Poisson Processes with Integrable Density

Bernard Picinbono, Cherif Bendjaballah

Poisson Processes with Integrable Density

Bernard Picinbono, *Life Fellow, IEEE,* and Cherif Bendjaballah

The pdf copy of the final published text can be obtained from the first author at the folowing address : bernard.picinbono@lss.supelec.fr

Reference of this text : IEEE Trans. Inf. Theory, Vol. 52, N 12 pp. 5606-5613, December 2006.

Abstract

When the density of a Poisson process is integrable various expressions published in the literature are incorrect. This is especially the case of the probability distribution of the distance between an origin and the following points of the process. The first purpose of this paper is to explain why the integrability of the density changes the situation. The second is to discuss various consequences of this fact on the probability distribution of random variables extracted from the process. Computer experiments are presented and are in excellent agreement with theoretical results. Some extensions of the same problem concerning renewal processes are discussed.

Index Terms

Detection and estimation, integration of the density, nonhomogeneous Poisson processes, point processes, renewal processes.

Manuscript received

B. P. is with the Laboratoire des Signaux et Systèmes (L2S), a joint laboratory of the C.N.R.S. and the École Supérieure d'Électricité, Plateau de Moulon, 3 rue Joliot–Curie 91192, Gif sur Yvette, France. The L2S is associated with the University of Paris–Sud, France. E-mail: bernard.picinbono@lss.supelec.fr, .

I. Introduction

Poisson processes play an important role in many areas of Applied Probability, Physics, and Information Sciences. They are defined and studied in various books corresponding to quite different approaches and among them we can cite [1], [2], [3], [4], [5]. There are also many papers discussing the use of Poisson processes in communication theory [6], [7], [8], [9], [10]. This is especially important in the context of optical communications at very low level of intensity. Indeed in this case the only information available is the time instants when photons are detected [11], [12].

The purpose of this correspondence is to show that the expressions of some probability distributions associated with a Poisson process are quite different depending on whether the density of this process is integrable or not. Various expressions given in the literature which are valid when the density is not integrable, become incorrect when the density is integrable and this situation is very common. The simplest example appears for Poisson processes in time when the density is limited to a given time interval, as in any physical situation. In order to understand where the problem is coming from, we present in Section II some known properties of Poisson processes with non-integrable density. In Section III the problem introduced by the fact that the density is integrable is discussed and explained. Some consequences on the probability distributions of distance between points are studied and various numerical calculations are presented in order to illustrate the results. Finally some extensions to renewal processes are presented and computer experiments on Poisson and renewal processes show an excellent agreement between theory and experiment.

II. Review of Known Results for Non-Integrable Density

Let us summarize the most important facts concerning Poisson processes used below.

A. Number of Points

A point process (PP) is a random distribution of points (in general time instants) x_i. It is a Poisson process if the random variables (RV) equal to the number of points in non-overlapping time intervals are independent. Such a process is completely characterized by its density $\lambda(x)$. In all that follows it is assumed that $\lambda(x) < +\infty$. It results from the independence assumption that the number $N(x_1, x_2)$ of points in the interval $[x_1, x_2[$ is a Poisson RV characterized by its mean value $m(x_1, x_2)$ given by

$$m(x_1, x_2) = \int_{x_1}^{x_2} \lambda(x)dx, \quad x_1 < x_2. \tag{1}$$

It is useful for what follows to introduce the mean value $m(x) = m(0, x)$ where 0 is an arbitrary origin. This yields

$$m(x_1, x_2) = m(x_2) - m(x_1). \tag{2}$$

In most cases the density $\lambda(x)$ is not integrable, which means that $m(+\infty)$ is infinite. This is especially true for stationary Poisson processes for which $\lambda(x)$ is constant and $m(x) = \lambda x$. We shall see that this property greatly simplifies the calculations.

B. Position of Points

Another approach consists in defining the PP by the position of its points [1], [6] - [10]. Let 0 be an arbitrary origin and X_i be the RV equal to the distance between the origin and the *i*th point of the Poisson process posterior to this origin. By construction we have $0 < X_i < X_j$ if $i < j$. Consider now the random vector $\mathbf{X}_n$ with components X_i, $1 \leq i \leq n$. The probability density function (PDF) of $\mathbf{X}_n$ is given by

$$p_n(\mathbf{x}_n) = \exp[-m(x_n)] \prod_{i=1}^{n} \lambda(x_i) \tag{3}$$

when $x_1 < x_2 < ... < x_n$ and zero otherwise. This expression appears in [1], [2], [3] and in similar forms in [6]-[10]. Note that for $n = 1$ this expression becomes

$$p_1(x) = \lambda(x) \exp[-m(x)], \tag{4}$$

which yields the classical exponential distribution when $\lambda(x) = \lambda$ or $m(x) = \lambda x$.

The principle of the proof of (3) appearing almost in the same form in [1], [2], [3], [6] is given in Appendix I where it is also shown that if $M \triangleq m(+\infty)$ is not finite the integral of (3) is 1, as expected for any PDF.

From (3) we deduce the marginal distribution $p_n(x_i)$ of X_i defined by

$$p_n(x_i) \triangleq \int p_n(\mathbf{x}_n) dx_1 ... dx_{i-1} dx_{i+1} ... dx_n. \tag{5}$$

It is shown analytically in Appendix II that $p_n(x_i)$ is independent of n and can be written as

$$p_n(x_i) = p(x_i) = \lambda(x_i) \exp[-m(x_i)] \frac{m^{i-1}(x_i)}{(i-1)!} \,. \tag{6}$$

This expression can be shown directly by using the property of independence of the number of points in non-overlapping intervals. Indeed the term $\lambda(x_i)dx_i$ is the probability of having at least one point in the interval $[x_i, x_i + dx_i[$ and the rest of the expression (6) is the probability of having $i - 1$ points in the interval $[0, x_i[$. It is easy to verify that if $\lambda(x)$ is not integrable, the integral of (6) is equal to 1.

Let us now consider the marginal PDF $p_n(x_i, x_j)$ $i < j < n$. From an obvious calculation or by using a direct reasoning similar to the one indicated above we deduce that $p_n(x_i, x_j) = p(x_i, x_j)$ with

$$p(x_i, x_j) = \lambda(x_i)\lambda(x_j) \exp[-m(x_j)] \frac{m^{i-1}(x_i)}{(i-1)!} \cdot \frac{m^{j-i-1}(x_i, x_j)}{(j-i-1)!} \,. \tag{7}$$

In particular for successive points we obtain

$$p(x_i, x_{i+1}) = \lambda(x_i)\lambda(x_{i+1}) \exp[-m(x_{i+1})] \, \frac{m^{i-1}(x_i)}{(i-1)!} \, . \tag{8}$$

From the marginal PDFs already calculated we deduce the conditional PDFs

$$p_n(x_j|x_i) = p(x_i, x_j)/p(x_i) = \lambda(x_j) \exp[-m(x_i, x_j)] \, \frac{m^{j-i-1}(x_i, x_j)}{(j-i-1)!} \, . \tag{9}$$

For successive indices this becomes

$$p_n(x_{i+1}|x_i) = p(x_i, x_{i+1})/p(x_i) = \lambda(x_{i+1}) \exp[-m(x_i, x_{i+1})]. \tag{10}$$

We have of course

$$\int_{x_i}^{\infty} p_n(x_{i+1}|x_i) dx_{i+1} = 1. \tag{11}$$

Markov Property

The conditional probability of the RV X_{i+1} conditional to a given value of $\mathbf{X}_i$, or $\mathbf{X}_i = \mathbf{x}_i$ is defined by $p(\mathbf{x}_{i+1})/p(\mathbf{x}_i)$. Using (3) yields that this ratio is equal to $p(x_{i+1}|x_i)$ given by (10). This shows that the successive RVs X_n are a Markov process with the transition probability (10).

C. Time Intervals

Another possible approach of a PP is to consider the distances between successive points, sometimes called *life times* when x is the time. These distances Y_i are related with the RVs X_i by $Y_i = X_i - X_{i-1}$ and $Y_1 = X_1$. These RVs can be considered as the values of a discrete time positive signal. By an obvious transformation we deduce the PDF of the Y_is which is

$$p(\mathbf{y}_n) = \lambda(y_1)\lambda(y_1 + y_2)...\lambda(y_1 + y_2... + y_n) \exp[-m(y_1 + y_2 + .. + y_n)]. \tag{12}$$

This is especially interesting in the stationary case where $\lambda(x) = \lambda$. In this situation the life times Y_i are a sequence of IID random variables with an exponential distribution defined by λ. This means that a stationary Poisson process is a renewal process with an exponential distribution [4].

However in the nonstationary case the PDF (12) cannot be factored and this means that the successive life times are not independent RVs. Then the property of independence of the number of points in distinct intervals characteristic of Poisson processes is no longer valid for the intervals between points. This means that a non-stationary Poisson process is not a non-stationary renewal process characterized by the fact that the life times Y_i are independent but not identically distributed.

It is easy to verify that the RVs Y_i do not, in general, constitute a Markov process. This shows that it is much more convenient to work with the distances X_i than with the life times Y_i.

III. Theoretical Results with Integrable Density

A. General Expressions

All the results of the previous section are consequences of (3). We shall see that this basic expression is no longer valid when the density $\lambda(x)$ is integrable. Thus we must find another expression replacing (3) in this situation.

Let us illustrate the problem in the very simple case of $n = 1$. The PDF $p_1(x_1)$ is given by (4) and its integral is obviously $1 - \exp(-M)$ where

$$M \triangleq \int_0^{+\infty} \lambda(x)dx < +\infty. \tag{13}$$

Thus the integral is equal to 1 only when M is infinite. As a consequence for finite M the function $p_1(x_1)$ of (4) is not a PDF.

This could be related with the concept of *defective* RVs introduced in [13], [14]. This concept, however, does not play any role in the following calculations. Readers interested in this question can look at p. 127 of [13] or p. 270 of [14].

The same situation appears for $p_n(\mathbf{x}_n)$ of (3) and its integral, denoted B_n, is no longer equal to 1 but to $I_n(0)$, where $I_n(x)$ is defined and calculated in Appendix III. This yields

$$B_n = 1 - \exp(-M)\left[1 + \frac{M^1}{(1)!} + ... + \frac{M^{n-1}}{(n-1)!}\right]. \tag{14}$$

This relation shows clearly that if M is infinite $B_n = 1$, and we return to the results of the previous section.

In order to obtain a normalized PDF it could be sufficient to replace (3) by

$$p_n(\mathbf{x}_n) = \frac{1}{B_n}\ \exp[-m(x_n)]\prod_{i=1}^{n}\lambda(x_i). \tag{15}$$

But this procedure is quite artificial and does not at all explain why the reasoning of Appendix I which shows (3) is not valid when the density is integrable.

For this purpose note first the meaning of the quantity B_n. Let N be the RV equal to the number of points of the Poisson process posterior to the origin 0. It is a Poisson RV with mean value M because of (13). As a consequence we deduce from (14) that

$$B_n = 1 - P[N \leq n-1] = P[N \geq n]. \tag{16}$$

Then the parameter B_n appearing in (15) is the probability that there are at least n points of the Poisson process posterior to the origin 0. When M is infinite this probability is of course equal to 1 because, whatever n, there are, with probability 1, at least n points of the Poisson process posterior to 0.

Let us now introduce the following time intervals.

$$\bar{I}_1 = [0, x_1[\ \ , \ \ \bar{I}_k = [x_{k-1} + dx_{k-1}, x_k[\ , \ 1 < k \leq n, \tag{17}$$

$$\bar{J}_k = [x_k, x_k + dx_k[\ , \ 1 \leq k \leq n. \tag{18}$$

Let $N(K)$ be the number of points of the Poisson process in any interval K. We can associate with the previous intervals the following events

$$\mathcal{A}_n = \bigcap_{k=1}^{n} [N(\bar{I}_k) = 0] \ \ , \ \ \mathcal{B}_n = \bigcap_{k=1}^{n} [N(\bar{J}_k) \geq 1]. \tag{19}$$

Let now $E_n(\mathbf{x}_n, d\mathbf{x}_n) = E_n$ be the event $\mathcal{A}_n \cap \mathcal{B}_n$. The principle of reasoning of Appendix I is to say that $p_n(\mathbf{x}_n)d\mathbf{x}_n = P[E_n]$. But the event E_n has no meaning if there is less than n points posterior to the origin 0. Thus $p_n(\mathbf{x}_n)d\mathbf{x}_n$ *is the conditional probability*

$$p_n(\mathbf{x}_n)d\mathbf{x}_n = P[E_n|(N \geq n)], \tag{20}$$

where N is the number of points posterior to the origin 0, i.e. belonging to the interval $[0, +\infty[$. This conditional probability can be written as

$$P[E_n|(N \geq n)] = P[E_n \cap (N \geq n)]/B_n, \tag{21}$$

because B_n is the probability that $N \geq n$. We also have

$$P[E_n \cap (N \geq n)] = P[E_n]P[(N \geq n)|E_n]. \tag{22}$$

But $P[(N \geq n)|E_n] = 1$ because if E_n is realized there are at least n points posterior to the origin. Finally we deduce from the Poisson assumption that

$$P[E_n] = P[E_n(\mathbf{x}_n, \mathbf{x}_n + d\mathbf{x}_n)] = \exp[-m(x_n)] \prod_{i=1}^{n} \lambda(x_i)d\mathbf{x}_n, \tag{23}$$

and inserting this expression in (21) yields (15). Thus the factor B_n, defined by (14) and appearing in (15) in order to ensure the normalization of the PDF, is not at all artificial but justified by the fact that $p_n(\mathbf{x}_n)$ is a conditional PDF. In order to specify this fact it could be convenient to write this PDF as $p_{n,\mathbf{X}_n}(\mathbf{x}_n|N \geq n)$ which is the conditional PDF of the random vector $\mathbf{X}_n$ with the condition that $N \geq n$. This notation, however, has a complicated form and for simplification we admit that all the PDFs noted p_n are conditional PDFs.

B. Marginal Probability Density Functions

These PDFs are obtained by inserting (15) into (5). It is clear that when $\lambda(t)$ is integrable, the integration with respect to $x_{i+1}, x_{i+2}, ..., x_n$ will change the result and then (6) is no longer valid.

For $i = n$ there is no such integration and the use of (3) or (15) will only introduce the factor $1/B_n$ ensuring the normalization. Thus we have

$$p_n(x_n) = \frac{1}{B_n}\lambda(x_n)\exp[-m(x_n)]\,\frac{m^{n-1}(x_n)}{(n-1)!}\,. \tag{24}$$

On the other hand if $i < n$ there are $n - i$ integrations with respect to the variables $x_k, i+1 \leq k \leq n = i + (n - i)$ and the result is

$$p_n(x_i) = \frac{1}{B_n}\,K_{i-1}(0, x_i)\lambda(x_i)I_{n-i}(x_i), \tag{25}$$

where $K_i(a, b)$ and $I_n(x)$ are defined in Appendices II and III respectively. By using (51) and (56) we obtain for $1 < i < n$

$$p_n(x_i) = \lambda(x_i)\,\frac{m^{i-1}(x_i)}{(i-1)!}\,\exp[-m(x_i)] \times h_n(x_i), \tag{26}$$

$$h_n(x_i) = \frac{1}{B_n}\left\{1 - \mathrm{e}^{-[M-m(x_i)]}c_{n-i}[a(x_i)]\right\}, \tag{27}$$

where a and $c_n(a)$ are defined by (54) and (55) respectively. For $i = n = 1$ the result is given by (24). Note that $p_n(x_i) = p(x_i)h_n(x_i)$, where $p(x_i)$ is given by (6), and corresponds to the case where $\lambda(x)$ is not integrable. This means that $h_n(x_i)$ is the correction factor due to the fact that $\lambda(x)$ has a finite integral. Note also that, contrary to (6), this PDF is no longer independent of n. Finally note that if $M \to \infty$ we return to (6).

It is especially interesting to study the PDF $p_n(x_1)$, written simply $p_n(x)$. It is given by

$$p_n(x) = \lambda(x)\mathrm{e}^{-m(x)}\frac{1}{B_n}\,\left\{1 - \mathrm{e}^{-[M-m(x)]}c_{n-1}[a(x)]\right\}. \tag{28}$$

It is clear that for a stationary Poisson process of density λ, M is infinite, which yields $p_n(x) = \lambda\exp(-\lambda x)$, PDF obviously independent of n.

Consider now the case of a time-limited Poisson process. This means that its density $\lambda(t)$ is bounded and zero outside an interval $[0, T]$. This ensures that the density is integrable. The same kind of assumption is introduced in [6] and [10]. In some sense this assumption corresponds to any experimental situation with Poisson processes because any experiment has a beginning (0) and an end (T). Note that if $n >> 1$ the PDF $p_n(x)$ satisfies $p_n(T) = 0$. Indeed it results from (54) that $a(T) = 0$ which implies $h_n(T) = 0$.

Let us now present numerical results calculated with some specific examples. First we assume that the density $\lambda(x)$ is constant in the interval $[0, T]$ and 0 otherwise. This situation is sometimes specified by the term of *semistationarity* because it corresponds to the case of a stationary Poisson process of density λ observed in a finite interval $[0,T]$. In this case the parameter M of the previous calculations is simply λT. This quantity is always finite but there are three different situations. If $\lambda T >> 1$ we are in the case where the expressions valid for stationary processes can be applied because the factor $\exp(-M)$ is almost equal to zero. In the opposite case if $\lambda T << 1$ the interval $[0,T]$ contains only 0 or 1 point and the problem disappears because the parameter B_n defined by (14) and used in (17) is arbitrarily small. It remains the intermediary situation which will now be analyzed. For simplification of the calculation we assume that $\lambda = T = 1 = \lambda T$.

The results of the calculation of $p_n(x)$ appear in Figure 1. The PDF $p_n(x)$ is represented in semilogarithmic coordinates for various values of n. As $p_1(x)$ is deduced from $p_0(x) \stackrel{\triangle}{=} \lambda \exp(-\lambda x)$ by a simple normalization factor, the corresponding curves (0 and 1) are parallel straight lines. Furthermore $p_1(x) > p_0(x)$ because the first is normalized in $[0,\infty[$ and the second in $[0,T]$. The other six curves correspond to $n = 2, 23, \cdots, 7$. It is easy to verify that they satisfy $p_n(T) = 0$ and also that they are normalized functions of x.

It is worth pointing out that when $n \to \infty$ the density $p_n(x)$ tends to a Dirac distribution. Indeed it is a PDF of a RV with mean value and variance tending to 0. This has an evident interpretation: if there is an infinite number of points in $[0,T]$ the first after the origin is at the origin.

In order to amplify the phenomenon let us now consider the case where the density of the Poisson process is zero outside the interval $[0,T[$ and in this interval is the increasing exponential function $\lambda_0[\exp(cx/T) - 1]$. Analogous functions are used in frequency modulation or to design some radar pulses. Using the previous expressions we calculate the PDF $p_n(x)$ for $1 \leq n \leq 7$. The results appear in Figure 2 with the parameters $\lambda_0 = 1$, $T = 1$, and $c = 1.25$. The density $\lambda(x)$ is also represented in this figure. The general property $p_n(T) = 0$ appears clearly in the figure. This is the main difference with $p_1(x)$ that does not satisfy this relation according to (24). The tendency to the Dirac distribution appears for the same reason as above.

Let us now consider the case of the two dimensional PDF $p_n(x_i, x_j)$. If $j = n$ we can use the same argument as previously and the result is

$$p_n(x_i, x_n) = \frac{1}{B_n}\, p(x_i, x_n), \tag{29}$$

where $p(x_i, x_j)$ is given by (7). On the other hand if $j < n$ there are $n - j$ integrations with respect to the variables x_k with $k \geq j + 1$ is such a way that

$$p_n(x_i, x_j) = p(x_i, x_j) h_n(x_j), \tag{30}$$

where $p(x_i, x_j)$ and $h_n(x_j)$ are given by (7) and (27) respectively.

For $j = i + 1$ we obtain

$$p_n(x_i, x_{i+1}) = p(x_i, x_{i+1})h_n(x_{i+1}), \tag{31}$$

where $p(x_i, x_{i+1})$ is given by (8).

As above this yields the conditional PDFs and in particular we have

$$p_n(x_{i+1}|x_i) = \lambda(x_{i+1})\mathrm{e}^{-m(x_i,x_{i+1})} \ \frac{h_n(x_{i+1})}{h_n(x_i)}. \tag{32}$$

If $M \to \infty$ the last term tends to 1 and we return to (10). Similar expression has recently been established for a simple case [15]. Finally it is easy to verify that

$$p_n(x_{i+1}|x_1, x_2, ..., x_i) = p_n(x_{i+1}|x_i), \tag{33}$$

which characterizes the Markov property.

C. Conditioning on fixed values of N

Instead of using the condition $N \geq n$ in order to arrive at the PDF given by (15), it is possible to introduce the condition $N = n$. To avoid confusion with the previous case we note $q_n(\mathbf{x}_n)$ the conditional PDF $p_n(\mathbf{x}_n|N = n)$. This requires of course that the density $\lambda(t)$ is integrable. This situation corresponds to the calculation presented in Section II of [6]. This assumption means that there is no point of the Poisson process posterior to x_n. A reasoning similar to the one presented in Appendix I yields the result appearing in Eq. (4) of [6], or with slightly different notations

$$q_n(\mathbf{x}_n) = \exp(-M)\prod_{i=1}^{n}\lambda(x_i) \tag{34}$$

for $x_1 < x_2 < .... < x_n$ and 0 otherwise. Let us show that this is not a PDF. Indeed the integral I of this function is

$$I = \exp(-M)\frac{M^n}{n!} = P(N = n) \tag{35}$$

which is not equal to 1. Using the same reasoning as for (20)-(23) we find that the true PDF is

$$q_n(\mathbf{x}_n) = \frac{n!}{M^n}\prod_{i=1}^{n}\lambda(x_i). \tag{36}$$

Let us present some consequences of this result. If $n = 1$ we obtain $q_1(x) = (1/M)\lambda(x)$. If furthermore we introduce the assumption of semistationarity which means that $\lambda(x) = \lambda$ for $0 \leq x \leq T$ and 0 otherwise we obtain $M = \lambda T$ and thus $q_1(x) = 1/T$. This is the well known result which states that if there is 1 point of a stationary Poisson process in an interval, this point is uniformly distributed in this interval. Finally the marginal PDF $q_n(x_1)$ defined as in (28) is

$$q_n(x) = \lambda(x)\frac{n}{M^n}\left[M - m(x)\right]^{n-1}. \tag{37}$$

In the semistationary case considered just above this becomes

$$q_n(x) = \frac{n}{T^n} [T - x]^{n-1} . \tag{38}$$

It can be verified that it is the PDF of the distance between the origin and the first point posterior to it when the RVs X_i are IID and uniformly distributed in $[0, T]$. This property of points of Poisson processes for a fixed value of N ($N = n$) is well known.

IV. Extensions to Renewal Processes

A renewal process (RP) is a PP in which the distances Y_i between successive points are positive IID random variables [4]. Such a process is then defined by the PDF $f(x)$ of these distances. We shall now consider RPs with a finite duration. We start from an origin considered as point of the process and we consider a sequence of IID RVs Y_i. The PP is limited to the interval $]0, T]$ and all the points outside of this interval are erased.

We limit our discussion to the calculation of the PDF $p_n(x)$ of the distance between the origin and the first point of the process posterior to this origin with the condition that there are at least n points in the interval $]0, T]$.

For this purpose let $\pi_n(x)$ and $b_n(x)$ be the probabilities of having n points or at least n points in $]0, x]$ respectively. They are related by

$$b_n(x) = 1 - \sum_{i=0}^{n-1} \pi_i(x) \tag{39}$$

and $b_0(x) = 1$. Finally let B_n be the probability that there are at least n points in $]0, T]$, or $B_n = b_n(T)$. It is shown in Appendix IV that for $n > 1$

$$p_n(x) = p_n(x|N \leq n) = (1/B_n) f(x) b_{n-1}(T - x), \tag{40}$$

where $f(x)$ is the PDF of the distance between successive points when there is no time limitation of the process. The division by B_n ensures that $p_n(x)$ is a normalized function. For $n = 1$ we have simply $p_1(x) = (1/B_1) f(x)$.

As $\pi_0(0) = 1$ and $\pi_k(0) = 0$ for $k > 0$, we deduce that $b_n(0) = 0$ and then $p_n(T) = 0$, which is the same result as the one indicated above for Poisson processes.

In order to illustrate this result let us consider the case of a renewal process sometimes called an Erlang process. It is obtained by deleting regularly one point over two of a stationary Poisson process. Thus the PDF function of the distances between points is $f(y) = \lambda^2 y \exp(-\lambda y)$.

The functions $p_n(x)$ are represented in Figure 3 for $\lambda = 2$ and $T = 1$. The value of λ is chosen in order that the mean distance between successive points is equal to 1, as in Figure 1. Note finally that $p_n(T) = 0$, as indicated above.

V. Computer experiments

We have seen in Section II. C that the intervals $\{Y_i\}$ between points of a PP constitute a positive discrete time signal. Conversely to any such signal we can associate a PP. Experiments on PPs can then be transformed into experiments on positive signals, and conversely.

If the signal $\{Y_i\}$ is positive and strictly white (sequence of IID positive RVs) the corresponding PP is a stationary renewal process. Then in order to generate such a PP characterized by the PDF $f(y)$ it suffices to realize a sequence of IID RVs with this PDF. If it is an exponential distribution, we obtain a Poisson process.

In order to realize such a PP in a computer experiment we can apply a general method used in [15] to construct a large variety of PPs. The sequence Y_i defining the renewal PP characterized by the PDF $f(y)$ satisfies the relation

$$U_i = \int_0^{Y_i} f(y)dy = F_Y(Y_i), \tag{41}$$

where $U_i \in [0,1]$ are RVs of uniform PDF $U(0,1)$ and $f(y)$ is a given PDF with the distribution function $F_Y(y)$. Then the RVs Y_i are deduced from the U_is by the relation $Y_i = F_Y^{-1}(U_i)$. In order to obtain an exponential distribution characterizing a Poisson process this becomes

$$Y_i = -\frac{1}{\lambda} \log\left(1 - U_i\right). \tag{42}$$

This method cannot be used directly to generate an Erlang renewal PP because the distribution function of the PDF $f(y) = \lambda^2 y \exp(-\lambda y)$ cannot be simply inverted. But another procedure is possible. It suffices to note that $f(y)$ is the PDF of the sum of two IID exponential RVs. Then the method starts now from two IID uniform RVs $U_{1,i}, U_{2,i} \in U[0,1]$. By applying twice (42) we obtain $S_{1,i} = -\frac{1}{\lambda}\log(1-U_{1,i})$ and $S_{2,i} = -\frac{1}{\lambda}\log(1-U_{2,i})$. These RVs are IID with an exponential distribution and their sum $Y_i = S_{1,i} + S_{2,i}$ has the PDF $f(y) = \lambda^2 y \exp(-\lambda y)$ defining a stationary Erlang PP. Note that because this sum the density of the corresponding Erlang PP is $\lambda/2$, as indicated at the end of the previous section.

Up to now our two processes (Poisson and Erlang) are strictly stationary and in order to compare experimental results with those established above we have to construct the algorithm realizing the condition introducing the conditional PDFs $p_n(x)$ presented in (28) and (40) and appearing in Figs. 1 and 3. This condition is that there are at least n points in the finite interval $[t_i, t_+T[$ posterior to each point of the stationary process. We must then eliminate all the samples Y_i such that $Y_i + Y_{i+1} + ... + Y_{i+n-1} > T$. This sum can easily be calculated recursively from the samples Y_i. In the experiments presented below we take $n = 2$. The condition used for $p_2(x)$ is then that there are at least 2 points in any interval

$[t_i, t_+T[$ and we must eliminate the samples Y_i such that $Y_i + Y_{i+1} > T$. For this purpose we calculate $V_i = Y_i + Y_{i+1}$ and the samples X_i corresponding to the conditional PDFs are given by

$$X_i = \frac{1}{2}\Big[1 + \text{sign}\,(T - V_i)\Big]Y_i, \tag{43}$$

where sign$(z) = \frac{z}{|z|}, z \neq 0$. These samples are correlated because of the term V_i which introduces a correlation between X_i and X_{i+1}. In order to suppress this correlation it suffices to take the samples $\bar{X}_i = (1/2)[1 + (-1)^i]X_i$. This is realized in the experiments described below. Note, however, that there is no apparent difference in the results when using X_i or $\bar{X}_i$, which means that the correlation between adjacent samples X_i does not play any role in the measurements of their histograms. These histograms yield the conditional PDF $p_2(x)$ analyzed above and appearing in Figs. 1 and 3 for Poisson and Erlang processes respectively.

The results presented in semilogarithmic coordinates appear in Fig. 4. The continuous curve in solid line are the transposition in semilogarithmic coordinates of the curves $p_2(x)$ of Figs. 1 and 3. The points of the figure represent the experimental results of the analysis by histograms, and then are an estimation of the theoretical PDFs. The number of samples analyzed if of the order of 10^7. The precision of the results decrease of course when this number is smaller.

The curve P1 corresponds to a stationary Poisson process. This means that there is no condition and the PDF is exponential. This curve is the curve 0 of Fig. 1. The curve P2 and E correspond to a Poisson process and an Erlang process respectively. For this last curve the calculation requires the expression of the probabilities $\pi_i(x)$ used in (39). It has recently been obtained in [16] and in the case of a constant density λ the result is

$$\pi_i(x) = \frac{[\mu(x)]^{2i}}{(2i)!}\Big[1 + \frac{\mu(x)}{2i+1}\Big]\exp[-\mu(x)]. \tag{44}$$

with $\mu(x) = \lambda x$ and $i = 2$.

This figure indicates an excellent agreement between theory and experimental results. The precision is a bit smaller in the case of Erlang distribution and this is due to the introduction of the sum of two RVs as explained above. Note finally that, as expected by the theory, the PDFs $p_2(x)$ are zero for $x = T$.

We have then an experimental confirmation of the main theoretical result of this paper stating that the PDFs $p_n(x)$ are conditional probability density functions.

Appendix A

Proof of (3) and calculation of its integral

Suppose first that $n = 1$. As x_1 is related to the first point of the process posterior to 0, the quantity $p(x_1)dx_1$ is the probability of finding 0 points in the interval $[0, x_1[$ and at least 1 point in the interval

$[x_1, x_1+dx_1[$. As these intervals are distinct, the number of points in them are independent Poisson RVs, which yields

$$p(x_1)dx_1 = \lambda(x_1)\exp[-m(x_1)]dx_1. \tag{45}$$

This is (3) for $n = 1$ or (4). Suppose now that $n = 2$. As the RV X_2 is related to the second point posterior to the origin 0, we deduce by the same reasoning that the probability $p(x_1, x_2)dx_1dx_2$ is the probability of finding 0 points in the intervals $[0, x_1[$ and $[x_1+dx_1, x_2[$ and at least 1 point in the intervals $[x_1, x_1 + dx_1[$ and $[x_2, x_2 + dx_2[$. This yields

$$p(x_1, x_2)dx_1dx_2 = \lambda(x_1)\lambda(x_2)\exp[-m(x_2)]dx_1dx_2, \tag{46}$$

which is (3) for $n = 2$. The extension to any value of n is obvious, which yields (3).

Let us now consider the calculation of the integral $I = \int p_n(\mathbf{x}_n)d\mathbf{x}_n$ where $p_n(\mathbf{x}_n)$ is given by (3). It can be written as

$$I = \int_0^\infty dx_1\lambda(x_1)\int_{x_1}^\infty dx_2\lambda(x_2)...\int_{x_{n-2}}^\infty dx_{n-1}\lambda(x_{n-1})J(x_{n-1}) \tag{47}$$

with

$$J(x_{n-1}) = \int_{x_{n-1}}^\infty dx_n\lambda(x_n)\exp[-m(x_n)]. \tag{48}$$

Noting that the derivative of $\exp[-m(x)]$ is $-\lambda(x)\exp[-m(x)]$ yields $J = \exp[-m(x_{n-1})]-\exp[-M] = \exp[-m(x_{n-1})]$ because M is assumed to be infinite. By repeating the same calculation we arrive to $I = \int_0^{+\infty} dx_1\lambda(x_1)\exp[-m(x_1)]$ which is equal to 1.

APPENDIX B
CALCULATION OF (5)

In (5) the integration with respect to $x_{i+1}, x_{i+2}, ..., x_n$ yields $\exp[-m(x_i)]$ as seen in Appendix I. As a result $p_n(x_i) = \lambda(x_i)\exp[-m(x_i)] \times K_{i-1}(x_i)$ where $K_{i-1}(x_i)$ is the integral

$$K_{i-1}(x_i) = \int_0^{x_i} dx_1\lambda(x_1)\int_{x_1}^{x_i} dx_2\lambda(x_2)...\int_{x_{i-2}}^{x_i} dx_{i-1}\lambda(x_{i-1}). \tag{49}$$

It is a particular case of the integral

$$K_i(a, b) \triangleq \int_a^b dx_1\lambda(x_1)\int_{x_1}^b dx_2\lambda(x_2)...\int_{x_{i-1}}^b dx_i\lambda(x_i). \tag{50}$$

Let us show by recursion that

$$K_i(a, b) = \frac{1}{i!}[m(b) - m(a)]^i. \tag{51}$$

This relation is obviously valid for $i = 1$. Suppose that it holds for $i - 1$. Let us show that it is valid for i. Using (50) and (51) yields

$$K_i(a, b) = \int_a^b dx_1 \lambda(x_1) \frac{1}{(i-1)!}[m(b) - m(x_1)]^{i-1}. \tag{52}$$

Noting that the derivative of $m(x)$ is $\lambda(x)$ yields (51) for i. As $p_n(x_i) = \lambda(x_i)\exp[-m(x_i)]K_{i-1}(0, x_i)$ and $m(0) = 0$, we obtain (6).

Appendix C
Calculation of $I_n(x)$

This is the integral

$$I_n(x) = \int_x^{+\infty} dx_1 \lambda(x_1) \int_{x_1}^{+\infty} dx_2 \lambda(x_2) ... \int_{x_{n-1}}^{+\infty} dx_n \lambda(x_n) \exp[-m(x_n)]. \tag{53}$$

Let introduce the following expressions

$$a(x) = M - m(x), \tag{54}$$

$$c_n(a) = 1 + \frac{a^1}{(1)!} + \frac{a^2}{(2)!} + ... + \frac{a^{n-1}}{(n-1)!}. \tag{55}$$

We shall show that

$$I_n(x) = \exp[-m(x)] - \exp(-M)c_n[a(x)]. \tag{56}$$

This relation is obviously valid for $n = 1$. Suppose now that it is valid for $n - 1$ and calculate $I_n(x)$. We can write

$$I_n(x) = \int_x^{+\infty} \lambda(x_1) I_{n-1}(x_1) dx_1. \tag{57}$$

From (56) we have to calculate two integrals. It is clear that

$$\int_x^{+\infty} \lambda(x_1) \exp[-m(x_1)] dx_1 = \exp[-m(x)] - \exp[-M] \tag{58}$$

because $\lambda(x_1)\exp[-m(x_1)]$ is the derivative of $-\exp[-m(x_1)]$. Furthermore

$$\int_x^{+\infty} \lambda(x_1) a^k(x_1) dx_1 = \frac{1}{k+1} a^{k+1}(x) \tag{59}$$

because the derivative of $a(x)$ is $-\lambda(x)$ and $a(+\infty) = 0$. Using (55) yields (56), which completes the proof. For $x = 0$, $m(x) = 0$, which yields (14).

APPENDIX D

CALCULATION OF $p_n(x)$ FOR A RENEWAL PROCESS

Let A_n be the event that there are at least n points in the interval $[0, T]$. Let E be the event that there is no point in $]0, x]$ and at least one point in $[x, x + dx[$. We have $p_n(x)dx = P(E|A_n)$ and

$$P(E|A_n) = \frac{P(E \cap A_n)}{P(A_n)} = \frac{P(E)P(A_n|E)}{P(A_n)}. \tag{60}$$

Noting that $P(E) = f(x)dx$ and that

$$P(A_n|E) = 1 - \sum_{i=0}^{n-2} \pi_i(T - x) \tag{61}$$

yields (40).

REFERENCES

[1] Snyder D. L., *Random Point Processes*, New York: Wiley, 1985.

[2] Snyder D. L. and Miller M. I. , *Random Point Processes in Time and Space*, New York: Springer, 1991.

[3] Blanc-Lapierrre A. and Picinbono B., *Fonctions Aléatoires*, Paris: Masson, 1991.

[4] Daley D.J. and Vere-Jones D., *An Introduction to the Theory of Point Processes, Vol I, Elementary Theory and Methods*, New-York: Springer, 2003.

[5] Lowen S. and Teich M. *Fractal-Based Point Processes*, New-York: Wiley, 2005.

[6] Bar-David I., "Communication under the Poisson regime," *IEEE Trans. Information Theory*, vol. IT-15, pp. 31-37, January 1969.

[7] Rubin I., "Regular point processes and their detection," *IEEE Trans. Information Theory*, vol. IT-18, pp. 547-557, September 1972.

[8] Rubin I., "Reduced memory likelihood processsing of point processes," *IEEE Trans. Information Theory*, vol. IT-20, pp. 729-738, November 1974.

[9] Fishman P. and Snyder D., "The statistical analysis of space-time point processes," *IEEE Trans. Information Theory*, vol. IT-22, pp. 257-274, May 1976.

[10] Hero A., "Timing estimation for a filtered Poisson process in Gaussian noise," *IEEE Trans. Information Theory*, vol. IT-37, pp. 92-106, January 1991.

[11] Saleh B. E. A., *Photoelectron Statistics*, Berlin: Springer-Verlag, 1978.

[12] Saleh B. E. A. and Teich M., *Fundamental of Photonics*, New York: Wiley, 1991.

[13] Feller W., *An Introduction to Probability Theory and its Applications, Vol I*, New-York: Wiley, 1962.

[14] Feller W., *An Introduction to Probability Theory and its Applications, Vol II*, New-York: Wiley, 1966.

[15] Bendjaballah C., "Correlation of time intervals for random point processes," *Journal of Optics B: Quantum and Semiclassical Optics*, vol. 7, pp. 215-223, August 2005.

[16] Bendjaballah C., "Nonclassical second-order photon statistics," *Journal of Physics B: At. Mol. Opt. Physics*, vol. 39, pp. 783-803, January 2006.

Figures captions

Figure 1. Probability density functions $p_n(x)$, $0 \leq n \leq 7$ for a time-limited Poisson process of constant density λ, $\lambda = T = 1$.

Figure 2. Probability density functions $p_n(x)$, $1 \leq n \leq 7$ for a time-limited Poisson process of density $\lambda(x) = \lambda_0[\exp(cx/T) - 1]$ (curve quoted λ), $\lambda_0 = T = 1$, $c = 1.25$.

Figure 3. Probability density functions $p_n(x)$, $1 \leq n \leq 7$ for an Erlang time-limited renewal process. PDF of the life time: $\lambda^2 y \exp(-\lambda y)$, $\lambda = 2$, $T = 1$.

Figure 4. Experimental measurements of $p_2(x)$ for conditions of Figures 1 and 3. Solid line: theory, dots: experiment.

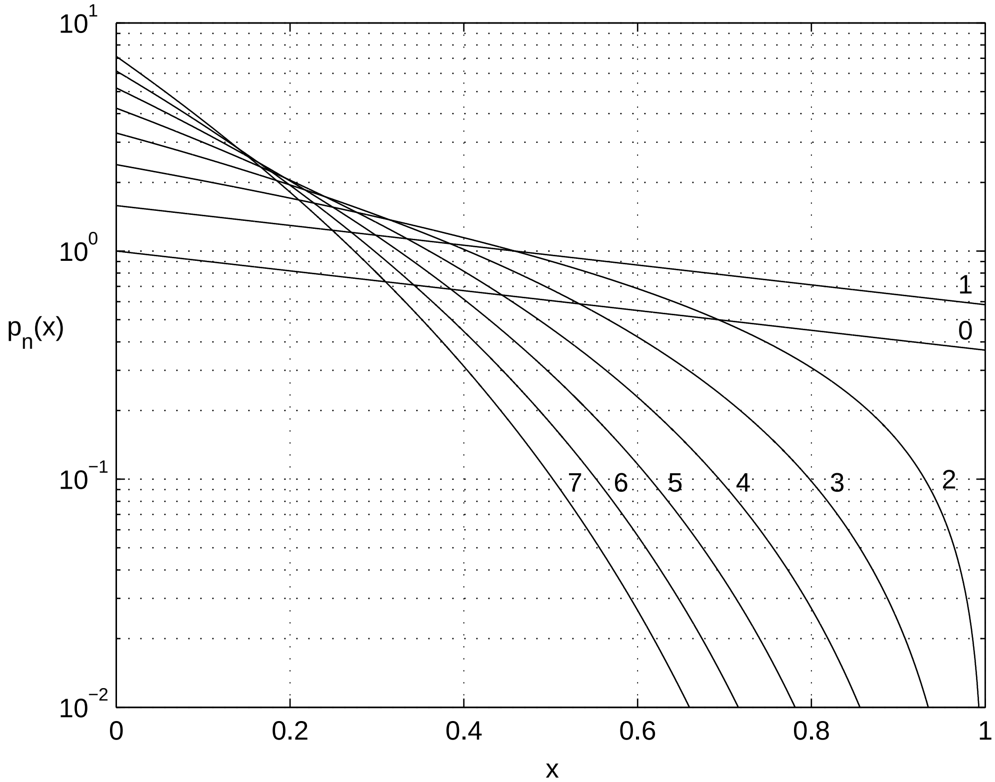

Fig. 1. .

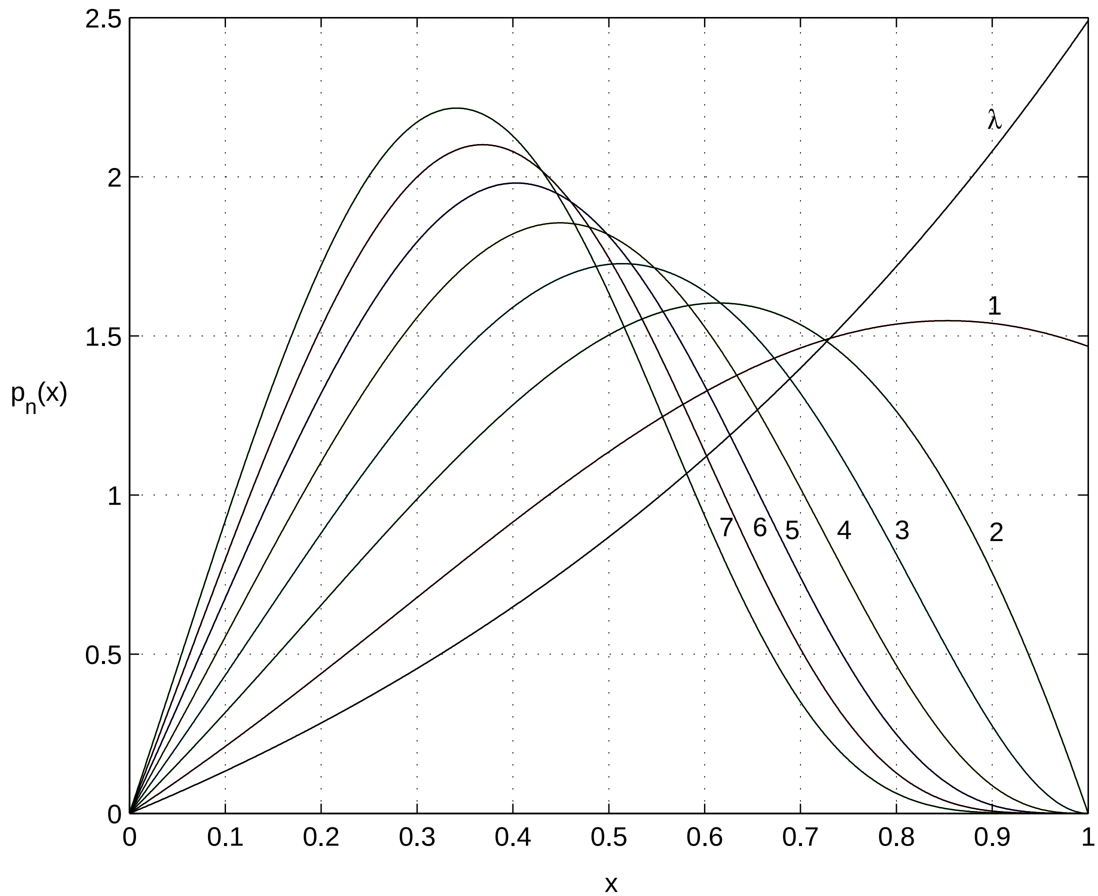

Fig. 2. .

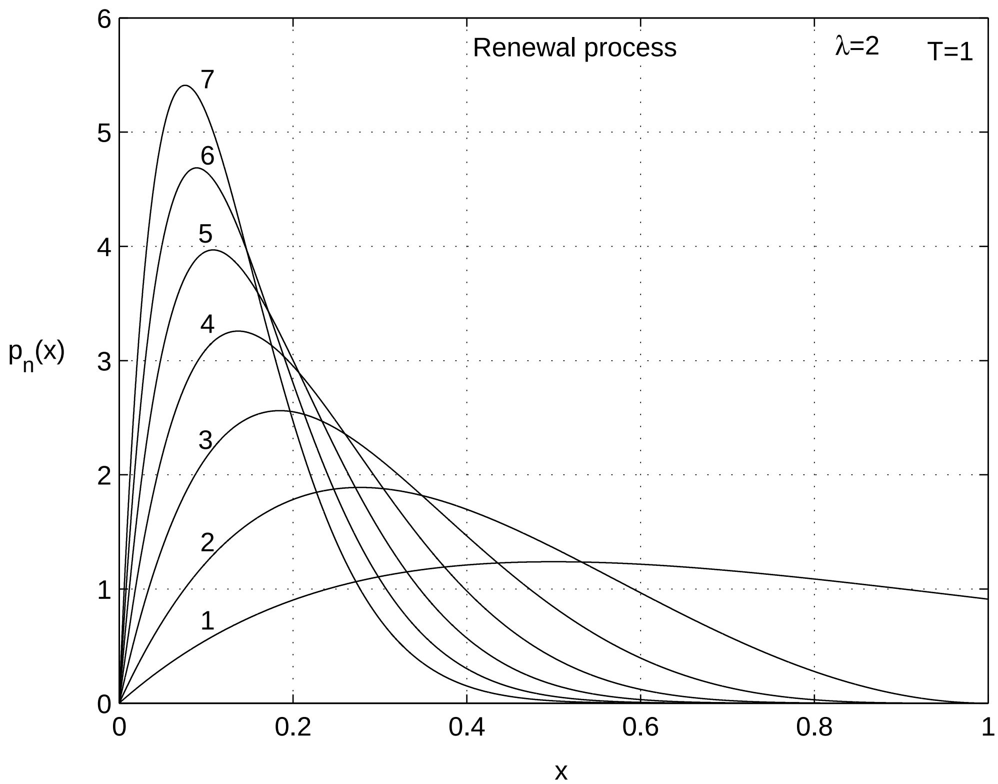

Fig. 3. .

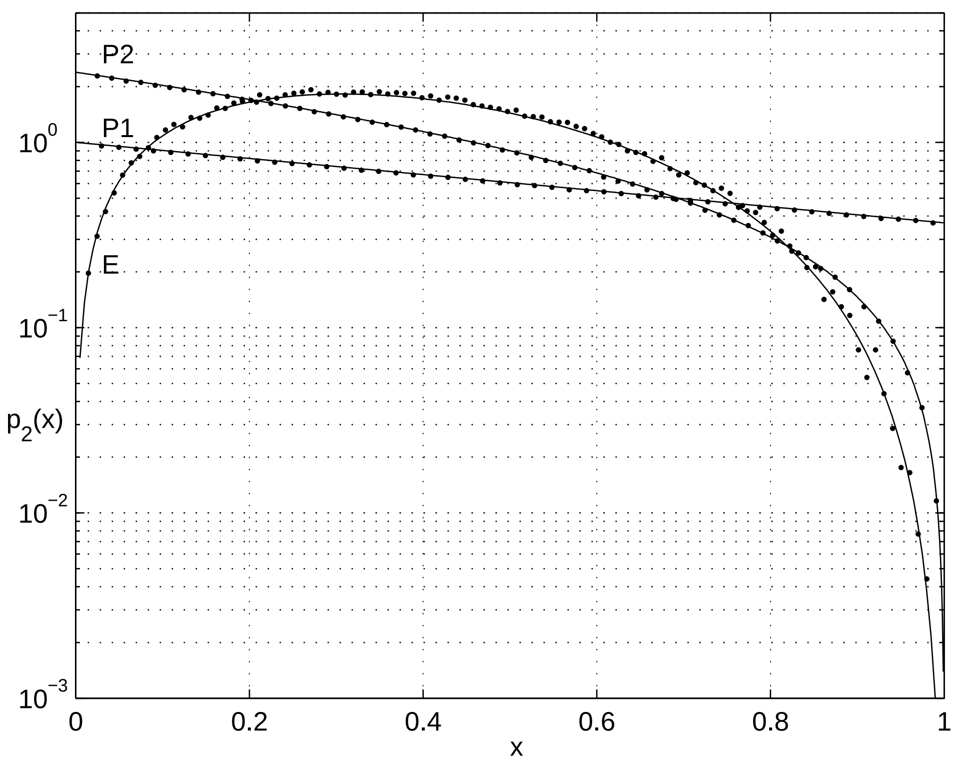

Fig. 4. .