An optimal reciprocally convex inequality and an augmented Lyapunov–Krasovskii functional for stability of linear systems with time-varying delay

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An optimal reciprocally convex inequality and a new Lyapunov-Krasovskii functional for stability analysis of linear systems with time-varying delay[☆]

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Abstract

This paper is concerned with stability of a linear system with a time-varying delay. First, an optimal reciprocally convex inequality is proposed. Compared with the extended reciprocally convex inequality recently reported, the optimal reciprocally convex inequality not only provides an optimal bound for the reciprocally convex combination, but also introduces less slack matrix variables. Second, a new Lyapunov-Krasovskii functional is tailored for the use of auxiliary function-based integral inequality. Third, based on the optimal reciprocally convex inequality and the new Lyapunov-Krasovskii functional, a stability criterion is derived for the system under study. Finally, two well-studied numerical examples are given to show that the obtained stability criterion can produce a larger upper bound of the time-varying delay than some existing methods.

Keywords: Time-delay systems, stability, reciprocally convex inequality, Lyapunov-Krasovskii functional.

1. Introduction

Consider the system with a time-varying delay described by

$$\begin{cases} \dot{x}(t) = Ax(t) + A_d x(t-h(t)) \\ x(\theta) = \phi(\theta), \quad \theta \in [-h_M, 0], \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the system state; A and A_d are real $n \times n$ constant matrices; $h(t)$ is the time-varying delay satisfying

$$0 \le h(t) \le h_M, \quad d_m \le \dot{h}(t) \le d_M < \infty \tag{2}$$

with h_M, d_m and d_M known scalars; and $\phi(\theta)$ is an initial condition. To begin with, for simplicity of presentation, we denote

$$\begin{cases} \rho_1(t) := \int_{t-h_M}^{t-h(t)} \frac{x(s)}{h_M - h(t)} ds, & \rho_2(t) := \int_{t-h_M}^{t-h(t)} \frac{(t-h(t)-s)x(s)}{(h_M-h(t))^2} ds \\ \rho_3(t) := \int_{t-h(t)}^{t} \frac{x(s)}{h(t)} ds, & \rho_4(t) := \int_{t-h(t)}^{t} \frac{(t-s)x(s)}{h^2(t)} ds. \end{cases} \tag{3}$$

The Lyapunov-Krasovskii functional method plus integral inequalities is regarded as a powerful tool for deriving a maximum upper bound h_M that the system (1) can tolerate and maintain stability (He *et al.* , 2007; Gu & Liu , 2009; Gu , 2013; Xu *et al.* , 2015). This method has gained increasing attention especially since the Jensen integral inequality is improved by the Wirtinger-based integral inequality (Seuret & Gouaisbaut , 2013), and much effort has been made in seeking less conservative stability criteria for the system (1), e.g. Zhang & Han (2015), Zhang *et al.* (2016), Zeng *et al.* (2015a), Kim (2016). It should be mentioned that the Wirtinger-based integral inequality is further improved by the auxiliary function-based integral inequality (Seuret & Gouaisbaut , 2015; Park *et al.* , 2015). However, it is found that, if taking some Lyapunov-Krasovskii functional, the stability criterion based on the auxiliary function-based integral inequality may be of the same conservatism as the one based on the Wirtinger-based integral inequality. To make it clear, we take the proof of Theorem 7 in Seuret & Gouaisbaut (2013) for example, where the Lyapunov-Krasovskii functional is chosen as

$$\tilde{V}(x_t, \dot{x}_t) = \tilde{x}^T(t) P \tilde{x}(t) + \tilde{V}_1(t, x_t) + V_2(t, \dot{x}_t) \tag{4}$$

where $\tilde{x}(t) := \mathrm{col}\{x(t), h(t)\rho_3(t), (h_M - h(t))\rho_1(t)\}$, and

$$\tilde{V}_1(t, x_t) = \int_{t-h(t)}^{t} x^T(s) Q x(s) ds + \int_{t-h_m}^{t} x^T(s) S x(s) ds \tag{5}$$

$$V_2(t, \dot{x}_t) = \int_{t-h_M}^{t} \int_{\theta}^{t} \dot{x}^T(s) R \dot{x}(s) ds d\theta \tag{6}$$

Applying the Wirtinger-based integral inequality, it is proven that (Seuret & Gouaisbaut , 2013)

$$\dot{\tilde{V}}(x_t, \dot{x}_t) \le \xi_1^T(t) \Phi(h(t), \dot{h}(t)) \xi_1(t), \tag{7}$$

where $\xi_1(t) = \mathrm{col}\{x(t), x(t-h(t)), x(t-h_M), \rho_3(t), \rho_1(t)\}$ with $\rho_1(t)$ and $\rho_3(t)$ defined in (3), and $\Phi(h(t), \dot{h}(t))$ is defined in Seuret & Gouaisbaut (2013). However, use the auxiliary function-based integral inequality (i.e. Lemma 1 on the next page) and the reciprocally convex approach (Park *et al.* , 2011) to get

$$\dot{\tilde{V}}(x_t, \dot{x}_t) \le \xi_1^T(t) \Phi(h(t), \dot{h}(t)) \xi_1(t) - \xi_2^T(t) \Psi \xi_2(t) \tag{8}$$

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where $\xi_2(t) := \mathrm{col}\{x(t-h(t))-x(t-h_M)-6\rho_1(t)+12\rho_2(t), x(t)-x(t-h(t))-6\rho_3(t)+12\rho_4(t)\}$ with $\rho_2(t)$ and $\rho_4(t)$ defined in (3), and $\mathit{\Psi} := \frac{1}{h_M}\left[\begin{smallmatrix} 5R & \tilde{S} \\ \tilde{S}^T & 5R \end{smallmatrix}\right] \geq 0$. Denote $\zeta_1(t) = \mathrm{col}\{\xi_1(t), 0, 0\}$, $\zeta_2(t) = \mathrm{col}\{\xi_1(t), \rho_4(t), \rho_2(t)\}$ and $\mathit{\Gamma} = \left[\begin{smallmatrix} 0 & I & -I & 0 & -6I & 0 & 12I \\ I & -I & 0 & -6I & 0 & 12I & 0 \end{smallmatrix}\right]$. Then (8) can be rewritten as

$$\dot{\tilde{V}}(x_t, \dot{x}_t) \leq \zeta_1^T(t)\mathrm{diag}\{\mathit{\Phi}(h(t), \dot{h}(t)), -I\}\zeta_1(t) - \zeta_2^T(t)\mathit{\Gamma}^T\mathit{\Psi}\mathit{\Gamma}\zeta_2(t) \tag{9}$$

Notice that $\zeta_1(t)$ and $\zeta_2(t)$ are *linearly independent* since $\zeta_1(t)$ does not include the vectors $\rho_2(t)$ and $\rho_4(t)$. Thus, the stability criteria derived from (7) and (9) both can be given by the same form as $\mathit{\Phi}(h(t), \dot{h}(t)) < 0$ due to $\mathit{\Psi} \geq 0$. Therefore, the use of the auxiliary function-based integral inequality may not reduce the conservatism of the obtained stability criterion.

From the above analysis, it is clear to see that, in order to derive less conservative stability criteria, one should construct a *proper* Lyapunov-Krasovskii functional such that the corresponding vector $\zeta_1(t)$ is linearly dependent on the vector $\zeta_2(t)$ in (9). To construct such a Lyapunov-Krasovskii functional is the first motivation of the study.

On the other hand, the reciprocally convex approach is widely used in the stability analysis of linear systems with time-varying delay. It is because, as stated in Park *et al*. (2011), the reciprocally convex approach can produce stability criteria with less decision variables while the conservatism will not be increased. Recently, the reciprocally convex inequality in Park *et al*. (2011) is extended in Seuret & Gouaisbaut (2016) by introducing four slack matrix variables. Although the extended reciprocally convex inequality is helpful for deriving a less conservative stability condition, the introduction of four slack matrix variables undoubtedly increases the computation complexity of the obtained stability criterion. How to reduce the slack matrix variables of the extended reciprocally convex inequality is a significant issue, which is the second motivation of the study.

In this paper, we focus on the stability analysis of linear systems with time-varying delay described by (1). First, an optimal reciprocally convex inequality is proposed, which provides an optimal bound for the reciprocally convex combination, while less slack matrix variables are introduced if compared with the extended reciprocally convex inequality (Seuret & Gouaisbaut , 2016). Second, a new Lyapunov-Krasovskii functional is tailored for the use of the auxiliary function-based integral inequality. On the one hand, the terms $\rho_2(t)$ and $\rho_4(t)$ appear in the derivative of the Lyapunov-Krasovskii functional, which means that the auxiliary function-based integral inequality can be used to formulate less conservative stability criteria; and on the other hand, the quadratic $\tilde{x}^T(t)P\tilde{x}(t)$ in (4) is deleted. Instead, $\tilde{V}_1(t, x_t)$ in (5) is augmented so that the relationship between $\rho_j(t)$ $(j = 1, \cdots, 4)$ and the other vectors is enhanced in the derivative of the Lyapunov-Krasovskii functional. Third, the optimal reciprocally convex inequality and the new Lyapunov-Krasovskii functional are employed to derive a new stability criterion for the system (1), whose effectiveness is demonstrated through two well-used numerical examples.

Notations: $\lambda_{\max}(Q)$ ($\lambda_{\min}(Q)$) stands for the maximum (minimum) eigenvalue of the matrix Q; $\mathrm{Sym}\{A\} = A + A^T$.

To end this section, we introduce some lemmas, which are useful in the stability analysis.

Lemma 1. *(Seuret & Gouaisbaut , 2015; Park* et al. *, 2015). For any $n \times n$ constant real matrix $R > 0$, two scalars r_1 and r_2 with $r_2 > r_1$, and a vector-valued function $\omega : [r_1, r_2] \to \mathbb{R}^n$ such that the integrations below are well defined, then*

$$\int_{r_1}^{r_2} \dot{\omega}^T(s)R\dot{\omega}(s)ds \geq \frac{1}{r_2 - r_1}\zeta^T(r_1, r_2)\tilde{R}\zeta(r_1, r_2) \tag{10}$$

where $\tilde{R} := diag\{R, 3R, 5R\}$ and $\zeta(r_1, r_2) := col\{\upsilon_0, \upsilon_1, \upsilon_2\}$ with $\upsilon_0 := \omega(r_2) - \omega(r_1)$ and

$$\begin{cases} \upsilon_1 := \omega(r_2) + \omega(r_1) - \frac{2}{r_2 - r_1}\int_{r_1}^{r_2}\omega(s)ds \\ \upsilon_2 := \upsilon_0 - \frac{6}{r_2 - r_1}\int_{r_1}^{r_2}\omega(s)ds + \frac{12}{(r_2 - r_1)^2}\int_{r_1}^{r_2}(r_2 - s)\omega(s)ds \end{cases} \tag{11}$$

Lemma 2. *(Kim , 2016). For a given quadratic function $\ell(s) = a_2 s^2 + a_1 s + a_0$, where $a_i \in \mathbb{R}$ $(i = 0, 1, 2)$, if the following inequalities hold*

$$(i).\ \ell(0) < 0; \quad (ii).\ \ell(h) < 0; \quad (iii).\ -h^2 a_2 + \ell(0) < 0 \tag{12}$$

one has $\ell(s) < 0$ for $\forall s \in [0, h]$.

Lemma 3. *(Fridman , 2014). The system (1) is asymptotically stable if the exists a quadratic Lyapunov-Krasovskii functional $V(t, \phi, \dot{\phi})$ such that for some $\varepsilon_i > 0$ $(i = 1, 2, 3)$*

$$\varepsilon_1\|\phi(0)\|^2 \leq V(t, \phi, \dot{\phi}) \leq \varepsilon_2\|\phi\|_W^2$$
$$\dot{V}(t, \phi, \dot{\phi}) \leq -\varepsilon_3\|\phi(0)\|^2$$

where $\|\phi\|_W^2 = \|\phi(0)\|^2 + \int_{-h_M}^0 \|\phi(s)\|^2 ds + \int_{-h_M}^0 \|\dot{\phi}(\theta)\|^2 d\theta$.

2. Main results

In this section, we will present our main results. An optimal reciprocally convex inequality and a new Lyapunov-Krasovskii functional are introduced, based on which a novel stability criterion is derived for the system (1).

2.1. An optimal reciprocally convex inequality

Recently, the reciprocally convex inequality proposed in Park *et al*. (2011) is extended by introducing some slack matrix variables, which is given as follows.

Lemma 4. *(Seuret & Gouaisbaut , 2016). Let $\mathcal{R}_1, \mathcal{R}_2 \in \mathbb{R}^{m\times m}$ be real symmetric positive definite matrices and $\varpi_1, \varpi_2 \in \mathbb{R}^m$ and a scalar $\alpha \in (0, 1)$. If there exist real symmetric matrices $X_1, X_2 \in \mathbb{R}^{m\times m}$ and real matrices $Y_1, Y_2 \in \mathbb{R}^{m\times m}$ such that*

$$\begin{bmatrix} \mathcal{R}_1 - X_1 & Y_1 \\ Y_1^T & \mathcal{R}_2 \end{bmatrix} \geq 0, \ \begin{bmatrix} \mathcal{R}_1 & Y_2 \\ Y_2^T & \mathcal{R}_2 - X_2 \end{bmatrix} \geq 0 \tag{13}$$

the following inequality holds

$$\mathscr{F}(\alpha) := \frac{1}{\alpha}\varpi_1^T\mathcal{R}_1\varpi_1 + \frac{1}{1-\alpha}\varpi_2^T\mathcal{R}_2\varpi_2 \geq \mathscr{G}(X_1, X_2) \tag{14}$$

$$\begin{aligned} \mathscr{G}(X_1, X_2) := {}& \varpi_1^T[\mathcal{R}_1 + (1-\alpha)X_1]\varpi_1 + \varpi_2^T(\mathcal{R}_2 + \alpha X_2)\varpi_2 \\ &+ 2\varpi_1^T[\alpha Y_1 + (1-\alpha)Y_2]\varpi_2 \end{aligned} \tag{15}$$

Lemma 4 presents a general lower bound $\mathscr{G}(X_1, X_2)$ for the reciprocally convex combination $\mathscr{F}(\alpha)$ by introducing four slack matrix variables, which bring additional degree of freedom if compared with the one in Park *et al.* (2011). However, more slack matrix variables usually lead to high computation complexity. The following analysis gives an optimal lower bound for $\mathscr{F}(\alpha)$ from the set $\{\mathscr{G}(X_1, X_2)|(X_1, X_2)$ satisfies (13)$\}$.

Theorem 1. *Let $\mathcal{R}_1, \mathcal{R}_2 \in \mathbb{R}^{m\times m}$ be real symmetric positive definite matrices and $\varpi_1, \varpi_2 \in \mathbb{R}^m$ and a scalar $\alpha \in (0,1)$. Then for any $Y_1, Y_2 \in \mathbb{R}^{m\times m}$, the following inequality holds*

$$\begin{aligned}\mathscr{F}(\alpha) \geq\ & \varpi_1^T[\mathcal{R}_1 + (1-\alpha)(\mathcal{R}_1 - Y_1\mathcal{R}_2^{-1}Y_1^T)]\varpi_1 \\ &+ \varpi_2^T[\mathcal{R}_2 + \alpha(\mathcal{R}_2 - Y_2^T\mathcal{R}_1^{-1}Y_2)]\varpi_2 \\ &+ 2\varpi_1^T[\alpha Y_1 + (1-\alpha)Y_2]\varpi_2 \end{aligned} \tag{16}$$

Proof. Since $\mathcal{R}_1 > 0$ and $\mathcal{R}_2 > 0$, the matrix inequalities in (13) are equivalent to, respectively,

$$\mathcal{R}_1 - X_1 - Y_1\mathcal{R}_2^{-1}Y_1^T \geq 0, \quad \mathcal{R}_2 - X_2 - Y_2^T\mathcal{R}_1^{-1}Y_2 \geq 0 \tag{17}$$

Denote $X_{10} = \mathcal{R}_1 - Y_1\mathcal{R}_2^{-1}Y_1^T$ and $X_{20} = \mathcal{R}_2 - Y_2^T\mathcal{R}_1^{-1}Y_2$. Then it follows from (17) that $X_{10} \geq X_1$ and $X_{20} \geq X_2$, which leads to $\mathscr{G}(X_{10}, X_{20}) \geq \mathscr{G}(X_1, X_2)$, where $\mathscr{G}(X_1, X_2)$ is defined in (15). Since (X_{10}, X_{20}) satisfies (13), by Lemma 4, one obtains

$$\mathscr{F}(\alpha) \geq \mathscr{G}(X_{10}, X_{20}) \geq \mathscr{G}(X_1, X_2) \tag{18}$$

which completes the proof.

Remark 1. Compared with Lemma 4, Theorem 1 provides an optimal lower bound $\mathscr{G}(X_{10}, X_{20})$ for the reciprocally convex combination $\mathscr{F}(\alpha)$. It is worth pointing out that the slack matrix variables X_1 and X_2 in (13) are *removed* from Theorem 1. Moreover, if one sets $Y_1 = Y_2 = S$, it is easy to show that $\mathscr{G}(X_{10}, X_{20})$ is *greater* than $\varpi_1^T\mathcal{R}_1\varpi_1 + 2\varpi_1^T S\varpi_2 + \varpi_2^T\mathcal{R}_2\varpi_2$, which is originally proposed in Park *et al.* (2011).

Remark 2. In Seuret & Gouaisbaut (2016), it is suggested to reduce the number of decision variables in Lemma 4 by imposing a constraint $X_1 = X_2$ on X_1 and X_2. However, in the case where the dimensions of $\mathcal{R}_1$ and $\mathcal{R}_2$ are not compatible, one cannot set $X_1 = X_2$; and in the case where $\mathcal{R}_1 = \mathcal{R}_2$, the constraint $X_1 = X_2$ only makes the lower bound $\mathscr{G}(X_1, X_1)$ deviate away from the optimal one $\mathscr{G}(X_{10}, X_{20})$ due to that, in most cases, X_{10} is not equal to X_{20} even though one sets $Y_1 = Y_2$.

2.2. An augmented Lyapunov-Krasovskii functional

In this section, we introduce a Lyapunov-Krasovskii functional candidate as

$$V(t, x_t, \dot{x}_t) = V_1(t, x_t) + h_M V_2(t, \dot{x}_t) \tag{19}$$

where $x_t := x(t+\theta)$, $\theta \in [-h_M, 0]$; $V_2(t, \dot{x}_t)$ is defined in (6); and

$$\begin{aligned} V_1(t, x_t) :=& \int_{t-h(t)}^{t} \eta_1^T(t,s)Q_1\eta_1(t,s)ds \\ &+ \int_{t-h_M}^{t-h(t)} \eta_2^T(t,s)Q_2\eta_2(t,s)ds \end{aligned} \tag{20}$$

where $Q_1 > 0$, $Q_2 > 0$ and $R > 0$ are to be determined, and

$$\begin{aligned} \eta_1(t,s) &:= \mathrm{col}\left\{\dot{x}(s), x(s), \eta_0(t), \textstyle\int_{t-h(t)}^{s} x(\theta)d\theta\right\} \\ \eta_2(t,s) &:= \mathrm{col}\left\{\dot{x}(s), x(s), \eta_0(t), \textstyle\int_{t-h_M}^{s} x(\theta)d\theta\right\} \\ \eta_0(t) &:= \mathrm{col}\left\{x(t), x(t-h(t)), x(t-h_M)\right\} \end{aligned}$$

It is not difficult to verify that there exist two constants $c_1 > 0$ and $c_2 > 0$ such that

$$c_1\|x_t(0)\|_2^2 \leq V(t, x_t, \dot{x}_t) \leq c_2\|x_t\|_W^2. \tag{21}$$

In fact, denote $\epsilon_0 = \min\{\lambda_{\min}(Q_1), \lambda_{\min}(Q_2)\}$. Then

$$\begin{aligned} V_1(t, x_t) &\geq \int_{t-h(t)}^{t} \epsilon_0 x^T(t)x(t)ds + \int_{t-h_M}^{t-h(t)} \epsilon_0 x^T(t)x(t)ds \\ &= h_M\epsilon_0 x^T(t)x(t) \triangleq c_1 x^T(t)x(t) \end{aligned} \tag{22}$$

On the other hand, denote $\epsilon_1 = \max\{\lambda_{\max}(Q_1), \lambda_{\max}(Q_2)\}$. Then

$$\begin{aligned} V_1(t, x_t) \leq\ & \epsilon_1 \int_{t-h_M}^{t} \left[\|\dot{x}(s)\|_2^2 + \|x(s)\|_2^2 + \|\eta_0(t)\|_2^2\right] ds \\ &+ \epsilon_1 \int_{t-h(t)}^{t}\int_{t-h(t)}^{s} \|x^T(\theta)\|_2 d\theta \int_{t-h(t)}^{s} \|x(\theta)\|_2 d\theta ds \\ &+ \epsilon_1 \int_{t-h_M}^{t-h(t)}\int_{t-h_M}^{s} \|x^T(\theta)\|_2 d\theta \int_{t-h_M}^{s} \|x(\theta)\|_2 d\theta ds \\ \leq\ & \epsilon_1 \int_{t-h_M}^{t} \left[\|\dot{x}(s)\|_2^2 + 3\|x(s)\|_2^2\right] ds + \epsilon_1 h_M \|x_t(0)\|_2^2 \\ &+ \epsilon_1 \int_{t-h(t)}^{t} h_M \int_{t-h_M}^{t} \|x(\theta)\|_2^2 d\theta ds \\ &+ \epsilon_1 \int_{t-h_M}^{t-h(t)} h_M \int_{t-h_M}^{t} \|x(\theta)\|_2^2 d\theta ds \\ \leq\ & \epsilon_1 h_M \|x_t(0)\|_2^2 + \epsilon_1 \int_{t-h_M}^{t} \left[\|\dot{x}(s)\|_2^2 + (h_M^2+3)\|x(s)\|_2^2\right] ds \\ V_2(t, \dot{x}_t) \leq\ & h_M\lambda_{\max}(R) \int_{t-h_M}^{t} \|\dot{x}(s)\|_2^2 ds \end{aligned}$$

which follows that

$$\begin{aligned} V(t, x_t, \dot{x}_t) \leq\ & \epsilon_1 h_M \|x_t(0)\|_2^2 + (h_M^2+3)\epsilon_1 \int_{t-h_M}^{t} \|x(s)\|_2^2 ds \\ &+ [h_M^2\lambda_{\max}(R) + \epsilon_1] \int_{t-h_M}^{t} \|\dot{x}(s)\|_2^2 ds \end{aligned}$$

Thus, there exists a constant $c_2 := \max\{\epsilon_1 h_M,\ h_M^2\lambda_{\max}(R) + \epsilon_1,\ (h_M^2+3)\epsilon_1\}$, such that $V(t, x_t) \leq c_2\|x_t\|_W^2$.

Compared with some existing Lyapunov-Krasovskii functionals, e.g. (4) and He *et al.* (2005), Seuret & Gouaisbaut (2013), Kim (2016), $V(t, x_t)$ in (19) has the following characteristics:

i) The quadratic term, say $x^T(t)Px(t)$ or $\tilde{x}^T(t)P\tilde{x}(t)$ in (4), is deleted. Instead, an augmented integral term $V_1(t, x_t)$ is introduced. As a result, the system states $x(s)$, $x(t)$, $x(t-d(t))$

and $x(t-h_M)$ are closely coupled by the matrices Q_1 and Q_2. Such coupling can enhance the relationship between $x(t)$ and the other delayed state vectors in the derivative of $V_1(t, x_t)$; and

ii) Taking the time-derivative of $V_1(t, x_t)$ yields two important integrals $\rho_2(t)$ and $\rho_4(t)$, which enable us to employ the integral inequality (10) to derive less conservative stability conditions.

2.3. A new stability criterion

In this section, based on Theorem 1 and the Lyapunov-Krasovskii functional (19), we establish and state a novel delay-dependent stability criterion for the system (1).

Proposition 1. *For given scalars h_M, d_m and d_M, the system (1) is asymptotically stable if there exist real matrices $Q_1 > 0$, $Q_2 > 0$, $R > 0$, Y_1 and Y_2 with appropriate dimensions such that*

$$\begin{bmatrix} \Upsilon_1(0,d)|_{d=d_m,d_M} & \Gamma_2^T Y_2^T \\ Y_2\Gamma_2 & -\tilde{R} \end{bmatrix} < 0, \quad \begin{bmatrix} \Upsilon(h_M,d)|_{d=d_m,d_M} & \Gamma_1^T Y_1 \\ Y_1^T\Gamma_1 & -\tilde{R} \end{bmatrix} < 0 \quad (23)$$

$$\begin{bmatrix} [-h_M^2\mathcal{G}_0(d) + \Upsilon_1(0,d)]|_{d=d_m,d_M} & \Gamma_2^T Y_2^T \\ Y_2\Gamma_2 & -\tilde{R} \end{bmatrix} < 0 \quad (24)$$

where $\tilde{R} = diag\{R, 3R, 5R\}$ and

$$\begin{aligned}
\Upsilon_1(h(t),\dot{h}(t)) :=& [\mathscr{C}_{11}+h(t)\mathscr{C}_{12}]^T Q_1[\mathscr{C}_{11}+h(t)\mathscr{C}_{12}] \\
&+ h_M^2\mathscr{C}_0^T R\mathscr{C}_0 - \mathscr{C}_5^T Q_2\mathscr{C}_5 - (1-\dot{h}(t))\mathscr{C}_2^T Q_1\mathscr{C}_2 - (2-\alpha)\Gamma_1^T\tilde{R}\Gamma_1 \\
&+ (1-\dot{h}(t))[\mathscr{C}_{41}+(h_M-h(t))\mathscr{C}_{42}]^T Q_2[\mathscr{C}_{41}+(h_M-h(t))\mathscr{C}_{42}] \\
&+ Sym\left\{\mathscr{D}_1^T Q_1[\mathscr{C}_{30}+h(t)\mathscr{C}_{31}+h^2(t)\mathscr{C}_{32}]\right\} \\
&+ Sym\left\{\mathscr{D}_2^T Q_2[\mathscr{C}_{60}+(h_M-h(t))\mathscr{C}_{61}+(h_M-h(t))^2\mathscr{C}_{62}]\right\} \\
&- (1+\alpha)\Gamma_2^T\tilde{R}\Gamma_2 - Sym\left\{\Gamma_1^T[\alpha Y_1+(1-\alpha)Y_2]\Gamma_2\right\} \quad (25)
\end{aligned}$$

$$\begin{aligned}
\mathcal{G}_0(\dot{h}(t)) :=& \mathscr{C}_{12}^T Q_1\mathscr{C}_{12} + (1-\dot{h}(t))\mathscr{C}_{42}^T Q_2\mathscr{C}_{42} \\
&+ Sym\left\{\mathscr{D}_1^T Q_1\mathscr{C}_{32} + \mathscr{D}_2^T Q_2\mathscr{C}_{62}\right\} \quad (26)
\end{aligned}$$

where $\mathscr{C}_0 := Ae_1 + A_d e_2$, $\alpha = (h_M - h(t))/h_M$ and

$$\begin{aligned}
&\mathscr{C}_{11} := col\{\mathscr{C}_0, e_1, e_1, e_2, e_3, 0\}, \quad \mathscr{C}_{12} := col\{0,0,0,0,0,e_6\} \\
&\mathscr{C}_2 := col\{e_8, e_2, e_1, e_2, e_3, 0\}, \quad \mathscr{C}_{30} := col\{e_1-e_2, 0,0,0,0,0\} \\
&\mathscr{C}_{31} := col\{0, e_6, e_1, e_2, e_3, 0\}, \quad \mathscr{C}_{32} := col\{0,0,0,0,0,e_7\} \\
&\mathscr{C}_{41} := col\{e_8, e_2, e_1, e_2, e_3, 0\}, \quad \mathscr{C}_{42} := col\{0,0,0,0,0,e_4\} \\
&\mathscr{C}_5 := col\{e_9, e_3, e_1, e_2, e_3, 0\}, \quad \mathscr{C}_{60} := col\{e_2-e_3, 0,0,0,0,0\} \\
&\mathscr{C}_{61} := col\{0, e_4, e_1, e_2, e_3, 0\}, \quad \mathscr{C}_{62} := col\{0,0,0,0,0,e_5\} \\
&\mathscr{D}_1 := col\{0, 0, \mathscr{C}_0, (1-\dot{h}(t))e_8, e_9, (\dot{h}(t)-1)e_2\} \\
&\mathscr{D}_2 := col\{0, 0, \mathscr{C}_0, (1-\dot{h}(t))e_8, e_9, -e_3\} \\
&\Gamma_1 := col\{e_2-e_3,\ e_2+e_3-2e_4,\ e_2-e_3-6e_4+12e_5\} \\
&\Gamma_2 := col\{e_1-e_2,\ e_1+e_2-2e_6,\ e_1-e_2-6e_6+12e_7\}
\end{aligned}$$

with e_i $(i = 1, 2, \cdots, 9)$ being the i-th $n \times 9n$ block-row vectors of the $9n \times 9n$ identity matrix.

Proof. Taking the time derivative of $V(t, x_t)$ in (19) along with the trajectory of the system (1) yields

$$\dot{V}(t,x_t) = \dot{V}_1(t,x_t) + h_M\dot{V}_2(t,x_t) \quad (27)$$

where

$$\begin{aligned}
\dot{V}_1(t,x_t) =& \eta_1^T(t,t)Q_1\eta_1(t,t) - \eta_2^T(t,t-h_M)Q_2\eta_2(t,t-h_M) \\
&- (1-\dot{h}(t))\eta_1^T(t,t-h(t))Q_1\eta_1(t,t-h(t)) \\
&+ (1-\dot{h}(t))\eta_2^T(t,t-h(t))Q_2\eta_2(t,t-h(t)) \\
&+ \int_{t-h(t)}^{t} 2\eta_1^T(t,s)Q_1\frac{\partial\eta_1(t,s)}{\partial t}ds \\
&+ \int_{t-h_M}^{t-h(t)} 2\eta_2^T(t,s)Q_2\frac{\partial\eta_2(t,s)}{\partial t}ds \quad (28)
\end{aligned}$$

$$h_M\dot{V}_2(t,x_t) = h_M^2\dot{x}^T(t)R\dot{x}(t) - h_M\int_{t-h_M}^{t}\dot{x}^T(s)R\dot{x}(s)ds \quad (29)$$

For simplicity, denote

$$\begin{aligned}
\xi(t) := col\{&x(t), x(t-h(t)), x(t-h_M), \rho_1(t), \rho_2(t), \rho_3(t), \\
&\rho_4(t), \dot{x}(t-h(t)), \dot{x}(t-h_M)\} \quad (30)
\end{aligned}$$

where $\rho_j(t)$ $(j = 1,2,3,4)$ are defined in (3). Then one has $\dot{x}(t) = \mathscr{C}_0\xi(t)$, and

$$\begin{aligned}
&\eta_1(t,t) = (\mathscr{C}_{11}+h(t)\mathscr{C}_{12})\xi(t), \quad \eta_1(t,t-h(t)) = \mathscr{C}_2\xi(t), \\
&\eta_2(t,t-h(t)) = [\mathscr{C}_{41}+(h_M-h(t))\mathscr{C}_{42}]\xi(t), \quad \eta_2(t,t-h_M) = \mathscr{C}_5\xi(t), \\
&\int_{t-h(t)}^{t} 2\eta_1^T(t,s)Q_1\frac{\partial\eta_1(t,s)}{\partial t}ds \\
&\quad = 2\xi^T(t)\mathscr{D}_1^T Q_1\left[\mathscr{C}_{30}+h(t)\mathscr{C}_{31}+h^2(t)\mathscr{C}_{32}\right]\xi(t), \quad (31) \\
&\int_{t-h_M}^{t-h(t)} 2\eta_2^T(t,s)Q_2\frac{\partial\eta_2(t,s)}{\partial t}ds = 2\xi^T(t)\mathscr{D}_2^T Q_2 \\
&\quad \times\left[\mathscr{C}_{60}+(h_M-h(t))\mathscr{C}_{61}+(h_M-h(t))^2\mathscr{C}_{62}\right]\xi(t). \quad (32)
\end{aligned}$$

Denote $\mathscr{I}(t) := h_M\int_{t-h_M}^{t}\dot{x}^T(s)R\dot{x}(s)ds$. Then

$$\mathscr{I}(t) = h_M\int_{t-h_M}^{t-h(t)}\dot{x}^T(s)R\dot{x}(s)ds + h_M\int_{t-h(t)}^{t}\dot{x}^T(s)R\dot{x}(s)ds$$

Applying Lemma 1 yields

$$\begin{aligned}
h_M\int_{t-h_M}^{t-h(t)}\dot{x}^T(s)R\dot{x}(s)ds &\geq \frac{1}{\alpha}(\Gamma_1\xi(t))^T\tilde{R}(\Gamma_1\xi(t)) \\
h_M\int_{t-h(t)}^{t}\dot{x}^T(s)R\dot{x}(s)ds &\geq \frac{1}{1-\alpha}(\Gamma_2\xi(t))^T\tilde{R}(\Gamma_2\xi(t))
\end{aligned}$$

where $\alpha = (h_M - h(t))/h_M$. Thus, apply (16) with $\mathcal{R}_1 = \mathcal{R}_2 = \tilde{R}$, $\varpi_1 = \Gamma_1\xi(t)$ and $\varpi_2 = \Gamma_2\xi(t)$ to obtain

$$\begin{aligned}
&\mathscr{I}(t) \geq \xi^T(t)\left[\Upsilon_0 - (1-\alpha)\Gamma_1^T Y_1\tilde{R}^{-1}Y_1^T\Gamma_1 - \alpha\Gamma_2^T Y_2^T\tilde{R}^{-1}Y_2\Gamma_2\right]\xi(t) \\
&\Upsilon_0 := (2-\alpha)\Gamma_1^T\tilde{R}\Gamma_1 + (1+\alpha)\Gamma_2^T\tilde{R}\Gamma_2 + Sym\left\{\Gamma_1^T[\alpha Y_1 + (1-\alpha)Y_2]\Gamma_2\right\}
\end{aligned}$$

To sum up, one has that

$$\dot{V}(t,x_t) \leq \xi^T(t)[\Upsilon_1(h(t),\dot{h}(t)) + \Upsilon_2(h(t))]\xi(t) \quad (33)$$

Table 1: The maximum admissible upper bound h_M for $d = -d_m = d_M$ for Example 1

Method \ d	0.1	0.5	0.8	NoDVs
Kim (2016)	4.753	2.429	2.183	$27n^2+4n$
Zeng *et al*. (2015)	4.788	3.055	2.615	$65n^2+11n$
Proposition 1	4.910	3.233	2.789	$54.5n^2+6.5n$

Table 2: The maximum admissible upper bound h_M for $d = -d_m = d_M$ for Example 2

Method \ d	0.1	0.2	0.5	0.8
Seuret (2013)	6.590	3.672	1.411	1.275
Kwon *et al*. (2014)	7.125	4.413	2.243	1.662
Proposition 1	7.230	4.556	2.509	1.940

where $\Upsilon_1(h(t), \dot{h}(t))$ is defined in (25), and

$$\Upsilon_2(h(t)) := (1-\alpha)\Gamma_1^T Y_1 \tilde{R}^{-1} Y_1^T \Gamma_1 + \alpha \Gamma_2^T Y_2^T \tilde{R}^{-1} Y_2 \Gamma_2$$

Notice that

$$\Upsilon_1(h(t), \dot{h}(t)) + \Upsilon_2(h(t)) = h^2(t)\mathcal{G}_0(\dot{h}(t)) + h(t)\mathcal{G}_1(\dot{h}(t)) + \mathcal{G}_2(\dot{h}(t))$$

where $\mathcal{G}_0(\dot{h}(t))$ is defined in (26); $\mathcal{G}_1(\dot{h}(t))$ and $\mathcal{G}_2(\dot{h}(t))$ are some proper real symmetric matrices irrespective of $h(t)$. Let us consider the quadratic function $\chi^T[\Upsilon_1(h(t), \dot{h}(t)) + \Upsilon_2(h(t))]\chi$ as

$$\chi^T[\Upsilon_1(h(t), \dot{h}(t)) + \Upsilon_2(h(t))]\chi = a_2 h^2(t) + a_1 h(t) + a_0 \quad (34)$$

where $a_0 = \chi^T \mathcal{G}_2(\dot{h}(t))\chi$, $a_1 = \chi^T \mathcal{G}_1(\dot{h}(t))\chi$ and $a_2 = \chi^T \mathcal{G}_0(\dot{h}(t))\chi$ with $\chi \in \mathbb{R}^{9n}$. If the linear matrix inequalities in (23) and (24) are satisfied, applying Lemma 2 and the Schur complement yields $\chi^T[\Upsilon_1(h(t), \dot{h}(t)) + \Upsilon_2(h(t))]\chi < 0$ for $h(t) \in [0, h_M]$ and $\dot{h}(t) \in [d_m, d_M]$. Let $\chi = \xi(t)$. Then one has $\dot{V}(t, x_t) \leq \xi^T(t)[\Upsilon_1(h(t), \dot{h}(t)) + \Upsilon_2(h(t))]\xi(t) < 0$ for $h(t) \in [0, h_M]$ and $\dot{h}(t) \in [d_m, d_M]$. Thus, applying Lemma 3, one can draw a conclusion that the system (1) is asymptotically stable. □

Remark 3. Proposition 1 presents a novel stability criteria based on the new Lyapunov-Krasovskii functional. From the proof, it is clear that in the estimation of $\dot{V}(t, x_t)$, Theorem 1 and Lemma 1 play a key role in deriving a tight upper bound for $\dot{V}(t, x_t)$. It is worth pointing out that Lemma 2 proposed in Kim (2016) provides a useful method to deal with the quadratic function (34) on the time-varying delay $h(t)$. Simulation results in the next section show that Proposition 1 can produce less conservative results than some existing approaches.

3. Numerical examples

Example 1. *Consider the system* (1) *with*

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \ A_d = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} \quad (35)$$

and the delay $h(t)$ is time-varying satisfying (2).

For comparison with some existing approaches, we calculate the maximum admissible upper bound h_M for different $d = -d_m = d_M$. For $d \in \{0.1, 0.5, 0.8\}$, applying the approaches in Kim (2016), Zeng *et al*. (2015) and Proposition 1, the obtained results are given in Tab. 1. Moreover, the number of decision variables (NoDVs) involved in solving the corresponding linear matrix inequalities is also listed in Tab. 1. From this table, one can see clearly that Proposition 1 delivers some larger upper bounds h_M for the time-varying delay $h(t)$ than those by Kim (2016) and Zeng *et al*. (2015). It should be mentioned that the number of decision variables required in Proposition 1 is smaller than that in Zeng *et al*. (2015).

Example 2. *Consider the system* (1) *with*

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \ A_d = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \quad (36)$$

and the delay $h(t)$ is time-varying satisfying (2).

For this example, in Seuret & Gouaisbaut (2013) and Kwon *et al*. (2014), the maximum admissible upper bound h_M of the time-varying delay $h(t)$ is calculated for $d = -d_m = d_M \in \{0.1, 0.2, 0.5, 0.8\}$ and the obtained results are listed in Tab. 2. However, applying Proposition 1 yields some larger upper bounds h_M, which are also given in this table. It is clear to see that for $d = 0.8$, the maximum upper bound h_M is improved by 16.73% and 52.16%, respectively, if compared with the ones in Kwon *et al*. (2014) and Seuret & Gouaisbaut (2013).

4. Conclusion

Stability of linear systems with time-varying delay has been revisited in this paper. By introducing an optimal reciprocally convex inequality and a new Lyapunov-Krasovskii functional, a novel stability criterion has been derived for the system under study. It has been shown that through two well-used numerical examples the obtained stability criterion can deliver larger upper bounds for the time-varying delay than some existing ones.

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