad (18)$$

$$u_{min} \leq \sigma z_{max} + e^{-\sigma\theta} d_{min} \quad (19)$$

Then, the conditions (18) and (19) are necessary for the $\mathcal{D}$-invariance of the interval $[z_{min}, z_{max}]$.

Inversely, if the conditions (18) and (19) are satisfied, u_1 and u_2 which fulfill (16), (17), (11) and (12), can be chosen as $u_1 = u_{max}$ and $u_2 = u_{min}$. We can see that the conditions (18) and (19) are necessary and sufficient for the $\mathcal{D}$-invariance of the interval $[z_{min}, z_{max}]$ for the system (6) verifying the following condition

$$z_{min} < z_{max} \quad (20)$$

4. PROPERTIES OF THE PROPOSED STRATEGY

4.1 Admissibility conditions of the control law

From the properties of $\mathcal{D}$-invariance, if the interval $\mathcal{Z} = [z_{min}, z_{max}]$ is $\mathcal{D}$-invariant for the closed-loop system, then the control law $u(t)$ evolves in the interval $[u_{min}, u_{max}]$, for any demand $d(t)$ evolving in the interval $[d_{min}, d_{max}]$.

In order to determine the admissibility conditions of the control law for the studied system expressed by (1), we apply a predictor based feedback structure. The predictor expressed by (5) can be written such as

$$y(t) = e^{\sigma\theta}(z(t) - \int_{t-\theta}^{t} e^{-\sigma(t-\tau)} u(\tau) d\tau) \quad (21)$$

We choose the values u_1 and u_2 which fulfill (11) and (12). One can see that $y(t)$ evolves in the interval

$$[e^{\sigma\theta}(z_{min} - \frac{1 - e^{-\sigma\theta}}{\sigma} u_2), e^{\sigma\theta}(z_{max} - \frac{1 - e^{-\sigma\theta}}{\sigma} u_1)] \quad (22)$$

if $u_1 \geq u_2$,

otherwise in the interval

$$[e^{\sigma\theta}(z_{min} - \frac{1 - e^{-\sigma\theta}}{\sigma} u_1), e^{\sigma\theta}(z_{max} - \frac{1 - e^{-\sigma\theta}}{\sigma} u_2)] \quad (23)$$

if $u_1 \leq u_2$.

We can then deduce that the constraint on $y(t)$ expressed by (2) is met if the following conditions are satisfied:

$$y_{min} \leq e^{\sigma\theta}(z_{min} - \frac{1 - e^{-\sigma\theta}}{\sigma} u_2) \quad (24)$$

$$y_{max} \geq e^{\sigma\theta}(z_{max} - \frac{1 - e^{-\sigma\theta}}{\sigma} u_1) \quad (25)$$

if $u_1 \leq u_2$,

otherwise

$$y_{min} \leq e^{\sigma\theta}(z_{min} - \frac{1 - e^{-\sigma\theta}}{\sigma} u_1) \quad (26)$$

$$y_{max} \geq e^{\sigma\theta}(z_{max} - \frac{1 - e^{-\sigma\theta}}{\sigma} u_2) \quad (27)$$

if $u_1 \geq u_2$.

Therefore, the interval $\mathcal{Z} = [z_{min}, z_{max}]$ is $\mathcal{D}$-invariant by the control law expressed either by(16) or (17), which verifies the constraint of $u(t)$ such that $u(t) \in [u_{min}, u_{max}]$, for all $d(t) \in \mathcal{D}$, for all $t \geq 0$, if and only if the following conditions are satisfied

$$u_{max} \geq u_1 \geq \sigma z_{min} + e^{-\sigma\theta} d_{max}$$

$$u_{min} \leq u_2 \leq \sigma z_{max} + e^{-\sigma\theta} d_{min}$$

4.2 Main results

Theorem 1. Being given a system of the form (1), there exists a command of the form (14) or (1), for which the *system is stable and the constraints (2) and (3) are fulfilled for any demand $d(t) \in \mathcal{D}$, if and only if the following conditions hold true*

$$\max(\frac{1-\mathrm{e}^{-\sigma\theta}}{\sigma}d_{max}+y_{min}, \frac{1-\mathrm{e}^{-\sigma\theta}}{\sigma}u_{min}+\mathrm{e}^{-\sigma\theta}y_{min}) \leq \frac{1}{\sigma}(u_{max}-\mathrm{e}^{-\sigma\theta}d_{max}) \quad (28)$$

$$\frac{1}{\sigma}(u_{min}-\mathrm{e}^{-\sigma\theta}d_{min}) \leq \min(\frac{1-\mathrm{e}^{-\sigma\theta}}{\sigma}d_{min}+y_{max}, \frac{1-\mathrm{e}^{-\sigma\theta}}{\sigma}u_{max}+\mathrm{e}^{-\sigma\theta}y_{max}) \quad (29)$$

$$\max(\frac{1-\mathrm{e}^{-\sigma\theta}}{\sigma}u_{min}+\mathrm{e}^{-\sigma\theta}y_{min}, \frac{1-\mathrm{e}^{-\sigma\theta}}{\sigma}d_{max}+y_{min}) < \min(\frac{1-\mathrm{e}^{-\sigma\theta}}{\sigma}u_{max}+\mathrm{e}^{-\sigma\theta}y_{max}, \frac{1-\mathrm{e}^{-\sigma\theta}}{\sigma}d_{min}+y_{max}) \quad (30)$$

Lemma 1
As seen in the last sections, we can deduce that the conditions (18), (19), (20) , (24), (25), (26) and (27) are sufficient for the admissibility of the affine and bang-bang control laws defined by u_1, u_2, z_{min} and z_{max}.
These conditions are expressed in terms of different parameters such θ, σ, u_1, u_2, z_{min}, z_{max}, y_{min}, y_{max}, d_{min}, d_{max}, u_{min} and u_{max}.

These parameters can be classified into different categories, first, there are intrinsic parameters to the system which are θ and σ, then, different parameters related to the specification which must be satisfied, y_{min}, y_{max}, d_{min}, d_{max}, u_{min} and u_{max}, finally, the parameters of the structure of control laws which are u_1, u_2, z_{min} and z_{max}.

4.3 Elimination of the parameters u_1, u_2, z_{min} and z_{max}

- Elimination of u_1

Resulting from the lemma 1, the conditions (16), (25) and (26) depend on u_1, which we want to extract from these conditions, using the mentioned expressions. From these conditions, we get the following one which is independent of the parameter u_1

$$z_{max}-\mathrm{e}^{-\sigma\theta}y_{max} \leq z_{min}-\mathrm{e}^{-\sigma\theta}y_{min}\ . \quad (31)$$

Conversely, if the last condition is fulfilled, one can choose u_1 such that the conditions (11), (25) and (26) are satisfied. The expression (31) is equivalent to the existence of a number u_1 which satisfies (11), (25) and (26). Using the conditions (16), (11) and (25), we can deduce the following relation.

$$\max(\frac{1-\mathrm{e}^{-\sigma\theta}}{\sigma}u_{min}, z_{max}-\mathrm{e}^{-\sigma\theta}y_{max}, \frac{1-\mathrm{e}^{-\sigma\theta}}{\sigma}(\sigma z_{min}+\mathrm{e}^{-\sigma\theta}d_{max})) \leq \frac{1-\mathrm{e}^{-\sigma\theta}}{\sigma}u_1 \quad (32)$$

From the expressions (11) et (26), we deduce the following condition:

$$\frac{1-\mathrm{e}^{-\sigma\theta}}{\sigma}u_1 \leq \min(\frac{1-\mathrm{e}^{-\sigma\theta}}{\sigma}u_{max}, z_{min}-\mathrm{e}^{-\sigma\theta}y_{min}) \quad (33)$$

Then, from the conditions (32) and (33), we find the condition (31)We conclude then the conditions mentioned below

$$\frac{1-\mathrm{e}^{-\sigma\theta}}{\sigma}u_{max} \geq z_{max}-\mathrm{e}^{-\sigma\theta}y_{max} \quad (34)$$

$$\frac{1-\mathrm{e}^{-\sigma\theta}}{\sigma}u_{max} \geq \frac{1-\mathrm{e}^{-\sigma\theta}}{\sigma}(\sigma z_{min}+\mathrm{e}^{-\sigma\theta}d_{max}) \quad (35)$$

$$z_{min}-\mathrm{e}^{-\sigma\theta}y_{min} \geq \frac{1-\mathrm{e}^{-\sigma\theta}}{\sigma}u_{min} \quad (36)$$

$$z_{min}-\mathrm{e}^{-\sigma\theta}y_{min} \geq \frac{1-\mathrm{e}^{-\sigma\theta}}{\sigma}(\sigma z_{min}+\mathrm{e}^{-\sigma\theta}d_{max}) \quad (37)$$

Lemma 2
Being given a system of the form (1). There exist a number u_1 that verifies the conditions (16), (11), (25) and (26) if and only if the conditions (31), (34), (35), (36) and (37) are fulfilled.
In this paragraph, we have operated on the conditions of the lemma 1 and we are interested to the elimniation of the parameter u_1, we have got conditions independant on u_1. Similarly, we will elimnate u_2 to get independant conditions on this parameter.

- Elimination of u_2

With a similar way as for u_1, and from the conditions (24) and (27) resulting from the lemma 1, we get the condition (31), and taken into account the expressions (24) and (27), we get the conditions as formulated below

$$\frac{1-\mathrm{e}^{-\sigma\theta}}{\sigma}u_2 \leq \min(\frac{1-\mathrm{e}^{-\sigma\theta}}{\sigma}u_{max}, z_{min}-\mathrm{e}^{-\sigma\theta}y_{min}, \frac{1-\mathrm{e}^{-\sigma\theta}}{\sigma}(\sigma z_{max}+\mathrm{e}^{-\sigma\theta}d_{min})) \quad (38)$$

$$\frac{1-\mathrm{e}^{-\sigma\theta}}{\sigma}u_2 \geq \max(\frac{1-\mathrm{e}^{-\sigma\theta}}{\sigma}u_{min}, z_{max}-\mathrm{e}^{-\sigma\theta}y_{max}) \quad (39)$$

Then, from the formulations (38) et (39), we get the conditions (19),(31), (34) and (36) that was got from the elimination of u_1, and the we get also the following conditions

$$\frac{1-\mathrm{e}^{-\sigma\theta}}{\sigma}(\sigma z_{max}+\mathrm{e}^{-\sigma\theta}d_{min}) \geq \frac{1-\mathrm{e}^{-\sigma\theta}}{\sigma}u_{min} \quad (40)$$

$$\frac{1-\mathrm{e}^{-\sigma\theta}}{\sigma}(\sigma z_{max}+\mathrm{e}^{-\sigma\theta}d_{min}) \geq z_{max}-\mathrm{e}^{-\sigma\theta}y_{max} \quad (41)$$

Lemma 3
Being given a system of the form (1). There exist a number u_2 that verifies the conditions (17), (12), (24) and (27) if and only if the conditions (31),(34), (36), (40) and (41) are fulfilled.

From the lemma 2 and 3, we can reformulate the lemma 1 using the conditions independent on the auxiliary parameters u_1 and u_2.

4.4 Elimination of z_{min} and z_{max}

In this section, as done for the elimination of u_1 and u_2, we will eliminate the auxiliary parameters z_{min} and z_{max}. We will operate on the conditions (31),(34), (35), (36), (37), (40) and (41), we get the following expressions

$$z_{max} - z_{min} \leq \mathrm{e}^{-\sigma\theta}(y_{max} - y_{min}) \quad (42)$$

$$z_{max} \leq \frac{1-\mathrm{e}^{-\sigma\theta}}{\sigma} u_{max} + \mathrm{e}^{-\sigma\theta} y_{max} \quad (43)$$

$$z_{min} \leq \frac{1}{\sigma}(u_{max} - \mathrm{e}^{-\sigma\theta} d_{max}) \quad (44)$$

$$z_{min} \geq \frac{1-\mathrm{e}^{-\sigma\theta}}{\sigma} u_{min} + \mathrm{e}^{-\sigma\theta} y_{min} \quad (45)$$

$$z_{min} \geq \frac{1-\mathrm{e}^{-\sigma\theta}}{\sigma} d_{max} + y_{min} \quad (46)$$

$$z_{max} \geq \frac{1}{\sigma}(u_{min} - \mathrm{e}^{-\sigma\theta} d_{min}) \quad (47)$$

$$z_{max} \leq \frac{1-\mathrm{e}^{-\sigma\theta}}{\sigma} d_{min} + y_{max} \quad (48)$$

Then, from the expressions (43) et (48), we deduce the formulation below

$$z_{max} \leq min(\frac{1-\mathrm{e}^{-\sigma\theta}}{\sigma} u_{max} + \mathrm{e}^{-\sigma\theta} y_{max}, \frac{1-\mathrm{e}^{-\sigma\theta}}{\sigma} d_{min} + y_{max}) \quad (49)$$

Similarly, for the condition on z_{min}, from the expressions (45) et (46), we get

$$z_{min} \geq max(\frac{1-\mathrm{e}^{-\sigma\theta}}{\sigma} u_{min} + \mathrm{e}^{-\sigma\theta} y_{min}, \frac{1-\mathrm{e}^{-\sigma\theta}}{\sigma} d_{max} + y_{min}) \quad (50)$$

4.5 Geometric interpretation of the obtained conditions

Based on the conditions mentioned above, those results have a geometric interpretation. Indeed, one can remark that the four inequalities (44), (47), (49) and (50) define a rectangle in the plane (z_{min}, z_{max}). Furthermore, the expressions (20) and (42) correspond to the definition of a strip located above the main diagonal $z_{min} = z_{max}$. The area obtained corresponds well to a polyhedron.

In order to obtain the necessary ans sufficient conditions of an admissible control law regardless of the intermediary parameters, we must eliminate the parameters z_{min} et z_{max} from the set of inequalities. Therefore, from the conditions (44) and (50), we obtain the first inequalities expressed by

$$\frac{1-\mathrm{e}^{-\sigma\theta}}{\sigma} d_{max} + y_{min} \leq \frac{1}{\sigma}(u_{max} - \mathrm{e}^{-\sigma\theta} d_{max}) \quad (51)$$

$$\frac{1-\mathrm{e}^{-\sigma\theta}}{\sigma} u_{min} + \mathrm{e}^{-\sigma\theta} y_{min} \leq \frac{1}{\sigma}(u_{max} - \mathrm{e}^{-\sigma\theta} d_{max}) \quad (52)$$

And from the conditions (47) et (49), we obtain

$$\frac{1}{\sigma}(u_{min} - \mathrm{e}^{-\sigma\theta} d_{min}) \leq \frac{1-\mathrm{e}^{-\sigma\theta}}{\sigma} u_{max} + \mathrm{e}^{-\sigma\theta} y_{max} \quad (53)$$

$$\frac{1}{\sigma}(u_{min} - \mathrm{e}^{-\sigma\theta} d_{min}) \leq \frac{1-\mathrm{e}^{-\sigma\theta}}{\sigma} d_{min} + y_{max} \quad (54)$$

Finally, we obtain the necessary and sufficient conditions depending only on the parameters of the studied system (u_{min}, u_{max}, θ, y_{min}, y_{max}, σ, d_{min}, d_{max}).

Lemma 4
Being given a system of the form (1). There exist two numbers z_{min} and z_{max} that verifies the conditions (20), (31), (34), (35), (36), (37), (40) and (41) if and only if the conditions (51),(52), (53) et (54) are fulfilled.

Corollary 1
Being given a dynamic system of the form (1), there exist numbers u_1, u_2, z_{min} and z_{max} that verifies the inequalities (18), (19), (20) , (24), (25), (26) and (27) if and only if the conditions (31),(34), (35),(36), (37), (40) and(41) are fulfilled.

5. DISCUSSIONS ON THE PROPOSED APPROACH

In this study, the first contribution compared to the work of our predecessors, consists in considering generic intervals for the variations of constraints and specifications. In the literature, the authors considered minimal boundary intervals for the stock level, the production order and the demand. In our case, we have defined these intervals by taking non-zero minimum values. The second contribution is to add the uncertainty constraint on the stock dynamics - which is the perishable rate of loss - in addition to the uncertainty about customer demand and production times. Finally, the originality of this work, from the point of view of production management, lies in the study of the conditions of existence of a control law. Indeed, before investing and choosing the production and storage units, someone has to study the constraints of dimensioning of the production system, under the existence of the control law.

As we assumed a time-independent static loss factor, further study is required to extend the proposed approach by considering a time-varying dynamic rate $\sigma(t)$, and study its impact on of the laws structure of orders as well as the conditions of their existences. Also, it would be interesting to consider the delay *theta* as variable, which is a great importance in distribution logistics.

6. CONCLUSION

The paper deals with the problem of perishable inventory control of supply chain subject to the demand variability under constraints. It focused in searching the conditions of existence of a control law which satisfy the demand while fulfilling the system constraints. A sufficient conditions of existence of two control laws are proposed. These conditions are based on the system parameters such as the perishability factor σ and the delay θ, in addition, the limits of the storage capacity y_{min}, y_{max}, then the limits of the demand d_{min}, d_{max}, and finally, the production orders levels u_{min}, u_{max}. The main advantage of the conditions proposed in this work is it permits the analysis of the existence of control laws before their conception and the the implementation in the space of parameters permits

This study opens perspectives at various levels. In our case, we consider two control laws. Further study is required to generalise these conditions for any type of control. Also, different hypotheses can be checked: uncertain delay and uncertain perishability factor which depend on time or date (date of expiry). Finally, one could consider a just-in-time strategies to regulate the perishable inventory systems.

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