GENERATING SINGULAR SIGNALS BY FILTERING BERNOULLI CORRELATED INPUTS

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Abstract

Singular random signals can be obtained at the output of some linear filters when the input is a Bernoulli white noise. It is shown that this whiteness assumption can be relaxed and some examples of colored Bernoulli signals generating singularity by filtering are presented. Computer experiments are realized in order to verify these results.

Statistical signal analysis, Signal and noise modeling, Non-Gaussian signals and noise, Markov processes., Prediction.

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I. Introduction

Singular random signals have been recently presented and analyzed in [1]. A signal x_k is said to be *singular* if the random variables (RV) x_k are singular for any k. This means that the marginal distribution function (DF) $F(x)$ of x_k is continuous but has a derivative equal to zero almost everywhere. Usually these RVs are considered as mathematical curiosities and one of the main purposes of [1] was to show that, on the contrary, they are very common. In particular the output x_k of a linear filter generated by the input signal w_k is singular if the filter satisfies some simple conditions concerning its poles and if w_k is discrete-valued and strictly white, or is a sequence of independent and identically distributed (IID) random variables (see p. 254 of [2]). This whiteness assumption is introduced in almost all the studies on this question [1], [3], [4], [5], [6]. Our purpose is to relax this assumption and to show that some correlated inputs can also generate singular signals.

II. Theoretical Analysis

The starting point of the analysis is the Lebesgue decomposition theorem. It says that the DF $F(x)$ of a random variable (RV) X can be expressed in an unique way as

$$F(x) = a_1 F_c(x) + a_2 F_d(x) + a_3 F_s(x), \quad a_i \geq 0, \quad a_1 + a_2 + a_3 = 1. \tag{1}$$

In this equation the three functions $F_i(x)$ are DFs and $F_c(x)$, $F_d(x)$, and $F_s(x)$ are the continuous, discrete and singular components of $F(x)$ respectively. If $a_1 = 1$, the RV is continuous and its PDF is the derivative of $F_c(x)$. If $a_2 = 1$, X is a discrete RV. If, finally, $a_3 = 1$, the RV X is singular.

The spectrum S_F of a RV X is the set of the points of increase of its DF $F(x)$. Its Lebesgue measure $L(S_F)$ is called the spectral measure (SM) of X. If the SM is zero, the continuous part in (1) is zero, or $a_1 = 0$. Then the RV is either discrete ($a_3 = 0$), or singular ($a_2 = 0$), or a mixture of a discrete and a singular parts. As a consequence in order to show that a RV is singular it suffices to show that its SM is zero and that there is no discrete part in the Lebesgue decomposition of its DF.

Let now $\mathcal{F}$ be a causal filter and h_k its impulse response. The output x_k generated by the input w_k is $x_k = \sum_{i=0}^{\infty} h_i w_{k-i}$. In order to simplify the presentation we shall study only symmetric Bernoulli input signals w_k. This means that w_k takes only the values ± 1 with equal probabilities. The extension to signals with more than two discrete values introduces only complexities of notations. The symmetry assumption of w_k implies that x_k is also symmetric, which means that x_k and $-x_k$ have the same DF which is the DF of the RV

$$X = \sum_{k=0}^{\infty} h_k w_k. \tag{2}$$

Let F_S be the set of filters satisfying the following conditions:

(a) They are dynamical, which means that they are causal, stable, and that their transfer function $H(z)$ is a rational function.

(b) They have an infinite impulse response (IIR), which means that there is at least one nonzero pole.

(c) Their poles are inside the circle of singularity introduced in [1]. It has the same center as the unit circle and its radius is 1/2.

The exponential filter $k_k = a^k$ belongs obviously to this set if $0 < |a| < 1/2$. The main result used below is contained in the following theorem partially indicated in [1].

Theorem 1. Let w_k be the input of a filter $\mathcal{F}$ belonging to F_S and x_k the corresponding output. If w_k is a Bernoulli symmetric signal then the SM of x_k is zero.

Proof. The filter $\mathcal{F}$ is defined by its transfer function $H(z)$ or its impulse response h_k. Let $\mathcal{F}'$ be the filter with the impulse response $g_k = 2^k h_k$. It is obvious that its transfer function is $G(z) = H(z/2)$. Then if the poles of $\mathcal{F}$ are z_i, those of $\mathcal{F}'$ are $2z_i$. The assumption that the poles of $\mathcal{F}$ are inside the circle of singularity implies therefore that the poles of $\mathcal{F}'$ are inside the unit circle, or that $\mathcal{F}'$ is stable and then $\sum_{k=0}^{\infty} |g_k| < +\infty$. Let X_N and R_N be the finite sum and the rest defined by $X_N = \sum_{k=0}^{N} h_k w_k$ and $R_N = \sum_{k=N+1}^{\infty} h_k w_k$. Since $w_k = \pm 1$ we have $|R_N| \leq \rho_N$ with $\rho_N = \sum_{k=N+1}^{\infty} |h_k|$. As X_N can take at the maximum 2^{N+1} values, the SM $\mathcal{S}$ of X satisfies $\mathcal{S} \leq 2^{N+1}\rho_N$. This is valid for all N. Then $\mathcal{S} \leq 2 \lim_{N\to\infty}(2^N \rho_N)$. But we have

$$2^N \rho_N = 2^N \sum_{k=N+1}^{\infty} |h_k| < \sum_{k=N+1}^{\infty} 2^k |h_k| = \sum_{k=N+1}^{\infty} |g_k|, \tag{3}$$

and the limit for $N \to \infty$ is 0 because $\sum_{k=0}^{\infty} |g_k| < +\infty$. This implies that the SM of X is 0, or $a_1 = 0$.

It is shown in [1] that if the input signal is white, which means that the w_k are IID symmetric Bernoulli RVs, then there is no discrete component in the DF of X, or $a_2 = 0$, and this implies that X is singular.

Our purpose is to investigate whether, (and if so, under what conditions), the whiteness assumption can be relaxed. We assume then that the w_ks are still symmetric Bernoulli, but not necessarily independent. Let X_N be the partial sum defined by

$$X_N = \sum_{k=0}^{N} h_k w_k. \tag{4}$$

It takes at the maximum 2^{N+1} distinct values v_i^N. From these values we can construct a tree of representation analyzed in [1]. To each value v_i^N at the step N we associate a node V_i^N of this tree and when passing from N to $N+1$ each node V_i^N generates two distinct nodes defined by . The nodes V_1^0 and V_2^0 defined by $-v_1^0 = v_2^0 = |h_0|$ are generated from an origin node V which does not correspond to a value of X_N. A node V_i^N is said to be *single* if it is generated by only one node $V_{j(i)}^{N-1}$. In the opposite case it is multiple. If all the nodes are single the number of distinct nodes V_i^N is 2^{N+1}, and we assume that this is satisfied in what follows. This assumption is verified for the exponential filter where $h_k = a^k$ when $0 < |a| < 1/2$ and also for all the filters such that there is no crossing between the branches of the tree, as analyzed in [1].

This assumption of single nodes depends uniquely on the filter. The set of filters $\mathcal{F}$ having this property is called F_D. Most of the following results are related to filters belonging to the set $F_{SD} = F_S \cap F_D$. The exponential filter $h_k = a^k$ belongs obviously to this set if $0 < |a| < 1/2$.

The assumption of single nodes means in particular that for any N and i there is a unique path going from V to V_i^N. Let $i_k^N(i)$, $0 \leq k \leq N$, be the indices j defining the nodes V_j^k of this path. These nodes can then be written $V_{i_k^N(i)}^k$ and the index $i_k^N(i)$ characterizes the unique node of the tree at the step k located on the path going from V to V_i^N.

The problem is to calculate the probabilities

$$p_N(i) \stackrel{\triangle}{=} P[x_N = v_i^N] \ , \ 1 \leq i \leq 2^{N+1}. \tag{5}$$

When the w_ks are IID this probability is $1/2^{N+1}$. When they are no longer independent, its calculation is much more complicated.

For this we introduce the conditional probability

$$p_N(i,j) \stackrel{\triangle}{=} p_N[x_N = v_i^N | x_{N-1} = v_j^{N-1}], \tag{6}$$

called also *transition probability* in the tree. It has two fundamental properties for the analysis that follows.

The first comes from its normalization specified for all j by $\sum_{i=1}^{2^{N+1}} p(i,j) = 1$. However a node V_j^{N-1} of the tree of construction generates only two nodes V_i^N characterized by the indices $i_+(j)$ and $i_-(j)$ and, according to (4), corresponding to the values $v_{i_\pm}^N(j) = v_j^{N-1} \pm h_N$. As a consequence for a given j there are only two terms in the previous sum and we have

$$p_N[i_+(j), j] + p_N[i_-(j), j)] = 1. \tag{7}$$

The second starts from the fact that for any node V_i^N there is only one node $V_{j(i)}^{N-1}$ at the step $N-1$ of the tree generating V_i^N at the step N. Thus $p_N(i,j)$ is zero except when $j = j(i)$, and the only nonzero values of $p_N(i,j)$ are

$$q_N(i) \stackrel{\triangle}{=} p_N[i, j(i)] \tag{8}$$

for $N > 0$ and $q_0(i) = 1/2$.

It results from (6) and from the unicity of the path between V and V_i^N that

$$p_N[(x_N = v_i^N) \cap (x_{N-1} = v_j^{N-1})] = p_{N-1}(j) \ . \ p_N(i,j) \ \delta[j - j(i)], \tag{9}$$

where $\delta[k]$ is the Kronecker delta symbol equal to 1 if $k = 0$ and 0 otherwise. By a summation on j, which contains only one term, we obtain

$$p_N(i) = p_{N-1}[j(i)] \ q_N(i). \tag{10}$$

By repeating this at all the nodes of the unique path between V and V_i^N characterized by the indices $i_k^N(i)$ we obtain

$$p_N(i) = \prod_{k=0}^{N} q_k[i_k^N(i)]. \tag{11}$$

When the RVs w_k are IID we have of course $q_N[i_k^N(i)] = 1/2$, and we find again that the values v_i^N have equal probabilities $1/2^{N+1}$.

The probabilities $p_N(i)$ of (11) are normalized, or $\sum_i p_N(i) = 1$, where the sum is extended to all the indices i from 1 to 2^{N+1}. This property is valid for $N = 0$ because $q_0(i) = 1/2$. Suppose that it is valid at the step $N-1$. Since each node V_j^{N-1} generates only two nodes $V_{i_-(j)}^N$ and $V_{i_+(j)}^N$ the result comes from (7).

The relation (11) is the basis of the following analysis of the singularity. Indeed if the SM of X is zero and if all the $p_N(i)$ tend to 0 when $N \to \infty$, there is no value v_i^∞ with a finite probability, and this means that the RV X cannot have a discrete component and then is singular. This can be specified by the following theorem.

Theorem 2. Let X be the RV defined by (2) where h_k is the impulse response of a filter belonging to F_{SD} and w_k a sequence of Bernoulli RVs. If the transition probabilities $q_N(i)$ defined by (8) satisfy

$$0 < q_N(i) < B < 1, \tag{12}$$

then the RV X is singular, or $a_1 = a_2 = 0$.

Proof. As $\mathcal{F}$ belongs to F_{SD} the SM of X is zero, according to Theorem 1. It remains to show that $a_2 = 0$. This is a direct consequence of (11) and (12) because $p_N(i) < (1/2)B^N$, which tends to zero when $N \to \infty$.

Comments. Note that this theorem introduces only a sufficient condition of singularity. Note also that it can be applied for white input because in this case $q_N(i) = 1/2$. The condition $\mathcal{F} \in F_{SD}$ is satisfied by a large class of filters and some are presented in [1]. However the question of characterizing *all* the dynamical filters belonging to F_{SD} remains open. As a matter of fact it is possible to extend this theorem to the case where some nodes are not single, but this introduces other conditions that cannot be presented in this paper.

III. Examples

Application of condition (12) requires the calculation of the probabilities $q_N(i)$ defined by (8) and related to the transitions probabilities $p_N(i,j)$ defined by (6). For this it is interesting to express these probabilities in terms of those of the RVs w_k. For this note first that for a given v_j^{N-1}, or a given node V_j^{N-1}, there is only one path coming from V to V_j^{N-1}, or one sequence of values of w_k, $0 < k \leq N-1$. Let us denote this sequence as $S_{N-1}(j)$. The transition probability $p_N(i,j)$ is simply related to the conditional probabilities of the w_k by

$$p_N[i_{\pm 1}(j), j] = P[w_N = \pm 1 | S_{N-1}(j)], \tag{13}$$

where $i_+(j)$ and $i_-(j)$ are the two indices i defining the two nodes V_i^N generated by the node V_j^{N-1}. Note that the probabilities (13) are now transition probabilities in time.

These expressions are especially interesting in the case where the input w_k is a Bernoulli symmetric Markovian signal. Let us remind that a symmetric Bernoulli Markovian signal of order P is a signal taking the values ± 1 with equal probabilities and which can be expressed as

$$w_k = f(w_{k-1}, w_{k-2}, ..., w_{k-P}, \mathbf{b}_k), \tag{14}$$

where $\mathbf{b}_k$ is a vectorial white noise. It is clear that the conditional DF of w_k at time k, conditionally to the whole past, depends only on $w_{k-1}, w_{k-2}, ..., w_{k-P}$. The function $f(.)$ and the noise $\mathbf{b}_k$ must satisfy conditions ensuring that if the w_ls, $k-P \leq l \leq k-1$ are symmetric Bernoulli signals, w_k has the same property. We shall present in the next Section examples of such signals that can be realized in computer experiments.

The Markov assumption means that the sequence $S_{N-1}(j)$ in (13) can be replaced by a sequence using only the past of order P. This can be written

$$p_N[i_{\pm}(j), j] = P[w_N = \pm 1 | w_{N-1}, ..., w_{N-P}]. \tag{15}$$

Since the RVs w_k take only the values ± 1, there are only at most 2^P distinct values of $p_N[i_{\pm}(j), j]$. If we assume that the signal is not predictable there is no value of p_N equal to 1. Then the bound of (12) is simply the maximum value of the finite number of transition probabilities $p_N(i,j)$. Then Theorem 2 can be applied and the RV X is singular.

It is interesting to present examples of signals for which Theorem 1 can be applied but not Theorem 2. In this case the spectral measure is zero, or $a_1 = 0$, but the RV X is not purely singular, or $a_3 \neq 1$. We shall first present a case where it is purely discrete, or $a_2 = 1$.

This especially appears when the symmetric Bernoulli signal w_k is predictable. A predictable signal of order P is defined by (14) where $\mathbf{b}_k = \mathbf{0}$. This means that at each k, w_k can be estimated without error in terms of its past P values. As a consequence w_k is also completely defined from its P first values, or

$$w_k = h(w_0, w_1, ..., w_{P-1}). \tag{16}$$

The simplest example of such a Bernoulli signal is $w_k = w_0(-1)^k$, where w_0 is a Bernoulli symmetric RV. Here $P = 1$.

It is clear that if w_k is predictable, then the RV X of (2) is discrete, or $a_2 = 1$. Indeed since the RVs w_i take only 2 values, X takes at the maximum 2^P distinct positive values which means that it is a discrete RV.

In the case of the signal $w_k = w_0(-1)^k$ used with the exponential filter defined by $h_k = a^k$, we have $X = w_0/(1+a)$ and it takes only the two values $\pm 1/(1+a)$ with equal probabilities.

It is clear that conditions of Theorem 2 are not satisfied for predictable signals because most of the transitions probabilities $p_N(i,j)$ are equal to 1 or 0 in such a way that there is only a finite number of paths going from V to infinity and giving values of X with nonzero probabilities.

Finally it is of interest to note that conditions of Theorem 1 can lead to a RV X with a DF mixture of discrete and singular part, or $a_1 = 0$ and $0 < a_2 < 1$. We shall present in the next section an example of such a situation.

IV. Experiments

Our purpose is now to verify the previous theoretical results by some computer experiments. The principle of these experiments is the same as the one used in [1]. In order to analyze the singularity of a signal x_k we realize histograms at different scales of a very large number of its values. The singularity is illustrated by the fact that the histograms introduce a fractal structure. This means that they have the same form whatever the scale of analysis. This shows, with the limited precision of any experiment, the lack of derivative.

In order to verify results of the previous section we shall introduce a procedure to generate by computer Bernoulli Markovian signals. They are defined as follows. Let u_k be a strictly white noise taking only the values 0 or 1 with probabilities $1-p$ and p respectively. Similarly let v_k be a Bernoulli strictly white noise taking the values ± 1 with the same probabilities 1/2 and independent of u_k. Consider now the signal

$$w_k = u_k w_{k-1} + \bar{u}_k v_k, \ , \ k \geq 1, \tag{17}$$

where $\bar{u}_k = 1 - u_k$. It has the form (14) where $P = 1$ and $\mathbf{b}_k = [u_k, v_k]$.

It is obvious that if w_{k-1} takes only the values ± 1 with probabilities 1/2, w_k has the same property. It can be shown that, whatever the values of w_1, w_k tends to have this property and that the covariance function γ_k of w_k is $p^{|k|}$, so it is an exponential covariance function. For $p = 0$, $u_k = 0$, and w_k is the white noise v_k with a zero covariance function. On the other hand if $p = 1$, $u_k = 1$ and $w_k = w_0$, which introduces constant covariance function or long range memory.

In the experiments presented below we use an exponential filter often mentioned in the previous analysis and also used in [1]. Its input-output relationship can be written in a recursive form with the recursion $x_k = ax_{k-1} + w_k$, where w_k is given by (17). In Fig. 1 we present histograms obtained when the input w_k is white, or when $p = 0$. The experiment is realized with 10^7 samples and $a = 1/3$. The fractal structure appears very clearly. In Fig. 2 we present the same histograms realized with correlated inputs defined by the linear Markovian model (17) with $p = 0.5$. The fractal structure remains valid but the main difference is the fact that the number of samples recorded in each cell of the histograms have much stronger variations than in the case of white inputs. This can be explained theoretically but the analysis cannot be presented here.

Note that when the input is white the histograms are symmetric with respect to their centers. This is no longer true when Markovian inputs are used, except for the histogram centered at 0 because X is symmetric and its histograms are symmetric with respect to 0. This is why only histograms of positive values of x_k are represented.

We shall now consider experiments in which the DF of the output x_k is a mixture of discrete and singular parts. For this consider again the signal $u_k = u_0(-1)^k$ and a white signal v_k taking the values ± 1 with equal probabilities and independent of u_0. Suppose now that w_k is equal either to u_k or to v_k for all k with the probabilities α and $1-\alpha$ respectively. This implies that its covariance function is $\gamma_k = \alpha(-1)^k + (1-\alpha)\delta[k]$. The DF of the RVs X is $F(x) = \alpha F_d(x) + (1-\alpha)F_s(x)$. Applying this signal again at the input to the exponential filter $h_k = a^k$ with $a < 1/2$ yields an output signal taking the values $\pm 1/(1+a)$ with probability α and being singular with probability $1-\alpha$.

Results of computer experiments on this signal are presented in Figures 3 and 4. In these experiments

$a = 1/3$, as in Figs. 1 and 2, and $\alpha = 0.5$. This implies that x_k takes the values ± 0.75 with probability 0.25, which means that its DF is discontinuous for $x = \pm 0.75$ and the amplitude of the discontinuity is 0.25. The first histogram of Fig. 3 exhibits these two discontinuities characterized by two lines at the points ± 0.75. One of these lines disappears in the second histogram of this figure and there is no longer a line in the last two histograms. The fractal structure introducing a singular component appears clearly in these histograms. In order to verify whether or not the discontinuity of the DF is an experimental artefact, we present in Fig. 4 four histograms isolating the point 0.75 with cells of decreasing widths. In the first histogram we observe still a residual contribution of the singular part of the DF. There is however a line at 0.75 and the fact that its amplitude is constant indicates clearly that there is effectively a discrete component. As the experiment uses 10^6 samples, the discontinuity corresponding to the probability 0.25 must be of the order of $2.5.10^5$, which clearly appears in the four histograms. This shows that the signal x_k is effectively a mixture of a discrete-valued signal and a singular signal.

V. CONCLUSION

When the input of a linear filter satisfying some conditions concerning essentially the location of its poles with respect to the circle of singularity is a Bernoulli white noise, the output can be singular. This assumption of whiteness can however be partially relaxed while maintaining the singularity of the output. Some sufficient conditions ensuring the singularity with colored inputs have been established. These conditions are obviously satisfied not only by white noise but also by a large class of correlated signals. It is especially the case of Markovian signals of finite order. The theoretical analysis also shows that the output generated by colored inputs can be a mixture of a discrete and a singular distribution. In particular if the input signal is predictable it was shown that the output of the filter is simply a discrete random variable. Computer experiments in order to verify the theoretical results have been realized and discussed. For this purpose a specific model of linear Markovian signal of order one was introduced and the experimental results are in complete agreement with the theory. Finally a model of Bernoulli input ensuring that the output contains a discrete and a singular part was introduced and here also the experimental results are in perfect agreement with the theory.

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Figures captions

Fig. 1. Histograms of x_k at different scales. $a = 1/3$; $p = 0$.

Fig. 2. Histograms of x at different scales. $a = 1/3$; $p = 0.5$.

Fig. 3. Mixture discrete-singular, $\alpha = 0.5$. Fractal part of the histogram.

Fig. 4. Mixture discrete-singular, $\alpha = 0.5$. Analysis of the neighborhood of 0.75.

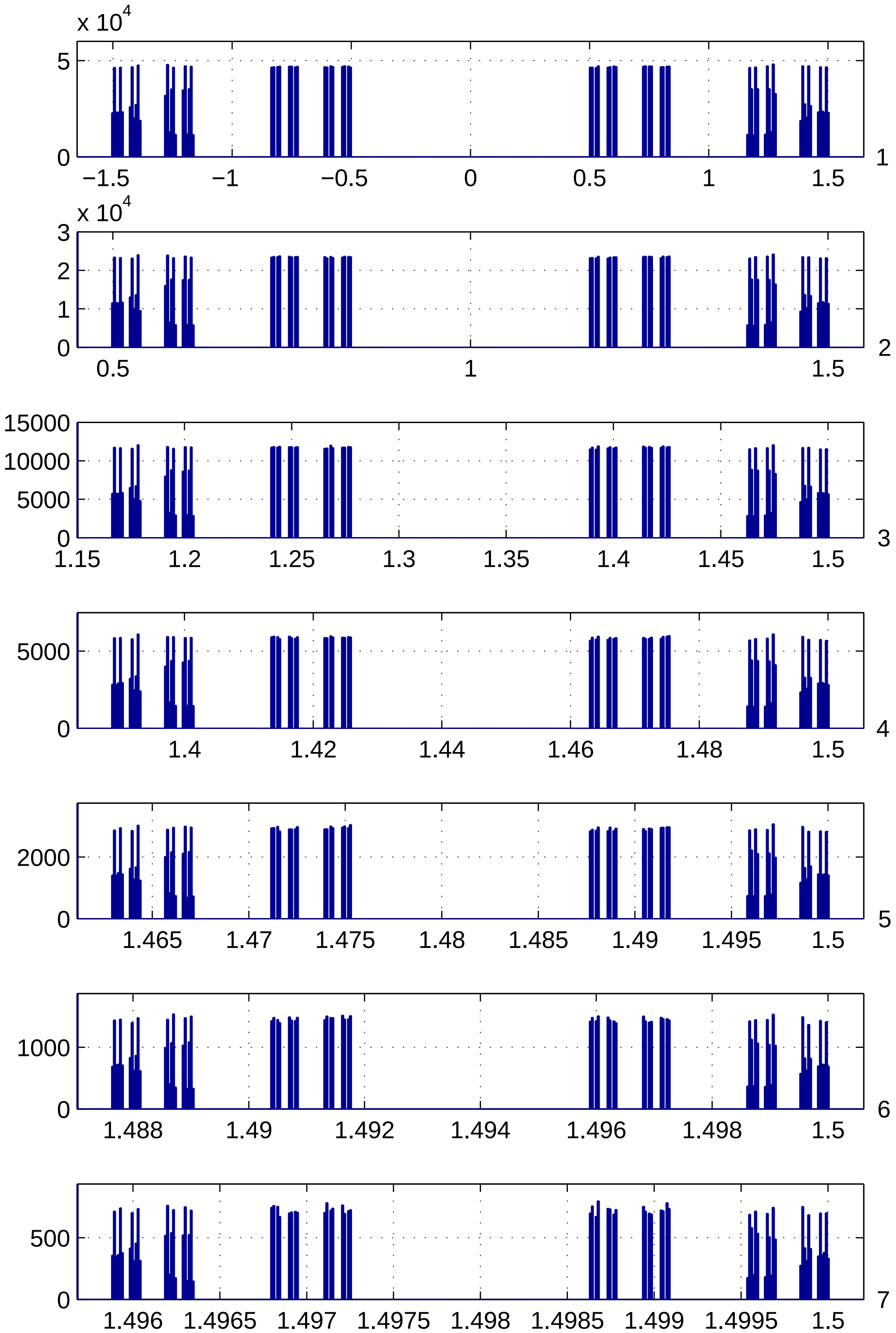
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Fig. 1. .

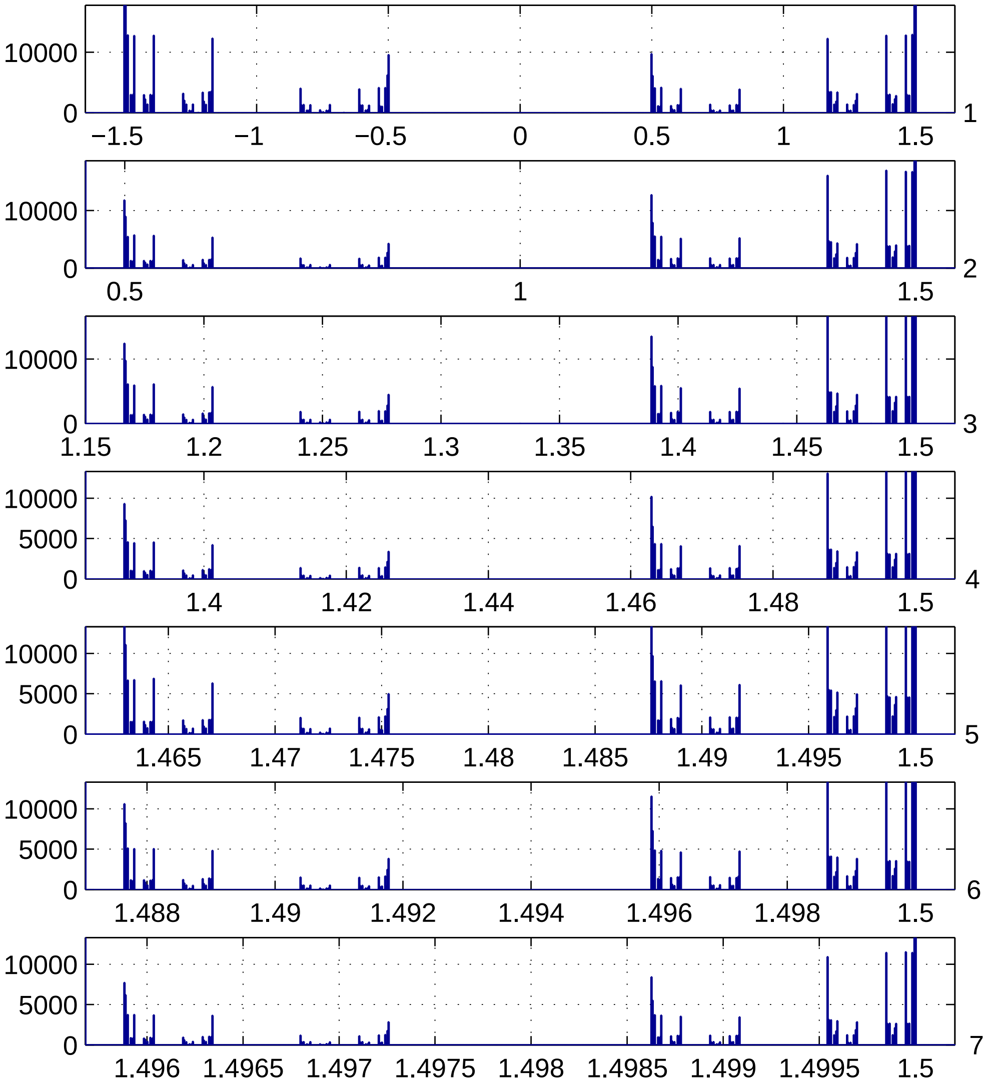
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Fig. 2. .

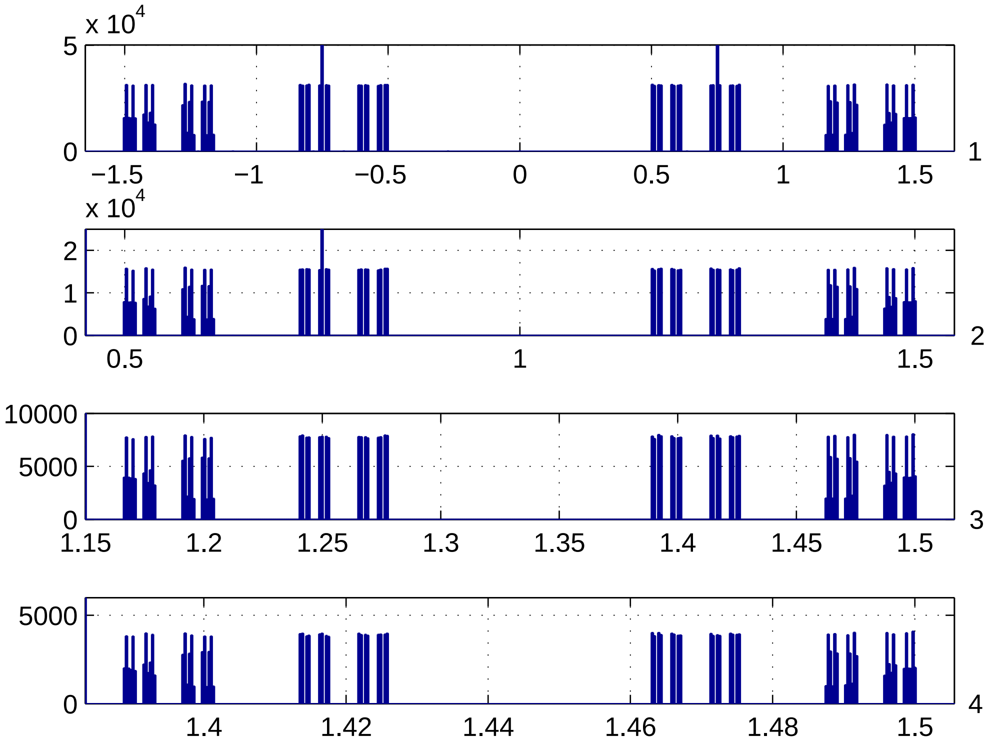
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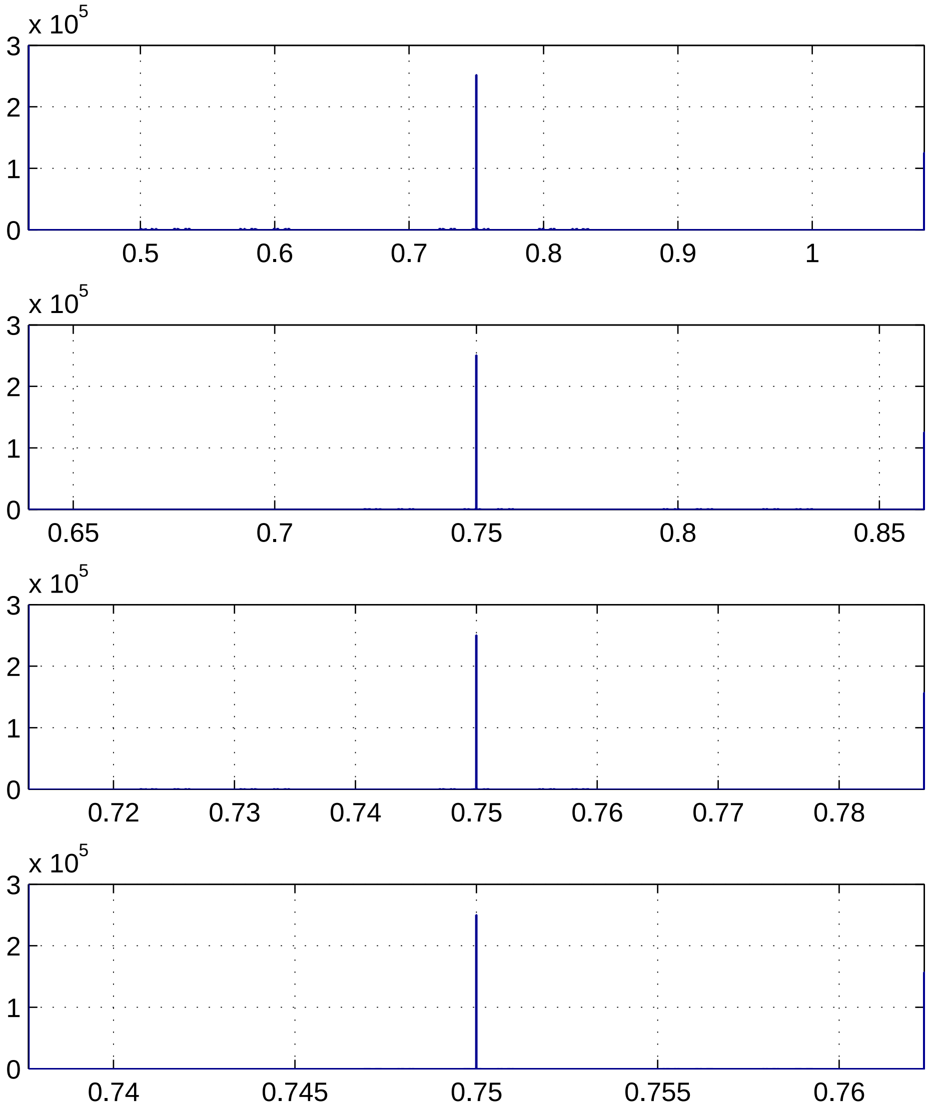
Fig. 3. .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0.75
0.76
0.77
0.78
x 10^5
0.74
0.745
0.75
0.755
0.76

Fig. 4. .