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**THE REGULARIZED TRACE FORMULA FOR THE
STURM-LIOUVILLE EQUATION WITH SPECTRAL
PARAMETER IN THE BOUNDARY CONDITIONS**

NAMIG J. GULIYEV

ABSTRACT. The regularized trace formula of first order for the Sturm–Liouville equation with spectral parameter in the boundary conditions is obtained.

Consider the boundary value problem

$$-y''(x) + q(x)y(x) = \lambda y(x), \quad x \in [0, \pi], \quad (1)$$

$$\lambda(y'(0) - hy(0)) = h_1 y'(0) - h_2 y(0), \quad (2)$$

$$\lambda(y'(\pi) + Hy(\pi)) = H_1 y'(\pi) + H_2 y(\pi), \quad (3)$$

where $q(x) \in \mathcal{W}_2^1[0, \pi]$ is a real-valued function, $h, h_1, h_2, H, H_1, H_2 \in \mathbb{R}$ and

$$\delta := hh_1 - h_2 > 0, \quad \rho := HH_1 - H_2 > 0.$$

Let $\varphi(x, \lambda)$ be the solution of (1) satisfying the initial conditions

$$\varphi(0, \lambda) = h_1 - \lambda, \quad \varphi'(0, \lambda) = h_2 - \lambda h.$$

We denote $s := \sqrt{\lambda}$. Since $\varphi(x, \lambda)$ satisfies the integral equation

$$\varphi(x, \lambda) = (h_1 - s^2) \cos sx + (h_2 - hs^2) \frac{\sin sx}{s} + \int_0^x \frac{\sin[s(x-t)]}{s} q(t) \varphi(t, \lambda) dt$$

and the estimate

$$\varphi(x, \lambda) = O(|s|^2 e^{|\operatorname{Im} s|x}),$$

one can obtain the following representations:

$$\varphi(x, \lambda) = -s^2 \cos sx - \left(h + \frac{1}{2} \int_0^x q(t) dt \right) s \sin sx + O(e^{|\operatorname{Im} s|x}), \quad (4)$$

$$\begin{aligned} \varphi'(x, \lambda) = & s^3 \sin sx - \left(h + \frac{1}{2} \int_0^x q(t) dt \right) s^2 \cos sx - \\ & - \left(h_1 + \frac{q(x) + q(0)}{4} + \frac{h}{2} \int_0^x q(t) dt + \frac{1}{8} \left(\int_0^x q(t) dt \right)^2 \right) s \sin sx - \\ & - \frac{s}{4} \int_0^x q'(t) \sin[s(x-2t)] dt + O(e^{|\operatorname{Im} s|x}). \quad (5) \end{aligned}$$

Denote by $\chi(\lambda)$ the characteristic function of (1)–(3):

$$\chi(\lambda) := (\lambda - H_1)\varphi'(\pi, \lambda) + (\lambda H - H_2)\varphi(\pi, \lambda).$$

From (4) and (5) we have

$$\begin{aligned} \chi(\lambda) := s^5 \sin s\pi - \omega s^4 \cos s\pi - \omega_1 s^3 \sin s\pi - \frac{s^3}{4} \int_0^\pi q'(t) \sin[s(\pi - 2t)] dt + \\ + O(|s|^2 e^{|\operatorname{Im} s|\pi}), \end{aligned} \quad (6)$$

where

$$\begin{aligned} \omega &:= h + H + \frac{1}{2} \int_0^\pi q(t) dt, \\ \omega_1 &:= h_1 + H_1 + hH + \frac{q(\pi) + q(0)}{4} + \frac{h + H}{2} \int_0^\pi q(t) dt + \frac{1}{8} \left(\int_0^\pi q(t) dt \right)^2. \end{aligned}$$

The eigenvalues of the problem (1)–(3) are real and coincide with the zeros of the function $\chi(\lambda)$. From (6) we obtain

$$s_n = n - 2 + \frac{\omega}{\pi(n - 2)} + \frac{\zeta_n}{n^2}, \quad \{\zeta_n\} \in l_2.$$

Then

$$\lambda_n = s_n^2 = (n - 2)^2 + \frac{2\omega}{\pi} + \frac{\xi_n}{n}, \quad \{\xi_n\} \in l_2. \quad (7)$$

Therefore

$$s_\lambda := \lambda_0 + \lambda_1 + \sum_{n=2}^{\infty} \left(\lambda_n - (n - 2)^2 - \frac{2\omega}{\pi} \right) < \infty. \quad (8)$$

The series (8) is called *the regularized trace of first order* (or *the first regularized trace*) for the problem (1)–(3). Analogously, one can define *the second, the third* etc. regularized traces. Note that the second regularized trace formula for the Sturm–Liouville problem with spectral parameter in a boundary condition has been found in [1].

Since $\chi(\lambda)$ is an entire function of order $1/2$, from Hadamard's theorem (see [2, Section 4.2]), using (6) we obtain

$$\chi(\lambda) = \pi(\lambda - \lambda_0)(\lambda - \lambda_1)(\lambda - \lambda_2) \prod_{n=3}^{\infty} \frac{\lambda_n - \lambda}{(n - 2)^2} \quad (9)$$

Let $\lambda = -\mu^2$. We calculate the sum s_λ of the series (8) by comparing the asymptotic expressions obtained from formulas (6) and (9) as $\mu \rightarrow \infty$. From the formula (9) we have

$$\begin{aligned} \chi(-\mu^2) &= -\pi(\mu^2 + \lambda_0)(\mu^2 + \lambda_1)(\mu^2 + \lambda_2) \prod_{n=3}^{\infty} \frac{\lambda_n + \mu^2}{(n - 2)^2} = \\ &= -\frac{(\mu^2 + \lambda_0)(\mu^2 + \lambda_1)(\mu^2 + \lambda_2) \operatorname{sh} \pi\mu}{\mu} \Phi(\mu), \end{aligned} \quad (10)$$

where

$$\Phi(\mu) = \prod_{n=1}^{\infty} \left(1 - \frac{n^2 - \lambda_{n+2}}{\mu^2 + n^2} \right).$$

We need the formulas (see [3])

$$\sum_{k=2}^{\infty} \frac{1}{k} \sum_{n=1}^{\infty} \frac{|n^2 - \lambda_{n+2}|^k}{(\mu^2 + n^2)^k} = o\left(\frac{1}{\mu^2}\right) \quad (11)$$

and

$$\sum_{n=1}^{\infty} \frac{1}{\mu^2 + n^2} = \frac{\pi}{2\mu} - \frac{1}{2\mu^2} + o\left(\frac{1}{\mu^2}\right). \quad (12)$$

Using (7) and (12) we have

$$\begin{aligned} \frac{1}{\mu^2} \sum_{n=1}^{\infty} \left(\lambda_{n+2} - n^2 - \frac{2\omega}{\pi} \right) \frac{n^2}{\mu^2 + n^2} &= \frac{1}{\mu^2} \sum_{n=1}^{\infty} \frac{n \xi_{n+2}}{\mu^2 + n^2} \leq \\ &\leq \frac{1}{\mu^2} \sqrt{\sum_{n=3}^{\infty} \xi_n^2} \sqrt{\sum_{n=1}^{\infty} \frac{n^2}{(\mu^2 + n^2)^2}} \leq \frac{\Omega}{\mu^2} \sqrt{\sum_{n=1}^{\infty} \frac{\mu^2 + n^2}{(\mu^2 + n^2)^2}} = \\ &= \frac{\Omega}{\mu^2} \sqrt{\sum_{n=1}^{\infty} \frac{1}{\mu^2 + n^2}} = o\left(\frac{1}{\mu^2}\right), \quad (13) \end{aligned}$$

where

$$\Omega := \sqrt{\sum_{n=3}^{\infty} \xi_n^2} < \infty.$$

From (11), (12) and (13) we calculate:

$$\begin{aligned} \ln \Phi(\mu) &= \sum_{n=1}^{\infty} \left(1 - \frac{n^2 - \lambda_{n+2}}{\mu^2 + n^2} \right) = - \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{n^2 - \lambda_{n+2}}{\mu^2 + n^2} \right)^k = \\ &= - \sum_{k=1}^{\infty} \frac{1}{k} \sum_{n=1}^{\infty} \left(\frac{n^2 - \lambda_{n+2}}{\mu^2 + n^2} \right)^k = - \sum_{n=1}^{\infty} \frac{n^2 - \lambda_{n+2}}{\mu^2 + n^2} + o\left(\frac{1}{\mu^2}\right) = \\ &= \frac{2\omega}{\pi} \sum_{n=1}^{\infty} \frac{1}{\mu^2 + n^2} + \sum_{n=1}^{\infty} \left(\lambda_{n+2} - n^2 - \frac{2\omega}{\pi} \right) \frac{1}{\mu^2 + n^2} + o\left(\frac{1}{\mu^2}\right) = \frac{\omega}{\mu} - \frac{\omega}{\pi\mu^2} + \\ &+ \frac{1}{\mu^2} \sum_{n=1}^{\infty} \left(\lambda_{n+2} - n^2 - \frac{2\omega}{\pi} \right) - \frac{1}{\mu^2} \sum_{n=1}^{\infty} \left(\lambda_{n+2} - n^2 - \frac{2\omega}{\pi} \right) \frac{n^2}{\mu^2 + n^2} + \\ &+ o\left(\frac{1}{\mu^2}\right) = \frac{\omega}{\mu} - \frac{\omega}{\pi\mu^2} + \frac{1}{\mu^2} \left(s_{\lambda} - \lambda_0 - \lambda_1 - \lambda_2 + \frac{2\omega}{\pi} \right) + o\left(\frac{1}{\mu^2}\right) = \\ &= \frac{\omega}{\mu} + \frac{1}{\mu^2} \left(s_{\lambda} - \lambda_0 - \lambda_1 - \lambda_2 + \frac{\omega}{\pi} \right) + o\left(\frac{1}{\mu^2}\right). \end{aligned}$$

Therefore

$$\Phi(\mu) = 1 + \frac{\omega}{\mu} + \frac{1}{\mu^2} \left(s_{\lambda} - \lambda_0 - \lambda_1 - \lambda_2 + \frac{\omega}{\pi} + \frac{\omega^2}{2} \right) + o\left(\frac{1}{\mu^2}\right)$$

and from (10)

$$\chi(-\mu^2) = \frac{e^{\pi\mu}}{2} \left(-\mu^5 - \omega\mu^4 + \mu^3 \left(s_\lambda + \frac{\omega}{\pi} + \frac{\omega^2}{2} \right) + o(\mu^3) \right). \quad (14)$$

The estimate (6) implies

$$\chi(-\mu^2) = \frac{e^{\pi\mu}}{2} (-\mu^5 - \omega\mu^4 + \omega_1\mu^3 + o(\mu^3)). \quad (15)$$

Comparing (14) and (15) we obtain the following trace formula:

$$\begin{aligned} \lambda_0 + \lambda_1 + \sum_{n=2}^{\infty} \left(\lambda_n - (n-2)^2 - \frac{2\omega}{\pi} \right) &= \\ &= h_1 + H_1 + \frac{q(\pi) + q(0)}{4} - \frac{h + H}{\pi} - \frac{h^2 + H^2}{2} - \frac{1}{2\pi} \int_0^\pi q(t) dt. \end{aligned}$$

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