Fully non-linear parabolic equations on compact Hermitian manifolds

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FULLY NON-LINEAR PARABOLIC EQUATIONS ON COMPACT HERMITIAN MANIFOLDS [1]

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Abstract

A notion of parabolic C-subsolutions is introduced for parabolic equations, extending the theory of C-subsolutions recently developed by B. Guan and more specifically G. Székelyhidi for elliptic equations. The resulting parabolic theory provides a convenient unified approach for the study of many geometric flows.

1 Introduction

Subsolutions play an important role in the theory of partial differential equations. Their existence can be viewed as an indication of the absence of any global obstruction. Perhaps more importantly, it can imply crucial a priori estimates, as for example in the Dirichlet problem for the complex Monge-Ampère equation [43, 18]. However, for compact manifolds without boundary, it is necessary to extend the notion of subsolution, since the standard notion may be excluded by either the maximum principle or cohomological constraints. Very recently, more flexible and compelling notions of subsolutions have been proposed by Guan [19] and Székelyhidi [50]. In particular, they show that their notions, called C-subsolution in [50], do imply the existence of solutions and estimates for a wide variety of fully non-linear elliptic equations on Hermitian manifolds. It is natural to consider also the parabolic case. This was done by Guan, Shi, and Sui in [21] for the usual notion of subsolution and for the Dirichlet problem. We now carry this out for the more general notion of C-subsolution on compact Hermitian manifolds, adapting the methods of [19] and especially [50]. As we shall see, the resulting parabolic theory provides a convenient unified approach to the many parabolic equations which have been studied in the literature.

Let (X, α) be a compact Hermitian manifold of dimension n, $\alpha = i\,\alpha_{\bar{k}j} dz^j \wedge d\bar{z}^k > 0$, and $\chi(z)$ be a real $(1,1)$- form,

$$\chi = i\,\chi_{\bar{k}j}(z) dz^j \wedge d\bar{z}^k.$$

If $u \in C^2(X)$, let $A[u]$ be the matrix with entries $A[u]^k{}_j = \alpha^{k\bar{m}}(\chi_{\bar{m}j} + \partial_j \partial_{\bar{m}} u)$. We consider the fully nonlinear parabolic equation,

$$\partial_t u = F(A[u]) - \psi(z), \tag{1.1}$$

[1]The first author was supported in part by the National Science Foundation under NSF Grant DMS-12-66033. The second author was supported by the CFM foundation and ATUPS travel grant.

where $F(A)$ is a smooth symmetric function $F(A) = f(\lambda[u])$ of the eigenvalues $\lambda_j[u]$, $1 \leq j \leq n$ of $A[u]$, defined on a open symmetric, convex cone $\Gamma \subset \mathbf{R}^n$ with vertex at the origin and containing the positive orthant Γ_n. We shall assume throughout the paper that f satisfies the following conditions:

(1) $f_i > 0$ for all i, and f is concave.
(2) $f(\lambda) \to -\infty$ as $\lambda \to \partial\Gamma$
(3) For any $\sigma < \sup_\Gamma f$ and $\lambda \in \Gamma$, we have $\lim_{t\to\infty} f(t\lambda) > \sigma$.

We shall say that a C^2 function u on X is admissible if the vector of eigenvalues of the corresponding matrix A is in Γ for any $z \in X$. Fix $T \in (0, \infty]$. To alleviate the terminology, we shall also designate by the same adjective functions in $C^{2,1}(X \times [0, T))$ which are admissible for each fixed $t \in [0, T)$. The following notion of subsolution is an adaptation to the parabolic case of Székelyhidi's [50] notion in the elliptic case:

Definition 1 *An admissible function $\underline{u} \in C^{2,1}(X \times [0, T))$ is said to be a (parabolic) C-subsolution of (1.1), if there exist constants $\delta, K > 0$, so that for any $(z, t) \in X \times [0, T)$, the condition*

$$f(\lambda[\underline{u}(z, t)] + \mu) - \partial_t \underline{u} + \tau = \psi(z), \quad \mu + \delta I \in \Gamma_n, \quad \tau > -\delta \tag{1.2}$$

implies that $|\mu| + |\tau| < K$. Here I denotes the vector $(1, \cdots, 1)$ of eigenvalues of the identity matrix.

We shall see below (§4.1) that this notion is more general than the classical notion defined by $f(\lambda([\underline{u}])) - \partial_t \underline{u}(z, t) > \psi(z, t)$ and studied by Guan-Shi-Sui [21]. A C-subsolution in the sense of Székelyhidi of the equation $F(A[u]) - \psi = 0$ can be viewed as a parabolic C-subsolution of the equation (1.1) which is time-independent. But more generally, to solve the equation $F(A[u]) - \psi = 0$ by say the method of continuity, we must choose a time-dependent deformation of this equation, and we would need then a C-subsolution for each time. The heat equation (1.1) and the above notion of parabolic subsolution can be viewed as a canonical choice of deformation.

To discuss our results, we need a finer classification of non-linear partial differential operators due to Trudinger [61]. Let Γ_∞ be the projection of Γ_n onto $\mathbf{R}^{n-1}$,

$$\Gamma_\infty = \{\lambda' = (\lambda_1, \cdots, \lambda_{n-1});\ \lambda = (\lambda_1, \cdots, \lambda_n) \in \Gamma \text{ for some } \lambda_n\} \tag{1.3}$$

and define the function f_∞ on Γ_∞ by

$$f_\infty(\lambda') = \lim_{\lambda_n \to \infty} f(\lambda', \lambda_n). \tag{1.4}$$

It is shown in [61] that, as a consequence of the concavity of f, the limit is either finite for all $\lambda' \in \Gamma_\infty$ or infinite for all $\lambda' \in \Gamma_\infty$. We shall refer to the first case as the *bounded case*,

and to the second case as the *unbounded case*. For example, Monge-Ampère flows belong to the unbounded case, while the J-flow and Hessian quotient flows belong to the bounded case. In the unbounded case, any admissible function, and in particular 0 if $\lambda[\chi] \in \Gamma$, is a C-subsolution in both the elliptic and parabolic cases. We have then:

Theorem 1 *Consider the flow* (1.1)*, and assume that f is in the unbounded case. Then for any admissible initial data u_0, the flow admits a smooth solution $u(z,t)$ on $[0,\infty)$, and its normalization $\tilde{u}$ defined by*

$$\tilde{u} := u - \frac{1}{V}\int_X u\,\alpha^n, \qquad V = \int_X \alpha^n, \tag{1.5}$$

converges in C^∞ to a function $\tilde{u}_\infty$ satisfying the following equation for some constant c,

$$F(A[\tilde{u}_\infty]) = \psi(z) + c. \tag{1.6}$$

The situation is more complicated when f belongs to the bounded case:

Theorem 2 *Consider the flow* (1.1)*, and assume that it admits a subsolution $\underline{u}$ on $X \times [0,\infty)$, but that f is in the bounded case. Then for any admissible data u_0, the equation admits a smooth solution $u(z,t)$ on $(0,\infty)$. Let $\tilde{u}$ be the normalization of the solution u, defined as before by (1.5). Assume that either one of the following two conditions holds.*

(a) *The initial data and the subsolution satisfy*

$$\partial_t \underline{u} \geq \sup_X (F(A[u_0]) - \psi); \tag{1.7}$$

(b) *or there exists a function $h(t)$ with $h'(t) \leq 0$ so that*

$$\sup_X (u(t) - h(t) - \underline{u}(t)) \geq 0 \tag{1.8}$$

and the Harnack inequality

$$\sup_X (u(t) - h(t)) \leq -C_1 \inf_X (u(t) - h(t)) + C_2 \tag{1.9}$$

holds for some constants $C_1, C_2 > 0$ independent of time.
Then $\tilde{u}$ converges in C^∞ to a function $\tilde{u}_\infty$ satisfying (1.6) for some constant c.

The essence of the above theorems resides in the a priori estimates which are established in §2. The C^1 and C^2 estimates can be adapted from the corresponding estimates for C-subsolutions in the elliptic case, but the C^0 estimate turns out to be more subtle. Following Blocki [1] and Székelyhidi [50], we obtain C^0 estimates from the Alexandrov-Bakelman-Pucci (ABP) inequality, using this time a parabolic version of ABP due to K. Tso [62]. However, it turns out that the existence of a C-subsolution gives only partial information

on the oscillation of u, and what can actually be estimated has to be formulated with some care, leading to the distinction between the cases of f bounded and unbounded, as well as Theorem 2.

The conditions (a) and especially (b) in Theorem 2 may seem impractical at first sight since they involve the initial data as well as the long-time behavior of the solution. Nevertheless, as we shall discuss in greater detail in section §4, Theorems 1 and 2 can be successfully applied to a wide range of parabolic flows on Hermitian manifolds previously studied in the literature, including the Kähler-Ricci flow, the Chern-Ricci flow, the J-flow, the Hessian flows, the quotient Hessian flows, and mixed Hessian flows. We illustrate this by deriving in §4 as a corollary of Theorem 2 a convergence theorem for a mixed Hessian flow, which seems new to the best of our knowledge. It answers a question raised for general $1 \leq \ell < k \leq n$ by Fang-Lai-Ma [12] (see also Sun [44, 45, 46, 48]), and extends the solution obtained for $k = n$ by Collins-Székelyhidi [7] and subsequently also by Sun [48, 49]:

Theorem 3 *Assume that (X, α) is a compact Kähler n-manifold, and fix $1 \leq \ell < k \leq n$. Fix a closed $(1,1)$-form χ which is k-positive and non-negative constants c_j, and assume that there exists a form $\chi' = \chi + i\partial\bar{\partial}\underline{u}$ which is a closed k-positive form and satisfies*

$$kc(\chi')^{k-1} \wedge \alpha^{n-k} - \sum_{j=1}^{\ell} j c_j (\chi')^{j-1} \wedge \alpha^{n-j} > 0, \tag{1.10}$$

in the sense of positivity of $(n-1, n-1)$-forms. Here the constant c is given by

$$c[\chi^k][\alpha^{n-k}] = \sum_{j=1}^{\ell} c_j [\chi^j][\alpha^{n-j}]. \tag{1.11}$$

Then the flow

$$\partial_t u = -\frac{\sum_{j=1}^{\ell} c_j \sigma_j(\lambda(A[u]))}{\sigma_k(\lambda(A[u]))} + c, \qquad u(\cdot, 0) = 0, \tag{1.12}$$

admits a solution for all time which converges smoothly to a function u_∞ as $t \to \infty$. The form $\omega = \chi + i\partial\bar{\partial}u_\infty$ is k-positive and satisfies the equation

$$c\,\omega^k \wedge \alpha^{n-k} = \sum_{j=1}^{\ell} c_j\, \omega^j \wedge \alpha^{n-j}. \tag{1.13}$$

Regarding the condition (a) in Theorem 2, we note that natural geometric flows whose long-time behavior may be very sensitive to the initial data are appearing increasingly frequently in non-Kähler geometry. A prime example is the Anomaly flow, studied in [33, 36, 37, 38, 13]. Finally, Theorem 2 will also be seen to imply as a corollary a theorem of Székelyhidi ([50], Proposition 26), and the condition for solvability there will be seen to correspond to condition (a) in Theorem 2. This suggests in particular that some additional conditions for the convergence of the flow cannot be dispensed with altogether.

2 A Priori Estimates

2.1 C^0 Estimates

We begin with the C^0 estimates implied by the existence of a C-subsolution for the parabolic flow (1.1). One of the key results of [50] was that the existence of a subsolution in the elliptic case implies a uniform bound for the oscillation of the unknown function u. In the parabolic case, we have only the following weaker estimate:

Lemma 1 *Assume that the equation* (1.1) *admits a parabolic C-solution on $X \times [0,T)$ in the sense of Definition 1, and that there exists a C^1 function $h(t)$ with $h'(t) \leq 0$ and*

$$\sup_X (u(\cdot,t) - \underline{u}(\cdot,t) - h(t)) \geq 0. \tag{2.1}$$

Then there exists a constant C depending only on χ, α, δ, $\|u_0\|_{C^0}$, and $\|i\partial\bar{\partial}\underline{u}\|_{L^\infty}$ so that

$$u(\cdot,t) - \underline{u}(\cdot,t) - h(t) \geq -C \quad \text{for all } (z,t) \in X \times [0,T). \tag{2.2}$$

Proof. First, note that by Lemma 6 proven later in §3, the function $\partial_t u$ is uniformly bounded for all time by a constant depending only on ψ and the initial data u_0. Integrating this estimate on $[0,\delta]$ gives a bound for $|u|$ on $X \times [0,\delta]$ depending only on ψ, u_0 and δ. Thus we need only consider the range $t \geq \delta$. Next, the fact that $\underline{u}$ is a parabolic subsolution and the condition that $h'(t) \leq 0$ imply that $\underline{u} + h(t)$ is a parabolic subsolution as well. So it suffices to prove the desired inequality with $h(t) = 0$, as long as the constants involved do not depend on $\partial_t \underline{u}$. Fix now any $T' < T$, and set for each t, $v = u - \underline{u}$, and

$$L = \min_{X \times [0,T']} v = v(z_0, t_0) \tag{2.3}$$

for some $(z_0, t_0) \in X \times [0, T']$. We shall show that L can be bounded from below by a constant depending only on the initial data u_0 and independent of T'. We can assume that $t_0 > 0$, otherwise we are already done. Let $(z_1, \cdots, z_n)$ be local holomorphic coordinates for X centered at z_0, $U = \{z; |z| < 1\}$, and define the following function on the set $\mathcal{U} = U \times \{t; -\delta \leq 2(t - t_0) < \delta\}$,

$$w = v + \frac{\delta^2}{4}|z|^2 + |t - t_0|^2, \tag{2.4}$$

where $\delta > 0$ is the constant appearing in the definition of subsolutions. Clearly w attains its minimum on $\mathcal{U}$ at (z_0, t_0), and $w \geq \min_{\mathcal{U}} w + \frac{1}{4}\delta^2$ on the parabolic boundary of $\mathcal{U}$. We can thus apply the following parabolic version of the Alexandrov-Bakelman-Pucci inequality, due to K. Tso ([62], Proposition 2.1, with the function u there set to $u = -w + \min_{\mathcal{U}} w + \frac{\delta^2}{4}$):

Let $\mathcal{U}$ be the subset of $\mathbf{R}^{2n+1}$ defined above, and let $w : \mathcal{U} \to \mathbf{R}$ be a smooth function which attains its minimum at $(0, t_0)$, and $w \geq \min_{\mathcal{U}} w + \frac{1}{4}\delta^2$ on the parabolic boundary of $\mathcal{U}$. Define the set

$$S := \left\{ (x,t) \in \mathcal{U} : \begin{array}{l} w(x,t) \leq w(z_0,t_0) + \frac{1}{4}\delta^2, \quad |D_x w(x,t)| < \frac{\delta^2}{8}, \text{ and} \\ w(y,s) \geq w(x,t) + D_x w(x,t).(y-x),\ \forall y \in U, s \leq t \end{array} \right\}. \tag{2.5}$$

Then there is a constant $C = C(n) > 0$ so that

$$C\delta^{4n+2} \leq \int_S (-w_t) \det(w_{ij}) dxdt.$$

Returning to the proof of Lemma 1, we claim that, on the set S, we have

$$|w_t| + \det\left(D^2_{jk} w\right) \leq C \tag{2.6}$$

for some constant depending only on δ, and $\|i\partial\bar{\partial}\underline{u}\|_{L^\infty}$. Indeed, let

$$\mu = \lambda[u] - \lambda[\underline{u}], \qquad \tau = -\partial_t u + \partial_t \underline{u}. \tag{2.7}$$

Along S, we have $D^2_{ij} w \geq 0$ and $\partial_t w \leq 0$. In terms of μ and τ, this means that $\mu + \delta \mathbf{I} \in \Gamma_n$ and $0 \leq -\partial_t w = \tau - 2(t - t_0) \leq \tau + \delta$. The fact that u is a solution of the equation (1.1) can be expressed as

$$f(\lambda[\underline{u}] + \mu) - \partial_t \underline{u} + \tau = \psi(z). \tag{2.8}$$

Thus the condition that $\underline{u}$ is a parabolic subsolution implies that $|\mu|$ and $|\tau|$ are bounded uniformly in (z, t). Since along S, we have $\det(D^2_{ij} w) \leq 2^n (\det(D^2_{\bar{k}j} w))^2$, it follows that both $|w_t|$ and $\det(D^2_{ij} w)$ are bounded uniformly, as was to be shown.

Next, by the definition of the points (x, t) on S, we have $w(x, t) \leq L + \frac{\delta^2}{4}$. Since we can assume that $|L| > \delta^2$, it follows that $w < 0$ and $|w| \geq \frac{|L|}{2}$ on S. Thus we can write, in view of (2.6), for any $p > 0$,

$$C_n \delta^{4n+2} \leq C \int_S dxdt \leq \left(\frac{|L|}{2}\right)^{-p} \int_S |w(x,t)|^p dxdt \leq \left(\frac{|L|}{2}\right)^{-p} \int_{\mathcal{U}} |w(x,t)|^p dxdt. \tag{2.9}$$

Next write

$$\begin{aligned} |w| &= -w = -v - \frac{\delta^2}{4}|z|^2 - (t - t_0)^2 \leq -v \\ &\leq -v + \sup_X v \end{aligned} \tag{2.10}$$

since $\sup_X v \geq 0$ by the assumption (2.1). Since $\lambda[u] \in \Gamma$ and the cone Γ is convex, it follows that $\Delta u \geq -C$ and hence

$$\Delta(v - \sup_X v) = \Delta u - \Delta \underline{u} \geq -A \tag{2.11}$$

for some constant A depending only on χ, α, and $\|i\partial\bar{\partial}\underline{u}\|_{L^\infty}$. The Harnack inequality applied to the function $v - \sup_X v$, in the version provided by Proposition 10, [50], implies that

$$\|v - \sup_X v\|_{L^p(X)} \leq C \tag{2.12}$$

for C depending only on (X, α), A, and p. Substituting these bounds into (2.9) gives

$$C\delta^{4n+2} \leq \left(\frac{|L|}{2}\right)^{-p} \int_{|t| < \frac{1}{2}\delta} \|\sup_X v - v\|^p_{L^p(X)} dt \leq C'\delta \left(\frac{|L|}{2}\right)^{-p} \tag{2.13}$$

from which the desired bound for L follows. Q.E.D.

2.2 C^2 Estimates

In this section we prove an estimate for the complex Hessian of u in terms of the gradient. The original strategy goes back to the work of Chou-Wang [6], with adaptation to complex Hessian equations by Hou-Ma-Wu [25], and to fully non-linear elliptic equations admitting a C-subsolution by Guan [19] and Székelyhidi [50]. Other adaptations to C^2 estimates can be found in [51], [32], [34], [67]. We follow closely [50].

Lemma 2 *Assume that the flow* (1.1) *admits a C-subsolution on $X \times [0,T)$. Then we have the following estimate*

$$|i\partial\bar\partial u| \leq \tilde C(1 + \sup_{X\times[0,T)} |\nabla u|^2_\alpha) \tag{2.14}$$

where $\tilde C$ depends only on $\|\alpha\|_{C^2}, \|\psi\|_{C^2}, \|\chi\|_{C^2}, \|\tilde u - \underline{\tilde u}\|_{L^\infty}, \|\nabla \underline{u}\|_{L^\infty}, \|i\partial\bar\partial \underline{u}\|_{L^\infty}, \|\partial_t u\|_{L^\infty}$, $\|\partial_t(u - \underline{u})\|_{L^\infty}$, and the dimension n.

Proof. Let $\mathcal{L} = -\partial_t + F^{k\bar l}\nabla_k\nabla_{\bar l}$. Denote $g = \chi + i\partial\bar\partial u$, then $A[u]^k{}_j = \alpha^{k\bar p} g_{\bar p j}$. We would like to apply the maximum principle to the function

$$G = \log\lambda_1 + \phi(|\nabla u|^2) + \varphi(\tilde v) \tag{2.15}$$

where $v = u - \underline{u}$, $\tilde v$ is the normalization of v, $\lambda_1 : X \to \mathbf{R}$ is the largest eigenvalue of the matrix $A[u]$ at each point, and the functions ϕ and φ will be specified below. Since the eigenvalues of $A[u]$ may not be distinct, we perturb $A[u]$ following the technique of [50], Proposition 13. Thus assume that G attains its maximum on $X \times [0,T']$ at some (z_0, t_0), with $t_0 > 0$. We choose local complex coordinates, so that z_0 corresponds to 0, and $A[u]$ is diagonal at 0 with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. Let $B = (B^i{}_j)$ be a diagonal matrix with $0 = B^1{}_1 < B^2{}_2 < \cdots < B^n{}_n$ and small constant entries, and set $\tilde A = A - B$. Then at the origin $\tilde A$ has eigenvalues $\tilde\lambda_1 = \lambda_1$, $\tilde\lambda_i = \lambda_i - B^i{}_i < \tilde\lambda_1$ for all $i > 1$.

Since all the eigenvalues of $\tilde A$ are distinct, we can define near 0 the following smooth function $\tilde G$,

$$\tilde G = \log\tilde\lambda_1 + \phi(|\nabla u|^2) + \varphi(\tilde v) \tag{2.16}$$

where

$$\phi(t) = -\frac{1}{2}\log(1 - \frac{t}{2P}), \quad P = \sup_{X\times[0,T']}(|\nabla u|^2 + 1) \tag{2.17}$$

and, following [51]

$$\varphi(t) = D_1 e^{-D_2 t} \tag{2.18}$$

for some large constants D_1, D_2 to be chosen later. Note that

$$\frac{1}{4P} \leq \phi' \leq \frac{1}{2P}, \qquad \phi'' = 2(\phi')^2 > 0. \tag{2.19}$$

The norm $|\nabla u|^2$ is taken with respect to the fixed Hermitian metric α on X, and we shall compute using covariant derivatives ∇ with respect to α. Since the matrix $B^j{}_m$ is constant in a neighborhood of 0 and since we are using the Chern unitary connection, we have $\nabla_{\bar{k}} B^j{}_m = 0$. Our conventions for the curvature and torsion tensors of a Hermitian metric α are as follows,

$$[\nabla_\beta, \nabla_\alpha] V^\gamma = R_{\alpha\beta}{}^\gamma{}_\delta V^\delta + T^\delta{}_{\alpha\beta} \nabla_\delta V^\gamma. \tag{2.20}$$

We also set

$$\mathcal{F} = \sum_i f_i(\lambda[u]). \tag{2.21}$$

An important observation is that there exists a constant C_1, depending only on $\|\psi\|_{L^\infty(X)}$ and $\|\partial_t u\|_{L^\infty(X\times[0,T))}$ so that

$$\mathcal{F} \geq C_1. \tag{2.22}$$

Indeed it follows from the properties of the cone Γ that $\sum_i f_i(\lambda) \geq C(\sigma)$ for each fixed σ and $\lambda \in \Gamma^\sigma$. When $\lambda = \lambda[u]$, σ must lie in the range of $\partial_t u + \psi$, which is a compact set bounded by $\|\partial_t u\|_{L^\infty(X\times[0,T))} + \|\psi\|_{L^\infty(X)}$, hence our claim.

2.2.1 Estimate of $\mathcal{L}(\log \tilde{\lambda}_1)$

Clearly

$$\mathcal{L} \log \tilde{\lambda}_1 = \frac{1}{\lambda_1}(F^{k\bar{k}} \tilde{\lambda}_{1,\bar{k}k} - \partial_t \tilde{\lambda}_1) - F^{k\bar{k}} \frac{|\tilde{\lambda}_{1,\bar{k}}|^2}{\lambda_1^2}. \tag{2.23}$$

We work out the term $F^{k\bar{k}} \tilde{\lambda}_{1,\bar{k}k} - \partial_t \tilde{\lambda}_1$ using the flow. The usual differentiation rules ([43]) readily give

$$\tilde{\lambda}_{1,\bar{k}} = \nabla_{\bar{k}} g_{\bar{1}1} \tag{2.24}$$

and

$$\tilde{\lambda}_{1,\bar{k}k} = \nabla_k \nabla_{\bar{k}} g_{\bar{1}1} + \sum_{p>1} \frac{|\nabla_{\bar{k}} g_{\bar{p}1}|^2 + |\nabla_{\bar{k}} g_{\bar{1}p}|^2}{\lambda_1 - \tilde{\lambda}_p} - \sum_{p>1} \frac{\nabla_k B^1{}_p \nabla_{\bar{k}} g_{\bar{p}1} + \nabla_k B^p{}_1 \nabla_{\bar{k}} g_{\bar{1}p}}{\lambda_1 - \tilde{\lambda}_p}. \tag{2.25}$$

while it follows from the flow that

$$\partial_t \tilde{\lambda}_1 = \partial_t u_{\bar{1}1} = F^{l\bar{k},s\bar{r}} \nabla_{\bar{1}} g_{\bar{k}l} \nabla_1 g_{\bar{r}s} + F^{k\bar{k}} \nabla_1 \nabla_{\bar{1}} g_{\bar{k}k} - \psi_{\bar{1}1}. \tag{2.26}$$

Thus

$$\begin{aligned} F^{k\bar{k}}\tilde{\lambda}_{1,\bar{k}k} - \partial_t\tilde{\lambda}_1 &= F^{k\bar{k}}(\nabla_k\nabla_{\bar{k}}g_{\bar{1}1} - \nabla_1\nabla_{\bar{1}}g_{\bar{k}k}) + F^{l\bar{k},s\bar{r}}\nabla_{\bar{1}}g_{\bar{k}l}\nabla_1 g_{\bar{r}s} - \psi_{\bar{1}1} \\ &\quad + F^{k\bar{k}}\sum_{p>1}\{\frac{|\nabla_{\bar{k}}g_{\bar{p}1}|^2 + |\nabla_{\bar{k}}g_{\bar{1}p}|^2}{\lambda_1 - \tilde{\lambda}_p} - \frac{\nabla_k B^1{}_p\nabla_{\bar{k}}g_{\bar{p}1} + \nabla_k B^p{}_1\nabla_{\bar{k}}g_{\bar{1}p}}{\lambda_1 - \tilde{\lambda}_p}\} \end{aligned}$$

A simple computation gives

$$\begin{aligned} \nabla_k\nabla_{\bar{k}}g_{\bar{1}1} - \nabla_1\nabla_{\bar{1}}g_{\bar{k}k} &= -2\mathrm{Re}(T^p_{k1}\nabla_{\bar{k}}g_{\bar{p}1}) + T\star\nabla\chi + R\star\nabla\bar{\nabla}u + T\star T\star\nabla\bar{\nabla}u \\ &\geq -2\mathrm{Re}(T^p_{k1}\nabla_{\bar{k}}g_{\bar{p}1}) - C_2(\lambda_1+1), \end{aligned} \tag{2.27}$$

where C_2 depending only on $\|\alpha\|_{C^2}$ and $\|\chi\|_{C^2}$. We also have

$$\begin{aligned} &\sum_{p>1}\{\frac{|\nabla_{\bar{k}}g_{\bar{p}1}|^2 + |\nabla_{\bar{k}}g_{\bar{1}p}|^2}{\lambda_1 - \tilde{\lambda}_p} - \frac{\nabla_k B^1{}_p\nabla_{\bar{k}}g_{\bar{p}1} + \nabla_k B^p{}_1\nabla_{\bar{k}}g_{\bar{1}p}}{\lambda_1 - \tilde{\lambda}_p}\} \\ &\geq \frac{1}{2}\sum_{p>1}\frac{|\nabla_{\bar{k}}g_{\bar{p}1}|^2 + |\nabla_{\bar{k}}g_{\bar{1}p}|^2}{\lambda_1 - \tilde{\lambda}_p} - C_3 \geq \frac{1}{2(n\lambda_1+1)}\sum_{p>1}|\nabla_{\bar{k}}g_{\bar{p}1}|^2 + |\nabla_{\bar{k}}g_{\bar{1}p}|^2 - C_3, \end{aligned} \tag{2.28}$$

where C_3 only depends on the dimension n, and the second inequality is due to the fact that $(\lambda_1 - \tilde{\lambda}_p)^{-1} \geq (n\lambda_1+1)^{-1}$, which follows itself from the fact that $\sum_i \lambda_i \geq 0$ and B was chosen to be small. Thus

$$\begin{aligned} &\nabla_k\nabla_{\bar{k}}g_{\bar{1}1} - \nabla_1\nabla_{\bar{1}}g_{\bar{k}k} + \sum_{p>1}\{\frac{|\nabla_{\bar{k}}g_{\bar{p}1}|^2 + |\nabla_{\bar{k}}g_{\bar{1}p}|^2}{\lambda_1 - \tilde{\lambda}_p} - \frac{\nabla_k B^1{}_p\nabla_{\bar{k}}g_{\bar{p}1} + \nabla_k B^p{}_1\nabla_{\bar{k}}g_{\bar{1}p}}{\lambda_1 - \tilde{\lambda}_p}\} \\ &\geq -2\mathrm{Re}(T^p_{k1}\nabla_{\bar{k}}g_{\bar{p}1}) + \frac{1}{2(n\lambda_1+1)}\sum_{p>1}|\nabla_{\bar{k}}g_{\bar{p}1}|^2 + |\nabla_{\bar{k}}g_{\bar{1}p}|^2 - C_2(\lambda_1+1) - C_3 \\ &\geq -C_4|\nabla_{\bar{k}}g_{\bar{1}1}| - C_5\lambda_1 - C_6 \end{aligned} \tag{2.29}$$

where we have used the positive terms to absorb all the terms $T^p_{k1}\nabla_{\bar{k}}g_{\bar{p}1}$, except for $T^1_{k1}\nabla_{\bar{k}}g_{\bar{1}1}$ and C_4, C_5, C_6 only depend on $\|\alpha\|_{C^2}, \|\chi\|_{C^2}, n$. Altogether,

$$F^{k\bar{k}}\tilde{\lambda}_{1,\bar{k}k} - \partial_t\tilde{\lambda}_1 \geq -C_4F^{k\bar{k}}|\nabla_{\bar{k}}g_{\bar{1}1}| + F^{l\bar{k},s\bar{r}}\nabla_{\bar{1}}g_{\bar{k}l}\nabla_1 g_{\bar{r}s} - \psi_{\bar{1}1} - C_5\mathcal{F}\lambda_1 - C_6\mathcal{F} \tag{2.30}$$

and we find

$$\mathcal{L}\log\tilde{\lambda}_1 \geq -F^{k\bar{k}}\frac{|\tilde{\lambda}_{1,\bar{k}}|^2}{\lambda_1^2} - \frac{1}{\lambda_1}F^{l\bar{k},s\bar{r}}\nabla_{\bar{1}}g_{\bar{k}l}\nabla_1 g_{\bar{r}s} - C_4\frac{1}{\lambda_1}F^{k\bar{k}}|\nabla_{\bar{k}}g_{\bar{1}1}| - C_7\mathcal{F}, \tag{2.31}$$

where we have bounded $\psi_{\bar{1}1}$ by a constant that can be absorbed in $C_6\mathcal{F}/\lambda_1 \leq C_6\mathcal{F}$, since $\lambda_1 \geq 1$ by assumption, and $\mathcal{F}$ is bounded below by a constant depending on $\|\psi\|_{L^\infty}$ and $\|\partial_t u\|_{L^\infty}$. The constant C_7 thus only depends on $\|\alpha\|_{C^2}, \|\chi\|_{C^2}, n$, $\|\partial_t u\|_{L^\infty}$ and $\|\psi\|_{C^2}$. In view of (2.24), this can also be rewritten as

$$\mathcal{L}\log\tilde{\lambda}_1 \geq -F^{k\bar{k}}\frac{|\tilde{\lambda}_{1,\bar{k}}|^2}{\lambda_1^2} - \frac{1}{\lambda_1}F^{l\bar{k},s\bar{r}}\nabla_{\bar{1}}g_{\bar{k}l}\nabla_1 g_{\bar{r}s} - C_4\frac{1}{\lambda_1}F^{k\bar{k}}|\tilde{\lambda}_{1,\bar{k}}| - C_7\mathcal{F}. \tag{2.32}$$

2.2.2 Estimate for $\mathcal{L}\phi(|\nabla u|^2)$

Next, a direct calculation gives

$$\begin{aligned}\mathcal{L}\phi(|\nabla u|^2) &= \phi'(F^{q\bar{q}}\nabla_q\nabla_{\bar{q}} - \partial_t)|\nabla u|^2 + \phi'' F^{q\bar{q}}\nabla_q|\nabla u|^2\nabla_{\bar{q}}|\nabla u|^2 \\ &= \phi'\{\nabla^j u(F^{q\bar{q}}\nabla_q\nabla_{\bar{q}} - \partial_t)\nabla_j u + \nabla^{\bar{j}} u(F^{q\bar{q}}\nabla_q\nabla_{\bar{q}} - \partial_t)\nabla_{\bar{j}} u\} \\ &\quad +\phi' F^{q\bar{q}}(|\nabla_q\nabla u|^2 + |\nabla_q\bar{\nabla} u|^2) + \phi'' F^{q\bar{q}}\nabla_q|\nabla u|^2\nabla_{\bar{q}}|\nabla u|^2. \end{aligned} \tag{2.33}$$

In view of the flow, we have

$$\nabla_j\partial_t u = F^{k\bar{k}}\nabla_j g_{\bar{k}k} - \psi_j, \quad \nabla_{\bar{j}}\partial_t u = F^{k\bar{k}}\nabla_{\bar{j}} g_{\bar{k}k} - \psi_{\bar{j}}. \tag{2.34}$$

It follows that

$$\begin{aligned}(F^{k\bar{k}}\nabla_k\nabla_{\bar{k}} - \partial_t)\nabla_{\bar{j}} u &= F^{k\bar{k}}(\nabla_k\nabla_{\bar{k}} u_{\bar{j}} - \nabla_{\bar{j}} g_{\bar{k}k}) + \psi_{\bar{j}} \\ &= F^{k\bar{k}}(-\nabla_{\bar{j}}\chi_{\bar{k}k} + \overline{T^p_{kj}}\nabla_{\bar{j}}\nabla_k u + R_{\bar{j}k}{}^{\bar{m}}{}_{\bar{k}}\nabla_{\bar{m}} u) + \psi_{\bar{j}} \end{aligned} \tag{2.35}$$

and hence, for small ε, there is a constant $C_8 > 0$ depending only on $\varepsilon, \|\chi\|_{C^2}, \|\alpha\|_{C^2}$ and $||\psi\|_{C^2}$ such that

$$\phi'\nabla^{\bar{j}} u(F^{q\bar{q}}\nabla_q\nabla_{\bar{q}} - \partial_t)\nabla_{\bar{j}} u \geq -C_8\mathcal{F} - \frac{\varepsilon}{P}F^{q\bar{q}}(|\nabla_q\nabla u|^2 + |\nabla_q\bar{\nabla} u|^2) \tag{2.36}$$

since we can assume that $\lambda_1 >> P = \sup_{X\times[0,T']}(|\nabla u|^2+1)$ (otherwise the desired estimate $\lambda_1 < CP$ already holds), and $(4P)^{-1} < \phi' < (2P)^{-1}$. Similarly we obtain the same estimate for $\phi'\nabla^j u(F^{q\bar{q}}\nabla_q\nabla_{\bar{q}} - \partial_t)\nabla_j u$. Thus by choosing $\varepsilon = 1/24$, we have

$$\mathcal{L}\phi(|\nabla u|^2) \geq -C_8\mathcal{F} + \frac{1}{8P}F^{q\bar{q}}(|\nabla_q\nabla u|^2 + |\nabla_q\bar{\nabla} u|^2) + \phi'' F^{q\bar{q}}\nabla_q|\nabla u|^2\nabla_{\bar{q}}|\nabla u|^2. \tag{2.37}$$

2.2.3 Estimate for $\mathcal{L}\tilde{G}$

The evaluation of the remaining term $\mathcal{L}\varphi(\tilde{v})$ is straightforward,

$$\mathcal{L}\varphi(\tilde{v}) = \varphi'(\tilde{v})(F^{k\bar{k}}\nabla_k\nabla_{\bar{k}}\tilde{v} - \partial_t\tilde{v}) + \varphi''(\tilde{v})F^{k\bar{k}}\nabla_k\tilde{v}\nabla_{\bar{k}}\tilde{v}. \tag{2.38}$$

Altogether, we have established the following lower bound for $\mathcal{L}\tilde{G}$,

$$\begin{aligned}\mathcal{L}\tilde{G} &\geq -F^{k\bar{k}}\frac{|\tilde{\lambda}_{1,\bar{k}}|^2}{\lambda_1^2} - \frac{1}{\lambda_1}F^{l\bar{k},s\bar{r}}\nabla_{\bar{1}} g_{\bar{k}l}\nabla_1 g_{\bar{r}s} - C_4\frac{1}{\lambda_1}F^{k\bar{k}}|\lambda_{1,\bar{k}}| - C_9\mathcal{F} \\ &\quad +\frac{1}{8P}F^{q\bar{q}}(|\nabla_q\nabla u|^2 + |\nabla_q\bar{\nabla} u|^2) + \phi'' F^{q\bar{q}}\nabla_q|\nabla u|^2\nabla_{\bar{q}}|\nabla u|^2 \\ &\quad +\varphi'(\tilde{v})(F^{k\bar{k}}\nabla_k\nabla_{\bar{k}}\tilde{v} - \partial_t\tilde{v}) + \varphi''(\tilde{v})F^{k\bar{k}}\nabla_k\tilde{v}\nabla_{\bar{k}}\tilde{v}, \end{aligned} \tag{2.39}$$

where C_4 and C_9 only depend on $\|\chi\|_{C^2}, \|\alpha\|_{C^2}, \|\psi\|_{C^2}, \|\partial_t u\|_{L^\infty}$ and the dimension n.

For a small $\theta > 0$ to be chosen hereafter, we deal with two following cases.

2.2.4 Case 1: $\theta\lambda_1 \leq -\lambda_n$

In this case, we have $\theta^2\lambda_1^2 \leq \lambda_n^2$. Thus we can write

$$\begin{aligned}\frac{1}{8P}F^{q\bar{q}}(|\nabla_q\nabla u|^2 + |\nabla_q\bar{\nabla}u|^2) &\geq \frac{F^{n\bar{n}}}{8P}|u_{\bar{n}n}|^2 = \frac{F^{n\bar{n}}}{8P}|\lambda_n - \chi_{\bar{n}n}|^2 \geq \frac{\mathcal{F}\lambda_n^2}{10nP} - \frac{C_{10}\mathcal{F}}{P} \\ &\geq \frac{\theta^2}{10nP}\mathcal{F}\lambda_1^2 - C_{10}\mathcal{F}, \end{aligned} \tag{2.40}$$

where C_{10} only depends on $\|\chi\|_{C^2}$. Next, it is convenient to combine the first and third terms in the expression for $\mathcal{L}\tilde{G}$,

$$-F^{k\bar{k}}\frac{|\tilde{\lambda}_{1,\bar{k}}|^2}{\lambda_1^2} - C_4\frac{1}{\lambda_1}F^{k\bar{k}}|\tilde{\lambda}_{1,\bar{k}}| \geq -\frac{3}{2}F^{k\bar{k}}\frac{|\tilde{\lambda}_{1,\bar{k}}|^2}{\lambda_1^2} - C_{11}\mathcal{F}. \tag{2.41}$$

where C_{11} only depends on C_4.

At a maximum point for $\tilde{G}$, we have $0 \geq \mathcal{L}\tilde{G}$. Combining the lower bound (2.39) for $\mathcal{L}\tilde{G}$ with the preceding inequalities and dropping the second and last terms, which are non-negative, we obtain

$$0 \geq \frac{\theta^2}{10nP}\mathcal{F}\lambda_1^2 - C_{12}\mathcal{F} - \frac{3}{2}F^{k\bar{k}}\frac{|\tilde{\lambda}_{1,\bar{k}}|^2}{\lambda_1^2} + \phi''F^{q\bar{q}}|\nabla_{\bar{q}}|\nabla u|^2|^2 + \varphi'(\tilde{v})(F^{k\bar{k}}\nabla_k\nabla_{\bar{k}}\tilde{v} - \partial_t\tilde{v}), \tag{2.42}$$

where $C_{12} = C_9 + C_{10} + C_{11}$, depending on $\|\chi\|_{C^2}, \|\alpha\|_{C^2}, \|\psi\|_{C^2}, \|\partial_t u\|_{L^\infty}$ and n. Since we are at a critical point of $\tilde{G}$, we also have $\nabla\tilde{G} = 0$, and hence

$$\frac{\tilde{\lambda}_{1,\bar{k}}}{\lambda_1} + \phi'\nabla_{\bar{k}}|\nabla u|^2 + \varphi'\partial_{\bar{k}}\tilde{v} = 0 \tag{2.43}$$

which implies

$$\begin{aligned}\frac{3}{2}F^{k\bar{k}}|\frac{\tilde{\lambda}_{1,\bar{k}}}{\lambda_1}|^2 &= \frac{3}{2}F^{k\bar{k}}|\phi'\nabla_{\bar{k}}|\nabla u|^2 + \varphi'\partial_{\bar{k}}\tilde{v}|^2 \leq 2F^{k\bar{k}}(\phi')^2|\nabla_{\bar{k}}|\nabla u|^2|^2 + 4F^{k\bar{k}}(\varphi')^2|\nabla_{\bar{k}}\tilde{v}|^2 \\ &\leq F^{k\bar{k}}\phi''|\nabla_{\bar{k}}|\nabla u|^2|^2 + C_{13}\mathcal{F}P, \end{aligned} \tag{2.44}$$

where C_{13} depending on $\|\tilde{v}\|_{L^\infty}$ and $\|\nabla\underline{u}\|_{L^\infty}$. Since $\varphi'(\tilde{v})$ is bounded in terms of $\|\tilde{v}\|_{L^\infty}$ and $\|\nabla\underline{u}\|_{L^\infty}$, and $|F^{k\bar{k}}\nabla_k\nabla_{\bar{k}}\tilde{v} - \partial_t\tilde{v}| \leq C_{14}\mathcal{F}\lambda_1 + C_{13}$, where C_{14} depending on $\|\partial_t v\|_{L^\infty}$ and $\|\partial\bar{\partial}\underline{u}\|_{L^\infty}$, we arrive at

$$0 \geq \frac{\theta^2}{10nP}\mathcal{F}\lambda_1^2 - C_{15}P\mathcal{F}, \tag{2.45}$$

where C_{15} depends on $\|\chi\|_{C^2}, \|\alpha\|_{C^2}, n, \|\psi\|_{C^2}, \|\partial\bar{\partial}\underline{u}\|_{L^\infty}, \|\nabla\underline{u}\|_{L^\infty}, \|\tilde{v}\|_{L^\infty}, \|\partial_t v\|_{L^\infty}$ and $\|\partial_t u\|_{L^\infty}$. This implies the desired estimate $\lambda_1 \leq \tilde{C}\,P$.

2.2.5 The key estimate provided by subsolutions

In the second case when $\theta\lambda_1 > -\lambda_n$, we need to use the following key property of subsolutions.

Lemma 3 *Let $\underline{u}$ be a subsolution of the equation (1.1) in the sense of Definition 1 with the pair (δ, K). Then there exists a constant $C = C(\delta, K)$, so that, if $|\lambda[u] - \lambda[\underline{u}]| > K$ with K in Definition 1, then either*

$$F^{pq}(A[u])(A^p{}_q[\underline{u}] - A^p{}_q[u]) - (\partial_t\underline{u} - \partial_t u) > C\,\mathcal{F} \tag{2.46}$$

or we have for any $1 \leq i \leq n$,

$$F^{ii}(A[u]) > C\,\mathcal{F}. \tag{2.47}$$

Proof. The proof is an adaptation of the one for the elliptic version [50, Proposition 6](see also [19] for a similar argument). However, because of the time parameter t which may tend to ∞, we need to produce explicit bounds which are independent of t. As in [50], it suffices to prove that

$$\sum_{i=1}^{n} f_i(\lambda[u])(\lambda_i[\underline{u}] - \lambda_i[u]) - (\partial_t\underline{u} - \partial_t u) > C\mathcal{F}. \tag{2.48}$$

For any $(z_0, t_0) \in X \times [0, T']$, since $\underline{u}$ is a C-subsolution as in Definition 1, the set

$$A_{z_0,t_0} = \{(w, s)|\, w + \frac{\delta}{2}I \in \overline{\Gamma}_n, s \geq -\delta,\ f(\lambda[\underline{u}(z_0, t_0)] + w) - \partial_t\underline{u}(z_0, t_0) + s \leq \psi(z_0)\}$$

is compact, and $A_{z_0,t_0} \subset B_{n+1}(0, K)$. For any $(w, s) \in A_{z_0,t_0}$, then the set

$$C_{w,s} = \{v \in \mathbf{R}^n | \exists r > 0,\, w + rv \in -\delta I + \Gamma_n,\ \ f(\lambda[\underline{u}(z_0, t_0)] + w + rv) - \partial_t\underline{u}(z_0, t_0) + s = \psi(z_0)\}$$

is a cone with vertex at the origin.

We claim that $C_{w,s}$ is stricly larger than Γ_n. Indeed, for any $v \in \Gamma_n$, we can choose $r > 0$ large enough so that $|w + rv| > K$, then by the definition of C-subsolution, at (z_0, t_0)

$$f(\lambda[\underline{u}] + w + rv) - \partial_t\underline{u} + s > \psi(z_0).$$

Therefore there exist $r' > 0$ such that $f(\lambda[\underline{u}]) + w + r'v) - \partial_t\underline{u} + s = \psi(z_0)$, hence $v \in C_{w,s}$. This implies that $\Gamma_n \subset C_{w,s}$. Now, for any pair (i, j) with $i \neq j$ and $i, j = 1, \ldots, n$, we choose $v^{(i,j)} := (v_1, \ldots, v_n)$ with $v_i = K + \delta$ and $v_j = -\delta/3$ and $v_k = 0$ for $k \neq i, j$, then we have $w + v^{(i,j)} \in -\delta\mathbf{1} + \Gamma_n$. By the definition of C-subsolution, we also have, at (z_0, t_0)

$$f(\lambda[\underline{u}]) + w + v^{(i,j)}) - \partial_t\underline{u} + s > \psi(z_0),$$

hence $v^{(i,j)} \in C_{w,s}$ for any pair (i,j).

Denote by $C^*_{w,s}$ the dual cone of $C_{w,s}$,

$$C^*_{w,s} = \{x \in \mathbf{R}^n : \langle x, y\rangle > 0,\ \forall y \in C_{w,s}\}.$$

We now prove that that there is an $\varepsilon > 0$ such that if $x = (x_1, \ldots, x_n) \in C^*_{w,s}$ is a unit vector, then $x_i > \varepsilon, \forall i = 1, \ldots n$. First we remark that $x_i > 0, \forall i = 1, n$ since $\Gamma_n \subset C_{w,s}$ Suppose that x_1 is the smallest element between x_i, then $\langle x, v^{(1,j)}\rangle > 0$, implies that $(K+\delta)x_1 \geq \frac{\delta}{3}x_j$, hence $(K+\delta)^2 x_1^2 \geq (\delta^2/9)x_j^2, \forall j = 2, \ldots, n$, so $n(K+\delta)^2 x_1^2 \geq \delta^2/9$. Therefore we can choose $\varepsilon = \frac{\delta^2}{9n(K+\delta)^2}$.

Fix $(z_1, t_1) \in X \times [0, T']$ such that at this point $|\lambda[u] - \lambda[\underline{u}]| > K$. Let $\mathcal{T}$ be the tangent plane to $\{(\lambda, \tau) |\, f(\lambda) + \tau = \sigma\}$ at $(\lambda[u(z_1,t_1)], -\partial_t u(z_1,t_1))$. There are two cases:

1) There is some point $(w,s) \in A_{z_1,t_1}$ such that at (z_1,t_1)

$$(\lambda[\underline{u}] + w, -\partial_t \underline{u} + s) \in \mathcal{T},$$

i.e

$$\nabla f(\lambda[u]).(\lambda[\underline{u}] + w - \lambda[u]) + (-\partial_t \underline{u} + s + \partial_t u) = 0. \tag{2.49}$$

Now for any $v \in C_{w,s}$, there exist $r > 0$ such that $f(\lambda[\underline{u}] + w + rv) - \partial_t \underline{u} + s = \psi(z)$, this implies that

$$\nabla f(\lambda[u]).(\lambda[\underline{u}] + w + rv - \lambda[u]) + (-\partial_t \underline{u} + s + \partial_t u) > 0,$$

so combing with (2.50) we get

$$\nabla f(\lambda[u]).v > 0.$$

It follows that at (z_1,t_1) we have $\nabla f(\lambda[u])(z,t) \in C^*_{w,s}$, so $f_i(\lambda[u]) \geq \varepsilon \nabla f(\lambda[u]), \forall i = 1, \ldots, n$, hence

$$f_i(\lambda[u]) > \frac{\varepsilon}{\sqrt{n}} \sum_p f_p(\lambda[u]), \forall i = 1, \ldots, n,$$

where

$$\varepsilon = \frac{\delta^2}{9n(K+\delta)^2}.$$

2) Otherwise, we observe that if $A_{z_1,t_1} \neq \emptyset$, then $(w_0, s_0) = (-\delta/2, \ldots, -\delta/2, -\delta) \in A_{z_1,t_1}$ and at (z_1,t_1), $(\lambda[\underline{u}] - w_0, -\underline{u}_t + s_0)$ must lie above $\mathcal{T}$ in the sense that

$$(\nabla f(\lambda[u]), 1).(\lambda[\underline{u}] + w_0 - \lambda[u], -\partial_t \underline{u} + s_0 + \partial_t u) > 0, \text{ at } (z_1,t_1). \tag{2.50}$$

Indeed, if it is not the case, using the monotonicity of f we can find $v \in \Gamma_n$ such that $(\lambda[\underline{u}] + w_0 + v, -\partial_t \underline{u} + s_0) \in \mathcal{T}$, so the concavity of $(\lambda, \tau) \mapsto f(\lambda) + \tau$ implies that $(w_0 + v, s_0)$ is in A_{z_1,t_1} and then satisfies the first case, this gives a contradiction. Now it follows from (2.50) that at (z_1,t_1)

$$\begin{aligned}(\nabla f(\lambda[u]), 1).(\lambda[\underline{u}] - \lambda[u], -\partial_t \underline{u} + \partial_t u) &\geq -\nabla f(\lambda[u]).w_0 - s_0 \\ &= (\delta/2)\mathcal{F} + \delta \geq (\delta/2)\mathcal{F},\end{aligned}$$

where $\mathcal{F} = \sum_i f_i(\lambda[u]) > 0$. This means

$$\sum_{i=1}^{n} f_i(\lambda[u])(\lambda[\underline{u}] - \lambda[u]) - (\partial_t \underline{u} - u_t) > (\delta/2)\mathcal{F} \tag{2.51}$$

as required.

Now if $A_{z_1,t_1} = \emptyset$, then at (z_1, t_1)

$$f(\lambda[\underline{u}] + w_0) - \partial_t \underline{u} + s_0 > \psi(z_1),$$

hence we also have that $(\lambda[\underline{u}] + w_0, -\partial_t \underline{u} + s_0)$ lies above $\mathcal{T}$ using the concavity of $(\lambda, \tau) \mapsto f(\lambda) + \tau$. By the same argument above, we also obtain the inequality (2.51).

So we get the desired inequalities. Q.E.D.

2.2.6 Case 2: $\theta\lambda_1 > -\lambda_n$

Set

$$I = \{i;\ F^{i\bar{i}} \geq \theta^{-1} F^{1\bar{1}}\}. \tag{2.52}$$

At the maximum point $\partial_{\bar{k}} \tilde{G} = 0$, and we can write

$$\begin{aligned} -\sum_{k\notin I} F^{k\bar{k}} \frac{|\tilde{\lambda}_{1,\bar{k}}|^2}{\lambda_1^2} &= -\sum_{k\notin I} F^{k\bar{k}} |\phi' \nabla_{\bar{k}} |\nabla u|^2 + \varphi' \partial_{\bar{k}} \tilde{v}|^2 \\ &\geq -2(\phi')^2 \sum_{k\notin I} F^{k\bar{k}} |\nabla_{\bar{k}} |\nabla u|^2|^2 - 2(\varphi')^2 \sum_{k\notin I} F^{k\bar{k}} |\nabla_{\bar{k}} \tilde{v}|^2 \\ &\geq -\phi'' \sum_{k\notin I} F^{k\bar{k}} |\nabla_{\bar{k}} |\nabla u|^2|^2 - 2(\varphi')^2 \theta^{-1} F^{1\bar{1}} P - C_{16}\mathcal{F}, \end{aligned} \tag{2.53}$$

where C_{16} depends on $\|\nabla \underline{u}\|_{L^\infty}$ and $\|\tilde{v}\|_{L^\infty}$. On the other hand,

$$-2\theta \sum_{k\in I} F^{k\bar{k}} \frac{|\tilde{\lambda}_{1,\bar{k}}|^2}{\lambda_1^2} \geq -2\theta\phi'' \sum_{k\in I} F^{k\bar{k}} |\nabla_{\bar{k}} |\nabla u|^2|^2 - 4\theta(\varphi')^2 \sum_{k\in I} F^{k\bar{k}} |\nabla_{\bar{k}} \tilde{v}|^2. \tag{2.54}$$

Choose $0 < \theta << 1$ such that $4\theta(\varphi')^2 \leq \frac{1}{2}\varphi''$. Then (2.39) implies that

$$\begin{aligned} 0 \geq\ & -\frac{1}{\lambda_1} F^{l\bar{k},s\bar{r}} \nabla_{\bar{1}} g_{\bar{k}l} \nabla_1 g_{\bar{r}s} - (1-2\theta) \sum_{k\in I} F^{k\bar{k}} \frac{|\tilde{\lambda}_{1,\bar{k}}|^2}{\lambda_1^2} \\ & - C\frac{1}{\lambda_1} F^{k\bar{k}} |\tilde{\lambda}_{1,\bar{k}}| + \frac{1}{8P} F^{q\bar{q}} (|\nabla_q \nabla u|^2 + |\nabla_q \bar{\nabla} u|^2) \\ & + \frac{1}{2}\varphi'' F^{k\bar{k}} |\nabla_{\bar{k}} \tilde{v}|^2 + \varphi'(F^{k\bar{k}} \nabla_k \nabla_{\bar{k}} \tilde{v} - \partial_t \tilde{v}) - 2(\varphi')^2 \theta^{-1} F^{1\bar{1}} P - C_{17}\mathcal{F}, \end{aligned} \tag{2.55}$$

where C_{17} depend on $\|\chi\|_{C^2}, \|\alpha\|_{C^2}, n, \|\psi\|_{C^2}, \|\partial_t u\|_{L^\infty}, \|\tilde{v}\|_{L^\infty}$ and $\|\nabla \underline{u}\|_{L^\infty}$. The concavity of F implies that

$$F^{l\bar{k},s\bar{r}}\nabla_{\bar{1}} g_{\bar{k}l}\nabla_1 g_{\bar{r}s} \leq \sum_{k\in I} \frac{F^{1\bar{1}} - F^{k\bar{k}}}{\lambda_1 - \lambda_k}|\nabla_1 g_{\bar{1}k}|^2 \tag{2.56}$$

since $\frac{F^{1\bar{1}}-F^{k\bar{k}}}{\lambda_1-\lambda_k} \leq 0$. Moreover, for $k \in I$, we have $F^{1\bar{1}} \leq \theta F^{k\bar{k}}$, and the assumption $\theta\lambda_1 \geq -\lambda_n$ yields

$$\frac{1-\theta}{\lambda_1 - \lambda_k} \geq \frac{1-2\theta}{\lambda_1}. \tag{2.57}$$

It follows that

$$\sum_{k\in I} \frac{F^{1\bar{1}} - F^{k\bar{k}}}{\lambda_1 - \lambda_k}|\nabla_1 g_{\bar{1}k}|^2 \leq -\sum_{k\in I} \frac{(1-\theta)F^{k\bar{k}}}{\lambda_1 - \lambda_k}|\nabla_1 g_{\bar{1}k}|^2 \leq -\frac{1-2\theta}{\lambda_1}\sum_{k\in I} F^{k\bar{k}}|\nabla_1 g_{\bar{1}k}|^2. \tag{2.58}$$

Combining with the previous inequalities, we obtain

$$\begin{aligned} 0 \;\geq\; & -(1-2\theta)\sum_{k\in I} F^{k\bar{k}}\frac{|\tilde{\lambda}_{1,\bar{k}}|^2 - |\nabla_1 g_{\bar{1}k}|^2}{\lambda_1^2} - C_{17}\mathcal{F} \\ & -\frac{C_4}{\lambda_1}F^{k\bar{k}}|\tilde{\lambda}_{1,\bar{k}}| + \frac{1}{8P}F^{q\bar{q}}(|\nabla_q\nabla u|^2 + |\nabla_q\bar{\nabla}u|^2) \\ & +\frac{1}{2}\varphi'' F^{k\bar{k}}|\nabla_{\bar{k}}\tilde{v}|^2 + \varphi'(F^{k\bar{k}}\nabla_k\nabla_{\bar{k}}\tilde{v} - \partial_t\tilde{v}) - 2(\varphi')^2\theta^{-1}F^{1\bar{1}}P. \end{aligned} \tag{2.59}$$

Since $\nabla_1 g_{\bar{1}k} = \tilde{\lambda}_{1,\lambda} + O(\lambda_1)$, we have

$$-(1-2\theta)\sum_{k\in I} F^{k\bar{k}}\frac{|\tilde{\lambda}_{1,\bar{k}}|^2 - |\nabla_1 g_{\bar{1}k}|^2}{\lambda_1^2} \geq -C_{18}\mathcal{F} \tag{2.60}$$

where C_{18} depends on $\|\chi\|_{C^2}$ and $\|\alpha\|_{C^2}$. Next, using again the equations for critical points, we can write

$$\begin{aligned} \frac{C_4}{\lambda_1}F^{k\bar{k}}|\tilde{\lambda}_{1,\bar{k}}| \;&=\; \frac{C_4}{\lambda_1}F^{k\bar{k}}|\phi'\nabla_{\bar{k}}|\nabla u|^2 + \varphi'\nabla_{\bar{k}}\tilde{v}| \qquad (2.61)\\ &\leq\; \frac{1}{2K^{\frac{1}{2}}}\sum F^{k\bar{k}}(|\nabla_{\bar{k}}\nabla_p u| + |\nabla_{\bar{k}}\nabla_{\bar{p}}u|) + C_\varepsilon|\varphi'|F^{k\bar{k}}|\nabla_{\bar{k}}\tilde{v}|^2 + \varepsilon C_{19}|\varphi'|\mathcal{F} + C_{20}\mathcal{F}, \end{aligned}$$

where C_{19} and C_{20} depend on C_4. Accordingly, the previous inequality implies

$$\begin{aligned} 0 \;\geq\; & \frac{1}{10K}F^{q\bar{q}}(|\nabla_q\nabla u|^2 + |\nabla_q\bar{\nabla}u|^2) + \frac{1}{2}\varphi'' F^{k\bar{k}}|\nabla_{\bar{k}}\tilde{v}|^2 + \varphi'(F^{k\bar{k}}\nabla_k\nabla_{\bar{k}}\tilde{v} - \partial_t\tilde{v}) \\ & -2(\varphi')^2\theta^{-1}F^{1\bar{1}}P - C_\varepsilon|\varphi'|F^{k\bar{k}}|\nabla_{\bar{k}}\tilde{v}|^2 - \varepsilon C_{19}|\varphi'|\mathcal{F} - C_{21}\mathcal{F}, \end{aligned} \tag{2.62}$$

where C_{21} depending only on $\|\chi\|_{C^2}, \|\alpha\|_{C^2}, n, \|\psi\|_{C^2}, \|\partial_t v\|_{C^0}, \|\tilde{v}\|_{L^\infty}, \|\partial_t u\|_{L^\infty}$ and $\|\nabla \underline{u}\|_{L^\infty}$. Finally we get

$$\begin{aligned} 0 \quad \geq \quad & F^{1\bar{1}}(\frac{\lambda_1^2}{20P} - 2(\varphi')^2\theta^{-1}P) + (\frac{1}{2}\varphi'' - C_\varepsilon|\varphi'|)F^{k\bar{k}}|\nabla_{\bar{k}}\tilde{v}|^2 \\ & -\varepsilon C_{19}|\varphi'|\mathcal{F} + \varphi'(F^{k\bar{k}}\nabla_k\nabla_{\bar{k}}\tilde{v} - \partial_t\tilde{v}) - C_{21}\mathcal{F}. \end{aligned} \tag{2.63}$$

We now apply Lemma 3. Fix δ and K as in Definition 1, if $\lambda_1 > K$, then there are two possibilities:

• Either $F^{k\bar{k}}(\underline{u}_{\bar{k}k} - u_{\bar{k}k}) + (\partial_t u - \partial_t \underline{u}) \geq \kappa\mathcal{F}$, for some κ depending only on δ and K, equivalently,

$$F^{k\bar{k}}\nabla_k\nabla_{\bar{k}}\tilde{v} - \partial_t\tilde{v} - \int_X \partial_t v\alpha^n \leq -\kappa\mathcal{F} + C_{22}\mathcal{F}, \tag{2.64}$$

where C_{22} depends on $\|\partial_t v\|_{L^\infty}$. Since $\varphi' < 0$, we find

$$\begin{aligned} 0 \quad \geq \quad & F^{1\bar{1}}(\frac{\lambda_1^2}{20P} - 2(\varphi')^2\theta^{-1}P) + (\frac{1}{2}\varphi'' - C_\varepsilon|\varphi'|)F^{k\bar{k}}|\nabla_{\bar{k}}\tilde{v}|^2 \\ & -C_{23}\mathcal{F} - \varepsilon C_{19}|\varphi'|\mathcal{F} - \varphi'\kappa\mathcal{F} \end{aligned} \tag{2.65}$$

with C_{23} depending only on $n, \|\chi\|_{C^2}, \|\alpha\|_{C^2}, \|\psi\|_{C^2}, \|\partial_t v\|_{L^\infty}, \|\tilde{v}\|_{L^\infty}, \|\partial_t u\|_{L^\infty}$ and $\|\nabla \underline{u}\|_{L^\infty}$. We first choose ε small enough so that $\varepsilon C_{19} < \kappa/2$, then D_2 large enough so that $\varphi'' > 2C_\varepsilon|\varphi'|$. We obtain

$$0 \geq F^{1\bar{1}}(\frac{\lambda_1^2}{20P} - 2(\varphi')^2\theta^{-1}P) - C_{23}\mathcal{F} - \frac{1}{2}\varphi'\kappa\mathcal{F}. \tag{2.66}$$

We now choose D_1 large enough (depending on $\|\tilde{v}\|_{L^\infty}$) so that $-C_{23} - \frac{1}{2}\varphi'\kappa > 0$. Then

$$\frac{\lambda_1^2}{20P} \leq 2(\varphi')^2\theta^{-1}P \tag{2.67}$$

and the desired upper bound for λ_1/P follows.

• Or $F^{1\bar{1}} \geq \kappa\mathcal{F}$. With D_1, D_2, and θ as above, the inequality (2.63) implies

$$0 \geq \kappa\mathcal{F}(\frac{\lambda_1^2}{20P} - 2(\varphi')^2\theta^{-1}P) - C_{24}\mathcal{F} - \varphi' F^{k\bar{k}}g_{\bar{k}k}, \tag{2.68}$$

with C_{24} depending only on $\|\chi\|_{C^2}, \|\alpha\|_{C^2}, n, \|\psi\|_{C^2}, \|\partial_t v\|_{L^\infty}, \|\tilde{v}\|_{L^\infty}, \|\partial_t u\|_{L^\infty}$, $\|\nabla \underline{u}\|_{L^\infty}$, and $\|i\partial\bar{\partial}\underline{u}\|_{L^\infty}$. Since $F^{k\bar{k}}g_{\bar{k}k} \leq \mathcal{F}\lambda_1$, we can divide by $\mathcal{F}P$ to get

$$0 \geq \kappa\frac{\lambda_1^2}{20P^2} - C_{25}(1 + \frac{1}{P} + \frac{\lambda_1}{P}) \tag{2.69}$$

with a constant C_{25} depending only on $\|\chi\|_{C^2}, \|\alpha\|_{C^2}, n, \|\psi\|_{C^2}, \|\tilde{v}\|_{L^\infty}, \|\partial_t v\|_{L^\infty}, \|\partial_t u\|_{L^\infty}$, $\|\nabla \underline{u}\|_{L^\infty}$, and $\|i\partial\bar{\partial}\underline{u}\|_{L^\infty}$. Thus we obtain the desired bound for λ_1/P.

It was pointed out in [50] that, under an extra concavity condition on f, C^2 estimates can be derived directly from C^0 estimates in the elliptic case, using a test function introduced in [39]. The same holds in the parabolic case, but we omit a fuller discussion.

2.3 C^1 Estimates

The C^1 estimates are also adapted from [50], which reduce the estimates by a blow-up argument to a key Liouville theorem for Hessian equations due to Székelyhidi [50] and Dinew and Kolodziej [10].

Lemma 4 *There exist a constant $C > 0$, depending on $\underline{u}$, $\|\partial_t u\|_{L^\infty(X\times[0,T))}$, $\|\tilde{u}\|_{L^\infty(X\times[0,T))}$ $\|\alpha\|_{C^2}, \chi, \psi$ and the constant $\tilde{C}$ in Lemma 2 such that*

$$\sup_{X\times[0,T)} |\nabla u|^2_\alpha \leq C. \tag{2.70}$$

Proof. Assume by contradiction that (2.70) does not hold. Then there exists a sequence $(x_k, t_k) \in X \times [0, T)$ with $t_k \to T$ such that

$$\lim_{k\to\infty} |\nabla u(t_k, x_k)|_\alpha = +\infty.$$

We can assume further that

$$R_k = |\nabla u(x_k, t_k)|_\alpha = \sup_{X\times[0,t_k]} |\nabla u(x, t)|_\alpha, \quad as \quad k \to +\infty,$$

and $\lim_{k\to\infty} x_k = x$.

Using localization, we choose a coordinate chart $\{U, (z_1, \ldots, z_n)\}$ centered at x, identifying with the ball $B_2(0) \subset \mathbf{C}^n$ of radius 2 centered at the origin such that $\alpha(0) = \beta$, where $\beta = \sum_j idz^j \wedge d\bar{z}^j$. We also assume that k is sufficiently large so that $z_k := z(x_k) \in B_1(0)$.

Define the following maps

$$\begin{aligned}
&\Phi_k : \mathbf{C}^n \to \mathbf{C}^n, \quad \Phi_k(z) := R_k^{-1} z + z_k, \\
&\tilde{u}_k : B_{R_k}(0) \to \mathbf{R}, \quad \tilde{u}_k(z) := \tilde{u}(\Phi_k(z), t_k) = \tilde{u}(R_k^{-1} z + z_k, t_k),
\end{aligned}$$

where $\tilde{u} = u - \int_X u\, \alpha^n$. Then the equation

$$u_t = F(A) - \psi(z),$$

implies that

$$f\left(R_k^2 \lambda[\beta_k^{i\bar{p}}(\chi_{k,\bar{p}j} + \tilde{u}_{k,\bar{p}j})]\right) = \psi(R_k^{-1} z + z_k) + u_t(\Phi_k(z), t_k), \tag{2.71}$$

where $\beta_k := R_k^2 \Phi_k^* \alpha$, $\chi_k := \Phi_k^* \chi$. Since $\beta_k \to \beta$, and $\chi_k(z, t) \to 0$, in C^∞_{loc} as $k \to \infty$, we get

$$\lambda[\beta_k^{i\bar{p}}(\chi_{k,\bar{p}j} + \tilde{u}_{k,\bar{p}j})] = \lambda(\tilde{u}_{k,\bar{j}i}) + O\left(\frac{|z|}{R_k^2}\right). \tag{2.72}$$

By the construction, we have

$$\sup_{B_{R_k}(0)} \tilde{u}_k \leq C, \quad \sup_{B_{R_k}(0)} |\nabla \tilde{u}_k| \leq C \tag{2.73}$$

where C depending on $\|\tilde{u}\|_{L^\infty}$, and

$$|\nabla \tilde{u}_k|(0) = R_k^{-1} |\nabla u_k|_\alpha(x_k) = 1.$$

Thanks to Lemma 2, we also have that

$$\sup_{B_{R_k}(0)} |\partial\bar{\partial}\tilde{u}_k|_\beta \leq C R_k^{-2} \sup_X |\partial\bar{\partial} u(., t_k)|_\alpha \leq C'. \tag{2.74}$$

As the argument in [50, 57], it follows from (2.73), (2.74), the elliptic estimates for Δ and the Sobolev embedding that for each given $K \subset \mathbf{C}^n$ compact, $0 < \gamma < 1$ and $p > 1$, there is a constant C such that

$$\|\tilde{u}_k\|_{C^{1,\gamma}(K)} + \|\tilde{u}_k\|_{W^{2,p}(K)} \leq C.$$

Therefore there is a subsequence of $\tilde{u}_k$ converges strongly in $C^{1,\gamma}_{loc}(\mathbf{C}^n)$, and weakly in $W^{2,p}_{loc}(\mathbf{C}^n)$ to a function v with $\sup_{\mathbf{C}^n}(|v| + |\nabla v|) \leq C$ and $\nabla v(0) \neq 0$, in particular v is not constant.

The proof can now be completed exactly as in [50]. The function v is shown to be a Γ-solution in the sense of Székelyhidi [50, Definition 15], and the fact that v is not constant contradicts Szekelyhidi's Liouville theorem for Γ-solutions [50, Theorem 20], which is itself based on the Liouville theorem of Dinew and Kolodziej [10]. Q.E.D.

2.4 Higher Order Estimates

Under the conditions on $f(\lambda)$, the uniform parabolicity of the equation (1.1) will follow once we have established an a priori estimate on $\|i\partial\bar{\partial}u\|_{L^\infty}$ and hence an upper bound for the eigenvalues $\lambda[u]$. However, we shall often not have uniform control of $\|u(\cdot, t)\|_{L^\infty}$. Thus we shall require the following version of the Evans-Krylov theorem for uniformly parabolic and concave equations, with the precise dependence of constants spelled out, and which can be proved using the arguments of Trudinger [60], and more particularly Tosatti-Weinkove [53] and Gill [17].

Lemma 5 *Assume that u is a solution of the equation (1.1) on $X \times [0, T)$ and that there exists a constant C_0 with $\|i\partial\bar{\partial}u\|_{L^\infty} \leq C_0$. Then there exist positive constants C and $\gamma \in (0, 1)$ depending only on α, χ, C_0 and $\|\psi\|_{C^2}$ such that*

$$\|i\partial\bar{\partial}u\|_{C^\gamma(X\times[0,T))} \leq C. \tag{2.75}$$

Once the C^γ estimate for $i\partial\bar{\partial}u$ has been established, it is well known that a priori estimates of arbitrary order follow by bootstrap, as shown in detail for the Monge-Ampère equation in Yau [65]. We omit reproducing the proofs.

3 Proof of Theorems 1 and 2

We begin with the following simple lemma, which follows immediately by differentiating the equation (1.1) with respect to t, and applying the maximum principle, which shows that the solution of a linear heat equation at any time can be controlled by its initial value:

Lemma 6 *Let $u(z,t)$ be a smooth solution of the flow (1.1) on any time interval $[0,T)$. Then $\partial_t u$ satisfies the following linear heat equation*

$$\partial_t(\partial_t u) = F^j{}_k \alpha^{k\bar{m}} \partial_j \partial_{\bar{m}} (\partial_t u) \tag{3.1}$$

and we have the following estimate for any $t \in [0,T)$,

$$\min_X (F(A[u_0]) - \psi) \leq \partial_t u(t,\cdot) \leq \max_X F(A[u_0] - \psi) \tag{3.2}$$

We can now prove a lemma which provides general sufficient conditions for the convergence of the flow:

Lemma 7 *Consider the flow (1.1). Assume that the equation admits a parabolic C-subsolution $\underline{u} \in C^{2,1}(X \times [0,\infty))$, and that there exists a constant C independent of time so that*

$$\mathrm{osc}_X u(t,\cdot) \leq C. \tag{3.3}$$

Then a smooth solution $u(z,t)$ exists for all time, and its normalization $\tilde{u}$ converges in C^∞ to a solution u_∞ of the equation (1.6) for some constant c.

In particular, if we assume further that $\|u\|_{L^\infty(X\times[0,\infty))} \leq C$ and for each $t > 0$, there exists $y = y(t) \in X$ such that $\partial_t u(y,t) = 0$, then u converges in C^∞ to a solution u_∞ of the equation (1.6) for the constant $c = 0$.

Proof of Lemma 7. We begin by establishing the existence of the solution for all time. For any fixed $T > 0$, Lemma 6 shows that $|\partial_t u|$ is uniformly bounded by a constant C. Integrating between 0 and T, we deduce that $|u|$ is uniformly bounded by $C\,T$. We can now apply Lemma 4, 2, 5, to conclude that the function u is uniformly bounded in C^k norm (by constants depending on k and T) for arbitrary k. This implies that the solution can be extended beyond T, and since T is arbitrary, that it exists for all time.

Next, we establish the convergence. For this, we adapt the arguments of Cao [2] and especially Gill [17] based on the Harnack inequality.

Since $\mathrm{osc}_X u(t,\cdot)$ is uniformly bounded by assumption, and since $\partial_t u$ is uniformly bounded in view of Lemma 6, we can apply Lemma 2 and deduce that the eigenvalues of the matrix $[\chi + i\partial\bar{\partial} u]$ are uniformly bounded over the time interval $[0,\infty)$. The

uniform ellipticity of the equation (3.5) follows in turn from the properties (1) and (2) of the function $f(\lambda)$. Next set

$$v = \partial_t u + A \tag{3.4}$$

for some large constant A so that $v > 0$. The function v satisfies the same heat equation

$$\partial_t v = F^{i\bar{j}} \partial_i \partial_{\bar{j}} v. \tag{3.5}$$

Since the equation (3.5) is uniformly elliptic, by the differential Harnack inequality proved originally in the Riemannian case by Li and Yau in [29], and extended to the Hermitian case by Gill [17], section 6, it follows that there exist positive constants C_1, C_2, C_3, depending only on ellipticity bounds, so that for all $0 < t_1 < t_2$, we have

$$\sup_X v(\cdot, t_1) \leq \inf_X v(\cdot, t_2) \left(\frac{t_2}{t_1}\right)^{C_2} \exp\left(\frac{C_3}{t_2 - t_1} + C_1(t_2 - t_1)\right). \tag{3.6}$$

The same argument as in Cao [2], section 2, and Gill [17], section 7, shows that this estimate implies the existence of constants C_4 and $\eta > 0$ so that

$$\mathrm{osc}_X v(\cdot, t) \leq C_4 e^{-\eta t} \tag{3.7}$$

If we set

$$\tilde{v}(z,t) = v(z,t) - \frac{1}{V}\int_X v\, \alpha^n = \partial_t u(z,t) - \frac{1}{V}\int_X \partial_t u\, \alpha^n = \partial_t \tilde{u}, \tag{3.8}$$

it follows that

$$|\tilde{v}(z,t)| \leq C_4 e^{-\eta t} \tag{3.9}$$

for all $z \in X$. In particular,

$$\partial_t(\tilde{u} + \frac{C_4}{\eta} e^{-\eta t}) = \tilde{v} - C_4 e^{-\eta t} \leq 0, \tag{3.10}$$

and the function $\tilde{u}(z,t) + \frac{C_4}{\eta} e^{-\eta t}$ is decreasing in t. By the assumption (3.3), this function is uniformly bounded. Thus it converges to a function $u_\infty(z)$. By the higher order estimates in section §2, the derivatives to any order of $\tilde{u}$ are uniformly bounded, so the convergence of $\tilde{u} + \frac{C_4}{\eta} e^{-\eta t}$ is actually in C^∞ The function $\tilde{u}(z,t)$ will also converge in C^∞, to the same limit $u_\infty(z)$. Now the function $\tilde{u}(z,t)$ satisfies the following flow,

$$\partial_t \tilde{u} = F(A[\tilde{u}]) - \psi(z) - \frac{1}{V}\int_X \partial_t u\, \alpha^n. \tag{3.11}$$

Taking limits, we obtain

$$0 = F(A[\tilde{u}_\infty]) - \psi(z) - \lim_{t\to\infty} \int_X \partial_t u\, \alpha^n \tag{3.12}$$

where the existence of the limit of the integral on the right hand side follows from the equation. Define the constant c as the value of this limit. This implies the first statement in Lemma 7.

Now we assume that $\|u\|_{L^\infty(X\times[0,\infty))} \leq C$ and for each $t \geq 0$, there exists $y = y(t) \in X$ such that $\partial_t u(y,t) = 0$. By the same argument above, we have

$$\mathrm{osc}_X \partial_t u(\cdot,t) \leq C_4 e^{-\eta t}, \tag{3.13}$$

for some $C_4, \eta > 0$. Since for each $t \geq 0$, there exists $y = y(t) \in X$ such that $\partial_t u(y,t) = 0$, we imply that for any $z \in X$,

$$|\partial_t u(z,t)| = |\partial_t u(z,t) - \partial_t u(y,t)| \leq \mathrm{osc}_X \partial_t u(\cdot,t) \leq C_4 e^{-\eta t}. \tag{3.14}$$

Therefore by the same argument above, the function $u(z,t) + \frac{C_4}{\eta} e^{-\eta t}$ converges in C^∞ and $\partial_t u$ converges to 0 as $t \to +\infty$. We thus infer that u converges in C^∞, to u_∞ satisfying the equation

$$F(A[\tilde{u}_\infty]) = \psi(z). \tag{3.15}$$

Lemma 7 is proved.

Proof of Theorem 1. Since f is unbounded, the function $\underline{u} = u_0$ is a C-subsolution of the flow. In view of Lemma 7, it suffices to establish a uniform bound for $\mathrm{osc}_X u(t,\cdot)$. But the flow can be re-expressed as the elliptic equation

$$F(A) = \psi + \partial_t u \tag{3.16}$$

where the right hand side $\psi + \partial_t u$ is bounded uniformly in t, since we have seen that $\partial_t u$ is uniformly bounded in t. Furthermore, because f is unbounded, the function $\underline{u} = u_0$ is a C-subsolution of (3.16). By the C^0 estimate of [50], the oscillation $\mathrm{osc}_X u(t,\cdot)$ can be bounded for each t by the C^0 norm of the right hand side, and is hence uniformly bounded. Q.E.D.

Proof of Theorem 2. Again, it suffices to establish a uniform bound in t for $\mathrm{osc}_X u(t,\cdot)$.

Consider first the case (a). In view of Lemma 6 and the hypothesis, we have

$$\partial_t \underline{u} \geq \partial_t u \tag{3.17}$$

on all of $X \times [0,\infty)$. But if we rewrite the flow (1.1) as

$$F(A) = \psi + \partial_t u \tag{3.18}$$

we see that the condition that $\underline{u}$ be a parabolic C-subsolution for the equation (1.1) together with (3.17) implies that $\underline{u}$ is a C-subsolution for the equation (3.18) in the elliptic

sense.. We can then apply Székelyhidi's C^0 estimate for the elliptic equation to obtain a uniform bound for $\mathrm{osc}_X u(t,\cdot)$.

Next, we consider the case (b). In this case, the existence of a function $h(t)$ with the indicated properties allows us to apply Lemma 1, and obtain immediately a lower bound,

$$u - \underline{u} - h(t) \geq -C \tag{3.19}$$

for some constant C independent of time. The inequality (1.9) implies than a uniform bound for $\mathrm{osc}_X u$.

4 Applications to Geometric Flows

Theorems 1 and 2 can be applied to many geometric flows. We should stress that they don't provide a completely independent approach, as they themselves are built on many techniques that had been developed to study these flows. Nevertheless, they may provide an attractive uniform approach.

4.1 A criterion for subsolutions

In practice, it is easier to verify that a given function $\underline{u}$ on $X \times [0,\infty)$ is a C-subsolution of the equation (1.1) using the following lemma rather than the original Definition 1:

Lemma 8 *Let $\underline{u}$ be a $C^{2,1}$ admissible function on $X \times [0,\infty)$, with $\|\underline{u}\|_{C^{2,1}(X\times[0,\infty))} < \infty$. Then $\underline{u}$ is a parabolic C-subsolution in the sense of Definition 1 if and only if there exists a constant $\tilde{\delta} > 0$ independent from (z,t) so that*

$$\lim_{\mu\to+\infty} f(\lambda[\underline{u}(z,t)] + \mu e_i) - \partial_t \underline{u}(z,t) > \tilde{\delta} + \psi(z) \tag{4.1}$$

for each $1 \leq i \leq n$. In particular, if $\underline{u}$ is independent of t, then $\underline{u}$ is a parabolic C-subsolution if and only if

$$\lim_{\mu\to+\infty} f(\lambda[\underline{u}(z,t)] + \mu e_i) > \psi(z). \tag{4.2}$$

Note that there is a similar lemma in the case of subsolutions for elliptic equations (see [50], Remark 8). Here the argument has to be more careful, not just because of the additional time parameter t, but also because the time interval $[0,\infty)$ is not bounded, invalidating certain compactness arguments.

Proof of Lemma 8. We show first that the condition (4.1) implies that $\underline{u}$ is a C-subsolution.

We begin by showing that the condition (4.1) implies that there exists $\epsilon_0 > 0$ and $M > 0$, so that for all $\epsilon \leq \epsilon_0$, all $\nu > M$, all (z,t), and all $1 \leq i \leq n$, we have

$$f(\lambda[\underline{u}(z,t)] - \epsilon I + \nu e_i) - \partial_t \underline{u}(z,t) > \frac{\tilde{\delta}}{4} + \psi(z). \tag{4.3}$$

This is because the condition (4.1) is equivalent to

$$f_\infty(\lambda'[\underline{u}(z,t)]) - \partial_t \underline{u}(z,t) > \tilde{\delta} + \psi(z). \tag{4.4}$$

Now the concavity of $f(\lambda)$ implies the concavity of its limit $f_\infty(\lambda')$ and hence the continuity of $f_\infty(\lambda')$. Furthermore, the set

$$\Lambda = \overline{\{\lambda[\underline{u}(z,t)], \forall (z,t) \in X \times [0,\infty)\}}, \tag{4.5}$$

as well as any of its translates by $-\epsilon I$ for a fixed ϵ small enough, is compact in Γ. So are their projections on $\mathbf{R}^{n-1}$. By the uniform continuity of continuous functions on compact sets, it follows that there exists $\epsilon_0 > 0$ so that

$$f_\infty(\lambda'[\underline{u}(z,t)] - \epsilon I) - \partial_t \underline{u}(z,t) > \frac{\tilde{\delta}}{2} + \psi(z) \tag{4.6}$$

for all (z,t) and all $\epsilon \leq \epsilon_0$. But f_∞ is the continuous limit of a sequence of monotone increasing continuous functions

$$f_\infty(\lambda' - \epsilon I) = \lim_{\nu \to \infty} f(\lambda - \epsilon I + \nu e_i). \tag{4.7}$$

By Dini's theorem, the convergence is uniform over any compact subset. Thus there exists $M > 0$ large enough so that $\nu > M$ implies that

$$f(\lambda[\underline{u}(z,t)] - \epsilon I + \nu e_i) > f_\infty(\lambda'[\underline{u}(z,t)] - \epsilon I) - \frac{\tilde{\delta}}{4} \tag{4.8}$$

for all (z,t) and all $\epsilon \leq \epsilon_0$. The desired inequality (4.3) follows from (4.6) and (4.8).

Assume now that $\underline{u}$ is not a C-subsolution. Then there exists ϵ_m, ν_m, τ_m, with $\epsilon_m \to 0$, $\nu_m \in -\epsilon_m I + \Gamma_n$, $\tau_m > -\epsilon_m$, and $|\tau_m| + |\nu_m| \to \infty$, so that

$$f(\lambda[\underline{u}(z_m,t_m)] + \nu_m) - \partial_t \underline{u}(z_m,t_m) + \tau_m = \psi(z_m,t_m). \tag{4.9}$$

Set $\nu_m = -\epsilon_m + \mu_m$, with $\mu_m \in \Gamma_n$. Then we can write

$$\begin{aligned} \tau_m &= -f(\lambda[\underline{u}(z_m,t_m)] - \epsilon_m I + \mu_m) + \partial_t \underline{u}(z_m,t_m) + \psi(z_m,t_m) \\ &\leq -f(\lambda[\underline{u}(z_m,t_m)] - \epsilon_m I) + \partial_t \underline{u}(z_m,t_m) + \psi(z_m,t_m) \end{aligned} \tag{4.10}$$

which is bounded by a constant. Thus we must have $|\nu_m|$ tending to $+\infty$, or equivalently, $|\mu_m|$ tending to $+\infty$.

By going to a subsequence, we may assume that there is an index i for which the i-th components μ_m^i of the vector μ_m tend to ∞ as $m \to \infty$. By the monotonicity of f in each component, we have

$$\begin{aligned} f(\lambda[\underline{u}(z_m,t_m)] - \epsilon_m I + \mu_m^i e_i) - \partial_t \underline{u}(z_m,t_m) &\leq f(\lambda[\underline{u}(z_m,t_m)] - \epsilon_m I + \mu_m) - \partial_t \underline{u}(z_m,t_m) \\ &= f(\lambda[\underline{u}(z_m,t_m)] + \nu_m) - \partial_t \underline{u}(z_m,t_m). \end{aligned}$$

In view of (4.3), the left hand side is $\geq \frac{\tilde{\delta}}{4} + \psi(z_m, t_m)$ for μ_m^i large and ϵ_m small enough. On the other hand, the equation (4.9) implies that the right hand side is equal to $\psi(z_m, t_m) - \tau_m$. Thus we obtain

$$\frac{\tilde{\delta}}{4} + \psi(z_m, t_m) \leq \psi(z_m, t_m) - \tau_m \leq \psi(z_m, t_m) + \epsilon_m. \tag{4.11}$$

Hence $\frac{\tilde{\delta}}{4} \leq \epsilon_m$, which is a contradiction, since $\epsilon_m \to 0$.

Finally, we show that if $\underline{u}$ is a subsolution, it must satisfy the condition (4.1). Assume otherwise. Then there exists an index i and a sequence $\delta_m \to 0$ and points (z_m, t_m) so that

$$\lim_{\nu \to \infty} f(\lambda[\underline{u}(z_m, t_m)] + \nu e_i) - \partial_t \underline{u}(z_m, t_m) \leq \delta_m + \psi(z_m). \tag{4.12}$$

Since f is increasing in ν, this implies that for any $\nu \in \mathbf{R}_+$, we have

$$f(\lambda[\underline{u}(z_m, t_m)] + \nu e_i) - \partial_t \underline{u}(z_m, t_m) \leq \delta_m + \psi(z_m). \tag{4.13}$$

For each $\nu \in \mathbf{R}_+$, define τ_m by the equation

$$f(\lambda[\underline{u}(z_m, t_m)] + \nu e_i) - \partial_t \underline{u}(z_m, t_m) + \tau_m = \psi_m. \tag{4.14}$$

The previous inequality means that $\tau_m \geq -\delta_m$, and thus the pair $(\tau_m, \mu = \nu e_i)$ satisfy the equation (1.2). Since we can take $\nu \to +\infty$, this contradicts the defining property of C-subsolutions. The proof of Lemma 8 is complete.

4.2 Székelyhidi's theorem

Theorem 2 can be applied to provide a proof by parabolic methods of the following theorem originally proved by Székelyhidi [50]:

Corollary 1 *Let (X, α) be a compact Hermitian manifold, and $f(\lambda)$ be a function satisfying the conditions (1-3) spelled out in §1 and in the bounded case. Let ψ be a smooth function on X. If there exists an admissible function u_0 with $F(A[u_0]) \leq \psi$, and if the equation $F(A[u]) = \psi$ admits a C-subsolution in the sense of [50], then the equation $F(A[u]) = \psi + c$ admits a smooth solution for some constant c.*

Proof of Corollary 1. It follows from Lemma 8 that a C-subsolution in the sense of [50] of the elliptic equation $F(A[u]) = \psi$ can be viewed as a time-independent parabolic C-subsolution $\underline{u}$ of the equation (1.1). Consider this flow with initial value u_0. Then

$$\partial_t \underline{u} = 0 \geq F(A[u_0]) - \psi. \tag{4.15}$$

Thus condition (a) of Theorem 2 is satisfied, and the corollary follows.

4.3 The Kähler-Ricci flow and the Chern-Ricci flow

On Kähler manifolds (X, α) with $c_1(X) = 0$, the Kähler-Ricci flow is the flow $\dot{g}_{\bar{k}j} = -R_{\bar{k}j}$. For initial data in the Kähler class $[\alpha]$, the evolving metric can be expressed as $g_{\bar{k}j} = \alpha_{\bar{k}j} + \partial_j \partial_{\bar{k}} \varphi$, and the flow is equivalent to the following Monge-Ampère flow,

$$\partial_t \varphi = \log \frac{(\alpha + i\partial\bar{\partial}\varphi)^n}{\alpha^n} - \psi(z) \tag{4.16}$$

for a suitable function $\psi(z)$ satisfying the compatibility condition $\int_X e^{\psi} \alpha^n = \int_X \alpha^n$. The convergence of this flow was proved by Cao[2], thus giving a parabolic proof of Yau's solution of the Calabi conjecture [65]. We can readily derive Cao's result from Theorem 1:

Corollary 2 *For any initial data, the normalization $\tilde{\varphi}$ of the flow (4.16) converges in C^∞ to a solution of the equation $(\alpha + i\partial\bar{\partial}\varphi)^n = e^{\psi}\alpha^n$.*

Proof of Corollary 2. The Monge-Ampère flow (4.16) corresponds to the equation (1.1) with $\chi = \alpha$, $f(\lambda) = \log \prod_{j=1}^n \lambda_j$, and Γ being the full octant Γ_n. It is straightforward that f satisfies the condition (1-3) in §1. In particular f is in the unbounded case, and Theorem 1 applies, giving the convergence of the normalizations $\tilde{u}(\cdot, t)$ to a smooth solution of the equation $(\alpha + i\partial\bar{\partial}\varphi)^n = e^{\psi + c}\alpha^n$ for some constant c. Integrating both sides of this equation and using the compatibility condition on ψ, we find that $c = 0$. The corollary is proved.

The generalization of the flow (4.16) to the more general set-up of a compact Hermitian manifold (X, α) was introduced by Gill [17]. It is known as the Chern-Ricci flow, with the Chern-Ricci tensor $Ric^C(\omega) = -i\partial\bar{\partial} \log \omega^n$ playing the role of the Ricci tensor in the Kähler-Ricci flow (we refer to [54, 55, 56, 59] and references therein). Gill proved the convergence of this flow, thus providing an alternative proof of the generalization of Yau's theorem proved earlier by Tosatti and Weinkove [52]. Generalizations of Yau's theorem had attracted a lot of attention, and many partial results had been obtained before, including those of Cherrier [3], Guan-Li [20], and others. Theorem 1 gives immediately another proof of Gill's theorem:

Corollary 3 *For any initial data, the normalizations $\tilde{\varphi}$ of the Chern-Ricci flow converge in C^∞ to a solution of the equation $(\alpha + i\partial\bar{\partial}\varphi)^n = e^{\psi + c}\alpha^n$, for some constant c.*

We note that there is a rich literature on Monge-Ampère equations, including considerable progress using pluripotential theory. We refer to [26, 11, 8, 23, 24, 41, 58, 59, 30, 31] and references therein.

4.4 Hessian flows

Hessian equations, where the Laplacian or the Monge-Ampère determinant of the unknown function u are replaced by the k-th symmetric polynomial of the eigenvalues of the Hessian

of u, were introduced by Caffarelli, Nirenberg, and Spruck [5]. More general right hand sides and Kähler versions were considered respectively by Chou and Wang [6] and Hou-Ma-Wu [25], who introduced in the process some of the key techniques for C^2 estimates that we discussed in §2. A general existence result on compact Hermitian manifolds was recently obtained by Dinew and Kolodziej [10], Sun [47], and Székelyhidi [50]. See also Zhang [66]. Again, we can derive this theorem as a corollary of Theorem 1:

Corollary 4 *Let (X, α) be a compact Hermitian n-dimensional manifold, and let χ be a positive real $(1,1)$-form which is k-positive for a given k, $1 \leq k \leq n$. Consider the following parabolic flow for the unknown function u,*

$$\partial_t u = \log \frac{(\chi + i\partial\bar\partial u)^k \wedge \alpha^{n-k}}{\alpha^n} - \psi(z). \tag{4.17}$$

Then for any admissible initial data u_0, the flow admits a solution $u(z,t)$ for all time, and its normalization $\tilde u(z,t)$ converge in C^∞ to a function $u_\infty \in C^\infty(X)$ so that $\omega = \chi + i\partial\bar\partial u_\infty$ satisfies the following k-Hessian equation,

$$\omega^k \wedge \alpha^{n-k} = e^{\psi + c}\alpha^n. \tag{4.18}$$

Proof of Corollary 4. This is an equation of the form (1.1), with $F = f(\lambda) = \log \sigma_k(\lambda)$, defined on the cone

$$\Gamma_k = \{\lambda;\, \sigma_j(\lambda) > 0,\ j = 1, \cdots, k\}, \tag{4.19}$$

where $\binom{n}{k}\sigma_k$ is the k-th symmetric polynomial in the components λ_j, $1 \leq j \leq n$. In our setting,

$$\sigma_k(\lambda[u]) = \frac{(\chi + i\partial\bar\partial u)^k \wedge \alpha^{n-k}}{\alpha^n}. \tag{4.20}$$

It follows from [43, Corollary 2.4] that $g = \sigma_k^{1/k}$ is concave and $g_i = \frac{\partial g}{\partial \lambda_i} > 0$ on Γ_k, hence $f = \log g$ satisfies the conditions (1-3) mentioned in §1.

The function $\underline{u} = 0$ is a subsolution of (4.17) and f is in the unbounded case since for any $\mu = (\mu_1, \cdots, \mu_n) \in \Gamma_k$, and any $1 \leq i \leq n$,

$$\lim_{s\to\infty} \log \sigma_k(\mu_1, \cdots, \mu_i + s, \cdots, \mu_n) = \infty. \tag{4.21}$$

The desired statement follows then from Theorem 1.

4.5 The J flow and quotient Hessian flows

The J-flow on Kähler manifolds was introduced independently by Donaldson [9] and Chen [4]. The case $n = 2$ was solved by Weinkove [63, 64], and the case of general dimension by

Song and Weinkove [42], who identified a necessary and sufficient condition for the long-time existence and convergence of the flow as the existence of a Kähler form χ satisfying

$$nc\chi^{n-1} - (n-1)\chi^{n-2} \wedge \omega > 0 \tag{4.22}$$

in the sense of positivity of $(n-1, n-1)$-forms. The constant c is actually determined by cohomology. Their work was subsequently extended to inverse Hessian flows on Kähler manifolds by Fang, Lai, and Ma [12], and to inverse Hessian flows on Hermitian manifolds by Sun [44]. These flows are all special cases of quotient Hessian flows on Hermitian manifolds. Their stationary points are given by the corresponding quotient Hessian equations. Our results can be applied to prove the following generalization to quotient Hessian flows of the results of [63, 64, 12], as well as an alternative proof of a result of Székelyhidi [50, Proposition 22] on the Hessian quotient equations. The flow (4.24) below has also been studied recently by Sun [46] where he obtained a uniform C^0 estimate using Moser iteration. Our proof should be viewed as different from all of these, since its C^0 estimate uses neither Moser iteration nor strict C^2 estimates $\mathrm{Tr}_\alpha \chi_u \leq C\, e^{u - \inf_X u}$.

Corollary 5 *Assume that (X, α) is a compact Kähler n-manifold, and fix $1 \leq \ell < k \leq n$. Fix a closed $(1,1)$-form χ which is k-positive, and assume that there exists a function $\underline{u}$ so that the form $\chi' = \chi + i\partial\bar\partial \underline{u}$ is closed k-positive and satisfies*

$$kc\,(\chi')^{k-1} \wedge \alpha^{n-k} - \ell (\chi')^{\ell-1} \wedge \alpha^{n-\ell} > 0 \tag{4.23}$$

in the sense of the positivity of $(n-1, n-1)$-forms. Here $c = \frac{[\chi^\ell] \cup [\alpha^{n-\ell}]}{[\chi^k] \cup [\chi^{n-k}]}$. Then for any admissible initial data $u_0 \in C^\infty(X)$, the flow

$$\partial_t u = c - \frac{\chi_u^\ell \wedge \alpha^{n-\ell}}{\chi_u^k \wedge \alpha^{n-k}} \tag{4.24}$$

admits a solution u for all time, and it converges to a smooth function u_∞. The form $\omega = \chi + i\partial\bar\partial u_\infty$ is k-positive and satisfies the equation

$$\omega^\ell \wedge \alpha^{n-\ell} = c\, \omega^k \wedge \alpha^{n-k}. \tag{4.25}$$

Proof of Corollary 5. The flow (4.24) is of the form (1.1), with

$$f(\lambda) = -\frac{\sigma_\ell(\lambda)}{\sigma_k(\lambda)},$$

defined on the cone

$$\Gamma_k = \{\lambda;\, \sigma_j(\lambda) > 0,\ j = 1, \cdots, k\}. \tag{4.26}$$

By the Maclaurin's inequality (cf. [43]), we have $\sigma_k^{1/k} \leq \sigma_\ell^{1/\ell}$ on Γ_k, hence $f(\lambda) \to -\infty$ as $\lambda \to \partial\Gamma_k$. It follows from [43, Theorem 2.16] that the function $g = (\sigma_k/\sigma_\ell)^{\frac{1}{(k-\ell)}}$ satisfies

$g_i = \frac{\partial g}{\partial \lambda_i} > 0$, $\forall i = 1, \ldots, n$ and g is concave on Γ_k. Therefore $f = -g^{-(k-\ell)}$ satisfies the condition (1-3) spelled out in §1. Moreover, f is in the bounded case with

$$f_\infty(\lambda') = -\frac{\ell\sigma_{\ell-1}(\lambda')}{k\sigma_{k-1}(\lambda'} \quad \text{where} \quad \lambda' \in \Gamma_\infty = \Gamma_{k-1}.$$

We can assume that $u_0 = 0$ by replacing χ (resp. u and $\underline{u}$) by $\chi + i\partial\bar{\partial}u_0$ (resp. $u - u_0$ and $\underline{u} - u_0$). The inequality (4.23) infers that $\underline{u}$ is a subsolution of the equation (4.24). Indeed, for any $(z,t) \in X \times [0,\infty)$, set $\mu = \lambda(B)$, $B^i{}_j = \alpha^{j\bar{k}}(\chi_{\bar{k}j} + \underline{u}_{\bar{k}j})(z,t)$. Since $\underline{u}$ is independent of t, it follows from Lemma 8 and the symmetry of f that we just need to show that for any $z \in X$ if $\mu' = (\mu_1, \cdots, \mu_{n-1})$ then

$$\lim_{s\to\infty} f(\mu', \mu_n + s) > -c. \tag{4.27}$$

This means

$$f_\infty(\mu') = -\frac{\ell\sigma_{\ell-1}(\mu')}{k\sigma_{k-1}(\mu')} > -c. \tag{4.28}$$

As in [50], we restrict to the tangent space of X spanned by by the eigenvalues corresponding to μ'. Then on this subspace

$$\sigma_j(\mu') = \frac{\chi^{j-1} \wedge \alpha^{n-j}}{\alpha^{n-1}} \tag{4.29}$$

for all j. Thus the preceding inequality is equivalent to

$$kc(\chi')^k \wedge \alpha^{n-k} - \ell(\chi')^{\ell-1} \wedge \alpha^{n-\ell} > 0. \tag{4.30}$$

By a priori estimates in Section 2, the solution exists for all times. We now use the second statement in Lemma 7 to prove the convergence. It suffices to check that u is uniformly bounded in $X \times [0, +\infty)$ and for all $t > 0$, there exists y such that $\partial_t u(y,t) = 0$. The second condition is straightforward since

$$\int_X \partial_t u \chi_u^k \wedge \alpha^{n-k} = 0.$$

For the uniform bound we make use of the following lemma

Lemma 9 *Let $\phi \in C^\infty(X)$ function and $\{\varphi_s\}_{s\in[0,1]}$ be a path with $\varphi(0) = 0$ and $\varphi(1) = \phi$. Then we have*

$$\int_0^1 \int_X \frac{\partial\varphi}{\partial s}\chi_\varphi^k \wedge \alpha^{n-k} ds = \frac{1}{k+1}\sum_{j=0}^{k} \int_X \phi\chi_\phi^j \wedge \chi^{k-j} \wedge \alpha^{n-k}, \tag{4.31}$$

so the left hand side is independent of φ. Therefore we can define the following functional

$$I_k(\phi) = \int_0^1 \int_X \frac{\partial\varphi}{\partial s}\chi_\varphi^k \wedge \alpha^{n-k} ds. \tag{4.32}$$

We remark that when $k = n$ and χ is Kähler, this functional is well-known (see for instance [64]). We discuss here the general case.

Proof of Lemma 9. Observe that

$$\int_0^1 \int_X \frac{\partial \varphi}{\partial s} \chi_\varphi^k \wedge \alpha^{n-k} ds = \sum_{j=1}^k \binom{k}{j} \int_0^1 \int_X \frac{\partial \varphi}{\partial s} (i\partial\bar{\partial}\varphi)^j \wedge \chi^{k-j} \wedge \alpha^{n-k} ds. \tag{4.33}$$

For any $j = 0, \ldots, k$ we have

$$\begin{aligned} \int_0^1 \int_X \frac{\partial \varphi}{\partial s} (i\partial\bar{\partial}\varphi)^j \wedge \chi^{k-j} \wedge \alpha^{n-k} ds &= \int_0^1 \frac{d}{ds} \left(\int_X \varphi (i\partial\bar{\partial}\varphi)^j \wedge \chi^{k-j} \wedge \alpha^{n-k} \right) ds \\ &\quad - \int_0^1 \int_X \varphi \frac{\partial}{\partial s} \left((i\partial\bar{\partial}\varphi)^j \wedge \chi^{k-j} \wedge \alpha^{n-k} \right) ds \\ &= \int_X \phi (i\partial\bar{\partial}\phi)^j \wedge \chi^{k-j} \wedge \alpha^{n-k} \\ &\quad - \int_0^1 \int_X \varphi \frac{\partial}{\partial s} \left((i\partial\bar{\partial}\varphi)^j \wedge \chi^{k-j} \wedge \alpha^{n-k} \right) ds \end{aligned} \tag{4.34}$$

We also have

$$\begin{aligned} \int_0^1 \int_X \varphi \frac{\partial}{\partial s} \left((i\partial\bar{\partial}\varphi)^j \wedge \chi^{k-j} \wedge \alpha^{n-k} \right) ds &= \int_0^1 \int_X j\varphi \left(i\partial\bar{\partial} \frac{\partial \varphi}{\partial s} \right) \wedge (i\partial\bar{\partial}\varphi)^{j-1} \wedge \chi^{k-j} \wedge \alpha^{n-k} ds \\ &= \int_0^1 \int_X j \frac{\partial \varphi}{\partial s} (i\partial\bar{\partial}\varphi)^j \wedge \chi^{k-j} \wedge \alpha^{n-k} ds, \end{aligned} \tag{4.35}$$

here we used in the second identity the integration by parts and the fact that χ and α are closed. Combining (4.34) and (4.35) yields

$$\int_0^1 \int_X \frac{\partial \varphi}{\partial s} (i\partial\bar{\partial}\varphi)^j \wedge \chi^{k-j} \wedge \alpha^{n-k} ds = \frac{1}{j+1} \int_X \phi (i\partial\bar{\partial}\phi)^j \wedge \chi^{k-j} \wedge \alpha^{n-k}. \tag{4.36}$$

Therefore (4.33) implies that

$$\begin{aligned} \int_0^1 \int_X \frac{\partial \varphi}{\partial s} \chi_\varphi^k \wedge \alpha^{n-k} ds &= \sum_{j=1}^k \binom{k}{j} \frac{1}{j+1} \int_X \phi (i\partial\bar{\partial}\phi)^j \wedge \chi^{k-j} \wedge \alpha^{n-k} \\ &= \sum_{j=1}^k \binom{k}{j} \frac{1}{j+1} \int_X \phi (\chi_\phi - \chi)^j \wedge \chi^{k-j} \wedge \alpha^{n-k} \\ &= \sum_{j=1}^k \binom{k}{j} \frac{1}{j+1} \int_X \sum_{p=0}^j \binom{j}{p} (-1)^{j-p} \phi \chi_\phi^p \wedge \chi^{k-p} \wedge \alpha^{n-k} \\ &= \sum_{p=0}^k \left(\sum_{j=p}^k \binom{k}{j} \frac{1}{j+1} \binom{j}{p} (-1)^{j-p} \right) \int_X \phi \chi_\phi^p \wedge \chi^{k-p} \wedge \alpha^{n-k}. \end{aligned} \tag{4.37}$$

By changing $m = j - p$, we get

$$\sum_{j=p}^{k} \binom{k}{j} \frac{1}{j+1} \binom{j}{p} (-1)^{j-p} = \binom{k}{p} \sum_{m=0}^{k-p} \frac{(-1)^m}{m+p+1} \binom{k-p}{m}. \tag{4.38}$$

The right hand side can be computed by

$$\begin{aligned} \binom{k}{p} \sum_{m=0}^{k-p} \frac{(-1)^m}{m+p+1} \binom{k-p}{m} &= \binom{k}{p} \int_0^1 (1-x)^{k-p} x^p dx \\ &= \binom{k}{p} p! \int_0^1 \frac{1}{(k-p+1)\dots k} (1-x)^k dx = \frac{1}{k+1}, \end{aligned}$$

where we used the integration by parts p times in the second identity. Combining this with (4.37) and (4.38) we get the desired identity (4.31). Q.E.D.

We now have for any $t^* > 0$, along the flow

$$I_k(u(t^*)) = \int_0^{t^*} \int_X \frac{\partial u}{\partial t} \chi_u^k \wedge \alpha^{n-k} = \int_0^{t^*} \left(c - \frac{\chi_u^\ell \wedge \alpha^{n-\ell}}{\chi_u^k \wedge \alpha^{n-k}} \right) \chi_u^k \wedge \alpha^{n-k} = 0.$$

As in Weinkove [63, 64], there exist $C_1, C_2 > 0$ such that for all $t \in [0, \infty)$,

$$0 \leq \sup_X u(., t) \leq -C_1 \inf_X u(., t) + C_2. \tag{4.39}$$

Indeed, in view of (4.31), $I_k(u) = 0$ along the flow implies that

$$\sum_{j=0}^{k} \int_X u \chi_u^j \wedge \chi^{k-j} \wedge \alpha^{n-k} = 0, \tag{4.40}$$

hence $\sup_X u \geq 0$ and $\inf_X u \leq 0$. For the right inequality in (4.39), we remark that there exists a positive constant B such that

$$\alpha^n \leq B \chi^k \wedge \alpha^{n-k}.$$

Therefore combining with (4.40) gives

$$\begin{aligned} \int_X u \alpha^n &= \int_X (u - \inf_X u) \alpha^n + \int_X \inf_X u \, \alpha^n \\ &\leq B \int_X (u - \inf_X u) \chi^k \wedge \alpha^{n-k} + \inf_X u \int_X \alpha^n \\ &= -B \sum_{j=1}^{k} \int_X u \chi_u^j \wedge \chi^{k-j} \wedge \alpha^{n-k} + \inf_X u \left(\int_X \alpha^n - B \int_X \chi^k \wedge \alpha^{n-k} \right) \\ &= -B \sum_{j=1}^{k} \int_X \left(u - \inf_X u \right) \chi_u^j \wedge \chi^{k-j} \wedge \alpha^{n-k} + \inf_X u \left(\int_X \alpha^n - B(k+1) \int_X \chi^k \wedge \alpha^{n-k} \right) \\ &\leq \inf_X u \left(\int_X \alpha^n - B(k+1) \int_X \chi^k \wedge \alpha^{n-k} \right) = -C_1 \inf_X u. \end{aligned}$$

Since $\Delta_\alpha u \geq -\mathrm{tr}_\alpha \chi \geq -A$, using the fact that the Green's function $G(.,.)$ of α is bounded from below we infer that

$$\begin{aligned} u(x,t) &= \int_X u\alpha^n - \int_X \Delta_\alpha u(y,t) G(x,y)\alpha^n(y) \\ &\leq -C_1 \inf_X u + C_2. \end{aligned}$$

Hence we obtain the Harnack inequality, $\sup_X u \leq -C_1 \inf_X u + C_2$.

Since we can normalize $\underline{u}$ by $\sup_X \underline{u} = 0$, the left inequality in (4.40) implies

$$\sup_X (u(\cdot,t) - \underline{u}(\cdot,t)) \geq 0.$$

It follows from Lemma 1 that

$$u \geq \underline{u} - C_3$$

for some constant C_3. This give a lower bound for u since $\underline{u}$ is bounded. The Harnack inequality in (4.39) implies then a uniform bound for u. Now the second statement in Lemma 7 implies the convergence of u. Q.E.D.

A natural generalization of the Hessian quotient flows on Hermitian manifolds is the following flow

$$\partial_t u = \log \frac{\chi_u^k \wedge \alpha^{n-k}}{\chi_u^\ell \wedge \alpha^{n-\ell}} - \psi \tag{4.41}$$

where $\psi \in C^\infty(X)$, the admissible cone is Γ_k, $1 \leq \ell < k \leq n$, and $\chi_u = \chi + i\partial\bar\partial u$. This flow was introduced by Sun [44] when $k = n$. We can apply Theorem 2 to obtain the following result, which is analogous to one of the main results in Sun [44], and analogous to the results of Song-Weinkove [42] and Fang-Lai-Ma [12] for $k = n$:

Corollary 6 *Let (X,α) be a compact Hermitian manifold and χ be a $(1,1)$-form which is k-positive. Assume that there exists a form $\chi' = \chi + i\partial\bar\partial \underline{u}$ which is k-positive, and satisfies*

$$k\,(\chi')^{k-1} \wedge \alpha^{n-k} - e^\psi\, \ell (\chi')^{\ell-1} \wedge \alpha^{n-\ell} > 0 \tag{4.42}$$

in the sense of the positivity of $(n-1,n-1)$-forms. Assume further that there exists an admissible $u_0 \in C^\infty(X)$ satisfying

$$e^\psi \geq \frac{\chi_{u_0}^k \wedge \alpha^{n-k}}{\chi_{u_0}^\ell \wedge \alpha^{n-\ell}} \tag{4.43}$$

Then the flow (4.41) admits a smooth solution for all time with initial data u_0. Furthermore, there exists a unique constant c so that the normalization

$$\tilde u = u - \frac{1}{[\alpha^n]} \int_X u\alpha^n \tag{4.44}$$

converges in C^∞ to a function u_∞ with $\omega_\infty = \chi + i\partial\bar\partial u_\infty$ satisfying

$$\omega_\infty^k \wedge \alpha^{n-k} = e^{\psi + c} \omega_\infty^\ell \wedge \alpha^{n-\ell}. \tag{4.45}$$

Proof of Corollary 6. This equation is of the form (1.1), with

$$F(A) = f(\lambda) = \log \frac{\sigma_k(\lambda)}{\sigma_\ell(\lambda)}, \text{ with } \lambda = \lambda(A), \tag{4.46}$$

defined on Γ_k. As in the proof of Corollary 5 we also have that f satisfies the conditions (1-3) mentioned in §1. Moreover, f is in the bounded case with

$$f_\infty(\lambda') = \log \frac{k\sigma_{k-1}(\lambda')}{\ell\sigma_{\ell-1}(\lambda')} \quad \text{where} \quad \lambda' \in \Gamma_\infty = \Gamma_{k-1}.$$

It suffices to verify that $\underline{u} = 0$ is a subsolution of the equation (4.41). For any $(z,t) \in X \times [0,\infty)$, set $\mu = \lambda(B)$, $B^i{}_j = \alpha^{j\bar{k}}\chi_{\bar{k}j}(z,t)$. Since $\underline{u}$ is independent of t, Lemma 8 implies that we just need to show that for any $z \in X$ if $\mu' = (\mu_1, \cdots, \mu_{n-1})$,

$$\lim_{s\to\infty} f(\mu', s) > \psi(z). \tag{4.47}$$

This means

$$f_\infty(\mu') = \log \frac{k\sigma_{k-1}(\mu')}{\ell\sigma_{\ell-1}(\mu')} > \psi(z), \tag{4.48}$$

where we restrict to the tangent space of X spanned by by the eigenvalues corresponding to μ'. As the argument in the proof of Corollary 5, this inequality is equivalent to

$$k\chi^k \wedge \alpha^{n-k} - \ell e^\psi \chi^{\ell-1} \wedge \alpha^{n-\ell} > 0. \tag{4.49}$$

Moreover, the condition (4.43) is equivalent to

$$0 = \underline{u} \geq F(A[u_0]) - \psi. \tag{4.50}$$

We can now apply Theorem 2 to complete the proof. Q.E.D

In the case of (X,α) compact Kähler, the condition on ψ can be simplified, and we obtain an alternative proof to the main result of Sun in [45]. We recently learnt that Sun [49] also provided independently another proof of [45] using the same flow as below:

Corollary 7 *Let (X,α) be Kähler and χ be a k-positive closed $(1,1)$-form. Assume that there exists a closed form $\chi' = \chi + i\partial\bar{\partial}\underline{u}$ which is k-positive, and satisfies*

$$k\,(\chi')^{k-1} \wedge \alpha^{n-k} - e^\psi\,\ell(\chi')^{\ell-1} \wedge \alpha^{n-\ell} > 0 \tag{4.51}$$

in the sense of the positivity of $(n-1,n-1)$-forms. Assume further that

$$e^\psi \geq c_{k,\ell} = \frac{[\chi^k] \cup [\chi^{n-k}]}{[\chi^\ell] \cup [\alpha^{n-\ell}]}. \tag{4.52}$$

Then for any admissible initial data $u_0 \in C^\infty(X)$, the flow (4.41) admits a smooth solution for all time. Furthermore, there exists a unique constant c so that the normalization

$$\tilde{u} = u - \frac{1}{[\alpha^n]} \int_X u\alpha^n \tag{4.53}$$

converges in C^∞ to a function u_∞ with $\omega_\infty = \chi + i\partial\bar{\partial} u_\infty$ satisfying

$$\omega_\infty^k \wedge \alpha^{n-k} = e^{\psi + c} \omega_\infty^\ell \wedge \alpha^{n-\ell}. \tag{4.54}$$

Proof of Corollary 6. By the same argument above, the admissible function $\underline{u} \in C^\infty(X)$ with $\sup_X \underline{u} = 0$ satisfying (4.51) is a C-subsolution. As explained in the proof of Corollary 5, we can assume that $u_0 = 0$.

We first observe that along the flow, the functional I_ℓ defined in Lemma 9 is decreasing. Indeed, using Jensen's inequality and then (4.52) we have

$$\begin{aligned} \frac{d}{dt} I_\ell(u) &= \int_X \frac{\partial u}{\partial t} \chi_u^\ell \wedge \alpha^{n-\ell} = \int_X \left(\log \frac{\chi_u^k \wedge \alpha^{n-k}}{\chi_u^\ell \wedge \alpha^{n-\ell}} - \psi \right) \chi_u^\ell \wedge \alpha^{n-\ell} \\ &\leq \log c_{k,\ell} \int_X \chi_u^\ell \wedge \alpha^{n-\ell} - \int_X \psi \chi_u^\ell \wedge \alpha^{n-\ell} \leq 0. \end{aligned} \tag{4.55}$$

Set

$$\hat{u} := u - h(t), \qquad h(t) = \frac{I_\ell(u)}{\int_X \chi^\ell \wedge \alpha^{n-\ell}}. \tag{4.56}$$

For any $t^* \in [0, \infty)$ we have

$$I_\ell(\hat{u}(t^*)) = \int_0^{t^*} \int_X \frac{\partial \hat{u}}{\partial t} \chi_u^\ell \wedge \alpha^{n-\ell} = \int_0^{t^*} \int_X \left(\frac{\partial u}{\partial t} - \frac{1}{\int_X \chi^\ell \wedge \alpha^{n-\ell}} \frac{d}{dt} I_\ell(u) \right) \chi_u^\ell \wedge \alpha^{n-\ell} = 0.$$

By the same argument in Corollary 5, we deduce that there exist $C_1, C_2 > 0$ such that

$$0 \leq \sup_X \hat{u}(., t) \leq -C_1 \inf_X \hat{u}(., t) + C_2, \tag{4.57}$$

for all $t \in [0, \infty)$. By our choice, $\sup_X \underline{u} = 0$, and (4.57) implies that

$$\sup_X (u - h(t) - \underline{u}) = \sup_X (\hat{u} - \underline{u}) \geq 0, \ \forall t \geq 0.$$

Since $I_\ell(u)$ is decreasing along the flow, we also have $h'(t) \leq 0$. Theorem 2 now gives us the required result. Q.E.D.

Similarly, we can consider the flow (1.1) with

$$\partial_t u = -\left(\frac{\chi_u^\ell \wedge \alpha^{n-\ell}}{\chi_u^k \wedge \alpha^{n-k}} \right)^{\frac{1}{k-\ell}} + \psi(z), \quad u(z,0) = 0, \tag{4.58}$$

where $1 \leq \ell < k \leq n$. When (X, α) is Kähler, ψ is constant and $k = n$, this is the inverse Hessian flow studied by Fang-Lai-Ma [12]. We can apply Theorem 2 to obtain another corollary which is analogous to the main result of Fang-Lai-Ma [12].

Corollary 8 *Let (X,α), and χ as in Corollary 6. Assume further that $\psi \in C^\infty(X, \mathbf{R}^+)$ and there exists a smooth function $\underline{u}$ with $\chi' = \chi + i\partial\bar\partial \underline{u}$ a k-positive $(1,1)$-form which satisfies*

$$k\psi^{k-\ell}(\chi')^{k-1}\wedge\alpha^{n-k} - \ell(\chi')^{n-\ell-1}\wedge\alpha^{n-\ell} > 0 \tag{4.59}$$

in the sense of positivity of $(n-1,n-1)$ forms, and

$$\psi^{k-\ell} \leq \frac{\chi^\ell\wedge\alpha^{n-\ell}}{\chi^k\wedge\alpha^{n-k}}. \tag{4.60}$$

Then the flow (4.58) exists for all time, and there is a unique constant c so that the normalized function $\tilde u$ converges to a function u_∞ with $\omega = \chi + i\partial\bar\partial u_\infty$ a k-positive form satisfying the equation

$$\omega^{n-\ell}\wedge\alpha^{n-\ell} = (\psi + c)^{k-\ell}\omega^k\wedge\alpha^{n-k}. \tag{4.61}$$

In particular, if (X,α) is Kähler, we assume further that χ is closed, then the condition (4.60) can be simplified as

$$\psi^{k-\ell} \leq c_{\ell,k} = \frac{[\chi^\ell]\cup[\alpha^{n-\ell}]}{[\chi^k]\cup[\alpha^{n-k}]}. \tag{4.62}$$

Proof of Corollary 8. This equation is of the form (1.1), with

$$F(A) = f(\lambda) = -\left(\frac{\sigma_\ell(\lambda)}{\sigma_k(\lambda)}\right)^{\frac{1}{k-\ell}}, \text{ with } \lambda = \lambda(A), \tag{4.63}$$

defined on Γ_k. As in Corollary 5, it follows from the Maclaurin's inequality, the monotonicity and concavity of $g = (\sigma_k/\sigma_\ell)^{\frac{1}{k-\ell}}$ (cf. [43]) that f satisfies the conditions (1-3) spelled out in §1. Moreover, f is in the bounded case with

$$f_\infty(\lambda') = -\left(\frac{\ell\sigma_{\ell-1}(\lambda')}{k\sigma_{k-1}(\lambda')}\right)^{\frac{1}{k-\ell}} \quad \text{where} \quad \lambda' \in \Gamma_\infty = \Gamma_{k-1}.$$

In addition, as the same argument in previous corollaries, the condition (4.59) is equivalent to that $\underline{u} = 0$ is a C-subsolution for (4.58). Moreover, the condition (4.60) implies that

$$0 = \underline{u} \geq F(A[0]) + \psi. \tag{4.64}$$

We can now apply Theorem 2 to get the first result.

Next, assume that (X,α) is Kähler and χ is closed. As in Corollary 7 and [12], the functional I_ℓ (see Lemma 9) is decreasing along the flow. Indeed, using (4.62),

$$\begin{aligned}\frac{d}{dt}I_\ell(u) &= \int_X \frac{\partial u}{\partial t}\chi_u^\ell\wedge\alpha^{n-\ell} = \int_X\left(-\left(\frac{\sigma_\ell(\lambda)}{\sigma_k(\lambda)}\right)^{\frac{1}{k-\ell}} + \psi\right)\chi_u^\ell\wedge\alpha^{n-\ell} \\ &\leq -\int_X\left(\frac{\sigma_\ell(\lambda)}{\sigma_k(\lambda)}\right)^{\frac{1}{k-\ell}}\chi_u^\ell\wedge\alpha^{n-\ell} + c_{\ell,k}^{\frac{1}{k-\ell}}\int_X\chi_u^\ell\wedge\alpha^{n-\ell}. \end{aligned}\tag{4.65}$$

Using the Hölder inequality, we get

$$\begin{aligned}
\int_X \chi_u^\ell \wedge \alpha^{n-\ell} &= \int_X \sigma_\ell\, \alpha^n = \int_X \left(\frac{\sigma_\ell}{\sigma_k^{1/(k-\ell+1)}}\right) \sigma_k^{\frac{1}{k-\ell+1}}\, \alpha^n \\
&\leq \left[\int_X \left(\frac{\sigma_\ell}{\sigma_k^{1/(k-\ell+1)}}\right)^{\frac{k-\ell+1}{k-\ell}} \alpha^n\right]^{\frac{k-\ell}{k-\ell+1}} \left(\int_X \sigma_k\, \alpha^n\right)^{\frac{1}{k-\ell+1}} \\
&= \left[\int_X \left(\frac{\sigma_\ell(\lambda)}{\sigma_k(\lambda)}\right)^{\frac{1}{k-\ell}} \chi_u^\ell \wedge \alpha^{n-\ell}\right]^{\frac{k-\ell}{k-\ell+1}} \left(\int_X \chi_u^k \wedge \alpha^{n-k}\right)^{\frac{1}{k-\ell+1}} \\
&= \left[\int_X \left(\frac{\sigma_\ell(\lambda)}{\sigma_k(\lambda)}\right)^{\frac{1}{k-\ell}} \chi_u^\ell \wedge \alpha^{n-\ell}\right]^{\frac{k-\ell}{k-\ell+1}} c_{\ell,k}^{\frac{-1}{k-\ell+1}} \left(\int_X \chi_u^\ell \wedge \alpha^{n-\ell}\right)^{\frac{1}{k-\ell+1}}.
\end{aligned}$$

This implies that

$$c_{\ell,k}^{\frac{1}{k-\ell}} \int_X \chi_u^\ell \wedge \alpha^{n-\ell} \leq \int_X \left(\frac{\sigma_\ell(\lambda)}{\sigma_k(\lambda)}\right)^{\frac{1}{k-\ell}} \chi_u^\ell \wedge \alpha^{n-\ell},$$

hence $dI_\ell(u)/dt \leq 0$.

For the rest of the proof, we follow the argument in Corollary 7, starting from the fact that $I_\ell(\hat{u}) = 0$ where

$$\hat{u} = u - \frac{I_\ell(u)}{\int_X \chi^\ell \wedge \alpha^{n-\ell}}.$$

Then we obtain the Harnack inequality

$$0 \leq \sup_X \hat{u}(.,t) \leq -C_1 \inf_X \hat{u}(.,t) + C_2, \tag{4.66}$$

for some constants $C_1, C_2 > 0$. Finally, Theorem 2 gives us the last claim. Q.E.D.

4.6 Flows with mixed Hessians σ_k

Our method can be applied to solve other equations containing many terms of σ_k. We illustrate this with the equation

$$\sum_{j=1}^{\ell} c_j \chi_u^j \wedge \alpha^{n-j} = c\chi_u^k \wedge \alpha^{n-k} \tag{4.67}$$

on a Kähler manifold (X, α), where $1 \leq \ell < k \leq n$, $c_j \geq 0$ are given non-negative constants, and $c \geq 0$ is determined by c_j by integrating the equation over X.

When $k = n$, It was conjectured by Fang-Lai-Ma [12] that this equation is solvable assuming that

$$nc\chi'^{n-1} - \sum_{k=1}^{n-1} kc_k \chi'^{k-1} \wedge \alpha^{n-k} > 0,$$

for some closed k-positive form $\chi' = \chi + i\partial\bar{\partial}v$. This conjecture was solved recently by Collins-Székelyhidi [7] using the continuity method. An alternative proof by flow methods is in Sun [48]. Theorem 3 stated earlier in the Introduction is an existence result for more general equations (4.67) using the flow (1.12) In particular, it gives a parabolic proof of a generalization of the conjecture due to Fang-Lai-Ma [12, Conjecture 5.1]. We also remark that the flow (1.12) was mentioned in Sun [44], but no result given there, to the best of our understanding.

Proof of Theorem 3. This equation is of the form (1.1), with

$$F(A) = f(\lambda) = -\frac{\sum_{j=1}^{\ell} c_j \sigma_j(\lambda)}{\sigma_k(\lambda)} + c,$$

defined on the cone Γ_k. As in the proof of Corollary 5, for any $j = 1, \ldots, \ell$, the function $-\sigma_j/\sigma_k$ on Γ_k satisfies the conditions (1-3) in §1, so does f. We also have that f is in the bounded case with

$$f_\infty(\lambda') = -\frac{\sum_{j=1}^{\ell} j c_j \sigma_{j-1}(\lambda)}{k\sigma_{k-1}(\lambda)} \quad \text{where} \quad \lambda' \in \Gamma_\infty = \Gamma_k.$$

Suppose $\chi' = \chi + i\partial\bar{\partial}\underline{u}$ with $\sup_X \underline{u} = 0$ satisfies

$$kc(\chi')^{k-1} \wedge \alpha^{n-k} - \sum_{j=1}^{\ell} j c_j (\chi')^{j-1} \wedge \alpha^{n-j} > 0.$$

By the same argument in Corollary 5, this is equivalent to that $\underline{u}$ is a C-subsolution of (1.12). Observe that for all $t^* > 0$,

$$\begin{aligned} I_k(u(t^*)) &= \int_0^{t^*} \int_X \frac{\partial u}{\partial t} \chi_u^k \wedge \alpha^{n-k} = \int_0^{t^*} \int_X \left(c - \frac{\sum_{j=1}^{\ell} c_j \sigma_j(\lambda)}{\sigma_k(\lambda)} \right) \chi_u^k \wedge \alpha^{n-k} \\ &= \int_0^{t^*} \left(c \int_X \chi_u^k \wedge \alpha^{n-k} - \sum_{j=1}^{\ell} c_j \int_X \chi_u^j \wedge \alpha^{n-j} \right) = 0. \end{aligned} \tag{4.68}$$

Therefore Lemma 9 implies that

$$\sum_{j=0}^{k} \int_X u \chi_u^j \wedge \chi^{k-j} \wedge \alpha^{n-k} = 0.$$

Therefore we can obtain the Harnack inequality as in Corollary 5:

$$0 \leq \sup_X u(.,t) \leq -C_1 \inf u(.,t) + C_2, \tag{4.69}$$

and $\inf_X u < 0$, for some positive constants C_1, C_2. Lemma 1 then gives a uniform bound for u. Since

$$\int_X \partial_t u \chi_u^k \wedge \alpha^{n-k} = 0,$$

for any $t > 0$, there exists $y = y(t)$ such that $\partial_t u(y,t) = 0$. The rest of the proof is the same to the proof of Corollary 5 where we used Lemma 7 to imply the convergence of the flow. Q.E.D.

We observe that equations mixing several Hessians seem to appear increasingly frequently in complex geometry. A recent example of particular interest is the Fu-Yau equation [14, 15, 32, 35] and its corresponding geometric flows [36].

4.7 Concluding Remarks

We conclude with a few open questions.

It has been conjectured by Lejmi and Székelyhidi [28] that conditions of the form (4.22) and their generalizations can be interpreted as geometric stability conditions. This conjecture has been proved in the case of the J-flow on toric varieties by Collins and Székelyhidi [7]. Presumably there should be similar interpretations in terms of stability of the conditions formulated in the previous section. A discussion of stability conditions for constant scalar curvature Kähler metrics can be found in [40].

It would also be very helpful to have a suitable geometric interpretation of conditions such as the one on the initial data u_0. Geometric flows whose behavior may behave very differently depending on the initial data include the anomaly flows studied in [33], [38], [13].

For many geometric applications, it would be desirable to extend the theory of subsolutions to allow the forms χ and ψ to depend on time as well as on u and ∇u.

Acknowledgements: The second-named author would like to thank his thesis advisor, Vincent Guedj for his constant support and encouragement, and also Yuxin Ge for some useful discussions. This work was begun when the second-named author was visiting the department of Mathematics of Columbia University, and completed when the first-named author was visiting the Institut de Mathematiques of Toulouse. Both authors would like to thank these institutions for their hospitality. They would also like to thank Nguyen Van Hoang for a careful reading of the paper.

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