Approximation of $|1 - L(z)|$ 2

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I. Introduction

Let us consider a Kalman filter (KF) based on an autoregressive model of order 2 (AR(2)) for tracking a complex scalar process denoted by $\alpha_{(k)}$. The equations of the AR(2) model are given in [1, Eqs (4), (9)-(11)] and those of the corresponding KF together with its steady state version in [1, Section III-A, III-B, III-C]. In the following, the notations introduced in these equations will be used. Let $\alpha(z)$ be the z-transform of $\alpha_{(k)}$. Let $L(z)$ be the z-transform of the impulse response of the steady state AR(2)-KF that gives the estimate of $\alpha(z)$, denoted by $\hat{\alpha}(z)$, with the observation as input (see Fig. 1).

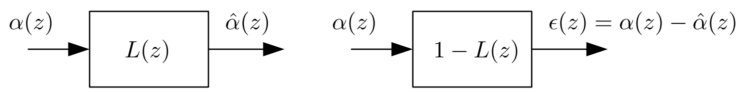

Fig. 1: Scheme of the steady state KF

This report is a self-content report that gives the expression for $|1 - L(z)|^2$, which is useful when calculating the mean square error of the estimate.

$L(z)$ is given by [1, Eq. (47)]:

$$L(z) = \frac{K_1 + a_2 K_2 z^{-1}}{1 + z^{-1}(a_2 K_2 - a_1(1 - K_1)) - a_2(1 - K_1)z^{-2}} \quad . \tag{1}$$

II. Expression for $|1 - L(z)|^2$

In this section, we calculate the expression for $|1 - L(z)|^2$ as a function of the KF parameters $\delta = 1 - r$, $\omega_{\text{AR(2)}}T = 2\pi f_{\text{AR(2)}}T$, fT and K_1. The only assumption that is used here is related to the fact that low normalized frequencies $fT \ll 1$ are considered, which leads to $z = e^{2i\pi fT} \simeq 1 + i2\pi fT$. In order to obtain the squared modulus of $(1 - L(z))$, we multiply $1 - L(z)$ by its conjugate $(1 - L(z))^*$, which yields:

$$|1 - L(e^{2i\pi fT})|^2 \simeq \frac{A(fT)^4 + B(fT)^2 + C}{D(fT)^4 + E(fT)^2 + F} \tag{2}$$

where A, B, C, D, E, F are defined as functions of δ, $\omega_{\text{AR(2)}}T$ and K_1 as follows:

$$\begin{aligned} A = {} & 64\pi^4\delta^2K_1^4 - 256\pi^4\delta^2K_1^3 + 384\pi^4\delta^2K_1^2 - 256\pi^4\delta^2K_1 + 64\pi^4\delta^2 - 64\pi^4\delta K_1^4 + 320\pi^4\delta K_1^3 - 576\pi^4\delta K_1^2 + 448\pi^4\delta K_1 - 128\pi^4\delta \\ & + 16\pi^4K_1^4 - 96\pi^4K_1^3 + 208\pi^4K_1^2 - 192\pi^4K_1 + 16\pi^4 \end{aligned} \tag{3}$$

$$\begin{aligned}
B = {} & 16\pi^2\delta^4K_1^4(\omega_{\mathrm{AR}(2)}T)^4 - 64\pi^2\delta^4K_1^4(\omega_{\mathrm{AR}(2)}T)^2 + 64\pi^2\delta^4K_1^4 - 64\pi^2\delta^4K_1^3(\omega_{\mathrm{AR}(2)}T)^4 + 256\pi^2\delta^4K_1^3(\omega_{\mathrm{AR}(2)}T)^2 - 256\pi^2\delta^4K_1^3 \\
& + 96\pi^2\delta^4K_1^2(\omega_{\mathrm{AR}(2)}T)^4 - 384\pi^2\delta^4K_1^2(\omega_{\mathrm{AR}(2)}T)^2 + 384\pi^2\delta^4K_1^2 - 64\pi^2\delta^4K_1(\omega_{\mathrm{AR}(2)}T)^4 + 256\pi^2\delta^4K_1(\omega_{\mathrm{AR}(2)}T)^2 - 256\pi^2\delta^4K_1 \\
& + 16\pi^2\delta^4(\omega_{\mathrm{AR}(2)}T)^4 - 64\pi^2\delta^4(\omega_{\mathrm{AR}(2)}T)^2 + 64\pi^2\delta^4 - 48\pi^2\delta^3K_1^4(\omega_{\mathrm{AR}(2)}T)^4 + 160\pi^2\delta^3K_1^4(\omega_{\mathrm{AR}(2)}T)^2 - 64\pi^2\delta^3K_1^4 + 208\pi^2\delta^3K_1^3(\omega_{\mathrm{AR}(2)}T)^4 \\
& - 704\pi^2\delta^3K_1^3(\omega_{\mathrm{AR}(2)}T)^2 + 320\pi^2\delta^3K_1^3 - 336\pi^2\delta^3K_1^2(\omega_{\mathrm{AR}(2)}T)^4 + 1152\pi^2\delta^3K_1^2(\omega_{\mathrm{AR}(2)}T)^2 - 576\pi^2\delta^3K_1^2 + 240\pi^2\delta^3K_1(\omega_{\mathrm{AR}(2)}T)^4 \\
& - 832\pi^2\delta^3K_1(\omega_{\mathrm{AR}(2)}T)^2 + 448\pi^2\delta^3K_1 - 64\pi^2\delta^3(\omega_{\mathrm{AR}(2)}T)^4 + 224\pi^2\delta^3(\omega_{\mathrm{AR}(2)}T)^2 - 128\pi^2\delta^3 + 52\pi^2\delta^2K_1^4(\omega_{\mathrm{AR}(2)}T)^4 - 144\pi^2\delta^2K_1^4(\omega_{\mathrm{AR}(2)}T)^2 \\
& + 16\pi^2\delta^2K_1^4 - 248\pi^2\delta^2K_1^3(\omega_{\mathrm{AR}(2)}T)^4 + 704\pi^2\delta^2K_1^3(\omega_{\mathrm{AR}(2)}T)^2 - 96\pi^2\delta^2K_1^3 + 436\pi^2\delta^2K_1^2(\omega_{\mathrm{AR}(2)}T)^4 - 1264\pi^2\delta^2K_1^2(\omega_{\mathrm{AR}(2)}T)^2 + 208\pi^2\delta^2K_1^2 \\
& - 336\pi^2\delta^2K_1(\omega_{\mathrm{AR}(2)}T)^4 + 992\pi^2\delta^2K_1(\omega_{\mathrm{AR}(2)}T)^2 - 192\pi^2\delta^2K_1 + 96\pi^2\delta^2(\omega_{\mathrm{AR}(2)}T)^4 - 288\pi^2\delta^2(\omega_{\mathrm{AR}(2)}T)^2 + 64\pi^2\delta^2 - 24\pi^2\delta K_1^4(\omega_{\mathrm{AR}(2)}T)^4 \\
& + 56\pi^2\delta K_1^4(\omega_{\mathrm{AR}(2)}T)^2 + 128\pi^2\delta K_1^3(\omega_{\mathrm{AR}(2)}T)^4 - 304\pi^2\delta K_1^3(\omega_{\mathrm{AR}(2)}T)^2 - 248\pi^2\delta K_1^2(\omega_{\mathrm{AR}(2)}T)^4 + 600\pi^2\delta K_1^2(\omega_{\mathrm{AR}(2)}T)^2 \\
& + 208\pi^2\delta K_1(\omega_{\mathrm{AR}(2)}T)^4 - 512\pi^2\delta K_1(\omega_{\mathrm{AR}(2)}T)^2 - 64\pi^2\delta(\omega_{\mathrm{AR}(2)}T)^4 + 160\pi^2\delta(\omega_{\mathrm{AR}(2)}T)^2 + 4\pi^2K_1^4(\omega_{\mathrm{AR}(2)}T)^4 - 8\pi^2K_1^4(\omega_{\mathrm{AR}(2)}T)^2 \\
& - 24\pi^2K_1^3(\omega_{\mathrm{AR}(2)}T)^4 + 48\pi^2K_1^3(\omega_{\mathrm{AR}(2)}T)^2 + 52\pi^2K_1^2(\omega_{\mathrm{AR}(2)}T)^4 - 104\pi^2K_1^2(\omega_{\mathrm{AR}(2)}T)^2 - 48\pi^2K_1(\omega_{\mathrm{AR}(2)}T)^4 + 96\pi^2K_1(\omega_{\mathrm{AR}(2)}T)^2 \\
& + 16\pi^2(\omega_{\mathrm{AR}(2)}T)^4
\end{aligned} \tag{4}$$

$$\begin{aligned}
C = {} & +4\delta^4K_1^4(\omega_{\mathrm{AR}(2)}T)^4 - 16\delta^4K_1^3(\omega_{\mathrm{AR}(2)}T)^4 + 24\delta^4K_1^2(\omega_{\mathrm{AR}(2)}T)^4 - 16\delta^4K_1(\omega_{\mathrm{AR}(2)}T)^4 + 4\delta^4(\omega_{\mathrm{AR}(2)}T)^4 - 12\delta^3K_1^4(\omega_{\mathrm{AR}(2)}T)^4 \\
& + 52\delta^3K_1^3(\omega_{\mathrm{AR}(2)}T)^4 - 84\delta^3K_1^2(\omega_{\mathrm{AR}(2)}T)^4 + 60\delta^3K_1(\omega_{\mathrm{AR}(2)}T)^4 - 16\delta^3(\omega_{\mathrm{AR}(2)}T)^4 + 13\delta^2K_1^4(\omega_{\mathrm{AR}(2)}T)^4 - 62\delta^2K_1^3(\omega_{\mathrm{AR}(2)}T)^4 \\
& + 109\delta^2K_1^2(\omega_{\mathrm{AR}(2)}T)^4 - 84\delta^2K_1(\omega_{\mathrm{AR}(2)}T)^4 + 24\delta^2(\omega_{\mathrm{AR}(2)}T)^4 - 6\delta K_1^4(\omega_{\mathrm{AR}(2)}T)^4 + 32\delta K_1^3(\omega_{\mathrm{AR}(2)}T)^4 - 62\delta K_1^2(\omega_{\mathrm{AR}(2)}T)^4 \\
& + 52\delta K_1(\omega_{\mathrm{AR}(2)}T)^4 - 16\delta(\omega_{\mathrm{AR}(2)}T)^4 + K_1^4(\omega_{\mathrm{AR}(2)}T)^4 - 6K_1^3(\omega_{\mathrm{AR}(2)}T)^4 + 13K_1^2(\omega_{\mathrm{AR}(2)}T)^4 - 12K_1(\omega_{\mathrm{AR}(2)}T)^4
\end{aligned} \tag{5}$$

$$D = 64\pi^4\delta^2K_1^2 - 128\pi^4\delta^2K_1 + 64\pi^4\delta^2 - 64\pi^4\delta K_1^2 + 192\pi^4\delta K_1 - 128\pi^4\delta + 16\pi^4K_1^2 - 64\pi^4K_1 + 32\pi^4 \tag{6}$$

$$\begin{aligned}
E = {} & 16\pi^2\delta^4K_1^2(\omega_{\mathrm{AR}(2)}T)^4 - 64\pi^2\delta^4K_1^2(\omega_{\mathrm{AR}(2)}T)^2 + 64\pi^2\delta^4K_1^2 - 32\pi^2\delta^4K_1(\omega_{\mathrm{AR}(2)}T)^4 + 128\pi^2\delta^4K_1(\omega_{\mathrm{AR}(2)}T)^2 - 128\pi^2\delta^4K_1 \\
& + 16\pi^2\delta^4(\omega_{\mathrm{AR}(2)}T)^4 - 64\pi^2\delta^4(\omega_{\mathrm{AR}(2)}T)^2 + 64\pi^2\delta^4 - 64\pi^2\delta^3K_1^3 - 64\pi^2\delta^3K_1^2(\omega_{\mathrm{AR}(2)}T)^4 + 224\pi^2\delta^3K_1^2(\omega_{\mathrm{AR}(2)}T)^2 + 128\pi^2\delta^3K_1(\omega_{\mathrm{AR}(2)}T)^4 \\
& - 448\pi^2\delta^3K_1(\omega_{\mathrm{AR}(2)}T)^2 + 192\pi^2\delta^3K_1 - 64\pi^2\delta^3(\omega_{\mathrm{AR}(2)}T)^4 + 224\pi^2\delta^3(\omega_{\mathrm{AR}(2)}T)^2 - 128\pi^2\delta^3 + 96\pi^2\delta^2K_1^3 + 96\pi^2\delta^2K_1^2(\omega_{\mathrm{AR}(2)}T)^4 \\
& - 304\pi^2\delta^2K_1^2(\omega_{\mathrm{AR}(2)}T)^2 - 96\pi^2\delta^2K_1^2 - 192\pi^2\delta^2K_1(\omega_{\mathrm{AR}(2)}T)^4 + 592\pi^2\delta^2K_1(\omega_{\mathrm{AR}(2)}T)^2 - 64\pi^2\delta^2K_1 + 96\pi^2\delta^2(\omega_{\mathrm{AR}(2)}T)^4 \\
& - 288\pi^2\delta^2(\omega_{\mathrm{AR}(2)}T)^2 + 64\pi^2\delta^2 - 48\pi^2\delta K_1^3 - 64\pi^2\delta K_1^2(\omega_{\mathrm{AR}(2)}T)^4 + 192\pi^2\delta K_1^2(\omega_{\mathrm{AR}(2)}T)^2 + 48\pi^2\delta K_1^2 + 128\pi^2\delta K_1(\omega_{\mathrm{AR}(2)}T)^4 \\
& - 352\pi^2\delta K_1(\omega_{\mathrm{AR}(2)}T)^2 - 64\pi^2\delta(\omega_{\mathrm{AR}(2)}T)^4 + 160\pi^2\delta(\omega_{\mathrm{AR}(2)}T)^2 + 8\pi^2K_1^3 + 16\pi^2K_1^2(\omega_{\mathrm{AR}(2)}T)^4 - 48\pi^2K_1^2(\omega_{\mathrm{AR}(2)}T)^2 - 32\pi^2K_1(\omega_{\mathrm{AR}(2)}T)^4 \\
& + 80\pi^2K_1(\omega_{\mathrm{AR}(2)}T)^2 + 16\pi^2(\omega_{\mathrm{AR}(2)}T)^4 - 32\pi^2(\omega_{\mathrm{AR}(2)}T)^2
\end{aligned} \tag{7}$$

$$\begin{aligned}
F = {} & 16\delta^4K_1^4 - 16\delta^4K_1^3(\omega_{\mathrm{AR}(2)}T)^2 - 32\delta^4K_1^3 + 4\delta^4K_1^2(\omega_{\mathrm{AR}(2)}T)^4 + 32\delta^4K_1^2(\omega_{\mathrm{AR}(2)}T)^2 + 16\delta^4K_1^2 - 8\delta^4K_1(\omega_{\mathrm{AR}(2)}T)^4 \\
& - 16\delta^4K_1(\omega_{\mathrm{AR}(2)}T)^2 + 4\delta^4(\omega_{\mathrm{AR}(2)}T)^4 - 32\delta^3K_1^4 + 48\delta^3K_1^3(\omega_{\mathrm{AR}(2)}T)^2 + 64\delta^3K_1^3 - 16\delta^3K_1^2(\omega_{\mathrm{AR}(2)}T)^4 - 96\delta^3K_1^2(\omega_{\mathrm{AR}(2)}T)^2 \\
& - 32\delta^3K_1^2 + 32\delta^3K_1(\omega_{\mathrm{AR}(2)}T)^4 + 48\delta^3K_1(\omega_{\mathrm{AR}(2)}T)^2 - 16\delta^3(\omega_{\mathrm{AR}(2)}T)^4 + 24\delta^2K_1^4 - 52\delta^2K_1^3(\omega_{\mathrm{AR}(2)}T)^2 - 40\delta^2K_1^3 + 24\delta^2K_1^2(\omega_{\mathrm{AR}(2)}T)^4 \\
& + 100\delta^2K_1^2(\omega_{\mathrm{AR}(2)}T)^2 + 16\delta^2K_1^2 - 48\delta^2K_1(\omega_{\mathrm{AR}(2)}T)^4 - 48\delta^2K_1(\omega_{\mathrm{AR}(2)}T)^2 + 24\delta^2(\omega_{\mathrm{AR}(2)}T)^4 - 8\delta K_1^4 + 24\delta K_1^3(\omega_{\mathrm{AR}(2)}T)^2 + 8\delta K_1^3 \\
& - 16\delta K_1^2(\omega_{\mathrm{AR}(2)}T)^4 - 40\delta K_1^2(\omega_{\mathrm{AR}(2)}T)^2 + 32\delta K_1(\omega_{\mathrm{AR}(2)}T)^4 + 16\delta K_1(\omega_{\mathrm{AR}(2)}T)^2 - 16\delta(\omega_{\mathrm{AR}(2)}T)^4 + K_1^4 - 4K_1^3(\omega_{\mathrm{AR}(2)}T)^2 \\
& + 4K_1^2(\omega_{\mathrm{AR}(2)}T)^4 + 4K_1^2(\omega_{\mathrm{AR}(2)}T)^2 - 8K_1(\omega_{\mathrm{AR}(2)}T)^4 + 4(\omega_{\mathrm{AR}(2)}T)^4
\end{aligned} \tag{8}$$

III. Approximations of A, B, C, D, E, F

The expression for $|1-L(z)|^2$ given in the previous section is valid for any scalar complex process $\alpha_{(k)}$ to be tracked as long as low normalized frequencies are considered. Now, in order to get a closed form expression, we restrict the application field by considering the additional assumptions made in [1], i.e., the Assumptions (i), (ii), (iii), (v), (vi), (vii), (viii), (ix) and (x) in [1, Section III.D]. In particular, we use the fact that (i) leads to $(\omega_{\mathrm{AR}(2)}T)^4 \ll (\omega_{\mathrm{AR}(2)}T)^3 \ll (\omega_{\mathrm{AR}(2)}T)^2 \ll (\omega_{\mathrm{AR}(2)}T) \ll 1$, (iii) leads to $\delta^4 \ll \delta^3 \ll \delta^2 \ll \delta \ll 1$, (viii) leads to $K_1^4 \ll K_1^3 \ll K_1^2 \ll K_1 \ll 1$, and (x) leads to $f_{\mathrm{AR}(2)}T \ll K_1$. This yields:

$$A \simeq -192\pi^4K_1 + 16\pi^4 \tag{9}$$

$$B \simeq 16\pi^2(\omega_{\mathrm{AR}(2)}T)^4 \tag{10}$$

$$C \simeq -12K_1(\omega_{\mathrm{AR}(2)}T)^4 \tag{11}$$

$$D \simeq -64\pi^4K_1 + 32\pi^4 \tag{12}$$

$$E \simeq -32\pi^2(\omega_{\mathrm{AR}(2)}T)^2 \tag{13}$$

$$F \simeq K_1^4 \tag{14}$$

Then, inserting Eqs (9)-(14) in (2), the following expression is obtained:

$$|1-L(e^{2i\pi fT})|^2 \simeq \frac{(-192\pi^4 K_1 + 16\pi^4)(fT)^4 + (16\pi^2(\omega_{\text{AR(2)}}T)^4)(fT)^2 - 12K_1(\omega_{\text{AR(2)}}T)^4}{(-64\pi^4 K_1 + 32\pi^4)(fT)^4 + (-32\pi^2(\omega_{\text{AR(2)}}T)^2)(fT)^2 + K_1^4}. \tag{15}$$

Now we provide a closed form approximation of the previous expression for fT close to $f_{\text{AR(2)}}T \ll 1$, which is in many cases the frequency range for which the values of the power spectral density of $\alpha_{(k)}$ dominate. This is typically the case for the Jakes' Doppler spectrum for instance. Using Assumptions (i), (viii) and (x) in [1, Section III.D] and assuming that $(fT)^4 \ll (fT)^2 \ll fT \leq f_{\text{AR(2)}}T \ll 1$, the following approximation of $|1-L(z)|^2$ is finally obtained in closed form:

$$|1-L(e^{2i\pi fT})|^2 \simeq \frac{16\pi^4(fT)^4}{K_1^4}. \tag{16}$$

References

[1] A. H. El Husseini, E. P. Simon, and L. Ros, "On the tuning of the second-order autoregressive model-based kalman filter for the estimation of slow fading channel described by the clarke model," *submitted to IEEE Transactions on Signal Processing*, 2018.