



HAL
open science

Approximation of $|1 - L(z)|^2$

Ali Houssam El Hussein, Eric Pierre Simon, Laurent Ros

► **To cite this version:**

Ali Houssam El Hussein, Eric Pierre Simon, Laurent Ros. Approximation of $|1 - L(z)|^2$. 2018.
hal-01700897v2

HAL Id: hal-01700897

<https://hal.science/hal-01700897v2>

Preprint submitted on 15 Oct 2018

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Approximation of $|1 - L(z)|^2$

Ali Houssam EL HUSSEINI*, Eric Pierre SIMON*, Laurent ROS†,

*University of Lille, UMR 8520 - IEMN, F-59655 Villeneuve d'Ascq, France

†Univ. Grenoble Alpes, CNRS Grenoble INP‡, GIPSA-lab 38000, Grenoble, France

‡ Institute of Engineering Univ. Grenoble Alpes

Email: *ali.elhousseini@ed.univ-lille1.fr, eric.simon@univ-lille1.fr, laurent.ros@gipsa-lab.grenoble-inp.fr*

I. INTRODUCTION

Let us consider a Kalman filter (KF) based on an autoregressive model of order 2 (AR(2)) for tracking a complex scalar process denoted by $\alpha(k)$. The equations of the AR(2) model are given in [1, Eqs (4), (9)-(11)] and those of the corresponding KF together with its steady state version in [1, Section 3.1, 3.2, 3.3]. In the following, the notations introduced in these equations will be used. Let $\alpha(z)$ be the z -transform of $\alpha(k)$. Let $L(z)$ be the z -transform of the impulse response of the steady state AR(2)-KF that gives the estimate of $\alpha(z)$, denoted by $\hat{\alpha}(z)$, with the observation as input (see Fig. 1).

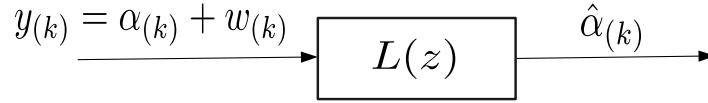


Fig. 1: Scheme of the steady state KF

This report is a self content report that gives the expression for $|1 - L(z)|^2$, which is useful when calculating the mean square error of the estimate.

$L(z)$ is given by [1, Eq. (31)]:

$$L(z) = \frac{K_1 + a_2 K_2 z^{-1}}{1 + z^{-1}(a_2 K_2 - a_1(1 - K_1)) - a_2(1 - K_1)z^{-2}} \quad (1)$$

II. EXPRESSION FOR $|1 - L(z)|^2$

In this section, we calculate the expression for $|1 - L(z)|^2$ as a function of the KF parameters $\delta = 1 - r$, $\omega_{\text{AR}(2)}T = 2\pi f_{\text{AR}(2)}T$, fT and K_1 . The only assumption that is used here is related to the fact that low normalized frequencies $fT \ll 1$ are considered, which leads to $z = e^{2i\pi fT} \simeq 1 + i2\pi fT$. In order to obtain the squared modulus of $(1 - L(z))$, we multiply $1 - L(z)$ by its conjugate $(1 - L(z))^*$, which yields:

$$|1 - L(e^{2i\pi fT})|^2 \simeq \frac{A(fT)^4 + B(fT)^2 + C}{D(fT)^4 + E(fT)^2 + F} \quad (2)$$

where A, B, C, D, E, F are defined as functions of $\delta, \omega_{\text{AR}(2)}T$ and K_1 as follows:

$$A = 64\pi^4 \delta^2 K_1^4 - 256\pi^4 \delta^2 K_1^3 + 384\pi^4 \delta^2 K_1^2 - 256\pi^4 \delta^2 K_1 + 64\pi^4 \delta^2 - 64\pi^4 \delta K_1^4 + 320\pi^4 \delta K_1^3 - 576\pi^4 \delta K_1^2 + 448\pi^4 \delta K_1 - 128\pi^4 \delta + 16\pi^4 K_1^4 - 96\pi^4 K_1^3 + 208\pi^4 K_1^2 - 192\pi^4 K_1 + 16\pi^4 \quad (3)$$

$$\begin{aligned}
B = & 16\pi^2\delta^4 K_1^4(\omega_{\text{AR}(2)}T)^4 - 64\pi^2\delta^4 K_1^4(\omega_{\text{AR}(2)}T)^2 + 64\pi^2\delta^4 K_1^4 - 64\pi^2\delta^4 K_1^3(\omega_{\text{AR}(2)}T)^4 + 256\pi^2\delta^4 K_1^3(\omega_{\text{AR}(2)}T)^2 - 256\pi^2\delta^4 K_1^3 \\
& + 96\pi^2\delta^4 K_1^2(\omega_{\text{AR}(2)}T)^4 - 384\pi^2\delta^4 K_1^2(\omega_{\text{AR}(2)}T)^2 + 384\pi^2\delta^4 K_1^2 - 64\pi^2\delta^4 K_1(\omega_{\text{AR}(2)}T)^4 + 256\pi^2\delta^4 K_1(\omega_{\text{AR}(2)}T)^2 - 256\pi^2\delta^4 K_1 \\
& + 16\pi^2\delta^4(\omega_{\text{AR}(2)}T)^4 - 64\pi^2\delta^4(\omega_{\text{AR}(2)}T)^2 + 64\pi^2\delta^4 - 48\pi^2\delta^3 K_1^4(\omega_{\text{AR}(2)}T)^4 + 160\pi^2\delta^3 K_1^4(\omega_{\text{AR}(2)}T)^2 - 64\pi^2\delta^3 K_1^4 + 208\pi^2\delta^3 K_1^3(\omega_{\text{AR}(2)}T)^4 \\
& - 704\pi^2\delta^3 K_1^3(\omega_{\text{AR}(2)}T)^2 + 320\pi^2\delta^3 K_1^3 - 336\pi^2\delta^3 K_1^2(\omega_{\text{AR}(2)}T)^4 + 1152\pi^2\delta^3 K_1^2(\omega_{\text{AR}(2)}T)^2 - 576\pi^2\delta^3 K_1^2 + 240\pi^2\delta^3 K_1(\omega_{\text{AR}(2)}T)^4 \\
& - 832\pi^2\delta^3 K_1(\omega_{\text{AR}(2)}T)^2 + 448\pi^2\delta^3 K_1 - 64\pi^2\delta^3(\omega_{\text{AR}(2)}T)^4 + 224\pi^2\delta^3(\omega_{\text{AR}(2)}T)^2 - 128\pi^2\delta^3 + 52\pi^2\delta^2 K_1^4(\omega_{\text{AR}(2)}T)^4 - 144\pi^2\delta^2 K_1^4(\omega_{\text{AR}(2)}T)^2 \\
& + 16\pi^2\delta^2 K_1^4 - 248\pi^2\delta^2 K_1^3(\omega_{\text{AR}(2)}T)^4 + 704\pi^2\delta^2 K_1^3(\omega_{\text{AR}(2)}T)^2 - 96\pi^2\delta^2 K_1^3 + 436\pi^2\delta^2 K_1^2(\omega_{\text{AR}(2)}T)^4 - 1264\pi^2\delta^2 K_1^2(\omega_{\text{AR}(2)}T)^2 + 208\pi^2\delta^2 K_1^2 \\
& - 336\pi^2\delta^2 K_1(\omega_{\text{AR}(2)}T)^4 + 992\pi^2\delta^2 K_1(\omega_{\text{AR}(2)}T)^2 - 192\pi^2\delta^2 K_1 + 96\pi^2\delta^2(\omega_{\text{AR}(2)}T)^4 - 288\pi^2\delta^2(\omega_{\text{AR}(2)}T)^2 + 64\pi^2\delta^2 - 24\pi^2\delta K_1^4(\omega_{\text{AR}(2)}T)^4 \\
& + 56\pi^2\delta K_1^4(\omega_{\text{AR}(2)}T)^2 + 128\pi^2\delta K_1^3(\omega_{\text{AR}(2)}T)^4 - 304\pi^2\delta K_1^3(\omega_{\text{AR}(2)}T)^2 - 248\pi^2\delta K_1^2(\omega_{\text{AR}(2)}T)^4 + 600\pi^2\delta K_1^2(\omega_{\text{AR}(2)}T)^2 \\
& + 208\pi^2\delta K_1(\omega_{\text{AR}(2)}T)^4 - 512\pi^2\delta K_1(\omega_{\text{AR}(2)}T)^2 - 64\pi^2\delta(\omega_{\text{AR}(2)}T)^4 + 160\pi^2\delta(\omega_{\text{AR}(2)}T)^2 + 4\pi^2 K_1^4(\omega_{\text{AR}(2)}T)^4 - 8\pi^2 K_1^4(\omega_{\text{AR}(2)}T)^2 \\
& - 24\pi^2 K_1^3(\omega_{\text{AR}(2)}T)^4 + 48\pi^2 K_1^3(\omega_{\text{AR}(2)}T)^2 + 52\pi^2 K_1^2(\omega_{\text{AR}(2)}T)^4 - 104\pi^2 K_1^2(\omega_{\text{AR}(2)}T)^2 - 48\pi^2 K_1(\omega_{\text{AR}(2)}T)^4 + 96\pi^2 K_1(\omega_{\text{AR}(2)}T)^2 \\
& + 16\pi^2(\omega_{\text{AR}(2)}T)^4 \quad (4)
\end{aligned}$$

$$\begin{aligned}
C = & +4\delta^4 K_1^4(\omega_{\text{AR}(2)}T)^4 - 16\delta^4 K_1^3(\omega_{\text{AR}(2)}T)^4 + 24\delta^4 K_1^2(\omega_{\text{AR}(2)}T)^4 - 16\delta^4 K_1(\omega_{\text{AR}(2)}T)^4 + 4\delta^4(\omega_{\text{AR}(2)}T)^4 - 12\delta^3 K_1^4(\omega_{\text{AR}(2)}T)^4 \\
& + 52\delta^3 K_1^3(\omega_{\text{AR}(2)}T)^4 - 84\delta^3 K_1^2(\omega_{\text{AR}(2)}T)^4 + 60\delta^3 K_1(\omega_{\text{AR}(2)}T)^4 - 16\delta^3(\omega_{\text{AR}(2)}T)^4 + 13\delta^2 K_1^4(\omega_{\text{AR}(2)}T)^4 - 62\delta^2 K_1^3(\omega_{\text{AR}(2)}T)^4 \\
& + 109\delta^2 K_1^2(\omega_{\text{AR}(2)}T)^4 - 84\delta^2 K_1(\omega_{\text{AR}(2)}T)^4 + 24\delta^2(\omega_{\text{AR}(2)}T)^4 - 6\delta K_1^4(\omega_{\text{AR}(2)}T)^4 + 32\delta K_1^3(\omega_{\text{AR}(2)}T)^4 - 62\delta K_1^2(\omega_{\text{AR}(2)}T)^4 \\
& + 52\delta K_1(\omega_{\text{AR}(2)}T)^4 - 16\delta(\omega_{\text{AR}(2)}T)^4 + K_1^4(\omega_{\text{AR}(2)}T)^4 - 6K_1^3(\omega_{\text{AR}(2)}T)^4 + 13K_1^2(\omega_{\text{AR}(2)}T)^4 - 12K_1(\omega_{\text{AR}(2)}T)^4 \quad (5)
\end{aligned}$$

$$D = 64\pi^4\delta^2 K_1^2 - 128\pi^4\delta^2 K_1 + 64\pi^4\delta^2 - 64\pi^4\delta K_1^2 + 192\pi^4\delta K_1 - 128\pi^4\delta + 16\pi^4 K_1^2 - 64\pi^4 K_1 + 32\pi^4 \quad (6)$$

$$\begin{aligned}
E = & 16\pi^2\delta^4 K_1^2(\omega_{\text{AR}(2)}T)^4 - 64\pi^2\delta^4 K_1^2(\omega_{\text{AR}(2)}T)^2 + 64\pi^2\delta^4 K_1^2 - 32\pi^2\delta^4 K_1(\omega_{\text{AR}(2)}T)^4 + 128\pi^2\delta^4 K_1(\omega_{\text{AR}(2)}T)^2 - 128\pi^2\delta^4 K_1 \\
& + 16\pi^2\delta^4(\omega_{\text{AR}(2)}T)^4 - 64\pi^2\delta^4(\omega_{\text{AR}(2)}T)^2 + 64\pi^2\delta^4 - 64\pi^2\delta^3 K_1^3 - 64\pi^2\delta^3 K_1^2(\omega_{\text{AR}(2)}T)^4 + 224\pi^2\delta^3 K_1^2(\omega_{\text{AR}(2)}T)^2 + 128\pi^2\delta^3 K_1(\omega_{\text{AR}(2)}T)^4 \\
& - 448\pi^2\delta^3 K_1(\omega_{\text{AR}(2)}T)^2 + 192\pi^2\delta^3 K_1 - 64\pi^2\delta^3(\omega_{\text{AR}(2)}T)^4 + 224\pi^2\delta^3(\omega_{\text{AR}(2)}T)^2 - 128\pi^2\delta^3 + 96\pi^2\delta^2 K_1^3 + 96\pi^2\delta^2 K_1^2(\omega_{\text{AR}(2)}T)^4 \\
& - 304\pi^2\delta^2 K_1^2(\omega_{\text{AR}(2)}T)^2 - 96\pi^2\delta^2 K_1^2 - 192\pi^2\delta^2 K_1(\omega_{\text{AR}(2)}T)^4 + 592\pi^2\delta^2 K_1(\omega_{\text{AR}(2)}T)^2 - 64\pi^2\delta^2 K_1 + 96\pi^2\delta^2(\omega_{\text{AR}(2)}T)^4 \\
& - 288\pi^2\delta^2(\omega_{\text{AR}(2)}T)^2 + 64\pi^2\delta^2 - 48\pi^2\delta K_1^3 - 64\pi^2\delta K_1^2(\omega_{\text{AR}(2)}T)^4 + 192\pi^2\delta K_1^2(\omega_{\text{AR}(2)}T)^2 + 48\pi^2\delta K_1^2 + 128\pi^2\delta K_1(\omega_{\text{AR}(2)}T)^4 \\
& - 352\pi^2\delta K_1(\omega_{\text{AR}(2)}T)^2 - 64\pi^2\delta(\omega_{\text{AR}(2)}T)^4 + 160\pi^2\delta(\omega_{\text{AR}(2)}T)^2 + 8\pi^2 K_1^3 + 16\pi^2 K_1^2(\omega_{\text{AR}(2)}T)^4 - 48\pi^2 K_1^2(\omega_{\text{AR}(2)}T)^2 - 32\pi^2 K_1(\omega_{\text{AR}(2)}T)^4 \\
& + 80\pi^2 K_1(\omega_{\text{AR}(2)}T)^2 + 16\pi^2(\omega_{\text{AR}(2)}T)^4 - 32\pi^2(\omega_{\text{AR}(2)}T)^2 \quad (7)
\end{aligned}$$

$$\begin{aligned}
F = & 16\delta^4 K_1^4 - 16\delta^4 K_1^3(\omega_{\text{AR}(2)}T)^2 - 32\delta^4 K_1^3 + 4\delta^4 K_1^2(\omega_{\text{AR}(2)}T)^4 + 32\delta^4 K_1^2(\omega_{\text{AR}(2)}T)^2 + 16\delta^4 K_1^2 - 8\delta^4 K_1(\omega_{\text{AR}(2)}T)^4 \\
& - 16\delta^4 K_1(\omega_{\text{AR}(2)}T)^2 + 4\delta^4(\omega_{\text{AR}(2)}T)^4 - 32\delta^3 K_1^4 + 48\delta^3 K_1^3(\omega_{\text{AR}(2)}T)^2 + 64\delta^3 K_1^3 - 16\delta^3 K_1^2(\omega_{\text{AR}(2)}T)^4 - 96\delta^3 K_1^2(\omega_{\text{AR}(2)}T)^2 \\
& - 32\delta^3 K_1^2 + 32\delta^3 K_1(\omega_{\text{AR}(2)}T)^4 + 48\delta^3 K_1(\omega_{\text{AR}(2)}T)^2 - 16\delta^3(\omega_{\text{AR}(2)}T)^4 + 24\delta^2 K_1^4 - 52\delta^2 K_1^3(\omega_{\text{AR}(2)}T)^2 - 40\delta^2 K_1^3 + 24\delta^2 K_1^2(\omega_{\text{AR}(2)}T)^4 \\
& + 100\delta^2 K_1^2(\omega_{\text{AR}(2)}T)^2 + 16\delta^2 K_1^2 - 48\delta^2 K_1(\omega_{\text{AR}(2)}T)^4 - 48\delta^2 K_1(\omega_{\text{AR}(2)}T)^2 + 24\delta^2(\omega_{\text{AR}(2)}T)^4 - 8\delta K_1^4 + 24\delta K_1^3(\omega_{\text{AR}(2)}T)^2 + 8\delta K_1^3 \\
& - 16\delta K_1^2(\omega_{\text{AR}(2)}T)^4 - 40\delta K_1^2(\omega_{\text{AR}(2)}T)^2 + 32\delta K_1(\omega_{\text{AR}(2)}T)^4 + 16\delta K_1(\omega_{\text{AR}(2)}T)^2 - 16\delta(\omega_{\text{AR}(2)}T)^4 + K_1^4 - 4K_1^3(\omega_{\text{AR}(2)}T)^2 \\
& + 4K_1^2(\omega_{\text{AR}(2)}T)^4 + 4K_1^2(\omega_{\text{AR}(2)}T)^2 - 8K_1(\omega_{\text{AR}(2)}T)^4 + 4(\omega_{\text{AR}(2)}T)^4 \quad (8)
\end{aligned}$$

III. APPROXIMATIONS OF A, B, C, D, E, F

The expression for $|1 - L(z)|^2$ given in the previous section is valid for any scalar complex process $\alpha_{(k)}$ to be tracked as long as low normalized frequencies are considered. Now, in order to get a closed form expression, we restrict the application field by considering the additional assumptions made in [1], i.e., the Assumptions (i), (ii), (iii), (v), (vi), (vii), (viii), (ix) and (x) in [1, Section III.D]. In particular, we use the fact that (i) leads to $(\omega_{\text{AR}(2)}T)^4 \ll (\omega_{\text{AR}(2)}T)^3 \ll (\omega_{\text{AR}(2)}T)^2 \ll (\omega_{\text{AR}(2)}T) \ll 1$, (iii) leads to $\delta^4 \ll \delta^3 \ll \delta^2 \ll \delta \ll 1$, (viii) leads to $K_1^4 \ll K_1^3 \ll K_1^2 \ll K_1 \ll 1$, and (x) leads to $f_{\text{AR}(2)}T \ll K_1$. This yields:

$$A \simeq -192\pi^4 K_1 + 16\pi^4 \quad (9)$$

$$B \simeq 16\pi^2(\omega_{\text{AR}(2)}T)^4 \quad (10)$$

$$C \simeq -12K_1(\omega_{\text{AR}(2)}T)^4 \quad (11)$$

$$D \simeq -64\pi^4 K_1 + 32\pi^4 \quad (12)$$

$$E \simeq -32\pi^2(\omega_{\text{AR}(2)}T)^2 \quad (13)$$

$$F \simeq K_1^4 \quad (14)$$

Then, inserting Eqs (9)-(14) in (2), the following expression is obtained:

$$|1 - L(e^{2i\pi fT})|^2 \simeq \frac{(-192\pi^4 K_1 + 16\pi^4)(fT)^4 + (16\pi^2(\omega_{\text{AR}(2)}T)^4)(fT)^2 - 12K_1(\omega_{\text{AR}(2)}T)^4}{(-64\pi^4 K_1 + 32\pi^4)(fT)^4 + (-32\pi^2(\omega_{\text{AR}(2)}T)^2)(fT)^2 + K_1^4}. \quad (15)$$

Now we provide a closed form approximation of the previous expression for fT close to $f_{\text{AR}(2)}T \ll 1$, which is in many cases the frequency range for which the values of the power spectral density of $\alpha_{(k)}$ dominate. This is typically the case for the Jakes' Doppler spectrum for instance. Using Assumptions (i), (viii) and (x) in [1, Section III.D] and assuming that $(fT)^4 \ll (fT)^2 \ll fT \leq f_{\text{AR}(2)}T \ll 1$, the following approximation of $|1 - L(z)|^2$ is finally obtained in closed form:

$$|1 - L(e^{2i\pi fT})|^2 \simeq \frac{16\pi^4(fT)^4}{K_1^4}. \quad (16)$$

REFERENCES

- [1] A. H. El Husseini, E. P. Simon, and L. Ros, "Second-order autoregressive model-based kalman filter for the estimation of slow fading channel described by the clarke model: optimal tuning and interpretations," *submitted to Elsevier Digital Signal Processing*, 2018.