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# OPTIMAL PERIODIC CONTROL FOR SCALAR DYNAMICS UNDER INTEGRAL CONSTRAINT ON THE INPUT – APPLICATION TO THE CHEMOSTAT MODEL

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**ABSTRACT.** This paper studies a periodic optimal control problem governed by a one-dimensional system, linear with respect to the control  $u(\cdot)$ , under an integral constraint on  $u(\cdot)$ . We give conditions for which the value of the cost function at steady state with a constant control  $\bar{u}$  can be improved by considering periodic control  $u(\cdot)$  with average value equal to  $\bar{u}$ . This leads to the so-called “over-yielding” met in several applications. With the use of the Pontryagin Maximum Principle, we provide the optimal synthesis of periodic strategies under the integral constraint. The results are illustrated on a problem of water quality in the chemostat model with periodic dilution rate, for various growth functions.

**KEY-WORDS.** Optimal Control, Pontryagin Maximum Principle, Periodic solutions, Chemostat Model, Over-yielding.

## 1. INTRODUCTION

In many applications, the control of dynamical models allows to drive the state of a system to a relevant operation point, typically a steady state optimizing the performances of the system. However, it may happen that a periodic operation of the system gives a better performance averaged on the period than a steady state (with a periodic control instead of a constant one). This question has already been investigated in the literature. In particular, the so-called  $\pi$ -criterion characterizes the existence of “best” periods. It consists first in determining an optimal steady state among constant controls, and then in checking on a linear-quadratic approximation if there exists a frequency of a periodic signal locally about the nominal constant one that could improve the cost (see [7, 6]). For instance, in [2, 3, 19], this method has been applied on the chemostat model, and it has been shown that its productivity can be improved with a periodic control when there is a delay in the dynamics. However, that there are relatively few theoretical works about global optimality of periodic control (apart [21] for the characterization of the value function under quite strong assumptions). Most of the existing works deal with local necessary conditions ([9, 15]), second order conditions ([8, 30, 16]) or approximations techniques ([13, 1]).

In the present work, we consider the problem of determining optimal periodic trajectories under an integral constraint on the control. Our objective is somewhat different than what has been described above. First of all, we do not necessarily consider nominal steady states optimizing the criterion. Secondly, we consider admissible trajectories under constraint, as following. For a given steady state  $\bar{x}$  and

its associated constant control  $\bar{u}$ , we consider the set of periodic trajectories with periodic controls having  $\bar{u}$  as average (which represents the integral constraint). This question has been motivated by applications for continuous transformation processes for which during each period of time of length  $T$ , an amount of input matter has to be transformed (which is represented by the integral constraint on the control). The question is to study if the quality of the product, in terms of the averaged concentration  $x(\cdot)$  during each time period, could be improved by the way the input matter is delivered during each time period (that is the choice of the control  $u(\cdot)$  satisfying the integral constraint). In this context, we say that a *over-yielding* occurs when the average of a  $T$ -periodic solution  $x(\cdot)$  is above  $\bar{x}$ . To our knowledge, this problem has not been yet addressed theoretically in the literature. From a mathematical view point, the integral constraint on the input brings two main difficulties

- (1) the existence of non-constant periodic trajectories with a control satisfying the integral constraint,
- (2) the characterization of a (globally) optimal control under both constraints of periodicity of the trajectory and the integral constraint on the input,

that we propose to tackle here for scalar dynamics in general framework. We show that convexity assumptions are playing an important role.

The paper is organized as follows. In Section 2, we formulate the problem and give a precise definition of over-yielding. We then provide assumptions on the dynamics and the cost function that guarantee or prevent over-yielding. In particular, we show that convexity is playing an important role. In section 3, we synthesize optimal periodic controls that improves the cost function compared to steady-state (see Theorem 3.6). In Section 4, we show how to relax the assumptions of Section 2 that are required on an invariant domain  $I$  of the dynamics, when these ones are fulfilled only locally about  $\bar{x}$ . This leads us to give a result similar to the one of Section 3 but for restrictive values of the period  $T$ . Finally, we illustrate on results in Section 5 in the context of the chemostat model (see for instance [29, 18]), considering different kind of growth functions available in the literature. The objective to study the impact of non-constant periodic inputs on the average water quality and, where appropriate, determine the best input profiles.

## 2. EXISTENCE OF OVER-YIELDING

Given two functions  $f, g : \mathbb{R} \rightarrow \mathbb{R}$  of class  $C^1$ , we consider the control system

$$(1) \quad \dot{x} = f(x) + ug(x),$$

where  $u$  is a control variable taking values in  $U := [-1, 1]$ . We suppose that the system satisfies the following hypotheses:

- (H1) There exists numbers  $a, b$  with  $a < b$  such that  $g$  is positive on the interval  $I := (a, b)$  with

$$f(a) - g(a) = 0 \quad \text{and} \quad f(b) + g(b) = 0.$$

- (H2) One has  $f - g < 0$  and  $f + g > 0$  on  $I$ .

**Remark 1.** Hypothesis (H1) implies that the interval  $I$  is invariant by (1) whereas Hypothesis (H2) is related with controllability properties of (1) (that be used in the next section for the synthesis of non-constant periodic trajectories). In the rest of the paper, we shall consider initial conditions in  $I$  only.

We define for  $x \in I$  the function

$$\psi(x) := -\frac{f(x)}{g(x)}$$

Notice that Hypotheses (H1)-(H2) imply that one has  $\psi(I) \subset U$ . Therefore, for any  $\bar{x} \in I$ , we can consider the control value  $\bar{u} := \psi(\bar{x})$  in  $U$ . Note that  $\bar{x}$  is an equilibrium of (1) for the constant control  $u = \bar{u}$ . In the sequel, we shall consider  $T$ -periodic solutions of (1), where  $T$  is a positive number, with a  $T$ -periodic control  $u(\cdot)$  that satisfies the integral constraint

$$(2) \quad \frac{1}{T} \int_0^T u(t) dt = \bar{u}$$

We define then the set  $\mathcal{U}_T$  of admissible controls as

$$(3) \quad \mathcal{U}_T := \{u \in L_1([0, T], U) \text{ s.t. (2) is fulfilled}\}$$

One has the following property.

**Lemma 2.1.** *Under Hypothesis (H1), any  $T$ -periodic solution  $x(\cdot)$  of (1) in  $I$  with  $u(\cdot) \in \mathcal{U}_T$  fulfills the property*

$$(4) \quad \int_0^T (\psi(x(t)) - \psi(\bar{x})) dt = 0$$

*Proof.* On the interval  $I$ , the function  $g$  is positive and one can write from equation (1)

$$\int_0^T \frac{\dot{x}(t)}{g(x(t))} dt = - \int_0^T \psi(x(t)) dt + \int_0^T u(t) dt$$

Define the function

$$h(x) := \int_{\bar{x}}^x \frac{d\xi}{g(\xi)}, \quad x \in I$$

and denote the function  $y(t) = h(x(t))$ . For any control function  $u(\cdot)$  that fulfills the constraint (2), one has then

$$y(T) - y(0) = - \int_0^T (\psi(x(t)) - \bar{u}) dt$$

where  $\bar{u} = \psi(\bar{x})$ . For any  $T$ -periodic solution  $x(\cdot)$  in  $I$ ,  $y(\cdot)$  is also  $T$ -periodic and one obtains the property (4).  $\square$

We assume now the equilibrium  $\bar{x}$  is asymptotically stable for the dynamics (1) in  $I$  with the constant control  $\bar{u}$ , requiring the following hypothesis.

( $\bar{H}$ ) The function  $\psi$  satisfies the property.

$$(\psi(x) - \psi(\bar{x}))(x - \bar{x}) > 0, \quad \forall x \in I \setminus \{\bar{x}\}$$

For convenience, we shall denote  $x(x_0, u(\cdot); t)$  the solution of (1) at time  $t$  for the initial condition  $x(0) = x_0 \in I$  and control  $u(\cdot) \in \mathcal{U}_T$ . In the following, we shall consider  $T$ -periodic solutions with the initial condition  $x(0) = \bar{x}$  (i.e. that are such that  $x(\bar{x}, u(\cdot); T) = \bar{x}$  for  $u(\cdot) \in \mathcal{U}_T$ ). We first show that Hypothesis ( $\bar{H}$ ) guarantees the existence of non-constant such solutions.

**Lemma 2.2.** *Under Hypotheses (H1)-( $\bar{H}$ ), there exists non-constant  $T$ -periodic solutions of (1) with  $x(0) = \bar{x}$  and  $u \in \mathcal{U}_T$ , for any  $T > 0$ .*

*Proof.* Consider the constant control  $u = \bar{u}$  and its associated dynamics in  $I$

$$\dot{x} = \bar{f}(x) := g(x)(\bar{u} - \psi(x)) = g(x)(\psi(\bar{x}) - \psi(x))$$

As the function  $g$  is positive on  $I$ , Hypothesis  $(\bar{H})$  implies that one has  $\bar{f} < 0$  on  $(\bar{x}, b)$ , and  $\bar{f} > 0$  on  $(a, \bar{x})$ . Therefore, one has the properties

$$(5) \quad \begin{aligned} x_0 \in (\bar{x}, \bar{b}) &\Rightarrow x(x_0, \bar{u}; T) < x_0 \\ x_0 \in (a, \bar{x}) &\Rightarrow x(x_0, \bar{u}; T) > x_0 \end{aligned}$$

Consider now any bounded  $T$ -periodic function  $v$  in  $L_1([0, +\infty), U)$  with

$$\int_0^T v(t) dt = 0,$$

and the control function

$$u_\epsilon(t) = \bar{u} + \epsilon v(t)$$

where  $\epsilon$  is a real number. Clearly,  $u_\epsilon(\cdot)$  satisfies the constraint (2) and for  $\epsilon$  small enough, one has  $u_\epsilon(t) \in U$  for any  $t \geq 0$ . Define then the function

$$\gamma(x_0, \epsilon) := x(x_0, u_\epsilon(\cdot); T) - x_0$$

for  $(x_0, \epsilon) \in I \times \mathbb{R}$ . By the Theorem of continuous dependency of the solutions of ordinary differential equations with respect to initial condition and parameter (see for instance [26],  $\gamma$  is a continuous function. From properties (5), one has

$$\begin{aligned} x_0 \in (\bar{x}, b) &\Rightarrow \gamma(x_0, 0) < 0 \\ x_0 \in (a, \bar{x}) &\Rightarrow \gamma(x_0, 0) > 0 \end{aligned}$$

and by continuity of  $\gamma$ , there exists three numbers  $\epsilon \neq 0$ ,  $x_0^+ \in (\bar{x}, b)$  and  $x_0^- \in (a, \bar{x})$  such that  $\gamma(x_0^+, \epsilon) < 0$  and  $\gamma(x_0^-, \epsilon) > 0$ . By the Mean Value Theorem, we deduce the existence of  $x_0 \in (x_0^-, x_0^+)$  such that  $\gamma(x_0, \epsilon) = 0$ , that is the existence of a  $T$ -periodic solution  $x(\cdot)$  of (1) with a non-constant control  $u(\cdot)$  that satisfies the constraint (2). From Lemma 2.1, such solution satisfies

$$\int_0^T (\psi(x(t)) - \psi(\bar{x})) dt = 0$$

which implies that the map  $t \mapsto \psi(x(t)) - \psi(\bar{x})$  cannot be of constant sign on  $[0, T]$ . Hypothesis  $(\bar{H})$  implies that  $x(t) - \bar{x}$  has to change its sign. Therefore there exists  $\bar{t} \in (0, T)$  with  $x(\bar{t}) = \bar{x}$  and then the control function  $\tilde{u}(\cdot) = u(\cdot + \bar{t})$  guarantees to have  $x(\bar{x}, \tilde{u}(\cdot); T) = \bar{x}$ .  $\square$

Now, let  $\ell : \mathbb{R} \rightarrow \mathbb{R}$  be a function of class  $C^1$  and consider the cost function

$$(6) \quad J_T(u) := \frac{1}{T} \int_0^T \ell(x_u(t)) dt,$$

where  $x_u$  is the solution of (1) with  $x_u(0) = \bar{x}$ , associated to a control  $u \in \mathcal{U}_T$ . Our aim in this work is to address the question of finding a periodic trajectory with  $x(0) = \bar{x}$  that has a better cost than the constant  $\bar{x}$ , with a ( $T$ -periodic) control of mean value  $\bar{u}$ . For this purpose, we introduce the following terminology.

**Definition 2.3.** Given  $T > 0$ , we say that (1) exhibits an *over-yielding* for the cost (6) if there exists a  $T$ -periodic solution  $x(\cdot)$  of (1) with  $x(0) = \bar{x}$  associated to a control  $u \in \mathcal{U}_T$  such that  $J_T(u) < \ell(\bar{x})$ .

Moreover, we aim at characterizing in the next Section the strategies realizing the minimum of the criterion (6) among such controls. The possibility of having an over-yielding relies on specific assumptions on the cost function and the dynamics, that we now introduce.

(H3)  $\ell : I \rightarrow \mathbb{R}$  is increasing and  $\gamma := \psi \circ \ell^{-1}$  is strictly convex increasing over  $\ell(I)$ .

**Remark 2.** Hypothesis (H3) implies that  $\psi$  is strictly increasing on  $I$  and therefore Hypothesis ( $\bar{H}$ ). Then, by Lemma 2.2, there exists  $T$ -periodic solutions  $x$  of (1) with  $x(0) = \bar{x}$  and  $u \in \mathcal{U}_T$ , different to  $\bar{x}$  when (H1)-(H2)-(H3) are fulfilled.

**Proposition 1.** *If (H1)-(H3) hold true, any non-constant  $T$ -periodic solution  $x(\cdot)$  of (1) with  $x(0) = \bar{x}$  and  $u \in \mathcal{U}_T$  satisfies  $J_T(u) < \ell(\bar{x})$ .*

*Proof.* Consider a  $T$ -periodic solution  $x(\cdot)$  with  $x(0) = \bar{x}$  associated to a control in  $\mathcal{U}_T$ . From Lemma 2.1, equality (4) is satisfied and we deduce

$$\int_0^T (\gamma(\ell(x(t))) - \gamma(\ell(\bar{x}))) dt = 0.$$

For a non-constant solution, we find by Jensen's inequality

$$\gamma\left(\frac{1}{T} \int_0^T \ell(x(t)) dt\right) < \frac{1}{T} \int_0^T \gamma(\ell(x(t))) dt = \gamma(\ell(\bar{x})).$$

Since  $\gamma$  is increasing over  $\ell(I)$ , we obtain

$$J_T(u) = \frac{1}{T} \int_0^T \ell(x(t)) dt < \ell(\bar{x}).$$

□

**Remark 3.** (i) The result of Proposition 1 applies in the simple case where  $\ell(x) = x$  and  $\psi$  is strictly convex and increasing over  $I$ .

(ii) If  $\psi$  is strictly convex and increasing over  $I$  and  $\ell$  is strictly concave increasing over  $I$ , the result of Proposition 1 also holds true (by a similar reasoning).

We now provide sufficient conditions for preventing any over-yielding.

(H4) There exists a  $C^1$  function  $\bar{\psi}$  such that

- (i)  $\bar{\psi} \geq \psi$  on  $I$  with  $\bar{\psi}(\bar{x}) = \psi(\bar{x})$
- (ii) the function  $\bar{\gamma} := \bar{\psi} \circ \ell^{-1}$  is concave increasing on  $\ell(I)$

**Proposition 2.** *If (H1)-(H4) hold true then no over-yielding is possible.*

*Proof.* We suppose by contradiction that there exists a periodic solution  $x(\cdot)$  associated to a control  $u \in \mathcal{U}_T$  such that

$$J_T(u) = \frac{1}{T} \int_0^T \ell(x(t)) dt < \ell(\bar{x}),$$

The function  $\bar{\gamma}$  being increasing on  $\ell(I)$ , we have

$$(7) \quad \bar{\gamma}\left(\frac{1}{T} \int_0^T \ell(x(t)) dt\right) < \bar{\gamma}(\ell(\bar{x})) = \bar{\psi}(\bar{x}) = \psi(\bar{x})$$

Using Jensen's inequality for  $\bar{\gamma}$ , we can write

$$(8) \quad \bar{\gamma}\left(\frac{1}{T} \int_0^T \ell(x(t)) dt\right) \geq \frac{1}{T} \int_0^T \bar{\gamma}(\ell(x(t))) dt$$

As one has  $\bar{\psi} = \bar{\gamma} \circ \ell \geq \psi$  over  $I$ , we get

$$(9) \quad \frac{1}{T} \int_0^T \bar{\gamma}(\ell(x(t))) dt \geq \frac{1}{T} \int_0^T \psi(x(t)) dt$$

Combining inequalities (7), (8), (9), we obtain

$$\psi(\bar{x}) > \frac{1}{T} \int_0^T \psi(x(t)) dt$$

which is a contradiction with the equality (4) given by Lemma 2.1.  $\square$

**Remark 4.** If  $\ell$  is increasing on  $I$  and  $\gamma$  is strictly concave increasing over  $\ell(I)$ , then one can show in the same way that no over-yielding is possible.

### 3. DETERMINATION OF OPTIMAL PERIODIC SOLUTIONS

In this Section, we assume that Hypotheses (H1)-(H2)-(H3) hold true, so that we know that over-yielding is possible, according to Proposition 1. For a given  $T > 0$ , we shall say that a solution  $x(\cdot)$  of (1) is  $T$ -admissible if it is  $T$ -periodic with  $x(0) = \bar{x}$  and  $u(\cdot) \in \mathcal{U}_T$ . We reformulate the control constraint (2) by considering the augmented dynamics

$$(10) \quad \begin{cases} \dot{x} &= f(x) + ug(x), \\ \dot{y} &= u, \end{cases}$$

together with the boundary conditions:

$$(11) \quad (x(0), y(0)) = (\bar{x}, 0) \quad \text{and} \quad (x(T), y(T)) = (\bar{x}, \bar{u}T)$$

The optimal control problem can be then stated as follows

$$(12) \quad \inf_{u \in \mathcal{U}} \int_0^T \ell(x(t)) dt \quad \text{s.t. } (x, y) \text{ satisfies (10) - (11),}$$

where  $\mathcal{U}$  is the set of measurable control functions  $u(\cdot)$  that take values in  $U$ .

**3.1. Application of the Pontryagin Maximum Principle.** We derive necessary optimality conditions using the Pontryagin Maximum Principle [27]. Let  $H : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  be the Hamiltonian associated to (12) :

$$H = H(x, y, \lambda_x, \lambda_y, \lambda_0, u) = \lambda_0 \ell(x) + \lambda_x f(x) + u(\lambda_x g(x) + \lambda_y),$$

where  $\lambda := (\lambda_x, \lambda_y)$  denotes the adjoint vector. Let  $u \in \mathcal{U}$  be an optimal control and  $(x, y)$  a solution of (10)-(11) associated to  $u$ . Then, there exists a scalar  $\lambda_0 \leq 0$  and an absolutely continuous map  $\lambda : [0, T] \rightarrow \mathbb{R}^2$  satisfying the adjoint equation

$$(13) \quad \begin{cases} \dot{\lambda}_x = -\lambda_0 \ell'(x(t)) - \lambda_x (f'(x(t)) + u(t)g'(x(t))) & \text{a.e. } t \in [0, T], \\ \dot{\lambda}_y = 0, \end{cases}$$

such that  $(\lambda_0, \lambda) \neq 0$ . The transversality condition implies  $\lambda_x(0) = \lambda_x(T)$ . Finally, the Hamiltonian condition writes

$$(14) \quad u(t) \in \arg \max_{\omega \in U} H(x(t), \lambda(t), \lambda_0, \omega) \quad \text{a.e. } t \in [0, T].$$

The switching function  $\phi(t) := \lambda_x(t)g(x(t)) + \lambda_y(t)$  provides the properties:

$$\begin{cases} \phi(t) > 0 & \Rightarrow u(t) = 1, \\ \phi(t) < 0 & \Rightarrow u(t) = -1, \\ \phi(t) = 0 & \Rightarrow u(t) \in [-1, 1]. \end{cases}$$

Since  $\lambda_x$  is  $T$ -periodic,  $\phi$  is also  $T$ -periodic ( $\lambda_y$  is constant). Moreover, if we differentiate  $\phi$  w.r.t  $t$ , we find that:

$$\dot{\phi}(t) = \lambda_x(t)(f(x(t))g'(x(t)) - f'(x(t))g(x(t))) - \lambda_0 \ell'(x(t))g(x(t)), \quad \text{a.e. } t \in [0, T].$$

An extremal trajectory is a quadruple  $(x, \lambda, \lambda_0, u)$  where  $(x, \lambda)$  satisfies the state-adjoint equations and  $u$  the Hamiltonian condition (14). We recall that a singular arc occurs if  $\phi$  vanishes on some time interval  $[t_1, t_2]$  with  $t_1 < t_2$ , and a switching time  $t_s \in (0, T)$  is such that  $x(\cdot)$  is not  $C^1$  at  $t_s$  (which implies  $\phi(t_s) = 0$ ).

**Lemma 3.1.** *Under Hypotheses (H1)-(H2)-(H3), there is no abnormal extremal i.e.  $\lambda_0 \neq 0$ .*

*Proof.* If  $\lambda_0 = 0$ , then  $\lambda_x$  cannot vanish from the adjoint equation. Otherwise  $\lambda_x$  would be zero over  $[0, T]$  and the switching function would be constant equal to  $\lambda_y$ . Since  $\lambda_y$  cannot be simultaneously equal to 0,  $\phi$  would be of constant sign over  $[0, T]$  implying that  $u = 1$  or  $u = 0$  over  $[0, T]$  and a contradiction with the periodicity of  $x$  (recall that  $f + g > 0$  and  $f - g < 0$  over  $I$ ). As a consequence,  $\lambda_x$  is of constant sign. Now, since  $\lambda_0 = 0$ , one has

$$\dot{\phi}(t) = \lambda_x(t)g(x(t))^2\psi'(x(t)), \quad t \in [0, T].$$

We deduce that  $\dot{\phi}$  is of constant sign (recall that  $\psi' > 0$ ), that is  $\phi$  monotone, and thus we have a contradiction with the periodicity of  $\phi$ . This ends the proof.  $\square$

Without any loss of generality, we may assume that  $\lambda_0 = -1$

**3.2. Properties of switching times.** Let us denote by  $x_m$  and  $x_M$  the minimum and maximum on  $[0, T]$  of a  $T$ -admissible solution  $x(\cdot)$ . Note that for any time  $t \in (0, T)$  such that  $x(t) \in \{x_m, x_M\}$ , then one has  $\phi(t) = 0$  (otherwise  $x(\cdot)$  would be monotone in a neighborhood of  $t$  implying a contradiction).

**Proposition 3.** *Under Hypotheses (H1)-(H2)-(H3), any extremal verifies the following properties.*

- (1) *At any switching time  $t_s \in (0, T)$ , one has  $x(t_s) \in \{x_m, x_M\}$ .*
- (2) *It has no singular arc.*

*Proof.* Let  $t_1, t_2$  in  $[0, T]$  be such that  $x(t_1) = x_m$  and  $x(t_2) = x_M$  with  $x_m, x_M$  in  $I$ . We deduce that  $\lambda_x(t_1)g(x_m) = \lambda_x(t_2)g(x_M) = -\lambda_y$ . Now, since  $H$  is conserved along any extremal trajectory (see for instance [11]), one has

$$H = -\ell(x_M) - \lambda_y \frac{f(x_M)}{g(x_M)} = -\ell(x_m) - \lambda_y \frac{f(x_m)}{g(x_m)},$$

implying that (recall that  $\gamma = \psi \circ \ell^{-1}$ )

$$(15) \quad \frac{1}{\lambda_y} = \frac{\psi(x_M) - \psi(x_m)}{\ell(x_M) - \ell(x_m)} = \frac{\gamma(\ell(x_M)) - \gamma(\ell(x_m))}{\ell(x_M) - \ell(x_m)}.$$

Suppose now that  $t_s$  is a switching time such that  $x(t_s) \in (x_m, x_M)$ . Using a similar computation as above, we find that

$$(16) \quad \frac{1}{\lambda_y} = \frac{\psi(x_M) - \psi(x(t_s))}{\ell(x_M) - \ell(x(t_s))} = \frac{\gamma(\ell(x_M)) - \gamma(\ell(x(t_s)))}{\ell(x_M) - \ell(x(t_s))}.$$

Since  $\gamma$  and  $\ell$  are respectively strictly convex and strictly increasing on  $[x_m, x_M]$ , (15) and (16) imply a contradiction, thus  $x(t_s) \in \{x_m, x_M\}$  as was to be proved.



Suppose now by a contradiction that there exists a time interval  $[t_1, t_2]$  such that  $\phi(t) = \dot{\phi}(t) = 0$  for  $t \in [t_1, t_2]$ . It is then easy to see that the trajectory is necessarily constant over  $[t_1, t_2]$  (indeed, combining  $\phi = \dot{\phi} = 0$  over  $[t_1, t_2]$  one finds that  $\lambda_y \psi'(x(t)) - \ell'(x(t)) = 0$  for  $t \in [t_1, t_2]$  implying that  $x$  must be constant over  $[t_1, t_2]$ ). Thus there exists  $x_s \in [x_m, x_M]$  such that  $x(t) = x_s$  for any time  $t \in [t_1, t_2]$ . Now, since the extremities of the singular arc  $t_1$  and  $t_2$  must be switching times, one must have  $x_s \in \{x_m, x_M\}$ . Suppose for instance that  $x_s = x_M$ . From the expression of  $\phi$  and  $\phi'$ , we deduce that

$$\frac{1}{\lambda_y} = \frac{\psi'(x_M)}{\ell'(x_M)} = \gamma'(\ell(x_M)),$$

which is a contradiction with (15) (since  $\gamma$  is strictly convex). We have a similar contradiction if  $x_s = x_m$ , which ends the proof.  $\square$

At this stage, we have thus proved that optimal trajectories are of bang-bang type (*i.e.* are concatenations of arcs with  $u = \pm 1$ ) such that at each switching time  $t_s$  one has  $x(t_s) \in \{x_m, x_M\}$ . By a similar reasoning as in the proof of Proposition 3, one can show that the number of switching times is finite. Moreover, since the switching function is  $T$ -periodic, the number of switching times is necessarily even (otherwise a switch will have to occur at  $x(T) = \bar{x}$  in contradiction with point 1 of Proposition 3). We focus now on solutions with two switches.

**3.3. Trajectories with two switches.** For a given  $T > 0$ , we consider trajectories  $x(\cdot)$  of (1) on  $[0, T]$  with  $x(0) = \bar{x}$  and a control defined by two switching times  $t_1, t_2$  with  $0 \leq t_1 < t_2 \leq T$ :

$$u(t) = \begin{cases} 1, & t \in [0, t_1) \\ -1, & t \in [t_1, t_2) \\ 1, & t \in [t_2, T) \end{cases}$$

These trajectories, that we shall call *BB trajectories*, will play an important role in the following. Note that under Hypotheses (H1)-(H2) a *BB trajectory* is characterized uniquely by its maximal and minimal values  $x_M = x(t_1)$  and  $x_m = x(t_2)$  in  $I$ . For convenience, we define on the interval  $I$  the function

$$\eta(x) = \frac{1}{f(x) + g(x)} - \frac{1}{f(x) - g(x)}$$

From Hypothesis (H2), note that  $\eta$  is a positive function on  $I$ .

**Lemma 3.2.** *Under Hypotheses (H1)-(H2), a BB trajectory is  $T$ -periodic when  $x_m, x_M$  satisfy*

$$(17) \quad \int_{x_m}^{x_M} \eta(x) dx = T$$

Moreover, the corresponding control satisfies the constraint (2) when  $x_m, x_M$  satisfy

$$(18) \quad \int_{x_m}^{x_M} \eta(x) \psi(x) dx = \bar{u}T$$

*Proof.* For  $t \in [0, t_1) \cup [t_2, T)$ , one has  $\dot{x} = f(x) + g(x) > 0$  and one can write

$$t_1 = \int_{\bar{x}}^{x_M} \frac{dx}{f(x) + g(x)}, \quad T - t_2 = \int_{x_m}^{x(T)} \frac{dx}{f(x) + g(x)}$$

Similarly for  $t \in [t_1, t_2)$ , one has  $\dot{x} = f(x) - g(x) < 0$  and

$$t_2 - t_1 = - \int_{x_m}^{x_M} \frac{dx}{f(x) - g(x)}$$

One then obtains

$$T = \int_{x_m}^{x(T)} \frac{dx}{f(x) + g(x)} - \int_{x_m}^{x_M} \frac{dx}{f(x) - g(x)} + \int_{\bar{x}}^{x_M} \frac{dx}{f(x) + g(x)}$$

and for a  $T$ -periodic solution,  $x(T) = \bar{x}$  gives exactly the property (17).

Proceeding with the same decomposition of the interval  $[0, T]$ , one can write

$$\int_0^T u(t)dt = \int_{\bar{x}}^{x_M} \frac{dx}{f(x) + g(x)} + \int_{x_m}^{x_M} \frac{dx}{f(x) - g(x)} + \int_{x_m}^{\bar{x}} \frac{dx}{f(x) + g(x)}$$

which gives the quality

$$\int_{x_m}^{x_M} \left( \frac{1}{f(x) + g(x)} + \frac{1}{f(x) - g(x)} \right) dx = \bar{u}T$$

when  $u(\cdot)$  fulfills (2). Finally, notice that one has

$$\frac{1}{f(x) + g(x)} + \frac{1}{f(x) - g(x)} = \eta(x)\psi(x)$$

and thus property (18) is satisfied.  $\square$

We first analyze the possibilities of satisfying the integral condition (17).

**Lemma 3.3.** *Under Hypotheses (H1)-(H2), for any  $T > 0$  there exists an unique function  $\beta_T : I \mapsto I$  that satisfies  $\beta_T(\alpha) > \alpha$  for any  $\alpha \in I$  and*

$$\int_{\alpha}^{\beta_T(\alpha)} \eta(x)dx = T, \quad \alpha \in I$$

Moreover  $\beta_T$  is  $C^1$ , increasing and bijective from  $I$  to  $I$ .

*Proof.* The function  $f+g$  is  $C_1$  and positive on  $I = (a, b)$  with  $(f+g)(b) = 0$ . It then verifies  $(f+g)(x) \leq K_+(b-x)$  for any  $x \in I$ , where  $K_+ = -\min_{x \in \bar{I}} (f+g)'(x) > 0$ . As the function  $\eta$  satisfies

$$\eta(x) > \frac{1}{f(x) + g(x)} \geq \frac{1}{K_+(b-x)} > 0, \quad x \in I,$$

one deduces that the map

$$\chi : (\xi_-, \xi_+) \mapsto \int_{\xi_-}^{\xi_+} \eta(x)dx$$

is such that for any  $\alpha \in I$ ,  $\chi(\alpha, \cdot)$  is increasing with  $\chi(\alpha, \alpha) = 0$  and  $\chi(\alpha, b) = +\infty$ . By the Theorem of implicit functions, there exists an unique map  $\beta_T : I \mapsto I$  such that  $\chi(\alpha, \beta_T(\alpha)) = T$  for any  $\alpha \in I$ . Moreover, one has

$$\beta_T'(\alpha) = \frac{\eta(\alpha)}{\eta(\beta_T(\alpha))} > 0, \quad \alpha \in I$$

The function  $\beta_T$  is thus increasing, and then admits limits  $\beta_T(a^+) \geq a$  and  $\beta_T(b^-) \leq b$  that verify  $\chi(a, \beta_T(a^+)) = T$  and  $\chi(b, \beta_T(b^-)) = T$ . As previously,  $f - g < 0$  on

$I$  with  $(f - g)(a) = 0$  implies that one has  $(f - g)(x) \geq K_-(x - a)$  for any  $x \in I$  (with  $K_- = \min_{x \in \bar{I}}(f - g)'(x) > 0$ ) and thus one has

$$\eta(x) > -\frac{1}{f(x) - g(x)} \geq \frac{1}{K_-(x - a)} > 0, \quad x \in I,$$

then, if  $\beta(a^+) > a$ , one should then have  $\chi(a, \beta_T(a^+)) = +\infty$  which is not possible. So one has  $\beta(a^+) = a$ . As the function  $\eta$  is positive on  $I$ , one has  $\beta_T(\alpha) > \alpha$  for any  $\alpha \in I$ , and we deduce the limit  $\beta_T(b^-) = b$ . This proves that  $\beta_T$  is bijective from  $I$  to  $I$ .  $\square$

We are ready now to show that there exists a unique  $BB$  trajectory that satisfies both integral conditions (17) and (18).

**Proposition 4.** *Under Hypotheses (H1)-(H2)-( $\bar{H}$ ), there exists unique  $x_m, x_M$  in  $I$  that verify (17) and (18). One has necessarily  $x_m < \bar{x} < x_M$ .*

*Proof.* From Lemma 3.3, the satisfaction of condition (17) implies to have  $x_M = \beta_T(x_m)$ . We thus have simply to show the uniqueness of  $x_m$  for the condition (18) to be fulfilled. Consider the function

$$F(\alpha) := \int_{\alpha}^{\beta_T(\alpha)} \eta(x)(\psi(x) - \psi(\bar{x}))$$

and notice that conditions (17) and (18) are both satisfied exactly when  $F(x_m) = 0$ . From Hypothesis ( $\bar{H}$ ) and properties of  $\beta_T$  given in Lemma 3.3, one has  $F(\alpha) > 0$  for any  $\alpha \in [\bar{x}, b)$ , and  $F(\alpha) < 0$  for any  $\alpha \in (a, \beta_T^{-1}(\bar{x})]$ . By the Mean Value Theorem, there exists  $x_m \in (\beta_T^{-1}(\bar{x}), \bar{x})$  such that  $F(x_m) = 0$ . Moreover, one has

$$F'(\alpha) = \eta(\beta(\alpha))(\psi(\beta(\alpha)) - \psi(\bar{x}))\beta'(\alpha) - \eta(\alpha)(\psi(\alpha) - \psi(\bar{x}))$$

As  $\beta_T$  is increasing and  $\psi$  satisfies ( $\bar{H}$ ), we obtain  $F'(\alpha) > 0$  for any  $\alpha < \bar{x}$  with  $\beta_T(\alpha) > \bar{x}$ , that is exactly for  $\alpha \in (\beta_T^{-1}(\bar{x}), \bar{x})$ , and we conclude about the existence and uniqueness of  $x_m, x_M$  in  $I$ , with  $x_m < \bar{x}$  and  $x_M > \bar{x}$ .  $\square$

It is worth to mention that  $x_m$  and  $x_M$  depend on the period  $T$ . In the next Lemma, we provide properties of  $x_m$  and  $x_M$  as functions of  $T$ .

**Lemma 3.4.** *Under Hypotheses (H1)-(H2)-( $\bar{H}$ ), the functions  $T \mapsto x_m(T)$  and  $T \mapsto x_M(T)$  are continuously differentiable, and respectively decreasing and increasing. Moreover, one has*

$$(19) \quad \lim_{T \rightarrow +\infty} x_m(T) = a \quad \text{and} \quad \lim_{T \rightarrow +\infty} x_M(T) = b.$$

*Proof.* For each  $T > 0$ , we know from Proposition 4 that there exists unique  $x_m(T), x_M(T)$  in  $I$  satisfying (17)-(18). Differentiating w.r.t.  $T$  the conditions (17) and (18), one obtains that the derivatives  $x_m', x_M'$  are solution of the linear system

$$\underbrace{\begin{bmatrix} \eta(x_M) & -\eta(x_m) \\ \eta(x_M)\psi(x_M) & -\eta(x_m)\psi(x_m) \end{bmatrix}}_X \begin{bmatrix} x_M' \\ x_m' \end{bmatrix} = \begin{bmatrix} 1 \\ \psi(\bar{x}) \end{bmatrix}$$

where  $\det(X) = \eta(x_M)\eta(x_m)(\psi(x_M) - \psi(x_m)) > 0$ . Then  $x_M'$ ,  $x_m'$  are given by the expressions

$$\begin{cases} x_M' &= \frac{\eta(x_m)(\psi(\bar{x}) - \psi(x_m))}{\det(X)} > 0 \\ x_m' &= \frac{\eta(x_m)(\psi(\bar{x}) - \psi(x_m))}{\det(X)} < 0 \end{cases}$$

From (17) and (18), one has

$$\frac{T}{2}(\bar{u} + 1) = \int_{x_m(T)}^{x_M(T)} \frac{dx}{f(x) + g(x)} < \int_a^{x_M(T)} \frac{dx}{f(x) + g(x)}$$

Taking the limit when  $T$  tends to  $+\infty$  in both side of this inequality, one obtains  $\lim_{T \rightarrow +\infty} x_M(T) = b$ . Similarly one can prove  $\lim_{T \rightarrow +\infty} x_m(T) = a$ .  $\square$

**3.4. Optimal solutions.** According to Proposition 4, for any  $T > 0$  there exists an unique *BB* trajectory  $\hat{x}_T(\cdot)$  that is  $T$ -admissible, generated by a control that we shall denote  $\hat{u}_T$ . Moreover, there exists a unique  $\bar{t} \in (0, T)$  such that  $\hat{x}_T(\bar{t}) = \bar{x}$ . Therefore, there exists exactly two  $T$ -admissible solutions  $\hat{x}_T(\cdot)$ ,  $\check{x}_T(\cdot)$  with two switches, given by  $\hat{u}_T$  and  $\check{u}_T$  with

$$\check{u}_T(t) = \hat{u}_T(t + \bar{t}), \quad t \geq 0$$

which have the same cost.

We study now the monotony of the cost  $J_T(\hat{u}_T)$  with respect to  $T$ .

**Lemma 3.5.** *Under Hypotheses (H1)-(H2)-(H3), one has*

$$S > T > 0 \Rightarrow J_S(\hat{u}_S) < J_T(\hat{u}_T)$$

*Proof.* Let us denote by  $x$ ,  $y$  the solutions of (1) corresponding to  $\hat{u}_T$  and  $\hat{u}_S$  respectively. Set  $x_M := x(t_1)$ ,  $x_m := x(t_2)$ ,  $y_M := x(s_1)$ ,  $y_m := x(s_2)$ . Recall that one has  $0 < t_1 < t_2 < T$ ,  $0 < s_1 < s_2 < S$  and from Lemma 3.4,  $x_M < y_M$ ,  $x_m > y_m$  with  $t_1 < s_1$ ,  $t_2 < s_2$ . Let us introduce the set  $E$  defined by

$$E := \{s \in [0, S] ; y(s) > x_M \text{ or } y(s) < x_m\}.$$

Let us also define the function  $\varphi : [0, T] \rightarrow [0, S] \setminus E$  by

$$\varphi(t) := \begin{cases} t & \text{if } t \in [0, t_1), \\ t + \delta_1 & \text{if } t \in [t_1, t_2), \\ t + \delta_2 & \text{if } t \in [t_2, T], \end{cases}$$

where  $\delta_1$ , resp.  $\delta_2$  is the time spent by  $y(\cdot)$  over  $x(\cdot)$ , resp. below  $x(\cdot)$ . By construction one has  $x(t) = y(\varphi(t))$  for  $t \in [0, T]$  and  $\varphi$  is bijective, thus  $\text{meas}(E) = S - T$ . Moreover, for any monotonic function  $\rho : I \rightarrow \mathbb{R}$  one has

$$\int_0^T \rho(x(t))dt = \int_0^T \rho(y(\varphi(t)))dt = \int_{[0, S] \setminus E} \rho(y(s))ds,$$

by considering the change of variable  $s = \varphi(t)$ . We then get

$$\int_0^T \ell(x(t))dt = \int_{[0, S] \setminus E} \ell(y(s))ds,$$

and

$$(20) \quad \int_0^T \gamma(\ell(x(t)))dt = \int_{[0, S] \setminus E} \gamma(\ell(y(s)))ds.$$

As both controls  $\hat{u}_T$  and  $\hat{u}_S$  satisfy the constraint (4), one has

$$\frac{1}{T} \int_0^T \gamma(\ell(x(t))) dt = \frac{1}{S} \int_0^S \gamma(\ell(y(s))) ds = \bar{u},$$

which implies

$$\frac{1}{S-T} \int_E \gamma(\ell(y(s))) ds = \bar{u}.$$

Let us now consider a function  $\hat{\gamma} : [\ell(y_m), \ell(y_M)] \rightarrow \mathbb{R}$  defined by

$$\hat{\gamma}(\xi) := \begin{cases} \ell(x_m) + \frac{\gamma(\ell(x_M)) - \gamma(\ell(x_m))}{\ell(x_M) - \ell(x_m)} (\xi - \ell(x_m)) & \text{for } \xi \in [\ell(x_m), \ell(x_M)] \\ \gamma(\xi) & \text{otherwise} \end{cases}$$

(see Figure 1). First, note that  $\hat{\gamma}$  is convex increasing and satisfies

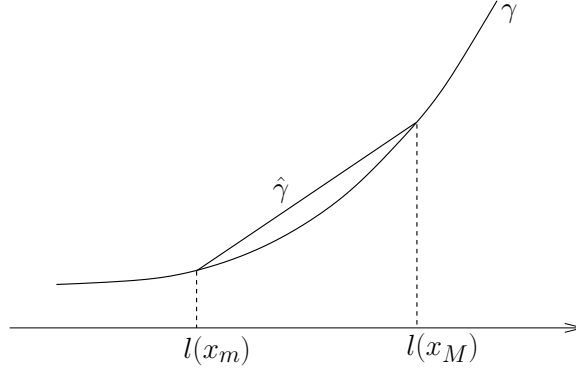


FIGURE 1. Functions  $\gamma$  and  $\hat{\gamma}$ .

$$(21) \quad \hat{\gamma}(\xi) > \gamma(\xi) \text{ for } \xi \in (\ell(x_m), \ell(x_M)).$$

As one has  $\gamma = \hat{\gamma}$  in  $[\ell(y_m), \ell(y_M)] \setminus [\ell(x_m), \ell(x_M)]$ , we also have

$$\frac{1}{S-T} \int_E \hat{\gamma}(\ell(y(s))) ds = \bar{u}.$$

By Jensen's inequality, we obtain

$$(22) \quad \frac{1}{S-T} \int_E \ell(y(s)) ds \leq \hat{\gamma}^{-1}(\bar{u}).$$

Now, since  $\hat{\gamma}$  is affine over  $[\ell(x_m), \ell(x_M)]$ , one obtains

$$\hat{\gamma} \left( \frac{1}{T} \int_0^T \ell(x(t)) dt \right) = \frac{1}{T} \int_0^T \hat{\gamma}(\ell(x(t))) dt > \frac{1}{T} \int_0^T \gamma(\ell(x(t))) dt = \bar{u}$$

using the property  $x(t) \in [x_m, x_M]$  for  $t \in [0, T]$ , (21) and (4). Therefore, one has

$$(23) \quad \frac{1}{T} \int_0^T \ell(x(t)) dt > \hat{\gamma}^{-1}(\bar{u}).$$

We get by (20), (22) and (23)

$$\begin{aligned}
\frac{1}{S} \int_0^S \ell(y(s)) ds &= \frac{1}{S} \int_E \ell(y(s)) ds + \frac{1}{S} \int_{[0, S] \setminus E} \ell(y(s)) ds \\
&\leq \frac{S-T}{S} \hat{\gamma}^{-1}(\bar{u}) + \frac{1}{S} \int_0^T \ell(x(t)) dt \\
&< \frac{S-T}{S} \frac{1}{T} \int_0^T \ell(x(t)) dt + \frac{T}{S} \frac{1}{T} \int_0^T \ell(x(t)) dt \\
&= \frac{1}{T} \int_0^T \ell(x(t)) dt,
\end{aligned}$$

which concludes the proof.  $\square$

We give now our main result.

**Theorem 3.6.** *Assume that Hypotheses (H1)-(H2)-(H3) are fulfilled. For any  $T > 0$ , there are two optimal trajectories, given by the two controls  $\hat{u}_T$  and  $\check{u}_T$ .*

*Proof.* Proposition 3 shows that an optimal trajectory consists in  $2n$  (with  $n \geq 1$ ) switches, that occur exactly at its maximal and minimal values. As (H3) implies  $(\bar{H})$ , Proposition 4 gives the uniqueness of a  $T$ -admissible  $BB$  trajectory, which amounts to state that there are exactly two extremals with two switches (i.e.  $n = 1$ ), given by the controls  $\hat{u}_T(\cdot)$  and  $\check{u}_T(\cdot)$ .

If  $n > 1$ , an optimal trajectory has to be  $T/n$ -periodic (as  $n$  switches have to occur for the same value  $x_m$  of  $x(\cdot)$  and the other  $n$  ones at the same value  $x_M$ ) with exactly two switches on  $[0, T/n]$ . Therefore its cost is equal to  $J(\hat{u}_{T/n})$ . By Lemma 3.5, one has  $J(\hat{u}_{T/n}) < J(\hat{u}_T)$ , which proves that only the case  $n = 1$  can be optimal.  $\square$

An interesting consequence of Lemma 3.5 is the monotony of the cost function evaluated at the optimal solution as a function of  $T$ .

**Proposition 5.** *The optimal criterion  $T \mapsto J_T(\hat{u}_T)$  is decreasing w.r.t.  $T$ .*

#### 4. RELAXING THE ASSUMPTIONS FOR LOCAL OVER-YIELDING

The previous Sections have shown the crucial role played by the monotony of the function  $\psi$  and the convexity of the function  $\gamma$  on the interval  $I$  (see Hypotheses  $(\bar{H})$  and (H3)). In the present Section, we consider situations for which these conditions are not fulfilled on the whole interval  $I$  but only on a neighborhood of  $\bar{x}$ . Typically, there could exist other values of  $\bar{x}$  satisfying  $\psi(\bar{x}) = \bar{u}$  (Hypothesis  $(\bar{H})$  is thus not fulfilled on  $I$ ) or  $\gamma$  could be convex only locally about  $\bar{x}$  (Hypothesis (H3) is thus not fulfilled on  $I$ ). The idea is then to restrict the values of the period  $T$  for characterizing (periodic) optimal solutions remaining in a neighborhood of  $\bar{x}$  (and presenting over-yielding).

For any  $T > 0$ , one can straightforwardly check that under Hypotheses (H1)-(H2) any  $T$ -periodic solution  $x(\cdot)$  of (1) with  $x(0) = \bar{x}$  and control  $u(\cdot)$  taking values in  $U$  verifies

$$(24) \quad x(t) \in [x_T^-, x_T^+], \quad \forall t \in [0, T],$$

where  $x_T^-, x_T^+$  are uniquely defined by the  $t^-, t^+$  in  $(0, T)$  such that

$$(25) \quad \begin{cases} x_T^- &= x(\bar{x}, -1; t^-) &= x(\bar{x}, 1; t^- - T), \\ x_T^+ &= x(\bar{x}, 1; t^+) &= x(\bar{x}, -1; t^+ - T) \end{cases}$$

Clearly, one has  $x_T^- < \bar{x} < x_T^+$  and  $(x_T^-, x_T^+) \rightarrow (\bar{x}, \bar{x})$  when  $T \rightarrow 0$ .

Therefore, one can revisit Proposition 4 as follows.

**Proposition 6.** *For any  $T > 0$  such that*

$$(26) \quad (\psi(x) - \psi(\bar{x}))(x - \bar{x}) > 0, \quad \forall x \in [x_T^-, x_T^+] \setminus \{\bar{x}\}$$

where  $x_T^-, x_T^+$  are given in (25), there exists unique  $x_m = x_m(T)$ ,  $x_M = x_M(T)$  in  $[x_T^-, x_T^+]$  that verify (17) and (18).

*Proof.* As we look for  $x_m, x_M$  in the interval  $[x_T^-, x_T^+]$ , we shall consider the function  $F$  defined in the proof of Proposition 4 on the interval  $[x_T^-, \beta_T^{-1}(x_T^+)]$  only. Notice that for any  $T > 0$  one has  $\beta_T(x_T^-) = \bar{x}$  and  $\beta_T(\bar{x}) = x_T^+$  (that is  $\beta_T^{-1}(x_T^+) = \bar{x}$ ). So  $x_m$  has to belong to the interval  $[x_T^-, \bar{x}]$ . For  $T > 0$  that fulfills (26), one has  $F(x_T^-) < 0$  and  $F(\bar{x}) > 0$ , and by the Mean value Theorem, we deduce that there exists  $x_m$  in  $(x_T^-, \bar{x})$  such that  $F(x_m) = 0$ , which amounts that have that pair  $x_m, x_M = \beta_T(x_m)$  satisfying (17) and (18). As in the proof of Proposition 4, the condition (26), instead of  $(\bar{H})$ , gives  $F' > 0$  on the interval  $(x_T^-, \bar{x})$  which provides the uniqueness of  $x_m$  on  $(x_T^-, \bar{x})$  (and thus the uniqueness of  $x_M$  as well).  $\square$

Then, one may wonder if is enough to require Hypothesis (H3) to be fulfilled on  $[x_m(T), x_M(T)]$  (instead of  $I$ ) to obtain the optimality of the controls  $\hat{u}_T, \check{u}_T$  as in Theorem 3.6. There could exist extremals with  $x(\cdot)$  taking values outside  $[x_m(T), x_M(T)]$ . We show that condition (26) is indeed enough to prevent this case from occurring.

**Proposition 7.** *Under Hypotheses (H1)-(H2), for any  $T > 0$  that verifies (26) and such that  $\psi$  is increasing on  $[x_m, x_M]$  where  $x_m = x_m(T)$ ,  $x_M = x_M(T)$  are given by Proposition 6, any  $T$ -admissible solution  $x(\cdot)$  verifies*

$$\hat{x} := \max_{t \in [0, T]} x(t) \leq x_M \quad \text{and} \quad \check{x} := \min_{t \in [0, T]} x(t) \geq x_m$$

*Proof.* Consider a  $T$ -admissible solution  $x(\cdot)$ . From the property (24), one has  $\hat{x} \leq x_T^+$  and  $\check{x} \geq x_T^-$ . Moreover, from condition (26) and Lemma 2.1, one deduces that the inequalities  $\hat{x} > \bar{x} > \check{x}$  have necessarily to be fulfilled. Let  $\hat{t} \in ]0, T[$  be such that  $x(\hat{t}) = \hat{x}$  and suppose that one has  $\hat{x} > x_M$ . One can assume, without loss of generality, that  $x(t) \geq \bar{x}$  is satisfied for any  $t \in [0, \hat{t}]$  (if not, consider  $t_0 = \sup\{t < \hat{t} \text{ s.t. } x(t) < \bar{x}\}$  and replace  $x(\cdot)$  by  $x(\cdot + t_0)$ ). Let  $A$  and  $B$  be the numbers defined by

$$A = \int_{\bar{x}}^{\hat{x}} \frac{dx}{f(x) + g(x)} \quad \text{and} \quad B = - \int_{\bar{x}}^{\hat{x}} \frac{dx}{f(x) - g(x)}$$

(which are positive by Hypothesis (H2)) that are the fastest times for a solution of (1) to reach, respectively,  $\hat{x}$  from  $\bar{x}$  (with the constant control  $u = 1$ ) and  $\bar{x}$  from  $\hat{x}$  (with the constant control  $u = -1$ ). Clearly, one has  $\hat{t} \geq A$  and  $T - \hat{t} > B$ .

We construct now a  $T$ -periodic solution  $\tilde{x}(\cdot)$  of (1) with  $\tilde{x}(0) = \bar{x}$  and a control  $\tilde{u}(\cdot)$  defined as follows

$$(27) \quad \tilde{u}(t) = \begin{cases} \bar{u} & t \in [0, \hat{t} - A[ \\ 1 & t \in [\hat{t} - A, \hat{t}[\cup[t^\dagger, T] \\ -1 & t \in [\hat{t}, t^\dagger[ \end{cases}$$

where  $t^\dagger$  is given by

$$t^\dagger = T - \int_{x^\dagger}^{\bar{x}} \frac{dx}{f(x) + g(x)}$$

with  $x^\dagger$  solution of  $\kappa(x^\dagger) = T - \hat{t}$ , the map  $\kappa(\cdot)$  being defined by

$$\kappa(\xi) := \int_{\xi}^{\bar{x}} \frac{dx}{f(x) + g(x)} - \int_{\xi}^{\hat{x}} \frac{dx}{f(x) - g(x)}$$

By Hypothesis (H2),  $\kappa(\cdot)$  is decreasing with

$$\kappa(x_m) = \int_{x_m}^{\hat{x}} \eta(x) dx - A > \int_{x_m}^{x_M} \eta(x) dx - \hat{t} = T - \hat{t}$$

and  $\kappa(\bar{x}) = B < T - \hat{t}$ . Therefore  $x^\dagger$  is uniquely defined with  $x^\dagger \in (x_m, \bar{x})$ . Moreover, one has

$$t^\dagger = \hat{t} - \int_{x^\dagger}^{\hat{x}} \frac{dx}{f(x) - g(x)} \in ]\hat{t}, T[$$

Expression (27) is thus well defined. See Fig. 2 as an illustration of the solution  $\tilde{x}(\cdot)$ .

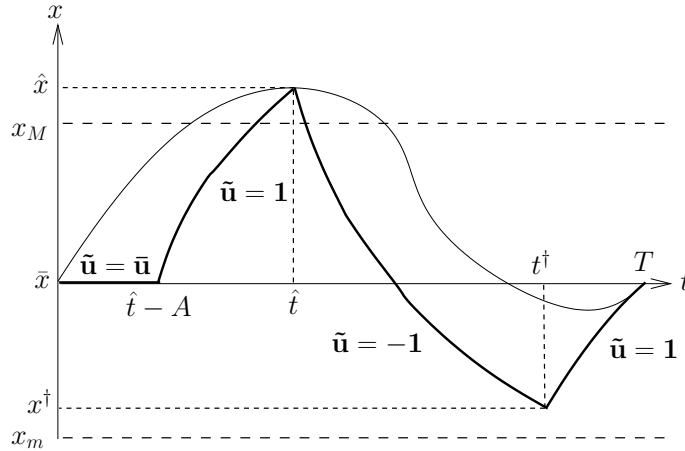


FIGURE 2. The solution  $\tilde{x}(\cdot)$  in thick line,  $x(\cdot)$  in thin line.

Clearly  $\tilde{x}(\cdot)$  reaches  $\hat{x}$  at  $\hat{t}$  and is below  $x(\cdot)$  on the interval  $[0, \hat{t}]$ . On  $[\hat{t}, t^\dagger]$ ,  $\tilde{x}(\cdot)$  has the fastest descent and therefore stays also below  $x(\cdot)$  on this interval. At  $t = t^\dagger$ , one has  $\tilde{x}(t^\dagger) = x^\dagger$ . Finally, from  $x = x^\dagger$  at time  $t^\dagger$  the constant control  $u = 1$  is the only one that allows to satisfy  $\tilde{x}(T) = \bar{x}$ . So, any periodic solution has to be above  $\tilde{x}(\cdot)$  on  $[t^\dagger, T]$ . We conclude that one has  $x(t) \geq \tilde{x}(t)$  for any  $t \in [0, T]$ . As



$\psi(x) > \psi(\bar{x})$  for  $x \in [x_M, \hat{x}]$  and  $\psi$  is increasing on  $[x_m, x_M]$  (by hypothesis), and as we have shown  $x(t) > x_m$  for any  $t \in [0, T]$ , one can write

$$\begin{aligned} \int_0^T (\psi(x(t)) - \psi(\bar{x}))dt &> \int_{\{t \in [0, T] | x(t) \leq x_M\}} (\psi(x(t)) - \psi(\bar{x}))dt \\ &\geq \int_{\{t \in [0, T] | x(t) \leq x_M\}} (\psi(\hat{x}(t)) - \psi(\bar{x}))dt \\ &= \int_{x^\dagger}^{x_M} (\psi(x) - \psi(\bar{x}))\eta(x)dx \end{aligned}$$

Finally, with  $x^\dagger > x_m$  and  $\eta > 0$  on  $I$ , one obtains

$$\int_0^T (\psi(x(t)) - \psi(\bar{x}))dt > \int_{x_m}^{x_M} (\psi(x) - \psi(\bar{x}))\eta(x)dx = 0$$

which is not possible according to Lemma 2.1. We then conclude that the inequality  $\hat{x} \leq x_M$  is satisfied. In a similar manner, one can prove the other inequality  $\check{x} \geq x_m$ .  $\square$

For periods  $T > 0$  that fulfill conditions of Proposition 7, we know that optimal solutions remains in the set  $[x_m(T), x_M(T)]$  and one obtains the same conclusion than Theorem 3.6 when Hypothesis (H3) is fulfilled on  $[x_m(T), x_M(T)]$  only.

**Theorem 4.1.** *Assume that Hypotheses (H1)-(H2) are fulfilled and consider  $T > 0$  such that*

- i)  $(\psi(x) - \psi(\bar{x}))(x - \bar{x}) > 0$  for any  $x \in [x_T^-, x_T^+] \setminus \{\bar{x}\}$ , where  $x_T^-, x_T^+$  are defined in (25),
- ii)  $\ell$  is increasing on  $[x_m, x_M]$  and  $\psi \circ \ell^{-1}$  is strictly convex increasing on  $[x_m, x_M]$ , where  $x_m = x_m(T)$ ,  $x_M = x_M(T)$  are given by Proposition 6

then, there are two optimal trajectories, given by the two controls  $\hat{u}_T$  and  $\check{u}_T$ .

*Proof.* Remark first that assumption ii) implies  $\psi$  to be increasing on  $[x_m, x_M]$ . Then, with assumption i) we know from Proposition 7 that any extremal is such that  $x(\cdot)$  belongs to  $[x_m, x_M]$ . With assumption ii) instead of Hypothesis (H3), the reader can easily check that the arguments of Theorem 3.6 apply in the same manner on  $[x_m, x_M]$  (instead of the whole interval  $I$ ), to prove that only the extremals with  $x(\cdot) = \hat{x}(\cdot)$  or  $x(\cdot) = \check{x}(\cdot)$  are optimal.  $\square$

## 5. APPLICATION TO THE CHEMOSTAT MODEL

Let us recall that the chemostat apparatus, invented simultaneously by [22] and [24] in the fiftens, is extensively used as an experimental device for studying the growth of micro-organisms. The mathematical model of the chemostat is often considered in the literature as a mathematical representation of the micro-organisms growth [29, 18], and not exclusively for the precise experimental chemostat device. It can be found in several real life situations, in natural environments, such as lakes, lagoons... [31], or in industrial applications such as waster-water treatment plants [14]. Originally, the word ‘‘chemostat’’ refers to a steady state operation and therefore many mathematical analyses address the long time behavior of the chemostat model under a constant input flow rate of substrate (which has also to be equal to the output flow for maintaining a constant volume of the water reservoir or tank).

Periodic flow rates in the chemostat model have been investigated in ecological contexts, typically for taking into account seasonality. In particular, it has been shown that periodic solutions could allow the coexistence of different species, while this is not possible in constant environment (see for instance [10, 25]). However, the impact of periodic inputs on the performances of the ecological functions of the ecosystem (resource conversion, biomass growth...) does not seem to have been studied in the ecological literature, even for the single species case. In industrial frameworks, it is known that periodic operations can impact positively or negatively the productivity of continuous cultures [17, 28]. The optimization of the bio-processes productivity has already been investigated with periodic controls, as mentioned in the introduction (see [2, 3, 19]). Our purpose here is different as we impose an integral constraint on the input flow rate, fixing the amount of nutrient that has to be delivered per period. Our objective is to compare temporal profiles that deliver the same amount of nutrient in terms of average biomass or nutrient concentrations in the chemostat model. Our study is two fold:

- (1) From an ecological view point, we investigate if a non constant flow rate can impact positively or negatively the average density of the consumers,
- (2) From a bio-process view point, we determine the best periodic flow rate maximizing the average water quality.

We consider then the classical chemostat model:

$$(28) \quad \begin{cases} \dot{b} &= \mu(s, b)b - Db \\ \dot{s} &= -\frac{1}{Y}\mu(s, b)b + D(s_{in} - s) \end{cases}$$

where  $b$  and  $s$  denote the concentrations, respectively of consumers (biomass) and resource (substrate). The parameters  $s_{in} > 0$  and  $Y > 0$  are the input concentration of nutrient and the biomass yield factor, respectively. The dilution rate  $D$  is the input variable, taking values within an interval  $[D_m, D_M]$  with  $0 \leq D_m < D_M$ . The function  $\mu$  is the specific growth function, which is  $C^1$ , non-negative and verifies  $\mu(0, b) = 0$  for any  $b$ . According to the literature, we distinguish two classes of growth functions:

- i)  $\mu$  does not depend on  $b$ . We assume that  $\mu$  is either increasing on  $[0, +\infty)$  or increasing on  $[0, \hat{s})$  and decreasing on  $(\hat{s}, +\infty)$ , with  $\hat{s} > 0$ . Typical instances of monotonic functions are given by the Hill expression [23]

$$\mu(s) := \mu_{\max} \frac{s^n}{K_s^n + s^n}$$

where  $\mu_{\max} > 0$ ,  $K_s > 0$  and  $n > 1$  are parameters. The well-known Monod function [22] corresponds to the particular value  $k = 1$  in this expression:

$$\mu(s) := \mu_{\max} \frac{s}{K_s + s}$$

A usual instance of non-monotonic growth functions is the Haldane expression [4]:

$$\mu(s) := \bar{\mu} \frac{s}{K_s + s + s^2/K_i}$$

where  $\bar{\mu}$ ,  $K_s$  and  $K_i$  are positive parameters. For this function, one has  $\hat{s} = \sqrt{K_s K_i}$ .

- ii)  $\mu$  is *density-dependent* i.e. it depends explicitly on density  $b$  of consumers. In that case,  $\mu$  is decreasing with respect to  $b$ , representing the crowding effect of consumers in competition for the common resource  $s$ . A prototype of such functions is the Contois kinetics [12]:

$$\mu(s, b) = \mu_{max} \frac{s}{s + Kb}$$

(where  $\mu_{max}$  and  $K$  are positive parameters, which is increasing w.r.t.  $s$ ).

Notice that without any loss of generality one can choose  $Y = 1$  (re-scaling the biomass concentration  $b$ ). Considering then the total density of matter  $m := b + s$ , one has

$$\dot{m} = D(t)(s_{in} - m)$$

Then, for any  $\bar{D} \in (D_m, D_M)$  and  $T$ -periodic  $D(\cdot)$  (where  $T$  is a positive number) such that

$$(29) \quad \frac{1}{T} \int_0^T D(t) dt = \bar{D} > 0$$

a  $T$ -periodic solution of (28) has to fulfill  $m(t) = s_{in}$  for any  $t$ . Therefore, we shall consider the dynamics on the  $m = b + s = s_{in}$  invariant domain:

$$(30) \quad \dot{s} = (s_{in} - s)(-\nu(s) + D),$$

with

$$\nu(s) = \mu(s, s_{in} - s)$$

We define the *break-even concentration* as

$$\lambda(D) := \inf\{s < s_{in} \text{ s.t. } \nu(s) < D\}$$

We fix a reference value  $\bar{D} \in (D_m, D_M)$  such that  $\bar{s} := \lambda(\bar{D}) < s_{in}$  and  $\bar{s} < \hat{s}$  for the non-monotonic case (we implicitly assume that the interval  $(D_m, D_M)$  allows this).  $s = \bar{s}$  is clearly an equilibrium of (30) for the constant control  $D = \bar{D}$ . Moreover, for all the instances of the function  $\mu$  we have described previously (Monod, Haldane, Hill and Contois), the function  $\nu$  is increasing about  $s = \bar{s}$ , which implies that  $s = \bar{s}$  is a locally asymptotically stable equilibrium.

We consider now the criterion

$$J_T(D(\cdot)) = \frac{1}{T} \int_0^T s(t) dt$$

for periodic solutions of (30) with  $s(0) = \bar{s}$  and inputs  $D(\cdot)$  that fulfill the integral constraint (29) with  $D(t) \in [D_m, D_M]$  for any  $t \in [0, T]$ . We first study if it is possible to have  $J_T(D(\cdot)) < \bar{s}$ . Notice that is exactly equivalent to have

$$\frac{1}{T} \int_0^T b(t) dt > \bar{b} = s_{in} - \bar{s}$$

(as one has  $m(t) = s(t) + b(t) = s_{in}$  for any  $t$ ). From an ecological view point, this amounts to study if a periodic environment allows to maintain a higher population of consumers  $b$  in average. In industrial waste-water treatments, the water quality is usually defined by the substrate  $s$  (considered as a pollutant) after separation of the biomass from the liquid, that has to be as small as possible. The question of interest is to determine the operating conditions that give the lowest concentration  $s$  averaged on a given period  $T$ . Here, we consider that one can play with the temporal distribution of the input flow rate of a water loaded with a high concentration of substrate  $s_{in}$ .

Let us begin by showing that Assumptions of the previous sections are fulfilled. We stick with the notation  $s$  (instead of  $x$ ) for the state variable as it is often used with the chemostat model. We posit

$$\begin{aligned} f(s) &:= \left( -\nu(s) + \frac{D_M + D_m}{2} \right) (s_{in} - s) \\ g(s) &:= \frac{D_M - D_m}{2} (s_{in} - s) \end{aligned}$$

and

$$u := \frac{2}{D_M - D_m} D - \frac{D_M + D_m}{D_M - D_m}$$

so that dynamics (30) has exactly the form (1) with  $u \in U$ . We take  $a = \lambda(D_m)$  and  $b = \lambda(D_M)$  which gives  $f(a) - g(a) = f(b) + g(b) = 0$ . As  $b \leq s_{in}$ ,  $g$  is clearly positive on  $(a, b)$ . Thus Hypothesis (H1) is fulfilled. Remark that for all the growth functions listed above, the function  $\nu$  is increasing on  $(a, b)$ , which implies that Hypothesis (H2) is also satisfied. The function  $\psi : I \rightarrow \mathbb{R}$  is given by the expression

$$\psi(s) = -\frac{f(s)}{g(s)} = \frac{2}{D_M - D_m} \nu(s) - \frac{D_M + D_m}{D_M - D_m}.$$

Let  $\bar{u}$  be the control associated to the value  $\bar{D}$ , which also satisfied  $\bar{u} = \psi(\bar{x})$ . Finally, our criterion amounts to choose the function  $l$  to be the identity function.

One can straightforwardly check the following Lemma concerning the Hill functions.

**Lemma 5.1.** *The Hill functions are increasing on  $\mathbb{R}_+$ , strictly convex on  $[0, s_c]$  and concave on  $[s_c, +\infty]$  where*

$$s_c := K_s \left( \frac{n-1}{n+1} \right)^{\frac{1}{n}}$$

**5.1. Study of the possibility of over-yielding.** We start by giving conditions for which an over-yielding is not possible.

**Proposition 8.** *An over-yielding is not possible in the following cases.*

- i) *The function  $\mu$  is of Monod type.*
- ii) *The function  $\mu$  is of Haldane type.*
- iii) *The function  $\mu$  is of Hill type with  $\lambda(D_m) \geq s_c$  or  $\bar{s} > K_s(n-1)^{1/n}$ .*
- iv) *The function  $\mu$  is of Contois type with  $K \leq 1$ .*

*Proof.* Remark first that one has  $\gamma = \psi$  (as  $l$  is identity) and the function  $\psi$  has the same monotony and convexity characteristics than the function  $\nu$ .

For the Monod kinetics, Hypothesis (H4) is fulfilled with  $\bar{\psi} = \psi$ , as  $\mu$  is concave increasing on  $\mathbb{R}_+$  (and thus  $\psi$  also). By Proposition 2 (and Remark 4), we conclude that an over-yielding is not possible.

For the Haldane function,  $\mu$  is concave on  $(0, \hat{s})$  and we can choose

$$(31) \quad \bar{\psi}(s) = \begin{cases} \psi(s), & s \leq \hat{s} \\ \psi(\hat{s}), & s > \hat{s} \end{cases}$$

so that Hypothesis (H4) is fulfilled. As for the Monod kinetics, an over-yielding is not possible.

According to Lemma 5.1, the Hill function  $\mu$  is concave and increasing on  $I$  when  $a \geq s_c$ , and as previously we conclude that an over-yielding is not possible. When  $a < s_c$ , we define the number  $s^\dagger$  as the smallest  $s \in \mathbb{R}_+$  such that the tangent to the graph of  $\mu$  at  $s$  is above the graph of  $\mu$  on  $\mathbb{R}_+$ . One can easily check that it exactly corresponds to the abscissa  $s^\dagger$  such that the tangent to the graph of  $\mu$  at  $s^\dagger$  passes by the origin, that is such that

$$\mu(s^\dagger) - \mu'(s^\dagger)s^\dagger = 0$$

which gives the expression  $s^\dagger = K_s(n-1)^{1/n}$ . We then consider the function (see Figure 3)

$$\bar{\psi}(s) = \begin{cases} \mu'(s^\dagger)s & \text{for } s \leq s^\dagger, \\ \psi(s) & \text{for } s > s^\dagger \end{cases}$$

which is concave increasing on  $\mathbb{R}_+$  and above the function  $\psi$ . When  $\bar{s} > s^\dagger$ , one has  $\bar{\psi}(\bar{s}) = \psi(\bar{s})$  and Hypothesis (H4) is then verified. As previously, we conclude that no over-yielding is possible.

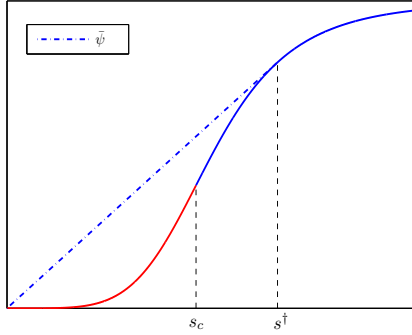


FIGURE 3. Example of graphs of functions  $\psi$  and  $\bar{\psi}$  for the Hill case with  $\mu_{max} = 2$ ,  $K_s = 2$  and  $n = 5$ .

For the Contois kinetics, one has

$$\nu(s) = \mu(s, s_{in} - s) = \mu_{max} \frac{s}{s + K(s_{in} - s)}, \quad s \in (0, s_{in})$$

from which one computes the expressions

$$(32) \quad \begin{aligned} \nu'(s) &= \mu_{max} \frac{s_{in}K}{(s + K(s_{in} - s))^2} \\ \nu''(s) &= \mu_{max} \frac{2s_{in}K(K-1)}{(s + K(s_{in} - s))^3} \end{aligned}$$

The function  $\nu$  is thus concave increasing on any interval  $I \subset (0, s_{in})$  when  $K \leq 1$ , preventing then any possibility of over-yielding.  $\square$

We focus now on situations for which over-yielding exists.

**Proposition 9.** *Over-yielding exists in the following cases.*

- i) *For the Hill functions with  $\lambda(D_M) \leq s_c$  or the Contois function with  $K > 1$ , any non constant  $T$ -periodic solution verifies  $J_T(D(\cdot)) < \bar{s}$ , whatever is  $T > 0$ .*

- ii) For the Hill function  $\bar{s} < s_c$ , any non constant  $T$ -periodic solution verifies  $J_T(D(\cdot)) < \bar{s}$  for  $T > 0$  small enough.

*Proof.* Remind first that that the function  $\gamma$  has the same monotony and convexity properties than the function  $\nu$ .

When the function  $\mu$  is of Hill type, we know by Lemma 5.1 that the function  $\nu$  (which is identical to  $\mu$ ) is strictly convex and increasing on  $[0, s_c)$  and concave on  $[K_c, +\infty)$ . So, when the interval  $I$  is included in  $(0, s_c)$ , Hypothesis (H3) is fulfilled and by Proposition 1, we obtain that any non-constant periodic solution has a better cost than the constant solution  $s = \bar{s}$ .

For the Contois function, the expressions of the two first derivatives of  $\nu$  given in (32) show that the function  $\nu$  is strictly convex and increasing on  $\mathbb{R}_+$  when  $K > 1$  and Proposition 1 applies again.

For the Hill function with  $\bar{s} < s_c$ , the function  $\nu$  is strictly convex only locally about  $\bar{s}$ . However,  $\nu$  is always increasing on  $\mathbb{R}_+$  and Proposition (6) ensures the existence and uniqueness of  $s_m(T)$  and  $s_M(T)$  for any  $T > 0$ . As already noticed in Section 4, one has  $(s_T^-, s_T^+) \rightarrow (\bar{s}, \bar{s})$  when  $T \rightarrow 0$  and thus one has also  $(s_m(T), s_M(T)) \rightarrow (\bar{s}, \bar{s})$ . Therefore, there exists  $\bar{T} > 0$  such that for  $T \in (0, \bar{T})$  one has  $[s_m(T), s_M(T)] \subset [0, s_c]$  and we conclude by Theorem (4.1) that the  $BB$  trajectories exhibit an over-yielding.  $\square$

## 5.2. Numerical illustrations of over-yielding.

5.2.1. *The Contois case.* From Proposition 9, we know that the  $BB$  trajectories are always optimal when  $K > 1$ . Figure 4 depicts these optimal trajectories for the optimal controls  $\hat{u}$  and  $\tilde{u}$  with the parameters values  $\mu_{max} = 1$ ,  $K = 2.5$ ,  $s_{in} = 3$  and the reference value  $\bar{D} \simeq 0.3874$  which corresponds to  $\bar{s} \simeq 1.8377$ .

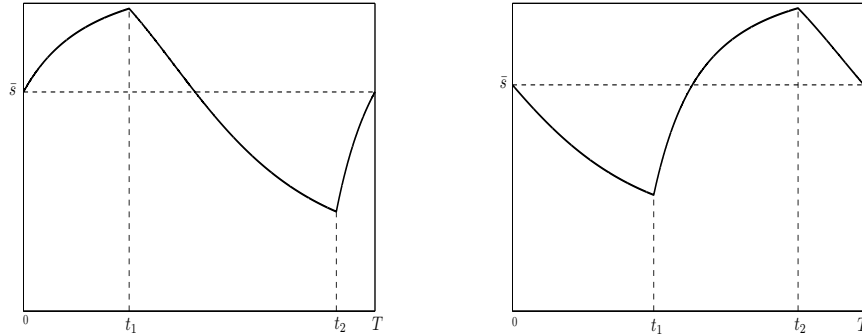


FIGURE 4. Optimal periodic trajectories for  $D_m = 0.1$ ,  $D_M = 0.8$  and  $T = 10$  in the Contois case ( $\hat{u}$  on the left;  $\tilde{u}$  on the right)

Figure 5 gives the evolution of the extreme values  $s_m$ ,  $s_M$  of the optimal trajectories and the optimal (average) cost as a function of the period  $T$ .

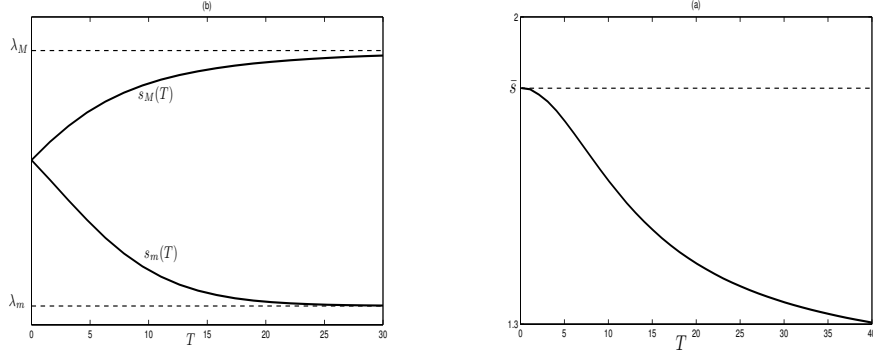


FIGURE 5. (a) Plot of  $s_m$  and  $s_M$  as function of  $T$ . (b) Plot of the optimal cost  $J_T(\hat{u}) = J_T(\tilde{u})$  as function of  $T$  (in the Contois case for  $D_{min} = 0.1$  and  $D_{max} = 0.8$ )

5.2.2. *The Hill case.* We have consider a Hill function for the parameters  $\mu_{max} = 3$ ,  $K_{s=1.9}$  and  $n = 2.5$  with  $s_{in} = 3$ . For these values, one computes  $s_c \simeq 1.3538$ . For the reference value  $\bar{D} \simeq 0.7221$  that gives  $\bar{s} = 1.2$ , we are in the conditions of point ii) of Proposition 9: the control strategies  $\hat{u}, \tilde{u}$  are optimal when the period  $T$  is not too large. Figure 6 gives  $BB$  trajectories, while Figure 7 presents extremes values and cost as functions of  $T$ .

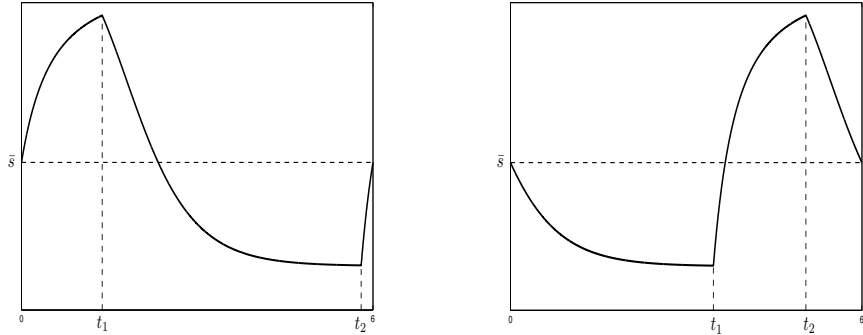


FIGURE 6.  $BB$  periodic trajectories for  $D_m = 0.2$ ,  $D_M = 2.4$  and  $T = 10$  in the Hill case ( $\hat{u}$  on the left;  $\tilde{u}$  on the right)

One can clearly see on Figure 7 (b) that  $BB$  trajectories do not provide overyielding for large  $T$  (because of too large excursions in the interval  $I$ ). However there exists a value  $\hat{T}$  of the period that gives the lowest average cost  $\hat{J}$  among  $BB$ -trajectories. We conjecture that the optimal cost for periods larger than  $\hat{T}$  cannot be lower than  $\hat{J}$ .

5.2.3. *Operating diagrams.* For a given reference value  $\bar{D}$ , one can play with two operating parameters:

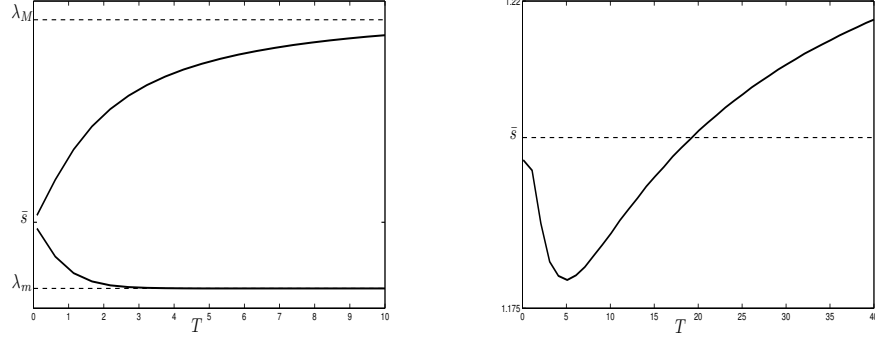


FIGURE 7. (a) Plot  $s_m$  and  $s_M$  as function of  $T$ . (b) Plot of the optimal cost  $J_T(\hat{u}) = J_T(\check{u})$  as function of  $T$  (in the Hill case for  $D_{min} = 0.2$  and  $D_{max} = 2.4$ )

- the amplitude  $\delta$  of the dilution rate  $D$  about  $\bar{D}$ :  $D \in [\bar{D} - \delta, \bar{D} + \delta]$  (which amounts to choose  $D_m = \bar{D} - \delta$  and  $D_M = \bar{D} + \delta$ ).
- the period  $T$ .

We consider the *operating diagram* in the  $(\delta, T)$  plane that gives the iso-values of the relative gain  $G_T$  provided by the *BB* trajectories compared to the constant control  $\bar{D}$ , where  $G_T$  is defined as

$$G_T = \frac{\bar{s} - J_T(\hat{u})}{\bar{s}}$$

This diagram can serve as a decision support tool for the practitioners to choose which characteristics of periodic operating conditions worth to be applied. Figure 8 illustrates this diagram for the Contois and Hill functions for the same parameters than for the previous Figures.

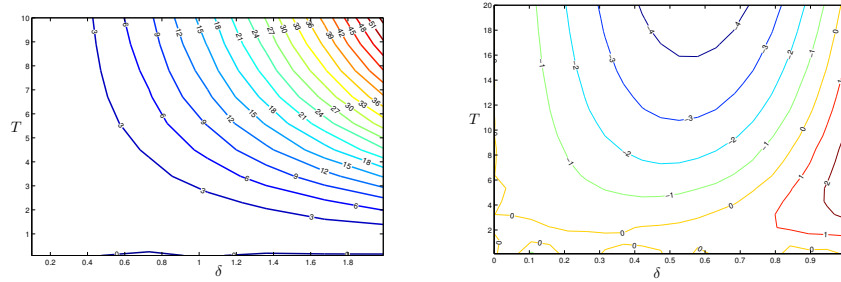


FIGURE 8. Operating diagrams in percentage of the optimal gain  $G_T$  with respect to  $(\delta, T)$  (Contois on the left, Hill on the right)

**5.3. A procedure to discriminate a Contois kinetics.** It often happens in microbiology that experimenters have to choose between several expressions for the growth function, to be identified on the experimental data. As already noticed in [20], periodic operations in the chemostat is a way to discriminate between models, playing with different frequencies of the dilution rate  $D(\cdot)$ . A typical situation is



the discrimination between a Monod expression, which is often used as a first try, and the density dependent Contois expression which amounts to consider that the affinity coefficient  $K_s$  in the Monod expression depends on the biomass concentration:  $K_s = Kb$ . When the magnitude of  $K_s$  can be roughly estimated and  $K_s/b$  is larger than one, the following procedure allows to discriminate between these two models.

**Procedure.**

- (1) Consider the chemostat at a (quasi) steady state for a nominal value  $D_0$  of the dilution rate (with  $D_m < D_0 < D_M$ ) at time 0. Let  $s_0$  be the corresponding value of  $s$  at steady state.

*Remark.* This ensures the chemostat's state to belong to the invariant set  $s + b = s_{in}$ .

- (2) Choose an arbitrary time  $t_1 > 0$  and apply the control

$$D(t) = \begin{cases} D_M & t \in [0, t_1] \\ D_m & t > t_1 \end{cases}$$

until the time  $\bar{t} > t_1$  such that  $s(\bar{t}) = s_0$  (which exists as  $D_m < D_0 < D_M$ ). Choose another arbitrary time  $t_2 > \bar{t}$  and apply the control

$$D(t) = \begin{cases} D_m & t \in [\bar{t}, t_2) \\ D_M & t > t_2 \end{cases}$$

until the time  $T > t_2$  such that  $s(T) = s_0$  (which exists as  $D_m < D_0 < D_M$ ).

- (3) Store the measurements history  $\{s(t)\}_{t \in [0, T]}$ .
- (4) Let  $\bar{D}$  be the mean value of the dilution rate during  $[0, T]$  :

$$\bar{D} = \frac{D_M(T + t_1 - t_2) + D_m(t_2 - t_1)}{T}$$

Apply for  $t > T$  the constant value  $\bar{D}$  and wait for the (quasi) steady state. Let  $\bar{s}$  be the corresponding value of  $s$  at steady state.

*Remark.* The trajectory  $s(\cdot)$  is a  $T$ -periodic solution with a control of mean value equal to  $\bar{D} = \mu(\bar{s})$ . From Lemma 2.1,  $\mu(\bar{s})$  is also equal to the mean value  $\mu(s(\cdot))$ . Therefore  $\bar{s}$  belongs to the interval  $[s(t_2), s(t_1)]$  and there exists  $t_0 > 0$  as the first time such that  $s(t_0) = \bar{s}$ .

- (5) With the data stored on  $[0, T]$ , determine the average value of  $s$  on  $[0, T]$ :

$$\tilde{s} = \frac{1}{T} \int_0^T s(t) dt$$

*Remark.* The average of the solution from  $s(t_0) = \bar{s}$  with the period control on  $[t_0, t_0 + T]$  is also equal to  $\tilde{s}$ .

- (6) If  $\tilde{s} < \bar{s}$ , we invalidate Monod (or Haldane) kinetics. Otherwise, we invalidate Contois (with  $K > 1$ ).

This simple procedure relies on the fact that only the Contois function with  $K > 1$  produces an over-yielding, and not the Monod function (see Propositions 8, 9). Note that this test does not require to test different frequencies.

## 6. CONCLUSION

In this work, we have provided sufficient conditions on the dynamics to allow over-yielding, and shown that under these conditions, there is exactly one optimal trajectory (up to a time translation) which is bang-bang with two switches on a period. This optimality result is global. We have also relaxed the hypotheses to prove the same optimality result globally, but for limited values of the period. Finally, we have illustrated our results on the chemostat model where the objective is to maximize the average water quality. Depending on the growth function, three kinds of situations could occur : no over-yielding at all, systematic over-yielding or over-yielding for moderated periods. To our best knowledge, such improvement in the context of the chemostat model is new.

Some of the techniques we have proposed here to cope with the integral constraint on the control variable, which is the main characteristic of the problem we have considered, could be deployed for systems in higher dimensions, and will be the matter of a future work.

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